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Economic Analysis and Modelling Using Fisher Chain Data

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Abstract

Following the 1993 recommendation of the United Nations System of National Accounts, Statistics Canada has switched to using the Fisher chain formula as the official measure to record real expenditure-based GDP in the national accounts in May 2001. This eliminates the substitution-bias problem associated with the Laspeyres formula and yields more accurate measures of economic growth. However, with Fisher chain quantity data, aggregate levels do not equal the arithmetic sum of components. In this paper, we discuss the properties of Fisher indexes, explore various approximation formulas for aggregation and subtraction using Fisher chain data, and derive output and price growth decomposition formulas. We also propose solutions to the difficulties that arise when modelling capital stock-flow accumulation rules and inventory investment.

Sommaire

À la suite de la recommandation des Nations-Unis de 1993 concernant le système de comptabilité nationale, Statistique Canada a choisi d'utiliser la formule idéale de Fisher à titre de mesure officielle pour enregistrer le PIB fondé sur les dépenses dans les Comptes nationaux en mai 2001. Ce changement élimine le problème de préférence de substitutions associé à la formule Laspeyres et améliore la précision de la mesure de la croissance de l'activité économique. Cependant, avec la chaîne de données de Fisher, l'agrégat n'égale pas à la somme arithmétique de ses composantes. Dans le présent document, nous discuterons des propriétés des index de Fisher, nous explorerons les diverses formules d'approximation pour l'agrégation et la soustraction à l'aide de la chaîne de données de Fisher, et nous dériverons les formules de décomposition pour les taux de croissance agrégés et les taux d'inflation en contributions des éléments. Nous proposerons aussi des solutions pour les difficultés qui surviennent lors de la modélisation des règles d'accumulation du capital national, de la circulation des capitaux ainsi que des investissements dans les stocks.

1. Introduction

One of the most fundamental and difficult tasks of any statistical agency is to properly separate price changes from changes in the quantity produced when recording economic activities. Before 2001, Statistics Canada used the Laspeyres formula to record activities in the Canadian system of national accounts. In the Laspeyres approach a base year is chosen such that the values of production of all commodities are evaluated at the prices prevailing in the base year. The advantage of the Laspeyres-type fixed-weight formula is its simplicity and ease of interpretation. Simply adding and subtracting components allows one to define new aggregates.

The Laspeyres formula, however, does have drawbacks. Because statistical agencies only change the base year once every few years, distortions are introduced when rapid relative price changes occur. In 1993, the United Nations System of National Accounts recommended that member countries switch to using Fisher chain indexes that update price and quantity weights every period. The U.S. Department of Commerce began to implement this recommendation in 1996, followed by Statistics Canada in May 2001.

The Fisher chain formula eliminates the substitution-bias problem. However, aggregate levels no longer equal the arithmetic sum of components. In this paper, we discuss the properties of the Fisher chain formula and the challenges faced when using these data. We then propose solutions to these challenges.

2. The Laspeyres formula and substitution bias

The Laspeyres formula values quantities in terms of a fixed set of base year prices. This fixed-weight measure represents the value of output at period t as if all prices had remained at their base-year levels.

This has many advantages. The level of GDP is constructed as the sum of its components. This additive property allows one to create various sub-aggregates by simple aggregation or subtraction of the required components, and shares of components of output add up to unity.

The fixed-weight formula does have undesirable properties. Because prices are fixed at the base year, the measure of real GDP growth is base-year dependent. This problem is particularly acute when some components of GDP are experiencing rapid price changes. Typically, economic agents substitute towards goods and services whose prices are declining. The further one moves from the base year, the nominal share of such goods in total expenditure will fall further below the base-year fixed-weight share. Too great a weight will thus be assigned to goods and services with falling prices and rising demand, and too little to those with rising prices and falling demand. This is the well-known problem of "substitution bias" associated with fixed-weight formulae.[1]

During the 1990s, the substitution bias was exacerbated by large computer price declines and associated increases in real demand. Frequent changes in the base year can resolve part of the substitution bias problem. This would move current prices closer to those in the base year, and gives a more accurate estimation of GDP growth. Changing base years, however, could cause other problems. For example, in each rebasing exercise, the usual practice of Statistics Canada was to preserve the post-rebasing and pre-rebasing growth rates of GDP components. This, however, caused the level of the sum of the GDP components to differ from the level of the aggregate GDP prior to the latest base year. To remedy this, Statistics Canada introduced "adjustment entries" for final demand categories to satisfy the level adding-up constraint. These adjustment entries, however, had no economic content or interpretation. Also, although the history of the growth rates of GDP components was preserved, the history of the levels of these components changed every time the base year is changed. Frequent base year changes meant frequent changes to history.

The treatment of new products that are introduced between base years is another problem faced by the fixed-weight formula. It is difficult to evaluate price differences between existing and new products if the period between changes in the base year is too long. This can add to the substitution-bias problem if there are large changes in the prices of new products.

[1] See Annex 1 for an algebraic example of substitution bias.

3. Fisher chain formula

In May 2001, Statistics Canada switched to using Fisher chain formula as the official measure to record real expenditure-based GDP in the national accounts. The formula uses a Fisher index (G_t^F) with weights from two adjacent time periods to calculate the change in real GDP (and its components):

$$G_t^F = \sqrt{\frac{\sum p_{t-1} q_t}{\sum p_{t-1} q_{t-1}} \times \frac{\sum p_t q_t}{\sum p_t q_{t-1}}}\ , \tag{3.1}$$

where the p's and q's are prices and quantities of elemental components of GDP. Note that the first term of the Fisher index is a Laspeyres index which uses last period's prices as weights and the second term is a Paasche index which uses the current period's prices as weights. The Fisher index is thus a geometric average of the Laspeyres and the Paasche indexes.

Given the Fisher index of GDP growth of equation (3.1), the level of real GDP (Q^F) at time t is constructed as:

$$Q_t^F = Q_0 G_1^F G_2^F \cdots G_t^F\ . \tag{3.2}$$

Q_0 is the value of real GDP in the reference year. If real GDP is expressed as an quantity index, then Q_0 is set equal to 100 (or 1) in the reference year. If real GDP is expressed as chained dollar, then Q_0 is set equal to the reference-year nominal GDP. Equation (3.2) defines the level of real GDP by setting it equal to its value in the reference year and then "chaining" it backward and forward from the reference year using its growth rates.

Similar to equation (3.2), the price index of Q is calculated using the Fisher formula of:

$$\pi_t^F = \sqrt{\frac{\sum q_{t-1} p_t}{\sum q_{t-1} p_{t-1}} \times \frac{\sum q_t p_t}{\sum q_t p_{t-1}}}\quad . \tag{3.3}$$

The Fisher chain price index is then defined as

$$P_t^F = P_0 \pi_1^F \pi_2^F \cdots \pi_t^F , \tag{3.4}$$

where P_0 is the value of the index in the reference year, usually set equal to 100.

The Fisher formula has important advantages over fixed-weight formulae. Since the current value of the index depends only on the current and last-period prices and quantities, there is no base year problem and no substitution bias. The Fisher formula updates price and quantity movements in each period. This means that the weights and the contribution of components to GDP growth are revised regularly, eliminating the substitution bias associated with fixed-weight formulae. New products can be incorporated more easily under the Fisher formula and the estimation of their contribution to GDP growth is more accurate.

4. Some basic properties of the Fisher chain formula

(a) The product of a Fisher chain index (or chained dollar) and the Fisher chain price index at period t, after adjusting for base-period nominal value, is equal to its nominal value at period t.

To see this, using equations (3.2) and (3.4) to give

$$Q_t^F \times P_t^F = (Q_0 \times G_1^F \times .....\times G_t^F) \times (P_0 \times \pi_1^F \times .....\times \pi_t^F)$$

$$= Q_0 \times P_0 \times \prod_{i=1}^{t} G_i^F \pi_i^F$$

$$= Q_0 \times P_0 \times \prod_{i=1}^{t} \left(\frac{\sum p_i q_i}{\sum p_{i-1} q_{i-1}} \right) \quad \text{(using equations (3.1) and (3.3))}$$

$$= Q_0 \times P_0 \times \frac{\sum p_t q_t}{\sum p_0 q_0} \quad = Q_0 \times P_0 \times \frac{Q_t^N}{Q_0^N}$$

$$= \frac{Q_0 \times P_0}{Q_0^N} \times Q_t^N \quad . \tag{4.1}$$

where $Q_t^N = \sum p_t q_t$; Q_0^N is the nominal value at t = 0, the reference period. The only difference between whether Q_t^F is a quantity index or chained dollar series when writing equation (4.1) is the normalizing factor (the first term on the right-hand side of the equation). If Q_t^F is a quantity index, then Q_0 equals 100; if Q_t^F is a chained dollar series, then Q_0 equals the nominal value in the reference period.

Note that the properties of equations (4.1) work well for annual data where the values of Q_0 is either set equal to 100 or to its annual nominal value and P_0 is set equal to 100. This is not true for quarterly data where no reference quarter exists. Hence, we have to adjust the values of Q_0 and P_0 properly in order to preserve the properties of (4.1). This requires setting the values of Q_0 and P_0 such that the four-quarter average of Q_t^F in the reference year is either equal to 100 or its nominal value in the reference year, and P_t^F equals 100.

Assuming "time *0*" is defined as the first quarter of the reference year and all time subscripts are quarterly. For the case where Q_t^F is a chain index, we require:

$$\frac{(Q_0 + Q_1 + Q_2 + Q_3)}{4} = \frac{Q_0(1 + G_1^F + G_1^F G_2^F + G_1^F G_2^F G_3^F)}{4} = 100 \quad .$$

Hence, $$Q_0 = \frac{100}{(1 + G_1^F + G_1^F G_2^F + G_1^F G_2^F G_3^F)/4} \tag{4.2}$$

and $$P_0 = \frac{100}{(1 + \pi_1^F + \pi_1^F \pi_2^F + \pi_1^F \pi_2^F \pi_3^F)/4} \quad . \tag{4.3}$$

For the case where Q_t^F is a chained dollar series, we require:

$$Q_0 = \frac{Q_0^N + Q_1^N + Q_2^N + Q_3^N}{1 + G_1^F + G_1^F G_2^F + G_1^F G_2^F G_3^F} \tag{4.4}$$

(b) The product of the change in the Fisher chain quantity index (or chained dollar) and the change in the Fisher chain price index is equal to the change in its nominal value.

This follows directly from equations (3.1) and (3.3) since:

$$G_t^F \times \pi_t^F = \sqrt{\frac{\sum p_{t-1} q_t}{\sum p_{t-1} q_{t-1}} \times \frac{\sum p_t q_t}{\sum p_t q_{t-1}}} \times \sqrt{\frac{\sum p_t q_{t-1}}{\sum p_{t-1} q_{t-1}} \times \frac{\sum p_t q_t}{\sum p_{t-1} q_t}}$$

$$= \frac{\sum p_t q_1}{\sum p_{t-1} q_{t-1}} = \frac{Q_t^N}{Q_{t-1}^N} \quad . \tag{4.5}$$

The right-hand side of equation (4.5) gives the gross growth rate of nominal Q.

(c) The Fisher chain price index of a chain variable Q is also its implicit price deflator.

The implicit price deflator is defined as:

$$IPD_t^F = \frac{Q_t^N}{Q_t^F} \times IPD_0 \quad ; \tag{4.6}$$

that is, the implicit price deflator is the ratio between the nominal value and the chained dollar value, normalized by a scale factor IPD_0 so that the annual average value in the reference year is equal to 100.

From equation (4.1), we have

$$Q_t^F \times P_t^F = \frac{Q_0 \times P_0}{Q_0^N} \times Q_t^N \quad .$$

Therefore, to show that $IPD_t^F = P_t^F$, we only need to show

$$IPD_0 = \frac{Q_0 \times P_0}{Q_0^N} \quad . \tag{4.7}$$

For annual data, we require $IPD_0 = 100$. Since $Q_0 = Q_0^N$ and $P_0 = 100$, it is obvious that equation (4.7) holds.

For quarterly data, we require that the four-quarter average of IPD_t^F in the reference year equals 100, that is, assuming "time *0*" is defined as the first quarter of the reference year and all time subscripts are quarterly, we require

$$\frac{(IPD_0^F + IPD_1^F + IPD_2^F + IPD_3^F)}{4} = (\frac{Q_0^N}{Q_0^F} + \frac{Q_1^N}{Q_1^F} + \frac{Q_2^N}{Q_2^F} + \frac{Q_3^N}{Q_3^F}) \times \frac{IPD_0}{4} = 100 \tag{4.8}$$

or this is equivalent to show that

$$(\frac{Q_0^N}{Q_0^F} + \frac{Q_1^N}{Q_1^F} + \frac{Q_2^N}{Q_2^F} + \frac{Q_3^N}{Q_3^F}) \times \frac{Q_0 \times P_0}{Q_0^N} / 4$$

$$= (\frac{1}{Q_0^F / Q_0} + \frac{Q_1^N / Q_0^N}{Q_1^F / Q_0} + \frac{Q_2^N / Q_0^N}{Q_2^F / Q_0} + \frac{Q_3^N / Q_0^N}{Q_3^F / Q_0}) \times \frac{P_0}{4} = 100 \quad . \tag{4.9}$$

Using (3.2) and (4.5), we have

$$(\frac{1}{Q_0^F / Q_0} + \frac{Q_1^N / Q_0^N}{Q_1^F / Q_0} + \frac{Q_2^N / Q_0^N}{Q_2^F / Q_0} + \frac{Q_3^N / Q_0^N}{Q_3^F / Q_0}) \times \frac{P_0}{4}$$

$$= (1 + \frac{G_1^F \pi_1^F}{G_1^F} + \frac{(G_1^F \pi_1^F)(G_2^F \pi_2^F)}{G_1^F G_2^F} + \frac{(G_1^F \pi_1^F)(G_2^F \pi_2^F)(G_3^F \pi_3^F)}{G_1^F G_2^F G_3^F}) \times \frac{P_0}{4}$$

$$= (1 + \pi_1^F + \pi_1^F \pi_2^F + \pi_1^F \pi_2^F \pi_3^F) \times \frac{P_0}{4} = 100 \qquad \text{(by equation (4.3))}$$

which shows (4.9) and hence (4.7) hold.

(d) Non-additive: With Fisher formula, the quantity of an aggregate is not equal to the arithmetic sum of its components. Therefore, the ratio of a component to the aggregate does not represent its share in the aggregate.

With the fixed-weight Laspeyres formula, an aggregate is equal to the sum of its components. This is easy to see. Let Y be an aggregate consisting of two components C and I. Then $C_t + I_t \equiv \sum p_0^C q_t^C + \sum p_0^I q_t^I = \sum_{i=C,I} p_0^i q_t^i \equiv Y_t$.

This additive property does not apply under the Fisher chain formula. This is clear from the definition of equations (3.1) and (3.2) where the Fisher chain aggregate is not equal to the sum of its components. This property of Fisher chain quantity invalidates the concept of "real share". In fact, the sum of the ratios of the components to the aggregate is not equal to unity. To show this, consider an example where the ratio of a Fisher chain component, say, investment in the ICT sector to the aggregate total investment in M&E is given by

$$\frac{ICT_t}{ME_t} = \frac{ICT_0 \times G_1^{ICT} \times G_2^{ICT} \times \ldots\ldots \times G_t^{ICT}}{ME_0 \times G_1^{ME} \times G_2^{ME} \times \ldots\ldots \times G_t^{ME}}$$

$$= \frac{ICT_0}{ME_0} \cdot \frac{G_1^{ICT}}{G_1^{ME}} \cdot \frac{G_2^{ICT}}{G_2^{ME}} \cdots\cdots \frac{G_t^{ICT}}{G_t^{ME}} \quad . \tag{4.10}$$

ICT_0 and ME_0 are nominal values of ICT_t and ME_t in the reference period, and the G's are Fisher growth rates. Equation (4.10) shows that the ratio represents the accumulation of relative growth rates only and does not have the conventional "share" interpretation. We can see that if the growth rates of the component are greater than those of the aggregate, then the ratio of the component to the aggregate can be greater than unity. This means that the component can be greater than the aggregate. This property is very different from the fixed-weight Laspeyres formula. Under the Fisher chain formula, we can only rank the relative importance of the components by their relative contribution to the growth of the aggregate.

5. Fisher aggregation and subtraction

As noted earlier, a Fisher aggregate is not equal to the arithmetic sum of its components. However, very often when analysing economic events we need to create sub-aggregates by eliminating one or more components of an aggregate or to create broader aggregates by combining different aggregates. In this section, we discuss three methods to resolve the aggregation and subtraction problems when using Fisher chain data: the Laspeyres index approximation, the "Fisher of Fishers", and the Tornqvist index approximation. All three are designed to use growth rates of Fisher data together with equation (3.2) to derive new Fisher aggregate levels. We compare the aggregates created using these three methods to actual data to verify their accuracy. Note that since we are using higher-level sub-aggregate data in our demonstrations and not elemental data, no approximation method can exactly reproduce original Fisher chain data from Statistics Canada.

5.1 The Laspeyres approximation method

Aggregation

Suppose we want to create a Fisher aggregate Y consisting of components $X^1, X^2, \cdots, X^n$. Because of the non-additive property of the Fisher chain formula, we cannot create Y by simply adding up the Xs. However, we can approximate Y by using Laspeyres indices of both Y and the Xs.

Note that the nominal value of Y is given by $Y_t = X_t^1 + X_t^2 + \cdots + X_t^n$. Assume that each component X is made up of elemental components whose quantity is represented by q with price p. Then:

$$X_t^i = \sum p_t^i q_t^i \quad , \quad i = 1, 2, \ldots, n \,. \tag{5.1}$$

We can write the Laspeyres quantity index for the aggregate Y as:

$$G_t^{Y^L} = \frac{\sum p_{t-1} q_t}{\sum p_{t-1} q_{t-1}} \tag{5.2}$$

where $\sum p_t q_t$ represents the sum of all the elemental components that are contained in $X^1, X^2, \cdots, X^n$. The Laspeyres chain quantity index for *Y* is defined as:

$$Y_t^L = Y_0 \times G_1^{Y^L} \times \cdots \times G_t^{Y^L} \tag{5.3}$$

where Y_0 is a scaling factor. Similarly, for the components $X^1, X^2, \cdots, X^n$, we can write the Laspeyres quantity index for each of the *X* as

$$G_t^{i\,L} = \frac{\sum p_{t-1}^i q_t^i}{\sum p_{t-1}^i q_{t-1}^i} = \frac{\sum p_{t-1}^i q_t^i}{X_{t-1}^i}, \qquad i = 1, 2, \ldots, n\,. \tag{5.4}$$

Multiplying both sides of equation (5.4) by the last period nominal share of X^i in *Y* and then summing the Laspeyres quantity indices for all the *X's* to give:

$$\sum_{i=1}^{n} G_t^{i\,L} \times \frac{X_{t-1}^i}{Y_{t-1}} = \sum_{i=1}^{n} \frac{\sum p_{t-1}^i q_t^i}{X_{t-1}^i} \times \frac{X_{t-1}^i}{Y_{t-1}}$$

$$= \frac{1}{Y_{t-1}} \sum_{i=1}^{n} \left(\sum p_{t-1}^i q_t^i\right).$$

$$= \frac{\sum p_{t-1} q_t}{\sum p_{t-1} q_{t-1}} = G_t^{Y\,L}\ . \tag{5.5}$$

Using equation (3.1), we can write the Fisher quantity index of *Y* as:

$$G_t^{Y^F} = \sqrt{\frac{\sum p_{t-1} q_t}{\sum p_{t-1} q_{t-1}} \times \frac{\sum p_t q_t}{\sum p_t q_{t-1}}}\,. \tag{5.6}$$

If $p_t = p_{t-1}$ in equation (5.6), then $G_t^{Y^F} = G_t^{Y^L}$. Therefore, we can approximate the Fisher quantity index of equation (5.6) using equation (5.5), substitute the Laspeyres quantity indices of $G_t^{i^L}$ by $G_t^{i^F}$ to obtain:

$$G_t^{Y^F} = \sum_{i=1}^{n} G_t^{i^F} \times \frac{X_{t-1}^{i}}{Y_{t-1}} \quad . \tag{5.7}$$

Equation (5.7) is the formula based on the Laspeyres approximation to create a Fisher chain quantity index such that $Y = X_1 + X_2 + \cdots + X_n$. It expresses the growth rate of the aggregate as a weighted sum of the growth rates of its components, where the weights are previous-period nominal shares. If movements in component prices are not drastic from period to period, then the product of a component's growth and its lagged nominal share is a good approximation of the component's contribution to the real growth of aggregate Y.[2]

Subtraction

We can also use the Laspeyres approximation method to create new variables by excluding components from existing aggregates. For example, based on the Fisher aggregation formula of equation (5.7), we can derive a Fisher subtraction formula for a new variable X^1 such that $X^1 = Y - X^2 - X^3 - \cdots - X^n$.

To show this, we rearrange equation (5.7) to yield:

$$G_t^{1^F} \times \frac{X_{t-1}^{1}}{Y_{t-1}} = G_t^{Y^F} - \sum_{i=2}^{n} G_t^{i^F} \times \frac{X_{t-1}^{i}}{Y_{t-1}} \quad . \tag{5.8}$$

Thus the real growth of component X^1 is:

$$G_t^{1^F} = (G_t^{Y^F} - \sum_{i=2}^{n} G_t^{i^F} \times \frac{X_{t-1}^{i}}{Y_{t-1}}) / (\frac{X_{t-1}^{1}}{Y_{t-1}}) . \tag{5.9}$$

[2] See Landefeld and Parker (1997), and Landefeld, Parker and Triplett (1995). We will discuss the contribution-to-growth issue in detail in Section 6.

Equation (5.9) is the formula for Fisher chain quantity index subtraction. Once we calculate $G_t^{1^F}$, we can use equation (3.2) to chain backward and forward to create the level series for X_t^1. Since the sum of nominal shares of all components equals 1, at times it is easier to write equation (5.9) as:

$$G_t^{1^F} = (G_t^{Y^F} - \sum_{i=2}^{n} G_t^{i^F} \times \frac{X_{t-1}^i}{Y_{t-1}}) / (1 - \sum_{i=2}^{n} \frac{X_{t-1}^i}{Y_{t-1}}) . \tag{5.10}$$

There is yet another way to calculate $G_t^{1^F}$. Note that the algebraic subtraction of $X^1 = Y - X^2 - X^3 - \cdots - X^n$ is equivalent to the algebraic aggregation of $X^1 = Y + (-X^2) + (-X^3) + \cdots + (-X^n)$. We can thus directly use the Fisher aggregation formula of (5.7) to calculate $G_t^{1^F}$. That is,

$$G_t^{1^F} = G_t^{Y^F} \times \frac{Y_{t-1}}{X_{t-1}^1} + \sum_{i=2}^{n} G_t^{i^F} \times \left(\frac{-X_{t-1}^i}{X_{t-1}^1} \right) , \tag{5.11}$$

where the nominal aggregation $X_{t-1}^1 = Y_{t-1} + (-X_{t-1}^2) + \cdots + (-X_{t-1}^n) = Y_{t-1} - \sum_{i=2}^{n} X_{t-1}^i$. In fact, equation (5.11) is equivalent to equation (5.10). From equation (5.11), we have:

$$\begin{aligned} G_t^{1^F} &= G_t^{Y^F} \times \frac{Y_{t-1}}{X_{t-1}^1} + \sum_{i=2}^{n} G_t^{i^F} \times \frac{-X_{t-1}^i}{Y_{t-1}} \times \frac{Y_{t-1}}{X_{t-1}^1} \\ &= (G_t^{Y^F} - \sum_{i=2}^{n} G_t^{i^F} \times \frac{X_{t-1}^i}{Y_{t-1}}) \times \frac{Y_{t-1}}{X_{t-1}^1} \\ &= (G_t^{Y^F} - \sum_{i=2}^{n} G_t^{i^F} \times \frac{X_{t-1}^i}{Y_{t-1}}) / (\frac{X_{t-1}^1}{Y_{t-1}}) , \end{aligned}$$

which is exactly the same as equation (5.10).

5.2 The "Fisher of Fishers" approximation method

Aggregation

Another way to aggregate is to use the Fisher quantity index of equation (3.1) directly. Because most users of data do not have access to elemental data on price and quantity but only their sub-aggregates, this method is commonly known as the “Fisher of Fishers” method.

Consider the same example of Section 5.1 where we want to create a chain quantity index of Y consisting of components $X^1, X^2, \cdots, X^n$. For convenience, we introduce the following notations:

$P_t^i \equiv$ the Fisher chain price of X^i at time t, for $i = 1,2,\cdots n$;

$Q_t^i \equiv$ the Fisher chained-dollars of X^i at time t, for $i = 1,2,\cdots n$;

$G_t^i = \dfrac{Q_t^i}{Q_{t-1}^i}$ is the real growth rate of X^i at time t, for $i = 1,2,\cdots n$.

P_t^i and Q_t^i are properly normalized so that the product of price and quantity is equal to its nominal value, that is,

$$X_t^i = P_t^i Q_t^i \ ; \qquad i = 1,2,\cdots n. \tag{5.12}$$

Using equation (3.1), we can approximate the growth rate of Y by the "Fisher of Fishers" method:

$$G_t^Y = \sqrt{\frac{\sum P_{t-1}^i Q_t^i}{\sum P_{t-1}^i Q_{t-1}^i} \times \frac{\sum P_t^i Q_t^i}{\sum P_t^i Q_{t-1}^i}}\ . \tag{5.13}$$

Using the equality $X_t^i = P_t^i Q_t^i$, for $i = 1,2,\cdots n$, we can write equation (5.13) as

$$G_t^Y = \sqrt{\frac{\sum P_{t-1}^i Q_{t-1}^i (\frac{Q_t^i}{Q_{t-1}^i})}{\sum P_{t-1}^i Q_{t-1}^i} \times \frac{\sum P_t^i Q_t^i}{\sum P_t^i Q_t^i (\frac{Q_{t-1}^i}{Q_t^i})}}$$

$$= \sqrt{\frac{\sum X_{t-1}^i G_t^i}{\sum X_{t-1}^i} \times \frac{\sum X_t^i}{\sum X_t^i / G_t^i}} = \sqrt{\frac{\sum X_{t-1}^i G_t^i}{Y_{t-1}} \times \frac{Y_t}{\sum X_t^i / G_t^i}}$$

$$= \sqrt{\frac{\sum (\frac{X_{t-1}^i}{Y_{t-1}}) G_t^i}{\sum (\frac{X_t^i}{Y_t}) / G_t^i}} \tag{5.14}$$

where the term X^i / Y is the nominal share of the i th component. Equation (5.14) is the formula for Fisher aggregation based on "Fisher of Fishers" approximation.

Subtraction

Consider the problem of constructing the sub-aggregate $X^1 = Y - X^2 - X^3 - \cdots - X^n$. We can derive the “Fisher of Fishers” method for subtraction from the aggregation formula of (5.14). We can rewrite (5.14) as:

$$(G_t^Y)^2 = \frac{\sum (\frac{X_{t-1}^i}{Y_{t-1}}) G_t^i}{\sum (\frac{X_t^i}{Y_t}) / G_t^i} = \frac{(\frac{X_{t-1}^1}{Y_{t-1}}) G_t^1 + \sum_{i=2}^n (\frac{X_{t-1}^i}{Y_{t-1}}) G_t^i}{(\frac{X_t^1}{Y_t}) / G_t^1 + \sum_{i=2}^n (\frac{X_t^i}{Y_t}) / G_t^i}. \tag{5.15}$$

Equation (5.15) is a quadratic function of G_t^1 of the form:

$$a_t (G_t^1)^2 + b_t G_t^1 + c_t = 0$$

where

$$a_t = \frac{X^1_{t-1}}{Y_{t-1}} = 1 - \sum_{i=2}^{n} \frac{X^i_{t-1}}{Y_{t-1}}, \quad b_t = \sum_{i=2}^{n} (\frac{X^i_{t-1}}{Y_{t-1}}) G^i_t - (G^Y_t)^2 \sum_{i=2}^{n} (\frac{X^i_t}{Y_t}) / G^i_t, \quad c_t = -(\frac{X^1_t}{Y_t})(G^Y_t)^2 .$$

The economically feasible solution is the non-negative root of the quadratic equation given by

$$G^1_t = \frac{-b_t + \sqrt{b_t^{\,2} - 4a_t c_t}}{2a_t} . \tag{5.16}$$

Equation (5.16) is the same formula proposed by the Macroeconomic Advisers Inc. for Fisher subtraction.[3]

Using equation (5.16), however, is rather cumbersome. An alternative, and simpler, way to do the Fisher subtraction of $X^1 = Y - X^2 - X^3 - \cdots - X^n$ is to directly use the Fisher aggregation of equation (5.14) and treat the subtraction as $X_1 = Y + (-X_2) + (-X_3) + \cdots + (-X_n)$. That is,

$$G^1_t = \sqrt{\frac{(\frac{Y_{t-1}}{X^1_{t-1}}) G^Y_t + \sum_{i=2}^{n} (\frac{-X^i_{t-1}}{X^1_{t-1}}) G^i_t}{(\frac{Y_t}{X^1_t}) / G^Y_t + \sum_{i=2}^{n} (\frac{-X^i_t}{X^1_t}) / G^i_t}} . \tag{5.17}$$

Analytically, equation (5.17) is not equivalent to (5.16), but both are good approximations. However, equation (5.17) is much easier to use and interpret, and avoids problems that are commonly associated with quadratic equations.

The "Fisher of Fishers" aggregation and subtraction formulas are applicable only to non-negative economic variables but not to variables with both negative and positive values such as net exports and changes in inventories. The formulas based on Laspeyres approximation, however, are applicable in both cases.

[3] See Ben Herzon (2000). Note that when using Fisher chain sub-aggregates rather than the elemental components, this formula provides a good approximation but can never exactly reproduce the original real growth of the sub-aggregate X^1.

5.3 The Tornqvist approximation method

The third approximate method to address the Fisher aggregation and subtraction problems is based on the Tornqvist index.[4]

The Tornqvist Quantity Index is defined as

$$\ln\left(\frac{Y_t^{TQ}}{Y_{t-1}^{TQ}}\right) = \sum \frac{1}{2}\left(\frac{p_t q_t}{\sum p_t q_t} + \frac{p_{t-1} q_{t-1}}{\sum p_{t-1} q_{t-1}}\right) \times \ln\left(\frac{q_t}{q_{t-1}}\right) \ . \qquad (5.18)$$

It expresses the (log) growth rate of an aggregate as a weighted sum of the (log) growth rates of its components, where the weights are averages of nominal shares in the current and previous periods.

Based on the Tornqvist approximation of equation (5.18), the formula for Fisher aggregation of $Y_t = X_t^1 + X_t^2 + \cdots + X_t^n$ is given by:

$$G_t^{Y\,F} = \sum_{i=1}^{n} \frac{1}{2}\left(\frac{X_{t-1}^{i}}{Y_{t-1}} + \frac{X_t^{i}}{Y_t}\right) \times G_t^{i\,F} \qquad (5.19)$$

and the formula for Fisher subtraction $X^1 = Y - X^2 - X^3 - \cdots - X^n$ derived from the equation (5.19) is:

$$G_t^{1\,F} = \left(G_t^{Y\,F} - \sum_{i=2}^{n} \frac{1}{2}\left(\frac{X_{t-1}^{i}}{Y_{t-1}} + \frac{X_t^{i}}{Y_t}\right) \times G_t^{i\,F}\right) \Bigg/ \left(\frac{1}{2}\left(\frac{X_{t-1}^{1}}{Y_{t-1}} + \frac{X_t^{1}}{Y_t}\right)\right). \qquad (5.20)$$

Since $Y_t = \sum_{i=1}^{n} X_t^i$, the average nominal share of X^1 can be expressed as:

$$\frac{1}{2}\left(\frac{X_{t-1}^{1}}{Y_{t-1}} + \frac{X_t^{1}}{Y_t}\right) = 1 - \sum_{i=2}^{n} \frac{1}{2}\left(\frac{X_{t-1}^{i}}{Y_{t-1}} + \frac{X_t^{i}}{Y_t}\right) .$$

Therefore, we can write the subtraction formula of equation (5.20) as:

[4] See Whelan (2000).

$$G_t^{1F} = \left(G_t^{YF} - \sum_{i=2}^{n} \frac{1}{2} \left(\frac{X_{t-1}^i}{Y_{t-1}} + \frac{X_t^i}{Y_t} \right) \times G_t^{iF} \right) \Bigg/ \left(1 - \sum_{i=2}^{n} \frac{1}{2} \left(\frac{X_{t-1}^i}{Y_{t-1}} + \frac{X_t^i}{Y_t} \right) \right). \tag{5.21}$$

Again, an alternative formula for Fisher subtraction $X^1 = Y - X^2 - X^3 - \cdots - X^n$ is the direct application of the Fisher aggregation formula of (5.19) to $X_1 = Y + (-X_2) + (-X_3) + \cdots + (-X_n)$, that is,

$$G_t^{1F} = \frac{1}{2} \left(\frac{Y_t}{X_t^1} + \frac{Y_{t-1}}{X_{t-1}^1} \right) \times G_t^{YF} + \sum_{i=2}^{n} \frac{1}{2} \left(\frac{-X_t^i}{X_t^1} + \frac{-X_{t-1}^i}{X_{t-1}^1} \right) \times G_t^{iF}. \tag{5.22}$$

Unlike the method based on Laspeyres approximation, formula (5.22) is not analytically equivalent to (5.21).

5.4 Comparing the accuracy of different approximation methods

We have described three approaches to derive the Fisher aggregation and subtraction formulas. Analytically, it is impossible to compare their approximation accuracies. Instead, we compare the empirical performance of these formulas using Fisher volume indexes and Fisher chained-dollar data from Statistics Canada. The sample period spans from 1981Q1 to 2000Q3.

For each approximation formula, we test its aggregation and subtraction accuracy against actual data. When we test for aggregation accuracy, we create the aggregate by using its components and compare the resulting growth rate and level to the actual aggregate. When we test for subtraction accuracy, we remove components from their aggregate and compare the resulting growth rate and level to the actual sub-aggregate data.

To judge the formulas' growth-rate approximation accuracy, we compare the percentage deviation of the approximated growth rates from the actual growth rates for various formulas, that is, we calculate $100(G_t - G_t^F)$ where G_t is the approximated growth rate. For tractability, we report only the average of the deviations for the full sample $\frac{1}{n}\sum_{t=1}^{n}100(G_t - G_t^F)$ and the average variation of the squared deviations

$\frac{1}{n}\sum_{t=1}^{n}(100(G_t - G_t^F))^2$.

To judge the ability of the formulas to approximate levels (both chain index and chained dollar) of actual data, we compare the percentage difference of calculated chain level relative to the original chain level for different approximation formulas, that is, we calculate $100(Y_t / Y_t^F - 1)$ where Y_t is the approximated level. Again, we only report the average of the differences for the whole sample $\frac{1}{n}\sum_{t=1}^{n}100(Y_t / Y_t^F - 1)$ and the average variation of the squared differences $\frac{1}{n}\sum_{t=1}^{n}(100(Y_t / Y_t^F - 1))^2$.

Table 1 (see Annex 2) reports the results of the Fisher aggregation accuracy of the three formulas. For example, for the variable GDP excluding statistical discrepancy and inventories, we create an equivalent variable by combining data from consumption expenditures, government expenditures, government and business investment, exports, and imports. The growth rates and levels of the resulting variable are then compared to the actual data. Results in Table 1 show that the Laspeyres approximation and the Fisher of Fishers are able to reproduce the actual data quite accurately while the Tornqvist approximation performs less well.

Table 2 (see Annex 2) reports the results of the accuracy of Fisher subtraction. For example, for the variable durable goods, we create an equivalent variable by removing semi-durable goods, non-durable goods, and services from total consumption expenditures. Again, as in Table 1, we compare the growth rates and levels of the resulting variable to the actual data. We also report the results of two different ways of doing the Fisher subtraction. The first method uses the direct subtraction of $X^1 = Y - X^2 - X^3 - \cdots - X^n$ while the other uses the negative aggregation method of $X_1 = Y + (-X_2) + (-X_3) + \cdots + (-X_n)$. Results show that both the Laspeyres approximation and the “Fisher of Fishers” are quite accurate in reproducing the actual data, whether by using the direct subtraction method or the negative aggregation method. Similar to the results in Table 1, the Tornqvist approximation performs less well. Table 3 (see Annex 2) reports the results of Fisher subtraction using chain index data.

Results of Tables 1 to 3 show that the Tornqvist approximation method is the least accurate. While the results based on the "Fisher of Fishers" are slightly better than the Laspeyres approximation in most cases, both methods provide good approximations. Therefore, the “Fisher of Fishers” is the preferred method for the purpose of more accurate calculation of an aggregate when all the components are positive. On the other hand, the method based on the Laspeyres approximation is simpler to use. For these reasons, the Laspeyres approximation is most widely used in practice[5] and this is also the method that we recommend for modelling purposes. For subtractions based on "Fisher of Fishers", we recommend the direct application of negative aggregation to the Fisher aggregation formula because it is equally accurate as the formula proposed by the Macroeconomic Advisers Inc. but much simpler to use.

6. Decomposition of inflation and contribution to growth

Under the Laspeyres formula, the contribution of a component’s inflation rate to the aggregate price inflation is given by the share-weighted component’s inflation rate. In Section 4, we showed that under the Fisher chain formula the ratio of a component to the aggregate does not represent the component’s share in the aggregate. Therefore, we need to devise a different formula to evaluate a component’s contribution to aggregate inflation. In this section, we derive a new method to decompose aggregate inflation rate when using Fisher chain data. We present an empirical example to demonstrate the efficacy of this method. As a corollary, we also present the derivations to decompose aggregate growth rate into its components’ contribution.

Consider an aggregate Q consists of n components indexed as $i = 1, \cdots, n$. From equation (4.5), we can write the nominal growth rate of the aggregate Q as

[5] See the whole issue of *Journal of Economic and Social Measurement*, vol. 24 (2), 1998. Also see papers mentioned in footnote 2.

$$\frac{Q_t^N}{Q_{t-1}^N} = \pi_t^F \cdot G_t^F \tag{6.1}$$

where $\pi_t^F \equiv \frac{P_t^F}{P_{t-1}^F}$ is the gross inflation rate of the aggregate, $G_t^F \equiv \frac{Q_t^F}{Q_{t-1}^F}$ is the gross growth rate of the aggregate , and Q^N is the nominal value and Q^F is the Fisher quantity of the aggregate. Using the definition of Fisher formulas, we can write

$$\pi_t^F = \sqrt{\frac{\sum q_t^i p_t^i}{\sum q_t^i p_{t-1}^i} \frac{\sum q_{t-1}^i p_t^i}{\sum q_{t-1}^i p_{t-1}^i}} = \sqrt{\frac{Q_t^N}{Q_{t-1}^N} \frac{\sum q_{t-1}^i p_t^i}{\sum q_t^i p_{t-1}^i}} \quad ; \; i = 1, \cdots, n .$$

and $$G_t^F = \sqrt{\frac{\sum p_t^i q_t^i}{\sum p_t^i q_{t-1}^i} \frac{\sum p_{t-1}^i q_t^i}{\sum p_{t-1}^i q_{t-1}^i}} = \sqrt{\frac{Q_t^N}{Q_{t-1}^N} \frac{\sum q_t^i p_{t-1}^i}{\sum q_{t-1}^i p_t^i}} \quad .$$

Thus, the Fisher formulas satisfy the condition

$$\frac{\pi_t^F}{G_t^F} = \frac{\sum q_{t-1}^i p_t^i}{\sum q_t^i p_{t-1}^i} \quad . \tag{6.2}$$

We can rewrite equation (6.2) as

$$\pi_t^F \sum q_t^i \; p_{t-1}^i = G_t^F \sum q_{t-1}^i \; p_t^i \; ,$$

or equivalently,

$$\pi_t^F \sum (q_{t-1}^i G_t^i) p_{t-1}^i = G_t^F \sum q_{t-1}^i (p_{t-1}^i \pi_t^i)$$

$$\pi_t^F \sum q_{t-1}^{N,i} G_t^i = G_t^F \sum q_{t-1}^{N,i} \pi_t^i = \sum q_{t-1}^{N,i} G_t^F \pi_t^i \tag{6.3}$$

where $\pi_t^i \equiv \frac{p_t^i}{p_{t-1}^i}$ is the gross inflation rate of component i , $G_t^i \equiv \frac{q_t^i}{q_{t-1}^i}$ is the gross growth of the quantity of component i , and $q_t^{N,i} \equiv q_t^i \times p_t^i$ is the nominal value of component i , at time t .

Similarly, we can rewrite equation (6.1) as

$$\pi_t^F G_t^F Q_{t-1}^N = Q_t^N \ ,$$

or equivalently,

$$\pi_t^F \sum (q_{t-1}^{N,i} G_t^F) = \sum q_t^i p_t^i = \sum (q_{t-1}^i G_t^i)(p_{t-1}^i \pi_t^i)$$

$$= \sum q_{t-1}^i \, p_{t-1}^i \, G_t^i \, \pi_t^i$$

$$= \sum q_{t-1}^{N,i} \, G_t^i \, \pi_t^i \ . \tag{6.4}$$

We can now use equations (6.3) and (6.4) to derive the formulas for decomposing the inflation rate and growth rate contribution.

Component's contribution to aggregate inflation rate

We can solve for π_t^F by adding equation (6.3) to (6.4) to give

$$\pi_t^F \left[\sum_{i=1}^n q_{t-1}^{N,i} (G_t^F + G_t^i) \right] = \sum_{i=1}^n q_{t-1}^{N,i} (G_t^F + G_t^i) \pi_t^i \quad .$$

Therefore

$$\pi_t^F = \frac{\sum_{i=1}^n q_{t-1}^{N,i} (G_t^F + G_t^i) \pi_t^i}{\sum_{i=1}^n q_{t-1}^{N,i} (G_t^F + G_t^i)} = \sum_{i=1}^n \frac{q_{t-1}^{N,i} (G_t^F + G_t^i)}{\sum_{i=1}^n q_{t-1}^{N,i} (G_t^F + G_t^i)} \cdot \pi_t^i \, ,$$

$$= \sum_{i=1}^n \frac{\left[1 + \frac{G_t^i}{G_t^F} \right] q_{t-1}^{N,i}}{\sum_{i=1}^n \left[1 + \frac{G_t^i}{G_t^F} \right] q_{t-1}^{N,i}} \cdot \pi_t^i \quad . \tag{6.5}$$

Subtracting 1 from both sides of equation (6.5) gives

$$\frac{\Delta P_t^F}{P_t^F} = \sum_{i=1}^{n} \frac{\left[1+\frac{G_t^i}{G_t^F}\right] q_{t-1}^{N,i}}{\sum_{i=1}^{n}\left[1+\frac{G_t^i}{G_t^F}\right] q_{t-1}^{N,i}} \cdot \frac{\Delta p_t^i}{p_t^i} \tag{6.6}$$

Equation (6.6) is the desired equation to decompose aggregate inflation rate into its components' contribution. It expresses the aggregate inflation rate as a weighted sum of its components' inflation rates where the weights are determined by the component's nominal share adjusted by relative chain growth rates.

Component's contribution to growth rate

We can also use equations (6.3) and (6.4) to derive an equation to decompose the aggregate chain growth rate. To show this, rewrite equation (6.3) as

$$\sum q_{t-1}^{N,i}\, G_t^F \pi_t^i = \pi_t^F \sum q_{t-1}^{N,i}\, G_t^i$$

and add the resulting equation to (6.4) to give

$$G_t^F \left[\sum_{i=1}^{n} q_{t-1}^{N,i}\left\{\pi_t^F + \pi_t^i\right\}\right] = \sum_{i=1}^{n} q_{t-1}^{N,i}\left\{\pi_t^F + \pi_t^i\right\} G_t^F .$$

Therefore, we have

$$G_t^F = \frac{\sum_{i=1}^{n} q_{t-1}^{N,i}\left\{\pi_t^F + \pi_t^i\right\} G_t^F}{\sum_{i=1}^{n} q_{t-1}^{N,i}\left\{\pi_t^F + \pi_t^i\right\}}$$

$$= \sum_{i=1}^{n} \frac{q_{t-1}^{N,i}\left\{\pi_t^F + \pi_t^i\right\}}{\sum_{i=1}^{n} q_{t-1}^{N,i}\left\{\pi_t^F + \pi_t^i\right\}} \cdot G_t^F \quad . \tag{6.7}$$

Subtracting 1 from both sides of (6.7) gives the desired equation:

$$\frac{\Delta Q_t^F}{Q_t^F} = \sum_{i=1}^{n} \frac{\left[1 + \dfrac{\pi_t^i}{\pi_t^F}\right] q_{t-1}^{N,i}}{\sum_{i=1}^{n} \left[1 + \dfrac{\pi_t^i}{\pi_t^F}\right] q_{t-1}^{N,i}} \cdot \frac{\Delta q_t^i}{q_{t-1}^i}. \tag{6.8}$$

Equation (6.8) decomposes the chain aggregate growth rate into a weighted sum of the chain growth rates of its components. Equations (6.6) and (6.8) are symmetric: equation (6.6) decomposes an aggregate inflation rate into a weighted sum of each components' inflation rate where the weights are determined by relative chain growth rates; equation (6.8) decomposes a chain aggregate growth rate into a weighted sum of its components' chain growth rates where the weights are determined by relative inflation rates.

If we rewrite equation (6.8) as

$$\frac{\Delta Q_t^F}{Q_{t-1}^F} = \sum_{i=1}^{n} \frac{\left[1 + \dfrac{\pi_t^i}{\pi_t^F}\right] p_{t-1}^i \cdot \Delta q_t^i}{\sum_{i=1}^{n} \left[q_{t-1}^i p_{t-1}^i + q_{t-1}^i p_{t-1}^i \cdot \dfrac{\pi_t^i}{\pi_t^F}\right]}$$

$$= \sum_{i=1}^{n} \frac{\left[p_{t-1}^i + \dfrac{p_t^i}{\pi_t^F}\right] \cdot \Delta q_t^i}{\sum_{i=1}^{n} \left[\left\{p_{t-1}^i + \dfrac{p_t^i}{\pi_t^F}\right\} q_{t-1}^i\right]} \quad , \tag{6.9}$$

we would obtain the growth decomposition method of Whelan (2000) and the BEA. Although equations (6.8) and (6.9) are equivalent, equation (6.8) has the advantage that it identifies the contribution explicitly as a weighted sum of individual chain growth rates, which gives a much clearer interpretation than equation (6.9).

Comparing equation (6.7) to the aggregation formula based on Laspeyres approximation of equation (5.7), we can see that equation (5.7) is a special case of equation (6.7) where all the component inflation rates are the same as the aggregate inflation, that is, $\pi_t^i = \pi_t^F$ for all i. Therefore, when the elemental components are used, the contribution to growth formula of (6.7) and its equivalent "Fisher of Fishers" aggregation formula (5.14) will hold exactly, while the Laspeyres formula of (5.7) is only an approximation.

Note that the contribution to growth of formulas (6.7) or (6.8) and the inflation decomposition formulas of (6.5) or (6.6) are derived using the elemental components. It is likely that these equations will not hold exactly when higher-level sub-aggregates are used as is the case in most macroeconomic analyses and modelling.

Empirical example of inflation decomposition

Next, we will empirically evaluate the efficacy of equation (6.6) using higher-level sub-aggregates. As an example, we decompose the inflation rate of aggregate machinery and equipment (M&E) into its contributing components. In order to simplify the results presented below, we further aggregate some components into higher-level sub-aggregates.[6] We combine automobiles, trucks, and other transportation equipment into "transport", agricultural machinery and industrial machinery aggregate into "machinery"; and furniture and other machinery and equipment into "rest of M&E". This simplifies the reporting results but at the expense of introducing additional aggregation errors into our decomposition.

Chart 1 presents the differences between actual annualized quarter-to-quarter M&E price inflation and the weighted sum of components' inflation rates based on equation (6.6). Note that the differences, or the contribution errors, are small and centre on zero.

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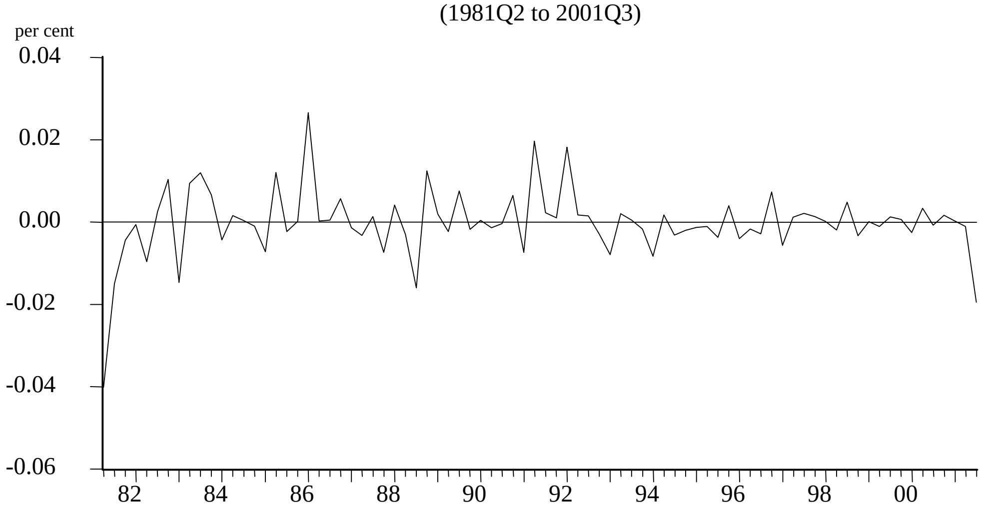
[6] The aggregation is constructed using Fisher-of-Fishers aggregation method.

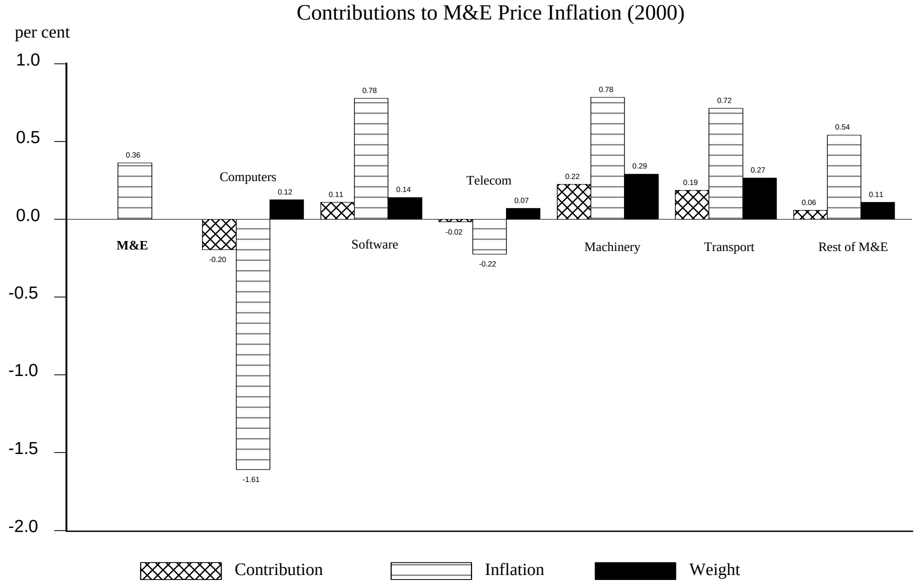
Chart 1
Annualized M&E Price Inflation Contribution Error
(1981Q2 to 2001Q3)



Average annualized M&E inflation = 0.4668 per cent
Average annualized sum of contribution errors = -0.00026 per cent

Chart 2 presents the decomposition of M&E price inflation for the year 2000. The components' contributions, weights, and inflation rates are included.

Chart 2
Contributions to M&E Price Inflation (2000)



7. Remodelling real identities and inventory investment

Under the Laspeyres formula, identities using quantity (or real) variables in large macroeconomic models such as the Canadian Economic and Fiscal Model (CEFM) of the Department of Finance are typically specified by simple arithmetic adding or subtracting of components. This cannot be done using Fisher chain formula.

Rewriting these real identities to conform to the properties of chain Fisher formula is quite straightforward. However, rewriting the equations for inventory investment is more difficult. In this section, we describe the basic framework for rewriting real identities, and the problems associated with re-specifying inventory investment. We discuss alternative methods that may bypass these problems and suggest a new specification for inventory investment.

7.1 A generic equation for real identities

A straightforward solution to the non-additivity problem is to rewrite all real aggregates in growth-rates rather than levels. Our preferred method is the Laspeyres approximation method as discussed in Section 5 because of its simplicity and high degree of accuracy.

Using the Laspeyres approximation method to specify real identities

Recall that the Laspeyres approximation method specifies the growth rate of a chained aggregate as the weighted sum of the growth rates of its components, where the weights are last-period nominal component shares. Specifically, suppose the chained aggregate ${Y_t}^F$ consists of components ${X_t^1}^F, {X_t^2}^F, \cdots, {X_t^n}^F$, we can approximate the growth rate of ${Y_t}^F$ as:

$$\frac{{Y_t}^F}{{Y_{t-1}}^F} \cong \theta_{t-1}^1 \cdot \frac{{X_t^1}^F}{{X_{t-1}^1}^F} + \theta_{t-1}^2 \cdot \frac{{X_t^2}^F}{{X_{t-1}^2}^F} + \cdots + \theta_{t-1}^n \cdot \frac{{X_t^n}^F}{{X_{t-1}^n}^F}, \tag{7.1}$$

where $\theta_t^i = X_t^i / Y_t$ is the nominal share of component X^i, for $i = 1, 2, \cdots, n$. Equation (7.1) is the basic framework with which we can re-specify real identities. Note that although equation (7.1) specifies Y_t^F in terms of its growth rate, we can always retrieve its level during simulation once we specify an initial value for Y_t^F.

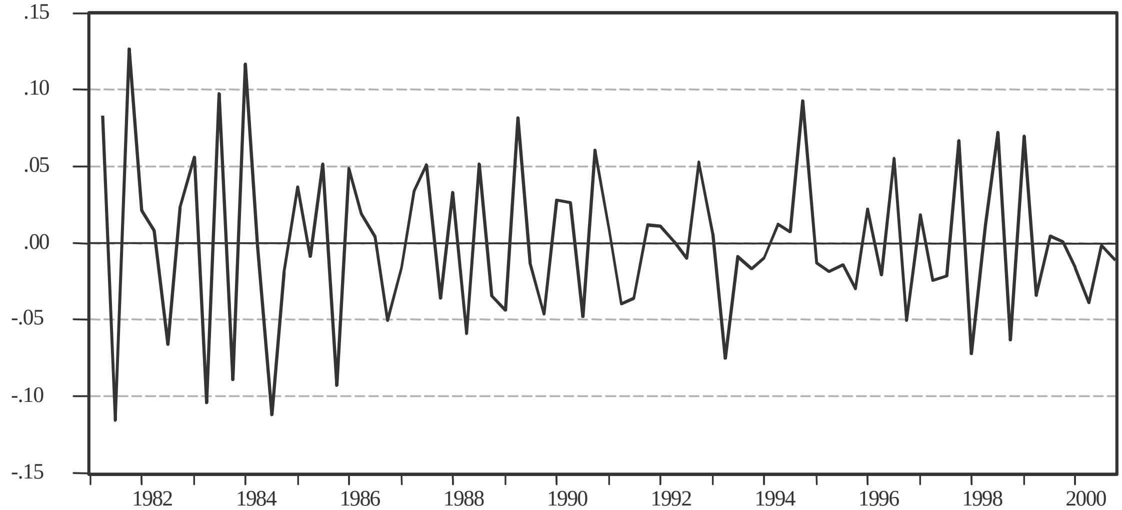
Since equation (7.1) is only an approximation, it will not reproduce Y_t^F exactly. In terms of model properties, it is convenient to have an exact reproduction of Y_t^F when simulating over history. In order to force the right-hand side equal to the left-hand side, we include an adjustment variable $Radj$ such that:

$$\frac{Y_t^F}{Y_{t-1}^F} = \theta_{t-1}^1 \cdot \frac{X_t^{1F}}{X_{t-1}^{1\,F}} + \theta_{t-1}^2 \cdot \frac{X_t^{2F}}{X_{t-1}^{2\,F}} + \cdots + \theta_{t-1}^n \cdot \frac{X_t^{nF}}{X_{t-1}^{n\,F}} + Radj_t \,, \tag{7.2}$$

where $Radj$ is the difference between the actual (gross) growth rate of Y_t^F and the weighted sum of the (gross) growth rates of the right-hand side variables. Hence, equation (7.2) holds exactly when simulating using historical data.

Chart 3 shows the adjustment series when we use equation (7.1) to approximate the growth rate of final domestic demand over the period 1981Q2 to 2000Q4. If the equation approximates the growth rate of final domestic demand well, then *Radj* should fluctuate around zero with no signs of bias.

Chart 3
Percentage Point Difference Between Actual and
Approximated Quarterly Growth Rates of Final Domestic Demand



The difference between the actual and estimated growth rate is generally less than 0.05 percentage points and has a mean value of -0.00018 percentage points with no sign of bias. Hence, the inclusion of this adjustment series during historical simulation is not expected to bias model properties.

We have tried to extract information from *Radj*. Our experience working with *Radj* suggests it contains no useful information that can be exploited. Experiences of other researchers[7] also support this conclusion.

A related modelling question is how to set the values of *Radj* over the forecast period. Since *Radj* shows no bias over history, we can set *Radj* equal to its mean value of zero over the forecast periods without affecting underlying forecast scenarios. Also, since *Radj* fluctuates between positive and negative values without any systematic pattern, it is reasonable to set *Radj* equal to zero starting in the first period of the forecast.

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[7] For example, Chris Varvares of the Macroeconomic Advisers and David Reifschneider of the Federal Reserve Board.

The estimation of nominal shares

Equation (7.2) requires nominal values of both Y and X^i for it to be operational over the forecast. Note that the nominal value of a variable is equal to the product of its chained quantity and chain price:

$$X_t^i = X_0^i \times X_t^{i^F} \times P_t^{i^F}, \text{ for } i = 1, 2, \cdots, n, \tag{7.3}$$

where X_0^i is a scaling constant and $P_t^{i^F}$ is the chain price of X^i. Since both quantities and prices are usually modeled in most macro models, the nominal value X^i is thus available over the forecast periods.

7.2 Remodelling investment in inventories

A more difficult modelling problem concerns the re-specification of inventory investment. The expenditure accounts include two inventory investment series: investment by business and investment by government. With Laspeyres data business investment in inventories is often used as a closing equation to preserve the GDP identity:

business investment in inventories = GDP - government investment in inventories - other components of GDP. (7.4)

With Fisher chain data, it would appear that equation (7.4) is another straightforward application of the Laspeyres approximation method. However, it turns out that a direct application of equation (7.1) is not possible because of the nature of the inventory investment data.

The construction of inventory investment data

Recall that the formula to calculate the change in a real aggregate Y is a Fisher index (G_t^F) that uses weights from two adjacent time periods:

$$G_t^F = \sqrt{\frac{\sum p_{t-1} q_t}{\sum p_{t-1} q_{t-1}} \times \frac{\sum p_t q_t}{\sum p_t q_{t-1}}}\,. \tag{7.5}$$

p and q are prices and quantities of elemental components of Y. Equation (7.5) is the general framework used by Statistics Canada to aggregate low-level elementals to create higher-level sub-components of GDP. A problem occurs, however, when constructing aggregate business and government inventory investment where the data fluctuate between positive and negative values, thereby causing the Laspeyres and Paasche indexes to switch signs. At times this results in taking the square roots of negative numbers in the Fisher calculation.[8] To bypass this problem, Statistics Canada applies the Fisher formula to inventory stocks instead. This causes no problem since stock values are always positive. In addition, Statistics Canada also constructs a "lagged" inventory stock that takes into account the effect of inventory valuation adjustment. Chained inventory investment series are then calculated as the difference between current-period and "lagged" stocks. Note that the "lagged" stock is not the current-period stock lagged one period.

Problems of applying the Laspeyres approximation method to investment in inventory

By construction, these inventory investment series are not Fisher indexes; hence the Laspeyres approximation method will not work well. A more serious problem is that some observations in the quarterly chained investment series have zero values, thereby preventing the calculation of their growth rates as required by the Laspeyres approximation method.

A potential way to bypass the zero-value problem is to model total inventory investment (business plus government) residually as a single item instead of separately. Specifically, we can rewrite the identity as:

$$GDP = C + I + G + X - M + YREE + INVT$$

where C = personal consumption expenditure; I = business and government gross fixed capital formation; G = government expenditure on goods and services; X = exports; M =

[8] Although the statistical discrepancy series also contains both negative and positive values, it is a single series and does not require using equation (5.5) to do the aggregation. Hence, this series poses no problem.

imports; *YREE* = statistical discrepancy; *INVT* = business and government inventory investment. We can apply equation (7.1) to the national accounts identity and write it as:

$$\frac{GDP_t^F}{GDP_{t-1}^F} = \theta_{t-1}^1 \cdot \frac{C_t^F}{C_{t-1}^F} + \cdots + \theta_{t-1}^n \cdot \frac{YREE_t^F}{YREE_{t-1}^F} + (1 - \sum_i^n \theta_{t-1}^i) \cdot \frac{INVT_t^F}{INVT_{t-1}^F} \quad . \qquad (7.6)$$

We can then solve for the (gross) growth rate of total inventories as

$$\frac{INVT_t^F}{INVT_{t-1}^F} = \frac{1}{(1 - \sum_i^n \theta_{t-1}^i)} \left[\frac{GDP_t^F}{GDP_{t-1}^F} - \left\{ \theta_{t-1}^1 \cdot \frac{C_t^F}{C_{t-1}^F} + \cdots + \theta_{t-1}^n \cdot \frac{YREE_t^F}{YREE_{t-1}^F} \right\} \right] . \qquad (7.7)$$

Once we calculate the growth rates of total inventory investment, they can be chained forward and backward to create a level series.

Simulation results, however, showed that equation (7.7) does not resolve the problem. Since investment in inventories can at times be very close to zero, a small approximation error by equation (7.7) can give rise to a very large estimation error in the growth rates. This translates in large errors in level estimates, substantially reducing the usefulness of equation (7.7). We conclude that a direct estimation of the level of investment in inventories based on the Fisher formula is not feasible.

As an alternative, one could model inventory investment using stocks. This has the advantage that because stocks are always positive, estimating their growth rates are much easier than those of investment flows. This approach, however, requires modelling the "lagged" stock (with associated inventory valuation adjustment), which may be more difficult than modeling investment flows.

Modelling inventories as contribution to GDP growth

Modelling the contribution of business and government inventory investment (either separately or as a group) to GDP growth poses fewer problems. We can modify equation (7.6) and write the growth rate of GDP (in percentage terms) as:

$$\frac{\Delta GDP_t}{GDP_{t-1}} = \theta^1_{t-1}.\frac{\Delta C_t}{C_{t-1}} + \cdots + \theta^{n-2}_{t-1}.\frac{\Delta YREE_t}{YREE_{t-1}} + \theta^{n-1}_{t-1}.\frac{\Delta INVG_t}{INVG_{t-1}} + \theta^{n}_{t-1}.\frac{\Delta INVB_t}{INVB_{t-1}} \tag{7.8}$$

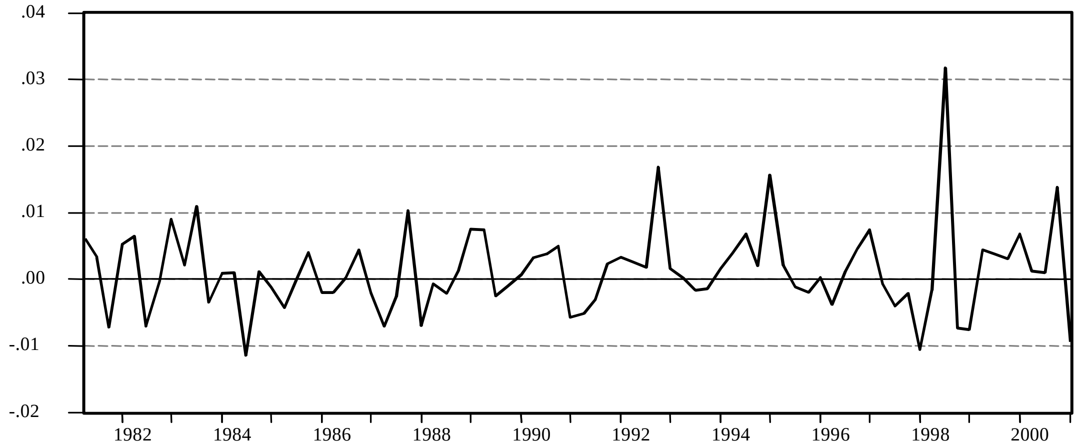
where *INVG* is government inventory investment and *INVB* is business inventory investment. Each term on the right-hand side of equation (7.8) represents the contribution of that item to GDP growth. We can invert equation (7.8) and write the growth contribution of business inventory investment equation as:

$$\theta^{n}_{t-1}.\frac{\Delta INVB_t}{INVB_{t-1}} = \frac{\Delta GDP_t}{GDP_{t-1}} - \left\{\theta^1_{t-1}.\frac{\Delta C_t}{C_{t-1}} + \cdots + \theta^{n-2}_{t-1}.\frac{\Delta YREE_t}{YREE_{t-1}} + \theta^{n-1}_{t-1}.\frac{\Delta INVG_t}{INVG_{t-1}}\right\} \tag{7.9}$$

Equation (7.9) is just another way of writing equation (7.7). The main difference is that with equation (7.7) we attempt to recover level estimates of inventory investment. With equation (7.9), we are concerned only with the growth contribution of inventory investment with no attempt to recover level estimates. This avoids the large level estimation errors we encountered when experimenting with equation (7.7). Another advantage of using equation (7.9) is that component contributions are available from Statistics Canada at the same time as the expenditure accounts data release. This eliminates the problem of calculating the growth contribution of inventory investment by government.

Since equation (7.9) is an approximation of actual output growth, we include an adjustment series (calculated as the difference between the actual and the approximated growth rates) so that equation (7.9) holds exactly over history. Chart 4 shows that the adjustment series fluctuates around zero with mean value of 0.0013 percentage points. This suggests that we can set the contribution from inventory investment by government to zero over the forecast periods.

Chart 4
Percentage Point Difference Between Actual and
Approximated Quarterly Growth Rates of Output



8. Stock-flow accumulation rules and depreciation rates of capital stocks

Stock-flow accumulation rules are common features of macroeconomic models. One example of such use is the accumulation of capital stocks using investment flows. Typically, current-period capital stock is specified as the sum of current-period investment flow plus last period's capital stock net of depreciation. However, the non-additive property of Fisher chain quantity invalidates this method. This creates two problems: first, how to accumulate capital stocks within a model such that they are consistent with investment flows; second, how to estimate historical aggregate depreciation rates for capital stocks.

In this section, we propose a method of accumulating capital stocks based on the Fisher aggregation of the Laspeyres approximation method. This allows us to bypass the non-additive property of the Fisher chain data. However, the problem of how to calculate chained aggregate depreciation rates remains unresolved. Based on available information, there is no conclusive and unique way to calculate chained depreciation rates. As a result, we resort to using the same calculation method as for Laspeyres data. The resulting rates are hence only approximations of the true underlying aggregate depreciation rates.

8.1 Aggregate depreciation rates

Original-dollar and Laspeyres constant-dollar depreciation rates

We will first describe briefly the construction of original-dollar and Laspeyres constant-dollar depreciation rates. This helps to highlight the difficulties encountered in calculating depreciation rates with chained data. We focus on these two capital stock measures because the traditional stock-flow accumulation rule holds for them.

Original-dollar capital stocks are unpublished Statistics Canada data. They are created by simple accumulation of investment series without taking into account the effect of price changes. As such, they are equivalent to the book value of companies' investment. Statistics Canada also publishes a current-dollar (or replacement cost) measure of capital stocks. However, these series are of limited use for our purpose here because the accumulation rule does not hold for them.[9]

Suppose an aggregate capital stock (K) and an investment (I) series each consists of n elemental components $K^1, K^2, ..., K^n$ and $I^1, I^2, ..., I^n$, respectively. For each component i, the capital stock is specified as

$$K_t^i = K_{t-1}^i - D_t^i + I_t^i \tag{8.1}$$

where D_t^i is economic depreciation and is given by

$$D_t^i = \delta_t^i K_{t-1}^i \text{ , for } i = 1,2,...,n \text{ .} \tag{8.2}$$

δ_t^i is the depreciation rate and is usually calculated based on the estimated service life of a capital stock. Using equation (8.2), we can rewrite equation (8.1) as the familiar accumulation rule of

$$K_t^i = K_{t-1}^i (1 - \delta_t^i) + I_t^i, \quad \text{for } i = 1,2,...,n. \tag{8.3}$$

[9] See Annex 3 for an explanation of why the accumulation rule does not hold for current-dollar capital stocks.

The aggregate capital stock (K), depreciation (D), and investment (I) are the simple sum of their components:

$$K_t = \sum_{i=1}^{n} K_t^i , \quad I_t = \sum_{i=1}^{n} I_t^i , \qquad D_t = \sum_{i=1}^{n} D_t^i = \sum_{i=1}^{n} \delta_t^i K_{t-1}^i . \tag{8.4}$$

We can use equations (8.1), (8.3), and (8.4) together to obtain

$$K_t = K_{t-1} - D_t + I_t . \tag{8.5}$$

Defining the aggregate depreciation rate as

$$\delta_t = D_t / K_{t-1} ,$$

the aggregate accumulation rule of equation (8.5) is thus given by:

$$K_t = (1 - \delta_t) K_{t-1} + I_t . \tag{8.6}$$

Note that the aggregate depreciation rate is equivalent to the weighted sum of individual depreciation rates:

$$\delta_t = \sum_{i=1}^{n} \left(\frac{K_{t-1}^i}{K_{t-1}}\right) \delta_t^i \tag{8.7}$$

where the weights are last-period capital stock shares. The additive property of equations (8.5) and (8.7) holds exactly for Laspeyres constant-dollar and original-dollar capital stocks.[10] In practice, most modellers calculate the aggregate depreciation rate not from (8.7) but by inverting equation (8.6) to get

$$\delta_t = \frac{I_t - (K_t - K_{t-1})}{K_{t-1}} . \tag{8.8}$$

[10] When using Laspeyres constant-dollar data, real capital stocks are used to calculate the weights.

Depreciation rate with Fisher chain data

Chained capital stocks are constructed using constant-dollar series. For each component i, constant-dollar capital stocks are constructed according to

$$K_t^{i\,R} = K_{t-1}^{i\quad R} - D_t^{i\,R} + I_t^{i\,R}, \quad \text{for } i = 1,2,\cdots,n \tag{8.9}$$

where real depreciation is defined as:

$$D_t^{i\,R} = \delta_t^{i\,R} K_{t-1}^{i\quad R}, \quad \text{for } i = 1,2,\cdots,n\,. \tag{8.10}$$

Chained aggregate capital stock (K_t^F), depreciation (D_t^F), and investment (I_t^F) are then constructed by applying Fisher formula separately to their respective elemental components. Because of the way K_t^F, D_t^F, and I_t^F are constructed, the identity of equation (8.5) will no longer hold and hence cannot be used as an accumulation rule. This also implies that equation (8.8) is not a valid way to calculate the implicit aggregate depreciation rate. In fact, the non-additive property implies that we cannot interpret the δ calculated by equation (8.8) with chained data as a depreciation rate.

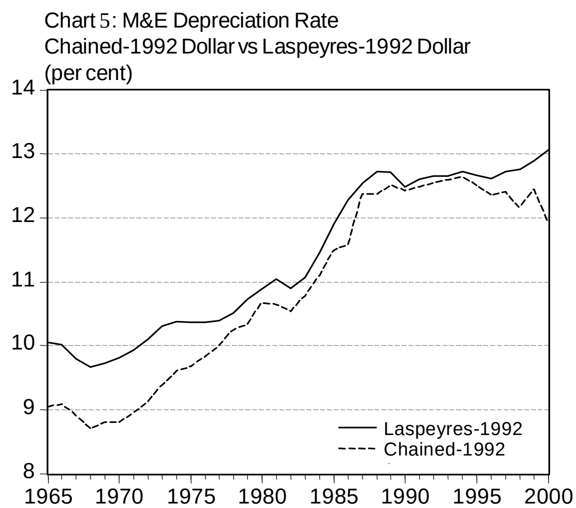
A more serious problem is that there is no unique way to calculate aggregate depreciation rates for chained capital stocks.[11] The general consensus is that any method used can only be an approximation of the underlying aggregate depreciation rates, and is therefore acceptable as long as it generates “reasonable” depreciation rates.

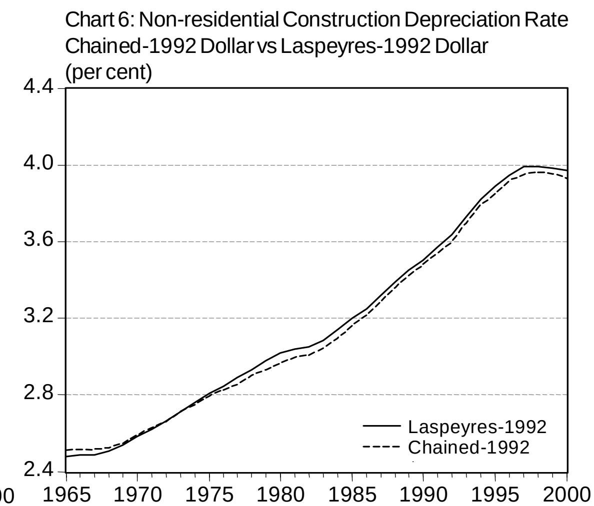
We have tried various ways to calculate aggregate depreciation rates and finally decided on the rule of[12]

$$\delta_t^F = \frac{D_t^F}{K_{t-1}^F}\,. \tag{8.11}$$

[11] Erwin Diewert of the University of British Columbia, Karl Whelan of the Federal Reserve Board, and members of the Capital Stock Division of Statistics Canada also recognised this is a problem encountered by macro-economists when using chained data. To our knowledge, no solution has yet been found.
[12] This rule is based on delay-depreciation data. For geometric-depreciation data, we recommend to use the rule of $\delta_t^F = D_t^F / (K_{t-1}^F + I_t^F / 2)$.

Note that this definition of depreciation rate is the same as that for the original-dollar and Laspeyres constant-dollar series. It is also similar to the way Statistics Canada calculates elemental depreciation. Using equation (8.11), we found that depreciation rates for chained M&E and non-residential construction capital stocks are similar to those calculated using Laspeyres constant-dollar series (see Charts 5 and 6). Since switching from Laspeyres to Fisher formula should not fundamentally affect the values of the aggregate depreciation rates, this suggests that equation (8.11) is a reasonable way to approximate aggregate depreciation rates.





As a demonstration, we have also use equation (8.8) to estimate implicit depreciation rates for the two chained capital stocks. They are intended to highlight the estimation errors that will occur if we were to ignore the fact that we have switched to chained data but continue to use equation (8.8). Charts 7 and 8 compare three different estimations of depreciation rates for M&E and non-residential construction capital stocks: using Laspeyres-1992 dollar series with equation (8.8), using chained-1992 dollar series with equation (8.11), and using chained-1992 dollar with equation (8.8) (labelled as equation (8.8) in the charts). As we can see the use of the traditional accumulation rule would tend to produce an ever-increasing depreciation rate.[13]

[13] See Tevlin, S. and Karl Whelan (2000) for similar results on calculating U.S. depreciation rates.

Chart 7: Different Estimations of
M&E Depreciation Rates
(per cent)

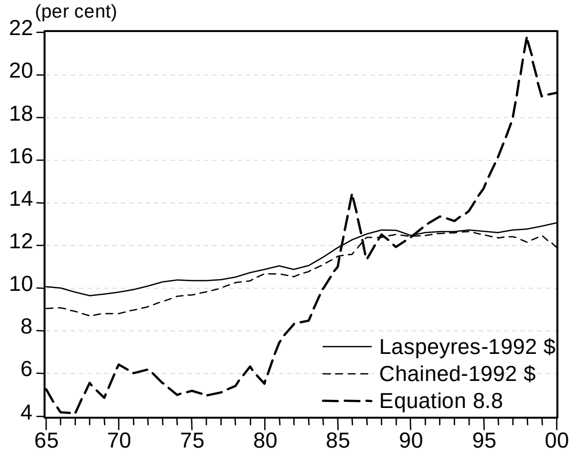
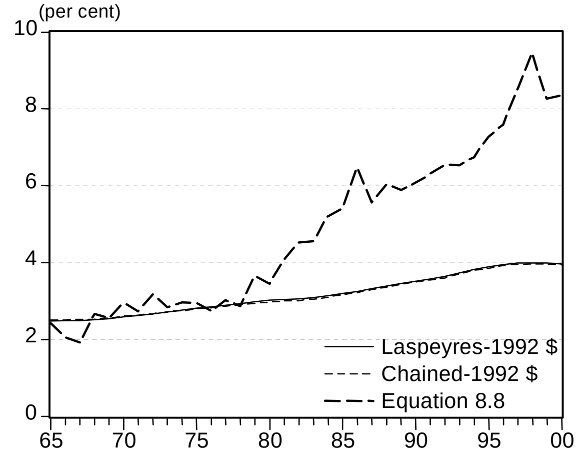


Chart 8: Different Estimations of
Non-residential Construction Depreciation Rates
(per cent)



8.2 Stock-flow accumulation rule

We have also devised an alternative method to replace equation (8.6) as an accumulation rule for chained capital stocks. It is based on the principle that the end-of-period chain capital stock (K_t^F) consists of two parts: the usable portion of existing stock after allowing for deterioration ($(1-\zeta_t^F)K_{t-1}^F$), and new investment (I_t^F). We can then apply the Laspeyres approximation method to model current-period growth of capital stock as the contribution of the growth of these two components. This contribution equation is the new accumulation rule for capital stocks.

Specifically, suppose Y is an aggregation of two components, X and Z . Although the level of Y is not additive for chain-type quantity series (denoted by the superscript F), that is, $Y^F \neq X^F + Z^F$, we can approximate the growth rate of the aggregate by the weighted sum of the growth rates of its components:

$$\frac{Y_t^F}{Y_{t-1}^F} = \theta_{t-1} \cdot \frac{X_t^F}{X_{t-1}^F} + (1-\theta_{t-1}) \cdot \frac{Z_t^F}{Z_{t-1}^F} \quad . \tag{8.12}$$

The weight $\theta_{t-1} = \frac{X_{t-1}^{N}}{Y_{t-1}^{N}}$ in equation (8.12) is given by the nominal share of X in Y in period *t-1*, where the superscript *N* denotes nominal values.

Since capital stock consists of current period investment and last period's capital stock net of depreciation, we can apply (8.12) to the capital stock series and write

$$\frac{K_t^F}{K_{t-1}^F} = \theta_{t-1} \cdot \frac{I_t^F}{I_{t-1}^F} + (1-\theta_{t-1}) \cdot \frac{(1-\zeta_t^F)K_{t-1}^F}{(1-\zeta_{t-1}^F)K_{t-2}^F}$$

$$= \frac{I_{t-1}^N}{K_{t-1}^N} \cdot \frac{I_t^F}{I_{t-1}^F} + \left(1 - \frac{I_{t-1}^N}{K_{t-1}^N}\right) \cdot \frac{(1-\zeta_t^F)K_{t-1}^F}{(1-\zeta_{t-1}^F)K_{t-2}^F} \ . \qquad (8.13)$$

Equation (8.13) expresses the growth rate of chain capital stock as a weighted sum of the growth rates of investment flow and last period's capital stock net of depreciation. Given investment and capital stock data, we can invert equation (8.13) to solve for the unknown parameter ζ. Once ζ is known, we can use equation (8.13) to accumulate capital stocks without violating the non-additive property of the Fisher formula.

It is easy to show that equation (8.13) holds exactly for Laspeyres constant-dollar data. Recall the accumulation rule for Laspeyres data is given by:

$$K_t = (1-\delta_t)K_{t-1} + I_t \ .$$

We can write this equivalently as

$$\frac{K_t}{K_{t-1}} = \frac{I_t}{K_{t-1}} + (1-\delta_t)$$

$$= \frac{I_{t-1}}{K_{t-1}} \cdot \frac{I_t}{I_{t-1}} + \frac{(1-\delta_{t-1})K_{t-2}}{K_{t-1}} \cdot \frac{(1-\delta_t)K_{t-1}}{(1-\delta_{t-1})K_{t-2}}$$

$$= \frac{I_{t-1}}{K_{t-1}} \cdot \frac{I_t}{I_{t-1}} + \frac{K_{t-1} - I_{t-1}}{K_{t-1}} \cdot \frac{(1-\delta_t)K_{t-1}}{(1-\delta_{t-1})K_{t-2}}$$

$$= \frac{I_{t-1}}{K_{t-1}} \cdot \frac{I_t}{I_{t-1}} + (1 - \frac{I_{t-1}}{K_{t-1}}) \cdot \frac{(1-\delta_t)K_{t-1}}{(1-\delta_{t-1})K_{t-2}} . \tag{8.14}$$

Equation (8.14) shows that a level equation can be rewritten into a growth-rate equation such that the growth rate of capital stock is determined by the weighted sum of investment growth and the last period capital growth rate net of depreciation. Note that unlike equation (8.12) the weights in equation (8.14) are expressed in real shares.

Conceptually, the ζ_t^F from equation (8.13) could approximate the Fisher aggregate depreciation rate. The choice of data used in the accumulation rule, however, prevents this interpretation. The current release of chained capital stocks has 1997 as the reference year. That means the values of chained capital stocks in 1997 are the same as their respective nominal stock values. For this exercise, Statistics Canada uses current-dollar capital stocks as the nominal measure. Therefore, for consistency, current-dollar capital stocks should be used to calculate nominal weights in the accumulation rule of equation (8.13). Note that in order for equation (8.13) to work properly, the following nominal identity has to hold:

$$K_t^N = K_{t-1}^N - D_t^N + I_t^N . \tag{8.15}$$

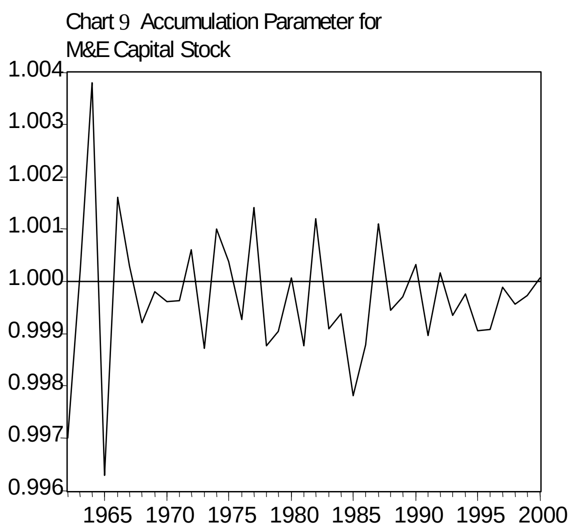
In this case, the ratio of I_t^N / K_t^N will have a share interpretation. However, equation (8.15) does not hold for current-dollar series because of the effect of lagged price adjustments when creating these series. This means we have to use a different nominal capital stock series to calculate nominal weights.

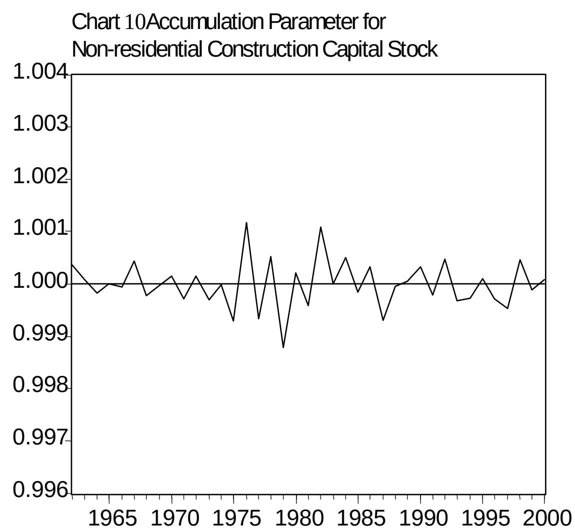
The only available alternative is to use original-dollar series. These series do satisfy equation (8.15), and the ratio I_t^N / K_t^N is indeed the share of investment in the capital stock. They are therefore chosen to calculate nominal weights in the accumulation rule. The problem is that they are not the nominal series used by Statistics Canada to set the reference-year values of chained capital stocks. This results in data inconsistencies in the accumulation rule that prevents interpreting ζ_t^F as the implicit aggregate depreciation rate. Therefore, ζ_t^F should only be interpreted as a parameter in the accumulation rule.

By setting $d_t = \dfrac{1-\zeta_t^F}{1-\zeta_{t-1}^F}$, we can rewrite equation (8.13) as:

$$\frac{K_t^F}{K_{t-1}^F} = \frac{I_{t-1}^N}{K_{t-1}^N} \cdot \frac{I_t^F}{I_{t-1}^F} + \left(1 - \frac{I_{t-1}^N}{K_{t-1}^N}\right) \cdot \frac{K_{t-1}^F}{K_{t-2}^F} \cdot d_t \ . \tag{8.16}$$

Since ζ_t^F is quite stable from one period to another, d_t should be close to unity. Charts 9 and 10 show the values of d_t for M&E and non-residential construction capital stocks.[14]





The historical values of d_t fluctuate around 1 with no bias. The values of both accumulation parameters average to 1 between 1962 and 2000.

For equation (8.16) to be operational over the forecast, we need forecast values of d_t and original-dollar capital stocks.[15] Since ζ_t^F is not likely to change substantially from quarter to quarter, d_t is expected to remain close to its average value of unity. Also, from Charts 9 and 10, it is very difficult to discern particular patterns for d_t . Hence, it is

[14] The calculation is based on chain-1992 data but adjusted to have 1997 as the reference year.

reasonable to set the d_t for both M&E and non-residential construction capital stocks equal to their historical average over the forecast.

For original-dollar capital stocks, we can apply the accumulation rule of (superscripts H represent original-dollar series)

$$K_t^H = K_{t-1}^H(1-\delta_t^H) + I_t^H \tag{8.17}$$

to accumulate them using forecast nominal investment. Over the historical period, the nominal depreciation rate δ_t^H can be calculated using equation (8.17). However, we still have to decide how to set δ_t^H over the forecast. One way is to set the growth rate of δ_t^H equal to the growth rate of the chained depreciation rate. Charts 11 to 14 compare the levels and growth rates of chained depreciation rates and δ_t^H for M&E and non-residential construction capital stocks. They show that the growth rates of chained depreciation rates and δ_t^H for both types of capital stocks are broadly consistent and hence it is reasonable to set their growth rates to be the same over the forecast.

Chart 11: M&E Depreciaton Rate
Chained-1992 Dollar vs Original
(per cent)

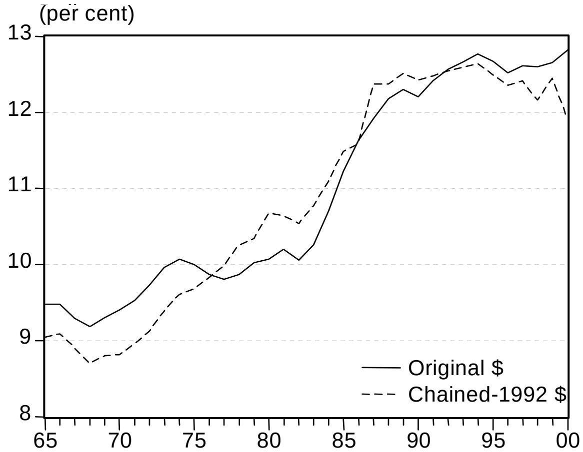
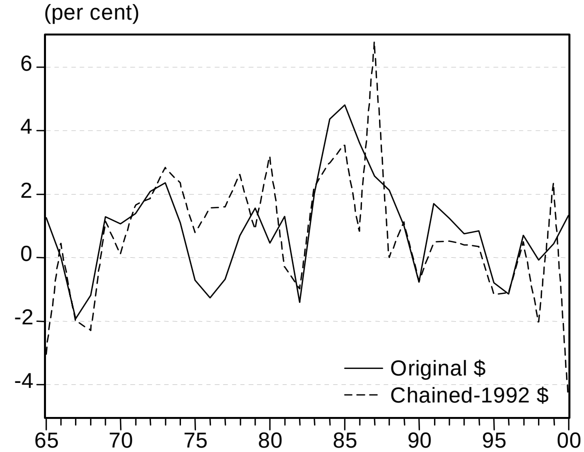


Chart 12: Growth Rates of M&E Depreciaton
Chained-1992 $ vs Original $
(per cent)



[15] There is no distinction between current-dollar and original-dollar investment flows.

Chart 13: Non-residential Construction Depreciaton Rate Levels - Chained-1992 Dollar vs Original Dollar (per cent)

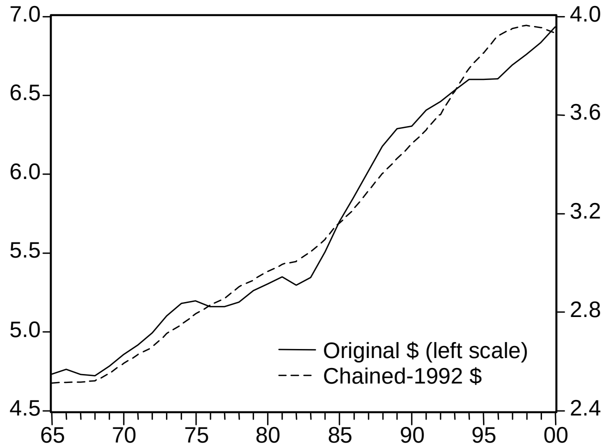
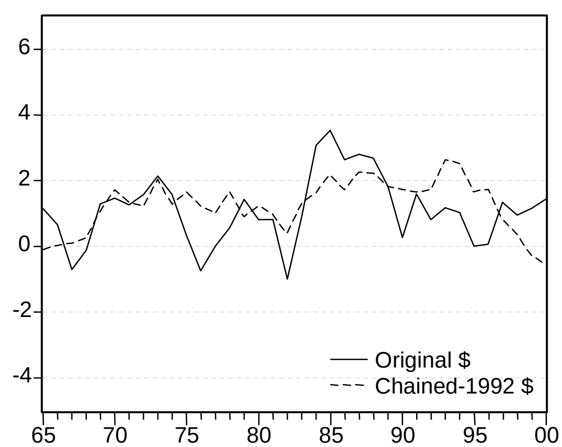


Chart 14: Growth Rates of Non-residential Construction Depreciaton Rates - Chained-1992 $ vs Original $ (per cent)



9. Conclusion

In principle, the Fisher chain formula provides a better measurement of economic activity than the Laspeyres formula. However, the non-additive property of Fisher chain data gives rise to difficulties and challenges when using this type of data for economic analysis and modelling. In this paper, we discuss the properties of Fisher indexes, explore various approximation formulas for aggregation and subtraction using Fisher chain data, and derive formulas to decompose aggregate output and price growth into contributing components. Empirical results show that while both the "Fisher of Fishers" and the Laspeyres approximation methods provide good approximates of actual data, the "Fisher of Fishers" performs slightly better. Therefore, we recommend using the “Fisher of Fishers” approximation method for the purpose of constructing aggregates and calculating contributions to growth. However, since the method based on the Laspeyres approximation is simpler to use and easier to interpret than the “Fisher of Fishers”, the Laspeyres method is the most widely used in practice. This is also the method that we would recommend for the purpose of modelling. Based on the Laspeyres approximation method, we have derived and proposed solutions to the problems in modelling the capital stock-flow accumulation rules and in modelling inventory investments.

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Annex 1

A Simple Algebraic Example of Substitution Bias[16]

Consider a simple national accounts identity of:

$$Y_t = C_t + I_t$$

$$= p_t^C \times C_t^R + p_t^I \times I_t^R \qquad \text{(A1.1)}$$

where Y_t, C_t, and I_t represent nominal output, consumption, and investment at period *t*; C_t^R and I_t^R represent physical quantities of consumption and investment at period *t*; p_t^C and p_t^I represent prices for consumption and investment at period *t*. Suppose the base period is set at 1992. We can write the price indexes for consumption and investment as:

$$\Phi_t^C = \frac{p_t^C}{p_{92}^C} \qquad \text{and} \qquad \Phi_t^I = \frac{p_t^I}{p_{92}^I}. \qquad \text{(A1.2)}$$

If $t = 1992$, then $\Phi_t^C = \Phi_t^I = 1$. Using equation (A1.2), the level of consumption measured in 1992 prices (or real consumption) is given by

$$C92_t = p_{92}^C \cdot C_t^R = p_{92}^C \cdot C_t^R \cdot \frac{p_t^C}{p_t^C} = \frac{p_t^C \cdot C_t^R}{\Phi_t^C} \quad . \qquad \text{(A1.3)}$$

Similarly, we can also define real investment as

$$I92_t = \frac{p_t^I \cdot I_t^R}{\Phi_t^I} \quad . \qquad \text{(A1.4)}$$

Using equations (A1.3) and (A1.4), we can write real output as

$$Y92_t = C92_t + I92_t$$

[16] This Annex draws on the work of Varvares et al (1998).

$$= p_{92}^{C} \cdot C_t^R + p_{92}^{I} \cdot I_t^R \qquad , \tag{A1.5}$$

and define the price deflator for output as:

$$\Phi_t^Y = \frac{p_t^C \cdot C_t^R + p_t^I \cdot I_t^R}{p_{92}^C \cdot C_t^R + p_{92}^I \cdot I_t^R} = \frac{C_t + I_t}{C92_t + I92_t} = \frac{Y_t}{Y92_t} \quad . \tag{A1.6}$$

The change in 1992-dollar real output is given by:

$$\frac{Y92_t}{Y92_{t-1}} = \frac{p_{92}^C \cdot C_t^R + p_{92}^I \cdot I_t^R}{p_{92}^C \cdot C_{t-1}^R + p_{92}^I \cdot I_{t-1}^R}$$

$$= \frac{p_{92}^C \cdot C_t^R}{Y92_{t-1}} + \frac{p_{92}^I \cdot I_t^R}{Y92_{t-1}}$$

$$= \frac{p_{92}^C \cdot C_t^R}{Y92_{t-1}} \times \frac{C92_{t-1}}{C92_{t-1}} + \frac{p_{92}^I \cdot I_t^R}{Y92_{t-1}} \times \frac{I92_{t-1}}{I92_{t-1}}$$

$$= \frac{p_{92}^C \cdot C_t^R}{Y92_{t-1}} \times \frac{C92_{t-1}}{p_{92}^C \cdot C_{t-1}^R} + \frac{p_{92}^I \cdot I_t^R}{Y92_{t-1}} \times \frac{I92_{t-1}}{p_{92}^I \cdot I_{t-1}^R}$$

$$= \left(\frac{C92_{t-1}}{Y92_{t-1}}\right) \times \frac{C_t^R}{C_{t-1}^R} + \left(\frac{I92_{t-1}}{Y92_{t-1}}\right) \times \frac{I_t^R}{I_{t-1}^R} \quad . \tag{A1.7}$$

Equation (A1.7) shows that the growth in real output is a weighted sum of the growth in real consumption and investment where the weights (in brackets) are last-period real shares.

Using (A1.4) to (A1.6), we can rewrite equation (A1.7) as

$$\frac{Y92_t}{Y92_{t-1}} = \left(\frac{C_{t-1}}{Y_{t-1}} \cdot \frac{\Phi_{t-1}^Y}{\Phi_{t-1}^C}\right) \frac{C_t^R}{C_{t-1}^R} + \left(\frac{I_{t-1}}{Y_{t-1}} \cdot \frac{\Phi_{t-1}^Y}{\Phi_{t-1}^I}\right) \frac{I_t^R}{I_{t-1}^R} \quad . \tag{A1.8}$$

Equation (A1.8) shows that the weights are dependent on the base period chosen since the ratio of the price deflators are base-period dependent. If the price of investment, for example, Φ_t^I falls faster relative to the aggregate price Φ_t^Y, the weight of investment increases in the calculation of output growth.

Typically, when commodities go through rapid price declines (increases), there is a tendency for economic agents to substitute towards (away from) those commodities. This results in the increase (decrease) in the quantity demanded for those commodities. Since prices are evaluated at the higher (lower) base-period levels, the rapid increase (decrease) in the quantity demanded will cause the Laspeyres formula to assign to those commodities higher (lower) contribution to overall economic growth. This effect is commonly refers to as substitution bias.

Annex 2

Table 1: Comparison of Fisher Aggregation

	Growth Rate						Chain Level					
	Mean			Variance			Mean			Variance		
	LP	FoF	TQ	LP	FoF	TQ	LP	FoF	TQ	LP	FoF	TQ
Chained-dollar												
GDP excluding statistical discrepancy and inventories	-0.001	0.000	0.005	0.000	0.000	0.001	0.024	0.001	-0.232	0.001	0.000	0.080
GDP excluding statistical discrepancy	0.002	0.003	0.102	0.002	0.001	0.352	0.076	0.018	-3.284	0.009	0.002	18.160
GDP	-0.002	-0.003	-0.102	0.002	0.001	0.350	-0.076	-0.018	3.562	0.009	0.002	21.763
Final domestic demand	0.000	0.000	0.007	0.000	0.000	0.000	-0.002	0.001	-0.199	0.000	0.000	0.064
Consumption	0.001	0.000	0.007	0.000	0.000	0.000	-0.015	0.000	-0.146	0.000	0.000	0.043
Business investment	0.000	0.000	0.036	0.000	0.000	0.003	-0.031	0.003	-1.026	0.002	0.000	1.786
Exports	0.000	0.000	0.008	0.000	0.000	0.000	-0.006	0.000	-0.245	0.000	0.000	0.102
Imports	0.000	0.000	0.013	0.000	0.000	0.001	-0.002	0.001	-0.319	0.000	0.000	0.176
Chain index												
Final domestic demand	-0.001	0.000	0.007	0.003	0.003	0.003	-0.018	-0.022	-0.077	0.002	0.002	0.032
Consumption	0.000	0.000	0.006	0.002	0.002	0.002	-0.023	-0.014	-0.098	0.002	0.001	0.034
Business investment	0.000	0.000	0.036	0.002	0.002	0.005	-0.031	-0.001	-0.353	0.003	0.001	0.861
Exports	-0.001	-0.001	0.007	0.004	0.004	0.005	0.005	-0.001	-0.060	0.002	0.002	0.046
Imports	0.000	0.000	0.013	0.004	0.004	0.005	0.008	0.009	-0.040	0.002	0.002	0.078

1. LP = Laspeyres approximation; FoF = Fisher of Fishers; TQ = Tornqvist approximation;
2. The aggregation uses the following relationships:
   GDP excluding statistical discrepancy and inventories = consumption + govt spending + govt investment + business investment + exports - imports;
   GDP excluding statistical discrepancy = GDP + (-statistical discrepancy);
   GDP = GDP excluding statistical discrepancy + statistical discrepancy;
   Final domestic demand = consumption + govt spending + govt investment + business investment;
   Consumption = durable goods + semi-durable + non-durable + services;
   Business investment = residential structures + non-residential structures + M&E;
   Exports = export of goods + export of services;
   Imports = import of goods + import of services.

Annex 2 (continued)

Table 2: Comparison of Fisher Subtraction with Chained-Dollar Data

	Growth Rate						Chained Level					
	Mean			Variance			Mean			Variance		
	LP	FoF	TQ	LP	FoF	TQ	LP	FoF	TQ	LP	FoF	TQ
Panel 1												
Consumption :	0.000	0.000	-0.012	0.000	0.000	0.000	0.002	-0.001	0.342	0.000	0.000	0.192
Durable goods	-0.005	0.000	-0.053	0.000	0.000	0.008	0.113	0.002	1.149	0.027	0.000	2.650
Semi-durable goods	-0.006	0.000	-0.068	0.000	0.000	0.012	0.152	0.003	1.531	0.049	0.000	4.695
Non-durable goods	-0.002	0.000	-0.025	0.000	0.000	0.002	0.056	0.001	0.556	0.007	0.000	0.623
Services	-0.001	0.000	-0.014	0.000	0.000	0.001	0.030	0.001	0.290	0.002	0.000	0.173
Business investment:	0.000	0.000	-0.040	0.000	0.000	0.004	0.014	-0.004	1.210	0.001	0.000	2.338
Non-res structure & equip	0.000	0.000	-0.033	0.000	0.000	0.003	0.038	-0.003	0.944	0.004	0.000	1.490
Residential structure	0.002	0.000	-0.072	0.002	0.000	0.014	0.097	-0.006	1.943	0.023	0.000	6.404
Government spending	0.000	0.000	-0.031	0.000	0.000	0.003	0.005	-0.004	0.904	0.000	0.000	1.324
Government investment	0.002	0.001	-0.257	0.006	0.000	0.157	0.067	-0.030	8.151	0.033	0.002	106.647
Export of goods	0.000	0.000	-0.009	0.000	0.000	0.000	0.007	-0.001	0.282	0.000	0.000	0.134
Export of services	0.000	0.000	-0.065	0.001	0.000	0.015	0.042	-0.003	2.014	0.004	0.000	6.940
Import of goods	0.000	0.000	-0.016	0.000	0.000	0.001	0.002	-0.001	0.390	0.000	0.000	0.263
Import of services	-0.001	0.000	-0.075	0.000	0.000	0.025	0.017	-0.004	1.875	0.001	0.000	6.185
Panel 2												
Consumption:	0.000	0.000	-0.012	0.000	0.000	0.000	0.002	-0.001	0.342	0.000	0.000	0.192
Durable goods	-0.005	0.000	-0.053	0.000	0.000	0.008	0.113	0.000	1.151	0.027	0.000	2.667
Semi-durable goods	-0.006	0.000	-0.069	0.000	0.000	0.012	0.152	0.003	1.536	0.049	0.000	4.718
Non-durable goods	-0.002	0.000	-0.025	0.000	0.000	0.002	0.056	0.002	0.557	0.007	0.000	0.625
Services	-0.001	0.000	-0.014	0.000	0.000	0.001	0.030	0.001	0.289	0.002	0.000	0.172
Business investment:	0.000	0.000	-0.040	0.000	0.000	0.004	0.014	-0.006	1.213	0.001	0.000	2.358
Non-res structure & equip	0.000	0.000	-0.033	0.000	0.000	0.003	0.038	-0.006	0.944	0.004	0.000	1.491
Residential structure	0.002	0.000	-0.072	0.002	0.000	0.014	0.097	0.005	1.948	0.023	0.000	6.417
Government spending	0.000	0.000	-0.031	0.000	0.000	0.002	0.005	-0.003	0.906	0.000	0.000	1.326
Government investment	0.002	0.001	-0.257	0.006	0.000	0.158	0.067	-0.034	8.193	0.033	0.003	107.580
Export of goods	0.000	0.000	-0.009	0.000	0.000	0.000	0.007	-0.001	0.281	0.000	0.000	0.133
Export of services	0.000	0.000	-0.066	0.001	0.000	0.016	0.042	-0.002	2.053	0.004	0.000	7.184
Import of goods	0.000	0.000	-0.016	0.000	0.000	0.001	0.002	-0.001	0.389	0.000	0.000	0.262
Import of services	-0.001	0.000	-0.076	0.000	0.000	0.026	0.017	-0.001	1.904	0.001	0.000	6.370

3. LP = Laspeyres approximation; FoF = Fisher of Fishers; TQ = Tornqvist approximation;
4. ***Panel 1*** uses subtraction formulas of equation (17) for LP, equation (23) for FoF, and equation (28) for TQ; ***Panel 2*** uses negative aggregation formulas of equation (18) for LP, equation (24) for FoF, and equation (29) for TQ;
5. Durable goods = consumption - semi-durable goods - non-durable goods - services; Non-residential structures and equipment = business investment - residential structures; Consumption = final domestic demand - govt spending - govt investment - business investment; Export of goods = exports - export services; Import of goods = imports - import services.

Annex 2 (continued)

Table 3: Comparison of Fisher Subtraction with Chain Index Data

	Growth Rate						Chained Level					
	Mean			Variance			Mean			Variance		
	LP	FoF	TQ	LP	FoF	TQ	LP	FoF	TQ	LP	FoF	TQ
Panel 1												
Consumption:	0.001	0.001	-0.011	0.009	0.009	0.009	-0.037	-0.031	0.071	0.006	0.005	0.084
Durable goods	-0.001	0.003	-0.049	0.128	0.131	0.129	-0.043	-0.111	0.539	0.078	0.072	1.675
Semi-durable goods	-0.001	0.005	-0.063	0.216	0.216	0.219	-0.047	-0.122	0.693	0.142	0.119	2.996
Non-durable goods	0.000	0.002	-0.023	0.030	0.030	0.030	-0.006	-0.041	0.263	0.020	0.016	0.413
Services	0.000	0.001	-0.013	0.009	0.009	0.009	-0.025	-0.043	0.131	0.006	0.006	0.118
Business investment:	0.001	0.001	-0.039	0.088	0.089	0.092	-0.062	-0.049	0.250	0.052	0.049	1.019
Non-res structure & equip	0.002	0.002	-0.032	0.005	0.005	0.007	-0.016	-0.043	0.239	0.005	0.005	0.652
Residential structure	0.004	0.001	-0.069	0.022	0.022	0.034	-0.012	-0.077	0.504	0.022	0.018	2.658
Government spending	0.002	0.002	-0.029	0.059	0.059	0.062	-0.029	-0.014	0.235	0.030	0.029	0.578
Government investment	0.013	0.034	-0.245	3.948	3.993	4.092	-0.182	-0.146	1.875	2.206	1.942	46.383
Export of goods	0.001	0.002	-0.008	0.006	0.006	0.006	-0.005	0.002	0.070	0.003	0.003	0.060
Export of services	0.012	0.013	-0.054	0.291	0.287	0.306	-0.037	-0.004	0.497	0.137	0.138	3.193
Import of goods	0.000	0.000	-0.016	0.005	0.005	0.007	-0.043	-0.043	0.017	0.005	0.005	0.110
Import of services	0.000	0.004	-0.075	0.124	0.121	0.161	-0.161	-0.197	0.135	0.100	0.106	2.961
Panel 2												
Consumption:	0.001	0.001	-0.011	0.009	0.009	0.009	-0.037	-0.031	0.071	0.006	0.005	0.084
Durable goods	-0.001	0.004	-0.049	0.128	0.131	0.129	-0.043	-0.113	0.545	0.078	0.072	1.694
Semi-durable goods	-0.001	0.006	-0.063	0.216	0.216	0.219	-0.047	-0.122	0.695	0.142	0.119	3.010
Non-durable goods	0.000	0.002	-0.023	0.030	0.030	0.030	-0.006	-0.040	0.263	0.020	0.016	0.414
Services	0.000	0.001	-0.013	0.009	0.009	0.009	-0.025	-0.043	0.131	0.006	0.006	0.118
Business investment:	0.001	0.001	-0.040	0.088	0.089	0.092	-0.062	-0.050	0.258	0.052	0.049	1.035
Non-res structure & equip	0.002	0.002	-0.032	0.005	0.005	0.007	-0.016	-0.046	0.241	0.005	0.005	0.655
Residential structure	0.004	0.001	-0.069	0.022	0.022	0.033	-0.012	-0.067	0.495	0.022	0.017	2.641
Government spending	0.002	0.002	-0.029	0.059	0.059	0.062	-0.029	-0.013	0.232	0.030	0.029	0.576
Government investment	0.013	0.034	-0.246	3.948	4.000	4.092	-0.182	-0.153	1.867	2.206	1.947	46.566
Export of goods	0.001	0.002	-0.008	0.006	0.006	0.006	-0.005	0.002	0.070	0.003	0.003	0.060
Export of services	0.012	0.013	-0.055	0.291	0.287	0.308	-0.037	-0.002	0.493	0.137	0.138	3.274
Import of goods	0.000	0.000	-0.016	0.005	0.005	0.007	-0.043	-0.043	0.018	0.005	0.005	0.110
Import of services	0.000	0.004	-0.076	0.124	0.121	0.164	-0.161	-0.196	0.124	0.100	0.106	3.036

6. LP = Laspeyres approximation; FoF = Fisher of Fishers; TQ = Tornqvist approximation;
7. ***Panel 1*** uses subtraction formulas of equation (17) for LP, equation (23) for FoF, and equation (28) for TQ; ***Panel 2*** uses negative aggregation formulas of equation (18) for LP, equation (24) for FoF, and equation (29) for TQ;
8. Durable goods = consumption - semi-durable goods - non-durable goods - services; Non-residential structures and equipment = business investment - residential structures; Consumption = final domestic demand - govt spending - govt investment - business investment; Export of goods = exports - export services; Import of goods = imports - import services.

Annex 2 (continued)
Table 4
Comparison of Depreciation Rates (per cent)

	Machinery and Equipment		Non-residential Construction	
	Chain-1992 dollar	Laspeyres-1992 dollar	Chain-1992 dollar	Laspeyres-1992 dollar
1962	8.50	9.57	2.51	2.46
1963	8.09	9.78	2.51	2.46
1964	9.33	9.98	2.52	2.47
1965	9.05	10.06	2.51	2.48
1966	9.09	10.01	2.51	2.49
1967	8.90	9.80	2.52	2.49
1968	8.70	9.66	2.52	2.51
1969	8.80	9.73	2.55	2.54
1970	8.81	9.81	2.59	2.58
1971	8.96	9.93	2.63	2.62
1972	9.12	10.09	2.66	2.66
1973	9.38	10.30	2.71	2.71
1974	9.60	10.38	2.75	2.76
1975	9.68	10.36	2.79	2.81
1976	9.83	10.36	2.83	2.85
1977	9.99	10.39	2.86	2.89
1978	10.25	10.51	2.90	2.93
1979	10.34	10.73	2.93	2.98
1980	10.67	10.89	2.97	3.02
1981	10.64	11.05	2.99	3.04
1982	10.54	10.89	3.01	3.05
1983	10.77	11.07	3.04	3.08
1984	11.09	11.45	3.09	3.14
1985	11.49	11.90	3.16	3.20
1986	11.58	12.28	3.22	3.25
1987	12.37	12.54	3.29	3.32
1988	12.37	12.72	3.36	3.39
1989	12.51	12.71	3.42	3.45
1990	12.42	12.48	3.48	3.50
1991	12.48	12.60	3.54	3.57
1992	12.54	12.65	3.60	3.64
1993	12.59	12.65	3.70	3.73
1994	12.64	12.73	3.79	3.82
1995	12.49	12.67	3.85	3.89
1996	12.35	12.62	3.92	3.95
1997	12.41	12.72	3.95	3.99
1998	12.16	12.76	3.96	3.99
1999	12.45	12.90	3.95	3.98
2000	11.92	13.06	3.93	3.97

Annex 3

Why the accumulation rule does not hold for current-dollar capital stock[17]

The fact that the construction of current-dollar capital stock incorporates the effect of price changes effectively prevents the accumulation rule to hold. Recall that the accumulation rule for Laspeyres constant dollar capital stock is specified as:

$$K_t^R = K_{t-1}^R + I_t^R - D_t^R . \tag{A3.1}$$

Current-dollar series are then constructed as:

$$K_t^C = K_t^R . P_t \ , \qquad K_{t-1}^C = K_{t-1}^R . P_{t-1} \ ,$$

$$D_t^C = D_t^R . P_t \ , \qquad I_t^C = I_t^R . P_t . \tag{A3.2}$$

Substitute (A3.2) into (A3.1) to give:

$$\frac{K_t^C}{P_t} = \frac{K_{t-1}^C}{P_{t-1}} + \frac{I_t^C}{P_t} - \frac{D_t^C}{P_t}$$

or $$K_t^C = K_{t-1}^C . \frac{P_t}{P_{t-1}} + I_t^C - D_t^C . \tag{A3.3}$$

If $P_t \neq P_{t-1}$ (which is true in most cases), the accumulation rule will not hold for current-dollar capital stocks.

[17] The derivation is based on information provided to us by Dr. Kuen Huang of the Capital Stock Division at Statistics Canada. Any errors, however, are the sole responsibility of the authors.