ISSN 1927-7032 (Print)
ISSN 1927-7040 (Online)

International Journal of Statistics and Probability

Vol. 10, No. 3 May 2021

CANADIAN CENTER OF SCIENCE AND EDUCATION

INTERNATIONAL JOURNAL OF STATISTICS AND PROBABILITY

An International Peer-reviewed and Open Access Journal for Statistics and Probability

International Journal of Statistics and Probability (ISSN: 1927-7032; E-ISSN: 1927-7040) is an open-access, international, double-blind peer-reviewed journal published by the Canadian Center of Science and Education. This journal, published **bimonthly** (January, March, May, July, September and November) in both **print and online versions**, keeps readers up-to-date with the latest developments in all areas of statistics and probability.

The scopes of the journal:

- Computational statistics
- Design of experiments
- Sample survey
- Statistical modelling
- Statistical theory
- Probability theory

The journal is included in:

- BASE
- Google Scholar
- JournalTOCs
- LOCKSS
- SHERPA/RoMEO
- Ulrich's

Copyright Policy

Copyrights for articles are retained by the authors, with first publication rights granted to the journal/publisher. Authors have rights to reuse, republish, archive, and distribute their own articles after publication. The journal/publisher is not responsible for subsequent uses of the work. Authors shall permit the publisher to apply a DOI to their articles and to archive them in databases and indexes such as EBSCO, DOAJ, and ProQuest.

Open-access Policy

We follow the Gold Open Access way in journal publishing. This means that our journals provide immediate open access for readers to all articles on the publisher's website. The readers, therefore, are allowed to read, download, copy, distribute, print, search, link to the full texts or use them for any other lawful purpose. The operations of the journals are alternatively financed by article processing charges paid by authors or by their institutions or funding agencies.
All articles published are open-access articles distributed under the terms and conditions of the Creative Commons Attribution license.

Submission Policy

Submission of an article implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the authorities responsible where the work was carried out. However, we accept submissions that have previously appeared on preprint servers (for example: arXiv, bioRxiv, Nature Precedings, Philica, Social Science Research Network, and Vixra); have previously been presented at conferences; or have previously appeared in other "non-journal" venues (for example: blogs or posters). Authors are responsible for updating the archived preprint with the journal reference (including DOI) and a link to the published articles on the appropriate journal website upon publication.

The publisher and journals have a zero-tolerance plagiarism policy. We check the issue using two methods: a plagiarism prevention tool (iThenticate) and a reviewer check. All submissions will be checked by iThenticate before being sent to reviewers.

We insist a rigorous viewpoint on the self-plagiarism. The self-plagiarism is plagiarism, as it fails to contribute to the research and science.

IJSP accepts both Online and Email submission. The online system makes readers to submit and track the status of their manuscripts conveniently. For any questions, please contact ijsp@ccsenet.org.

Online Available: http://ijsp.ccssent.org

ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

Editorial Team

Editor-in-Chief

Chin-Shang Li, University of California, Davis, USA

Associate Editors

Anna Grana', University of Palermo, Italy

Gane Samb Lo, University Gaston Berger, Senegal

Getachew Asfaw Dagne, University of South Florida, USA

Vyacheslav M. Abramov, Swinburne University of Technology, Australia

Editorial Assistant

Wendy Smith, Canadian Center of Science and Education, Canada

Reviewers

Abdullah Smadi, Jordan
Abouzar Bazyari, Iran
Adekola Lanrewaju Olumide, Nigeria
Adeyeye Awogbemi, Nigeria
Afsin Sahin, Turkey
Ali Reza Fotouhi, Canada
Bibi Abdelouahab, Algeria
Carla Santos, Portugal
Carla J. Thompson, USA
Carolyn Huston, Australia
Daoudi Hamza, Algeria
Doug Lorenz, USA
Emmanuel Akpan, Nigeria
Emmanuel John Ekpenyong, Nigeria
Encarnación Alvarez-Verdejo, Spain
Faisal Khamis, Canada
Félix Almendra-Arao, México
Frederic Ouimet, USA
Gabriel A Okyere, Ghana
Gennaro Punzo, Italy
Gerardo Febres, Venezuela
Habib ur Rehman, Thailand
Ivair R. Silva, Brazil
Jacek Bialek, Poland
Jingwei Meng, USA
Kartlos Kachiashvili, Georgia
Kassim S. Mwitondi, UK
Keshab R. Dahal, USA
Krishna K. Saha, USA
Man Fung LO, Hong Kong
Mingao Yuan, USA
Mohamed Hssikou, Morocco
Mohamed Salem Abdelwahab Muiftah, Libya
Mohieddine Rahmouni, Tunisia
Nahid Sanjari Farsipour, Iran
Navin Chandra, India
Noha Youssef, Egypt
Olusegun Michael Otunuga, USA
Pablo José Moya Fernández, Spain
Philip Westgate, USA
Pourab Roy, USA
Priyantha Wijayatunga, Sweden
Qingyang Zhang, USA
Renisson Neponuceno de Araujo Filho, Brazil
Reza Momeni, Iran
Robert Montgomery, USA
Sajid Ali, Pakistan
Samir Khaled Safi, Palestine
Sharandeep Singh, India
Shatrunjai Pratap Singh, USA

Shuling Liu, USA
Sohair F. Higazi, Egypt
Soukaina Douissi, Morocco
Subhradev Sen, India
Tomás R. Cotos-Yáñez, Spain
Vilda Purutcuoglu, Turkey
Wei Zhang, USA
Weizhong Tian, USA
Wojciech Gamrot, Poland
Xiangchun Yu, China
Yuvraj Sunecher, Mauritius
Zaixing Li, China

International Journal of Statistics and Probability; Vol. 10, No. 3; May 2021
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

Contents

International Journal of Statistics and Probability; Vol. 10, No. 3; May 2021
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

Empirical Likelihood Ratio Test for Seemingly Unrelated Regression Models

Chuanhua Wei[1] & Xiaoxiao Ma[1]

[1] School of Science, Minzu University of China, Beijing 100081, China

Correspondence: Correspondence: Chuanhua Wei, School of Science, Minzu University of China, Beijing 100081, P.R.China.

Received: November 26, 2020 Accepted: January 23, 2021 Online Published: March 4, 2021

doi:10.5539/ijsp.v10n3p1 URL: https://doi.org/10.5539/ijsp.v10n3p1

Abstract

This paper considers the problem of testing independence of equations in a seemingly unrelated regression model. A novel empirical likelihood test approach is proposed, and under the null hypothesis it is shown to follow asymptotically a chi-square distribution. Finally, simulation studies and a real data example are conducted to illustrate the performance of the proposed method.

Keywords: seemingly unrelated regression, empirical likelihood, independence

1. Introduction

The Seemingly Unrelated Regression (SUR) of Zellner (1962) is an important tool to analyze multiple equations with correlated disturbances. SUR models have been studied extensively by statistician and econometrician and applied in many areas, more details can be found in Srivastava and Giles (1987) and Fiebig (2001).

Due to the correlation of the model errors in regression equations, the SUR model allows one to estimate the regression coefficients more efficiently than each of the regression equations is estimated separately with the correlation is ignored. It is by now clear that for the traditional linear SUR model, the Generalized Least Squares (GLS) estimator is more efficient than its Ordinary Least Squares (OLS) counterpart. They are equivalence if the error covariance of the SUR model is diagonal. Therefore, the problem of testing independence of equations of a SUR model is important. Many testing approaches have been proposed for this problem. Breusch and Pagan (1980) proposed a Lagrange multiplier test statistic. Dufour and Khalaf (2002) extended the exact independence test method of Harvey and Phillips (1982) to the multi-equation framework. Tsay (2004) constructed a multivariate independent test statistic for SUR model with serially correlated errors.

Different to the above methods, we propose a empirical likelihood test statistic. The empirical likelihood of Owen (1988,1990) is an effective nonparametric inference method. More references can be found in Owen (2001).

The paper is organized as follows. The empirical log-likelihood ratio test statistic is given in Section 2. Section 3 conducts some simulation studies to illustrate the performance of the proposed method. An empirical example is also provided to demonstrate the usefulness of this test. Finally, conclusion is given in Section 4. The Appendix provides the proofs of the main results.

2. Test Statistic and Its Properties

Consider the following SUR model that comprises the p regression equations

$$\mathbf{Y}_i = \mathbf{X}_i\boldsymbol{\beta}_i + \boldsymbol{\varepsilon}_i, \quad i = 1, 2, \cdots, p, \tag{2.1}$$

with $\mathbf{Y}_i = (y_{i1}, y_{i2}, \cdots, y_{in})^{\mathrm{T}}$ is a $n \times 1$ vector of responses, $\mathbf{X}_i = (\mathbf{x}_{i1}, \mathbf{x}_{i2}, \cdots, \mathbf{x}_{in})^{\mathrm{T}}$ is a full-rank $n \times q_i$ matrix of regressors with $\mathbf{x}_{ik}^{\mathrm{T}} = (x_{ik1}, x_{ik2}, \cdots, x_{ikq_i})$, $\boldsymbol{\beta}_i = (\beta_{i1}, \beta_{i2}, \cdots, \beta_{iq_i})^{\mathrm{T}}$ is a vector of q_i-dimensional unknown parameters, and $\boldsymbol{\varepsilon}_i = (\varepsilon_{i1}, \varepsilon_{i2}, \cdots, \varepsilon_{in})^{\mathrm{T}}$ is a $n \times 1$ error vector with $E\varepsilon_{ik} = 0, k = 1, 2, \cdots, n$.

The model (2.1) can be rewritten in vectors and matrixces,

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \tag{2.2}$$

where

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \\ \vdots \\ \mathbf{Y}_p \end{bmatrix}, \mathbf{X} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{X}_p \end{bmatrix}, \boldsymbol{\beta} = \begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \\ \vdots \\ \boldsymbol{\beta}_p \end{bmatrix}, \boldsymbol{\varepsilon} = \begin{bmatrix} \boldsymbol{\varepsilon}_1 \\ \boldsymbol{\varepsilon}_2 \\ \vdots \\ \boldsymbol{\varepsilon}_p \end{bmatrix}$$

so that $\mathbf{X}$ is a $(np) \times q$ matrix, $\mathbf{Y}$ and $\boldsymbol{\varepsilon}$ each have dimension $(np) \times 1$ and $\boldsymbol{\beta}$ has dimension $q \times 1$, with $q = \sum_{i=1}^{p} q_i$. The basic assumptions underlying the disturbances of model (2.1) are

$$E(\varepsilon_{ik}\varepsilon_{jl}) = \begin{cases} \sigma_{ij}, & k = l, \\ 0, & \text{otherwise,} \end{cases}$$

for $1 \le i, j \le p$ and $1 \le k, l \le n$. Then, we have $Var(\boldsymbol{\varepsilon}_i) = E\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i^{\mathrm{T}} = \sigma_{ii}\mathbf{I}_n$, and $Cov(\boldsymbol{\varepsilon}_i, \boldsymbol{\varepsilon}_j) = E\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_j^{\mathrm{T}} = \sigma_{ij}\mathbf{I}_n$, with $\mathbf{I}_n$ is the identity matrix of order n. Therefore, the $np \times 1$ disturbance vector $\boldsymbol{\varepsilon}$ has the following variance-covariance matrix

$$\boldsymbol{\Omega} = E(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^{\mathrm{T}}) = \boldsymbol{\Sigma} \otimes \mathbf{I}_n, \tag{2.3}$$

with

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{bmatrix}.$$

We consider the problem of testing independence of p equations in model (2.1), which may be expressed as $H_0 : \sigma_{ij} = 0$ for $1 \le i < j \le p$, or equivalently

$$H_0 : \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11} & 0 & \cdots & 0 \\ 0 & \sigma_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{pp} \end{bmatrix}. \tag{2.4}$$

Letting $\mathbf{U}_k = (\varepsilon_{1k}\varepsilon_{2k}, \varepsilon_{1k}\varepsilon_{3k}, \cdots, \varepsilon_{1k}\varepsilon_{pk}, \varepsilon_{2k}\varepsilon_{3k}, \cdots, \varepsilon_{2k}\varepsilon_{pk}, \cdots, \varepsilon_{(p-1)k}\varepsilon_{pk})^{\mathrm{T}}, k = 1, 2, \cdots, n$, it is obvious that there are $N = \frac{p(p-1)}{2}$ elements in $\mathbf{U}_k$. For example, for the two equations case $p = 2$, we have $\mathbf{U}_k = \varepsilon_{1k}\varepsilon_{2k}$ and $N = 1$, for the three equations case $p = 3$, $\mathbf{U}_k = (\varepsilon_{1k}\varepsilon_{2k}, \varepsilon_{1k}\varepsilon_{3k}, \varepsilon_{2k}\varepsilon_{3k})^{\mathrm{T}}$ and $N = 3$. It is obvious that testing for diagonality of the $\boldsymbol{\Sigma}$ is equivalent to testing whether $E\mathbf{U}_k = \mathbf{0}, k = 1, 2, \cdots, n$. By Owen(1990), this can be done using the empirical likelihood method. Let $p_1, p_2, \cdots, p_n$ be nonnegative numbers summing to unity. Then the corresponding empirical log-likelihood ratio can be defined as

$$\bar{l}_n = -2\max\left\{\sum_{k=1}^{n}\log(np_k) : \sum_{k=1}^{n} p_k\mathbf{U}_k = 0, p_k \ge 0, \sum_{k=1}^{n} p_k = 1\right\}. \tag{2.5}$$

However, $\varepsilon_{ik}'s$ in $\mathbf{U}_k$ are unknown, then $\bar{l}_n$ cannot be used directly. To solve the problem, we can replace ε_{ik} by its estimator

$$\hat{\varepsilon}_{ik} = y_{ik} - \mathbf{x}_{ik}^{\mathrm{T}}\hat{\boldsymbol{\beta}}_i,$$

with $\hat{\boldsymbol{\beta}}_i = (\mathbf{X}_i^{\mathrm{T}}\mathbf{X}_i)^{-1}\mathbf{X}_i^{\mathrm{T}}\mathbf{Y}_i$ is the least-squares estimator of the coefficients contained in the ith equation of model (1.1). Then, use $\hat{\varepsilon}_{ik}$ to replace ε_{ik} in $\mathbf{U}_k$, the estimated empirical log-likelihood ratio is then defined by

$$l_n = -2\max\left\{\sum_{k=1}^{n}\log(np_k) : \sum_{k=1}^{n} p_k\boldsymbol{\xi}_k = 0, p_k \ge 0, \sum_{k=1}^{n} p_k = 1\right\}, \tag{2.6}$$

where $\boldsymbol{\xi}_k = (\hat{\varepsilon}_{1k}\hat{\varepsilon}_{2k}, \hat{\varepsilon}_{1k}\hat{\varepsilon}_{3k}, \cdots, \hat{\varepsilon}_{1k}\hat{\varepsilon}_{pk}, \hat{\varepsilon}_{2k}\hat{\varepsilon}_{3k}, \cdots, \hat{\varepsilon}_{2k}\hat{\varepsilon}_{pk}, \cdots, \hat{\varepsilon}_{(p-1)k}\hat{\varepsilon}_{pk})$.

By the Lagrange multiplier technique, the empirical log-likelihood ratio can be represented as

$$l_n = 2\sum_{k=1}^{n}\log\left(1 + \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{\xi}_k\right), \tag{2.7}$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_n)^T$ is the solution of the equation

$$\frac{1}{n}\sum_{k=1}^{n}\frac{\boldsymbol{\xi}_k}{1 + \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{\xi}_k} = 0. \tag{2.8}$$

The following theorem indicates that l_n is asymptotically distributed as a χ^2-distribution.

Theorem 2.1. Suppose the assumptions 1-2 given in Appendix hold, under the null hypothesis, as $n \to \infty$, we have

$$l_n \xrightarrow{D} \chi^2_N,$$

where χ^2_N is a χ^2-distribution with $N = \frac{p(p-1)}{2}$ degrees of freedom.

Remark 2.1 For the testing problem (2.4), Breusch and Pagan (1980) proposed a Lagrange multiplier test statistic. This is based upon the sample correlation coefficients of the OLS residuals:

$$\mathrm{LM} = n \sum_{i=1}^{p-1} \sum_{j=i}^{n} \hat{\rho}_{ij}^2,$$

where $\hat{\rho}_{ij}$ is the sample estimate of the pair-wise correlation of the residuals. Specifically,

$$\hat{\rho}_{ij} = \hat{\rho}_{ji} = \frac{\sum\limits_{k=1}^{n} \hat{\varepsilon}_{ik}\hat{\varepsilon}_{jk}}{(\sum\limits_{k=1}^{n} \hat{\varepsilon}_{ik}^2)^{1/2}(\sum\limits_{k=1}^{n} \hat{\varepsilon}_{jk}^2)^{1/2}}$$

Under the null hypothesis, LM has an asymptotic χ^2_N distribution, too.

3. Numerical Studies

3.1 Simulation Studies

In this subsection, we conducted some simulations to illustrate the finite sample properties of the proposed test procedure. In our simulations, the data are generated from the following SUR model

$$y_{ik} = x_{ik1}\beta_{i1} + x_{ik2}\beta_{i2} + \varepsilon_i, i = 1, 2, 3, k = 1, 2, \cdots, n$$

where $x_{ik1} \sim N(0, 1)$, $x_{ik2} \sim U(-2, 2)$, and $x_{ik3} \sim N(2, 1)$. The parameters are set as $\beta_{11} = 1, \beta_{12} = 2, \beta_{21} = 2, \beta_{22} = 3, \beta_{31} = -1, \beta_{32} = 3$. The model error $\varepsilon_{ik} \sim N(0, \sigma_{ii})$ and

$$\mathrm{Cov}[(\varepsilon_{1k}, \varepsilon_{2k}, \varepsilon_{3k})^{\mathrm{T}}] = \begin{bmatrix} \sigma_{11} & \rho_{12}\sqrt{\sigma_{11}\sigma_{22}} & \rho_{13}\sqrt{\sigma_{11}\sigma_{33}} \\ \rho_{12}\sqrt{\sigma_{11}\sigma_{22}} & \sigma_{22} & \rho_{23}\sqrt{\sigma_{22}\sigma_{33}} \\ \rho_{13}\sqrt{\sigma_{11}\sigma_{33}} & \rho_{23}\sqrt{\sigma_{22}\sigma_{33}} & \sigma_{33} \end{bmatrix}.$$

where $\sigma_{11} = 0.25, \sigma_{22} = 0.64, \sigma_{33} = 0.49$.

In order to examine the empirical size of the proposed empirical likelihood (EL) test and the Lagrange multiplier (LM) test statistic, we set $\boldsymbol{\rho} = (\rho_{12}, \rho_{13}, \rho_{23}) = (0, 0, 0)$, and $n = 30, 50, 100, 150, 200, 300, 400$, 1000 replications were run and the rejection rate under a given significance level $\alpha(0.01, 0.05, 0.10)$ was computed as the empirical size of the test, and the results are reported in Table 3.1. From the results, we can see that the empirical size of the proposed EL test is quite large for small samples. The size distortion of the LM test is smaller than that of the EL test for small samples. The sizes of both the EL test and the LM test converge to the correct nominal levels when n grows, as would be expected. The fact that the size distortion of the EL test is relatively large indicates that the approximation of the finite sample distribution in small samples using the asymptotic χ^2 is relatively poor. The phenomenon was also reported by Dong and Giles (2007), Liu *et al.* (2008) and Liu *et al.* (2011) in other testing problems. According to Owen (2001), this may be improved by using Fisher's F-distribution, or Bartlett correction, or bootstrap sample.

To assess the power of the EL and the LM tests, we took the values of $\boldsymbol{\rho}$ to be each of the following values, (0.1,0,0), (0,0.5,0), (0,0,-0.9), (0.2,0.3,0), (-0.5,0,0.4), (0,-0.5,-0.8), (0.1,-0.2,0.1), (0.1,0.2,0.8), (-0.5,0.4,-0.6), and $n = 30, 50$. Results are presented in Table 3.2. we can see that the power of the EL is bigger than that of the LM test for significance levels of 10%, 5%, and 1%.

Table 3.1. Empirical sizes of EL and LM tests

n	$\alpha =$	0.01	$\alpha =$	0.05	$\alpha =$	0.10
	EL	LM	EL	LM	EL	LM
30	0.067	0.008	0.142	0.048	0.215	0.095
50	0.032	0.014	0.106	0.040	0.155	0.096
100	0.021	0.009	0.067	0.049	0.121	0.090
150	0.013	0.012	0.053	0.050	0.107	0.104
200	0.011	0.009	0.053	0.047	0.119	0.096
300	0.011	0.012	0.047	0.047	0.111	0.102
400	0.011	0.009	0.053	0.051	0.104	0.095

Table 3.2. Power comparison of the EL test with the LM test

n	$\boldsymbol{\rho}$	$\alpha =$	0.01	$\alpha =$	0.05	$\alpha =$	0.10
		EL	LM	EL	LM	EL	LM
30	(0.1,0,0)	0.092	0.011	0.183	0.061	0.258	0.126
	(0,0.5,0)	0.594	0.286	0.765	0.579	0.832	0.691
	(0,0,-0.9)	1.000	1.000	1.000	1.000	1.000	1.000
	(0.2,0.3,0)	0.307	0.096	0.508	0.289	0.606	0.406
	(-0.5,0,0.4)	0.858	0.578	0.96	0.858	0.977	0.924
	(0,-0.5,-0.8)	1.000	1.000	1.000	1.000	1.000	1.000
	(0.1,0.2,-0.1)	0.183	0.036	0.308	0.144	0.419	0.231
	(0.1,0.2,0.8)	0.993	0.963	0.997	0.998	0.999	0.998
	(-0.5,0.4,-0.6)	0.936	0.898	0.978	0.959	0.986	0.989
50	(0.1,0,0)	0.062	0.019	0.158	0.086	0.247	0.154
	(0,0.5,0)	0.800	0.659	0.923	0.874	0.962	0.934
	(0,0,-0.9)	1.000	1.000	1.000	1.000	1.000	1.000
	(0.2,0.3,0)	0.460	0.258	0.616	0.512	0.741	0.655
	(-0.5,0,0.4)	0.988	0.941	0.996	0.991	0.997	0.994
	(0,-0.5,-0.8)	1.000	1.000	1.000	1.000	1.000	1.000
	(0.1,0.2,-0.1)	0.226	0.062	0.396	0.232	0.489	0.360
	(0.1,0.2,0.8)	0.999	1.000	1.000	1.000	1.000	1.000
	(-0.5,0.4,-0.6)	0.992	0.991	1.000	1.000	1.000	0.999

3.2 A Real Example

Baltagi and Griffin (1983) considered the following gasoline demand equation

$$\ln \frac{\mathrm{Cas}}{\mathrm{Car}} = \alpha + \beta_1 \ln \frac{\mathrm{Y}}{\mathrm{N}} + \beta_1 \ln \frac{\mathrm{P_{MG}}}{\mathrm{P_{GDP}}} + \beta_1 \ln \frac{\mathrm{Car}}{\mathrm{N}} + u,$$

where Gas/Car is motor gasoline consumption per auto, Y/N is real per capita income, PMG/PGDP is real motor gasoline price and Car/N denotes the stock of cars per capita. This panel consists of annual observations across 18 OECD countries, covering the period 1960-78. The data for this example can be found in package **plm** of the open source software **R**. Baltagi (2008) (P 244) considered the problem of testing the independence of the first two countries: Austria and Belgium. The observed values of Breusch-Pagan (1980) Lagrange multiplier test statistic and the Likelihood Ratio test statistic for this problem are 0.947 and 1.778, respectively. The observed value of the proposed empirical likelihood test statistic is 2.343. All the three test statistics are distributed as χ_1^2 under the null hypothesis, and do not reject the null hypothesis.

4. Conclusion

This paper proposes a novel approach for the independence test for the disturbances of the SUR models based on the empirical-likelihood method. The proposed test statistic under the null hypothesis is shown to has an asymptotic chi-square distribution. Compared to the Lagrange multiplier test statistic, the simulation experiment demonstrates that the proposed method performs satisfactorily. Furthermore, our approach can be applied to the case that the model errors of one equation of SUR model are correlated.

Appendix: Proof of the main results

We begin with the following assumptions required to derive the main results. These assumptions are quite mild and can be easily satisfied.

Assumption 1. $E(\mathbf{x}_{ik}\varepsilon_{ik}) = \mathbf{0}$ for $1 \le i \le p, 1 \le k \le n$.

Assumption 2. $E(\mathbf{X}_i^{\mathrm{T}}\mathbf{X}_i)$ is nonsingular, $1 \le i \le p$.

In order to prove that main results, we first introduce several lemmas.

Lemma 1 Under the assumptions 1-2 and the null hypothesis, we have

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\boldsymbol{\xi}_k \xrightarrow{D} N(\mathbf{0}, \boldsymbol{\Omega}),$$

with $\sigma_k^2 = \sigma_{kk}$ and

$$\boldsymbol{\Omega} = \begin{bmatrix} \sigma_1^2\sigma_2^2 & 0 & \cdots & 0 \\ 0 & \sigma_1^2\sigma_3^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{p-1}^2\sigma_p^2 \end{bmatrix}.$$

Proof: By the result of Tsay (2004), we can obtain Lemma 1.

Lemma 2 Under the assumptions 1-2 and the null hypothesis, we have

$$\frac{1}{n}\sum_{k=1}^{n}\boldsymbol{\xi}_k\boldsymbol{\xi}_k^{\mathrm{T}} \xrightarrow{p} \boldsymbol{\Omega}.$$

Proof: Firstly, we consider $\frac{1}{n}\sum_{k=1}^{n}\hat{\varepsilon}_{ik}\hat{\varepsilon}_{jk}\hat{\varepsilon}_{sk}\hat{\varepsilon}_{lk}$, one element of $\frac{1}{n}\sum_{k=1}^{n}\boldsymbol{\xi}_k\boldsymbol{\xi}_k^{\mathrm{T}}$. Let $e_{ik} = \mathbf{x}_{ik}^{\mathrm{T}}(\boldsymbol{\beta}_i - \hat{\boldsymbol{\beta}}_i)$, by the definition of $\boldsymbol{\xi}_k$, we have

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n}\hat{\varepsilon}_{ik}\hat{\varepsilon}_{jk}\hat{\varepsilon}_{sk}\hat{\varepsilon}_{lk} &= \frac{1}{N}\sum_{k=1}^{n}(e_{ik}+\varepsilon_{ik})(e_{jk}+\varepsilon_{jk})(e_{sk}+\varepsilon_{sk})(e_{lk}+\varepsilon_{lk}) \\ &= \frac{1}{n}\sum_{k=1}^{n}\varepsilon_{ik}\varepsilon_{jk}\varepsilon_{sk}\varepsilon_{lk} + \sum_{i=1}^{15} I_i. \end{aligned}$$

We let $I_1 = \frac{1}{n}\sum_{k=1}^{n} e_{ik}e_{jk}e_{sk}e_{lk}$, By Lemma 3 in Owen (1990), we have

$$\begin{aligned} |I_1| &= \frac{1}{n}\sum_{k=1}^{n}|\mathbf{x}_{ik}^{\mathrm{T}}(\boldsymbol{\beta}_i - \hat{\boldsymbol{\beta}}_i)\mathbf{x}_{jk}^{\mathrm{T}}(\boldsymbol{\beta}_j - \hat{\boldsymbol{\beta}}_j)\mathbf{x}_{sk}^{\mathrm{T}}(\boldsymbol{\beta}_s - \hat{\boldsymbol{\beta}}_s)\mathbf{x}_{lk}^{\mathrm{T}}(\boldsymbol{\beta}_l - \hat{\boldsymbol{\beta}}_l)| \\ &\le ||\boldsymbol{\beta}_i - \hat{\boldsymbol{\beta}}_i||||\boldsymbol{\beta}_j - \hat{\boldsymbol{\beta}}_j||||\boldsymbol{\beta}_s - \hat{\boldsymbol{\beta}}_s||||\boldsymbol{\beta}_l - \hat{\boldsymbol{\beta}}_l||\frac{1}{n}\sum_{k=1}^{n}||\mathbf{x}_{ik}||||\mathbf{x}_{jk}||||\mathbf{x}_{sk}||||\mathbf{x}_{lk}|| \\ &\le ||\boldsymbol{\beta}_i - \hat{\boldsymbol{\beta}}_i||||\boldsymbol{\beta}_j - \hat{\boldsymbol{\beta}}_j||||\boldsymbol{\beta}_s - \hat{\boldsymbol{\beta}}_s||||\boldsymbol{\beta}_l - \hat{\boldsymbol{\beta}}_l|| \max_{1\le i\le p, 1\le k\le n}||\mathbf{x}_{ik}||^4 \\ &= O_p(n^{-1/2})O_p(n^{-1/2})O_p(n^{-1/2})O_p(n^{-1/2})o_p(n^2) \\ &= o_p(1). \end{aligned}$$

Hence, $I_1 = o_p(1)$. By the similar way, we can prove that $I_i = o_p(1), i = 2, 3, \cdots, 15$. Thus,

$$\frac{1}{n}\sum_{k=1}^{n}\hat{\varepsilon}_{ik}\hat{\varepsilon}_{jk}\hat{\varepsilon}_{sk}\hat{\varepsilon}_{lk} = \frac{1}{n}\sum_{k=1}^{n}\varepsilon_{ik}\varepsilon_{jk}\varepsilon_{sk}\varepsilon_{lk} + o_p(1),$$

and

$$\frac{1}{n}\sum_{k=1}^{n}\boldsymbol{\xi}_k\boldsymbol{\xi}_k^{\mathrm{T}} = \frac{1}{n}\sum_{k=1}^{n}\mathbf{U}_k\mathbf{U}_k^{\mathrm{T}} + o_p(1).$$

Finally, under the null hypothesis, and by the law of large numbers, we have

$$\frac{1}{n}\sum_{k=1}^{n}\boldsymbol{\xi}_k\boldsymbol{\xi}_k^{\mathrm{T}} \xrightarrow{p} \boldsymbol{\Omega}.$$

Lemma 3 Under the Assumptions 1-2, we have

$$\max_{1\le k\le n}\|\boldsymbol{\xi}_k\| = o_p(n^{1/2}),$$

where $\|\cdot\|$ is the Euclidean norm with $\|\mathbf{a}\| = (a_1^2+\cdots+a_k^2)^{1/2}$ and $\mathbf{a} = (a_1,\cdots,a_k)^{\mathrm{T}}$.

Proof:

$$\begin{aligned}\max_{1\le k\le n}|\hat{\varepsilon}_{ik}\hat{\varepsilon}_{jk}| &= \max_{1\le k\le n}\left|\left[\mathbf{x}_{ik}^{\mathrm{T}}(\boldsymbol{\beta}_i-\hat{\boldsymbol{\beta}}_i)+\varepsilon_{ik}\right]\left[\mathbf{x}_{jk}^{\mathrm{T}}(\boldsymbol{\beta}_j-\hat{\boldsymbol{\beta}}_j)+\varepsilon_{jk}\right]\right| \\ &\le \left(\|\boldsymbol{\beta}_i-\hat{\boldsymbol{\beta}}_i\|\max_{1\le k\le n}\|\mathbf{x}_{ik}\|\right)\times\left(\|\boldsymbol{\beta}_j-\hat{\boldsymbol{\beta}}_j\|\max_{1\le k\le n}\|\mathbf{x}_{ik}\|\right)+\max_{1\le k\le n}|\varepsilon_{ik}\varepsilon_{jk}| \\ &\quad+\left(\|\boldsymbol{\beta}_i-\hat{\boldsymbol{\beta}}_i\|\max_{1\le k\le n}\|\mathbf{x}_{ik}\|\right)\max_{1\le k\le n}|\varepsilon_{ik}|+\left(\|\boldsymbol{\beta}_j-\hat{\boldsymbol{\beta}}_j\|\max_{1\le k\le n}\|\mathbf{x}_{ik}\|\right)\max_{1\le k\le n}|\varepsilon_{jk}|.\end{aligned}$$

By Lemma 3 in Owen (1990), we can prove that

$$\max_{1\le i\le p,1\le k\le n}\|\mathbf{x}_{ik}\| = o_p(n^{1/2}),\ \max_{1\le i\le p,1\le k\le n}|\varepsilon_{ik}| = o_p(n^{1/2}),\ \max_{1\le k\le n}|\varepsilon_{ik}\varepsilon_{jk}| = o_p(n^{1/2}),$$

Combining $\|\boldsymbol{\beta}_i-\hat{\boldsymbol{\beta}}_i\| = O_p(n^{-1/2})$ and $\|\boldsymbol{\beta}_k-\hat{\boldsymbol{\beta}}_k\| = O_p(n^{-1/2})$, we have

$$\max_{1\le k\le n}|\hat{\varepsilon}_{ik}\hat{\varepsilon}_{jk}| = o_p(n^{1/2}).$$

Therefore, we have

$$\max_{1\le k\le n}\|\boldsymbol{\xi}_k\| = o_p(n^{1/2}).$$

Proof of Theorem 2.1. Using the same strategy as the proof of Theorem 3.2 in Owen (1991), we can prove that

$$\|\boldsymbol{\lambda}\| = O_p(n^{-1/2}). \tag{a.1}$$

It follows from Lemma 3 and (a.1) that

$$\max_{1\le k\le n}|\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{\xi}_k| = O_p(n^{-1/2})o_p(n^{1/2}) = o_p(1).$$

Hence, by Taylor's expansion, we have

$$l_n = 2\sum_{k=1}^{n}\log(1+\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{\xi}_k) = 2\sum_{k=1}^{n}\left(\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{\xi}_k-\frac{1}{2}(\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{\xi}_k)^2\right)+r_n, \tag{a.2}$$

with

$$|r_n| \le C\|\boldsymbol{\lambda}\|^3\max_k\|\boldsymbol{\xi}_k\|\sum_{k=1}^{n}\|\boldsymbol{\xi}_k\|^2 = o_p(1)\quad .$$

Based on the equation (2.8), by Lemma 3 and (a.1), we have

$$\boldsymbol{\lambda} = \left(\sum_{k=1}^{n}\boldsymbol{\xi}_k\boldsymbol{\xi}_k^{\mathrm{T}}\right)^{-1}\sum_{k=1}^{n}\boldsymbol{\xi}_k+o_p(n^{-1/2}), \tag{a.3}$$

and

$$\sum_{k=1}^{n}\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{\xi}_i = \sum_{k=1}^{n}\left(\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{\xi}_k\right)^2+o_p(1). \tag{a.4}$$

By (a.1)-(a.4), we know that

$$\begin{aligned}l_n &= \sum_{k=1}^{n}\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{\xi}_i\boldsymbol{\xi}_k^{\mathrm{T}}\boldsymbol{\lambda}+o_p(1) \\ &= \left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\boldsymbol{\xi}_k\right)^{\mathrm{T}}\left(\frac{1}{n}\sum_{k=1}^{n}\boldsymbol{\xi}_k\boldsymbol{\xi}_k^{\mathrm{T}}\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\boldsymbol{\xi}_k\right)+o_p(1).\end{aligned}$$

Finally, combining Lemmas 1 and 2, we have $l_n \xrightarrow{D} \chi_N^2$ as $n\to\infty$. The theorem is then proved.

Conflicts of Interest: The author declares no conflict of interest.

References

Baltagi, B. H. (2008). Econometrics (Fourth Edition), Springer, Berlin Heidelberg.

Baltagi, B. H., & Griffin, J. M. (1983). Gasoline Demand in the OECD: An Application of Pooling and Testing Procedures. *European Economic Review, 22*, 117-137. https://doi.org/10.1016/0014-2921(83)90077-6

Breusch, T. S., & Pagan, A. R. (1980). The Lagrange Multiplier Test and Its Applications to Model Specification in Econometrics. *Review of Economic Studies, 47*, 239-253. https://doi.org/10.2307/2297111

Dong, L. B., & Giles, D. A. (2007). An Empirical Likelihood Ratio Test for Normality. *Communications in Statistics-Simulation and Computation, 36*(1), 197-215. https://doi.org/10.1080/03610910601096544

Dufour, J. M., & Khalaf, L. (2002). Exact tests for contemporaneous correlation of disturbances in seemingly unrelated regressions. *Journal of Econometrics, 106*, 143-170. https://doi.org/10.1016/S0304-4076(01)00093-8

Fiebig, D. G. (2001). Seemingly Unrelated Regression, Chapter 5 in Baltagi, B. H. (ed.), A Companion to Theoretical Econometrics (Blackwell: Massachusetts).

Harvey, A. C., & Phillips, G. D. A. (1982). Testing for contemporaneous correlation of disturbances in systems of regression equations. *Bulletin of Economic Research, 34,* 79-81. https://doi.org/10.1111/j.1467-8586.1982.tb00438.x

Liu, F., & Chen, G. M., & Chen, M. (2008). Testing serial correlation in partial linear errors-in-variables models based on empirical likelihood. *Communications in Statistics-Theory and Methods, 37*(12), 1905-1918. https://doi.org/10.1080/03610920801893780

Liu, X. H., & Wang, Z. Z., & Hu, X. M., & Wang, G. F. (2011). Testing serial correlation in partially linear single-index errors-in-variables models. *Communications in Statistics-Theory and Methods, 40,* 2554-2573. https://doi.org/10.1080/03610926.2010.489174

Owen, A. B. (1988). Empirical likelihood ratio confidence intervals for a single functional. *Biometrika, 75*, 237-249. https://doi.org/10.1093/biomet/75.2.237

Owen, A. B. (1990). Empirical likelihood ratio confidence regions. *The Annals of Statistics, 18*, 90-120. https://doi.org/10.1214/aos/1176347494

Owen, A. B. (1991). Empirical likelihood for linear models. *The Annals of Statistics, 19,* 1725-1747. https://doi.org/10.1214/aos/1176348368

Owen, A. B. (2001). Empirical Likelihood. Chapman and Hall, NewYork.

Srivastava, V. K., & Giles, D. E. A. (1987). Seemingly Unrelated Regression Equations Models: Estimation and Inference, Marcel Dekker, New York.

Tsay, W. J. (2004). Testing for contemporaneous correlation of disturbances in seemingly unrelated regressions with serial dependence. *Economics Letters, 83*, 69-76. https://doi.org/10.1016/j.econlet.2003.10.005

Zellner, A. (1962). An efficient method of estimating seemingly unrelated regressions and tests for aggregation bias. *Journal of the American Statistical Association, 57*, 348-368. https://doi.org/10.1080/01621459.1962.10480664

Copyrights

Copyright for this article is retained by the author(s), with first publication rights granted to the journal.

This is an open-access article distributed under the terms and conditions of the Creative Commons Attribution license (http://creativecommons.org/licenses/by/4.0/).

International Journal of Statistics and Probability; Vol. 10, No. 3; May 2021
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

Lehmann Type II Frechet Poisson Distribution: Properties, Inference and Applications as a Life Time Distribution

Adebisi Ade Ogunde[1], Gbenga Adelekan Olalude[2], Oyebimpe Emmanuel Adeniji[1], Kayode Balogun[3]

[1]Department of Statistics, University Of Ibadan, Oyo State, Nigeria

[2]Department of Statistics, Federal Polytechnic Ede, Osun State, Nigeria

[3]Department of statistics, Federal School of Statistics, Ibadan, Oyo State, Nigeria

Correspondence: A. A. Ogunde, Department of Statistics, University Of Ibadan, Oyo State, Nigeria.

Received: November 11, 2020 Accepted: December 28, 2020 Online Published: March 24, 2021

doi:10.5539/ijsp.v10n3p8 URL: https://doi.org/10.5539/ijsp.v10n3p8

Abstract

A new generalization of the Frechet distribution called Lehmann Type II Frechet Poisson distribution is defined and studied. Various structural mathematical properties of the proposed model including ordinary moments, incomplete moments, generating functions, order statistics, Renyi entropy, stochastic ordering, Bonferroni and Lorenz curve, mean and median deviation, stress-strength parameter are investigated. The maximum likelihood method is used to estimate the model parameters. We examine the performance of the maximum likelihood method by means of a numerical simulation study. The new distribution is applied for modeling three real data sets to illustrate empirically its flexibility and tractability in modeling life time data.

Keywords: Lehmann Type II Frechet Poisson distribution, stress-strength parameter, generating functions, order statistics, stochastic ordering

1. Introduction

Frechet distribution which is also known as Inverse Weibull distribution belong to the class of Type II extreme value distribution was developed by Frechet (1924) is a very useful distribution for modeling life time data. The Frechet distribution is one of the important distributions in extreme value theory and has several applications which include: floods, horse racing, accelerated life testing, earthquakes, sea waves, rainfall and wind speeds. For more studies on the properties and applications of Frechet distribution, see Kotz and Nadarajah (2000), also Harlow (2002). This distribution can be used to analyse life time data that exhibits decreasing increasing or constant failure rate. However, models with complex hazard rate shapes such as bathtub, unimodal and other shapes are often encountered in real life time data analysis which may include mortality studies, reliability analysis etc., which the Frechet distribution may not provide a reasonable parametric fit when used for modeling complex phenomenon. Several modifications have been made to improve its parametric fits, some of which are: Beta-Exponential Frechet was developed and studied by Mead et al. (2017). The properties of Transmuted Frechet was investigated by Mahmoud and Mandour (2013), Transmuted Exponentiated Frechet was studied by Elbatal et al. (2014), Krishna et al. (2013) developed and studied Marshall-Olkin Frechet distribution, gamma extended Frechet distribution was studied by Silva et al. (2013) and the exponentiated Frechet distribution was studied by Nadarajah and Kotz (2003). The Odd Lindley Frechet Distribution was studied by Korkmaz et al. (2017), alpha power transformed Frechet was studied by Suleman et al. (2019) and Mead and Abd-Eltawab (2014) studied Kumaraswamy Frechet. Afify et al. (2016a) studied Weibull Frechet, Kumaraswamy Marshall-Olkin Frechet distribution was developed by Afify et al. (2016b), Kumaraswamy transmuted Marshall-Olkin Frechet was studied by Yousof et al. (2016), Beta Transmuted Frechet distribution was developed and studied by Afify et al. (2016c). Yousof et al. (2018b) developed and studied the Topp Leone Generated Frechet distribution and Odd log-logistic Frechet was studied by Yousof et al. (2018a).

In recent times, several new families of distribution have been developed by compounding the Poisson distribution with many other univariate continuous distributions to provide a more flexible and tractable distribution for modeling lifetime failure data. Lu and Shi (2012) developed and studied the Weibull Poisson distribution. The exponential Poisson was studied by Francisco et al. (2020). The exponential Weibull-Poisson distribution which generalises the Weibull-Poisson was studied by Mahmoudi and Sepahdar (2013) and the two parameter Poisson-exponential distribution with increasing failure was studied by Cancho et al. (2011). The exponentiated exponential-Poisson distribution was derived and studied by Barreto-Souza and Cribari-Neto (2009). The Kumaraswamy Lindley-Poisson distribution which generalises the Lindley-Poisson distribution was studied by Pararai et al. (2015), Mohamed and Rezk (2019) developed and studied the properties and applications of the extended Poisson-Frechet distribution.

Suppose that a random variable X follows a Frechet distribution, having a cumulative distribution function (cdf) and probability density function (pdf), respectively given as:

$$G(x;\gamma,\omega) = e^{-\gamma x^{-\omega}} \tag{1}$$

And

$$g(x;\gamma,\omega) = \gamma\omega x^{-\omega-1} e^{-\gamma x^{-\omega}} \tag{2}$$

Where $\gamma > 0$and $\omega > 0$. γ is a scale parameter and ω is a shape parameter.

2. Frechet Poisson Distribution

Suppose that the failure time of each subsystem has the Frechet model defined by pdf and cdfin (1) and (2). Given N, let $Z_j$denote the failure time of the j^{th} subsystem which are independently and identically distributed random variable from Frechet distribution. Taking N to be distributed according to the truncated Poisson random variable with probability mass function (pmf)

$$P(N=n) = \frac{v^n}{n!\,(e^v-1)}; \qquad n = 1,2, \;.\;.\;\;., v > 0 \tag{3}$$

Suppose that the failure time of each subsystem has the Frechet distribution defined by the cdf given in equation (1).

$$X = \min_{j}^{N} \{Z_j\}$$

Unconditional cdf of X given N is

$$\bar{H}_{FP}(x;\gamma,\omega,v) = 1 - P(X > x/N) = 1 - P(Z_1 > x)^N = 1 - \left[1 - e^{-\gamma x^{-\omega}}\right]^N \tag{4}$$

The equation (4) above is the exponentiated Frechet distribution.

So, the unconditional cdf of X (for $x > 0$) is given by

$$\bar{H}_{FP}(x;\gamma,\omega,v) = \frac{1}{(e^v-1)}\sum_{n=1}^{\infty}\frac{v^n\left[1-\left[1-e^{-\gamma x^{-\omega}}\right]^n\right]}{n!}$$

The cdf of Frechet Poisson (FP) distribution is given by

$$\bar{H}_{FP}(x;\gamma,\omega,v) = \frac{1-e^{-ve^{-\gamma x^{-\omega}}}}{1-e^{-v}}, \qquad x > 0, \gamma, \omega, v > 0 \tag{5}$$

The FP density function is given by

$$\bar{h}_{FP}(x;\gamma,\omega,v) = \frac{\gamma\omega v x^{-\omega-1} e^{-\gamma x^{-\omega}} e^{-ve^{-\gamma x^{-\omega}}}}{1-e^{-v}}, \qquad x > 0, \gamma, \omega, v > 0 \tag{6}$$

Where $v, \gamma > 0$and $\omega > 0$. γ is a scale parameter, v and ω are shape parameters

2.1 Lehman Type II Frechet Poisson Distribution

In this sub-section, we present the Lehman Type II Frechet-Poisson (LFP) distribution, and derive some of its properties which include: cdf, pdf, hazard function $(h(x))$, reversed hazard function $(H(x))$, quantile function and sub-models.

The Lehman type II distribution is a hybrid of the generalised exponentiated distribution developed by Cordeiro et al. (2013). Given $\bar{H}(x)$ to be an arbitrary baseline cdf in the interval (0,1). The cdf $\bar{H}(x)$, called the Exponentiated-G (EG)distribution has the cdf

$$F(x;\alpha,\beta) = [1 - \{1 - \bar{H}(x)\}^{\alpha}]^{\beta} \tag{7}$$

Where $\alpha > 0$ and $\beta > 0$ are two additional shape parameters which exhibits tractable properties especially for simulations, since the quantile function takes a simple form given by

$$x = Q_{\bar{H}}\left(\left\{1-\left(1-u^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}\right\}\right) \tag{8}$$

Where $Q_{\bar{H}}(u)$ is the baseline quantile function. The two extra shape parameters can control both tail weight and entropy of EG distribution. The expression in (7) can be splitted into two generalised distribution called the Lehman type I and the Lehman type II distribution by respectively taking $\alpha = 1$ and $\beta = 1$. The distribution function of Lehman type I and Lehman type II are given respectively by

$$F(x;\beta) = \{\bar{H}(x)\}^{\beta} \tag{9}$$

and

$$F(x;\alpha) = 1-\{1-\bar{H}(x)\}^{\alpha} \tag{10}$$

Where $\bar{H}(x)$ is the baseline distribution. For the purpose of this study $\bar{H}(x)$is the cdf of FP distribution. Thus, the goal of this study is to develop another generalization of the Frechet Poisson distribution called the Lehman Type II Frechet Poisson distribution with a wider scope of applications that may be used in modeling real life time data which may include applications in medicine, reliability, aeronautical engineering, weather forecasting and other extreme conditions with a better fit than the Frechet Poisson distribution.

By taking $\bar{H}(x)$ as the cdf of FP distribution in equation (10), we obtain the cdf of Lehman Type II Frechet Poisson (LFP) distribution. The cdf of four-parameter LFP distribution is given by

$$F_{LFP}(x) = 1-\left\{1-\frac{1-e^{-ve^{-\gamma x^{-\omega}}}}{1-e^{-v}}\right\}^{\alpha} \tag{11}$$

for $x > 0, \alpha > 0, v > 0, \omega > 0\ and\ \gamma > 0$. The corresponding density of LFP distribution is given by

$$f_{LFP}(x) = \frac{\alpha v\gamma\omega x^{-\omega-1}e^{-\gamma x^{-\omega}}e^{-ve^{-\gamma x^{-\omega}}}}{1-e^{-v}}\left\{1-\frac{1-e^{-ve^{-\gamma x^{-\omega}}}}{1-e^{-v}}\right\}^{\alpha-1} \tag{12}$$

for $x > 0, \alpha > 0, v > 0, \omega > 0, \gamma > 0$.

Plots of the pdf of LFP distribution are given below in figure 1 and figure 2 for arbitrary values of $,\alpha, v, \omega and \gamma$.

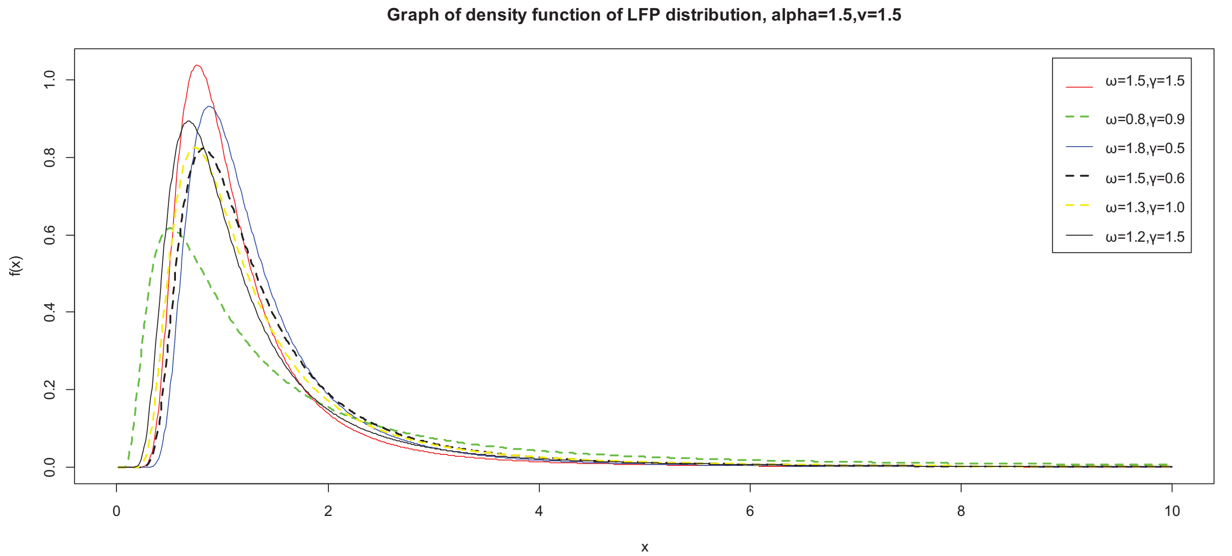


Figure 1. Plot of the pdf of LFP distribution for different values of ω, γ keeping the values of α, v constant at 1.5

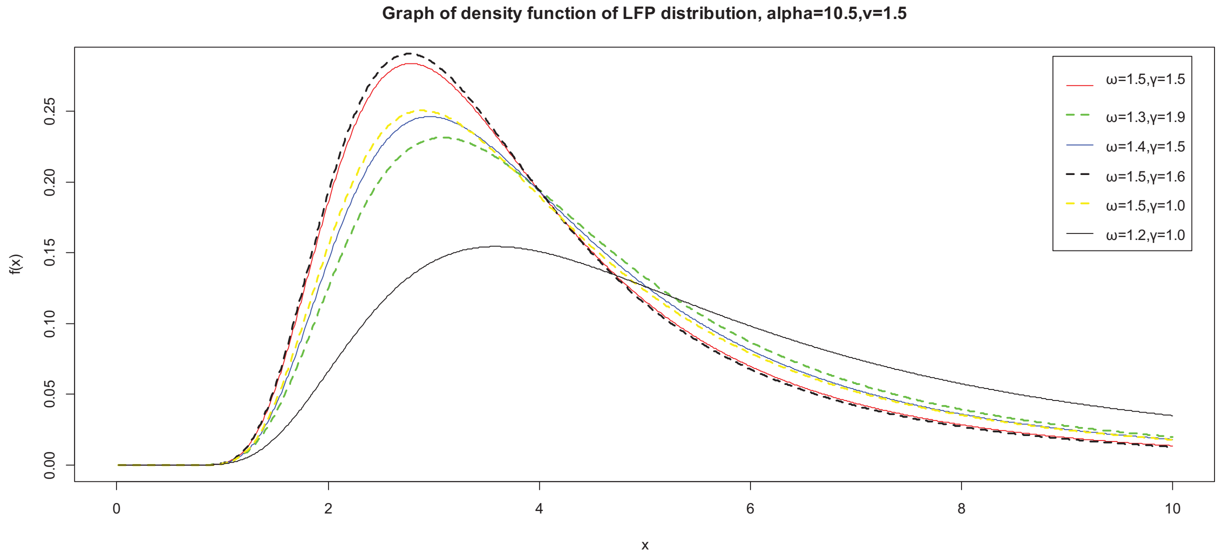


Figure 2. Plot of the pdf of LFP distribution for different values of ω, γ keeping the values of α, ν constant at 10.5 and 1.5 respectively

From figure 1 and 2 above, LFP distribution can be viewed as a suitable model for fitting a unimodal and right skewed data

2.2 Survival and the Hazard Function

The survival function of the LFP distribution is given by:

$$S_{LFP}(x) = 1 - F_{LFP}(x) = \left\{1 - \frac{1 - e^{-\nu e^{-\gamma x^{-\omega}}}}{1 - e^{-\nu}}\right\}^{\alpha} \tag{13}$$

for $x > 0, \alpha > 0, \nu > 0, \omega > 0$ and $\gamma > 0$. The graph of the survival function for various values of the parameters $\alpha, \nu, \gamma,$ and ω is given in figure 3.

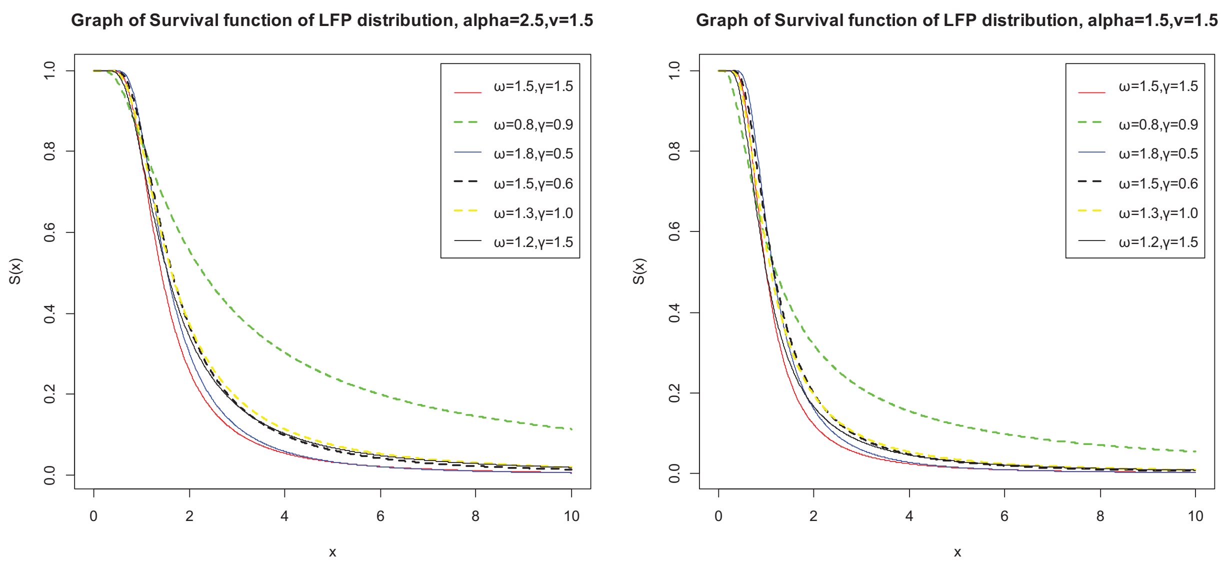


Figure 3. Plot of the survival function of LFP distribution for different values of ω, γ keepingthe values of α, ν constant

The hazard and the reverse hazard function of the LFP distribution are respectively given by

$$h_{LFP}(x) = \frac{f_{LFP}(x;\alpha,\gamma,\nu,\omega)}{1 - F_{LFP}(x;\alpha,\gamma,\nu,\omega)} \tag{14}$$

Putting equations (12) and (13) in (14), we have

$$h_{LFP}(x) = \frac{\alpha\nu\gamma\omega x^{-\omega-1} e^{-\gamma x^{-\omega}} e^{-\nu e^{-\gamma x^{-\omega}}}}{(1-e^{-\nu})\left\{1 - \frac{1-e^{-\nu e^{-\gamma x^{-\omega}}}}{1-e^{-\nu}}\right\}} \tag{15}$$

and

$$\bar{H}_{LFP}(x) = \frac{f_{LFP}(x;\alpha,\gamma,\nu,\omega)}{F_{LFP}(x;\alpha,\gamma,\nu,\omega)} \tag{16}$$

Putting equations (11) and (12) in (16), gives

$$\bar{H}_{LFP}(x) = \frac{\frac{\alpha\nu\gamma\omega}{1-e^{-\nu}} x^{-\omega-1} e^{-\gamma x^{-\omega}} e^{-\nu e^{-\gamma x^{-\omega}}} \left\{1 - \frac{1-e^{-\nu e^{-\gamma x^{-\omega}}}}{1-e^{-\nu}}\right\}^{\alpha-1}}{1 - \left\{1 - \frac{1-e^{-\nu e^{-\gamma x^{-\omega}}}}{1-e^{-\nu}}\right\}^{\alpha}} \tag{17}$$

for $x > 0, \alpha > 0, \nu > 0, \omega > 0$ *and* $\gamma > 0$. The graph of the hazard function for various values of the parameters $\alpha, \nu, \gamma,$ and ω is given in figure 4 and 5.

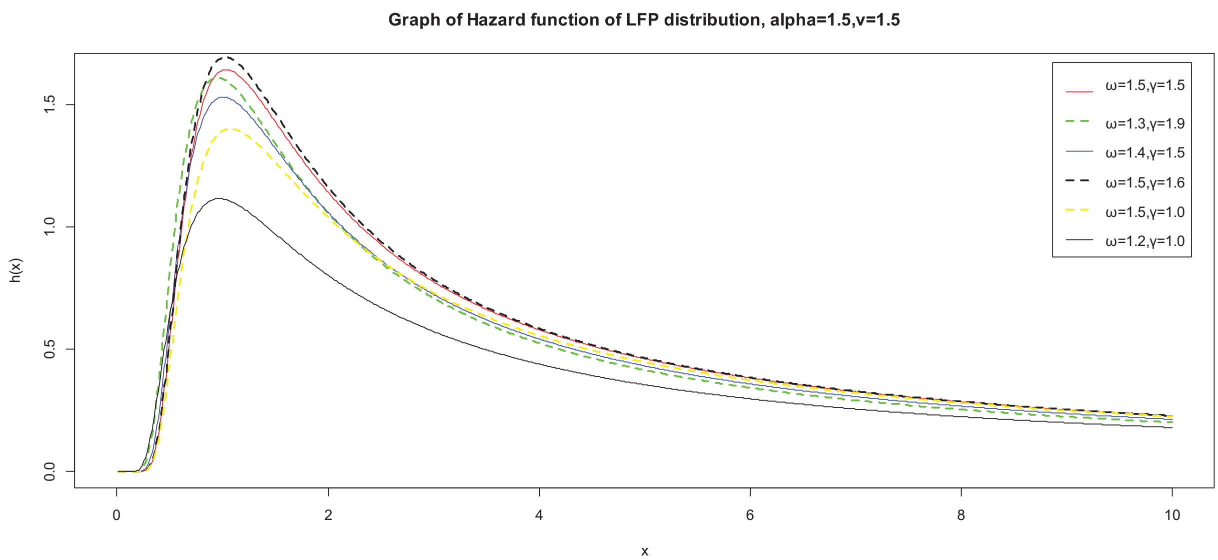


Figure 4. Plot of the hazard function of LFP distribution for different values of ω, γ keeping the values of α, ν constant at 1.5 and 1.5 respectively

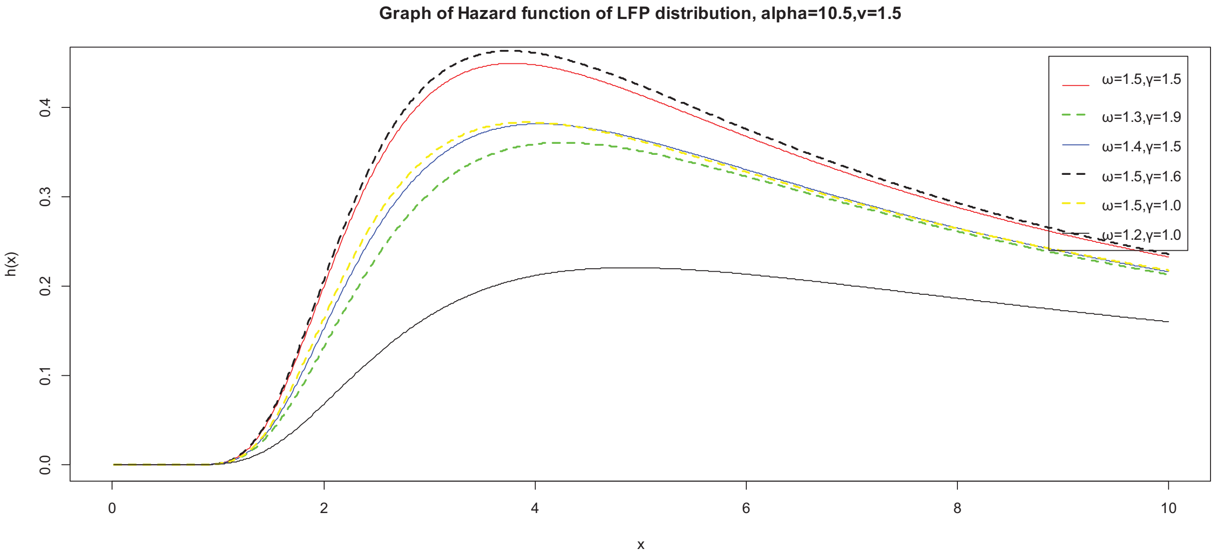


Figure 5. Plot of the survival function of LFP distribution for different values of ω, γ keeping the values of α, ν constant at 10.5 and 1.5 respectively

The graph of the hazard function in figures 4 and 5 for different values of the parameters exhibits various shapes such as monotonically decreasing, increasing, increasing-decreasing and upside down bathtub shapes. This feature indicates the flexibility of LFP distribution and its suitability in modeling monotonic and non-monotonic hazard behaviour which are often encountered in real life situations.

2.3 Some Sub-models of the LFP Distribution

In this sub-section, we give the sub-models of LFP distribution for selected values of the parameters α, ν, γ and ω are presented

- ✓ When $\gamma = 1$, we obtain the Lehmann Type II Inverted Weibull Poisson distribution.
- ✓ When $\omega = 1$, we obtain the Lehmann Type II Inverse exponential Poisson distribution which is given in equation (49).
- ✓ When $\alpha = 1$, we obtain the Frechet Poisson distribution which is given in equation (50).
- ✓ When $\alpha = \gamma = 1$, we obtain the Inverted Weibull Poisson distribution whichpdf is given in equation (51).
- ✓ When $\alpha = \omega = 1$, we obtain the Inverse exponential Poisson distribution which is given in equation (52).
- ✓ When $v = \gamma = 1$, we obtain the Lehmann Type II Inverted Weibull distribution.
- ✓ When $v = \omega = 1$, we obtain the Lehmann Type II Inverse exponential distribution
- ✓ When $v = 1$, we obtain the Lehmann Type II Frechet distribution.
- ✓ When $\alpha = v = 1$, we obtain the Frechet distribution.

2.4 Expansion of the Density Function

Considering the binomial series expansion given by

$$(1-w)^{m-1} = \sum_{j=0}^{\infty} \binom{m-1}{j} (-1)^j w^j; \;\; m > 0, |w| < 1 \tag{18}$$

Thus we have:

$$\left\{1 - \frac{1 - e^{-\nu e^{-\gamma x^{-\omega}}}}{1 - e^{-\nu}}\right\}^{\alpha-1} = \sum_{i=0}^{\infty} (-1)^i \binom{\alpha-1}{i} \left(\frac{1}{1-e^{-\nu}}\right)^i \left(1 - e^{-\nu e^{-\gamma x^{-\omega}}}\right)^i$$

Subsequently,

$$\left(1-e^{-ve^{-\gamma x^{-\omega}}}\right)^{i}=\sum_{j=0}^{i}(-1)^{j}\binom{i}{j}\left(e^{-ve^{-\gamma x^{-\omega}}}\right)^{j}$$

Then we have,

$$f_{LFP}=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{\alpha-1}{i}\binom{i}{j}(-1)^{i+j}\left(\frac{1}{1-e^{-\nu}}\right)^{i+1}\alpha\gamma\omega x^{-\omega-1}e^{-\gamma x^{-\omega}}\left(e^{-ve^{-\gamma x^{-\omega}}}\right)^{j+1} \tag{19}$$

Since,

$$e^{m}=\sum_{k=0}^{\infty}\frac{m^{k}}{k!} \tag{20}$$

Applying equation (20) to equation (19), we obtain

$$f_{LFP}=\sum_{i=j}^{\infty}\sum_{j,k=0}^{\infty}\binom{\alpha-1}{i}\binom{i}{j}(-1)^{i+j+k}\left(\frac{1}{1-e^{-\nu}}\right)^{i+1}(j+1)^{k}\,\nu^{k+1}\alpha\gamma\omega(k+1)x^{-\omega-1}\frac{e^{-\gamma(k+1)x^{-\omega}}}{(k+1)!}$$

Therefore,

$$f_{LFP}=\sum_{i=j}^{\infty}\sum_{j,k=0}^{\infty}\Gamma_{i,j,k}(\alpha,\nu)\gamma\omega(k+1)x^{-\omega-1}\frac{e^{-\gamma(k+1)x^{-\omega}}}{(k+1)!} \tag{21}$$

Where,

$$\Gamma_{i,j,k}(\alpha,\nu)=\binom{\alpha-1}{i}\binom{i}{j}(-1)^{i+j+k}\left(\frac{1}{1-e^{-\nu}}\right)^{i+1}(j+1)^{k}\nu^{k+1}\alpha$$

Finally we have,

$$f_{LFP}=\sum_{i=j}^{\infty}\sum_{j,k=0}^{\infty}\Gamma_{i,j,k}(\alpha,\nu)\,g(x;\gamma(k+1),\omega)$$

2.5 Quantile Function

The quantile function of the LFP distribution is obtained by solving the equation $F(X_u)=u$, where $0<u<1.$ Then we obtain

$$X_u=\left[-\frac{1}{\gamma}log\left\{-\frac{1}{\nu}\left[\log(1-(1-e^{-\nu})\left\{1-\left(1-u\right)^{\frac{1}{\alpha}}\right)\right\}\right]\right\}\right]^{-\frac{1}{\omega}} \tag{22}$$

Classical measures of skewness and kurtosis may be difficult to obtain due to non-existence of higher moments in several heavy tailed distributions. When such a situation occurs, the quantile measures can be considered. The Bowley (B)skewness; Kenny and Keeping (1962) is one of the foremost measures of skewness that is based on quantile of a distribution. It is given by

$$B=\frac{q_{\frac{3}{4}}-2q_{\frac{2}{4}}+q_{\frac{1}{4}}}{q_{\frac{3}{4}}-q_{\frac{2}{4}}} \tag{23}$$

Consequently, the coefficient of Kurtosis can be obtained using Moor's (1988) approach to estimating kurtosis which is based on octiles of a distribution and is given by

$$M=\frac{q_{\frac{7}{8}}-q_{\frac{3}{8}}-q_{\frac{5}{8}}+q_{\frac{1}{8}}}{q_{\frac{6}{8}}-q_{\frac{4}{8}}} \tag{24}$$

It is of noteworthy that the two measures are more robust to outliers.

Table 1 given below represent the various values of Bowley Skewness and Moors Kurtosis for given values of the parameters taking $\nu = 2.3$ and $\alpha = 1.5$

γ, ω	$q_{\frac{1}{4}}$	$q_{\frac{2}{4}}$	$q_{\frac{3}{4}}$	$q_{\frac{1}{8}}$	$q_{\frac{3}{8}}$	$q_{\frac{7}{8}}$	B	M
0.5,1.2	0.0903	0.3017	1.3621	0.0440	0.1654	5.0189	0.6683	3.3897
1.0,2.1	0.2095	0.5269	1.5341	0.1169	0.3355	3.6548	0.5207	1.9488
1.5,2.1	0.4070	0.9584	2.5329	0.2348	0.6316	5.5003	0.4812	1.6988
2.0,3.5	0.5269	1.0901	2.4018	0.3251	0.7680	4.4039	0.3992	1.2758
3.5,6.2	1.1713	2.1648	4.1056	0.7686	1.6152	6.5881	0.3228	0.9622
10.0,15.5	5.8348	9.3509	14.8654	4.1586	7.4923	20.5708	0.2212	0.6188
20.5,20.5	21.5328	33.4270	51.194	15.6469	27.207	68.7588	0.1980	0.5480

From Table 1, we can conclude that the LFP distribution can be used to model data that skewed to the right (positively skewed) with various degree of kurtosis (Kleptokurtic, mesokurtic and leptokurtic).

3. Moments

In this section, we obtain the moment of the LFP distribution. Momentplays important role in statistical analysis, most especially in determining the structural properties of a distribution such as skewness, kurtosis, dispersion, mean etc.

Theorem 1. Let a random variable X follows the Lehmann type II Frechet Poisson distribution, the r^{th} moment of LFP distribution is given by

$$\mu_r' = \sum_{i=j}^{\infty} \sum_{j,k=0}^{\infty} \Gamma_{i,j,k}(\alpha, \nu) \frac{\gamma\omega}{(k+1)!} \{\gamma(k+1)\}^{\frac{r}{\omega}} \Gamma\left(1 - \frac{r}{\omega}\right)$$

Proof: let X be a random variable from LFP distribution, ther^{th}moment is given by

$$E(X^r) = \mu_r' = \int_{-\infty}^{\infty} x^r f_{LFP}(x) dx \tag{25}$$

Substitute for $f_{LFP}(x)$ in equation (25), we have

$$\mu_r' = \sum_{i=j}^{\infty} \sum_{j,k=0}^{\infty} \Gamma_{i,j,k}(\alpha, \nu) \frac{\gamma\omega}{(k+1)!} \int_{-\infty}^{\infty} x^r (k+1) x^{-\omega-1} e^{-\gamma(k+1)x^{-\omega}} dx \tag{26}$$

By letting

$$B(x) = \int_{-\infty}^{\infty} x^r (k+1) x^{-\omega-1} e^{-\gamma(k+1)x^{-\omega}} dx \tag{27}$$

Taking, $m = \gamma(k+1)x^{-\omega}$,$dx = -\frac{1}{\omega}\{\gamma(k+1)\}^{\frac{1}{\omega}} m^{-\frac{1}{\omega}-1} dm$ and putting it in equation (27), we have

$$B(x) = \{\gamma(k+1)\}^{\frac{r}{\omega}} \Gamma\left(1 - \frac{r}{\omega}\right) \tag{28}$$

It then follows that the r^{th} moment of LFP distribution is given as

$$\mu_r' = \sum_{i=j}^{\infty}\sum_{j,k=0}^{\infty} \Gamma_{i,j,k}(\alpha,\nu)\frac{\gamma\omega}{(k+1)!}\{\gamma(k+1)\}^{\frac{r}{\omega}}\Gamma\left(1-\frac{r}{\omega}\right) \tag{29}$$

$$r<\omega, where\ \Gamma(w)=\int_0^{\infty} p^{w-1}e^{-w}dw\ is\ the\ complementary\ incomplete\ gamma\ function$$

Table 2 given below represents the first four moments, Variance(σ^2), the Coefficient of Variation (λ_{cv}), Coefficient of Skewness(λ_{sk}) and Coefficient of Kurtosis (λ_{ku})for arbitrary values of the parameters of LFP distribution taking a fixed value of $\gamma = 3.0$ and $\omega = 5.5$ for Table 2 and for Table 3, we fixed $\alpha = 0.1, \nu = 0.5$.

$$\sigma^2 = \mu_2' - (\mu_1')^2$$

$$\lambda_{cv} = \frac{\sqrt{\mu_2}}{\mu_1'} = \frac{\sqrt{\mu_2' - (\mu_1')^2}}{\mu_1'}$$

$$\lambda_{sk} = \frac{\mu_3}{(\sigma^2)^{\frac{3}{2}}} = \frac{\mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^2}{\sqrt[\frac{3}{2}]{\mu_2' - (\mu_1')^2}}$$

$$\lambda_{ku} = \frac{\mu_4}{(\sigma^2)^2} = \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1' - 3(\mu_1')^2}{\{\mu_2' - (\mu_1')^2\}^2}$$

Table 2 and Table 3 represent the first four moments, σ^2, λ_{cv}, λ_{sk} and λ_{ku} of LFP distribution.

Table 2. First four moments, σ^2, λ_{cv}, λ_{sk} and λ_{ku} of LFP distribution

ν, α	μ_1'	μ_2'	μ_3'	μ_4'	σ^2	CV	λ_{sk}	λ_{ku}
2.5,1.5	1.1251	1.2909	1.5117	1.8373	0.0251	0.1408	−78.9055	−78.0228
2.0,1.0	1.2357	1.6062	2.2626	3.7434	0.0793	0.2279	−28.5627	−17.8667
3.0,2.0	1.0705	1.1586	1.2691	1.4087	0.0126	0.1049	−112.9874	−137.3718
5.5,4.0	0.9758	0.9557	0.9396	0.9272	0.0035	0.0606	223.3868	−111.9412
10.5,10.5	0.9076	0.8251	0.7515	0.6855	0.0014	0.0412	2909.2	−10609.4

Table 3. First four moments, σ^2, λ_{cv}, λ_{sk} and λ_{ku} of LFP distribution

γ, ω	μ_1'	μ_2'	μ_3'	μ_4'	σ^2	CV	λ_{sk}	λ_{ku}
0.1,4.5	0.5261	0.2822	0.1547	0.0869	0.0054	0.0000	662.4362	−6111.172
0.3,6.5	0.7576	0.5791	0.4470	0.3487	0.0051	0.0943	765.3136	−3668.017
0.8,10.5	0.9236	0.8559	0.7960	0.7429	0.0029	0.0583	835.9249	−1652.537
1.5,15.0	0.9860	0.9738	0.9634	0.9547	0.0016	0.0406	426.4312	−209.6875
5.0,20.5	1.0507	1.1632	1.1632	1.2257	0.0592	0.2316	−20.5070	102.1894

- ✓ It can be observed from Table 2 and Table 3 that the LFP distribution can be to model data that skewed to the right (positively skewed) or left (negatively skewed) with various degree of kurtosis.

3.1 Moment Generating Function

Moment generating function is a very useful function that can be used to describe certain properties of the distribution. The moment generating function of *LFP* distribution is given in the following theorem.

Theorem 2. Let X follows the *LFP* distribution, the moment generating function, $M_X(t)$ is

$$M_X(t)=\sum_{i=j}^{\infty}\sum_{j,k,r=0}^{\infty}\Gamma_{i,j,k}(\alpha,\nu)\frac{\gamma\omega}{(k+1)!}\frac{t^r}{r!}\{\gamma(k+1)\}^{\frac{r}{\omega}}\Gamma\left(1-\frac{r}{\omega}\right),\quad t\in\mathbb{R},r<w$$

Proof: The moment generating function of a random variable X is given by

$$M_X(t)=\boldsymbol{E}(e^{Tx})=\int_{-\infty}^{\infty}e^{tx}f_{LFP}(x)dx, \tag{30}$$

Where $f_{LFP}(x)$ is given in equation (21). Using series expansion in (20), we have

$$M_X(t)=\int_{-\infty}^{\infty}\sum_{r=0}^{\infty}\frac{t^r}{r!}x^rf(x)dx=\sum_{r=0}^{\infty}\frac{t^r}{r!}E(X^r) \tag{31}$$

Using $E(X^r)$ given in equation (29) in equation (31), we have

$$M_X(t)=\sum_{i=j}^{\infty}\sum_{j,k,r=0}^{\infty}\Gamma_{i,j,k}(\alpha,\nu)\frac{\gamma\omega}{(k+1)!}\frac{t^r}{r!}\{\gamma(k+1)\}^{\frac{r}{\omega}}\Gamma\left(1-\frac{r}{\omega}\right),\quad t\in\mathbb{R},r<w \tag{32}$$

It could be observed from the series expansion of (32) that moments are the coefficient of $\frac{t^r}{r!}$.

3.2 Incomplete Moment

The incomplete moment can be used to estimate the mean deviation, median deviation and the measures of inequalities such as the Bonferroni and Lorenz curves. The incomplete moment of Lehmann type II Frechet Poisson distribution is given in the following theorem.

Theorem 3. Let X follows the *LFP* distribution, the incomplete moment, $\varphi(t)$ is

$$\varphi(t)=\sum_{i=j}^{\infty}\sum_{j,k=0}^{\infty}\Gamma_{i,j,k}(\alpha,\nu)\frac{\gamma\omega}{(k+1)!}\{\gamma(k+1)\}^{\frac{r}{\omega}}\Gamma\left\{\left(1-\frac{r}{\omega}\right);\gamma(k+1)t^{-\omega}\right\},\quad r<\omega$$

Proof: The incomplete moment of Lehmann type II Frechet Poisson distribution is given by

$$\varphi(t)=\int_0^t x^r f_{LFP}(x)dx$$

$$=\sum_{i=j}^{\infty}\sum_{j,k=0}^{\infty}\Gamma_{i,j,k}(\alpha,\nu)\frac{\gamma\omega}{(k+1)!}\int_0^t x^r(k+1)x^{-\omega-1}e^{-\gamma(k+1)x^{-\omega}}dx$$

By letting

$$C(x)=\int_0^t x^r(k+1)x^{-\omega-1}e^{-\gamma(k+1)x^{-\omega}}dx \tag{33}$$

Taking, $m=\gamma(k+1)x^{-\omega}$, $dx=-\frac{1}{\omega}\{\gamma(k+1)\}^{\frac{1}{\omega}}m^{-\frac{1}{\omega}-1}dm$ and putting it in equation (33), we have

$$C(x) = \{\gamma(k+1)\}^{\frac{r}{\omega}}\Gamma\left\{\left(1-\frac{r}{\omega}\right);\gamma(k+1)x^{-\omega}\right\} \quad (34)$$

Then we have,

$$\varphi(t) = \sum_{i=j}^{\infty}\sum_{j,k=0}^{\infty}\Gamma_{i,j,k}(\alpha,\nu)\frac{\gamma\omega}{(k+1)!}\{\gamma(k+1)\}^{\frac{r}{\omega}}\left\{\left(1-\frac{r}{\omega}\right);\gamma(k+1)t^{-\omega}\right\} \qquad r<\omega \quad (35)$$

$$where\ \Gamma(w,q) = \int_{q}^{\infty} p^{w-1}e^{-w}dw\ is\ the\ complementary\ incomplete\ gamma\ function$$

3.3 Mean Deviation and Median Deviation, Bonferroni and Lorenz Curves

The amount of spread in a population can be obtained using deviation from the mean and median. The mean deviation about the mean and the median of the LFP distribution are expressed as

$\phi_1(x) = 2\mu F_{LFP} - 2\mu + 2J(\mu)$, and $\phi_1(x) = -\mu + 2J(M)$,

Where

$$J(\mu) = \int_{\mu}^{\infty} x f_{LFP}(x)dx$$

$$= \sum_{i=j}^{\infty}\sum_{j,k=0}^{\infty}\Gamma_{i,j,k}(\alpha,\nu)\frac{\gamma\omega}{(k+1)!}\{\gamma(k+1)\}^{\frac{r}{\omega}}\Gamma\left\{\left(1-\frac{r}{\omega}\right);\gamma(k+1)\mu^{-\omega}\right\}, \quad r<\omega$$

Bonferroni and Lorenz curves are given as

$$B(p) = \frac{1}{p\mu}\int_{0}^{q} x f_{LFP}(x)dx = \{\mu - J(q)\},$$

And

$$L(p) = \frac{1}{\mu}\int_{0}^{q} x f_{LFP}(x)dx = \{\mu - J(q)\},$$

Where

$$J(q) = \sum_{i=j}^{\infty}\sum_{j,k=0}^{\infty}\Gamma_{i,j,k}(\alpha,\nu)\frac{\gamma\omega}{(k+1)!}\{\gamma(k+1)\}^{\frac{r}{\omega}}\Gamma\left\{\left(1-\frac{r}{\omega}\right);\gamma(k+1)q^{-\omega}\right\}, \quad r<\omega$$

3.4 Renyi Entropy

The Renyi Entropy measures the uncertainty in a distribution as defined by Renyi (1961). The Renyi entropy of LFP distributionis given in the following theorem.

Theorem 4. Let X follows the LFP distribution, the Renyi Entropy, $I_\theta(x)$is

$$I_\theta(x) = \frac{1}{1-\theta}\left\{log\left(H^l\omega^{\theta-1}\{\gamma(k+\theta)\}^{\frac{1-\theta(\omega+1)}{\omega}}\Gamma\left\{1+\frac{(\theta-1)(\omega+1)}{\omega}\right\}\right)\right\}, \theta>0, \theta\neq 1$$

Proof: By definition

$$I_\theta(x) = \frac{1}{1-\theta}\left\{log\left(\int_{-\infty}^{\infty} f^{\theta}{}_{LFP}(x)dx\right)\right\}, \quad \theta>0, \theta\neq 1 \quad (36)$$

From equation (11)

$$f^{\theta}{}_{LFP}(x) = \left[\frac{\alpha v\gamma\omega x^{-\omega-1}e^{-\gamma x^{-\omega}}e^{-ve^{-\gamma x^{-\omega}}}}{1-e^{-v}}\left\{1-\frac{1-e^{-ve^{-\gamma x^{-\omega}}}}{1-e^{-v}}\right\}^{\alpha-1}\right]^{\theta}$$

Applying binomial series expansion given in equation (18), we have

$$f^{\theta}{}_{LFP}(x) = H^{l}\omega^{\theta}x^{\theta(-\omega-1)}e^{-\gamma x^{-\omega}(k+\theta)}$$

Where

$$H^{l} = (v\alpha\gamma)^{\theta}\sum_{i=j}^{\infty}\sum_{j,k}^{\infty}\binom{\theta(\alpha+1)}{i}\binom{i}{j}(-1)^{i+j+k+1}\left(\frac{1}{1-e^{-v}}\right)^{i+1}\frac{v^{k+\theta}}{k!}(i+\theta)^{k}$$

Consequently,

$$\int_{0}^{\infty} f^{\theta}{}_{LFP}(x)dx = H^{l}\omega^{\theta}\int_{-\infty}^{\infty} x^{\theta(-\omega-1)}e^{-\gamma x^{-\omega}(k+\theta)}dx \qquad (37)$$

By letting $c = \gamma x^{-\omega}(k+\theta) and\ dx = -\frac{1}{\omega}c^{-\frac{1}{\omega}-1}\{\gamma(k+\theta)\}^{\frac{1}{\omega}}dc$ and putting it in equation (37), we have

$$\int_{0}^{\infty} f^{\theta}{}_{LFP}(x)dx = H^{l}\omega^{\theta-1}\{\gamma(k+\theta)\}^{\frac{1-\theta(\omega+1)}{\omega}}\Gamma\left\{1+\frac{(\theta-1)(\omega+1)}{\omega}\right\} \qquad (38)$$

Putting equation (38) in (36), we have

$$I_{\theta}(x) = \frac{1}{1-\theta}\left\{log\left(H^{l}\omega^{\theta-1}\{\gamma(k+\theta)\}^{\frac{1-\theta(\omega+1)}{\omega}}\Gamma\left\{1+\frac{(\theta-1)(\omega+1)}{\omega}\right\}\right)\right\} \qquad (39)$$

It should be noted that Renyi entropy tends to Shannon entropy as $\theta \longrightarrow 1$.

3.5 Stress-strength Parameter

Suppose X_1 and X_2 are two continuous and independent random variables, where $X_1 \sim LFP(\alpha_1, v_1, \gamma, \omega)$ and $X_2 \sim LFP(\alpha_2, v_2, \gamma, \omega)$, then an expression for the stress-strength parameter can be obtained using the relation given by

$$Ḱ = \int_{-\infty}^{\infty} f_1(x;\alpha_1,v_1,\gamma,\omega)F_2(x;\alpha_2,v_2,\gamma,\omega)\,dx \qquad (40)$$

Using the pdf and the cdf of LFP in the expression above, the strength-stress parameter, Ḱ, can be obtained as

$$Ḱ = q_1 - q_2$$

Where

$$q_1 = \alpha_1 v_1\gamma\omega\int_{-\infty}^{\infty}\frac{\omega x^{-\omega-1}e^{-\gamma x^{-\omega}}e^{-v_1e^{-\gamma x^{-\omega}}}}{1-e^{-v_1}}\left\{1-\frac{1-e^{-v_1e^{-\gamma x^{-\omega}}}}{1-e^{-v_1}}\right\}^{\alpha_1-1}dx$$

$$= F_1(x;\alpha_1,v_1,\gamma,\omega)$$

and

$$q_2 = \alpha_1 v_1\gamma\omega\int_{-\infty}^{\infty}\frac{\omega x^{-\omega-1}e^{-\gamma x^{-\omega}}e^{-v_1e^{-\gamma x^{-\omega}}}}{1-e^{-v_1}}\left\{1-\frac{1-e^{-v_1e^{-\gamma x^{-\omega}}}}{1-e^{-v_1}}\right\}^{\alpha_1-1}\left\{1-\frac{1-e^{-v_2e^{-\gamma x^{-\omega}}}}{1-e^{-v_2}}\right\}^{\alpha_2} \qquad (41)$$

Using equations (18) and (20) in (41)

$$q_2 = \alpha_1 v_1 \gamma\omega \sum_{i=j}^{\infty}\sum_{k=l}^{\infty}\sum_{j,l,m=0}^{\infty} \binom{\alpha_1 - 1}{i}\binom{i}{j}\binom{\alpha_2}{k}\binom{k}{l}(-1)^{i+j+k+l+m}\frac{\left(v_1 + v_{1j} + v_{2l}\right)^m}{m!\,(1-e^{-v})^{i+k+1}}$$

Finally, we have

$$\acute{K} = F_1(x) - \alpha_1 v_1 \gamma\omega \sum_{i=j}^{\infty}\sum_{k=l}^{\infty}\sum_{j,l,m=0}^{\infty} \binom{\alpha_1 - 1}{i}\binom{i}{j}\binom{\alpha_2}{k}\binom{k}{l}(-1)^{i+j+k+l+m}\frac{\left(v_1 + v_{1j} + v_{2l}\right)^m}{m!\,(1-e^{-v})^{i+k+1}}$$

3.6 Order Statistics

Suppose that $X_1, X_2, \ldots, X_n$ is a random sample of size n from a continuous pdf, $f(x)$. Let $x_{1:n} < x_{2:n} < \ldots x_{n:n}$ represent the corresponding order statistics. If $X_1, X_2, \ldots, X_n$ is a random sample from LFP distribution, it then follows from equations (11) and (12) that the *pdf* of the m^{th} order statistic, say $Z_m = X_{m:n}$ is given by

$$f_m(z_m) = \frac{n!\, f_{LFP}(x)}{(m-1)!\,(n-m)!}\sum_{p=0}^{n-m}\binom{n-m}{p}(-1)^p [F_{LFP}(x)]^{p+m-1}$$

$$= \frac{n!\,\alpha}{(m-1)!\,(n-m)!}\sum_{p=0}^{n-m}\sum_{q,s,u=0}^{\infty}\sum_{r=s}^{\infty}\binom{n-m}{p}\binom{p+m-1}{q}\binom{\alpha(p+1)-1}{r}\binom{r}{s}$$

$$\times (-1)^{p+q+r+s+u}\left(\frac{1}{1-e^{-v}}\right)^{r+1} v^{u+1}(s+1)^u(u+1)\gamma\omega x^{-\omega-1}\frac{e^{-\gamma x^{-\omega}(u+1)}}{(u+1)!}$$

$$= \sum_{p,q,r,s,u=0}^{\infty}\psi_{p,q,r,s,u}(\alpha, v)\, g\big(x;\omega,\gamma(u+1)\big)$$

Where

$$\psi_{p,q,r,s,u}(\alpha,v) = \sum_{p,q,s,u}^{\infty}\sum_{r=s}^{\infty}\binom{n-m}{p}\binom{p+m-1}{q}\binom{\alpha(p+1)-1}{r}\binom{r}{s}\frac{1}{(u+1)!}$$

$$\times (-1)^{p+q+r+s+u}\left(\frac{1}{1-e^{-v}}\right)^{r+1} v^{u+1}(s+1)^u(u+1)$$

And $g\big(x;\omega,\gamma(u+1)\big)$ is the Frechet pdf with parameters $\omega > 0$and $\gamma(u+1) > 0$. Thus, we can define the distribution of the m^{th} order statistics as a linear combination of the Frechet distribution.

3.7 Stochastic Ordering

In this section, we examine the stochastic and reliability properties of LFP distribution. Stochastic ordering has applications in many field of study such as survival analysis, insurance, actuarial and management sciences, finance, reliability and survival analysis Shaked and Shanthikumar (2007).

Suppose X and Z are two random variables with cdf's G and F respectively. Survival functions $\bar{G} = 1 - G$ and $\bar{F} = 1 - F$ and their corresponding densities g and f. Then X is said to be smaller than Z in stochastic order $(X \leq_{st} Z)$ if $\bar{G}(x) \leq \bar{F}(x)$for all $x \geq 0$; in likelihood ratio order $(X \leq_{lr} Z)$ if $g(x)/f(x)$ is decreasing in all $x \geq 0$; hazard rate order $(X \leq_{hr} Z)$ if $\bar{G}(x)/\bar{F}(x)$ is decreasing in all $x \geq 0$; reversed hazard rate order $(X \leq_{rhr} Z)$ if $G(x)/F(x)$ is decreasing in all $x \geq 0$. These four stochastic orders are related to each other as

$$X \leq_{rhr} Z \Leftarrow X \leq_{lr} Z \Rightarrow X \leq_{hr} Z \Rightarrow X \leq_{st} Z$$

Theorem: let $X \sim LFP(\alpha_1, v_1, \gamma, \omega)$ and$Z \sim LFP(\alpha_2, v_2, \gamma, \omega)$. If $\alpha_1 < \alpha_2$ and $v_1 < v_2$, then

$$X \le_{rhr} Z \Leftarrow X \le_{lr} Z \Rightarrow X \le_{hr} Z \Rightarrow X \le_{st} Z$$

Proof: $\frac{g(x)}{f(x)} = \frac{\alpha_1 \nu_1 e^{-\nu_1 e^{-\gamma t^{-\omega}}}}{1-e^{-\nu_1}} \left\{1 - \frac{1-e^{-\nu_1 e^{-\gamma x^{-\omega}}}}{1-e^{-\nu_1}}\right\}^{\alpha_1 - 1} \div \frac{\alpha_2 \nu_2 e^{-\nu_2 e^{-\gamma x^{-\omega}}}}{1-e^{-\nu_2}} \left\{1 - \frac{1-e^{-\nu_2 e^{-\gamma x^{-\omega}}}}{1-e^{-\nu_2}}\right\}^{\alpha_2 - 1}$

$$\boldsymbol{log}\left[\frac{g(x)}{f(x)}\right] = log\left(\frac{\alpha_1}{\alpha_2}\right) + log\left(\frac{\nu_1}{\nu_2}\right) + log\left(\frac{1-e^{-\nu_2}}{1-e^{-\nu_1}}\right) + (\nu_2 - \nu_1)e^{-\gamma x^{-\omega}} +$$

$$(\alpha_1 - 1)log\left(\left\{1 - \frac{1-e^{-\nu_1 e^{-\gamma x^{-\omega}}}}{1-e^{-\nu_1}}\right\}\right) - (\alpha_2 - 1)log\left(\left\{1 - \frac{1-e^{-\nu_2 e^{-\gamma x^{-\omega}}}}{1-e^{-\nu_2}}\right\}\right)$$

$$\frac{d}{dx}\left(log\left[\frac{g(x)}{f(x)}\right]\right) = x^{-\omega-1}e^{-\gamma x^{-\omega}}\left\{(\nu_2 - \nu_1)\gamma\omega - \frac{\nu_1 e^{-\nu_1 e^{-\gamma x^{-\omega}}}}{\left(e^{-\nu_1 e^{-\gamma x^{-\omega}}} - e^{-\nu_1}\right)} + \frac{\nu_2 e^{-\nu_2 e^{-\gamma x^{-\omega}}}}{\left(e^{-\nu_2 e^{-\gamma x^{-\omega}}} - e^{-\nu_2}\right)}\right\}$$

Hence, for

$$\alpha_1 < \alpha_2 \text{ and } \nu_1 < \nu_2$$

It follows that

$$X \le_{rhr} Z \Leftarrow X \le_{lr} Z \Rightarrow X \le_{hr} Z \Rightarrow X \le_{st} Z$$

4. Maximum Likelihood Estimation

The log-likelihood function $l(\underline{x}/\varpi) = log\left(L(\underline{x}/\varpi)\right)$ of the *LPF* distribution is given by

$$l(\underline{x}/\varpi) = nlog(\alpha\nu\gamma\omega) - (\omega+1)\sum_{i=1}^{n} log(x_i) - \gamma\sum_{i}^{n} x_i^{-\omega} - \nu\sum_{i=1}^{n} e^{-\gamma x_i^{-\omega}}$$

$$(\alpha - 1)\sum_{i=1}^{n}\left[1 - \frac{1-e^{-\nu e^{-\gamma x_i^{-\omega}}}}{1-e^{-\nu}}\right] \quad (43)$$

The partial derivatives of the log-likelihood function with respect to the model parameters $(\alpha, \gamma, \upsilon, \omega)$yield the score vector and are obtained as

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\left[1 - \frac{1-e^{-\upsilon e^{-\gamma x_i^{-\omega}}}}{1-e^{-\upsilon}}\right] \quad (44)$$

$$\frac{\partial l}{\partial \gamma} = \frac{n}{\gamma} - \sum_{i=1}^{n} x_i^{-\omega} + \upsilon\gamma\sum_{i=1}^{n} x_i^{-\omega} e^{-\gamma x_i^{-\omega}} + (\alpha - 1)\upsilon\sum_{i=1}^{n}\frac{\left(x_i^{-\omega}e^{-\gamma x_i^{-\omega}}e^{-\upsilon e^{-\gamma x_i^{-\omega}}}\right)}{\left(e^{-\upsilon e^{-\gamma x_i^{-\omega}}} - e^{-\upsilon}\right)} \quad (45)$$

$$\frac{\partial l}{\partial \upsilon} = \frac{n}{\upsilon} - \sum_{i=1}^{n} e^{-\gamma x_i^{-\omega}} + (\alpha - 1)\sum_{i=1}^{n}\frac{e^{-\upsilon}\left(1-e^{-\upsilon e^{-\gamma x_i^{-\omega}}}\right) + \upsilon e^{-\gamma x_i^{-\omega}} e^{-\upsilon e^{-\gamma x_i^{-\omega}}}(1-e^{-\upsilon})}{\left(e^{-\upsilon e^{-\gamma x_i^{-\omega}}} - e^{-\upsilon}\right)(1-e^{-\upsilon})} \quad (46)$$

And

$$\frac{\partial l}{\partial \omega} = \frac{n}{\omega} - \sum_{i=1}^{n} log(x_i) + \gamma\sum_{i}^{n} x_i^{-\omega}\, log(x_i) + (\alpha - 1)\upsilon\gamma\sum_{i=1}^{n}\frac{x_i^{\omega} log(x_i) e^{-\gamma x_i^{-\omega}} e^{-\upsilon e^{-\gamma x_i^{-\omega}}}}{\left(e^{-\upsilon e^{-\gamma x_i^{-\omega}}} - e^{-\upsilon}\right)} \quad (47)$$

The equations (44), (45), (46) and (47) are non-normal equations which cannot be solved by setting the above partial derivatives to zero, therefore the parameters $\alpha, \gamma, v, \omega$ must be found using the iterative methods. The maximum likelihood estimate of the parameters, denoted by $\widehat{\varpi}$ is obtained by solving the nonlinear equation $\left(\frac{\partial l}{\partial \alpha}, \frac{\partial l}{\partial \gamma}, \frac{\partial l}{\partial v}, \frac{\partial l}{\partial \omega}\right)^T = 0$, using a numerical method such as Newton-Raphson procedure, Trapezoidal techniques etc. The Fisher information is given by $\mathrm{I}(\varpi) = \left[I_{\theta_i, \theta_j}\right]_{4\times4} = E\left(-\frac{\partial^2 l}{\partial \theta_i, \partial \theta_j}\right), i, j = 1,2,3,4$ can be numerically obtained by using R or MATLLAB software. For the purpose of this study we make use of Adequacy model in R, the Fisher information matrix $n\mathrm{I}(\varpi)$ can be approximated by

$$\Omega(\widehat{\varpi}) \approx \left[-\frac{\partial^2 l}{\partial \theta_i \partial \theta_j}\bigg|_{\varpi=\widehat{\varpi}}\right]_{4\times4}, \quad i, j = 1,2,3,4 \tag{48}$$

For a given set of observations, the matrix given in equation (48) is obtained after convergence of the Newton-Raphson procedure in R or MATLAB.

The multivariate Normal distribution $N_4\left(\underline{0}, \Omega((\widehat{\varpi})^{-1})\right)$, with mean vector $\underline{0} = (0,0,0,0)^T$, can be used to construct the confidence interval and the confidence regions for the model parameters and for the hazard and the survival functions. The approximate $100(1-\boldsymbol{\Delta})\%$ two-sided confidence intervals forα, γ, v *and* ω are given by:

$\hat{\alpha} \pm Z_{\frac{\Delta}{2}}\sqrt{\mathrm{I}_{\alpha\alpha}^{-1}(\widehat{\varpi})}$, $\hat{v} \pm Z_{\frac{\Delta}{2}}\sqrt{\mathrm{I}_{vv}^{-1}(\widehat{\varpi})}$, $\hat{\gamma} \pm Z_{\frac{\Delta}{2}}\sqrt{\mathrm{I}_{\gamma\gamma}^{-1}(\widehat{\varpi})}$, *and* $\widehat{\omega} \pm Z_{\frac{\Delta}{2}}\sqrt{\mathrm{I}_{\omega\omega}^{-1}(\widehat{\varpi})}$,

Respectively, where $\mathrm{I}_{\alpha\alpha}^{-1}(\widehat{\varpi}), \mathrm{I}_{vv}^{-1}(\widehat{\varpi})$, $\mathrm{I}_{\gamma\gamma}^{-1}(\widehat{\varpi})$ and $\mathrm{I}_{\omega\omega}^{-1}(\widehat{\varpi})$ are diagonal elements of $\mathrm{I}_n^{-1}(\widehat{\varpi})$, and $Z_{\frac{\Delta}{2}}$ is the upper $\frac{\Delta}{2}^{th}$ percentile of the distribution of the standard normal.

5. Application

In this section, we demonstrate the applicability and flexibility of the LFP distribution in modeling real life data using three life data sets. The method of maximum likelihood is used to estimate the model parameters; also, we carried out a Monte Carlo simulation for different parameter values coupled with different sample sizes.

5.1 Monte Carlo Simulation

A simulation study is carried out in order to test the performance of the MLEs for estimating *LFP* model parameters. We consider two different sets of parameters $\alpha = 0.4, v = 0.6, \gamma = 0.3, \omega = 0.5$ and $\alpha = 0.5, v = 1.6, \gamma = 0.5, \omega = 0.5$. For each parameter combination, we simulate data from LFP model with different sample sizes$n = 50, n = 100, n = 150$ *and* $n = 200$, taking from a population size $N = 1000$. Table 4 list the Absolute bias (AB), standard error (SE) and the mean square error (MSE). According to the simulation result the mean square error decay to zero as the sample size increases as expected.

Table 4. Monte Carlo Simulation Results for LFP distribution

Par	n	AB	SE	MSE	Par	AB	SE	MSE
$\alpha = 0.4$	50	1.2931	3.0523	10.9886	$\alpha = 0.5$	1.2338	3.0033	10.5421
	100	1.0742	2.0149	5.2137		1.1873	2.0772	5.7244
	150	1.3460	1.6685	4.5956		1.4545	1.7065	5.0277
	200	1.1821	1.3739	3.2850		1.2836	1.3856	3.5675
$v = 0.6$	50	0.0049	0.2922	0.0854	$v = 1.6$	1.0050	0.2834	1.0903
	100	0.3409	0.1450	0.1372		1.3486	0.1453	1.8345
	150	0.2981	0.1257	0.1047		1.3053	0.1255	1.7196
	200	0.2705	0.1079	0.0848		1.231	0.1445	1.5362
$\gamma = 0.3$	50	0.4086	0.9149	1.0040	$\gamma = 1.5$	0.9433	1.3902	2.7948
	100	0.2486	0.1197	0.0761		0.7897	0.8137	1.2857
	150	0.2454	0.0926	0.0688		0.8244	0.5973	1.0364
	200	0.1846	0.1523	0.0573		0.5952	0.6236	0.7431
$\omega = 0.5$	50	0.9691	0.2707	1.0125	$\omega = 0.5$	0.7709	0.7458	1.1505
	100	0.5074	0.4870	0.4946		2.2749	1.3907	7.1092
	150	0.4625	0.3345	0.3257		2.1381	0.94`12	4.5715
	200	0.2879	0.2531	0.1469		1.6586	0.69871	3.2370

5.2 Application to Real Life Data

In this section, the LFP distribution is applied to three real life data sets in order to demonstrate the usefulness, tractability and applicability of the model. We fit the density of the LFP distribution and compare it performance with its sub-models which includes: Lehmann type II Inverse Exponential Poisson ($LIEP$) distribution, Frechet Poisson (FP) distribution, Inverted Weibull Poisson (IWP) distribution, Inverse Exponential Poisson (IEP) distribution, Frechet (F) distribution. The density of the competing models is given by:

$$f_{LIEP}(x) = \frac{\alpha v \gamma x^{-2} e^{-\gamma x^{-1}} e^{-ve^{-\gamma x^{-1}}}}{1-e^{-v}} \left\{1 - \frac{1-e^{-ve^{-\gamma x^{-1}}}}{1-e^{-v}}\right\}^{\alpha-1}, x>0; \alpha, v, \gamma > 0 \quad (49)$$

$$f_{FP}(x;\gamma,\omega,v) = \frac{\gamma \omega v x^{-\omega-1} e^{-\gamma x^{-\omega}} e^{-ve^{-\gamma x^{-\omega}}}}{1-e^{-v}}, \qquad x>0;\ \gamma,\omega,v>0 \quad (50)$$

$$f_{IWP}(x;\omega,v) = \frac{\omega v x^{-\omega-1} e^{-\gamma x^{-\omega}} e^{-ve^{-x^{-\omega}}}}{1-e^{-v}}, \qquad x>0; \omega, v>0 \quad (51)$$

$$f_{IEP}(x;\gamma,v) = \frac{\gamma v x^{-2} e^{-\gamma x^{-\omega}} e^{-ve^{-\gamma x^{-1}}}}{1-e^{-v}}, \qquad x>0;\ \gamma, v>0 \quad (52)$$

We consider various measures of the goodness-of-fit including the Akaike Information Criterion ($AIC = 2k - 2l$), Consistent Akaike Information Criterion ($CAIC = AIC + \frac{2k(k+1)}{n-k-1}$), Bayesian Information Criterion ($BIC = klog(n) - 2l$), Hannan-Quinn information criterion ($HQIC = -2l + 2klog\{log(n)\}$), Anderson Darling Statistic ($A^* = \left(\frac{9}{4n^2} + \frac{3}{4n} + 1\right)\left\{n + \frac{1}{n}\sum_{j=1}^{n}(2j-1)log[z_i(1-z_n-j+1)]\right\}$), Kolmogorov Smirnoff ($KS$) ,Crammer Von-Misses ($W^* = \left(\frac{1}{2n}+1\right)\left\{\sum_j^n \left(z_i - \frac{2j-1}{2n}\right)^2 + \frac{1}{12n}\right\}$) statistic and the Probability value (PV), where n is the number of observations, $z_i = F(y_i)$, k is the is the number of estimated parameters and $y_i{}'$s are the ordered observations. The smaller these statistics are, the better the fit of the model to the data except for PV which must be the largest among the competing models. Upper tail percentiles of the asymptotic distributions of these goodness-of-fit statistics were tabulated in Nichols and Padgett (2006). We provide the estimates of the parameters of the distributions, standard errors (in parenthesis), confidence interval (in curly bracket).

The first data set represents the remission times (in months) of a random sample of 128 bladder cancer patients. For previous study see Lee and Wang (2003). The second data consists of 101 observations obtained from a failure time in hours of Kevlar 49/epoxy strands with pressure at 90% and had been studied by Andrews and Herzberg (2012). For the third data set, we consider the number of failures for the air conditioning system of jet airplanes which were reported by Cordeiro and Lemonte (2011) and Huang and Oluyede (2014). The three data sets are given in Table 5 given below.

Table 5. Cancer data, Kevlar 49/Epoxy data, and Air condition failure data

$Data\ 1$	$Data\ 2$	$Data\ 3$
0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26,3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70,5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41,7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66,11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53,12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69	0.01, 0.01, 0.02, 0.02,0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09,0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.20,0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.40, 0.42,0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68,0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85,0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10,1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40,1.43, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58,1.60, 1.63, 1.64, 1.80, 1.80, 1.81, 2.02, 2.05, 2.14, 2.17,2.33, 3.03, 3.03, 3.34, 4.20, 4.69, 7.89	194, 413, 90, 74, 55, 23, 97, 50, 359, 50, 130, 487, 57, 102, 15, 14, 10, 57, 320, 261, 51, 44, 9, 254, 493, 33, 18,209, 41, 58, 60, 48, 56, 87, 11, 102, 12, 5, 14, 14, 29, 37, 186, 29, 104, 7, 4, 72, 270, 283, 7, 61, 100, 61, 502, 220,120, 141, 22, 603, 35, 98, 54, 100, 11, 181, 65, 49, 12, 239, 14, 18, 39, 3, 12, 5, 32, 9, 438, 43, 134, 184, 20, 386, 182,71, 80, 188, 230, 152, 5, 36, 79, 59, 33, 246, 1, 79, 3, 27, 201, 84, 27, 156, 21, 16, 88, 130, 14, 118, 44, 15, 42, 106, 46,230, 26, 59, 153, 104, 20, 206, 5, 66, 34, 29, 26, 35, 5, 82, 31, 118, 326, 12, 54, 36, 34, 18, 25, 120, 31, 22, 18, 216, 139, 67, 310, 3, 46, 210, 57, 76, 14, 111, 97, 62, 39, 30, 7, 44, 11, 63, 23, 22, 23, 14, 18, 13, 34, 16, 18, 130, 90, 163, 208, 1, 24, 70, 16, 101, 52, 208, 95, 62, 11, 191, 14, 71

The Exploratory data analysis of the three sets of data is given in Table 6. From this table, we can conclude that the three sets of data are over-dispersed, positively skewed and kleptokurtic. From the Total Time on Test (TTT) plot, figure 4 (Diagram 1) indicates that the first data set exhibits unimodal failure rate, figure 5 (Diagram 1) indicates that the second data set exhibits a bathtub failure rate and figure 6 (Diagram 1) indicates that the third data set exhibits decreasing failure rate.

Table 6. Exploratory data Analysis of the tree failure data

	$Data\ 1$	$Data\ 2$	$Data\ 3$
$Minimum$	0.08	0.010	1.00
$First\ quartile$	3.348	0.240	20.750
$meadian$	6.395	0.800	54.00
$Mean$	9.366	1.025	92.07
$Third\ quartile$	11.840	1.450	118.00
$Maximum$	79.050	7.890	603.0
$Variance$	110.425	1.253	11645.933
$Skewness$	3.326	3.047	2.157
$Kurtosis$	16.154	14.475	5.192

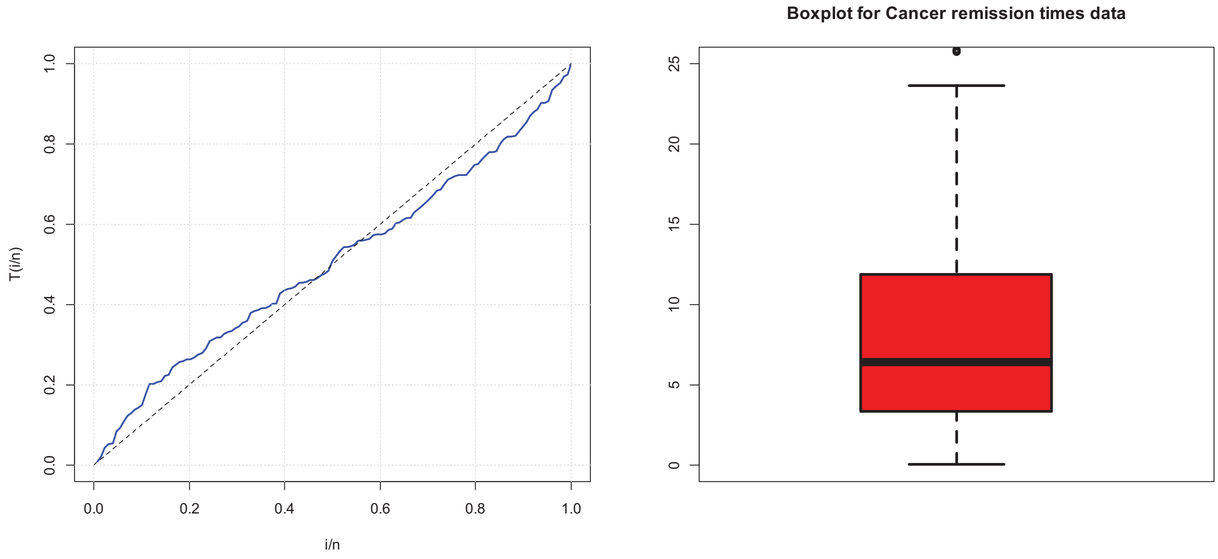


Diagram I

Diagram II

Figure 4. TTT plot and the Boxplot for Cancer data

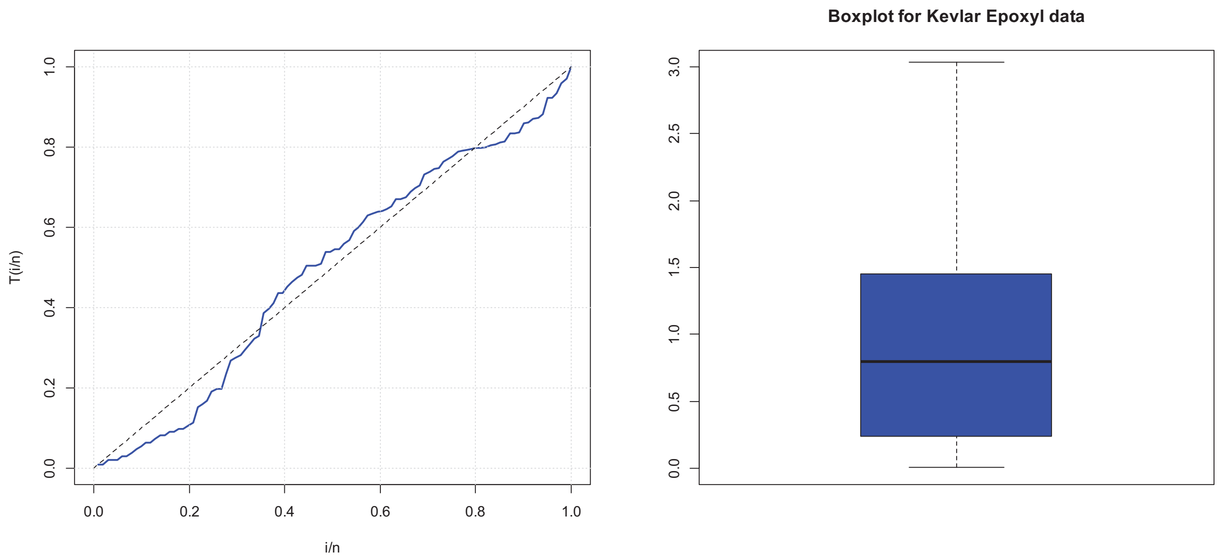
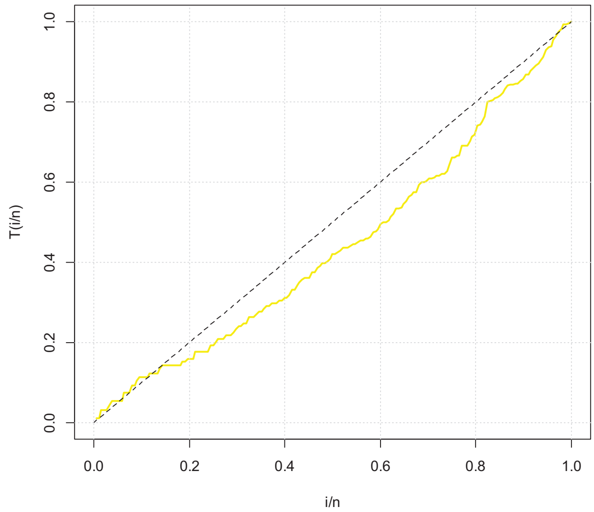


Figure 5. TTT plot and the Boxplot for Kevlar Epoxy data



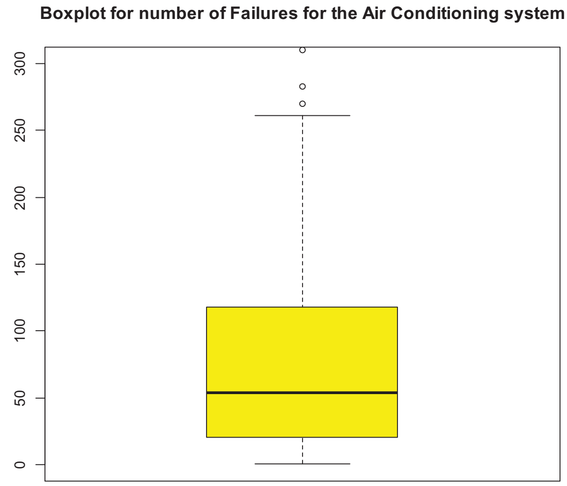


Figure 6. TTT plot and the Boxplot for Air condition data

Table 7. MLE's, Standard error (in parenthesis) and 95% Confidence interval (in curly bracket) for cancer remission data

$Model$	α	ν	γ	ω
LFP	7.0604(0.5734) {5.9365,8.1843}	0.2434(0.0310) {0.1826,0.3042}	7.0686(3.2671) {0.6651,13.4721}	9.1513(4.1383) {1.0402,17.2624}
$LIEP$	0.8570(0.1865) {0.4915,1.2225}	−6.7737(1.1381) {4.543,9.0044}	1.5282(0.230) {1.0774,1.979}	− −
FP	− −	0.6419(0.1435) {0.3606,0.9232}	1.0052(0.0580) {0.8915,1.1189}	−6.4163(1.2857) {−8.9363, −3.8963}
IWP	− −	0.9992(0.0556) {0.890,1.108}	− −	−4.4354(0.5281) {4.4354,0.5281}
IEP	− −	0.6363(0.1451) {0.3519,0.9207}	-6.4676(1.3156) {−9.0462, −3.889}	− −
F	− −	− −	2.4304(0.2193) {2.0005,2.8602}	0.7523(0.0424) {0.6692,0.8354}

Table 8. Measures of goodness-of-fit of Cancer remission data

MLE	$-ll$	AIC	$CAIC$	BIC	$HQIC$	K	A^*	W^*	PV
LFP	412.127	832.253	832.578	843.662	836.889	0.0363	0.0459	0.4314	0.9505
$LIEP$	423.207	852.413	852.607	860.970	855.889	0.1623	1.3297	0.2044	0.0023
FP	427.135	860.270	860.463	868.826	863.746	0.0944	2.5375	0.4031	0.2038
IWP	429.306	862.612	862.708	868.316	864.929	0.1077	2.5378	0.4028	0.1022
IEP	427.132	858.265	858.360	863.968	860.582	0.0965	2.5244	0.4007	0.1839
F	444.001	892.002	892.098	897.706	894.319	0.410	11.1780	1.9946	2.2e-16

Table 9. MLE's, Standard error (in parenthesis) and 95% Confidence interval (in curly bracket) for49/Epoxy data

$Model$	α	ν	γ	ω
LFP	0.5899(0.1914) {0.2148,0.9650}	0.4416(0.0406) {0.362,0.5212}	−5.0361(1.1548) {−7.2995, −2.7727}	9.8611(3.3049) {3.3835,16.3387}
$LIEP$	0.0563(0.0131) {0.0306,0.0820}	−4.1031(0.8534) {−5.7758, −2.4304}	0.7924(0.1229) {0.5515,1.0333}	− −
FP	− −	2.6433(0.3657) {1.9265,3.3601}	0.2594(0.0296) {0.2014,0.3174}	15.9277(5.6611) {4.8319,27.0235}
IWP	− −	0.4468(0.0245) {0.3988,0.4948}	− −	2.7522(0.4095) {1.9496,3.5548}
IEP	− −	0.0670(0.0122) {0.0431,0.0909}	−4.5136(0.7695) {−6.0218, −3.0054}	− −
F	− −	− −	0.4206(0.0585) {0.3059,0.5353}	0.6141(0.0424) {0.5310,0.6972}

Table 10. Measures of goodness-of-fit for 49/Epoxy data

MLE	$-ll$	AIC	$CAIC$	BIC	$HQIC$	K	A^*	W^*	PV
LFP	107.494	222.988	223.404	233.448	227.222	0.1133	2.3507	0.4410	0.1495
$LIEP$	130.812	267.623	267.871	275.469	270.799	0.9411	3.4488	0.6431	2.2e-16
FP	114.193	234.386	234.633	242.231	237.562	0.1649	3.5739	0.6649	0.0082
IWP	131.263	266.525	266.648	271.756	268.642	0.2040	6.1356	1.1365	0.0005
IEP	132.207	268.414	268.537	273.645	270.532	0.2399	5.7642	1.0745	1.78e-05
F	132.441	268.882	269.004	274.112	270.999	0.4176	10.5943	1.9962	9.992 e-16

Table 11. MLE's, Standard error (in parenthesis) and 95% Confidence interval (curly bracket) for air condition failure data

$Model$	α	ν	γ	ω
LFP	6.2447(2.9967) {0.3712,12.1182}	0.4016(0.0459) {0.3116,0.4916}	−4.7004(1.6543) {−7.9428, −1.4580}	26.5895(16.9552) {−6.6427,59.8217}
$LIEP$	6.6206(1.5056) {3.6696,9.5716}	−5.1296(0.9439) {−6.9796, −3.2796}	1.0924(0.1248) {0.8478,1.3370}	− −
FP	− −	5.1506(1.4678) {2.2737,8.0275}	0.9394(0.0488) {0.8437,1.0351}	−5.0641(1.1124) {−7.2444, −2.8838}
IWP	− −	0.8072(0.0378) {0.7331,0.8813}	− −	−14.6451(1.7261) {−18.0283, −11.2619}
IEP	− −	6.1265(1.3240) {3.5315,8.7215}	−5.1192(0.9989) {−7.077, −3.1614}	− −
F	− −	− −	11.2205(1.3427) {8.5888,13.8522}	0.7466(0.0374) {0.6733,0.8199}

Table 12.Measures of goodness-of-fit for air condition failure data

MLE	$-ll$	AIC	$CAIC$	BIC	$HQIC$	KS	A^*	W^*	PV
LFP	1033.17	2074.35	2074.57	2087.29	2079.59	0.0397	0.2675	0.0341	0.9288
$LIEP$	1048.30	2102.61	2102.74	2112.32	2106.54	0.9872	3.0343	0.4337	$2.2e - 16$
FP	1047.84	2101.68	2101.81	2111.38	2105.61	0.0676	1.9501	0.2955	0.3572
IWP	1055.17	2114.34	2114.40	2120.81	2116.96	0.0866	2.6848	0.4048	0.1189
IEP	1048.601	2101.20	2101.27	2107.67	2103.82	0.0803	2.0998	0.3205	0.1772
F	1061.42	2126.84	2126.90	2133.31	2129.46	0.4025	11.1658	1.9067	$2.2e - 16$

Based on the values obtained which were recorded in Tables 8, 10 and 12, it is clear that the Lehman Type II Frechet Poisson distribution provide the best fit for the three real life data considered having possessed the smallest AIC, CAIC, BIC, HQIC, A^{*}, W^{*} and the largest PV among all others competing models. Figures 7, 8 and 9 also illustrate the flexibility of Lehmann Type II Frechet Poisson distribution in modeling real life data.

6. Conclusion

This work examined the flexibility, tractability and applicability of Lehmann Type II Frechet Poisson distribution. Some structural properties of the newly developed distribution are derived and population parameters are obtained using maximum likelihood estimation method. Simulation study and three real life data were used to illustrate the model usefulness in modeling life data. Among other competing models considered the Lehmann Type II Frechet Poisson distribution provides the best fit. We recommend that further studies should be carried by using different estimation methods such as moment method, least square method etc. and compare the performance of the estimation techniques.

Conflicts of interest

The authors want to declare that there is no conflict of interest during and after the preparation of the manuscript.

Acknowledgement

The authors wish to express their sincere appreciations to the anonymous referees for their suggestions, comments and contributions that help us to improve more on the work.

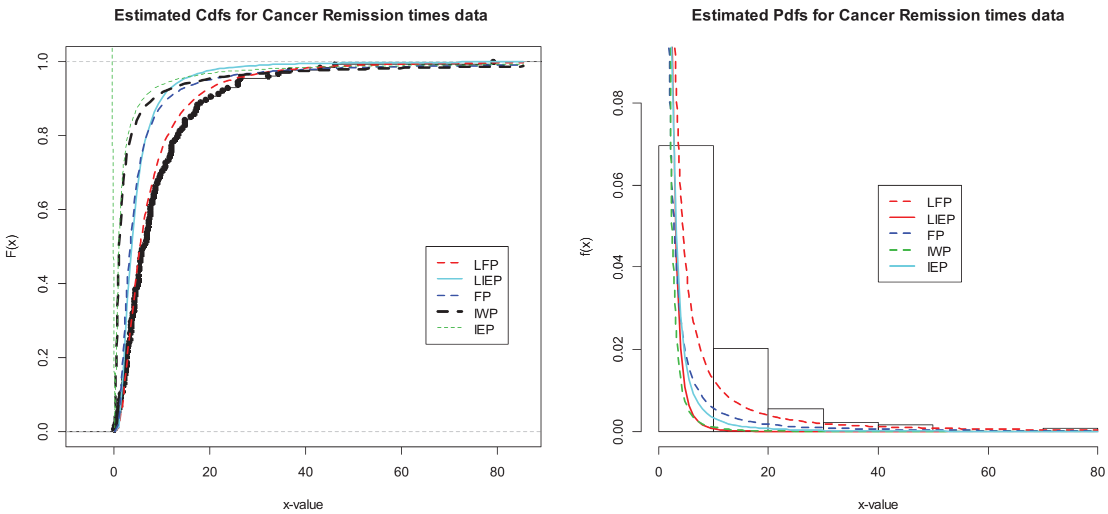


Figure 7. Estimated *pdf* and *cdf* function and other competing models for cancer remission data

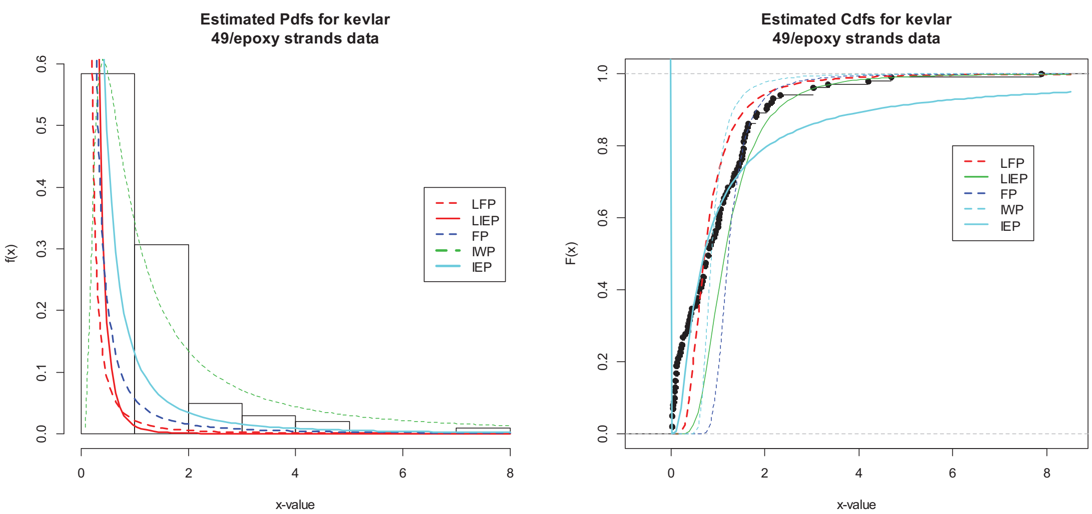


Figure 8. Estimated *pdf* and *cdf* function and other competing models for Kevlar 49/epoxy data

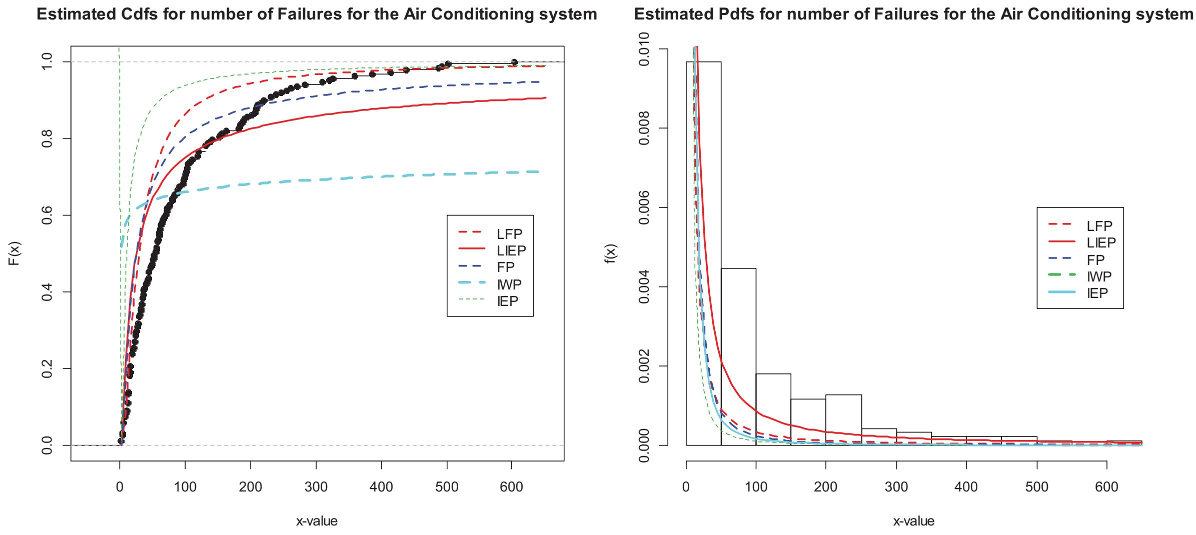


Figure 9. Estimated *cdf* and *pdf* function and other competing models for air condition failure data

References

Afify, A. Z., Yousof, H. M., Cordeiro, G. M., & Ahmad, M. (2016a). The Kumaraswamy Marshall-Olkin Frechet distribution with applications. *Journal of Islamic Countries Society of Statistical Sciences, 2*, 1-18.

Afify, A. Z., Yousof, H. M., Cordeiro, G. M., Ortega, E. M. M., & Nofal, Z. M. (2016b). The Weibull Frechet distribution and its applications. *Journal of Applied Statistics*, *43*, 2608- 2626. https://doi.org/10.1080/02664763.2016.1142945

Afify, A. Z., Yousof, H. M., & Nadarajah, S. (2016c). The beta transmuted-H family of distributions: properties and applications. *Statistics and its Inference*, *10*, 505-520. https://doi.org/10.4310/SII.2017.v10.n3.a13

Andrews, D. F., & Herzberg, A. M. (2012). Data: a collection of problems from many fields for the student and research worker. *Springer Science and Business Media.*

Barreto-Souza, W., & Cribari-Neto, F. A. (2009*)*. Generalization of the exponential- Poisson distribution*. Statistics and Probability Letters*, *79*, 2493-2500. https://doi.org/10.1016/j.spl.2009.09.003

Cancho, V. G., Louzada-Neto, F., & Barriga, G. D. C. (2011). The Poisson-exponential lifetime distribution. *Computational Statistics and Data Analysis*, *55*, 677-686. https://doi.org/10.1016/j.csda.2010.05.033

Cordeiro, G. M., & Lemonte A. J. (2011). The exponentiated generalized inverse Gaussian distribution. *Statistics and Probability Letters*, *81*, 506-517. https://doi.org/10.1016/j.spl.2010.12.016

Cordeiro, G. M., E., Ortega, E. M. M., & Cunha, D. C. C. (2013). The exponentiated generalized class of distributions. *Journal of Data Science*, *11*, 1–27, 2013. https://doi.org/10.6339/JDS.201301_11(1).0001

Elbatal, I., Asha, G., & Raja, V. (2014). Transmuted exponentiated Frechet distribution: properties and applications. *Journal of Statistics Applications and Probability*, *3*(3), 379-394.

Francisco, L., Pedro, L. R., & Paulo, H. F. (2020*)*. Exponential Poisson distrinbution: Estimation and Application to rainfall and aircraft data with zero occurrence. *Communication in Statistics-Simulation and Computation*, *49*(4), 1024-1043. https://doi.org/10.1080/03610918.2018.1491988

Frechet, M. (1924). Sur la loi des erreurs d'observation, *Bulletin de la Societe Mathematique de Moscou*, *33*, 5-8.

Harlow, D. G. (2002*)*. Applications of the Frechet distribution function. *International Journal of Material Production Technology*, *17*, 482-495. https://doi.org/10.1504/IJMPT.2002.005472

Huang, S., & Oluyede, B. O. (2014). Exponentiated kumaraswamy-dagum distribution with applications to income and lifetime data. *Journal of Statistical Distributions and Applications*, *1*(1), 1-20. https://doi.org/10.1186/2195-5832-1-8

Kenney, J. F., & Keeping, E. S. (1962). Mathematics of Statistics*, Part 1, 3rd edition.Van Nostrand, New Jersey.*

Korkmaz, M. C. Yousof, H. M., & Ali, M. M. (2017). Some Theoretical and Computational Aspects of the Odd Lindley

Frechet Distribution. *Journal of Statisticians: Statistics and Actuarial Sciences*, *2*, 129-140.

Kotz, S., & Nadarajah, S. (2000). Extreme Value Distributions*: Theory and Applications,* Imperial College Press, London. https://doi.org/10.1142/p191

Krishna, E., Jose, K. K., Alice, T., & Ristic, M. M. (2013). The Marshall-Olkin Frechet Distribution. *Communications in Statistics - Theory and Methods*, *42*, 4091-4107. https://doi.org/10.1080/03610926.2011.648785

Lee, E. T., & Wang, J. W. (2003). *Statistical methods for survival data analysis (3rd ed.)*, New York: Wiley. https://doi.org/10.1002/0471458546

Lu, W., & Shi, D. (20012). A New Compounding Lifetime Distribution: the Weibull- Poisson distribution, *Journal of Applied Statistics*, *39*, 21-38. https://doi.org/10.1080/02664763.2011.575126

Mahmoudi, E., & Sepahdar, A. (2013). Exponentiated Weibull-Poisson distribution:Model, Properties and Applications. *Mathematics and Computers in Simulation*, *92*, 76- 97. https://doi.org/10.1016/j.matcom.2013.05.005

Mahmoud. M. R., & Mandouh, R. M. (2013). On the transmuted Frechet distribution. *Journal of Applied Sciences Research, 9*(10), 5553-5561.

Mead, M. E., & Abd-Eltawab, A. R. (2014). A note on Kumaraswamy Frechet distribution with applications. *Australian Journal of Basic and Applied Sciences*, *8*(15), 294-300.

Mead, M. E., Afify, A. Z., Hamedani, G. G., & Gosh, I. (2017). The beta exponential Frechet distribution with applications. *Austrian Journal of Statistics*, *46*(1), 41-63. https://doi.org/10.17713/ajs.v46i1.144

Mohamed, G. K., & Rezk, H. (2019). *The* extended Frechet distribution and its Applications*. Pakistan Journal of Statistics Operation and Research, 15*(4), 905-919. https://doi.org/10.18187/pjsor.v15i4.2660

Moors, J. J. (1988). A quantile alternative for kurtosis*. Journal of Royal Statistical Society. series D., The statistician, 37*, 25-32. https://doi.org/10.1080/03610926.2016.1231816

Nadarajah, S., & Kotz, S. (2003). The exponentiated Frechet distribution. Interstat *Electronic Journal,* 1-7.

Nichols, M. D., & Paggett, W. A. (2006). Bootstrap control chart for Weibull percentiles*. Quality and Reliability Engineering International*, *22*, 141–151. https://doi.org/10.1002/qre.691

Pararai, M., Oluyedede, B. O., & Warahena-Liyanage, G. (2015). Kumaraswamy Lindley-Poisson distribution: theory and applications. *Asian Journal of Mathematics and Applications*, *2015*.

Renyi, A. (1961*).* On measures of entropy and information. *Proceedings of the 4th Berkeley Symposiumon Mathematical Statistics and Probability, vol. I,* University of California Press, Berkeley. pp. 547-561.

Silva, R. V. D., de Andrade, T. A., Maciel, D., Campos, R. P., & Cordeiro, G. M. (2013). A new lifetime model: The gamma extended Frechet distribution. *Journal of Statistical Theory and Applications, 12,* 39-54. https://doi.org/10.2991/jsta.2013.12.1.4

Nasiru, S., Mwita, P. N., & Ngesa, O. (2019). Alpha power transformed Frechet distribution. *Applied Mathematics & Information Sciences*, *13*(1), 129-141. https://doi.org/10.18576/amis/130117

Yousof, H. M., Afify, A. Z., Abd El Hadi, N. E., Hamedani, G. G., & Butt, N. S. (2016). On six-parameter Fréchet distribution: properties and applications. *Pakistan Journal of Statistics and Operation Research*, 281-299. https://doi.org/10.18187/pjsor.v12i2.1327

Yousof, H. M., Altun, E., & Hamedani, G. G. (2018a). A new extension of Frechet distribution with regression models, residual analysis and characterizations. *Journal of Data Science, 16*, 743-770. https://doi.org/10.6339/JDS.201810_16(4).00005

Yousof, H. M., Jahanshahi, S. M., Ramires, T. G., Aryal, G. R., & Hamedani, G. G. (2018b). A new distribution for extreme values: regression model, characterizations and applications. *Journal of Data Science*, *16*, 677-706. https://doi.org/10.6339/JDS.201810_16(4).00002

Copyrights

Copyright for this article is retained by the author(s), with first publication rights granted to the journal.

This is an open-access article distributed under the terms and conditions of the Creative Commons Attribution license (http://creativecommons.org/licenses/by/4.0/).

International Journal of Statistics and Probability; Vol. 10, No. 3; May 2021
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

Analysis of Change Points With Bayes Factor, Thresholds, and CUSUM

Lucila Muñiz Merino[1], Bulmaro Juárez Hernández[1] & Hugo A. Cruz Suárez[1]

[1] Facultad de Ciencias Físico Matemáticas, Universidad Autónoma de Puebla, Puebla, México

Correspondence: Facultad de Ciencias Físico Matemáticas, Universidad Autónoma de Puebla, Puebla, México, Postal address San Claudio, Cd. Universitaria, Jardines de San Manuel, 72572, Puebla, Pue

Received: January 21, 2021 Accepted: March 3, 2021 Online Published: March 24, 2021

doi:10.5539/ijsp.v10n3p31 URL: https://doi.org/10.5539/ijsp.v10n3p31

Abstract

In this work, an analysis of change points is made with the Bayes factor, thresholds, and cumulative sum (CUSUM) statistics methods. For the analysis of change points with the Bayes factor, Poisson data were simulated; the threshold method was worked with a regression and data of the National Institute of Statistics, Geography and Informatics (INEGI) of Mexico and coronavirus were used for the CUSUM.

Keywords: Bayes factor, CUSUM, thresholds

1. Introduction

When a change point is mentioned, the first question that comes to mind is: what is a change point? Chen and Gupta (2012) defined it as the site, or point in time t, in a succession of data $\{x_{t_i}\}$ $i = 1, \dots, n$ observed and ordered with respect to time, in such a way that these observations follow a distribution F_1, before a point, and in another point after it, the distribution is F_2. That is, from the statistical point of view, the succession of observations shows an inhomogeneous behaviour.

Under the classical and Bayesian approaches, the change point problem is considered one of the central problems of statistical inference, since it interrelates the statistical control theory, the hypothesis tests (when detecting if there is any change in the sequence of observed random variables), and the estimation theory (when estimating the number of changes and their corresponding locations). This under the classical and Bayesian approaches.

The change point problems originally appear in the quality control and generally can be found in the mathematical modeling of various disciplines such as Environmental Science, Epidemiology, Seismic Signal Processes, Economics, Finance, Geology, Medicine, Biology, Physics, etc. According to Chen and Gupta (2012), the change point problem is, in general, visualized as follows:

Let $X_1, X_2, \dots, X_n$ be a succession of independent random vectors (or variables) with probability distribution functions $F_1, F_2, \dots, F_n$, respectively. The change point problem is to test the null hypothesis, so the problem of the point of change consists of testing the null hypothesis H_0 about the non-existence of change against the alternative, H_a that there is at least one change point:

$H_0 : F_1 = F_2 = \ldots = F_n$

vs

$H_{\text{a}} : F_1 = \cdots = F_{(k_1)} \neq F_{(k_{1+1})} = \cdots = F_{(k_2)} \neq F_{(k_{2+1})} = \cdots = F_{(k_q)} \neq F_{(k_{q+1})} = \cdots = F_n.$

Where $1 < k_1 < k_2 < \cdots < k_q < n$, q is the unknown number of change points and $k_1, k_2, \cdots, k_q$ are the respective unknown positions that have to be estimated. If the distributions $F_1, F_2, \dots, F_n$ become a common parametric family $F(\theta)$, where $\theta \in \mathbb{R}^p$, then the problem of change points is to test the null hypothesis H_0 about the non-existence of change in the parameters $\theta_i, i = 1, \dots, n$ of the population against the alternative H_a that there is at least one change point:
$H_0 : \theta_1 = \theta_2 = \cdots = \theta_n = \theta$ (unknown)

vs

$H_{\text{a}} : \theta_1 = \cdots = \theta_{(k_1)} \neq \theta_{(k_{1+1)})} = \cdots = \theta_{(k_2)} \neq \theta_{(k_{2+1})} = \cdots = \theta_{(k_q)} \neq \theta_{(k_{q+1})} = \cdots = \theta_n.$

where q and $k_1, k_2, \dots, k_q$ must be estimated. Together, these hypotheses reveal the inference aspects of the change points in order to determine if there is any change point in the process, as well as to estimate their number and their respective positions.

The objective of this work is to develop algorithms and programs to apply some procedures to detect change points, particularly the Bayes factor, the threshold and the cumulative sum statistics (CUSUM) methods.

In this work, some procedures were programmed in order to obtain change points, for this the Bayes factor was used to detect temporal and spatiotemporal change points. The homogeneous Poisson process was also used since data was simulated with this process. Multiple temporal-change points and spatiotemporal change points were detected with the Bayes factor. Changes were also detected with the threshold method and with the cumulative sum (CUSUM) method. Finally, the Buishand range, the Pettitt and the standard normal homogeneity tests were applied, which confirmed the change points detected by the cumulative sum method. For the application of the Bayes factor and the threshold methods, the concepts presented in Altieri (2015) were taken as the basis, Taylor (2000)'s article was used for the cumulative sum method.

In this work, the programs were elaborated with some instructions from the INLA package in Gómez (2020), Blangiardo and Cameletti (2015), except for the standard normal homogeneity, Pettit and Buishand ranges tests, for which the Trend package (from R) of Pohlert (2020) was used.

Simulated data from a Poisson process as well as cement database, INEGI, and coronavirus data were used. The INEGI and cement data were taken from an example of regression in Gómez (2020), and coronavirus data was taken from the page *coronavirus.gob.mx.*

2. Materials and Methods

2.1 Bayes Factor

To detect the change points, the Bayes factor was used. The definition of the Bayes factor will be given below. The change points are determined according to the values in Table 1.

If there is a problem in which you must choose between two possible models, based on an observed data set D, the plausibility of the two different models M_1 and M_2, parameterized by parameter vectors θ_1 and θ_2, can be measured using the Bayes factor, which is defined as:

$$B = \frac{P(D|M_1)}{P(D|M_2)} = \frac{\int_{\theta_1} P(D|\theta_1, M_1)\Pi(\theta_1|M_1)d\theta_1}{\int_{\theta_2} P(D|\theta_2, M_2)\Pi(\theta_2|M_2)d\theta_1}$$

where $P(D|M_1)$ is called the marginal likelihood or integrated likelihood. This is similar to what is done in the likelihood ratio tests but now, instead of maximizing the likelihood, the Bayes factor performs a weighted average on the distribution of the parameters.

A value of $B > 1$ means that the data supports M_1 more than M_2.

In the case of the Bayes factor, Jeffreys established an interpretation scale of B, this is shown in Table 1.

Table 1. Interpretation scale of B, according to Jeffreys

B	Strength of the evidence in favor of M_1
$B \leq 1$	Negative supports M_2
$1 < B \leq 3$	Very scarce
$3 < B \leq 10$	Substantial
$10 < B \leq 30$	Strong
$30 < B \leq 100$	Very strong
> 100	Decisive

Another way to consider the Bayes factor is as follows:

Suppose two hypotheses H_0 and H_1 such that the a priori probabilities are $f_0 = P(H_0)$ and $f_1 = P(H_1)$.

After observing a sample, the a posteriori probabilities of both hypotheses are $\alpha_0 = P(H_0|x)$ y $\alpha_1 = P(H_1|x)$. The Bayes factor in favor of H_0 is defined as:

$$B = \frac{\frac{\alpha_0}{\alpha_1}}{\frac{f_0}{f_1}} = \frac{\alpha_0 f_1}{\alpha_1 f_0}.$$

Thus, the Bayes factor represents the a posteriori plausibility divided by the a priori plausibility and reports the changes in our beliefs introduced by data. This has the property of being almost objective and partially eliminates the influence of the a priori distribution.

As an example suppose the simple contrast:

$$H_0 : \theta = \theta_0 \quad vs \quad H_a : \theta = \theta_1.$$

We have that the a posteriori distributions are:

$$\alpha_0 = P(H_0|\mathbf{x}) = \frac{f_0 L(\theta_0|\mathbf{x})}{f_0 L(\theta_0|\mathbf{x}) + f_1 L(\theta_1|\mathbf{x})},$$

$$\alpha_1 = P(H_1|\mathbf{x}) = \frac{f_1 L(\theta_1|\mathbf{x})}{f_0 L(\theta_0|\mathbf{x}) + f_1 L(\theta_1|\mathbf{x})}.$$

So, the Bayes factor is:

$$B = \frac{\alpha_0 f_1}{\alpha_1 f_0} = \frac{f_0 L(\theta_0|\mathbf{x}) f_1}{f_1 L(\theta_1|\mathbf{x}) f_0} = \frac{L(\theta_0|\mathbf{x})}{L(\theta_1|\mathbf{x})},$$

which coincides with the likelihood ratio, so that the a priori distribution would not influence the Bayes factor, in this case.

Thus, the Bayes factor for the change point, when it is divided into two segments, is given by the likelihood ratio:

$$\frac{L_0}{L_1} = \frac{Q_1 Q_2}{L_1},$$

where Q_1 is the likelihood of segment 1 and Q_2 is the likelihood of segment 2, under the alternative hypothesis; and L_1 is the likelihood under the null hypothesis.

Applying the logarithm, we have

$$B = log\{Q_1\} + log\{Q_2\} - log\{L_1\}.$$

2.1.1 Homogeneous Poisson Process

To find the change points, first is simulated a homogeneous Poisson process and after of obtained data is worked on. The homogeneous Poisson process is defined as follows:

Definition 1: A collection of random variables $\{N(t) : t \geq 0\}$, defined in a probability space (ω, F, P) is said to be a Poisson process (homogeneous) with intensity $\lambda > 0$ if it satisfies the following properties:

i)$P(N(0) = 0) = 1$, this is, $N(0)$ it is always 0.

ii) For all $0 < s < t$, $N(t) - N(s)$ has Poisson distribution of parameter $\lambda(t - s)$.

iii) For all $0 \leq t_1 < \ldots < t_n, n \geq 1$ (that is, to say for all finite set of times), the random variables $N(t_n) - N(t_{n-1}), \ldots, N(t_2) - N(t_1), N(t_1) - N(0), N(0)$, are independent. This is known as the property of independent increments.

2.1.2 Bisection Method

The bisection method was used to determine the change points with the Bayes factor. This method consists of dividing the data in half and looking for a change point with the Bayes factor as well as with the likelihoods corresponding to the null and alternative hypotheses. Then we go to the left side of the data and it is also divided in half; subsequently, the right side is divided, and so on. We continue going successively to the left and to the right, dividing each side in half and looking for change points with the Bayes factor.

2.2 Threshold Method

The following is an example of change points determined by the threshold method, explained in Altieri (2015), which consists of associating a posteriori probability to the data and defining a threshold, in such a way that, if the probability is less than that threshold, it is said that there is a change point.

An example of how to associate probabilities is given below, using de following multiple linear regression:

$$Y_i = \beta_0 + \sum_{j=1}^{4} \beta_j X_{j,i} + \epsilon_i.$$

For this example, we took the cement database in G´mez (2020). In the model, Y_i represents the evolved heat of the observation i and $X_{j,i}$ is the proportion of the component j in the observation i. The parameter β_0 is an intercept and β_j, $j = 1, \ldots, 4$ are coefficients associated with the covariates. Finally, ϵ_i, $i = 1, 2, \ldots, n$ is an error term with a Gaussian distribution having zero mean and precision τ.

2.2.1 Cement Data

The cement database used is shown in Table 2.

Table 2. Regression data

X_1	X_2	X_3	X_4	Y	X_1	X_2	X_3	X_4	Y
7	26	6	60	78.5	1	31	22	44	72.5
1	29	15	52	74.3	2	54	18	22	93.1
11	56	8	20	104.3	21	47	4	26	115.9
11	31	8	47	87.6	1	40	23	34	83.8
7	52	6	33	95.9	11	66	9	12	113.3
11	55	9	22	109.2	10	68	8	12	109.4
3	71	17	6	102.7					

2.3 Cumulative Sum Method

The Cumulative Sum method consists of the following:

1.- The average is calculated $\frac{x_1+x_2+\ldots+x_{32}}{32}$.

2. The cumulative sum starts at zero $S_0 = 0$.

3. Other cumulative sums of the form $S_i = S_{i-1} + (X_i - \bar{X})$ are calculated for $i = 2, \ldots, 32$

$S_0 = 0,$

$S_1 = S_0 + (X_1 - \bar{X}),$

$S_2 = S_1 + (X_2 - \bar{X}),$

$\vdots$

$S_{32} = S_{31} + (X_{32} - \bar{X}).$

Bootstrapping is also performed, but an estimator of the magnitude of the change is required before that. An option that works well regardless of distribution and despite multiple changes is $Sdiff$, which is the maximum difference of S_i and the minimum of S_i, as can be seen below:

$Sdiff = Smax - Smin$ where $Smax = max_{i=0,\ldots,33} S_i$, $Smin = min_{i=0,\ldots,33} S_i$.

The magnitude of the change is $Sdiff$. A positive value of Sdiff indicates that there was a change from low to high, which means that traffic accidents changed. The latter is the topic of the problem addressed. Next, a bootstrap analysis of a single routine is performed. The procedure is the next:

1. A bootstrap sample of 32 units, denoted by $X_1^o, X_2^o, \ldots, X_{32}^o$, was generated, randomly rearranging the 32 original values. This is called sampling without replacement.

2. Based on the bootstrap samples, the CUSUM bootstrap is calculated. This is denoted by $S_0^o, S_1^o, \ldots, S_{32}^o$.

3. The maximum, the minimum and the CUSUM bootstrap difference, denoted by $S^o max$, $S^o min$, and $S^o diff$ are calculated.

4. It is determined if the bootstrap difference $S^o diff$ is less than the difference $Sdiff$.

A bootstrap analysis consists of performing a large number of bootstraps and counting the number of bootstraps for which

$S^o diff$ is less than $S diff$. Let N be the number of bootstrap samples performed and let X be the bootstrap for which $S^o diff < S diff$. Then the confidence of the change occurred is calculated as a percentage as follows:

Confidence level=$100\frac{X}{N}\%$.

2.3.1 Traffic Accident Data

The cumulative sum method was applied to traffic accident data in 32 cities, which are the states of the Mexican Republic. The data, which correspond to the number of traffic accidents, were obtained from INEGI page and are presented in Table 3:

Table 3. Number of traffic accidents

City	Accidents	City	Accidents	City	Accidents
1	74	11	188	22	146
2	60	12	56	23	69
3	26	13	36	24	113
4	24	14	351	25	278
5	94	15	49	26	208
6	22	16	171	27	41
7	78	17	68	28	136
8	270	18	42	29	80
9	223	19	229	30	140
10	104	20	36	31	37
11	188	21	206	32	83

2.3.2 Coronavirus Data

Coronavirus data corresponding to May 2020, obtained from the *coronavirus.gob.mx* page, were used to detect the change points, applying the cumulative sum method. Three groups of data were formed, corresponding to eight, seven, and five days, respectively. In each database, it was detected where the change from minor to major occurred, that is, where a higher number of contagion cases occurred in the country. In the database they are divided into infections of men and women; however, for the analysis of changes, the total contagion cases (men plus women) were considered. Table 4 shows the data used.

Table 4. Coronavirus infection data

Date	Data	Date	Data	Date	Data
1	1225	9	1270	16	1483
2	1241	10	1161	17	1145
3	1121	11	2512	18	2747
4	2298	12	2428	19	2261
5	1840	13	2516	20	2083
6	2024	14	2353		
7	2085	15	2564		
8	2071				

3. Results and Discussion

3.1 Analysis of Multiple Change Points With the Bayes Factor

In order to detect, analyze, and compare points of change, a program was developed to detect 5 points of change. A set of 60 data with 6 different values for the parameter λ were used, which were obtained from a Poisson simulation process. This exercise is intended to detect the changes generated for the different values of λ. In addition, a comparison was made with the results obtained when applying an a priori uniform, a log-gamma, and a Gaussian.

A program was generated in R to detect multiple points of change in time. In total, for five change points, 60 data were used; the smallest segment was four points. A homogeneous Poisson process with different λ was simulated and the a posterior distribution was approached with the R INLA package, which performs the approximation through Taylor series. Bayes factor and the bisection method were used to detect the change points.

A uniform a priori, log gamma and a Gaussian were used. Five change points are shown in Figure 1.

For the following problem, 60 data are used and the smallest division was reached, which was four data. So, there are fifteen divisions as shown in Table 5.

The change points for a uniform a priori with $\lambda = 2, 1, 4, 7, 6, 1$ and in the data divisions or segments $8, 7, 15, 15, 8, 7$, were as shown in Table 5, the change points are the first four and the seventh.

With the same change points and the a priori loggamma with parameters 0.01 and 0.01, the result is that shown in Table 5. It can be seen that compared to the uniform, the log-likelihoods of the loggamma are smaller, however they also detect the change points.

With the a priori Gaussian with zero mean and precision parameter 0.001, for the same number of change points as the uniform and the log gamma, the five change points were detected; these are the first four and the seventh, in Table 5. It can be seen that the value of the log-likelihood of the seventh change point is less than those of the following positions, however.

The five change points in Table 5 were detected with the Bayes factor B, mentioned above. The values for B indicate the strength of the evidence in favor of the change point hypothesis. Va-lues less than 1 indicate that there is no change point. The data in bold, which correspond to the change points, were detected in the following way: value 1 in the middle of the data, value 2 in data 15, value 3 in data 45, value 4 in data 8, and value 7 in data 53. As can be seen, the values of the change points are not equal to each other or to the rest. On the other hand, where a change point did not occur, there were values equal to each other, for example, values 5 and 6, as well as 8, 10, and 14, and in the same way as 9, 11, 13, and 15; therefore, all these values indicate the non-existence of change points. The strength of the evidence in favor of the alternative hypothesis, indicated by applying the bisection method for gradual detection of the change points, is strong for the first point of change and decreases for the following ones; notice this in Table 5. The change points are shown in

Table 5. Change points

Number	uniform	loggamma	Gaussian
1	**49.565011**	**43.644575**	**42.972975**
2	**24.392282**	**25.829232**	**26.592785**
3	**13.940830**	**26.041694**	**28.329433**
4	**16.000633**	**9.990938**	**11.641769**
5	12.478724	2.387813	4.477951
6	12.478724	2.387813	4.477951
7	**12.572475**	**4.498905**	**5.523642**
8	5.701123	1.835601	5.801758
9	4.732525	1.384310	4.912561
10	5.701123	1.835601	5.801758
11	4.732525	1.384310	4.912561
12	5.701123	1.835601	5.801758
13	4.732525	1.384310	4.912561
14	5.701123	1.835601	5.801758
15	4.732525	1.384310	4.912561

Figure 1 and are at 8, 15, 30, 45, and 53 divisions.

3.1.1 Six Change Points With the Bayes Factor

With the same uniform model, six change points were detected for $\lambda = 2, 1, 4, 7, 6, 1, 2$, in the data divisions or segments of $8, 7, 15, 15, 8, 4, 3$. The change points detected are the first four, the seventh, and the last. These are shown in Tables 6 and 7.

With the a priori logarithm of gamma, six change points were also detected, which are the first four, the seventh and the last in Table 6. The difference with the uniform is that the log likelihood is smaller, as can be seen.

For the a priori Gaussian, six change points were also detected. These are shown below in log likelihood values. The results are somewhat similar to those from the uniform. The change points correspond to the first four, the seventh, and the last in Table 6. The difference between Gaussian and uniform is that the last change point is less than the previous log likelihood values, where there is no change point.

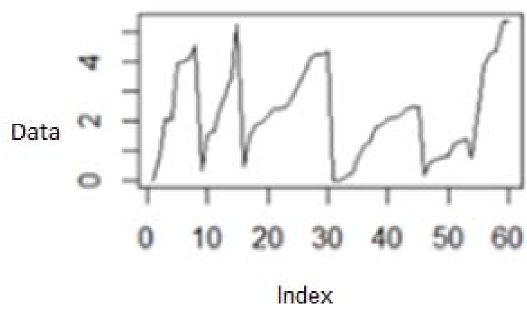


Figure 1. Five change points graph

With the beta logit (with parameters 2 and 2), six change points were well detected, being the same as for the uniform and the gamma logit: the first four, the seventh, and the last. No big differences are visualized with respect to the uniform.

For the a priori truncated Normal with mean 0 and precision 0.001, the likelihood is smaller compared to the uniform. This is very similar to the loggamma, since six change points were also detected: the first four, the seventh, and the last in Tables 6 and 7.

In a similar way to the previous problem, the observed change points are shown in tables 6 and 7. These values are not repeated among themselves or among the rest. Instead, the rest of the values are repeated, for example, 5 and 6, as well as 8, 10, 12 and 14, and also 9, 11, and 13; therefore, all these values do not indicate change points, even when they are greater than 1.

Table 6. Six change points

Num	uniform	loggamma	Gaussian
1	**37.382628**	**44.717137**	**38.258157**
2	**23.107912**	**28.913661**	**26.506732**
3	**22.987741**	**27.851420**	**26.529322**
4	**16.466681**	**5.045601**	**11.798824**
5	12.478724	2.387813	4.477951
6	12.478724	2.387813	4.477951
7	**16.555598**	**3.062915**	**12.572375**
8	5.701123	1.835601	5.801758
9	4.732525	1.384310	4.912561
10	5.701123	1.835601	5.801758
11	4.732525	1.384310	4.912561
12	5.701123	1.835601	5.801758
13	4.732525	1.384310	4.912561
14	5.701123	1.835601	5.801758
15	**4.733385**	**1.720949**	**4.280097**

The change points are shown in Figure 2. These are given in 8, 15, 30, 45, 53, and 57 divisions.

Programming was done in R. The program is presented in appendix A as Program A1.

An algorithm was created for the problem of detecting six change points with the Bayes factor. This algorithm is presented below:

Algorithm 1: it detects six change points.

Input the INLA function returns the variables **mp1, mp2**, and **mp** which contain data simulated with an AR1 autoregressive model and data from a Poisson family for **datos A**, **datos B** and **datos**.

Output the vector **respuesta** keeps the likelihoods of 15 divisions.

Initialize vectors are initialized **mp1, mp2**, and mp which contain the simulated data with an AR1 autoregressive model

Table 7. Six change points

Num	logit beta	normal truncated
1	**49.664317**	**51.841263**
2	**30.567246**	**16.016904**
3	**18.174484**	**33.658036**
4	**9.944739**	**7.969972**
5	4.633706	2.186421
6	4.633706	2.186421
7	**11.248933**	**1.984581**
8	4.076750	1.881876
9	3.794214	1.783849
10	4.076750	1.881876
11	3.794214	1.783849
12	4.076750	1.881876
13	3.794214	1.783849
14	4.076750	1.881876
15	**4.405551**	**2.260447**

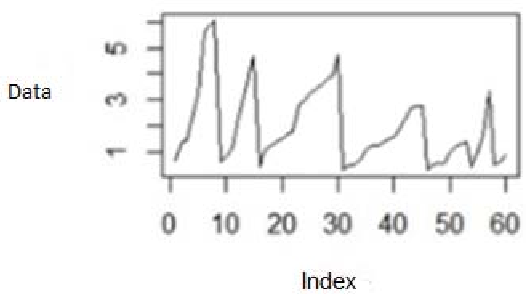


Figure 2. Six change points graph

and a Poisson family for **datos A**, **datos A**, and **datos**.

Return {respuesta} vector that keeps the verosimilitudes of the divisions.

The **datos** vector is divided into 1 to 30 and 31 to 60, and it is assigned to **datos A** and **datos B**.

i) The likelihood for **datos A** and **datos B** is calculated and stored in the **respuesta** vector,

ii) Vectors **datos A** and **datos B** are halved and the data are assigned to **datos C** and **datos D**,

iii) The likelihood fo **datos C** and **datos D** is calculated.

iv Steps **i)** and **ii)** are repeated for the new divisions of the data until the division of only two data.

v) The results obtained are recorded.

3.1.2 Space-Time Analysis of Multiple Change Points With the Bayes Factor

For multiple change points in space-time, what was done was assigning positions. The first one was assigned in the center and the rest around it in ascending and counterclockwise order. The a posteriori was approximated with the R-INLA package. The a priori uniform, the Bayes factor, and the bisection method were also used. The simulation of values was done for a homogeneous Poisson process; four random values of λ (between 1 and 15) were used for each position, so three change points were estimated. The values of λ are given in each row of Table 8.

And the resulting change points are the first three of Tables 9, 10, 11, and 12. These are in 15, 30, and 45 divisions.

The behavior of the values is similar to that of the first problem presented in this work. The change points indicated in bold in tables 9, 10, 11, and 12 are not repeated, unlike the rest. According to the strength of the evidence for Bayes factor B, values less than 1 indicate that there is no point of change. These values are observed in table 10, position 6, value 3, and in table 11, position 9, value 2.

Table 8. Lambda values

	[1]	[2]	[3]	[4]
[1]	10	3	5	13
[2]	8	12	6	5
[3]	14	8	13	5
[4]	5	4	7	12
[5]	12	7	6	8
[6]	12	9	1	6
[7]	1	13	5	3
[8]	11	14	1	3
[9]	1	13	5	15
[10]	13	1	7	4
[11]	5	11	8	9
[12]	10	3	7	15
[13]	1	13	12	6
[14]	7	1	11	12
[15]	13	12	4	1
[16]	5	6	12	7

Table 9. Change points space-time

	[1]	[2]	[3]	[4]
1	**33.693643**	**44.955163**	**35.315124**	**41.732270**
2	**27.795301**	**25.226235**	**28.387088**	**19.482977**
3	**28.727677**	**21.122882**	**27.775696**	**28.697886**
4	12.478724	12.478724	12.478724	12.478724
5	12.478724	12.478724	12.478724	12.478724
6	12.478724	12.478724	12.478724	12.478724
7	12.478724	12.478724	12.478724	12.478724
8	5.701123	5.701123	5.701123	5.701123
9	4.732525	4.732525	4.732525	4.732525
10	5.701123	5.701123	5.701123	5.701123
11	4.732525	4.732525	4.732525	4.732525
12	5.701123	5.701123	5.701123	5.701123
13	4.732525	4.732525	4.732525	4.732525
14	5.701123	5.701123	5.701123	5.701123
15	4.732525	4.732525	4.732525	4.732525

The Bayes factor method detects the change points well. Its weakness is that it only detects changes up to a division of four, but not when the amount of data is lower.

Programming was performed in R. It is shown in appendix A as program A2.

An algorithm was built to detect 3 space-time points of change in 16 positions. This algorithm is shown below:

Algorithm 2: it detects 3 space-time points of change

Input the INLA function returns the variables **mp1, mp2,** and **mp** which contain data simulated with an AR1 autoregressive model and data from a Poisson family for **datos A**, **datos B** and **datos**.

Output the **matrizrespuesta** keeps the likelihoods of 5 divisions in 16 positions.

Initialize vector **mp1, mp2,** and **mp** are initialized. These contain data simulated with an AR1 autoregressive model and data from a Poisson family for **datos A**, **datos B** and **datos**

n=16 means 16 positions.

for i in 1:n

Table 10. Change points space-time

	[5]	[6]	[7]	[8]
1	**33.574630**	**61.47789451**	**52.546434**	**44.358894**
2	**27.549217**	**27.37020690**	**18.324691**	**21.142981**
3	**28.777261**	-0.08807506	**20.678559**	**1.851282**
4	12.478724	12.47872437	12.478724	12.478724
5	12.478724	12.47872437	12.478724	12.478724
6	12.478724	12.47872437	12.478724	12.478724
7	12.478724	12.47872437	12.478724	12.478724
8	5.701123	5.70112304	5.701123	5.701123
9	4.732525	4.73252533	4.732525	4.732525
10	5.701123	5.70112304	5.701123	5.701123
11	4.732525	4.73252533	4.732525	4.732525
12	5.701123	5.70112304	5.701123	5.701123
13	4.732525	4.73252533	4.732525	4.732525
14	5.701123	5.70112304	5.701123	5.701123
15	4.732525	4.73252533	4.732525	4.732525

Table 11. Change points space-time

	[9]	[10]	[11]	[12]
1	**71.06948660**	**35.332035**	**34.663739**	**28.888859**
2	-0.08807506	**28.183967**	**26.321000**	**25.185406**
3	**19.83004426**	**27.173106**	**28.558468**	**28.850767**
4	12.47872437	12.478724	12.478724	12.478724
5	12.47872437	12.478724	12.478724	12.478724
6	12.47872437	12.478724	12.478724	12.478724
7	12.47872437	12.478724	12.478724	12.478724
8	5.70112304	5.701123	5.701123	5.701123
9	4.73252533	4.732525	4.732525	4.732525
10	5.70112304	5.701123	5.701123	5.701123
11	4.73252533	4.732525	4.732525	4.732525
12	5.70112304	5.701123	5.701123	5.701123
13	4.73252533	4.732525	4.732525	4.732525
14	5.70112304	5.701123	5.701123	5.701123
15	4.73252533	4.732525	4.732525	4.732525

4 **lambdas** from a 1-15 Poisson process are randomly calculated and they are included in the vector **vlambda**.

The four vectors simulated by change points, corresponding to the four lambdas, are included in the vector **datos** which has space for 60 data

end for

for i in 1:n

The vector **datos** is divided into 1 to 30 and 31 to 60 and assigned to **datos A** and **datos B**.

i) The likelihood for each vector in **datos A** and **datos B** is calculated and stored in the **matrizrespuesta**,

ii) The vector **datos A** and **datos B** are halved and data are assigned to **datos C** and **datos D**,

iii) The likelihoods for **datos C** and **datos D** are calculated.

iv) Steps **i)** and **ii)** are repeated for the new divisions of the data until the division of only two data.

v) The results obtained are recorded.

end for

return returns the **matrizrespuesta**. This matrix contains the likelihoods.

Table 12. Change points space-time

	[13]	[14]	[15]	[16]
1	**55.319548**	**36.493439**	**53.350957**	**37.905475**
2	**7.263903**	**27.399136**	**14.610914**	**25.226621**
3	**28.683089**	**26.696299**	**17.792607**	**28.209964**
4	12.478724	12.478724	12.478724	12.478724
5	12.478724	12.478724	12.478724	12.478724
6	12.478724	12.478724	12.478724	12.478724
7	12.478724	12.478724	12.478724	12.478724
8	5.701123	5.701123	5.701123	5.701123
9	4.732525	4.732525	4.732525	4.732525
10	5.701123	5.701123	5.701123	5.701123
11	4.732525	4.732525	4.732525	4.732525
12	5.701123	5.701123	5.701123	5.701123
13	4.732525	4.732525	4.732525	4.732525
14	5.701123	5.701123	5.701123	5.701123
15	4.732525	4.732525	4.732525	4.732525

3.2 Threshold Method

For the example of the change point, the variable X_1 in Table 2 of Section 2.2.1 was used. Probabilities of the a posteriori distribution were assigned to data of X_1. A threshold of $3.041631e-07$ was set, so data having a probability less than this value were considered as change points.

The program A3 in Appendix A was developed in R. The results were given in zeros and ones:

0 0 1 1 0 1 0 0 0 1 0 1 1

The change occurs where the number 1 begins to be more constant, that is, where the a posteriori probability assigned to the data is less than the defined threshold, indicating that a point of change exists. The latter happens in position ten and three, with a threshold of 3.041631e-07.

An algorithm was created which detects multiple points of change by the threshold method. This algorithm is presented below:

Algorithm 3: it detects multiple change points by the threshold method

Input cement matrix, datos matrix, treshold, regression in ml.

Output Vector **menores** include zeros and ones; 1 means that the a posteriori probability is less than threshold y 0 means that the a posteriori probability is greater than threshold.

Inizialite n=13 is the amount of data.

For h in 1:n

The a posteriori probability is calculated for each data.

End For

For h in 1:n

if the a posteriori probability is less than threshold, then a 1 is stored in the vector **menores else** if it is greater a zero is stored.

End For

Print menores

3.3 Cumulative Sum Method

3.3.1 With Traffic Accident Data

The algorithm of Section 2.3 was applied to the traffic accident data of Section 2.3.1, Table 3. According to the magnitude of the change, the minimum sum was −439.6875, and it was found in the data located in the seventh position. The maximum sum, the minimum sum and the difference were:

Smax=183.875
Smin=-439.6875
Sdiff=623.5625

Confidence Level for a hundred sample bootstrap=$100\frac{X}{N}\% = 83\%$.

It can be seen in Figure 3 that the change point is at value 7, where it changes from the smallest number of traffic accidents to a higher number.

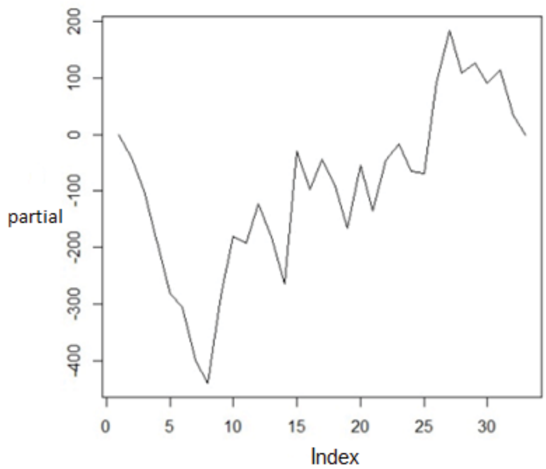


Figure 3. Graph with change point at 7

What the algorithm establishes was applied and the program A4 in appendix A was developed.

An algorithm was created with the cumulative sum method, which detects a single point of change. This algorithm is shown below:

Algorithm 4: it detects a single change point

Input the matrix **datos** contain data for the analysis

Output parciales are cumulative data sums, **rparciales** are cumulative bootstrap sums, **rSdif** are the resampling differences.

Inizialite n=33 they are 33 data

m=100 is resampling of size 100 with 33 data

For i in 2:n

The cumulative sum is calculated and stored in **parciales**

End For

The maximum**Smax=max(parciales)**, ,minimum **Smin=min(parciales)** and difference **Sdif=Smax-Smin** are calculated

For k in 2:n

The resampling cumulative sum is calculated and stored in**rparciales**

End For

It is calculated the maximum of **rparciales rSmax=max(rparciales)**

mimimum **rSmin=min(rparciales)**, and difference **rSdif=rSmax-rSmin**

return rSdif returns the resampling difference vector.

For j in 1:m a bootstrap resampling of size 100 with 33 data is performed

The differences are calculated for each resampling **diferencias[m] = remuestreo(sample(datos, 32))**

End for

For j in 1:m

If If the resampling differences are less than **Sdif** then the normal differences are stored in the vector **respuesta**.

End for

The confidence Interval is calculated:

Nconfianza=100*sum(respuestas)/length(respuestas)

Other tests applied with traffic accident data

The Buishand, Pettitt and standard normal homogeneity tests were performed on the same traffic accident data. For all tests, the result was the change point at position 7. It was used in R the trend library in Pohlert (2020), using the following code:

```
library(trend)
data < -c(74, 60, 26, 24, 94, 22, 78, 270, 223, 104, 188, 56, 36
, 351, 49, 171, 68, 42, 229, 36, 206, 146, 69,
113, 278, 208, 41, 136, 80, 140, 37, 83)
y < -ts(data, start = c(1900), freq = 1)
x < -ts(data, start = c(1900), freq = 1)
buishand ranges test
br.test(y, m = 20000)
pettit test
pettitt.test(x)
standard normal homogeneity test
snh.test(x, m = 20000)
```

3.3.2 With Coronavirus Data

The coronavirus data from Section 2.3.2 in Table 4 were used. The infection data are shown in a bar graph, in Figure 4. The bar graph shows how the number of infections decreases and increases again. Increases or changes occurred at May

Figure 4. Graph of contagion

4, May 11 an May 18, for the first, second and last group, respectively. This is observed in the CUSUM cumulative sums graph shown in Figure 5. For each cumulative sum, the statistics and the percentage are those shown in Table 13.

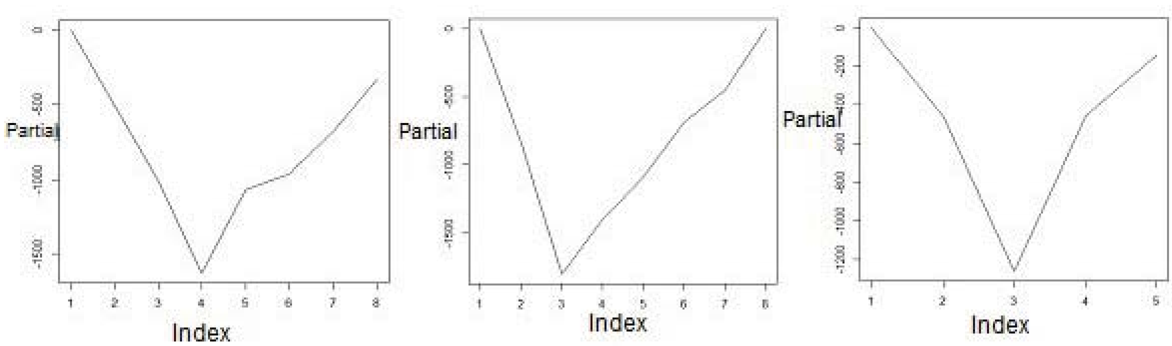


Figure 5. Data groups 1, 2 and 3

The R program applied for the analysis of coronavirus data, was the same as that used for the traffic accidents data.

With the CUSUM method only one change point is detected, however changes are detected even when the database is small, which is the case with the coronavirus data.

Table 13. Statistics and percentage

Group 1	Group 2	Group 3
Sdiff=1627.375	Sdiff=1798.714	Sdiff=1259.6
Smax=0	Smax=0	Smax=0
Smin=-1627.375	Smin=-1798.714	Smin=-1259.6
ConfidenceL=89%	83%	61%

4. Conclusion

Programs were developed for the Bayes factor, thresholds, and CUSUM methods. In the case of the Bayes factor method, the purpose was to numerically detect the points of change and compare the results with those of the a priori distributions used. For the threshold and CUSUM methods, the objective was to detect the change points. Coronavirus, car traffic, and regression (cement) databases were used, as well as a database simulated with a Poisson process.

Pettitt, standard normal homogeneity, and Buishand ranges methods were used. The R Trend package, which contains these methods, was applied to detect the change points.

It can be concluded that the threshold and CUSUM methods detect change points well. The difference is that the threshold method allows to detect multiple points of change in a regression.

The CUSUM method only detects a point of change, as do Pettitt, standard normal homogeneity, and Buishand ranges methods. Using the CUSUM method, changes are detected even when the database is small, as happened with the coronavirus data.

The Bayes factor method detected the change points well. Its weakness is that it can only detect changes up to a division of four, but when the amount of data in the division is lower, the change points are no longer detected.

References

Altieri, L. (2015). A bayesian changepoint analysis on spatio-temporal point processes (Doctoral dissertation). University of Bologna. Italy. https://doi.org/10.1016/j.spasta.2015.05.005

Blangiardo, M., & Cameletti, M. (2015). *Spatial and Spatio-temporal Bayesian Models with R-INLA*. John Wiley & Sons.

Chen, J., & Gupta, A. K. (2012). *Parametric Statistical Changepoint Analysis: With Applications to Genetics, Medicine, and Finance* (2nd ed.). Birkhauser. https://doi.org/10.1007/978-0-8176-4801-5

Gómez, R. V. (2020). *Bayesian Inference with INLA* (1st ed.). Champman & Hall.

Pohlert, T. (2020). *Non-parametric trend tests and change-point detection.* R Package version 3.0.

Taylor, W. A. (2000). Change-point analysis: A powerful new tool for detecting changes. Baxter Healthcare Corporation, Round Lake, IL.

Appendix A

Program A1

```
rm(list = ls())
library(poisson)
library(INLA)
vector <-hpp.event.times(2, 15, num.sims = 1, t0 = 0)
plot(vector)
vector1 <-hpp.event.times(1, 15, num.sims = 1, t0 = 0)
vector2 <-hpp.event.times(4, 15, num.sims = 1, t0 = 0)
vector4 <-hpp.event.times(6, 15, num.sims = 1, t0 = 0)
datos <-c(vector, vector1, vector2, vector4)
plot(datos, type = "l")
datosA = datos[1 : 30]
datosB = datos[31 : 60]
datos
datosA
datosB
```

```
vec_res = rep(0, 20)
factor_b < -function(datosA, datosB, datos){
p1 < -data.frame("num" = seq(1, length(datosA), 1), "datosA" = datosA)
p2 < -data.frame("num" = seq(1, length(datosB), 1), "datosB" = datosB)
mp1 < -inla(num f(datosA, model = "ar1"), data = p1, family = "poisson")
mp2 < -inla(num f(datosB, model = "ar1"), data = p2, family = "poisson")
midf < -data.frame("num" = seq(1, length(datos), 1), "datos" = datos)
mp < -inla(num f(datos, model = "ar1"), data = midf, family = "poisson")
respuesta < -as.numeric(mp1$mlik[1, 1]) + as.numeric(mp2$mlik[1, 1])
-as.numeric(mp$mlik[1, 1])
return(respuesta)
}
vec_res[1] = factor_b(datosA, datosB, datos)
vec_res
datosC = datosA[1 : 15]
datosD = datosA[16 : 30]
vec_res[2] = factor_b(datosC, datosD, datosA)
vec_res
datosE = datosB[1 : 15]
datosF = datosB[16 : 30]
vec_res[3] = factor_b(datosE, datosF, datosB)
vec_res
datosG = datosC[1 : 8]
datosH = datosC[9 : 15]
vec_res[4] = factor_b(datosG, datosH, datosC)
vec_res
datosI = datosD[1 : 8]
datosJ = datosD[9 : 15]
vec_res[5] = factor_b(datosI, datosJ, datosD)
vec_res
datosK = datosE[1 : 8]
datosL = datosE[9 : 15]
vec_res[6] = factor_b(datosK, datosL, datosE)
vec_res
datosM = datosF[1 : 8]
datosN = datosF[9 : 15]
vec_res[7] = factor_b(datosM, datosN, datosF)
vec_res
datosO = datosG[1 : 4]
datosP = datosG[5 : 8]
vec_res[8] = factor_b(datosO, datosP, datosG)
vec_res
datosQ = datosH[1 : 4]
datosR = datosH[5 : 7]
vec_res[9] = factor_b(datosQ, datosR, datosH)
vec_res
datosS = datosI[1 : 4]
datosT = datosI[5 : 8]
vec_res[10] = factor_b(datosS, datosT, datosI)
vec_res
datosU = datosJ[1 : 4]
datosV = datosJ[5 : 7]
vec_res[11] = factor_b(datosU, datosV, datosJ)
vec_res
datosX = datosK[1 : 4]
datosY = datosK[5 : 8]
vec_res[12] = factor_b(datosX, datosY, datosK)
```

```
vec_res
datosW = datosL[1 : 4]
datosZ = datosL[5 : 7]
vec_res[13] = factor_b(datosW, datosZ, datosL)
vec_res
datosAA = datosM[1 : 4]
datosBB = datosM[5 : 8]
vec_res[14] = factor_b(datosAA, datosBB, datosM)
vec_res
datosCC = datosN[1 : 4]
datosDD = datosN[5 : 7]
vec_res[15] = factor_b(datosCC, datosDD, datosN)
vec_res
```

To use another a priori function, it is changed in the program by the following instruction, where the a priori distribution to use and its parameters are included, since the INLA package has the a priori distributions to use.

```
prec.prior < −list(prec = list(prior = ”logtgaussian”, param = c(0, 0.001)))
mp1 < −inla(num f(datosA, model = ”ar1”, hyper = prec.prior), data = p1, family = ”poisson”)
```

Program A2

```
rm(list = ls())
library(poisson)
library(INLA)
esp_t = matrix(0, 60, 16)
matriz_resp = matrix(0, 15, 16)
mlambdas = matrix(0, 16, 4)
factor_b < −function(datosA, datosB, datos){
p1 < −data.frame(”num” = seq(1, length(datosA), 1), ”datosA” = datosA)
p2 < −data.frame(”num” = seq(1, length(datosB), 1), ”datosB” = datosB)
mp1 < −inla(num f(datosA, model = ”ar1”), data = p1, family = ”poisson”)
mp2 < −inla(num f(datosB, model = ”ar1”), data = p2, family = ”poisson”)
midf < −data.frame(”num” = seq(1, length(datos), 1), ”datos” = datos)
mp < −inla(num f(datos, model = ”ar1”), data = midf, family = ”poisson”)
respuesta < −as.numeric(mp1$mlik[1, 1]) + as.numeric(mp2$mlik[1, 1])
−as.numeric(mp$mlik[1, 1])
return(respuesta)
}
for(i in 1 : 16){
vlambda = sample(1 : 15, 4)
vector < −hpp.event.times(vlambda[4], 15, num.sims = 1, t0 = 0)
plot(vector)
mlambdas[i, ] = vlambda
vector1 < −hpp.event.times(vlambda[1], 15, num.sims = 1, t0 = 0)
vector2 < −hpp.event.times(vlambda[2], 15, num.sims = 1, t0 = 0)
vector4 < −hpp.event.times(vlambda[3], 15, num.sims = 1, t0 = 0)
datos < −c(vector, vector1, vector2, vector4)
esp_t[, i] = datos
}
for(indice in 1 : 16){
datosA = esp_t[1 : 30, indice]
datosB = esp_t[31 : 60, indice]
datos = esp_t[, indice]
datosA
datosB
matriz_resp[1, indice] = factor_b(datosA, datosB, datos)
datosC = datosA[1 : 15]
datosD = datosA[16 : 30]
```

```
matriz_resp[2, indice] = factor_b(datosC, datosD, datosA)
datosE = datosB[1 : 15]
datosF = datosB[16 : 30]
matriz_resp[3, indice] = factor_b(datosE, datosF, datosB)
datosG = datosC[1 : 8]
datosH = datosC[9 : 15]
matriz_resp[4, indice] = factor_b(datosG, datosH, datosC)
datosI = datosD[1 : 8]
datosJ = datosD[9 : 15]
matriz_resp[5, indice] = factor_b(datosI, datosJ, datosD)
datosK = datosE[1 : 8]
datosL = datosE[9 : 15]
matriz_resp[6, indice] = factor_b(datosK, datosL, datosE)
datosM = datosF[1 : 8]
datosN = datosF[9 : 15]
matriz_resp[7, indice] = factor_b(datosM, datosN, datosF)
datosO = datosG[1 : 4]
datosP = datosG[5 : 8]
matriz_resp[8, indice] = factor_b(datosO, datosP, datosG)
datosQ = datosH[1 : 4]
datosR = datosH[5 : 7]
matriz_resp[9, indice] = factor_b(datosQ, datosR, datosH)
datosS = datosI[1 : 4]
datosT = datosI[5 : 8]
matriz_resp[10, indice] = factor_b(datosS, datosT, datosI)
datosU = datosJ[1 : 4]
datosV = datosJ[5 : 7]
matriz_resp[11, indice] = factor_b(datosU, datosV, datosJ)
datosX = datosK[1 : 4]
datosY = datosK[5 : 8]
matriz_resp[12, indice] = factor_b(datosX, datosY, datosK)
datosW = datosL[1 : 4]
datosZ = datosL[5 : 7]
matriz_resp[13, indice] = factor_b(datosW, datosZ, datosL)
datosAA = datosM[1 : 4]
datosBB = datosM[5 : 8]
matriz_resp[14, indice] = factor_b(datosAA, datosBB, datosM)
datosCC = datosN[1 : 4]
datosDD = datosN[5 : 7]
matriz_resp[15, indice] = factor_b(datosCC, datosDD, datosN)
indice = indice + 1
}
matriz_resp
mlambdas
```

Program A3

```
library(INLA)
cement
m1 < −inla(y x1 + x2 + x3 + x4, data = cement)
dato < −c(7, 1, 11, 11, 7, 11, 3, 1, 2, 21, 1, 11, 10)
m1$marginals.fixed$x1
1 − inla.pmarginal(0, m1$marginals.fixed$x1)
probas = seq(1, 13)
for(h in 1 : 13){
probas[h] = inla.pmarginal(dato[h]+0.2, m1$marginals.fixed$x1)−inla.pmarginal(dato[h]−0.2, m1$marginals.fixed$x1)
}
menores = seq(1, 13)
```

```
umbral = 3.041631e − 07
for(h  in  1 : 13){
menores[h] = probas[h] < umbral
}
menores
```

Program A4

```
datos < −c(74, 60, 26, 24, 94, 22, 78, 270, 223, 104, 188, 56, 36, 351, 49, 171, 68, 42, 229, 36, 206,
146, 69, 113, 278, 208, 41, 136, 80, 140, 37, 83)
promd < −mean(datos)
promd
parciales < −seq(1, 33)
parciales[1] = 0
for(iin2 : 32){
parciales[i] = parciales[i − 1] + datos[i − 1] − promd
}
parciales
Smax = max(parciales)
Smin = min(parciales)
Sdif = Smax − Smin
Sdif
Smax
Smin
plot(parciales, type = ”l”)
remuestreo < −function(rdatos){
rpromd < −mean(rdatos)
rparciales < −seq(1, 33)
rparciales[1] = 0
for(k  in  2 : 33){
rparciales[k] = rparciales[k − 1] + rdatos[k − 1] − rpromd
}
rSmax = max(rparciales)
rSmin = min(rparciales)
rSdif = rSmax − rSmin
return(rSdif)
}
diferencias = seq(1, 100)
for(m  in  1 : 100){
diferencias[m] = remuestreo(sample(datos, 32))
}
diferencias
respuestas < −seq(1, 100)
for(m  in  1 : 100){
respuestas[m] = diferencias[m] < Sdif
}
respuestas
diferencias
Sdif
NConfianza = 100 ∗ sum(respuestas)/length(respuestas)
NConfianza
```

Copyrights

Copyright for this article is retained by the author(s), with first publication rights granted to the journal.

This is an open-access article distributed under the terms and conditions of the Creative Commons Attribution license (http://creativecommons.org/licenses/by/4.0/).

International Journal of Statistics and Probability; Vol. 10, No. 3; May 2021
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

Statistical Properties of a New Bathtub Shaped Failure Rate Model With Applications in Survival and Failure Rate Data

Muhammad Z. Arshad[1*], Muhammad Z. Iqbal[1], Alya Al Mutairi[2]

[1] Department of Mathematics and Statistics, University of Agriculture, Faisalabad 54000, Pakistan

[2] Department of Mathematics, Faculty of Science, Taibah University, Medina 4321, Saudi Arabia

Correspondence: Muhammad Z. Arshad, Department of Mathematics and Statistics, University of Agriculture, Faisalabad 54000, Pakistan.

Received: February 21, 2021 Accepted: March 15, 2021 Online Published: March 24, 2021

doi:10.5539/ijsp.v10n3p49 URL: https://doi.org/10.5539/ijsp.v10n3p49

Abstract

In this study, we proposed a flexible lifetime model identified as the modified exponentiated Kumaraswamy (MEK) distribution. Some distributional and reliability properties were derived and discussed, including explicit expressions for the moments, quantile function, and order statistics. We discussed all the possible shapes of the density and the failure rate functions. We utilized the method of maximum likelihood to estimate the unknown parameters of the MEK distribution and executed a simulation study to assess the asymptotic behavior of the MLEs. Four suitable lifetime data sets we engaged and modeled, to disclose the usefulness and the dominance of the MEK distribution over its participant models.

Keywords: Kumaraswamy distribution, bathtub shaped hazard rate function, maximum likelihood estimation, order statistics, hydrology, reliability engineering, petroleum engineering

2000 Mathematics Subject Classification: 60E05, 62P30, 62P12

1. Introduction

In this world of science, the significance of probability distributions has an imperative role to elucidate the real-world random phenomenon. In this scenario, Kumaraswamy (1980) proposed a much better choice against the beta distribution, the Kumaraswamy distribution. It is defined over the interval bounded in (0, 1)). Several characteristics like uni-anti-modal, uni-modal, decreasing, increasing, or constant failure rate, which the Kumaraswamy distribution and the beta distribution shared alike. For details, readers are referred to as Jones (2009). He highlighted some significant and common features of Kumaraswamy distribution involved simple normalizing constant, uncomplicated explicit expressions for the density function, distribution function, order statistics, and quantile function. Beta and Kumaraswamy distributions, both are the special cases of the generalized beta distribution see McDonald (1984), Ali *et al.* (2017), and Mukhtar *et al.* (2019). To model in hydrology, atmosphere temperature, clinical trials, engineering, and geology, among other real word random phenomena, Kumaraswamy distribution considers a far better choice than beta distribution.

Let X be a random variable follow by the Kumaraswamy distribution. The associated cumulative distribution function (CDF) and corresponding probability density function (PDF) with two shape parameters $(\alpha, \beta > 0)$ with$0 < x < 1$, are given by, respectively

$$P(x;\alpha,\beta) = \int_0^x p(x)dx = \alpha\beta\int_0^x x^{\alpha-1}(1-x^{\alpha})^{\beta-1}dx = 1-\ (1-x^{\alpha})^{\beta},$$

$$p(x;\alpha,\beta) = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\,.$$

The capability of Kumaraswamy distribution was raised by Cordeiro and de Castro (2011) in introducing a new generalized class, called the Kumaraswamy-G (short Kum-G) family. The cumulative distribution function (CDF) and probability density function (PDF) of the Kum-G family, is defined by, respectively

$$F(x;\alpha,\beta,\xi) = 1-\ \left(1-G^{\alpha}(x;\xi)\right)^{\beta},$$

$$f(x;\alpha,\beta,\xi)=\alpha\beta\,g(x;\xi)G^{\beta-1}(x;\xi)\big(1-G^{\alpha}(x;\xi)\big)^{\beta-1}.$$

where $G(x;\xi)$ is CDF of arbitrary baseline model based on the parametric vector ξ with $\alpha,\beta>0$ are the two shape parameters, respectively. Let $g(x;\xi)=dG(x;\xi)/dx$ is the probability density function of any baseline model.

To study further modifications and generalizations using the Kum-G family, see the exemplar work of Bourguignon *et al.* (2013). They developed the Kumaraswamy Pareto (KP) distribution and discussed their vital characteristics and explored their application to the hydrological data. Lemonte *et al.* (2013) developed two versions of the Kumaraswamy distribution named (i) exponentiated Kumaraswamy distribution, and (ii) Log Exponentiated Kumaraswamy distribution. They derived numerous mathematical and reliable characters and discussed the application with the assistance of Log Exponentiated Kumaraswamy distribution. Alizadeh *et al.* (2015) developed the Kumaraswamy version of the Marshall-Olkin (1997) family. Afify *et al.* (2016) initiated the Kumaraswamy version of Marshall-Olkin Fréchet distribution (Krishna *et al*. (2013)) and explored their application in the medical science and reliability engineering data. Ibrahim (2017) developed the Kumaraswamy version of the power function distribution and explored their application in medical science data. Bursa and Ozel (2017) discussed the exponentiated version of Kumaraswamy power function distribution and explored their application in the metrology data. Mahmoud *et al.* (2018) developed a five-parameter Kumaraswamy edition of the exponentiated Fréchet distribution. They explored twenty-seven models and explored their application in reliability engineering data. Nawaz *et al*. (2018) generalized Kappa distribution via Kumaraswamy G class with the intention that it would be a better alternative to the generalized Kappa distribution and exploring their application in the hydrology data. Silva *et al.* (2019) proposed the exponentiated Kum-G class and explored their application in the reliability engineering data. Cribari-Neto and Santos (2019), introduced an interesting work according to some specific nature of data included exactly zero, exactly one, or both the cases were involved known as the inflated Kumaraswamy distributions. This distribution was the mixture of Kumaraswamy and Bernoulli distributions.

This article is organized in the following sections. We define the linear expressions, shapes, quantile function, reliability, and other mathematical measures in Section 2. The estimation of the model parameters by the method of maximum likelihood and simulation results is performed in Section 3. Applications to real data sets are discussed in Section 4 to illustrate the importance and flexibility of the proposed model and finally, some conclusions are reported in Section 5.

1.1 New Model

The new model is based on the Type II Half Logistic G family of distributions attributed to Hassan *et al*. (2017) with associated CDF is given as follows:

$$F(x;\phi,\zeta)=1-\int_0^{-log W(x;\zeta)}\frac{2\phi e^{-\phi t}}{(1+e^{-\phi t})^2}dt=\frac{2[W(x;\zeta)]^{\phi}}{1+[W(x;\zeta)]^{\phi}},\tag{1}$$

where $W(x;\zeta)$ is any arbitrary baseline model based on $\zeta\in\Omega$, and $\phi>0$ is a shape parameter with $x>0$.

For deep understanding, we suggest the reader see some notable efforts including Balakrishnan (1985), extended half logistic distribution by Altun *et al*. (2018), type II half logistic exponential by Elgarhy *et al*. (2019), Kumaraswamy inverse Lindley distribution by Hemeda *et al*. (2020), Al-Marzouki *et al*. (2021), and among others.

The new model is:

(i) flexible enough and bounded in (0, 1) interval,

(ii) exhibits a bathtub-shaped failure rate function,

(iii) offers more realistic and rationalized results specifically on the complex skewed symmetric and sophisticated random phenomena,

(iv) provides consistently a better fit over its competitors as shown in the application section using four real data sets,

(v) provides simple and uncomplicated CDF, PDF, and likelihood functions.

Formally, a random variable X is said to follow the modified exponentiated Kumaraswamy (MEK) distribution if the baseline model $W(x;\alpha,\beta,\gamma)$ by Lemonte *et al*. (2013) with associated CDF,

$$W(x;\alpha,\beta,\gamma)=\big(1-(1-x^{\alpha})^{\beta}\big)^{\gamma},\tag{2}$$

is placed in equation (1) with ϕ=1. The associated CDF with three shape parameters $\alpha,\beta,\gamma>0$ and the corresponding PDF is given by respectively

$$F(x;\alpha,\beta,\gamma)=\frac{2}{1+(1-(1-x^{\alpha})^{\beta})^{-\gamma}}, \tag{3}$$

$$f(x;\alpha,\beta,\gamma)=\frac{2\alpha\beta\gamma x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\left(1-(1-x^{\alpha})^{\beta}\right)^{-\gamma-1}}{(1+(1-(1-x^{\alpha})^{\beta})^{-\gamma})^{2}}, \tag{4}$$

2. Distributional Properties

2.1 Linear Representation

Linear combination provides a much informal approach to discuss the CDF and PDF than the conventional integral computation when determining the mathematical properties. For this, we consider the following binomial expansion:

$$(1-z)^{\beta}=\sum_{i=0}^{\infty}(-1)^{i}\binom{\beta}{i}z^{i}, |z|<1,$$

From Equation (3), linear expression of CDF is given by

$$F(x)=2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\binom{-1}{i}\binom{-\gamma i}{j}\binom{\beta j}{k}x^{\alpha k}. \tag{5}$$

From Equation (4), linear expression of PDF is given by

$$f(x)=2\alpha\beta\gamma x^{\alpha-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{j}\binom{-2}{i}\binom{-\gamma i-\gamma-1}{j}(1-x^{\alpha})^{\beta j+\beta-1}. \tag{6}$$

$$f(x)=2\alpha\beta\gamma\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\binom{-2}{i}\binom{-\gamma i-\gamma-1}{j}\binom{\beta j+\beta-1}{k}x^{\alpha k+\alpha-1}. \tag{7}$$

Expression in Equation (6) will be quite helpful in the forthcoming computations of various mathematical properties of the MEK distribution.

2.2 Shapes

Different plots of density and failure rate functions of the MEK distribution are displayed in Figures 1 and 2, for various choices of the parameters. Possible shapes of the density function including increasing, decreasing, symmetric, and upside-down bathtub shapes and, Figure 2 illustrates the increasing, decreasing, U - shaped, and upside-down bathtub-shaped failure rate function.

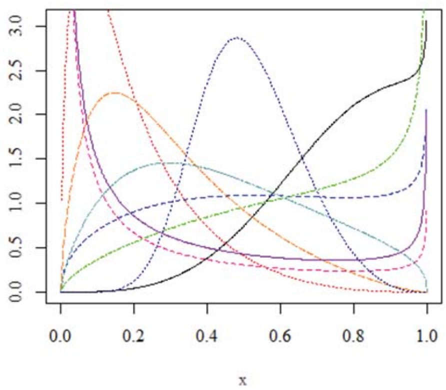


Figure 1. Plot of the density function for Parameters Black($\alpha = 4.7, \beta = 0.9, \gamma = 0.9$),Blue($\alpha = 0.8, \beta = 0.8, \gamma = 1.8$),Red($\alpha = 0.9, \beta = 3.7, \gamma = 1.7$), Green($\alpha = 1.0, \beta = 0.6, \gamma = 1.6$), chocolate1($\alpha = 1.1, \beta = 2.5, \gamma = 1.5$), Cadet blue($\alpha = 1.2, \beta = 1.4, \gamma = 1.4$), Darkviolet ($\alpha = 1.3, \beta = 0.5, \gamma = 0.3$), Deeppink($\alpha = 1.4, \beta = 0.6, \gamma = 0.2$), Navy($\alpha = 1.5, \beta = 3.7, \gamma = 5.1$)

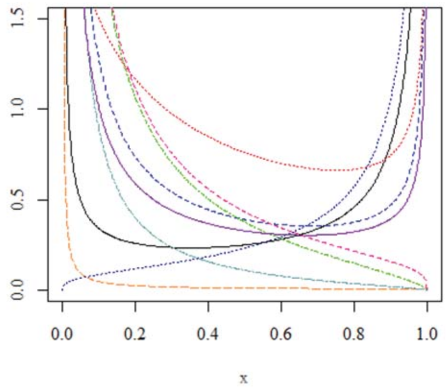


Figure 2. Plot of the failure rate function for Parameters Black($\alpha = 0.1, \beta = 0.1, \gamma = 1.5$), Blue($\alpha = 1.1, \beta = 0.3, \gamma = 0.3$),Red($\alpha = 2.1, \beta = 0.5, \gamma = 0.4$),Green($\alpha = 0.1, \beta = 1.7, \gamma = 0.5$), chocolate1($\alpha = 0.01, \beta = 0.9, \gamma = 0.7$), Cadet blue($\alpha = 0.3, \beta = 1.7, \gamma = 0.8$), Darkviolet ($\alpha = 0.2, \beta = 0.5, \gamma = 0.9$), Deeppink($\alpha = 0.5, \beta = 1.3, \gamma = 1.1$), Navy($\alpha = 1.1, \beta = 0.1, \gamma = 1.2$)

2.3 Quantiles

Hyndman and Fan (1996) introduced the concept of quantile function. The p^{th} quantile function of $X \sim \text{MEK}(x; \alpha, \beta, \gamma)$ with $\alpha, \beta, \gamma > 0$, is obtained by inverting the CDF mention in Equation (3). Quantile function is defined by $p = F(x_p) = P(X \le x_p),\ p \in (0,1)$.

Quantile function of *X* is given by

$$x_p = \left(1 - \left(1 - \left(\frac{p}{2-p}\right)^{1/\gamma}\right)^{1/\beta}\right)^{1/\alpha}. \tag{8}$$

One may obtain 1st quartile, median and 3rd quartile of *X* by setting *p* = 0.25, 0.5, and 0.75 in Equation (8) respectively. Henceforth, to generate random numbers, we assume that CDF (5) follows uniform distribution *u*= *U* (0, 1).

2.4 Skewness, Kurtosis, and Mean Deviation

The Skewness and kurtosis of MEK distribution can be calculated by the following two useful measures

$$B = \frac{Q_{0.75} + Q_{0.25} - 2Q_{0.50}}{Q_{0.75} - Q_{0.25}}, \quad \text{and} \quad M = \frac{Q_{0.375} - Q_{0.125} - Q_{0.625} + Q_{0.875}}{Q_{0.75} - Q_{0.25}},$$

by Bowley (1920) and Moors (1988) respectively. These descriptive measures, based on quartiles and octiles, provide more robust estimates than the traditional skewness and kurtosis measures. Moreover, these measures are almost less reactive to outliers and work more effectively for the distributions, deficient in moments. The following Table-1, presents some results of the first four moments about the origin, variance, skewness, and kurtosis of MEK distribution for some choices of parameters place in S-I($\alpha = 1.07, \beta = 5, \gamma = 1.1$), S-II($\alpha = 1.1, \beta = 5, \gamma = 1.07$), S-III($\alpha = 1.09, \beta = 5, \gamma = 1.1$), S-IV($\alpha = 1.1, \beta = 5, \gamma = 1.09$), S-V($\alpha = 1.1, \beta = 5, \gamma = 1.1$), S-VI($\alpha = 1.1, \beta = 1.1, \gamma = 5$), S-VII($\alpha = 1.1, \beta = 1.2, \gamma = 5$), S-VIII($\alpha = 1.1, \beta = 1.3, \gamma = 5$), S-IX($\alpha = 1.01, \beta = 5, \gamma = 1.3$), and S-X($\alpha = 1.02, \beta = 5, \gamma = 1.4$). The behavior of variance, skewness, and kurtosis has decreasing trend as per the results indicate in Table-1.

Table 1. Some results of moments, variance, skewness, and kurtosis

μ'_s	S-I	S-II	S-III	S-IV	S-V
μ'_1	2.2094	2.1831	2.1857	2.1772	2.1743
μ'_2	4.4551	4.3292	4.3346	4.2921	4.2741
μ'_3	9.8507	9.4252	9.4155	9.2805	9.2114
μ'_4	23.176	21.849	21.709	21.294	21.033
Variance	0.8224	0.7061	0.6777	0.6404	0.6082
Skewness	1.0973	1.0949	1.0885	1.0893	1.0867
Kurtosis	1.1677	1.1658	1.1554	1.1558	1.1514
μ'_s	S-VI	S-VII	S-VIII	S-IX	S-X
μ'_1	2.1743	2.1487	2.1282	2.2315	2.1974
μ'_2	4.2741	4.1162	3.9881	4.5144	4.3161
μ'_3	9.2113	8.6180	8.1575	9.9463	9.2237
μ'_4	21.033	18.903	17.364	22.952	20.505
Variance	0.6081	0.3139	0.0613	0.6987	0.3811
Skewness	1.0867	1.0649	1.0491	1.0752	1.0581
Kurtosis	1.1514	1.1157	1.0917	1.1262	1.1007

2.5 Reliability Characteristics

One of the imperative roles of probability distribution in reliability engineering is to analyze and predicts the life of a component. Numerous reliability measures for the MEK distribution are discussed here. One may explain the reliability function as the probability of a component that survives till the time *x* and analytically it is written as$R(x) = 1 - F(x)$.

Reliability function of *X* is given by

$$R(x) = 1 - \frac{2}{1 + (1-(1-x^{\alpha})^{\beta})^{-\gamma}}. \tag{9}$$

In reliability theory, a significant contribution of a function, most of the time considers as a failure rate function or hazard rate function, and sometimes it is called the force of mortality. Time depended this function is used to measure the failure rate of a component in a particular period *x* and mathematically it is written as $h(x) = f(x)/R(x)$.

Hazard rate function of *X* is given by

$$h(x) = \frac{2\alpha\beta\gamma x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\left(1-(1-x^{\alpha})^{\beta}\right)^{-\gamma-1}}{\left((1+(1-(1-x^{\alpha})^{\beta})^{-\gamma})\right)(1+(1-(1-x^{\alpha})^{\beta})^{-\gamma}-2)}, x > 0. \tag{10}$$

The conditional survivor function is the probability that a component whose life says *x*, survives in an additional interval at *z*. It can be written as $R(^{z}/_{x}) = P(X > z + x/X > t) = \frac{R(X > z+x)}{P(X > x)} = \frac{R(x+z)}{R(x)}$.

Conditional survivor function of *X* is given by

$$R(^{z}/_{x}) = \frac{\left((1-(1-(x+z)^{\alpha})^{\beta}\right)^{-\gamma} - 1\big)\left(1+\left(1-(1-x^{\alpha})^{\beta}\right)^{-\gamma}\right)}{(1+(1-(1-(x+z)^{\alpha})^{\beta})^{-\gamma})(1+(1-(1-x^{\alpha})^{\beta})^{-\gamma}-1)}.$$

Most of the time, it is assumed that the mechanical components/parts of some systems follow the bathtub-shaped failure rate phenomena. For this, several well-established and useful reliability measures are available in the literature to discuss the significance of EM distribution. the cumulative hazard rate function is expressed by $h_c(x) = -\log\big(R(x)\big)$.

Cumulative hazard rate function of *X* is given by

$$h_c(x) = -\log\left(1 - \frac{2}{1 + (1 - (1 - x^{\alpha})^{\beta})^{-\gamma}}\right)$$

The reverse hazard rate function is expressed by $h_r(x) = f(x)/R(x)$.

Reverse hazard rate function of *X* is given by

$$\frac{2\alpha\beta\gamma x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\left(1-(1-x^{\alpha})^{\beta}\right)^{-\gamma-1}\left(1+\left(1-(1-x^{\alpha})^{\beta}\right)^{-\gamma}\right)}{(1+(1-(1-x^{\alpha})^{\beta})^{-\gamma})^2((1-(1-x^{\alpha})^{\beta})^{-\gamma}-1)}.$$

Mills ratio is expressed by $M(x) = R(x)/f(x)$.

Mills ratio of *X* is given by

$$\frac{\left(1+\left(1-(1-x^{\alpha})^{\beta}\right)^{-\gamma}\right)^2\left(\left(1-(1-x^{\alpha})^{\beta}\right)^{-\gamma}-1\right)}{2\alpha\beta\gamma x^{\alpha-1}(1-x^{\alpha})^{\beta-1}(1-(1-x^{\alpha})^{\beta})^{-\gamma-1}(1+(1-(1-x^{\alpha})^{\beta})^{-\gamma})}.$$

Odd function is expressed by $O(x) = F(x)/R(x)$.

Odd function of *X* is given by

$$O(x) = \frac{2}{((1-(1-x^{\alpha})^{\beta})^{-\gamma}-1)}$$

We may develop the linear expressions for reliability characteristics, mention in section 1.2. The reliability and hazard rate functions of *X* are given by

$$R^*(x) = 1 - 2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\binom{-1}{i}\binom{-\gamma i}{j}\binom{\beta j}{k}x^{\alpha k},$$

and

$$h^*(x) = \frac{2\alpha\beta\gamma\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\binom{-2}{i}\binom{-\gamma i-\gamma-1}{j}\binom{\beta j+\beta-1}{k}x^{\alpha k+\alpha-1}}{1-2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\binom{-1}{i}\binom{-\gamma i}{j}\binom{\beta j}{k}x^{\alpha k}}$$

2.6 Limiting Behavior

Here we study the limiting behavior of distribution function, density function, reliability function, and failure rate function of the MEK distribution present in Equations (3), (4), (9), and (10) at $x \to 0$ and $x \to 1$.

Proposition-1

Limiting behavior of distribution function, density function, reliability function, and failure rate function of the MEK distribution at $x \to 0$ *is followed by*

$$F(x) \sim 0,$$
$$f(x) \sim 0,$$
$$R(x) \sim 1,$$
$$h(x) \sim 0.$$

Proposition-2

Limiting behavior of distribution function, density function, reliability function, and failure rate function of the MEK distribution at $x \to 1$ *is followed by*

$$F(x) \sim 1,$$
$$f(x) \sim 0,$$
$$R(x) \sim 0,$$
$$h(x) \sim \text{Indeterminate}.$$

The above limiting behaviors of distribution, density, reliability, and failure rate functions illustrate that there is no effect of parameters on the tail of the MEK distribution.

2.7 Moments and Its Associated Measures

Moments have a remarkable role in the discussion of distribution theory, to study the significant characteristics of a probability distribution.

Theorem 1: If $X \sim$MEK $(x;\alpha,\beta,\gamma)$, for $\alpha,\beta,\gamma>0$, then the *r*-th ordinary moment (say μ'_r) of *X* is given by

$$\mu'_r = 2\beta\gamma\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{-2}{i}\binom{-\gamma i-\gamma-1}{j}B\left(\frac{r}{\alpha}+1,\beta(j+1)\right)$$

Proof: μ'_r can be written by following Equation (6), as

$$\mu'_r = 2\beta\gamma\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{-2}{i}\binom{-\gamma i-\gamma-1}{j}\int_0^1 x^r\left(\alpha x^{\alpha-1}(1-x^{\alpha})^{\beta j+\beta-1}\right)dx,$$

by simple computation on the prior expression leads to the final form of the *r*-th ordinary moment and it is given by

$$\mu'_r = 2\beta\gamma\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{-2}{i}\binom{-\gamma i-\gamma-1}{j}B\left(\frac{r}{\alpha}+1,\beta(j+1)\right), \tag{11}$$

where $B(x;\alpha,\beta)=\int_0^x t^{\alpha}(1-t)^{\beta-1}dt$ and $\alpha,\beta,\gamma>0$ are the beta function and shape parameters, control the tail behavior of *X*, respectively.

The derived expression in Equation (11) provides a supportive and useful role in the development of numerous statistics. For instance: to deduce the mean of *X,* place *r* =1 in Equation (11) and it is given by

$$\mu'_1 = 2\beta\gamma\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{-2}{i}\binom{-\gamma i-\gamma-1}{j}B\left(\frac{1}{\alpha}+1,\beta(j+1)\right).$$

The higher-order ordinary moments of *X* approximating to 2nd, 3rd, and 4th, can be formulated by setting *r* = 2, 3, and 4 in Equation (11) respectively. Further to discuss the variability in *X,* the Fisher index F.I = $(Var(X)/E(X))$ plays a supportive role. One may perhaps further determine the well-established statistics for instance: skewness $(\beta_1=\mu_3^2/\mu_2^3)$, kurtosis $(\beta_2=\mu_4/\mu_2^2)$, and mode = $\left(\sqrt{\beta_1}(\beta_2+3)\mathrm{SD}/\left(2(5\beta_2-6\beta_1-9)\right)\right)$ of *X* by integrating Equation (11).

Moment generating function $M_X(t)$ can be presented by

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\mu'_r.$$

Moment generating function of *X* is followed by equation (9)

$$M_X(t)=2\beta\gamma\sum_{r=0}^{\infty}\frac{t^r}{r!}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{-2}{i}\binom{-\gamma i-\gamma-1}{j}B\left(\frac{r}{\alpha}+1,\beta(j+1)\right).$$

A well-established recurrence relationship between the ordinary moments (μ'_r) and central moments (μ_s) to derive the cumulants is $\mu_s=\sum_{k=0}^{s}\binom{s}{k}(-1)^k(\mu'_1)^s\mu'_{s-k}$. Hence, the first four cumulants are: $K_1=\mu'_1$, $K_2=\mu'_2-\mu_1'^2$, $K_3=\mu'_3-3\mu'_2\mu'_1+2\mu_1'^3$, and $K_4=\mu'_4-4\mu'_3\mu'_1-3\mu_2'^2+12\mu'_2\mu_1'^2-6\mu_1'^4$.

The *s*-th central moment (μ_s) of *X* is given by

$$\mu_s=\sum_{k=0}^{s}\binom{s}{k}(-1)^k\left(\frac{\left(2\beta\gamma\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j\binom{-2}{i}\binom{-\gamma i-\gamma-1}{j}}{B\left(\frac{1}{\alpha}+1,\beta(j+1)\right)}\right)^s}{2\beta\gamma\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j\binom{-2}{i}\binom{-\gamma i-\gamma-1}{j}}{B\left(\frac{s-k}{\alpha}+1,\beta(j+1)\right)}}\right).$$

2.8 Incomplete Moments

Incomplete moments are classified into lower incomplete moments and upper incomplete moments. Lower incomplete moments are defined as $M_r(v)=E_{X\le v}(x^r)=\int_0^l x^r f(x)dx.$

Lower incomplete moments of *X* is given by

$$M_r(v)=2\beta\gamma\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{-2}{i}\binom{-\gamma i-\gamma-1}{j}B_l\left(\frac{r}{\alpha}+1,\beta(j+1)\right).$$

Upper incomplete moments are defined as $M_s^*(u)=E_{X>u}(x^r)=\int_l^1 x^r f(x)dx$ or more convenient, it can be written as $M_s^*(u)=\int_0^1 x^r f(x)dx-\int_0^l x^r f(x)dx.$

Upper incomplete moments of X is given by

$$M_s^*(u)=2\beta\gamma\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{-2}{i}\binom{-\gamma i-\gamma-1}{j}\begin{pmatrix}B\left(\frac{r}{\alpha}+1,\beta(j+1)\right)-\\B_l\left(\frac{r}{\alpha}+1,\beta(j+1)\right)\end{pmatrix}.$$

Let be the residual life (RL) function $m_n(w)=E[(X-w)^n/X\le w]=\frac{1}{S(w)}\int_w^1(x-w)^n f(x)dx$ has the *n*-th moment

$m_n(w)=\frac{1}{1-F(w)}\sum_{r=0}^{n}\binom{n}{r}(-w)^{n-r}\left(\int_0^1 x^r f(x)dx-\int_0^w x^r f(x)dx\right).$

Residual life function X is given by

$$m_n(w)=\frac{2\beta\gamma\sum_{r=0}^{n}\binom{n}{r}(-w)^{n-r}}{1-F(w)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j\binom{-2}{i}}{\binom{-\gamma i-\gamma-1}{j}}\begin{pmatrix}B\left(\frac{r}{\alpha}+1,\beta(j+1)\right)-\\B_w\left(\frac{r}{\alpha}+1,\beta(j+1)\right)\end{pmatrix}.$$

The life expectancy or mean residual life (MRL) function, $m_1(w)$, *of X, follows from the above equation with n = 1.*

Let be the reverse residual life (RRL) function $R_n(w)=E[(w-X)^n/X\le w]=\frac{1}{F(w)}\int_0^1(w-x)^n f(x)dx$ has the *n*-th moment. $R_n(w)=\frac{1}{F(w)}\sum_{r=0}^{n}\binom{n}{r}(-1)^r w^{n-r}\int_0^1 x^r f(x)dx.$

Reverse residual life (RRL) function of *X* is given by

$$R_n(w)=\frac{2\beta\gamma}{F(w)}\left(\sum_{r=0}^{n}\binom{n}{r}(-1)^r w^{n-r}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j\binom{-2}{i}}{\binom{-\gamma i-\gamma-1}{j}}\right)B_w\left(\frac{r}{\alpha}+1,\beta(j+1)\right).$$

The mean waiting time or mean inactivity time of X, follows from the above Equation with n = 1.

Kayid and Izadkhah (2014) defined, strong mean inactivity time (SMIT). It can be written as $M(t)=t^2-\frac{1}{f(t)}\int_0^t x^2 f(x)dx$ for g , $t>0$.

Strong mean inactivity time of *X* is given by

$$M(t)=t^2-\frac{\left(\left(1+\left(1-(1-x^{\alpha})^{\beta}\right)^{-\gamma}\right)^2\right)\left(2\beta\gamma\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\begin{matrix}(-1)^j\binom{-2}{i}\binom{-\gamma i-\gamma-1}{j}\\ B_t\left(\frac{2}{\alpha}+1,\beta(j+1)\right)\end{matrix}\right)}{2\alpha\beta\gamma x^{\alpha-1}(1-x^{\alpha})^{\beta-1}(1-(1-x^{\alpha})^{\beta})^{-\gamma-1}}.$$

Mean past lifetime (MPL) for the conditional random variable $(x-X/X\le x)$ is given by $k(x)=E(x-X/X\le x)$. It can be written as $k(x)=x-\frac{\int_0^t xf(x)dx}{F(x)}$.

Mean past life time of X is given by

$$k(x)=x-\frac{1}{2}\left(1+\left(1-(1-x^{\alpha})^{\beta}\right)^{-\gamma}\right)\left(2\beta\gamma\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\begin{matrix}(-1)^j\binom{-2}{i}\binom{-\gamma i-\gamma-1}{j}\\ B_t\left(\frac{1}{\alpha}+1,\beta(j+1)\right)\end{matrix}\right).$$

2.9 Order Statistics

In reliability analysis and life testing of a component in quality control, order statistics (OS) and moments have noteworthy consideration. Let $X_1,X_2,\ldots,X_n$ be a random sample of size *n* follows to the MEK distribution and $X_{(1)}<X_{(2)}<\ldots<X_{(n)}$ be the corresponding order statistics. The random variables$X_{(i)}$, $X_{(1)}$, and $X_{(n)}$ be the *i*th, minimum, and maximum order statistics of X.

The PDF of $X_{(i)}$is given by

$$f_{(i)}(x)=\frac{1}{B(i,n-i+1)!}\left(F(x)\right)^{i-1}\left(1-F(x)\right)^{n-i}f(x),\qquad i=1,2,3,\ldots,n.$$

By incorporating Equations (3) and (4), the PDF of $X_{(i)}$takes the form

$$f_{(i:n)}(x)=\frac{\alpha\beta\gamma}{B(i,n-i+1)!}\left(\begin{matrix}\left(\frac{2}{1+(1-(1-x^{\alpha})^{\beta})^{-\gamma}}\right)^{i-1}\\ \left(1-\frac{2}{1+(1-(1-x^{\alpha})^{\beta})^{-\gamma}}\right)^{n-i}\\ \left(\frac{2\alpha\beta\gamma x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\left(1-(1-x^{\alpha})^{\beta}\right)^{-\gamma-1}}{(1+(1-(1-x^{\alpha})^{\beta})^{-\gamma})^2}\right)\end{matrix}\right).$$

The last equation is quite helpful in computing the *w*-th moment order statistics of the MEK distribution. Further, the minimum and maximum order statistics of X follow directly from the above equation with *i*=1 and *i*= *n*, respectively.

The *w*-th moment order statistics, $E(X_{OS}^{w})$, of *X* is

$$E(X_{OS}^{w})=\frac{2\alpha\beta\gamma}{B(i,n-i+1)!},\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{j+l}(2)^{\alpha}\binom{n-i}{j}\binom{\alpha}{k}\binom{\beta}{l}B\left(\frac{r}{\alpha}+1,\beta l+1\right).\qquad(12)$$

2.10 Entropy

When a system is quantified by disorderedness, randomness, diversity, or uncertainty, in general, it is known as entropy.

Rényi (1961) entropy of *X* is described by

$$H_{\zeta}(X)=\frac{1}{1-\zeta}log\int_0^1 f^{\zeta}(x)dx\ ,\qquad \zeta>0\ and\ \zeta\neq 1.\qquad(13)$$

First, we simplify $f(x)$ in terms of $f^{\zeta}(x)$, we get

$$f^{\zeta}(x)=(2\alpha\beta\gamma)^{\zeta}x^{\zeta(\alpha-1)}(1-x^{\alpha})^{\zeta(\beta-1)}\left(1-(1-x^{\alpha})^{\beta}\right)^{-\zeta(\gamma+1)}\left(1+\left(1-(1-x^{\alpha})^{\beta}\right)^{-\gamma}\right)^{2\zeta}.$$

$$f^{\zeta}(x) = (2\alpha\beta\gamma)^{\zeta}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{j}\binom{-2\zeta}{i}\binom{-\lambda}{j}x^{\zeta(\alpha-1)}(1-x^{\alpha})^{\beta j+\zeta(\beta-1)},$$

by shifting the above equation in Equation (13), we get

$$H_{\zeta}(X) = \frac{1}{1-\zeta}log\left((2\alpha\beta\gamma)^{\zeta}\int_{0}^{1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{j}\binom{-2\zeta}{i}\binom{-\lambda}{j}x^{\zeta(\alpha-1)}(1-x^{\alpha})^{\beta j+\zeta(\beta-1)}dx\right),$$

by solving simple mathematics on the prior equation we will be provided the reduced form of the Rényi entropy for X and it is given by

$$H_{\zeta}(X) = \frac{1}{1-\zeta}log\left((2\alpha\beta\gamma)^{\zeta}\right)\frac{1}{\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\tau_{i,j}\,B\left(\frac{\zeta(\alpha-1)+1}{\alpha},\beta j+\zeta(\beta-1)+1\right), \tag{14}$$

where $\lambda = \zeta(\gamma+1)+i\gamma, \tau_{i,j} = (-1)^{j}\binom{-2\zeta}{i}\binom{-\lambda}{j}$.

The quadratic entropy is a special case of Rényi entropy, called quadratic Rényi entropy (QRE). It has a wide range of applications in economics, signal processing, and physics. It is obtained by substituting ζ by 2 in Equation (14).

A generalization of the Boltzmann-Gibbs entropy is the η – entropy. Although in physics, it is referred to as the Tsallis entropy. Tsallis (1988) entropy / η – entropy is described by

η – entropy is described by

$$H_{\eta}(X) = \frac{1}{\eta-1}\left(1-\int_{0}^{1}f^{\eta-1}(x)dx\right), \qquad \eta > 0\ and\ \eta \neq 1.$$

η – entropy of X is given by

$$H_{\eta}(X) = \frac{1}{\eta-1}\left(\begin{array}{c}1-\left((2\alpha\beta\gamma)^{\eta-1}\right)\frac{1}{\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\tau_{i,j}\\ B\left(\frac{(\eta-1)(\alpha-1)+1}{\alpha},\beta j+(\eta-1)(\beta-1)+1\right)\end{array}\right). \tag{15}$$

Mathai and Haubold (2013) generalized the classical Shannon entropy known as $\phi -$ entropy. It is presented by

$$H_{\phi}(X) = \frac{1}{\phi-1}\left(\int_{0}^{1}f^{2-\phi}(x)dx-1\right), \qquad \phi > 0\ and\ \phi \neq 1.$$

$\phi -$ entropy of X is given by

$$H_{\phi}(X) = \frac{1}{\phi-1}\left(\left(\begin{array}{c}\left((2\alpha\beta\gamma)^{\phi}\right)\frac{1}{\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\tau_{i,j}\\ B\left(\frac{(2-\phi)(\alpha-1)+1}{\alpha},\beta j+(2-\phi)(\beta-1)+1\right)\end{array}\right)-1\right). \tag{16}$$

Another generalized version of the Shannon entropy is the $\bar{\varphi}$ - entropy. It is presented by

$$H_{\bar{\varphi}}(X) = \frac{1}{\bar{\varphi}-1}\left(1-\int_{0}^{1}f^{\bar{\varphi}}(x)dx\right), \qquad \bar{\varphi} \neq 1.$$

$\bar{\varphi}$ - entropy of X is given by

$$H_{\bar{\varphi}}(X)=\frac{1}{\bar{\varphi}-1}\left(1-\left(((2\alpha\beta\gamma)^{\bar{\varphi}})\frac{1}{\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\tau_{i,j}\,B\left(\frac{\bar{\varphi}(\alpha-1)+1}{\alpha},\beta j+\bar{\varphi}(\beta-1)+1\right)\right)\right). \quad (17)$$

Havrda and Charvat (1967) introduced $\omega-$ entropy measure. It is presented by

$$H_{\omega}(X)=\frac{1}{2^{1-\omega}-1}\left(\int_{0}^{1}f^{\omega}(x)dx-1\right), \qquad \omega>0\ and\ \omega\neq 1.$$

$\omega-$ entropy of X is given by

$$H_{\omega}(X)=\frac{1}{2^{1-\omega}-1}\left(\left(((2\alpha\beta\gamma)^{\omega})\frac{1}{\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\tau_{i,j}\,B\left(\frac{\omega(\alpha-1)+1}{\alpha},\beta j+\omega(\beta-1)+1\right)\right)-1\right). \quad (18)$$

where $\lambda=\zeta(\gamma+1)+i\gamma,\tau_{i,j}=(-1)^{j}\binom{-2\zeta}{i}\binom{-\lambda}{j}$.

3. Estimation

In this section, we utilize the method of maximum likelihood estimation which provides the maximum information about the unknown model parameters.

By Equation (4), the likelihood function, $L(\vartheta)=\prod_{i=1}^{n}f(x_i;\alpha,\beta,\gamma)$,of the MEK distribution is:

$$L\,(\vartheta)=(2\alpha\beta\gamma)^{n}\prod_{i=1}^{n}\frac{{x_i}^{\alpha-1}(1-{x_i}^{\alpha})^{\beta-1}\left(1-(1-{x_i}^{\alpha})^{\beta}\right)^{-\gamma-1}}{(1+(1-(1-{x_i}^{\alpha})^{\beta})^{-\gamma})^{2}}.$$

The log-likelihood function,$l(\vartheta)$, reduces to

$$l(\vartheta)=\begin{pmatrix} n(log2+log\alpha+log\beta+log\gamma)+(\alpha-1)\sum_{i=1}^{n}log(x_i)+(\beta-1)\sum_{i=1}^{n}log(1-{x_i}^{\alpha})- \\ (\gamma+1)\sum_{i=1}^{n}log\left(1-(1-{x_i}^{\alpha})^{\beta}\right)-2\sum_{i=1}^{n}log\left(1+\left(1-(1-{x_i}^{\alpha})^{\beta}\right)^{-\gamma}\right)\end{pmatrix}.$$

The maximum likelihood estimates (MLEs) of the MEK model parameters can be obtained by maximizing the last equation for α,β, and γ, or by solving the following nonlinear Equations,

$$\frac{\partial l}{\partial\alpha}=\begin{pmatrix}\frac{n}{\alpha}+\sum_{i=1}^{n}log(x_i)+\sum_{i=1}^{n}log(x_i)-(\beta-1)\sum_{i=1}^{n}\frac{{x_i}^{\alpha}logx_i}{1-{x_i}^{\alpha}}- \\ (\gamma-1)\sum_{i=1}^{n}\frac{\beta{x_i}^{\alpha}logx_i(1-{x_i}^{\alpha})^{\beta-1}}{1-(1-{x_i}^{\alpha})^{\beta}} \\ +2\sum_{i=1}^{n}\frac{\beta\gamma{x_i}^{\alpha}logx_i(1-{x_i}^{\alpha})^{\beta-1}\left(1-(1-{x_i}^{\alpha})^{\beta}\right)^{-\gamma-1}}{1+(1-(1-{x_i}^{\alpha})^{\beta})^{-\gamma}}\end{pmatrix},$$

$$\frac{\partial l}{\partial\beta}=\begin{pmatrix}\frac{n}{\beta}+\sum_{i=1}^{n}log(1-{x_i}^{\alpha})-(1+\gamma)\sum_{i=1}^{n}\frac{(1-{x_i}^{\alpha})^{\beta}log(1-{x_i}^{\alpha})}{1-(1-{x_i}^{\alpha})^{\beta}} \\ +2\sum_{i=1}^{n}\frac{\gamma log(1-{x_i}^{\alpha})(1-{x_i}^{\alpha})^{\beta}\left(1-(1-{x_i}^{\alpha})^{\beta}\right)^{-\gamma-1}}{1+(1-(1-{x_i}^{\alpha})^{\beta})^{-\gamma}}\end{pmatrix},$$

$$\frac{\partial l}{\partial \gamma}=\begin{pmatrix} \frac{n}{\gamma}-\sum_{i=1}^{n} log\left(1-(1-x_i{}^{\alpha})^{\beta}\right)+ \\ 2\sum_{i=1}^{n}\frac{\left(\left(1-(1-x_i{}^{\alpha})^{\beta}\right)^{-\gamma}\right)log\left(1-(1-x_i{}^{\alpha})^{\beta}\right)}{1+(1-(1-x_i{}^{\alpha})^{\beta})^{-\gamma}} \end{pmatrix}.$$

The last three non-linear Equations do not provide the analytical solution for MLEs and the optimum value of α, β, and γ. The Newton-Raphson is considered an appropriate algorithm which plays a supportive role in such kind of MLEs. For numerical solutions, the R statistical software (package name, *Adequacy-Model*) is preferred to estimate the MEK distribution parameters.

3.1 Simulation Study

In this section, to observe the performance of MLE's, the following algorithm is adopted.

Step-1: A random sample $x_1, x_2, x_3, \ldots, x_n$ of sizes n = 25, 50, and 100 are generated from Equation (5).

Step-2: Each sample is replicated 1000 times.

Step-3: The required results are obtained based on the different combinations of the parameters place in S-XI, S-XII, and S-XIII.

Step-4: Gradual decrease in S.Es and pretty close ML estimates to the true parameters for the increases of sample size help out to declare that the method of maximum likelihood estimation works quite well for MEK distribution.

4. Application

In this section, we report the flexibility and potentiality of the MEK distribution by modeling in various disciplines of applied sciences. For this, we consider four suitable lifetime data sets. The ***first dataset*** presents the 20 observations of flood including 0.265, 0.269, 0.297, 0.315, 0.3235, 0.338, 0.379, 0.379, 0.392, 0.402, 0.412, 0.416, 0.418, 0.423, 0.449, 0.484, 0.494, 0.613, 0.654, 0.74, discussed by Dumonceaux and Antle (1973). The ***second dataset*** discussed by Caramanis *et al*. (1983) and Mazumdar and Gaver (1984). They estimated the unit capacity factors by comparing two different algorithms called SC16 and P3. The observations are 0.853, 0.759, 0.866, 0.809, 0.717, 0.544, 0.492, 0.403, 0.344, 0.213, 0.116, 0.116, 0.092, 0.070, 0.059, 0.048, 0.036, 0.029, 0.021, 0.014, 0.011, 0.008, 0.006. The ***third dataset*** refers to 20 mechanical parts failure times. This data set was analyzed by Murthy *et al*. (2004) and the observations are 0.067, 0.068, 0.076, 0.081, 0.084, 0.085, 0.085, 0.086, 0.089, 0.098, 0.098, 0.114, 0.114, 0.115, 0.121, 0.125, 0.131, 0.149, 0.160, 0.485 and finally the ***forth dataset*** refers to the measurement on 48 samples of petroleum rock obtained from petroleum reservoirs. This data was discussed by Cordeiro and Brito (2012) and the observations are: 0.0903296, 0.2036540, 0.2043140, 0.2808870, 0.1976530, 0.3286410, 0.1486220, 0.1623940, 0.2627270, 0.1794550, 0.3266350, 0.2300810, 0.1833120, 0.1509440, 0.2000710, 0.1918020, 0.1541920, 0.4641250, 0.1170630, 0.1481410, 0.1448100, 0.1330830, 0.2760160, 0.4204770, 0.1224170, 0.2285950, 0.1138520, 0.2252140, 0.1769690, 0.2007440, 0.1670450, 0.2316230, 0.2910290, 0.3412730, 0.4387120, 0.2626510, 0.1896510, 0.1725670, 0.2400770, 0.3116460, 0.1635860, 0.1824530, 0.1641270, 0.1534810, 0.1618650, 0.2760160, 0.2538320, 0.2004470.

Some descriptive statistics are presented in Table 3. The MEK distribution is compared with its competing models (mention in Table-4), based on some criteria called, -Log-likelihood (-LL), Bayesian information criterion (BIC), Cramer-Von Mises (W*), Anderson-Darling (A*), and Kolmogorov Smirnov (K-S) test statistics. Tables 5-8, confirm the parameter estimates and their standard errors (in parenthesis) and the goodness-of-fit criteria, respectively. The MEK distribution is a better fit among all competitors, based on the results in Tables 5-8. Further, fitted density and distribution functions, Kaplan-Meier survival, and Probability- Probability (PP) plots are presented in Figures 3-6, respectively, provide close fits to the four datasets.

Table 2. Average MLEs and Standard Errors (in parenthesis)

	S-XI ($\alpha = 0.1, \beta = 0.5, \gamma = 0.5$) Parameter estimate (Standard Error)			S-XII ($\alpha = 0.2, \beta = 0.5, \gamma = 0.7$) Parameter estimate (Standard Error)		
n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$
25	0.1846 (0.2493)	0.4643 (0.2518)	0.4647 (0.5686)	0.2146 (0.5654)	0.6297 (0.3091)	0.8348 (1.9998)
50	0.0964 (0.1442)	0.5622 (0.2283)	0.5111 (0.7047)	0.1405 (0.2377)	0.5124 (0.1945)	0.8625 (1.2555)
100	0.0971 (0.0964)	0.5005 (0.1516)	0.4974 (0.4481)	0.1949 (0.2040)	0.5225 (0.1407)	0.6845 (0.6362)

	S-XIII ($\alpha = 1.1, \beta = 1.7, \gamma = 0.2$) Parameter estimate (Standard Error)		
n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$
25	1.4605 (0.0020)	1.4963 (0.8981)	0.1374 (0.0236)
50	1.2108 (0.0935)	1.5576 (0.1588)	0.1690 (0.0243)
100	1.1019 (0.0249)	1.6998 (0.0424)	0.2000 (0.0175)

Table 3. Descriptive Information

Data set	Minimum	Median	Mean	Maximum	Skewness	Kurtosis
Flood data	0.011	0.041	0.045	0.125	1.1672	4.324
Unit capacity data	0.006	0.116	0.288	0.866	0.718	1.974
Failure times data	0.067	0.098	0.121	0.485	3.585	15.203
Petroleum rock data	0.090	0.198	0.218	0.464	1.169	4.109

Table 4. Competitive Models

Abbr.	Model	Parameters/ variable Range	Reference
L-I	$G(x) = x^{\alpha}$	$\alpha > 0, 0 < x < 1$	Lehmann (1953)
L-II	$G(x) = 1 - (1 - x)^{\alpha}$	$\alpha > 0, 0 < x < 1$	Lehmann (1953)
TL	$G(x) = (2x - x^2)^{\alpha}$	$\alpha > 0, 0 < x < 1$	Topp and Leone (1955)
Kum	$G(x) = 1 - (1 - x^{\alpha})^{\beta}$	$\alpha, \beta > 0, 0 < x < 1$	Kumaraswamy (1980)
GPF	$G(x) = 1 - \left(\frac{g - x}{g - m}\right)^{\alpha}$	$\alpha > 0, m \le x \le g$	Saran and Pandey (2004)
EK	$G(x) = \left(1 - (1 - x^{\alpha})^{\beta}\right)^{\gamma}$	$\alpha, \beta, \gamma > 0, 0 < x < 1$	Lemonte *et al.* (2013)
WPF	$G(x) = 1 - e^{-\alpha\left(\frac{x^{\beta}}{g^{\beta} - x^{\beta}}\right)^{\gamma}}$	$\alpha, \beta, \gamma > 0, 0 < x \le g$	Tahir *et al.* (2014)
KPF	$G(x) = 1 - \left(1 - \left(\frac{x}{g}\right)^{\alpha\beta}\right)^{\gamma}$	$\alpha, \beta, \gamma > 0, 0 < x \le g$	Ibrahim (2017)
MT-II	$G(x) = e^{x^{\alpha} ln2} - 1$	$\alpha > 0, 0 < x < 1$	Muhammad (2017)

Topp-Leone (TL), Kumaraswamy (Kum), Lehmann -I and Lehmann-II (L-I, L-II), generalized power function (GPF), exponentiated Kumaraswamy (EK), Weibull power function (WPF), Kumaraswamy power function (KPF), and Mustapha Type-II (MT-II).

Table 5. Parameter Estimates and Standard Errors (parenthesis) for Flood data

Model	Parameters (Standard Errors)			Information Criterion				
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	-LL	BIC	W*	A*	K-S
MEK	**0.766 (0.474)**	**4.537 (1.501)**	**25.346 (31.929)**	**-15.903**	**-22.820**	**0.059**	**0.369**	**0.142**
EK	0.684 (0.393)	5.002 (1.496)	35.178 (46.797)	-15.514	-22.041	0.074	0.454	0.161
K	3.363 (0.603)	11.792 (5.361)	-	-12.866	-19.741	0.166	0.972	0.211
TL	2.244 (0.502)	-	-	-7.367	-11.739	0.119	0.712	0.335
L-I	1.114 (0.249)	-	-	-0.112	2.771	0.122	0.731	0.394
L-II	1.727 (0.386)	-	-	-2.512	-2.027	0.128	0.764	0.413
MT-II	0.852 (0.211)	-	-	1.247	5.489	0.131	0.782	0.388
GPF	1.579 (0.353)	-	-	-16.277	-29.559	0.131	0.728	0.224
WPF	30.814 (16.071)	11.045 (20.466)	0.319 (0.590)	-13.264	-17.540	0.146	0.868	0.198
KPF	1.386 (173.04)	1.693 (211.35)	1.865 (0.572)	-9.884	-10.780	0.303	1.717	0.263

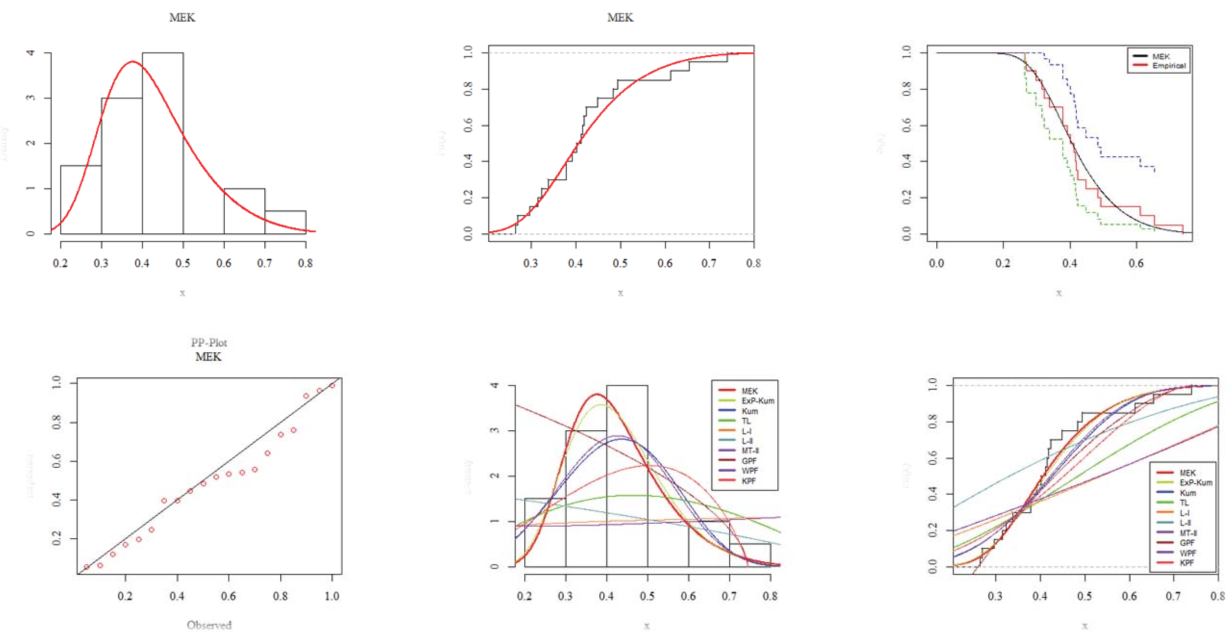


Figure 3. The Empirical Fitted PDF, CDF, Kaplan-Meier Survival, and PP-Plots of the MEK distribution for flood data

Table 6. Parameter Estimates and Standard Errors (parenthesis) for Unit Capacity Factors data

Model	Parameters (Standard Errors)			Information Criterion				
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	-LL	BIC	W*	A*	K-S
MEK	**1.411 (9.021)**	**0.957 (0.436)**	**0.435 (2.767)**	**-10.151**	**-10.897**	**0.090**	**0.579**	**0.151**
EK	0.065 (0.117)	1.185 (0.235)	9.781 (20.034)	-9.849	-10.292	0.103	0.648	0.169
K	0.504 (0.129)	1.186 (0.326)	-	-9.671	-13.071	0.108	0.682	0.179
TL	0.594 (0.124)	-	-	-8.115	-13.095	0.119	0.746	0.169
L-I	0.454 (0.095)	-	-	-9.485	-15.833	0.107	0.675	0.189
L-II	1.989 (0.415)	-	-	-4.383	-5.630	0.112	0.703	0.347
MT-II	0.371 (0.086)	-	-	-8.921	-14.708	0.117	0.732	0.199
GPF	1.185 (0.247)	-	-	-3.516	-3.897	0.114	0.683	0.411
WPF	2.285 (1.167)	1.105 (0.679)	0.551 (0.244)	-9.234	-9.061	0.095	0.616	0.155
KPF	1.389 (72.029)	0.287 (14.865)	0.737 (0.187)	-11.752	-14.099	0.128	0.767	0.211

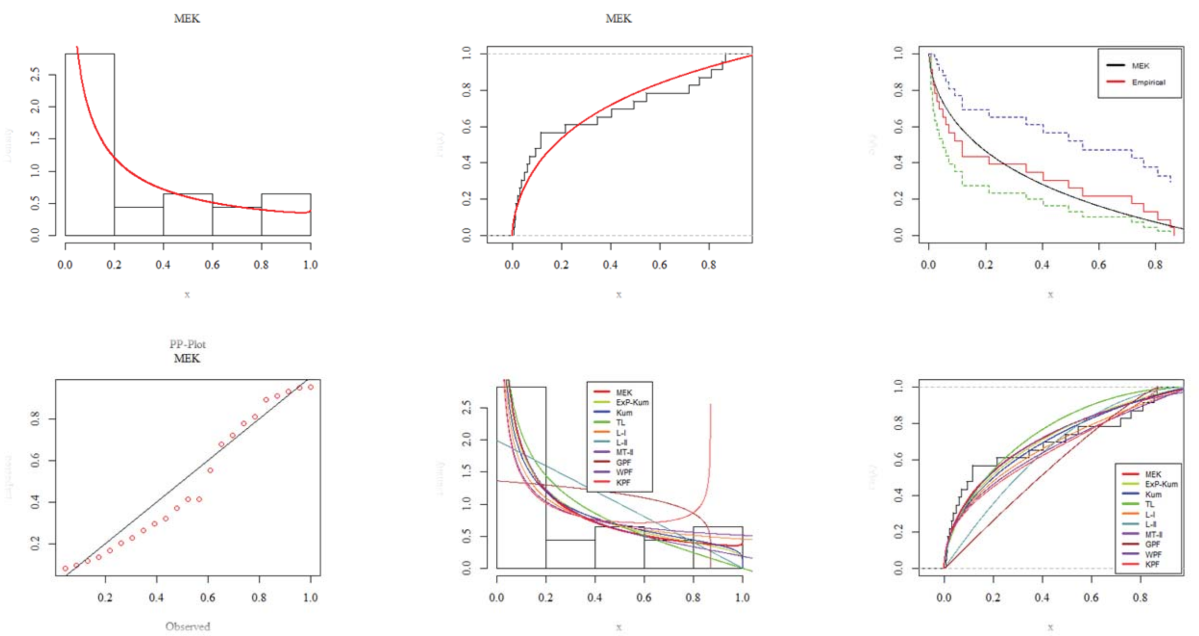


Figure 4. The Empirical Fitted PDF, CDF, Kaplan-Meier Survival, PP-Plots of the MEK distribution for unit capacity factors data

Table 7. Parameter Estimates and Standard Errors (parenthesis) for Failure Times data

Model	Parameters (Standard Errors)			Information Criterion				
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	-LL	BIC	W*	A*	K-S
MEK	**0.568 (0.199)**	**10.893 (3.789)**	**40.053 (42.362)**	**-34.540**	**-60.094**	**0.143**	**1.053**	**0.166**
EK	0.517 (0.165)	11.897 (4.019)	63.739 (63.007)	-33.551	-58.115	0.172	1.232	0.168
K	1.587 (0.244)	21.868 (10.210)	-	-25.648	-60.094	0.143	1.053	0.166
TL	0.625 (0.139)	-	-	-13.742	-24.490	0.339	2.156	0.484
L-I	0.448 (0.100)	-	-	-8.558	-14.121	0.321	2.063	0.510
L-II	7.341 (1.641)	-	-	-22.593	-42.191	0.369	2.314	0.398
MT-II	0.340 (0.084)	-	-	-7.097	-11.197	0.339	2.153	0.500
GPF	3.135 (0.701)	-	-	-26.208	-50.417	0.416	2.501	0.426
WPF	25.321 (10.981)	8.698 (30.616)	0.189 (0.664)	-26.422	-43.857	0.397	2.452	0.264
KPF	1.053 (87.439)	0.959 (79.636)	2.224 (0.682)	-19.137	-29.286	0.762	4.159	0.370

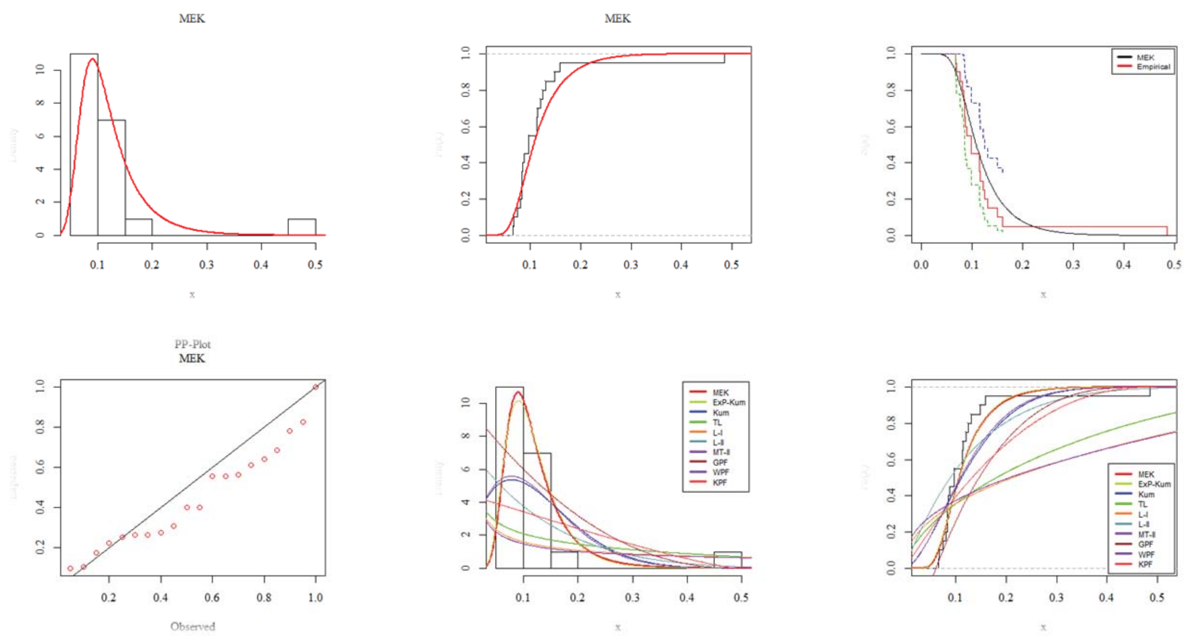


Figure 5. The Empirical Fitted PDF, CDF, Kaplan-Meier Survival, PP-Plots of the MEK distribution for failure times data

Table 8. Parameter Estimates and Standard Errors (parenthesis) for Petroleum Rock data

Model	Parameters (Standard Errors)			Information Criterion				
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	-LL	BIC	W*	A*	K-S
MEK	**0.756 (0.450)**	**8.525 (3.693)**	**21.870 (31.023)**	**-58.371**	**-105.12**	**0.038**	**0.232**	**0.089**
EK	0.727 (0.271)	9.439 (2.784)	24.699 (24.178)	-57.859	-104.10	0.058	0.346	0.108
K	2.719 (0.294)	44.667 (17.587)	-	-52.491	-97.241	0.208	1.280	0.153
TL	0.989 (0.143)	-	-	-21.166	-38.461	0.119	0.721	0.368
L-I	0.630 (0.091)	-	-	-6.011	-8.152	0.114	0.690	0.429
L-II	3.965 (0.572)	-	-	-30.221	-56.569	0.128	0.778	0.359
MT-II	0.479 (0.077)	-	-	-25.54	-1.238	1.225	0.743	0.424
GPF	1.788 (0.258)	-	-	-52.703	-101.534	0.232	1.442	0.156
WPF	42.995 (15.791)	8.774 (28.625)	0.313 (1.021)	-52.741	-93.869	0.200	1.225	0.149
KPF	1.441 (90.546)	1.405 (88.274)	2.632 (0.555)	-46.042	-80.471	0.417	2.545	0.186

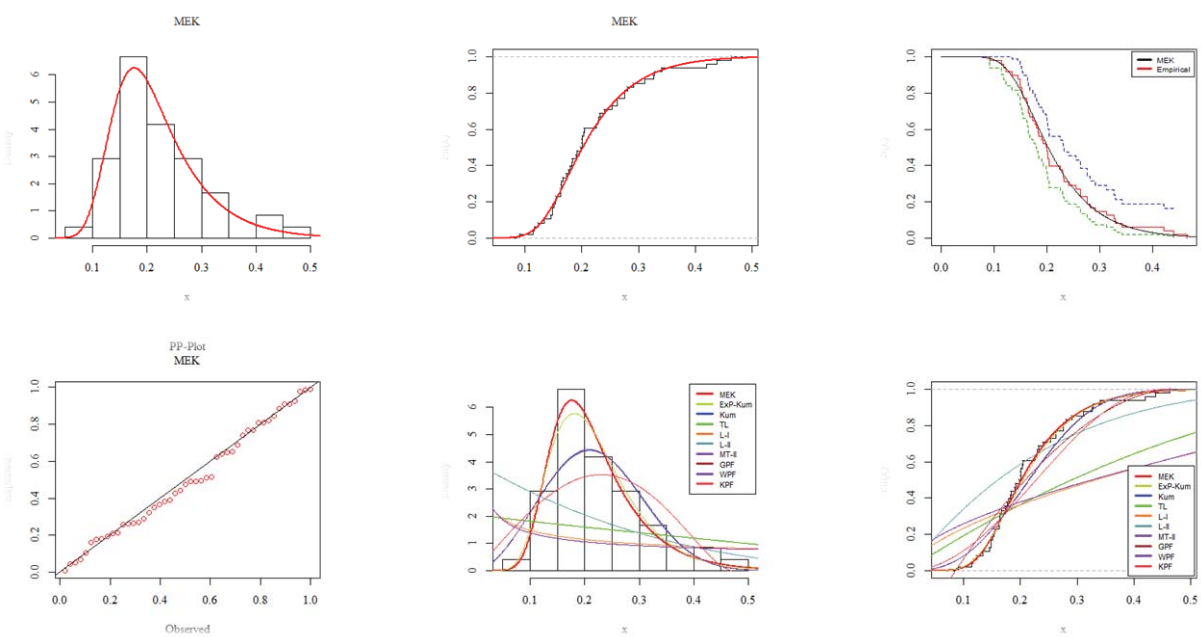


Figure 6. The Empirical Fitted PDF, CDF, Kaplan-Meier Survival, PP-Plots of the MEK distribution for petroleum rock data

5. Conclusion

In this article, we developed a flexible lifetime model that demonstrated the increasing, decreasing, and upside-down bathtub-shaped density and failure rate functions. The proposed model is referred to as the modified exponentiated Kumaraswamy (MEK) distribution. Numerous mathematical and reliability measures were derived and discussed. For estimation of the model parameters, we followed the method of maximum likelihood and executed a simulation study to observe the asymptotic behavior of MLEs. The MEK distribution explored its dominance by modeling in four-lifetime datasets and we hope it will be considered as a choice against the baseline model.

Acknowledgment

The authors declare no conflict of interest in preparing this article, authors thank the referee for constructive comments

References

Afify, A. Z., Yousof, H. M., Cordeiro, G. M., Nofal, Z. M., & Ahmed, A. N. (2016). The Kumaraswamy Marshall-Olkin Fréchet distribution with applications. *Journal of ISOSS*, *2*, 41-58. Retrieved from https://www.bu.edu.eg/portal/uploads/Commerce/Statistics,Math%20&%20Insurances/1567/publications/Ahmed%20Zakarya%20Mohammad%20Afify_KMOFr_Vol2(2)-151-168.pdf

Ali, A., Sultan, Z., & Al Mutairi, A. (2017). The Extended Weibull-Geometric Distribution: Properties and Application. *Journal of the North for Basic and Applied Sciences*, *347*(5686), 1-17. DOI:01.61821/8891400

Alizadeh, M., Tahir, M. H., Cordeiro, G. M., Mansoor, M., Zubair, M., & Hamedani, G. (2015). The Kumaraswamy marshal-Olkin family of distributions. *Journal of the Egyptian Mathematical Society*, *23*(3), 546-557. https://doi.org/10.1016/j.joems.2014.12.002

Altun, E., Khan, M. N., Alizadeh, M., Ozel, G., & Butt, N. S. (2018). Extended half-logistic distribution with theory and lifetime data application. *Pakistan Journal of Statistics and Operation Research, 14*(2), 319–331.

Bowley, A. L. (1920). Elements of Statistics, 4th Edn (New York, Charles Scribner). *Bowley4Elements of Statistics1920*.

Balakrishnan, N. (1985). Order statistics from the half logistic distribution. *Journal of Statistical Computation and Simulation*, *20*, 287–309.

Bursa, N., & Gamze, O. Z. E. L. (2017). The exponentiated Kumaraswamy-power function distribution. *Hacettepe Journal of Mathematics and Statistics*, *46*(2), 277-292. https://doi.org/10.15672/HJMS.2017.411

Bourguignon, M., Silva, R. B., Zea, L. M., & Cordeiro, G. M. (2013). The Kumaraswamy Pareto distribution. *Journal of Statistical Theory and Applications*, *12*(2), 129-144. https://doi.org/10.2991/jsta.2013.12.2.1

Cribari-Neto, F., & Santos, J. (2019). Inflated Kumaraswamy distributions. *Anais da Academia Brasileira de Ciências*, *91*(2). https://doi.org/10.1590/0001-3765201920180955

Cordeiro, G. M., & dos Santos Brito, R. (2012). The beta power distribution. *Brazilian journal of probability and statistics*, *26*(1), 88-112. https://doi.org/10.1214/10-BJPS124

Cordeiro, G. M., & de Castro, M. (2011). A new family of generalized distributions. *Journal of statistical computation and simulation*, *81*(7), 883-898. https://doi.org/10.1080/00949650903530745

Caramanis, M., Stremel, J., Fleck, W., & Daniel, S. (1983). Probabilistic production costing: an investigation of alternative algorithms. *International Journal of Electrical Power & Energy Systems*, *5*(2), 75-86. https://doi.org/10.1016/0142-0615(83)90011-X

Dumonceaux, R., & Antle, C. E. (1973). Discrimination between the log-normal and the Weibull distributions. *Technometrics*, *15*(4), 923-926. https://doi.org/10.2307/1267401

Elgarhy, M., Haq, M. A., & Perveen, I. (2019). Type II Half Logistic Exponential Distribution with Applications. *Annals of Data Science, 6*(2), 245–247. https://doi.org/10.1007/s40745-018-0175-y

Hassan, A. S., Elgarhy, M., & Shakil, M. (2017). Type II half logistic family of distributions with applications. *Pakistan Journal of Statistics and Operation Research*, *13*(2), 245–264. https://doi.org/10.18187/pjsor.v13i2.1560

Hyndman, R. J., & Fan, Y. (1996). Sample quantiles in statistical packages. *The American Statistician*, *50*(4), 361-365. https://doi.org/10.1080/00031305.1996.10473566

Hemeda, S. E., Pramanik, S., Sudhansu S., &Maiti, S. S. (2020). Kumaraswamy Inverse Lindley Distribution with

Stress-Strength Reliability. *Gazi Journal of Engineering Sciences*, *6*(3), 255-264. https://dx.doi.org/10.30855/gmbd.2020.03.09

Havrda, J., & Charvat, F. S. (1967). Quantification method of classification processes: Concept of structural-entropy, *Kybernetika*, *3*, 30-35.

Ibrahim, M. (2017). The Kumaraswamy power function distribution. *J. Stat. Appl. Probab*, *6*, 81-90. doi:10.18576/jsap/060107

Jones, M. C. (2009). Kumaraswamy's distribution: A beta-type distribution with some tractability advantages. *Statistical methodology*, *6*(1), 70-81. https://doi.org/10.1016/j.stamet.2008.04.001

Krishna, E., Jose, K. K., & Ristić, M. M. (2013). Applications of Marshall–Olkin Fréchet distribution. *Communications in Statistics-Simulation and Computation*, *42*(1), 76-89. https://doi.org/10.1016/j.stamet.2008.04.001

Kumaraswamy, P. (1980). A generalized probability density function for double-bounded random processes. *Journal of hydrology*, *46*(1-2), 79-88. https://doi.org/10.1016/0022-1694(80)90036-0

Kayid, M., & Izadkhah, S. (2014). Mean inactivity time function, associated orderings, and classes of life distributions. *IEEE Transactions on Reliability*, *63*(2), 593-602. https://doi.org/10.1109/TR.2014.2315954

Lehmann, E. L. (1953). The power of rank tests, *Ann. Math. Stat*., 23-43.

Lemonte, A. J., Barreto-Souza, W., & Cordeiro, G. M. (2013). The exponentiated Kumaraswamy distribution and its log-transform. *Brazilian Journal of Probability and Statistics*, *27*(1), 31-53. https://doi.org/10.1214/11-BJPS149

Mazumdar, M., & Gaver, D. P. (1984). On the computation of power-generating system reliability indexes. *Technometrics*, *26*(2), 173-185. https://doi.org/10.1080/00401706.1984.10487942.

Muhammad, M. (2017). A new lifetime model with a bounded support. *Asian Research Journal of Mathematics*, 1-11. https://doi.org/10.9734/ARJOM/2017/35099

Mukhtar, S., Ali, A., & Alya, A. M. (2019). Mc-Donald modified Burr-III distribution: properties and applications. *Journal of Taibah University for Science*, *13*(1), 184-192. https://doi.org/10.1080/16583655.2018.1553501

Mahmoud, M. M., Aryal, G., Afify, A. Z., & Ahmad, M. (2018). The Kumaraswamy Exponentiated Fréchet Distribution. *Pak. J. Statist., 34*(3), 177-193.

Murthy, D.N.P., Xie, M., & Jiang, R. (2004).Weibull Models, John Wiley & Sons Inc., Hoboken, NJ.

McDonald, J. B. (1984). Some generalized functions for the size distribution of income. *Econometrica, 52*, 647-663.

Moors, J. J. A. (1998). A quantile alternative for kurtosis. *Journal of the Royal Statistical Society D, 37*, 25–32.

Mathai, A. M., & Haubold, H. J. (2013). On a Generalized Entropy Measure Leading to the Pathway Model with a Preliminary Application to Solar Neutrino Data. *Entropy*, *15*, 4011-4025.

Nawaz, T., Hussain, S., Ahmad, T., Naz, F., & Abid, M. (2018). Kumaraswamy generalized Kappa distribution with application to stream flow data. *Journal of King Saud University-Science*, *32*(1), 1-11. https://doi.org/10.1016/j.jksus.2018.04.005

Rényi, A. (1961). On measures of entropy and information, In: Proceedings of the 4th Fourth Berkeley Symposium on Mathematical Statistics and Probability, *University of California Press, Berkeley*, 547- 561.

Saran, J., & Pandey, A. (2004). Estimation of parameters of a power function distribution and its characterization by k-th record values. *Statistica*, *64*(3), 523-536. retrieved from file:///C:/Users/academic99/Desktop/56-Article%20Text-215-1-10-20071015.pdf

Silva, R., Silva, F. G., Ramos, M., Cordeiro, G., Marinho, P., & Andrade, T. (2019). The exponentiated Kumaraswamy-G class: general properties and application. *Revista Colombiana de Estadística*, 1-33.

Tsallis, C. (1988). Possible generalization of Boltzmann-Gibbs statistics. *Journal of Statistical Physics*, *52*(1), 479-487.

Tahir, M. H., Alizadeh, M., Mansoor, M., Cordeiro, G. M., & Zubair, M. (2014). The Weibull-power function distribution with applications. *Hacettepe Journal of Mathematics and Statistics*, *45*(1), 245-265. https://doi.org/10.15672/HJMS.2014428212

Topp, C. W., & Leone, F. C. (1955). A family of J-shaped frequency functions. *Journal of the American Statistical*

Association, *50*(269), 209-219. https://doi.org/10.1080/02664760802230583

Copyrights

Copyright for this article is retained by the author(s), with first publication rights granted to the journal.

This is an open-access article distributed under the terms and conditions of the Creative Commons Attribution license (http://creativecommons.org/licenses/by/4.0/).

International Journal of Statistics and Probability; Vol. 10, No. 3; May 2021
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

R-squared of a Latent Interaction in Structural Equation Model: A Tutorial of Using R

Lu Qin[1], Jihong Zhang[2], Xinya Liang[3], and Qianqian Pan[4]

[1] Department of Psychology, Howard University

[2] Department of Educational Measurement and Statistics, University of Iowa

[3] Department of Educational Statistics & Research Method, University of Arkansas

[4] Faculty of Education, The University of Hong Kong

Correspondence: Lu Qin, Howard University, Minor Hall 111, 2400 6th St NW, Washington DC, 20059

Received: March 4, 2021 Accepted: April 9, 2021 Online Published: April 13, 2021

doi:10.5539/ijsp.v10n3p69 URL: https://doi.org/10.5539/ijsp.v10n3p69

Abstract

M*plus* (Muthén & Muthén, 1998 - 2017) is one popular statistical software to estimate the latent interaction effects using the *latent moderated structural equation* approach (LMS). However, the variance explained by a latent interaction that supports the interpretation of estimation results is not currently available from the M*plus* output. To relieve human computations and to facilitate interpretations of latent interaction effects in social science research, we developed two functions (LIR & LOIR) in the R package IRmplus to calculate the R-squared of a latent interaction above and beyond the first-order simple main effects in Structural Equation Modeling. This tutorial provides a step-by-step guide for applied researchers to estimating a latent interaction effect in M*plus*, and to obtaining the R-squared of a latent interaction effect using the LIR & LOIR functions. Example data and syntax are available online.

Keywords: R-squared, latent interactions, R package, IRmplus, M*plus*

M*plus* (Muthén & Muthén, 1998 - 2017) is one popular statistical software for estimating various latent variable models (Hallquist & Wiley, 2018). It has a built-in function to estimate latent interaction effects using the *latent moderated structural equation* approach (LMS; Klein & Moosbrugger, 2000; Muthen & Muthen, 1998-2017). Comparing to the product indicator approach that models the latent interactions using the products of observed indicators of exogenous latent factors (Kenny & Judd, 1984), the LMS approach estimates the latent interactions by approximating a mixture of conditional distributions of observed indicators (Kelava et al., 2011). When latent factors and observed indicators are multivariate normally distributed, the LMS approach provides unbiased estimates of latent interaction effects (Kelava & Nagengast, 2012; Cham, West, Ma, & Aiken, 2012). However, this approach is limited in that the M*plus* output does not provide model fit measures, R-squared estimation, or standardized parameter estimates. Obtaining these quantities requires to run additional analyses or use hand computations. For example, the model fit comparison by the log-likelihood ratio test (LRT; Neyman & Pearson, 1933) can be conducted using the function compareModels in the R package MplusAutomation (Hallquist & Wiley, 2018). To obtain standardized parameter estimates, one may first standardize all variables in the dataset, and then perform an analysis in M*plus* based on the standardized variables (Maslowsky, Jager, & Hemken, 2015). Nonetheless, for the R-squared of latent interactions, manual computations are necessary, although equations of the R-squared estimation have been presented in Maslowsky and Hemken (2015). Using and reporting the R-squared of latent interaction effects remains a challenge for applied researchers due to the computational complexity.

To relieve human computations and to facilitate interpretations of latent interaction effects in practice, we developed two functions (LIR and LOIR) in an R package IRmplus to calculate the variance explained by the latent interaction above and beyond the first-order simple main effects in latent variable modeling. R is a leading programming software that supports data analysis and statistical modeling (R Core Team, 2017), which has been widely used in social science studies.

In this paper, we briefly introduce the computation of the R-squared of a latent interaction and the IRmplus package, followed by two examples using the LIR and LOIR functions in the IRmplus package. The strengths and limitations are discussed at the end.

A Brief Overview of the R-squared of Latent Interaction

Interactions between two latent variables are often estimated in structural equation models (SEM). SEM allows for testing a variety of hypothesized models to explain the relationships among a set of latent factors and observed variables (e.g., Bollen, 1989; Ullman & Bentler, 2003). SEM is composed of a measurement model and a structural model (Schumacker & Lomax, 2004). The measurement model examines the associations between latent factors and observed indicators. Let i be an i^{th} individual, p be a number of observed indicators, and m be a number of latent factors. The measurement model is expressed in Equation (1) as:

$$\mathbf{y}_\mathrm{i} = \boldsymbol{\nu} + \boldsymbol{\Lambda}\boldsymbol{\omega}_i + \boldsymbol{\varepsilon}_\mathrm{i} \quad (1)$$

where $\mathbf{y}_\mathrm{i}$ is a $p \times 1$ vector of observed indicators, $\boldsymbol{\nu}$ is a $p \times 1$vector of intercepts, $\boldsymbol{\Lambda}$ is a $p \times m$ matrix of factor loadings, $\boldsymbol{\omega}_\mathrm{i}$ is a $m \times 1$ vector of latent factors, and $\boldsymbol{\varepsilon}_\mathrm{i}$ is a $p \times 1$ vector of measurement errors that assumes a multivariate normal distribution with a mean vector of $\mathbf{0}$ and a diagonal matrix of $\boldsymbol{\Psi}$.

The structural model explains the relationships among latent factors or among latent factors and observed covariates. The two-way latent interactions can be estimated in the structural model under two scenarios: (1) between latent factors, and (2) between a latent factor and an observed covariate, detailed in the following scenarios.

Scenario 1: Structural Model with a Two-way Interaction between Latent Factors

Consider a model with one endogenous variable η_i and two exogenous variables $({\xi_1}_i, {\xi_2}_i)$, denoted as $\boldsymbol{\omega}_\mathrm{i} = (\eta_i, {\xi_1}_i, {\xi_2}_i)$. A two-way latent interaction term between two exogenous variables, $\xi_1\xi_2$, is also included, as depicted in Figure 1.

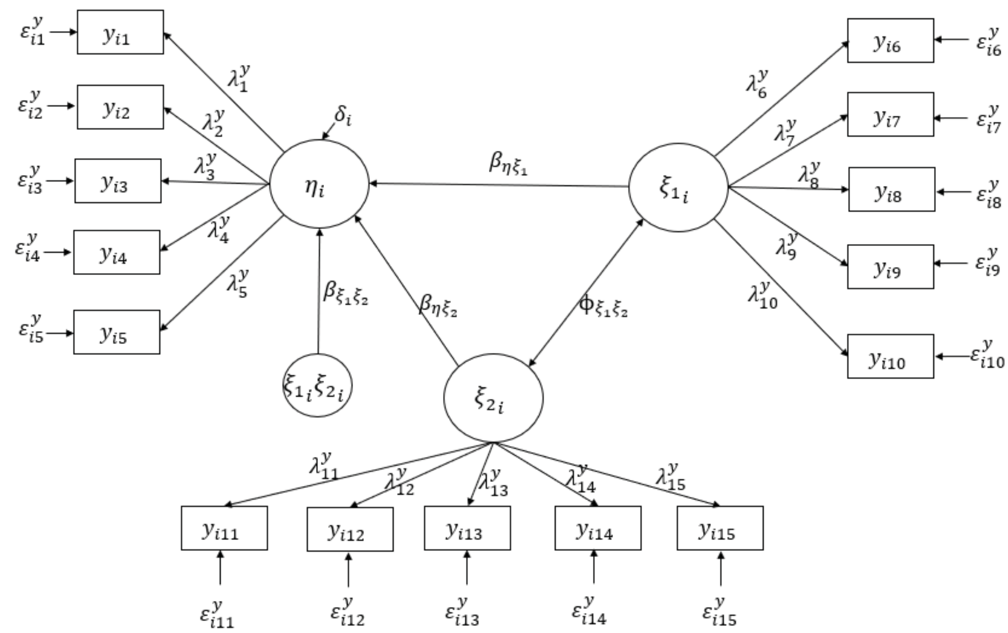


Figure 1. Diagram of a Structural Equation Model: Two-way Interaction between Latent Factors

Note. Structural equation model with one latent interaction effect between two latent factors. Each of η, ξ_1, and ξ_2 has five observed indicators (e.g., $y_{i1}, \dots, y_{i5}; y_{i6}, \dots, y_{i10}; y_{i11}, \dots, y_{i15}$) as a measurement model. Note that the latent interaction term ${\xi_1}_i{\xi_2}_i$ is produced with LMS approach here.

The structural part of this model is expressed in Equation (2) as:

$$\eta_i = \alpha + \beta_{\eta\xi_1}{\xi_1}_i + \beta_{\eta\xi_2}{\xi_2}_i + \beta_{\xi_1\xi_2}{\xi_1}_i{\xi_2}_i + \delta_i, \quad (1)$$

where α is the intercept of the endogenous latent factor η_i,

$\beta_{\eta\xi_1}$ is the regression coefficient assessing the effect of the exogenous latent factor ${\xi_1}_i$ on the endogenous latent factor η_i,

$\beta_{\eta\xi_2}$ is the regression coefficient assessing the effect of the latent moderator ${\xi_2}_i$ on the endogenous latent factor η_i,

$\beta_{\xi_1\xi_2}$ is the regression coefficient measuring the latent interaction effect ${\xi_1}_i{\xi_2}_i$ (typically estimated by the LMS approach) on the endogenous latent factor η_i,

δ_i is the factor disturbance of η_i, assumed a normal distribution with a mean of 0 and a variance of σ_δ^2.

Klein and Moosbrugger (2000) proposed the LMS approach to directly claim latent interaction in the structural equation in SEM. More detailed technical introduction of the LMS approach can be found in the Klein and Moosbrugger (2000), Klein and Muthén (2007), Kelava et al. (2011), and Preacher, Zhang, and Zyphur (2016). Given that the latent interaction is assumed to have no covariance with the first-order simple main effects (Klein & Moorusberg, 2011), the R-squared of a latent interaction can be calculated in the following two steps (Maslowsky, Jager, & Hemken, 2015).

The first step is to compute the R-squared of the simple main effects $R^2_{\eta 0}$ without the latent interaction as follows:

$$R^2_{\eta 0} = \frac{\beta^2_{\eta\xi_1}\sigma^2_{\xi_1}+\beta^2_{\eta\xi_2}\sigma^2_{\xi_2}+2\beta_{\eta\xi_1}\beta_{\eta\xi_2}}{\beta^2_{\eta\xi_1}\sigma^2_{\xi_1}+\beta^2_{\eta\xi_2}\sigma^2_{\xi_2}+2\beta_{\eta\xi_1}\beta_{\eta\xi_2}+\sigma^2_{\delta}}, \quad (3)$$

where $\sigma^2_{\xi_1}$ is the variance of the exogenous latent factor ${\xi_1}_i$,

$\sigma^2_{\xi_2}$ is the variance of the latent moderator ${\xi_2}_i$,

σ^2_{δ} is the disturbance variance of the endogenous latent factor η_i.

The second step is to compute the R-squared of $R^2_{\eta 1}$ including both simple main effects and a latent interaction effect as follows:

$$R^2_{\eta 1} = \frac{\beta^2_{\eta\xi_1}\sigma^2_{\xi_1}+\beta^2_{\eta\xi_2}\sigma^2_{\xi_2}+2\beta_{\eta\xi_1}\beta_{\eta\xi_2}+\beta^2_{\xi_1\xi_2}(\sigma^2_{\xi_1}\sigma^2_{\xi_2}+\left(\sigma^2_{\xi_1\xi_2}\right)^2)}{\beta^2_{\eta\xi_1}\sigma^2_{\xi_1}+\beta^2_{\eta\xi_2}\sigma^2_{\xi_2}+2\beta_{\eta\xi_1}\beta_{\eta\xi_2}+\beta^2_{\xi_1\xi_2}(\sigma^2_{\xi_1}\sigma^2_{\xi_2}+\left(\sigma^2_{\xi_1\xi_2}\right)^2)+\sigma^2_{\delta}} \quad (4)$$

where $\sigma^2_{\xi_1\xi_2}$ is the covariance between the exogenous latent factor ${\xi_1}_i$ and the latent moderator ${\xi_2}_i$.

Lastly, the R-squared of the latent interaction is computed as: $R^2_{\eta 1} - R^2_{\eta 0}$, which indicates the additional proportion of variances explained by a two-way interaction above and beyond the simple main effects.

Scenario 2: Structural Model with a Two-way Interaction between a Latent Factor and an Observed Covariate

Consider a model with an endogenous latent factor η, an exogenous latent factor ξ, an observed covariate Z, and a latent interaction term ξZ produced by the LMS approach, as depicted in Figure 2.

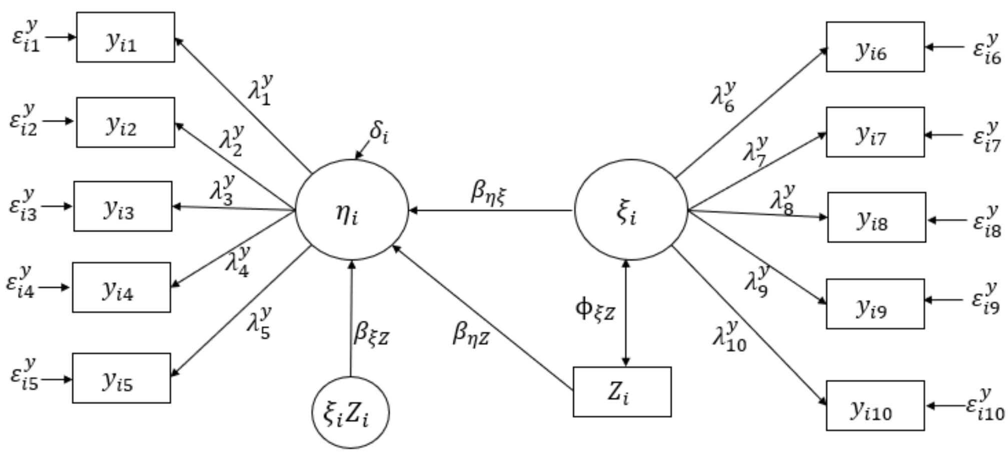


Figure 2. Diagram of a Structural Equation Model: Two-way Interaction between a Latent Factor and an Observed Indicator

Note. Structural equation model with one latent interaction effect between an exogenous latent factor and an observed covariate. Each of η and ξ has five observed indicators (e.g., $y_{i1}, \dots, y_{i5}; y_{i6}, \dots, y_{i10}$) as a measurement model. Note that the latent interaction term $\xi_i Z_i$ is produced with LMS approach here.

In this case, the latent factor vector is: $\boldsymbol{\omega}_{\mathrm{i}} = (\eta_i, \xi_i)$. The structural model including the two-way interaction between a latent factor and an observed covariate can be expressed as:

$$\eta_i = \alpha + \beta_{\eta\xi} Z_i + \beta_{\eta Z} Z_i + \beta_{\xi Z}\xi_i Z_i + \delta_i, \quad (5)$$

where $\beta_{\eta\xi}$ is the regression coefficient assessing the effect of the exogenous latent factor ξ_i on the endogenous latent factor η_i,

$\beta_{\eta Z}$ is the regression coefficient assessing the effect of the observed covariate Z_i on the endogenous latent factor η_i,

$\beta_{\xi Z}$ is the regression coefficient measuring the latent interaction effect $\xi_i Z_i$ on the endogenous latent factor η_i.

Similar to the computation between latent factors in Equations (3) and (4), the R-squared estimation of a latent interaction between a latent factor and a covariate is also computed as $R^2_{\eta 1} - R^2_{\eta 0}$, with ξ replaced by Z when computing $R^2_{\eta 1}$ and $R^2_{\eta 0}$, respectively:

$$R^2_{\eta 0} = \frac{\beta^2_{\eta\xi}\sigma^2_{\xi} + \beta^2_{\eta Z}\sigma^2_Z + 2\beta_{\eta\xi}\beta_{\eta Z}}{\beta^2_{\eta\xi}\sigma^2_{\xi} + \beta^2_{\eta Z}\sigma^2_Z + 2\beta_{\eta\xi}\beta_{\eta Z} + \sigma^2_{\delta}}, \quad (2)$$

$$R^2_{\eta 1} = \frac{\beta^2_{\eta\xi}\sigma^2_{\xi} + \beta^2_{\eta Z}\sigma^2_Z + 2\beta_{\eta\xi}\beta_{\eta Z} + \beta^2_{\xi Z}(\sigma^2_{\xi}\sigma^2_Z + (\sigma^2_{\xi Z})^2)}{\beta^2_{\eta\xi}\sigma^2_{\xi} + \beta^2_{\eta Z}\sigma^2_Z + 2\beta_{\eta\xi}\beta_{\eta Z} + \beta^2_{\xi Z}(\sigma^2_{\xi}\sigma^2_Z + (\sigma^2_{\xi Z})^2) + \sigma^2_{\delta}}, \quad (7)$$

where σ^2_Z is the variance of the observed covariate Z_i,

$\sigma^2_{\xi Z}$ is the covariance between the exogenous latent factor ξ_i and the observed indicator Z_i.

The IRmplus Package

Two-way latent interactions are common in social science research and the modeling of latent interactions has brought increasing attention. The LIR and LOIR functions in the IRmplus package were developed to compute the R-squared of a single two-way latent interaction in the SEM given the M*plus* output, following Equations (1) through (7). The LMS approach, implemented using the XWITH command in M*plus*, is one popular method to estimate the two-way interactions, whereas the M*plus* output lacks the effect size estimates unless fitting to a dataset with all variables standardized. To employ IRmplus, Two M*plus* outputs are necessary: one for the model without the latent interaction, and one for the model including the latent interaction. IRmplus reads needed parameter estimates from the M*plus* outputs and uses them as the input to compute latent interaction effects. Table 1 presents two functions, LIR and LOIR, included in the IRmplus package, that compute the R-squared of a latent interaction for the two scenarios discussed above.

Table 1. Functions included in the IRmplus package

Function	Approach	Details
LIR	LMS	Compute R^2 of an interaction effect between two latent factors in SEM model in M*plus*
LOIR	LMS	Compute R^2 of an interaction effect between a latent factor and an observed variable in SEM model in M*plus*

The LIR and LOIR functions were developed to compute the R-squared of a latent interaction one at a time. If multiple two-way interactions exist in the structural model, the LIR and/or LOIR functions can be executed multiple times to obtain the unique proportion of variances explained by the individual latent interaction.

The LIR and LOIR functions include six main arguments. The "M0" reads the M*plus* output containing only simple main effects. The "M1" reads the M*plus* output containing both simple main effects and latent interaction effects. The "endogenous" is the endogenous latent factor shown on the M*plus* output. The "exogenous" is the exogenous latent factor shown on the M*plus* output. The "moderator" is the moderator variable shown on the M*plus* output, which can be a latent factor or an observed covariate in the structural model. The "interaction" is the interaction term produced by the XWITH function and shown on the M*plus* output. The two examples below present the M*plus* and R scripts for computing the latent interaction R^2 using the IRmplus package.

Two-way Interaction between Latent Factors

To demonstrate the utility of functions in the IRmplus package, we simulated two data sets to provide step-by-step guide to computing the two-way interaction between latent factors in a SEM model[1]. The first dataset is read as "Example1.dat". The data generation model contains three latent factors and 15 observed indicators $(y_1, \ldots, y_{15})$, where

[1] Examples (data and M*plus* code) are available at https://github.com/luluqinqin/IRmplus/tree/master/Examples.

y_1 to y_5 measure the exogenous latent factor F1, y_6 to y_{10} measure the latent moderator F2, and y_{11} to y_{15} measure the endogenous latent factor F3. In the structural model, F3 is regressed on F1, F2, and the interaction term between F1 and F2. We use F in this section to label latent variables because this is a common label M*plus* reads for the latent variables. The F1, F2, and F3 here correspond to the ξ_1, ξ_2, and η in Equations (2) through (4).

Because IRmplus is published on GitHub, the devtools package is required before installing and loading the IRmplus package from GitHub[2]. The IRmplus package is built upon the MplusAutomation, stringr, tidyverse, and stringi packages and it only needs to be installed once. However, loading the packages (library()) is needed every time when the R program starts.

```
install.packages("devtools")
library(devtools)
install_github("luluqinqin/IRmplus")
library(IRmplus)
```

Next, we need to prepare the M*plus* syntax following a two-step procedure. First, a three-factor SEM model without interactions is fitted, where the endogenous latent factor F3 is regressed on the exogenous latent factors F1 and F2. The syntax is presented below and needs to be saved as an external M*plus* input file (e.g., “SEM_NoINT.inp”).

```
TITLE: 3 Factor SEM-Without Interaction;
DATA: FILE = "example1.dat";
VARIABLE:
NAMES ARE ID y1-y15;
USEVARIABLES ARE y1-y15;
ANALYSIS:
TYPE = RANDOM;
ALGORITHM = INTEGRATION;
MODEL:
F1 BY y1 y2 y3 y4 y5;
F2 BY y6 y7 y8 y9 y10;
F3 BY y11 y12 y13 y14 y15;
F3 ON F1 F2;
OUTPUT: SAMPSTAT;
```

Second, we prepare the syntax for the three-factor SEM model with a two-way latent interaction to estimate the interaction effect between exogenous latent factors F1 and F2. The syntax is shown below and saved as another external M*plus* input file (e.g., “SEM_INT.inp”).

```
TITLE: 3 Factor SEM-With Interaction;
DATA: FILE = "example1.dat";
VARIABLE:
NAMES ARE ID y1-y15;
USEVARIABLES ARE y1-y15;
ANALYSIS:
TYPE = RANDOM;
ALGORITHM = INTEGRATION;
MODEL:
F1 BY y1 y2 y3 y4 y5;
```

2 Please make sure the toolchain bundle “Rtools” (https://cran.r-project.org/bin/windows/Rtools/) is installed in the R before installing the devtools package.

```
F2 BY y6 y7 y8 y9 y10;
F3 BY y11 y12 y13 y14 y15;
Inter | F1 XWITH F2;
F3 ON F1 F2 Inter;
OUTPUT: SAMPSTAT;
```

After running the two M*plus* input files, we obtain two M*plus* output files (e.g., "sem_noint.out", "sem_int.out"), which will serve as the inputs for running the LIR function. Because the interaction is between two latent factors, the LIR function from the IRmplus is used to calculate the R-squared of the latent interaction between F1 and F2 following the command below.

```
LIR (M0 = "sem_noint.out", M1 = "sem_int.out", endogenous = "F3", exogenous = "F1", moderator = "F2", interaction = "INTER")
> 0.127
```

In the LIR function arguments, the exogenous latent factor is "F1", the moderator is "F2", the endogenous latent factor is "F3", and the interaction term "INTER" is produced by XWITH function in M*plus* syntax. It is to note that the R script is case-sensitive so that the IRmplus arguments (e.g., "INTER") need to be the same as that shown on the M*plus* output. In this example, the LIR function returns the R-squared of a latent interaction as 0.127, which indicates that the two-way latent interaction explains around 13% additional variances above and beyond the simple main effects of exogenous latent factors.

Two-way Interaction between a Latent Variable and an Observed Covariate

To use the LOIR function in the IRmplus package for the computation of the R-squared of the latent interaction between a latent factor and an observed covariate using the second simulated dataset "Example2.dat". The data generation model contains 15 observed indicators measuring three latent factors (the same measurement model as that in example 1) and one binary covariate (Gender). In the structural model, F3 is regressed on F1, F2, Gender, and the interaction term between F1 and Gender. The F1, F3, and Gender here correspond to the ξ, η, and Z in Equations (5) through (7).

We only present the MODEL command in the M*plus* syntax below, as other commands are similar to those in the first example. The same two-step procedure is followed. First, the model with only the main effects is fitted and the syntax is saved (e.g., "SEM2_NoINT.inp).

```
MODEL
F1 BY y1 y2 y3 y4 y5;
F2 BY y6 y7 y8 y9 y10;
F3 BY y11 y12 y13 y14 y15;
F3 ON F1 F2 gender;
```

Second, the model with an interaction between the latent variable F1 and the observed variable gender is fitted and saved (e.g., "SEM2_INT.inp").

```
MODEL
F1 BY y1 y2 y3 y4 y5;
F2 BY y6 y7 y8 y9 y10;
F3 BY y11 y12 y13 y14 y15;
inter2 | F1 XWITH gender;
F3 ON F1 F2 gender inter2;
F1 WITH gender;
```

The WITH function in the M*plus* syntax above is used to request the covariance estimate between an exogenous latent factor and an observed covariate. Although "F1 WITH gender" and "gender WITH F1" both provide the same covariance estimate, the LOIR function only supports the syntax starting with the exogenous latent factor, which is "F1 WITH gender" in this example.

In the LOIR function arguments, the exogenous latent factor is "F1", the moderator is "GENDER", the endogenous latent factor is "F3", and the interaction term is "INTER2". To match the variable names printed on the M*plus* output, the moderator and interaction term are both capitalized in the LOIR function. The R-squared estimation is computed as

0.102, indicating 10% of the variances are attributable to the latent interaction term.

```
LOIR (M0 = "sem2_noint.out", M1 = "sem2_int.out", endogenous = "F3", exogenous = "F1", moderator = "GENDER", interaction = "INTER2")
> 0.102
```

Discussion

The attractive features of the LIR and LOIR functions in the IRmplus package include that it minimizes errors due to human misjudges (e.g., output misinterpretation/hand calculation), and promotes the applications and interpretations of latent interactions in latent variable modeling. However, the IRmplus package is limited in some ways that would serve as our future research directions. It requires multiple executions of the LIR and/or LOIR functions when multiple two-way interactions are presented in complex modeling. Future developments of the IRmplus package include exploring the calculation of the effect size for the 3-way latent interaction, estimation of interaction effects using other estimation approaches (e.g., product indicator approach), and automatic computation of multiple latent interaction terms.

Conclusions

The IRmplus package connects two popular statistical programs, M*plus* and R, to provide an effective computation of latent interactions between two latent variables, or between a latent variable and an observed indicator. The current version of IRmplus (v1.0) supports outputs from M*plus* version 8. We will continue incorporating more advanced latent variable models and developing new functions to support newly methodological developments. We hope that the package will be a practical tool to assist researchers to better understand the impact of latent interactions. The authors are grateful for any feedback and suggestions.

References

Bollen, K. A. (1989). *Structural equations with latent variables*. New York: Wiley. https://doi.org/10.1002/9781118619179

Cham, H., West, S. G., Ma, Y., & Aiken, L. S. (2012). Estimating latent variable interactions with nonnormal observed data: A comparison of four approaches. *Multivariate Behavioral Research*, *47*(6), 840-876. https://doi.org/10.1080/00273171.2012.732901

Hallquist, M. N., & Wiley, J. F. (2018). MplusAutomation: An R package for facilitating large-scale latent variable analyses in Mplus. *Structural Equation Modeling: A Multidisciplinary Journal*, *25*(4), 621-638. https://doi.org/10.1080/10705511.2017.1402334

Kelava, A., & Nagengast, B. (2012). A Bayesian model for the estimation of latent interaction and quadratic effects when latent variables are non-normally distributed. *Multivariate Behavioral Research*, *47*(5), 717-742. https://doi.org/10.1080/00273171.2012.715560

Kelava, A., Werner, C. S., Schermelleh-Engel, K., Moosbrugger, H., Zapf, D., Ma, Y., Cham, H., Aiken, L. S., & West, S. G. (2011). Advanced nonlinear latent variable modeling: Distribution analytic LMS and QML estimators of interaction and quadratic effects. *Structural Equation Modeling: A Multidisciplinary Journal*, *18*(3), 465-491. https://doi.org/10.1080/10705511.2011.582408

Kenny, D. A., & Judd, C. M. (1984). Estimating the nonlinear and interactive effects of latent variables. *Psychological bulletin*, *96*(1), 201-210. https://doi.org/10.1037/0033-2909.96.1.201

Klein, A., & Moosbrugger, H. (2000). Maximum likelihood estimation of latent interaction effects with the LMS method. *Psychometrika*, *65*(4), 457-474. https://doi.org/10.1007/BF02296338

Klein, A. G., & Muthén, B. O. (2007). Quasi-maximum likelihood estimation of structural equation models with multiple interaction and quadratic effects. *Multivariate Behavioral Research*, *42*(4), 647-673. https://doi.org/10.1080/00273170701710205

Maslowsky, J., Jager, J., & Hemken, D. (2015). Estimating and interpreting latent variable interactions: A tutorial for applying the latent moderated structural equations method. *International Journal of Behavioral Development*, *39*(1), 87-96. https://doi.org/10.1177/0165025414552301

Muthén, L. K., & Muthén, B. O. (2017). 1998–2017. Mplus user's guide. Muthén & Muthén: Los Angeles, CA.

Neyman, J., & Pearson, E. S. (1933). On the problem of the most efficient tests of statistical hypotheses. *Philosophical Transactions of the Royal Society of London A, 231*(694-706), 289-337. https://doi.org/10.1098/rsta.1933.0009

Preacher, K., Zhang, Z., & Zyphur, M. J. (2016). Multilevel structural equation models for assessing moderation within and across levels of analysis. *Psychological Methods, 21*(2), 189-205. https://doi.org/10.1037/met0000052

R Core Team. (2017). *R: A language and environment for statistical computing*. R Foundation for Statistical Computing. Vienna, Austria.

Schumacker, R. E., & Lomax, R. G. (2004). *A beginner's guide to structural equation modeling*. Lawrence Erlbaum Associates, New Jersey, London. https://doi.org/10.4324/9781410610904

Ullman, J. B., & Bentler, P. M. (2003). Structural equation modeling. *Handbook of psychology*, 607-634. https://doi.org/10.1002/0471264385.wei0224

Copyrights

Copyright for this article is retained by the author(s), with first publication rights granted to the journal.

This is an open-access article distributed under the terms and conditions of the Creative Commons Attribution license (http://creativecommons.org/licenses/by/4.0/).

International Journal of Statistics and Probability; Vol. 10, No. 3; May 2021
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

Generalized Mean-Field Fractional BSDEs With Non-Lipschitz Coefficients

Qun Shi

Correspondence: School of Mathematics and Statistics, Jiangxi Normal University, Nanchang, Jiangxi, P.R.China

Received: February 4, 2021 Accepted: April 9, 2021 Online Published: April 13, 2021

doi:10.5539/ijsp.v10n3p77 URL: https://doi.org/10.5539/ijsp.v10n3p77

Abstract

In this paper we consider one dimensional generalized mean-field backward stochastic differential equations (BSDEs) driven by fractional Brownian motion, i.e., the generators of our mean-field FBSDEs depend not only on the solution but also on the law of the solution. We first give a totally new comparison theorem for such type of BSDEs under Lipschitz condition. Furthermore, we study the existence of the solution of such mean-field FBSDEs when the coefficients are only continuous and with a linear growth.

Keywords: backward stochastic differential equation, continuous coefficients, comparison theorem, fractional Brownian motion, mean-field

Mathematical subject classification: 60H05, 60H07.

1. Introduction

General backward stochastic differential equations driven by a Brownian motion were first studied by Pardoux and Peng (1992). Later Pardoux and Zhang (1998) introduced the generalized BSDEs,i.e. BSDEs with an additional term-an integral with respect to an increasing process. Backward stochastic differential equations driven by a fractional Brownian motion with $H \in (1/2, 1)$ were first considered by Biagini,Hu,Øksendal and Sulem (2002), where they studied the stochastic maximal principle in the framework of a fractional Brownian motion. By adapting the four-step scheme introduced by Ma,Protter and Yong(1994) and the so-called S-transform, Bender (2005) stuided BSDEs driven by a fractional Brownian motion with $H \in (0, 1)$. Indeed, throughout a backward parabolic PDE, he constructed an explicit solution of a kind of linear fractional BSDE. Hu and Peng (2009) were the first to study nonlinear BSDEs governed by a fractional Brownian motion.

It is well known that backward stochastic differential equation provided stochastic representation of solution of some classes of partial differential equations of second order. With the help of backward stochastic differential equations with respect to a Brownian motion and a Poisson random measure, some authors generalized this result to integro-partial differential equations. The pioneer result on BSDEs, established by Pardoux and Peng (1990) require Lipschitz condition on the drift of the equation. Sow study on BSDE with jumps, established by Sow (2014) require non-Lipschitz coefficients and application to large deviations.

Mathematical mean-field approaches play an important role in many fields, among them, finance and game theory. Since the pioneering of Lasry and Lions (2007) the research on mean-field has attracted a lot of researchers. Buckdahn, Djehiche, Li and Peng (2009) studied a type of mean field problem by a purely stochastic approach and introduced a new type of BSDE which they called mean-field BSDE. Buckdahn, Li and Peng (2009) obtained the existence and the uniqueness of the solution of the mean-field BSDEs when the coefficient f is Lipschitz, and the terminal condition ξ is a square integrable random variable. They also got a comparison theorem. Later, more and more works have been studied on mean-field SDEs and BSDEs, see Buckdahn, Li and Peng (2009), Buckdahn, Li, Peng and Rainer (2017), Hao and Li (2016), Li (2017), Li and Min (2016). Du, Li and Wei (2011) considered a special type of one dimensional mean-field BSDEs with coefficients which are continuous and have a linear growth. They got the existence of the minimal solution. Recently, Juan (2018) considered general mean-field BSDEs with continuous coefficients. Our aim in the present work is to extend result to generalized mean-field BSDEs driven by fractional Brownian motion with continuous coefficients.

Let us recall that, for $H \in (0, 1)$, a fBm $(B^H(t))_{t\geqslant 0}$ with Hurst parameter H is a continuous and centered Gaussian process with covariance

$$E\left[B^H(t)B^H(s)\right] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \quad t, s \geqslant 0.$$

For $H = 1/2$, the fBm is a standard Brownian motion. If $H > 1/2$, then $B^H(t)$ has a long-range dependence, which means that for $r(n) := cov(B^H(1), B^H(n+1) - B^H(n))$, we have $\sum_{n=1}^{\infty} r(n) = \infty$. Moreover, B^H is self-similar, i.e. $B^H(at)$ has the same law as $a^H B^H(t)$ for any $a > 0$. Since there are many models of physical phenomena and finance which exploit the self-similarily and the long-range dependence, fBm are a very useful tool to characterize such type of problems.

However, since fBm are not semimartingales nor Markov processes when $H \neq 1/2$, we can not use the classical theory of stochastic calculus to define the fractional stochastic integral. In essence, two different integration theories with respect to fractional Brownian motion have been defined and studied. The first one, originally due to Young (1936), concerns the pathwise Riemann-Stieljes integral which exists if the integrand has Hölder continuous paths of order $\alpha > 1 - H$. But it turn out that this integral has the properties comparable to the Stratonovich integral, which leads to difficulties in applications. The second one concerns the divergence operator (Skorohod integral), define as the adjoint of the derivative operator in the framework of the Malliavin calculus. This approach was introduced by Decreusefond and Uštuňel (1998).

Concerning the study of BSDEs in the fractional framework, the major problem is the absence of a martingale representation type theorem with respect to fBm. For the first time, Hu and Peng (2009) overcome this problem, in the case $H > 1/2$.

We now introduce a class of reflected diffusion processes with standard Brownian motion. Let G be an open connected subset of R^d , which is such that for some $l \in C^2(\mathbb{R}^d)$, $G = \{x : l(x) > 0\}$, $\partial G = \{x : l(x) = 0\}$ and $|\nabla l(x)| = 1$ for $x \in \partial G$. Note that at any boundary point $x \in \partial G$, $\nabla l(x)$ is a unit normal vector to the boundary, pointing towards to the interior of G. If drift coeffcient and diffusion coeffcient satisfying some Lipschitz , then it follows from the results in Lions and Sznitman (1984) (see also Saisho (1987)) that for each $x \in \partial G$, there exists a unique pair of progressively measurable continuous processes (η_t, Λ_t), such that

$$\eta_t = \eta_0 + \int_0^t b(s)ds + \int_0^t \sigma(s)dB_s + \int_0^t \nabla l(\eta_s)d\Lambda_s, \quad 0 \leqslant t \leqslant T,$$

$$\Lambda_t = \int_0^t \mathbf{1}_{\eta_s \in \partial G} d\Lambda_s, \quad \Lambda_. \quad is \quad a \quad nondecreasing \quad process.$$

the existence of such a problem driven by fBm was shown in Ferrante and Rovira (2013) and a set $D = (0, +\infty)$.

In this paper we study the generalized mean-field BSDEs driven by fBm with Hurst parameter $H > 1/2$. We prove that kind of equation has an adapted solution under continuous coefficients. The paper is organized as follows. In section 2 we give some definitions and results about fractional stochastic integral which will be needed throughout the paper. Section 3 contains the definition of the generalized BSDEs driven by fBm and assumptions. In section 4, we will prove comparison theorem for the generalized mean-field FBSDE. Finally, section 5 is devoted to prove the main theorem of the paper.

2. Fractional Stochastic Calculus

Denote, for given $H \in (1/2, 1)$, $\phi(x) = H(2H-1)|x|^{2H-2}, x \in \mathbb{R}$. Let ξ and η be measurable functions on $[0, T]$. Define

$$\langle \xi, \eta \rangle_t = \int_0^t \int_0^t \phi(u-v)\xi(u)\eta(v)dudv$$

and $\|\xi\|_t^2 = \langle \xi, \xi \rangle_t$.Note that , for any $t \in [0, T]$, $\langle \xi, \eta \rangle_t$ is a Hilbert scalar product.

Let $\mathcal{H}$ be the completion of the measurable functions such that $\|\xi\|_t^2 < \infty$. The elements of $\mathcal{H}$ may be distributions (refer to Pipiras and Taqqu (2000)).

Let $(\xi_n)_n$ be a sequence in $\mathcal{H}$ such that $\langle \xi_i, \xi_j \rangle_T = \delta_{ij}$. By $\mathcal{P}_T$ denote the set of all polynomials of fractional Brownian motion in $[0, T]$, i.e. it contains all elements of the form

$$F(\omega) = f\left(\int_0^T \xi_1(t)dB_t^H, \cdots, \int_0^T \xi_k(t)dB_t^H\right),$$

where f is a polynomial function of k variables. The Malliavin derivative operator D_s^H of an element $F \in \mathcal{P}_T$ is defined as follows:

$$D_s^H F = \sum_{i=1}^k \frac{\partial f}{\partial x_i}\left(\int_0^T \xi_1(t)dB_t^H, \cdots, \int_0^T \xi_k(t)dB_t^H\right) \cdot \xi_i(s), \quad s \in [0, T].$$

Since the divergence operator D^H is closable from $L^2(\Omega, \mathcal{F}, P)$ to $(\Omega, \mathcal{F}, \mathcal{H})$, By $\mathbb{D}_{1,2}$ denote the Banach space be the a completion of $\mathcal{P}_T$ with the following norm: $\|F\|_{1,2}^2 = E|F|^2 + E\|D_s^H F\|_T^2$.

Now we also introduce another derivative

$$\mathbb{D}_t^H F = \int_0^T \phi(t-s) D_s^H F ds.$$

The following results are well known, refer to Duncan and Hu (2000), Hu (2005).

Theorem 2.1. *(Hu (2005),Proposition 6.25) Let $F : (\Omega, \mathcal{F}, P) \to \mathcal{H}$ be a stochastic process such that*

$$E\left(\|F\|_T^2 + \int_0^T \int_0^T |\mathbb{D}_s^H F_t|^2 dsdt\right) < \infty.$$

Then, the Itô-type stochastic integral denoted by $\int_0^T F_s dB_s^H$ exists in $L^2(\Omega, \mathcal{F}, P)$. Moreover, $E\left(\int_0^T F_s dB_s^H\right) = 0$ and

$$E\left(\int_0^T F_s dB_s^H\right)^2 = E\left(\|F\|_T^2 + \int_0^T \int_0^T \mathbb{D}_s^H F_t \mathbb{D}_t^H F_s dsdt\right).$$

Theorem 2.2. *(Hu (2005),Proposition 10.3) Let f, g:$[0,T] \to \mathbb{R}$ be deterministic continuous functions. If*

$$X_t = X_0 + \int_0^t g(s)ds + \int_0^t f(s)dB_s^H, \quad t \in [0,T],$$

where X_0 is a constant and $F \in C^{1,2}([0,T] \times \mathbb{R})$, then

$$F(t,X_t) = F(0,X_0) + \int_0^t \frac{\partial F}{\partial s}(s,X_s)ds + \int_0^t \frac{\partial F}{\partial x}(s,X_s)dX_s + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(s,X_s)\frac{d}{ds}(\|f\|_s^2)ds, \quad t \in [0,T].$$

Theorem 2.3. *(Hu (2005),Proposition 11.1) Let $f_i(s), g_i(s), i = 1,2$ are in $\mathbb{D}_{1,2}$ and $E\int_0^T (|f_i(s)| + |g_i(s)|)ds < \infty$. Assume that $\mathbb{D}_t^H f_1(s)$ and $\mathbb{D}_t^H f_2(s)$ are continuously differential with respect to $(s,t) \in [0,T]\times[0,T]$ for almost all $\omega \in \Omega$. Suppose that*

$$E\left(\int_0^T \int_0^T |\mathbb{D}_t^H f_i(s)|^2 dsdt\right) < \infty.$$

For $i = 1,2$, denote

$$X_i(t) = \int_0^t g_i(s)ds + \int_0^t f_i(s)dB_s^H, \quad t \in [0,T],$$

Then

$$\begin{aligned} X_1(t)X_2(t) &= \int_0^t X_1(s)g_2(s)ds + \int_0^t X_1(t)f_2(s)dB_s^H + \int_0^t X_2(s)g_1(s)ds + \int_0^t X_2(t)f_1(s)dB_s^H \\ &+ \int_0^t \mathbb{D}_s^H X_1(s)f_2(s)ds + \int_0^t \mathbb{D}_s^H X_2(s)f_1(s)ds. \end{aligned}$$

3. Generalized Fractional BSDE

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leqslant t\leqslant T}, P)$, $T > 0$, be a complete stochastic basis, and $\mathcal{F}_{t^+} = \cap_{\delta>0}\mathcal{F}_{t+\delta} = \mathcal{F}_t$. Suppose that the filtration is generated by d-dimensional fractional Brownian motion $(B_t^H)_{0\leqslant t\leqslant T}$, and $T > 0$ is an arbitrarily fixed time horizon. We suppose that there is a sub-σ-field $\mathcal{F}_0 \subset \mathcal{F}$, $\mathcal{F}_0$ includes all P-null subsets of $\mathcal{F}$, such that

i) the fractional Brownian motion B^H is independent of $\mathcal{F}_0$, and

ii) $\mathcal{F}_0$ is "rich enough", i.e.,$\mathcal{P}_2(\mathbb{R}^k) = \{P_\vartheta, \vartheta \in L^2(\mathcal{F}_0;\mathbb{R}^k)\}$, $k \geqslant 1$.

Recall that $\mathcal{P}_2(\mathbb{R}^k)$ is the set of the probability measures on $(\mathbb{R}^k, B(\mathbb{R}^k))$ with finite second moment. Here $B(\mathbb{R}^k)$ denotes the Borel σ-field over $\mathbb{R}^k$. By $\mathcal{F} = (\mathcal{F}_t), t \in [0,T]$, we denote the filtration generated by B^H, completed and augmented by $\mathcal{F}_0$.

The space $\mathcal{P}_2(\mathbb{R}^d)$ is endowed with the 2-Wasserstein metric

$$W_2(\mu,\nu) = \inf_{\pi\in\prod(\mu,\nu)} \left\{ \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \pi(dx,dy) \right)^{1/2} \right\},$$

where $\prod(\mu,\nu)$ is the family of all couplings of μ and ν, i.e., $\pi \in \prod(\mu,\nu)$ if and only if π is a measure on $\mathbb{R}^d\times\mathbb{R}^d$ with marginals $\mu,\nu\in\mathcal{P}_2(\mathbb{R}^d)$.

Assume that
- η_0 is a given constant;
- $b,\sigma : [0,T]\to\mathbb{R}$ are continuous deterministic, σ is differentiable and $\sigma_t \neq 0, t\in[0,T]$.

Note that, since $\|\sigma\|_t^2 = H(2H-1)\int_0^t\int_0^t |u-v|^{2H-2}\sigma(u)\sigma(v)dudv$, we have

$$\frac{d}{dt}(\|\sigma\|_t^2) = 2\sigma(t)\widehat{\sigma}(t) > 0, \quad where \quad \widehat{\sigma}(t) = \int_0^t \phi(t-v)\sigma(v)dv, \quad 0\leqslant t\leqslant T.$$

Remark 3.1. *(Remark 6 by Maticiuc and Nie (2015))*
There exists a suitable constant $M>0$ which is only dependent H such that

$$\frac{t^{2H-1}}{M} \leqslant \frac{\widehat{\sigma}(t)}{\sigma(t)} \leqslant Mt^{2H-1}, \quad 0\leqslant t\leqslant T.$$

since

$$\begin{aligned}\widehat{\sigma}(t) &= \int_0^t \phi(t-v)\sigma(v)dv = H(2H-1)\int_0^t (t-v)^{2H-2}\sigma(v)dv = H(2H-1)\int_0^1 (t(1-u))^{2H-2}\sigma(tu)tdu \\ &= H(2H-1)t^{2H-1}\int_0^1 (1-u)^{2H-2}\sigma(tu)du,\end{aligned}$$

then by continuity of σ, we get the remark.

We now introduce a class of reflected processes. Let G be an open connected subset of $\mathbb{R}^d$, which is such that for some $l\in C^2(\mathbb{R}^d)$, $G=\{x : l(x)>0\}$, $\partial G = \{x : l(x)=0\}$ and $|\nabla l(x)|=1$ for $x\in\partial G$. Note that at any boundary point $x\in\partial G$, $\nabla l(x)$ is a unit normal vector to the boundary, pointing towards to the interior of G. Let $\eta_0\in G$ and (η_t,Λ_t) be a solution of the following reflected SDE with respect to fractional Brownian motion

$$\eta_t = \eta_0 + \int_0^t b(s)ds + \int_0^t \sigma(s)dB_s^H + \int_0^t \nabla l(\eta_s)d\Lambda_s, \quad 0\leqslant t\leqslant T, \tag{1}$$

By a solution of (1), we mean a pair of processes such that $\eta_. \in G$, Λ is a nondecreasing process, $\Lambda_0 = 0$, and $\int_0^T(\eta_t - a)d\Lambda_s \leqslant 0$ for any $a\in G$,

$$\Lambda_t = \int_0^t \mathbf{1}_{\eta_s\in\partial G}d\Lambda_s.$$

The existence of such a problem was shown in Lions and Sznitman (2007) for a standard Brownian motion.

Remark 3.2. *This problem is solved in Ferrante and Rovira (2009) for a fractional Brownian motion and a set $G=(0,\infty)$.*

Given a final time $T>0$, a final condition ξ, which is a $\mathcal{F}_T$ measurable real valued random variable and the functions

$$f : [0,T]\times\Omega\times\mathbb{R}\times\mathcal{P}_2(\mathbb{R}^{1+d})\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}, \quad g : [0,T]\times\Omega\times\mathbb{R}\times\mathcal{P}_2(\mathbb{R})\times\mathbb{R}\to\mathbb{R},$$

we consider the following generalized BSDE with respect to fBm with parameters (ξ,f,g,Λ) (short name GFBSDE) whose generators depend on both the solution (Y,Z) and the law of (Y,Z), the law of Y, respectivity, i.e.

$$Y_t = \xi + \int_t^T f(s,\eta_s,P_{(Y_s,Z_s)},Y_s,Z_s)ds + \int_t^T g(s,\eta_s,P_{(Y_s)},Y_s)d\Lambda_s - \int_t^T Z_s dB_s^H, \quad 0\leqslant t\leqslant T. \tag{2}$$

in order to give a probabilistic formula for the solution of a system of elliptic PDEs, this requires the new term–an integral with respect to a increasing process in this equation (2) which is independent of Z_s, the local time of the diffusion on the boundary.

Next we introduce the following sets:
• $C^{1,2}_{pol}([0,T]\times\mathbb{R})$ is the space of all $C^{1,2}$ functions over $[0,T]\times\mathbb{R}$, which together with their derivatives are of polynomial growth,
• $\mathcal{V}_{[0,T]} = \left\{Y = \psi(\cdot,\eta) : \psi \in C^{1,2}_{pol}([0,T]\times\mathbb{R}),\ \frac{\partial\psi}{\partial t}\quad is\quad bounded\quad, t\in[0,T]\right\}$,
• $\widetilde{\mathcal{V}}^H_{[0,T]}$ the completion of $\mathcal{V}_{[0,T]}$ under the following norm (where $\beta > 0$)

$$||Y||_\beta = \left(\int_0^T t^{2H-1}E[e^{\beta\Lambda_t}|Y_t|^2]dt\right)^{1/2} = \left(\int_0^T t^{2H-1}E[e^{\beta\Lambda_t}|\psi(t,\eta_t)|^2]dt\right)^{1/2},$$

We assume that the coefficients f and g of the GFBSDE are continuous functions and satisfy the following assumption ($H1$):
(***H*1.1**) Linear growth: There exists $K \geqslant 0$, such that

$$|f(t,\eta,\mu,y,z)| \leqslant K(1 + W_2(\mu,\delta_0) + |y| + |\eta| + |z|), dtdP - a.e \quad for \quad all \quad (\eta,\mu,y,z),$$

$$|g(t,\eta,\nu,y)| \leqslant K(1 + W_2(\nu,\delta_0) + |y| + |\eta|), dtdP - a.e \quad for \quad all \quad (\eta,\nu,y).$$

where δ_0 is the Dirac measure with mass at $0 \in \mathbb{R}^{1+d}$ or $0 \in \mathbb{R}^d$.
(***H*1.2**) Lipschitz in (μ,y,z): i.e. there exists a constant $C \in \mathbb{R}^+$ such that for all $\mu_1,\mu_2 \in \mathcal{P}_2(\mathbb{R}^{1+d})$, $\nu_1,\nu_2 \in \mathcal{P}_2(\mathbb{R}^d)$ and all $y_1,y_2 \in \mathbb{R}$, $z_1,z_2 \in \mathbb{R}^d$,

$$|f(s,\eta,\mu_1,y_1,z_1) - f(s,\eta,\mu_2,y_2,z_2)| \leqslant C(W_2(\mu_1,\mu_2) + |y_1 - y_2| + |z_1 - z_2|)\ \ dsdP - a.e.$$

$$|g(s,\eta,\nu_1,y_1) - g(s,\eta,\nu_2,y_2)| \leqslant C(W_2(\nu_1,\nu_2) + |y_1 - y_2|)\ \ dsdP - a.e.$$

(***H*1.3**) A progressively measurable continuous, non-decreasing processes Λ_t has continuous density function.
(***H*1.4**) There exists $\beta > 0$ and a function ψ with bounded derivative such that $\xi = \psi(\eta_T)$, $E(e^{\beta\Lambda_T}|\xi|^2) < \infty$ and the integrability condition holds

$$E\left(\int_0^T e^{\beta\Lambda_s}(1 + E[(Y_s,Z_s)^2])ds + \int_0^T e^{\beta\Lambda_s}|\eta_s|^2 ds + \int_0^T e^{\beta\Lambda_s}(1 + E[(Y_s)^2])d\Lambda_s\right) < \infty.$$

4. Comparison Theorem for General Mean-Field Fractional BSDEs

Definition 4.1. *A binary of processes $(Y_t,Z_t)_{0\leqslant t\leqslant T}$ is called a solution to (2), if $(Y_t,Z_t) \in \widetilde{\mathcal{V}}^{1/2}_{[0,T]} \times \widetilde{\mathcal{V}}^H_{[0,T]}$ and satisfies (2).*

Lemma 4.2. *Assume X is a mean nonzero Gaussian with nonzero covariance, if for two continuous functions $k_1(x), k_2(x)$ such that $k_1(X) = k_2(X)$, then $k_1(x) = k_2(x)$ for all $x \in \mathbb{R}$.*

Proof. Let $f_X(x)$ denote the density function of X, we have

$$f_X(x) = \frac{1}{\sqrt{2\pi}\theta}e^{-\frac{(x-\alpha)^2}{2\theta^2}},$$

where α denote mean, θ^2 denote variance. Since $k_1(X) = k_2(X)$, take expectation in both sides of this equality, we get

$$\int_{-\infty}^{+\infty}(k_1(x) - k_2(x))f_X(x)dx = 0,$$

by density of $C_0^\infty(\mathbb{R})$ in $C(\mathbb{R})$ and $f_X(x) \geqslant 0$ for all $x \in \mathbb{R}$, consequently $k_1(x) = k_2(x)$ for all $x \in \mathbb{R}$. □

Lemma 4.3. *Assume that h_1, h_2 and $h_3 \in C^{0,1}_{pol}([0,T]\times\mathbb{R})$ such that*

$$\int_0^t h_1(s,\eta_s)ds + \int_0^t h_2(s,\eta_s)dB^H_s + \int_0^t h_3(s,\eta_s)d\Lambda_s = 0, \quad 0 \leqslant t \leqslant T.$$

Then we have

$$h_1(s,x) = h_2(s,x) = h_3(s,x) = 0, \quad 0 \leqslant s \leqslant T, \quad x \in \mathbb{R}.$$

Proof. To simplify notation, we let $\eta_0 = b(t) = 0$ for all $t \in [0, T]$ in (1). Similary to Hu (2005) Theorem 12.3, we have

$$\begin{aligned} h_1(s,\eta_s) &= Eh_1(s,\eta_s) + \int_0^s \left(\int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s,\eta_s) dy \right) \sigma(u) dB_u^H \\ &+ \int_0^s \left(\int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s,\eta_s) dy \right) \nabla l(\eta_u) d\Lambda_u, \end{aligned}$$

where

$$p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

and

$$p_{u,s}(x) = p_{\|\sigma\|_s - \|\sigma\|_u}(x).$$

Thus, by stochastic Fubini theorem

$$\begin{aligned} \int_0^t h_1(s,\eta_s) ds &= \int_0^t Eh_1(s,\eta_s) ds + \int_0^t \int_0^s \left(\int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s,\eta_s) dy \right) \sigma(u) dB_u^H ds \\ &+ \int_0^t \int_0^s \left(\int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s,\eta_s) dy \right) \nabla l(\eta_u) d\Lambda_u ds \\ &= \int_0^t Eh_1(s,\eta_s) ds + \int_0^t \sigma(u) \left(\int_u^t \int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s,\eta_s) dy ds \right) dB_u^H \\ &+ \int_0^t \nabla l(\eta_u) \left(\int_u^t \int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s,\eta_s) dy ds \right) d\Lambda_u \\ &= \int_0^t Eh_1(s,\eta_s) ds + \int_0^t [h_2(u,\eta_u) + \sigma(u) \left(\int_u^t \int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s,\eta_s) dy ds \right)] dB_u^H \\ &+ \int_0^t [h_3(u,\eta_u) + \nabla l(\eta_u) \left(\int_u^t \int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s,\eta_s) dy ds \right)] d\Lambda_u \\ &- \int_0^t h_2(u,\eta_u) dB_u^H - \int_0^t h_3(u,\eta_u) d\Lambda_u, \end{aligned}$$

Thus from assumption, we have

$$\int_0^t Eh_1(s,\eta_s) ds = 0,$$

$$\int_0^t \left[h_2(u,\eta_u) + \sigma(u) \int_u^t \int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s,\eta_s) dy ds \right] dB_u^H = 0,$$

$$\int_0^t \left[h_3(u,\eta_u) + \nabla l(\eta_u) \int_u^t \int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s,\eta_s) dy ds \right] d\Lambda_u = 0.$$

But $h_2(u,\eta_u) + \sigma(u) \int_u^t \int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s,\eta_s) dy ds$ and $h_3(u,\eta_u) + \nabla l(\eta_u) \int_u^t \int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s,\eta_s) dy ds$ are $\mathcal{F}_u$ adapted (since these are a function of η_u). So from Theorem12.1 of Hu (2005), we see that

$$h_2(u,\eta_u) + \sigma(u) \int_u^t \int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s,\eta_s) dy ds = 0,$$

$$h_3(u,\eta_u) + \nabla l(\eta_u) \int_u^t \int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s,\eta_s) dy ds = 0.$$

In our situation, (η_u, Λ_u) is a solution of reflected SDE with respect to fractional Brownian motion

$$\eta_u = \int_0^s \sigma(u) dB_u^H + \int_0^s \nabla l(\eta_u) d\Lambda_u, \quad 0 \leqslant u \leqslant s,$$

where Λ is a nondecreasing process, and

$$\Lambda_u = \int_0^s \mathbf{1}_{\eta_u \in \partial G} d\Lambda_u.$$

Although η_u is not center Gaussian process, but by Lemma 4.2, we have

$$h_2(u,z) + \sigma(u)\int_u^t \int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s,z) dy ds = 0, \tag{3}$$

$$h_3(u,z) + \nabla l(z)\int_u^t \int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s,z) dy ds = 0. \tag{4}$$

for all $z \in \mathbb{R}$. Next, the step as same as Lemma 4.2 of Hu (2005),and consequently $h_1(s,z) = 0$ for all $0 \leqslant s \leqslant T$, $z \in \mathbb{R}$. Finally, Bringing $h_1(s,z) = 0$ into the formulas (3) and (4), $h_2(u,z) = 0, h_3(u,z) = 0$ are then an immediate consequence for all $0 \leqslant s \leqslant T$, $z \in \mathbb{R}$. □

It is well known following Lemma (refer to Hu (2005)).

Lemma 4.4. *Let $(Y_t, Z_t)_{0\leqslant t \leqslant T}$ be a solution of the GFBSDE (2). Then we have the stochastic representation*

$$\mathbb{D}_t^H Y_t = \frac{\widehat{\sigma}(t)}{\sigma(t)} Z_t, \quad 0 \leqslant t \leqslant T,$$

Proposition 4.5. *Let $\xi \in L^2(\Omega, \mathcal{F}_T; \mathbb{R})$. Assume $(H1)$ holds. Then there exists a unique solution of (2). Moreover, for all $t \in [0, T]$,*

$$E\left(e^{\beta\Lambda_s}|Y_t|^2 + \int_t^T e^{\beta\Lambda_s} s^{2H-1}|Z_s|^2 ds + \int_t^T e^{\beta\Lambda_s}|Y_s|^2 d\Lambda_s\right) \leqslant K\Theta(t,T),$$

where

$$\Theta(t,T) := E\left(e^{\beta\Lambda_T}|\xi|^2 + 2\int_t^T e^{\beta\Lambda_s}(1 + E[(Y_s, Z_s)^2])ds + \int_t^T e^{\beta\Lambda_s}|\eta_s|^2 ds + 2\int_t^T e^{\beta\Lambda_s}(1 + E[(Y_s)^2])d\Lambda_s\right).$$

Proof. First we will show the second part of the above theorem. Assume that (Y, Z) is a solution of (5). By K we will denote a constant which may vary from line to line. From the Itô formula

$$\begin{aligned}
& e^{\beta\Lambda_t}|Y_t|^2 + 2\int_t^T e^{\beta\Lambda_s}(\mathbb{D}_s^H Y_s) Z_s ds + \beta\int_t^T e^{\beta\Lambda_s}|Y_s|^2 d\Lambda_s \\
= \;& e^{\beta\Lambda_T}|\xi|^2 + 2\int_t^T e^{\beta\Lambda_s}|Y_s| f(s, \eta_s, P_{(Y_s,Z_s)}, Y_s, Z_s) ds + 2\int_t^T e^{\beta\Lambda_s}|Y_s| g(s, \eta_s, P_{(Y_s)}, Y_s) d\Lambda_s \\
& +2\int_t^T e^{\beta\Lambda_s}|Y_s| Z_s dB_s^H.
\end{aligned}$$

By linear growth of f and g, for all $\mu \in \mathcal{P}_2(\mathbb{R}^{1+d})$, $\nu \in \mathcal{P}_2(\mathbb{R}^d)$, we have

$$\begin{aligned}
2|yf(t,\eta,\mu,y,z)| &\leqslant 2K|y|(1 + W_2(\mu,\delta_0) + |\eta| + |y| + |z|) \\
&\leqslant (2K^2 + 2K + \frac{MK^2}{s^{2H-1}})|y|^2 + |\eta|^2 + \frac{1}{M}s^{2H-1}|z|^2 + (1 + W_2(\mu,\delta_0))^2
\end{aligned}$$

$$2|yg(t,\eta,\nu,y)| \leqslant 2K|y|(1 + W_2(\nu,\delta_0) + |\eta| + |y|) \leqslant (2K + 2K^2)|y|^2 + |\eta|^2 + (1 + W_2(\nu,\delta_0))^2$$

There, we can write

$$\begin{aligned}
& E\left(e^{\beta\Lambda_t}|Y_t|^2 + \frac{2}{M}\int_t^T e^{\beta\Lambda_s} s^{2H-1}|Z_s|^2 ds + \beta\int_t^T e^{\beta\Lambda_s}|Y_s|^2 d\Lambda_s\right) \\
\leqslant \;& E(e^{\beta\Lambda_T}|\xi|^2) + 2E\int_t^T e^{\beta\Lambda_s}|Y_s| f(s, \eta_s, P_{(Y_s,Z_s)}, Y_s, Z_s) ds + 2E\int_t^T e^{\beta\Lambda_s}|Y_s| g(s, \eta_s, P_{(Y_s)}, Y_s) d\Lambda_s
\end{aligned}$$

$$\begin{aligned}
&\leqslant E(e^{\beta\Lambda_T}|\xi|^2) + E\int_t^T (2K^2 + 2K + \frac{MK^2}{s^{2H-1}} + 1)e^{\beta\Lambda_s}|Y_s|^2 ds + (2K + 2K^2)E\int_t^T e^{\beta\Lambda_s}|Y_s|^2 d\Lambda_s\\
&+ E\int_t^T e^{\beta\Lambda_s}(|\eta_s|^2)ds + \frac{1}{M}E\int_t^T s^{2H-1}e^{\beta\Lambda_s}|Z_s|^2 ds\\
&+ E\int_t^T e^{\beta\Lambda_s}(1 + W_2(P_{(Y_s,Z_s)},\delta_0))^2 ds + E\int_t^T e^{\beta\Lambda_s}(1 + W_2(P_{(Y_s)},\delta_0))^2 d\Lambda_s\\
&\leqslant \Theta(t,T) + E\int_t^T (2K^2 + 2K + \frac{MK^2}{s^{2H-1}})e^{\beta\Lambda_s}|Y_s|^2 ds + (2K + 2K^2)E\int_t^T e^{\beta\Lambda_s}|Y_s|^2 d\Lambda_s\\
&+ \frac{1}{M}E\int_t^T s^{2H-1}e^{\beta\Lambda_s}|Z_s|^2 ds
\end{aligned}$$

Choosing $\beta \geqslant (2K + 2K^2 + 1)$, we get

$$\begin{aligned}
&E\left(e^{\beta\Lambda_t}|Y_t|^2 + \frac{1}{M}\int_t^T e^{\beta\Lambda_s}s^{2H-1}|Z_s|^2 ds + \int_t^T e^{\beta\Lambda_s}|Y_s|^2 d\Lambda_s\right)\\
\leqslant\ & \Theta(t,T) + E\int_t^T (2K^2 + 2K + \frac{MK^2}{s^{2H-1}})e^{\beta\Lambda_s}|Y_s|^2 ds.
\end{aligned}$$

By Gronwall's inequality,

$$Ee^{\beta\Lambda_t}|Y_t|^2 \leqslant \Theta(t,T)\exp\left\{(2K^2 + 2K)(T-t) + MK^2\frac{T^{2H-1} - t^{2H-1}}{2-2H}\right\}$$

and also get

$$E\left(\int_t^T e^{\beta\Lambda_s}s^{2H-1}|Z_s|^2 ds + \int_t^T e^{\beta\Lambda_s}|Y_s|^2 d\Lambda_s\right) \leqslant C\Theta(t,T).$$

Now we will prove the existence and uniqueness of the solution of (5). The method used here is the fixed point theorem. We will show that the mapping $\Gamma : \widetilde{\mathcal{V}}^{1/2}_{[0,T]} \times \widetilde{\mathcal{V}}^{H}_{[0,T]} \to \widetilde{\mathcal{V}}^{1/2}_{[0,T]} \times \widetilde{\mathcal{V}}^{H}_{[0,T]}$ given by $(X,W) \to \Gamma(X,W) = (Y,Z)$ is a contraction, where (Y,Z) is a solution of the following generalized BSDE:

$$Y_t = \xi + \int_t^T f(s,\eta_s,P_{(X_s,W_s)},X_s,W_s)ds + \int_t^T g(s,\eta_s,P_{(X_s)},X_s)d\Lambda_s - \int_t^T Z_s dB^H_s$$

Let $k \in \mathbb{N}$ and $t_i = \frac{i-1}{k}T, i = 1,\cdots,k+1$. First we will show that Γ is a contraction on $\widetilde{\mathcal{V}}^{1/2}_{[t_k,T]} \times \widetilde{\mathcal{V}}^{H}_{[t_k,T]}$. Take $X, X' \in \widetilde{\mathcal{V}}^{1/2}_{[t_k,T]}$, $W, W' \in \widetilde{\mathcal{V}}^{H}_{[t_k,T]}$, let $\Gamma(X,W) = (Y,Z)$, $\Gamma(X',W') = (Y',Z')$ and let $\overline{Y} = Y - Y'$, $\overline{Z} = Z - Z'$, $\overline{X} = X - X'$, $\overline{W} = W - W'$. From Itô formula, for $t \in [t_k,T]$, we have

$$\begin{aligned}
&E\left(e^{\beta\Lambda_t}|\overline{Y}_t|^2 + 2\int_t^T e^{\beta\Lambda_s}(\mathbb{D}^H_s\overline{Y}_s)\overline{Z}_s ds + \beta\int_t^T e^{\beta\Lambda_s}|\overline{Y}_s|^2 d\Lambda_s\right)\\
=\ & 2E\int_t^T e^{\beta\Lambda_s}|\overline{Y}_s|(f(s,\eta_s,P_{(X_s,W_s)},X_s,W_s) - f(s,\eta_s,P_{(X'_s,W'_s)},X'_s,W'_s))ds\\
+\ & 2E\int_t^T e^{\beta\Lambda_s}|\overline{Y}_s|(g(s,\eta_s,P_{(X_s)},X_s) - g(s,\eta_s,P_{(X'_s)},X'_s))d\Lambda_s
\end{aligned}$$

Note that $2|\overline{y}_s||f(s,\eta_s,\mu_1,x_s,w_s) - f(s,\eta_s,\mu_2,x'_s,w'_s)| \leqslant 2C|\overline{y}_s|(|\overline{x}_s| + |\overline{w}_s| + W_2(\mu_1,\mu_2))$.

$2|\overline{y}_s||g(s,\eta_s,\nu_1,x_s) - g(s,\eta_s,\nu_2,x'_s)| \leqslant 2C|\overline{y}_s|(W_2(\nu_1,\nu_2) + |\overline{x}_s|) \leqslant \frac{C^2}{\alpha}|\overline{y}_s|^2 + 2\alpha|\overline{x}_s|^2 + 2\alpha W_2^2(\nu_1,\nu_2)$ for some $\alpha > 0$.

Choose $\beta = \frac{C^2}{\alpha} + 1$. Then by the Schwartz inequality we obtain

$$\begin{aligned}
&E\left(e^{\beta\Lambda_t}|\overline{Y}_t|^2 + \frac{2}{M}\int_t^T e^{\beta\Lambda_s}s^{2H-1}|\overline{Z}_s|^2 ds + \int_t^T e^{\beta\Lambda_s}|\overline{Y}_s|^2 d\Lambda_s\right)\\
=\ & 2KE\int_t^T e^{\beta\Lambda_s}|\overline{Y}_s|(|\overline{X}_s| + |\overline{W}_s|)ds + \alpha E\int_t^T e^{\beta\Lambda_s}|\overline{X}_s|^2 d\Lambda_s
\end{aligned}$$

$$\leqslant \quad 2K\int_t^T \left(Ee^{\beta\Lambda_s}|\overline{Y}_s|^2\right)^{1/2}\left(Ee^{\beta\Lambda_s}(|\overline{X}_s|+|\overline{W}_s|)^2\right)^{1/2}ds+\alpha E\int_t^T e^{\beta\Lambda_s}|\overline{X}_s|^2 d\Lambda_s.$$

Denote $\varphi(t)=\left(Ee^{\beta\Lambda_s}|\overline{Y}_s|^2\right)^{1/2}$ and $\psi(t)=\alpha E\int_t^T e^{\beta\Lambda_s}|\overline{X}_s|^2 d\Lambda_s$ which is nonincrease. Then by above

$$\varphi(t)^2\leqslant 2K\int_t^T\varphi(t)\left(Ee^{\beta\Lambda_s}(|\overline{X}_s|+|\overline{W}_s|)^2\right)^{1/2}ds+\psi(t),\quad t\in[t_k,T].$$

Applying Lemma 20 in Maticiuc and Nie (1994) to the above inequality we get

$$\varphi(t)\leqslant\sqrt{2}K\int_t^T\left(Ee^{\beta\Lambda_s}(|\overline{X}_s|+|\overline{W}_s|)^2\right)^{1/2}ds+\sqrt{\psi(t)},\quad t\in[t_k,T].$$

and therefore for $t\in[t_k,T]$

$$Ee^{\beta\Lambda_s}|\overline{Y}_s|^2\leqslant 4K^2\left(\int_t^T\left(Ee^{\beta\Lambda_s}(|\overline{X}_s|+|\overline{W}_s|)^2\right)^{1/2}ds\right)^2+2\psi(t),$$

Integrate of both sides on $[t_k,T]$ of above inequality, we can compute

$$\begin{aligned}\int_{t_k}^T\varphi(s)^2ds &\leqslant 2\psi(t_k)(T-t_k)+4K^2\int_{t_k}^T\left(\int_t^T\left(Ee^{\beta\Lambda_s}(|\overline{X}_s|+|\overline{W}_s|)^2\right)^{1/2}ds\right)^2dt\\ &\leqslant 2\psi(t_k)(T-t_k)+8K^2(T-t_k)\left(\int_{t_k}^T(Ee^{\beta\Lambda_s}|\overline{X}_s|^2)^{1/2}ds\right)^2\\ &+ 8K^2(T-t_k)\left(\int_{t_k}^T\left(\frac{1}{s^{2H-1}}Ee^{\beta\Lambda_s}s^{2H-1}|\overline{W}_s|^2\right)^{1/2}ds\right)^2\\ &\leqslant 2\psi(t_k)(T-t_k)+8K^2(T-t_k)^2E\int_{t_k}^T e^{\beta\Lambda_s}|\overline{X}_s|^2ds\\ &+ 8K^2(T-t_k)\int_{t_k}^T\frac{1}{s^{2H-1}}dsE\int_{t_k}^T e^{\beta\Lambda_s}s^{2H-1}|\overline{W}_s|^2ds\\ &:= C\cdot(T-t_k)\widetilde{\Theta}(t_k,T),\end{aligned}$$

and similarly

$$\int_{t_k}^T\frac{1}{s^{2H-1}}\varphi(s)^2ds\quad\leqslant\quad\frac{C}{2-2H}\cdot(T^{2-2H}-t_k^{2-2H})\cdot\widetilde{\Theta}(t_k,T),$$

where

$$\widetilde{\Theta}(t_k,T)=E\left(\int_{t_k}^T e^{\beta\Lambda_s}s^{2H-1}|\overline{W}_s|^2ds+\int_{t_k}^T e^{\beta\Lambda_s}|\overline{X}_s|^2(ds+d\Lambda_s)\right).$$

Using above inequalities, we deduce

$$\begin{aligned}&E\left(\int_{t_k}^T e^{\beta\Lambda_s}s^{2H-1}|\overline{Z}_s|^2ds+\int_{t_k}^T e^{\beta\Lambda_s}|\overline{Y}_s|^2(ds+d\Lambda_s)\right)\\ \leqslant\ & E\int_{t_k}^T e^{\beta\Lambda_s}|\overline{Y}_s|^2ds+C\alpha E\int_{t_k}^T e^{\beta\Lambda_s}|\overline{X}_s|^2d\Lambda_s+CE\int_{t_k}^T e^{\beta\Lambda_s}\frac{1}{\alpha}|\overline{Y}_s|^2(2+\frac{1}{s^{2H-1}})ds\\ +\ & CE\int_{t_k}^T e^{\beta\Lambda_s}\alpha(|\overline{X}_s|^2+s^{2H-1}|\overline{W}_s|^2)ds\\ \leqslant\ & C\cdot(T-t_k)\widetilde{\Theta}(t_k,T)+\frac{C}{\alpha}\int_{t_k}^T\varphi(s)(2+\frac{1}{s^{2H-1}})ds+C\alpha\widetilde{\Theta}(t_k,T)\\ \leqslant\ & C\left(\alpha+(2+\frac{1}{\alpha})(T-t_k))+\frac{1}{\alpha}(T^{2-2H}-t_k^{2-2H})\right)\widetilde{\Theta}(t_k,T)\end{aligned}$$

Choosing α such that $C\alpha \leqslant 1/4$ and taking k large enough that $C(\alpha+2)(T-t_k)/\alpha \leqslant 1/4$ and $C(T^{2-2H}-t_k^{2-2H})/\alpha \leqslant 1/4$, we obtain

$$\begin{aligned} & E\left(\int_{t_k}^T e^{\beta\Lambda_s} s^{2H-1}|\overline{Z}_s|^2 ds + \int_{t_k}^T e^{\beta\Lambda_s}|\overline{Y}_s|^2(ds+d\Lambda_s)\right) \\ \leqslant\ & \frac{3}{4}\widetilde{\Theta}(t_k,T) \end{aligned}$$

Thus Γ is contraction operator in $\widetilde{\mathcal{V}}^{1/2}_{[t_k,T]} \times \widetilde{\mathcal{V}}^{H}_{[t_k,T]}$, and (Y^m, Z^m) is a Cauchy sequence in $\widetilde{\mathcal{V}}^{1/2}_{[t_k,T]} \times \widetilde{\mathcal{V}}^{H}_{[t_k,T]}$, where $(Y^0, Z^0) \in \widetilde{\mathcal{V}}^{1/2}_{[t_k,T]} \times \widetilde{\mathcal{V}}^{H}_{[t_k,T]}$, and for $m \geqslant 0$

$$\begin{aligned} Y_t^{m+1} &:= \xi + \int_t^T f(s,\eta_s,P_{(Y_s^m,Z_s^m)},Y_s^m,Z_s^m)ds + \int_t^T g(s,\eta_s,P_{(Y_s^m)},Y_s^m)d\Lambda_s \\ &- \int_t^T Z_s^{m+1} dB_s^H. \end{aligned}$$

Then there exists $(Y,Z) \in \widetilde{\mathcal{V}}^{1/2}_{[t_k,T]} \times \widetilde{\mathcal{V}}^{H}_{[t_k,T]}$ being a limit of (Y^m, Z^m), i.e.

$$\lim_{m\to+\infty} E\left(e^{\beta\Lambda_t}|Y_t^m - Y_t|^2 + \int_{t_k}^T e^{\beta\Lambda_s}(|Y_s^m - Y_s|^2 + s^{2H-1}|Z_s^m - Z_s|^2)ds\right) = 0,$$

$$\lim_{m\to+\infty} E\left(\int_{t_k}^T e^{\beta\Lambda_s}|Y_s^m - Y_s|^2 d\Lambda_s\right) = 0,$$

Therefore for any $t \in [t_k, T]$,

$$\begin{aligned} & \lim_{m\to\infty}\left(-Y_t^{m+1} + \xi + \int_t^T f(s,\eta_s,P_{(Y_s^m,Z_s^m)},Y_s^m,Z_s^m)ds + \int_t^T g(s,\eta_s,P_{(Y_s^m)},Y_s^m)d\Lambda_s\right) \\ =\ & -Y_t + \xi + \int_t^T f(s,\eta_s,P_{(Y_s,Z_s)},Y_s,Z_s)ds + \int_t^T g(s,\eta_s,P_{(Y_s)},Y_s)d\Lambda_s \end{aligned}$$

in $L^2(\Omega,\mathcal{F},P)$ and $Z^m\mathbf{1}_{[t,T]} \to Z\mathbf{1}_{[t,T]}$ in $L^2(\Omega,\mathcal{F},\mathcal{H})$. We show (Y,Z) that satisfies (5) on $[t_k,T]$. The next step is to solve the equation on $[t_{k-1},t_k]$. With the same arguments, repeating the above technique we obtain a uniqueness of the solution of generalized BSDE with respect to fBm on the whole interval $[0,T]$. □

Now we would like to study the comparison theorem. From the counter examples in Borkowska (2013) (see the example 3.1 and 3.2 therein) and example 2.1 in Juan, Hao and Zhang (2018) (only need to simple modify), we know that if the driver f depends on the law of Z or is not increasing with respect to the law of Y, we usually do not have the comparison theorem. Now we give two examples here.

Example: Let $d = 1$. We consider

$$Y_t^i = \xi^i + \int_t^T E[|Z_s^i|]ds - \int_t^T Z_s^i dB_s^H, \quad i = 1,2. \quad 0 \leqslant t \leqslant T.$$

For $\xi^2 = 0, (Y^2, Z^2) = (0,0)$, in particular, $Y_0^2 = 0$. We consider two cases for ξ^1.
(i) For $\xi^1 := -((B_T^H)^+)^2 \leqslant 0$, $Z_t^1 := E[D_t^H[\xi^1]|\mathcal{F}_t] = -2E[(B_T^H)^+|\mathcal{F}_t] \leqslant 0$. Thus $E[|Z_t^1|] = E[-Z_t^1] = 2E[(B_T^H)^+] = 2\int_0^\infty x\frac{1}{\sqrt{2\pi}T^H}e^{-\frac{x^2}{2T^{2H}}}dx = \frac{2T^H}{\sqrt{2\pi}}$, $t \in [0,T]$. And $Y_0^1 = E[\xi^1] + \int_t^T E[|Z_s^1|]ds = -\frac{T^{2H}}{2} + \frac{2T^{H+1}}{\sqrt{2\pi}} > 0$, for $T > (\frac{\sqrt{2\pi}}{4})^{\frac{1}{1-H}}$, i.e. for $T > (\frac{\sqrt{2\pi}}{4})^{\frac{1}{1-H}}$, $Y_0^1 > 0 = Y_0^2$, although $\xi^1 \leqslant 0 = \xi^2, P-a.s.$
(ii) For $\xi^1 := -e^{-B_T^H} < 0$, $Z_t^1 := E[D_t^H[\xi^1]|\mathcal{F}_t] = E[e^{-B_T^H}|\mathcal{F}_t] > 0, t \in [0,T]$. Thus $E[|Z_t^1|] = E[Z_t^1] = E[e^{-B_T^H}] = \int_R e^{-x}\frac{1}{\sqrt{2\pi}T^H}e^{-\frac{x^2}{2T^{2H}}}dx = \frac{1}{\sqrt{2\pi}T^H}\int_R e^{-\frac{1}{2T^{2H}}(x+T^{2H})^2}dxe^{\frac{T^{2H}}{2}} = e^{\frac{T^{2H}}{2}}$, $t \in [0,T]$, and $Y_0^1 = E[\xi^1] + \int_t^T E[|Z_s^1|]ds = -e^{\frac{T^{2H}}{2}} + Te^{\frac{T^{2H}}{2}} > 0$, for $T > 1$, i.e. for $T > 1$, $Y_0^1 > 0 = Y_0^2$, although $\xi^1 < 0 = \xi^2, P-a.s.$

We consider now the mean-field BSDE as follows

$$Y_t = \xi + \int_t^T f(s,\eta_s,P_{Y_s},Y_s,Z_s)ds + \int_t^T g(s,\eta_s,P_{Y_s},Y_s)d\Lambda_s - \int_t^T Z_s dB_s^H, \quad 0 \leqslant t \leqslant T. \tag{5}$$

Theorem 4.6. *(Comparison theorem) Let* $(f_i, g_i) = (f_i(s,\omega,\eta,\mu,y,z), g_i(s,\omega,\eta,\nu,y))$, $i = 1,2$, *be two pair drivers satisfying the assumption (H1.4). Moreover, we suppose*

(i) One of the both coefficients pairs satisfies Lipschitz in (μ, y, z) *and* (ν, y).

(ii) One of the both coefficients pairs satisfies: for all $\theta_1, \theta_2 \in L^2(\Omega, \mathcal{F}; \mathbb{R})$, *and all* $(s,\eta,y,z) \in [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d$, $f_i(s,\eta,P_{\theta_1},y,z) - f_i(s,\eta,P_{\theta_2},y,z) \leqslant L(E[((\theta_1-\theta_2)^+)^2])^{1/2}$,

$g_i(s,\eta,P_{\theta_1},y) - g_i(s,\eta,P_{\theta_2},y) \leqslant L(E[((\theta_1-\theta_2)^+)^2])^{1/2}$.

Let $\xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}_T; \mathbb{R})$ *and denote by* (Y^1, Z^1) *and* (Y^2, Z^2) *the solution of the mean-field BSDE (6) with data* (ξ_1, f_1, g_1) *and* (ξ_2, f_2, g_2), *respectively. Then, if* $\xi_1 \leqslant \xi_2, P-a.s.$, $f_1(s,\eta,\mu,y,z) \leqslant f_2(s,\eta,\mu,y,z), dsdP-a.e.$, *and* $g_1(s,\eta,\nu,y) \leqslant g_2(s,\eta,\nu,y), dsdP-a.e.$ *for all* (η,μ,ν,y,z), *it holds that also* $Y_s^1 \leqslant Y_s^2$, *for all* $s \in [0,T], P-a.s.$

Proof. Without loss of generality, we assume that (i) and (ii) are satisfied by (f_1, g_1). Let us put $\overline{f_s} := f_1(s,\eta,P_{Y_s^1},Y_s^1,Z_s^1) - f_2(s,\eta,P_{Y_s^2},Y_s^2,Z_s^2)$, $\overline{g_s} := g_1(s,\eta,P_{Y_s^1},Y_s^1) - g_2(s,\eta,P_{Y_s^2},Y_s^2)$, and $\overline{Z_s} := Z_s^1 - Z_s^2$, $\overline{Y_s} := Y_s^1 - Y_s^2$. From Itô-Tanakas formula applied to $(\overline{Y_s}^+)^2$, we have

$$E[(\overline{Y_s}^+)^2] + E\int_t^T \frac{d}{ds}\|\overline{Z_r}\|_s^2 1_{(\overline{Y_s}>0)}ds = 2E\int_t^T \overline{Y_s}^+ 1_{(\overline{Y_s}>0)}\overline{f_s}ds + 2E\int_t^T \overline{Y_s}^+ 1_{(\overline{Y_s}>0)}\overline{g_s}d\Lambda_s,$$

Notice that, since (f_1, g_1) is Lipschitz continuous and $f_1 \leqslant f_2, g_1 \leqslant g_2$, we have

$$\begin{aligned}
& E[(\overline{Y_s}^+)^2] + E\int_t^T \frac{d}{ds}(\|\overline{Z_r}\|_s^2)1_{(\overline{Y_s}>0)}ds \\
\leqslant\ & 2E\int_t^T \overline{Y_s}^+ 1_{(\overline{Y_s}>0)}\Big(f_1(s,\eta,P_{Y_s^1},Y_s^1,Z_s^1) - f_2(s,\eta,P_{Y_s^2},Y_s^1,Z_s^1) + C|\overline{Y_s}| + C|\overline{Z_s}|\Big)ds \\
& +2E\int_t^T \overline{Y_s}^+ 1_{(\overline{Y_s}>0)}\Big(g_1(s,\eta,P_{Y_s^1},Y_s^1) - g_2(s,\eta,P_{Y_s^2},Y_s^1) + C|\overline{Y_s}|\Big)d\Lambda_s,
\end{aligned}$$

Moreover, as for all $\theta_1, \theta_2 \in L^2(\Omega, \mathcal{F}; \mathbb{R})$ and $(s,\eta,y,z) \in [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d$,

$$f_1(s,\eta,P_{\theta_1},y,z) - f_1(s,\eta,P_{\theta_2},y,z) \leqslant L(E[((\theta_1-\theta_2)^+)^2])^{1/2},$$

$$g_1(s,\eta,P_{\theta_1},y) - g_1(s,\eta,P_{\theta_2},y) \leqslant L(E[((\theta_1-\theta_2)^+)^2])^{1/2}.$$

we have

$$\begin{aligned}
& E[(\overline{Y_s}^+)^2] + E\int_t^T \frac{d}{ds}(\|\overline{Z_r}\|_s^2)1_{(\overline{Y_s}>0)}ds \\
\leqslant\ & CE\int_t^T \overline{Y_s}^+ 1_{(\overline{Y_s}>0)}\Big((E[(\overline{Y_s}^+)^2])^{1/2} + |\overline{Y_s}| + |\overline{Z_s}|\Big)ds \\
& +CE\int_t^T \overline{Y_s}^+ 1_{(\overline{Y_s}>0)}\Big((E[(\overline{Y_s}^+)^2])^{1/2} + |\overline{Y_s}|\Big)d\Lambda_s,
\end{aligned}$$

by Remark 3.1, we obtain, there exists a suitable constant $M > 0$,

$$\frac{2}{M}s^{2H-1}|\overline{Z_s}|^2 \leqslant \frac{d}{ds}(\|\overline{Z_r}\|_s^2) \leqslant 2Ms^{2H-1}|\overline{Z_s}|^2,$$

Thus

$$\begin{aligned}
& E[(\overline{Y_s}^+)^2] + \frac{2}{M}E\int_t^T s^{2H-1}|\overline{Z_s}|^2 1_{(\overline{Y_s}>0)}ds \\
\leqslant\ & CE\int_t^T \overline{Y_s}^+ 1_{(\overline{Y_s}>0)}\Big((E[(\overline{Y_s}^+)^2])^{1/2} + |\overline{Y_s}| + |\overline{Z_s}|\Big)ds \\
& +CE\int_t^T \overline{Y_s}^+ 1_{(\overline{Y_s}>0)}\Big((E[(\overline{Y_s}^+)^2])^{1/2} + |\overline{Y_s}|\Big)d\Lambda_s \\
\leqslant\ & CE\int_t^T (\overline{Y_s}^+)^2 ds + CE\int_t^T (\overline{Y_s}^+)^2 s^{1-2H}ds + CE\int_t^T |\overline{Z_s}|^2 s^{2H-1}1_{(\overline{Y_s}>0)}ds + CE\int_t^T (\overline{Y_s}^+)^2 d\Lambda_s
\end{aligned}$$

$$\leqslant \quad CE\int_t^T (\overline{Y_s}^+)^2(1+p(s)+s^{1-2H})ds + CE\int_t^T |\overline{Z_s}|^2 s^{2H-1}1_{(\overline{Y_s}>0)}ds,$$

the last inequality applies assumption ($H1.4$). Choose suitable M, such that $\frac{2}{M}-C>0$, then we have

$$E[(\overline{Y_s}^+)^2] \leqslant CE\int_t^T (\overline{Y_s}^+)^2(1+p(s)+s^{1-2H})ds,$$

From Gronwall's inequality, $E(\overline{Y_s}^+)^2 = 0$, $s\in[0,T]$, i.e. $Y_s^1 \leqslant Y_s^2$, $P-a.s$, $s\in[0,T]$. □

5. General Mean-Field Fractional BSDEs Under Continuous Coefficients

We assume that the coefficients f and g of the GFBSDE are continuous functions and satisfy the following assumption ($H2$):

(***H*2.1**) Linear growth: There exists $K\geqslant 0$, such that

$$|f(t,\eta,\mu,y,z)| \leqslant K(1+W_2(\mu,\delta_0)+|y|+|\eta|+|z|), dtdP-a.e \quad for \quad all \quad (\eta,\mu,y,z),$$

$$|g(t,\eta,\nu,y)| \leqslant K(1+W_2(\nu,\delta_0)+|y|+|\eta|), dtdP-a.e \quad for \quad all \quad (\eta,\nu,y).$$

where δ_0 is the Dirac measure with mass at $0\in\mathbb{R}^{1+d}$ or $0\in\mathbb{R}^d$.

(***H*2.2**) Monotonicity in μ: for all $\theta_1,\theta_2\in L^2(\Omega,\mathcal{F};\mathbb{R})$, and all $(\eta,y,z)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d$,

$$f(s,\eta,P_{\theta_2},y,z) \leqslant f(s,\eta,P_{\theta_1},y,z), \quad dtdP-a.e, whenever \quad \theta_2\leqslant\theta_1,$$

$$g(s,\eta,P_{\theta_2},y) \leqslant g(s,\eta,P_{\theta_1},y), \quad dtdP-a.e, whenever \quad \theta_2\leqslant\theta_1.$$

(***H*2.3**) For a.e.$(s,\omega)\in[0,T]\times\Omega$, $f(s,\omega,\cdot,\cdot,\cdot,\cdot)$, $g(s,\omega,\cdot,\cdot,\cdot)$ are continuous with a continuity modulus $\rho:\mathbb{R}^+\to\mathbb{R}^+$ for μ:

$$|f(s,\omega,\eta,\mu_1,y,z)-f(s,\omega,\eta,\mu_2,y,z)| + |g(s,\omega,\eta,\nu_1,y)-g(s,\omega,\eta,\nu_2,y)| \leqslant \rho(W_2(\mu_1,\mu_2)).$$

Here ρ is supposed to be increasing and such that $\rho(0+)=0$.

Remark 5.1. *(**H2.2**) is equivalent to the following condition:*

*(**H2.2′**): For all $\mu_1,\mu_2\in\mathcal{P}_2(\mathbb{R})$, $(s,\eta,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d$, it holds $f(s,\eta,\mu_2,y,z)\leqslant f(s,\eta,\mu_1,y,z)$, whenever the distribution functions F_{μ_1}, F_{μ_2} satisfy $F_{\mu_1}\leqslant F_{\mu_2}$. Recall that $F_\mu(x)=\mu((-\infty,x])$, $x\in\mathbb{R}$, $\mu\in\mathcal{P}_2(\mathbb{R})$.*

*Indeed, if we let $\mu_1=P_{\theta_1}$, $\mu_2=P_{\theta_2}$, then from $\theta_2\leqslant\theta_1$, P-a.s., we get $F_{\mu_1}\leqslant F_{\mu_2}$, and (**H2.2′**) implies $f(s,\eta,P_{\theta_2},y,z)\leqslant f(s,\eta,P_{\theta_1},y,z)$. This shows that (**H2.2′**) ⇒ (**H2.2**).*

*In order to show that (**H2.2**)⇒(**H2.2′**): We consider $\mu_1,\mu_2\in\mathcal{P}_2(\mathbb{R})$, with $F_{\mu_1}\leqslant F_{\mu_2}$. Let ξ be a random variable uniformly distributed on $[0,1]$, and let $F_{\mu_i}^{-1}$ be the left inverse function of F_{μ_i}. Then $\theta_2:=F_{\mu_2}^{-1}(\xi)\leqslant F_{\mu_1}^{-1}(\xi)=:\theta_1$, and $P_{\theta_1}=\mu_1$, $P_{\theta_2}=\mu_2$. From (**H2.2**) we get $f(s,\eta,P_{\theta_2},y,z)\leqslant f(s,\eta,P_{\theta_1},y,z)$.*

Before proving the main theorem in this paper, we need the following lemma which gives the approximation of continuous functions by the Lipschitz functions and it was presented by Lepeltier and Martin (1997). we have to introduce a new method to study the relationship between two measures, we define

$$W_{2,+}(\mu,\nu) = \inf_{\pi\in\prod(\mu,\nu)}\left\{\left(\int_{\mathbb{R}^d\times\mathbb{R}^d}|(x-y)^+|^2\pi(dx,dy)\right)^{1/2}\right\},$$

where $\prod(\mu,\nu)$ is the family of all couplings of μ and ν, i.e., $\pi\in\prod(\mu,\nu)$ if and only if π is a measure on $\mathbb{R}^d\times\mathbb{R}^d$ with marginals $\mu,\nu\in\mathcal{P}_2(\mathbb{R}^d)$.

The following Lemma is a modified based on Lemma 3.1 in Li, Liang and Zhang (2018).

Lemma 5.2. *Let $f : [0,T]\times\Omega\times\mathbb{R}\times\mathcal{P}_2(\mathbb{R}^{1+d})\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ be a continuous function in (η,μ,y,z) and satistying $(H2)$, Then the sequence of functions*

$$f_n(s,\omega,\eta,\mu,y,z) := ess \inf_{(\zeta,\nu,r,b)\in\mathbb{R}\times\mathcal{P}_2(\mathbb{R}^{1+d})\times\mathbb{R}\times\mathbb{R}^d}\{f(s,\omega,\zeta,\nu,r,b) + nW_{2,+}(\mu,\nu) + n|\eta-\zeta| + n|y-r| + n|z-b|\}$$

is well defined for $n \geqslant K$ and has the following properties

(i) Linear growth: for all $(s,\omega,\eta,\mu,y,z) \in [0,T]\times\Omega\times\mathbb{R}\times\mathcal{P}_2(\mathbb{R}^{1+d})\times\mathbb{R}\times\mathbb{R}^d$, $|f_n(s,\omega,\eta,\mu,y,z)| \leqslant C(1 + W_2(\mu,\delta_0) + |\eta| + |y| + |z|)$;

(ii) Monotonicity in μ: $f_n(s,\omega,\eta,\mu_2,y,z) \leqslant f_n(s,\omega,\eta,\mu_1,y,z)$, for $\mu_1,\mu_2 \in \mathcal{P}_2(\mathbb{R}^{1+d})$ with $F_{\mu_2} \geqslant F_{\mu_1}$, for all $(s,\omega,\eta,y,z) \in [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d, n \geqslant 1$;

(iii) Monotonicity in n: for any $(s,\omega,\eta,\mu,y,z) \in [0,T]\times\Omega\times\mathbb{R}\times\mathcal{P}_2(\mathbb{R}^{1+d})\times\mathbb{R}\times\mathbb{R}^d$, $n \leqslant m$, $f_n(s,\omega,\eta,\mu,y,z) \leqslant f_m(s,\omega,\eta,\mu,y,z)$;

(iv) Lipschtiz condition: for any $(s,\omega,\eta,\mu,y,z) \in [0,T]\times\Omega\times\mathbb{R}\times\mathcal{P}_2(\mathbb{R}^{1+d})\times\mathbb{R}\times\mathbb{R}^d$, $|f_n(s,\omega,\eta,\mu,y,z) - f_n(s,\omega,\eta_1,\mu_1,y_1,z_1)| \leqslant n(W_2(\mu,\mu_1) + |\eta-\eta_1| + |y-y_1| + |z-z_1|)$;

(v) Strong convergence: If $(\eta_n,\mu_n,y_n,z_n) \to (\eta,\mu,y,z)$ in $\mathbb{R}\times\mathcal{P}_2(\mathbb{R}^{1+d})\times\mathbb{R}\times\mathbb{R}^d$ as $n\to\infty$, then $f_n(s,\omega,\eta_n,\mu_n,y_n,z_n) \to f(s,\omega,\eta,\mu,y,z)$ as $n \to \infty$.

From Lemma 5.2, for fixed s, we consider the sequence $f_n(s,\omega,\eta,\mu,y,z)$, and $g_n(s,\omega,\nu,\mu,y)$ $n \geqslant 1$, related to f and g, respectively. Also consider $h(s,\omega,\eta,\mu,y,z) = K(1 + W_2(\mu,\delta_0) + |\eta| + |y| + |z|)$. It is obvious now that f_n and h are F-progressively measurable functions which are Lipschitz in (μ,y,z), uniformly in (s,ω). For $\xi \in L^2(\Omega,\mathcal{F}_T;\mathbb{R})$ we know from Proposition 4.5, for $n \geqslant K$, that the following mean-field BSDEs have a unique adapted solution

$$Y_t^n = \xi + \int_t^T f_n(s,\eta_s,P_{Y_s^n},Y_s^n,Z_s^n)ds + \int_t^T g_n(s,\eta_s,P_{Y_s^n},Y_s^n)d\Lambda_s - \int_t^T Z_s^n dB_s^H, \quad 0 \leqslant t \leqslant T. \tag{6}$$

$$U_t = |\xi| + \int_t^T h(s,\eta_s,P_{U_s},U_s,V_s)ds + \int_t^T q(s,\eta_s,P_{U_s},U_s)d\Lambda_s - \int_t^T V_s dB_s^H, \quad 0 \leqslant t \leqslant T. \tag{7}$$

From Lemma 5.2, we know that (f_n,g_n) and (h,q) satisfy the assumptions of Proposition 4.5, therefore we have

$$-U_s \leqslant Y_s^m \leqslant Y_s^n \leqslant U_s, P-a.s, \quad s\in[0,T], \quad for \quad all \quad n \geqslant m \geqslant K. \tag{8}$$

The following two Lemmas have been implied in Proposition 4.5.

Lemma 5.3. *There exists a conatant C which depends on K,T and $E[e^{\beta\Lambda_T}\xi^2]$, such that*

$$E\left(e^{\beta\Lambda_s}|Y_t^n|^2 + \int_t^T e^{\beta\Lambda_s}s^{2H-1}|Z_s^n|^2ds + \int_t^T e^{\beta\Lambda_s}|Y_s^n|^2d\Lambda_s\right) \leqslant C,$$

$$E\left(e^{\beta\Lambda_s}|U_t|^2 + \int_t^T e^{\beta\Lambda_s}s^{2H-1}|V_s|^2ds + \int_t^T e^{\beta\Lambda_s}|U_s|^2d\Lambda_s\right) \leqslant C.$$

Lemma 5.4. $(Y^n,Z^n), n \geqslant 1$, *converges in $\mathcal{V}_{[0,T]}^{1/2}\times\mathcal{V}_{[0,T]}^H$.*

Now ,we give the main result of this paper:

Theorem 5.5. *Let $\xi \in L^2(\Omega,\mathcal{F}_T;\mathbb{R})$. Assume $(H2)$ holds. Then equation*

$$Y_t = \xi + \int_t^T f(s,\eta_s,P_{Y_s},Y_s,Z_s)ds + \int_t^T g(s,\eta_s,P_{Y_s},Y_s)d\Lambda_s - \int_t^T Z_s^n dB_s^H, \quad 0 \leqslant t \leqslant T. \tag{9}$$

has an adapted solution (Y,Z). Also, there is a minimal solution (Y^*,Z^*) of (9), in the sense that for any other solution (Y,Z) of (9), we have $Y_s^* \leqslant Y_s$, $s\in[0,T]$, P-a.s. Moreover, for all $t\in[0,T]$,

$$E\left(e^{\beta\Lambda_s}|Y_t|^2 + \int_t^T e^{\beta\Lambda_s}s^{2H-1}|Z_s|^2ds + \int_t^T e^{\beta\Lambda_s}|Y_s|^2d\Lambda_s\right) \leqslant C\Theta(t,T),$$

where

$$\Theta(t,T) := E\left(e^{\beta\Lambda_T}|\xi|^2 + 2\int_t^T e^{\beta\Lambda_s}(1 + E[(Y_s,Z_s)^2])ds + \int_t^T e^{\beta\Lambda_s}|\eta_s|^2ds + 2\int_t^T e^{\beta\Lambda_s}(1 + E[(Y_s)^2])d\Lambda_s\right).$$

Proof. From (8) we have $Y^{n_0} \leqslant Y^n \leqslant U$ for all $n \geqslant n_0 \geqslant K$. Moreover, $Y^n \to Y$ converges in $\mathcal{V}^{1/2}_{[0,T]}$, On the other hand, also $Z^n \to Z$ in $\mathcal{V}^H_{[0,T]}$.

Hence, thanks to (i) and (v) in Lemma 5.2, we get

$$f_n(s, \eta_s, P_{Y^n_s}, Y^n_s, Z^n_s) \to f(s, \eta_s, P_{Y_s}, Y_s, Z_s), \quad n \to \infty,$$

$$g_n(s, \eta_s, P_{Y^n_s}, Y^n_s) \to g(s, \eta_s, P_{Y_s}, Y_s). \quad n \to \infty.$$

Thus

$$E\left(\int_t^T e^{\beta\Lambda_s}|f_n(s, \eta_s, P_{Y^n_s}, Y^n_s, Z^n_s) - f(s, \eta_s, P_{Y_s}, Y_s, Z_s)|^2 ds\right) \to 0, as \quad n \to \infty,$$

and

$$E\left(\int_t^T e^{\beta\Lambda_s}(g_n(s, \eta_s, P_{Y^n_s}, Y^n_s) - g(s, \eta_s, P_{Y_s}, Y_s))d\Lambda_s\right)^2 \to 0, as \quad n \to \infty.$$

From Theorem 2.1, Lemma 4.5 and remark 3.2, we can get

$$\begin{aligned}
E(\int_t^T e^{\beta\Lambda_s}(Z^n_s - Z_s)dB^H_s)^2 &= E\left(\int_t^T e^{2\beta\Lambda_s}(Z^n_s - Z_s)^2 ds + \int_t^T \int_t^T \mathbb{D}^H_r(Z^n_s - Z_s)\mathbb{D}^H_s(Z^n_r - Z_r)drds\right) \\
&= E\left(\int_t^T e^{2\beta\Lambda_s}(Z^n_s - Z_s)^2 ds + 2\int_t^T \int_s^T \mathbb{D}^H_r(Z^n_s - Z_s)\mathbb{D}^H_s(Z^n_r - Z_r)drds\right) \\
&\leqslant E\left(\int_t^T e^{2\beta\Lambda_s}(Z^n_s - Z_s)^2 ds + 2M^2\int_t^T \int_s^T (sr)^{2H-1}(Z^n_s - Z_s)(Z^n_r - Z_r)drds\right), \quad as \quad n \to \infty.
\end{aligned}$$

On the other hand, from the BSDE (6) we can prove similarly that $E[\int_0^T |Y^n_t - Y^m_t|dt] \to 0$ as $n, m \to \infty$. Therefore, Y has a continuous version, i.e. $Y \in \mathcal{V}^{1/2}_{[0,T]}$ and $E[\int_0^T |Y^n_t - Y_t|dt] \to 0$ as $n \to \infty$. Thus, taking the limit in (6), we get that (Y, Z) solves (9).

Let $(\widehat{Y}, \widehat{Z}) \in \mathcal{V}^{1/2}_{[0,T]} \times \mathcal{V}^H_{[0,T]}$ be any solution of (9). From the comparison theorem we get that $Y^n_s \leqslant \widehat{Y}_s, s \in [0, T], P - a.s.$, for all $n \geqslant 1$, and therefore $Y_s \leqslant \widehat{Y}_s$, $s \in [0, T]$, P-a.s., that is, Y is the minimal solution of (9). □

Acknowledgments

This project was sponsored by National Natural Science Foundation of China (11901257) and the Fundamental Research Funds for the Jiangxi Normal University.

References

Bender, C. (2005). Exiplicit solutions of a calss linear fractional BSDEs. *Syst. Control Lett., 54*, 671-680.

Borkowska, K. J. (2013). Generalized BSDEs driven by fractional Brownian motion. *Stat.Prob.Lett., 83*, 805-811.

Biagini, F., Hu, Y., Øksendal, B., & Sulem, A. (2002). A stochastic maximal principle for processes driven by fractional Brownian motion. *Stoch. Process. Appl., 100*, 233-253.

Buckdahn, R., Djehiche, B., Li, J., & Peng, S. (2009). Mean-field backward stochastic differential equations. *A limit approach, Ann. Probab., 37*(4), 1524-1565.

Buckdahn, R., Li, J., & Peng, S. (2009). Mean-field backward stochastic differential equations and related partial differential equations. *Stochastic Process Appl., 119*, 3133-3154.

Buckdahn, R., Li, J., Peng, S., & Rainer, C. (2017). Mean-field stochastic differential equations and associated PDEs. *Ann. Probab., 45*(2), 824-878.

Decreusefond, L., & Uštuňel, A. S. (1998). Stoachstic analysis of the fractional Brownian motion. *Potential Anal., 10*, 177-214.

Duncan, T. E., Hu, Y., & Pasik-Duncan, B. (2000). Stochastic Calculus for fractional Brownian motion. I. Theory. *SIAM. J. Control Option, 38*, 582-621.

Du, H., Li., J., & Wei, Q. (2011). Mean-field backward stochastic differential equations with continuous coefficients, in: Proceedings of the 30th Chinese Control Conference, Yantai, China, July 22-24.

Ferrante, M., & Rovira, C. (2013).stochastic differential with non-negatively constraints driven by fractional Brownian motion. *J. Evol. Equ., 13*, 617-632.

Hu, Y. (2005). Integral transformations and anticipative calculus for fractional Brownian motion. *Mem. Amer. Math. Soc., 825*, viii+127.

Hu, Y., & Peng, S. (2009). Backward stochastic differential equations driven by fractional Brownian motion. *SIAM J. Control Optim., 48*, 1675-1700.

Hao., T., & Li, J. (2016). Mean-field SDEs with jumps and nonlocal integral-PDEs. *NoDEA Nonlinear Differential Equations Appl., 23*(2), 1-51.

Juan, L., & Min, H. (2016). Weak solutions of mean-field stochastic differential equations and application to zero-sum stochastic differential games. *SIAM J. Control Optim., 54*(3), 1826-1858.

Juan, L. (2018). Mean-field forward and backward SDEs with jumps and Associated nonlocal quasi-linear integral-PDEs. *Stochastic Process. Appl., 128*(9), 3118-3180.

Juan, L., Hao, L., & Xiao, Z. (2018). General mean-field BSDEs with continuous coefficients. *Journal of Mathematical Analysis and Applications, 466*, 264-280.

Lions, P. L., & Sznitman, A. S. (1984). Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math., 37,* 511-537.

Lepeltier, J. P., & Martin, J. S. (1997). Backward stochastic differential equations with continuous coefficient. *Statist. Probab. Lett., 32,* 425-430.

Lasry, J. M., & Lions, P. L. (2007). Mean field games. *Jpn. J. Math, 2*(1), 229-260.

Maticiuc, L., & Nie, T. (2015). Fractional backward stochastic differential equations and Fractional backward variational inequalities. *J. Theoret. Probab., 28*, 337-395.

Ma, J., Protter, P., & Yong, J.(1994). Solving forward-backward stochastic differential equations explicitly-a four step scheme. *Probeb. Theo. Relat. Fields, 98*, 339-359 .

Nualart, D. (2006). *Malliavin Calculus and Related Topics* (2nd ed.). Springer-Verlag.

Pardoux, E., & Peng, S. (1992). Backward SDE and quasilinear parabolic PDEs. In: Rozovskii, B. L., Sowers, R. Stochastic Partial Differential Equations and their Applications, LNCIS 176. Springer, New York, 200-217.

Pardoux, E., & Zhang, S. (1998). Generalized BSDEs and non-Neumann boundary value problems. *Probab. Theory Related. Fields, 110*, 535-558.

Pipiras, V., & Taqqu, M. S. (2000). Integration questions related to fractional Brownian motion. *Probab. Theory Related Fields, 118,* 251-291.

Pardoux, é., & Peng, S. (1990). Adapted solution of a backward stochastic differential equation. *Systems Control Letters, 14*, 55-61.

Sow, A. B. (2014). BSDE with jumps and non Lipschitz coefficients: application to large deviations. *Brazilian J. Probab. Stat., 8*, 96-108.

Saisho, Y. (1987). Stochastic differential equations for multidimensional domains with refecting boundary, Probab. *Theory Related Fields, 74*, 455-477.

Young, L. C. (1936). An inquality of the Hölder type connected with Stietjies integration. *Acta Math., 67*, 251-282.

Copyrights

Copyright for this article is retained by the author(s), with first publication rights granted to the journal.

This is an open-access article distributed under the terms and conditions of the Creative Commons Attribution license (http://creativecommons.org/licenses/by/4.0/).

International Journal of Statistics and Probability; Vol. 10, No. 3; May 2021
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

On the Log-Logistic Distribution and Its Generalizations: A Survey

Abdisalam Hassan Muse[1], Samuel M. Mwalili[1,2], Oscar Ngesa[1,2,3]

[1] Mathematics Department, Pan African University, Institute of Basic Sciences, Technology, and Innovation, Kenya

[2] Statistics and Actuarial Sciences Department, Jomo Kenyatta University of Agriculture and Technology, Kenya

[3] Mathematics and Informatics Department, Taita-Taveta University, Kenya

Correspondence: Abdisalam Hassan Muse, Mathematics Department (Statistics Option), Pan African University, Institute of Basic Sciences, Technology, and Innovation, (PAUSTI), P.O. Box 62000 00200 Nairobi, Kenya.

Received: February 4, 2021 Accepted: April 7, 2021 Online Published: April 20, 2021

doi:10.5539/ijsp.v10n3p93 URL: https://doi.org/10.5539/ijsp.v10n3p93

Abstract

In this paper, we present a review on the log-logistic distribution and some of its recent generalizations. We cite more than twenty distributions obtained by different generating families of univariate continuous distributions or compounding methods on the log-logistic distribution. We reviewed some log-logistic mathematical properties, including the eight different functions used to define lifetime distributions. These results were used to obtain the properties of some log-logistic generalizations from linear representations. A real-life data application is presented to compare some of the surveyed distributions.

Keywords: log-logistic distribution, log-logistic generalizations, generalized classes of distributions, construction of new families, censored data, survival analysis

1. Introduction

The log-logistic distribution, also known as Fisk distribution in economics, is one of the important continuous probability distributions with a heavy tail defined by one scale (or one rate) and one shape parameters. The log-logistic distribution is a distribution with a non-negative random variable whose logarithm has the very popular logistic distribution. It was initially introduced to model population growth by (Verhulst, 1838). It is often applied to model random lifetimes, and hence has applications in time-to-event analysis. So; if the original data of variable is $x_1, x_2, x_3, \ldots$, etc., then $log\,(x_1), log(x_2),\ log(x_3), \ldots$, etc. follow logistic distribution. A logarithmic transformation on the logistic. The LL distribution is similar in shape to the 2-parameter log-normal distribution but it is more suitable for use in the time-to-event data analysis since it has heavier tails than the 2-parameter log-normal. The good thing for log-logistic distribution is that it has greater mathematical tractability when dealing with incomplete (or censored) data and also its cumulative distribution can be written in closed form. Log-logistic distribution is particularly applicable to model heavy tailed data in business, medicine, economics, income, wealth, and social sciences. It can also be found in modeling non-monotone (i.e., unimodal) hazard functions.

The log-logistic distribution has various important properties compared to many other parametric distributions used in the field of survival and reliability analysis: (i) it is cumulative distribution function (cdf) has an explicit closed-from expression, which is very useful for analyzing time-to-event data with incomplete information (e.g. censoring and truncation); (ii) it has a similar shape of pdf and hazard function as the log-normal distribution but has heavier-tails and the tail properties are what the inference is based on; (iii) it has a non-monotonic hazard function: the hazard function is unimodal when shape parameter is greater than 1 and is decreasing monotonically when shape parameter is less than or equal to 1; this is what makes to be different from the Weibull distribution; (iv) it has the potential for analysis of time-to-event data whose rate increases initially and decreases later; (v) it is also used to analyse the skewed data; (vi) the LL distribution can be adopted as the basis of an accelerated failure time (AFT) model by allowing the scale parameter α to differ between groups (Reath et al. 2018), (vii) it has also closed under the proportional odds model; and the last but not the least (viii) The generalization of the LL distribution has an attractive feature of being a member to both AFT and Proportional hazard (PH) models. These important properties are what makes that the log-logistic distribution can be viewed as a simple while useful parametric model which can be widely used in many different disciplines, including demography for modeling population growth (Verhulst, 1838); economics for the distribution of wealth or income inequality (Fisk, 1961); engineering for reliability analysis (Ashkar & Mahdi, 2003); and hydrology for modelling stream flow rates and precipitation (Rowinski et al. 2002) and many other fields.

Some other authors who discussed and studied the properties and applications of LL distribution are (Kleiber and Kotz, 2003) studied the application of LL distribution in economics. Collatt (2003) discussed the application of LL distribution in health science for modeling the time following for heart transplantation, Tahir et al (2014) discussed it is useful for modeling censored data usually common in survival and reliability experiments. Other authors who studied the applications of LL distribution are (Prentice, 1976) (Prentice and Kalbfleisch, 1979); (Bennett, 1983); (Singh and George, 1988); (Nandram, 1989); (Diekmann, 1992); (Bacon, 1993); (Little et al. 1994); (BRÜEDERL and Diekmann, 1995); (Gupta et al. 1999) among others. The LL distribution has been widely used in different fields such as actuarial science, economics, survival analysis, reliability analysis, hydrology and engineering. In some cases, the log-logistic distribution is verified to be a good alternative to the log-normal distribution for modeling censored data in survival and reliability analysis due to its mathematical simplicity and it is characterizing increasing hazard rate function. However due to the symmetry of the log-logistic model, it may be poor when the hazard rate is heavily tailed or skewed. Therefore, there is an increasing trend in generalization of the baseline LL distribution by adding an extra shape parameter to the parent (or baseline) distribution or by using other generalization techniques. In the statistical literature, proposing new probability distributions is rich and growing rapidly and various are the papers extending the LL distribution designed to serve as statistical models for a wide range of real lifetime applications with does not follow any of the existing probability distributions.

The remainder of the paper is organized as follows. Section 2 reviews the LL distribution, and the two common parametrization methods for LL distribution and some mathematical properties of the LL distribution are discussed. Section 3 discussed the extensions of the LL distribution. Section 4 methods for generating new families of continuous probability distributions and we cite telegraphically twenty distributions obtained by different generated families and compounding methods on the log-logistic distribution. Section 5 presents the estimation of the parameters. A real-life data application is presented in section 6. Section 7 we discussed the censored data and the G- families of the distributions. concluding remarks and the summary of the work is presented in section 8. Finally, Section 9 we highlight some future works after the survey.

2. Log-logistic Distribution

There are several different parametrizations of the distribution in use. In this study we focused on the two common ones; scale parametrization and rate parametrization. There are several functions related to continuous probability distributions. In this study, we focused on those functions which are related to lifetime distributions as a random variable. The most common ones are; cumulative distribution function (cdf), probability density function (pdf), survival (reliability function), hazard (failure) rate function (HR), cumulative hazard rate (CHR) function, cumulative hazard rate average function (HRA), and the conditional survival function (CSF). The good thing for these functions is that they completely describe the distribution of lifetime, and if you know any of these functions, it is easy to determine the others.

For a random variable X has a log-logistic distribution having shape parameter $\beta > 0,$ and scale parameter $\alpha > 0,$ denoted by $X \sim LL\,(\alpha, \beta)$. The cdf, pdf, survivor function, hazard (failure) rate function, cumulative hazard rate function, reversed hazard rate function, the hazard rate average function, and the conditional survival functions are given by respectively:

$$F(x;\ \alpha,\beta) = \frac{1}{1+\left(\frac{x}{\alpha}\right)^{-\beta}} = \frac{\left(\frac{x}{\alpha}\right)^{\beta}}{1+\left(\frac{x}{\alpha}\right)^{\beta}} = = \frac{x^{\beta}}{\alpha^{\beta}+x^{\beta}} \tag{1}$$

$$f(x;\ \alpha,\beta) = \frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta-1}}{\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{2}} \tag{2}$$

$$s(x;\ \alpha,\beta) = 1-F(x;\ \alpha,\beta) = \frac{\left(\frac{x}{\alpha}\right)^{-\beta}}{1+\left(\frac{x}{\alpha}\right)^{-\beta}} = \frac{1}{1+\left(\frac{x}{\alpha}\right)^{\beta}} \tag{3}$$

$$h(x;\ \alpha,\beta) = \frac{f(x;\ \alpha,\beta)}{S(x;\ \alpha,\beta)} = \frac{f(x;\ \alpha,\beta)}{1-F(x;\ \alpha,\beta)} = \frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta-1}}{1+\left(\frac{x}{\alpha}\right)^{\beta}}, \tag{4}$$

$$H(x) = -logR(x) = \int_0^x h(x)dx, \tag{5}$$

$$r(x;\ \alpha,\beta) = \frac{f(x;\ \alpha,\beta)}{F(x;\ \alpha,\beta)} == \frac{\left[\frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta-1}}{\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^2}\right]}{\left[\frac{\left(\frac{x}{\alpha}\right)^{\beta}}{1+\left(\frac{x}{\alpha}\right)^{\beta}}\right]} = \frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{-1}}{1+\left(\frac{x}{\alpha}\right)^{\beta}} \tag{6}$$

$$HRA(x) = \frac{H(x)}{x} = \frac{\int_0^x h(x)dx.}{x}, x > 0, \tag{7}$$

$$CSF = P(X > x+t|X > t) = R(x|t) = \frac{R(x+t)}{R(x)}, \qquad t > 0, \qquad x > 0\ R(.) > 0, \tag{8}$$

Where $F(x;\ \alpha,\beta)$is the CDF of x analogous to H(x) in HRA(x).

2.1 Alternative Parametrization

An alternative parametrization is given by applying the rate parameter which is commonly used in some families like the exponential distribution (is the reciprocal of the scale parameter α) $\rho = \frac{1}{\alpha}$

Therefore, without loss of generality the cumulative density function, probability density function, survivor function and hazard function of the LL distribution are, respectively

$$F(x) = \frac{1}{1+(\rho x)^{-\beta}}, \tag{9}$$

$$f(x) = \frac{\beta\rho(\rho x)^{\beta-1}}{(1+(\rho x)^{\beta})^2}, \tag{10}$$

$$S(x) = \frac{1}{1+(\rho x)^{\beta}}, \tag{11}$$

$$h(x) = \frac{\beta\rho(\rho x)^{\beta-1}}{1+(\rho x)^{\beta}}, \tag{12}$$

Where$\rho > 0$ and $\beta > 0$ are the unknown parameters, $x > 0$ is the support of the distribution, and $\beta > 0$. ρ is the rate parameter and β is the shape parameter, that shows as that log-logistic distribution is monotone decreasing when $\beta \leq 1$, and is unimodal when $\beta >1$. If T has a LL distribution, then Y = log T has a logistic distribution.

2.2 Mean Residual Life Function

The mean residual life (MRL) function has been widely used in survival and reliability analysis because of its easy interpretability and large area of application. The MRL function computes the expected remaining survival time of a subject given that a component has survived or not failed until time t.

Suppose that $F(0) = 0$ and $\mu \equiv E(X) = \int_0^\infty S(x)dx < \infty$. Then the MRL function for continuous X is computed by

$$m(t|\theta) = E(T-t|T > t) = \frac{\int_t^\infty (x-t)f(x|\theta)dx}{S(t|\theta)} = \frac{\int_t^\infty S(x|\theta)dx}{S(t|\theta)} \tag{13}$$

Where θ represents the parameter vector of (α,β) and $S(.)$ is the survival (reliability) function. and $m(t|\theta) \equiv 0$, whenever $S(t|\theta) = 0$.

For continuous distributions with finite mean, the survival function is defined through the MRL function:

$$S(t|\theta) = \frac{m(0)}{m(t|\theta)} exp - \left[\int_0^t \frac{1}{m(x|\theta)} dx\right] \tag{14}$$

Consider the reliability or survival function of the LL distribution with scale and shape parameters. The mean of the LL distribution is only finite when the shape parameter is greater than 1, thus the mean residual life function is only defined when $\beta > 1$. The mrl for the LL distribution is easily obtained by

$$\int_t^\infty S(x|\theta) = \int_t^\infty \frac{1}{1+(px)^k}dx = \frac{1}{pk}\int_{A(t)}^1 z^{\frac{1}{k}-1}(1-z)^{-1/k}dz = \frac{1}{pk}\left[B\left(\frac{1}{k},1-\frac{1}{k}\right) - B_{A(t)}\left(\frac{1}{k},1-\frac{1}{k}\right)\right],$$

Where

$$A(t) = \frac{(pt)^k}{\{1+(pt)^k\}}$$

And

$$B(p,q) = \int_0^x y^{p-1}(1-y)^{q-1}dy.$$

Hence the MRL function is given by

$$m(t|\theta) = \frac{1}{pk}\left[B\left(\frac{1}{k},1-\frac{1}{k}\right) - B_{A(t)}\left(\frac{1}{k},1-\frac{1}{k}\right)\right]\{1+(pt)^k\},\ \ k>1. \tag{15}$$

The hazard rate function can also be defined by the MRL function

$$h(t|\theta) = \frac{1+m'(t|\theta)}{m(t|\theta)} \tag{16}$$

The critical point t^* of the hazard rate is given by

$$t^* = \frac{(k-1)^{\frac{1}{k}}}{p},$$

The hazard rate increases to its maximum at the point t^* and then steadily decreases. The mean residual life function has the reverse shape of the hazard rate function. This fact was proved by (Gupta and Akman, 1995).

2.3 The Quantile Functions

The quantile functions play a central role in statistical and data analysis. Generally, a probability distribution can be defined either in terms of the distribution function or by the quantile function (Midhu et al. 2013).

The quantile function (inverse CDF) of LL distribution is;

$$X_{q=} F^{-1}(p;\ \alpha,\beta) = \alpha\left(\frac{p}{1-p}\right)^{1/\beta},p\ \in [0,1), \tag{17}$$

It follows that the

Lower quartile is

$$X_{q_1} = 3^{-1/\beta}\alpha, \tag{18}$$

Medium is

$$Medium = X_{q_2} = \alpha, \tag{19}$$

Upper quartile is

$$X_{q_3} = 3^{1/\beta}\alpha, \tag{20}$$

2.4 Moments and the Moment Generating Function

Many important properties and features of a probability distribution can be obtained through its moments, such as mean, variance, kurtosis, and skewness. The essential moment functions, such as the moment generating function, rth moment, rth central moment, are presented.

Theorem 4.3.2: If $T \sim LL\ (\alpha,\beta)$, then the moment generating function, 1st moment, 2nd moment, and rth moments are given, respectively by

$$M_X(t) = \sum\nolimits_{n=0}^{\infty} \frac{(t\alpha)^n}{n!}.\boldsymbol{B}\left(\frac{\beta+n}{\beta},\frac{\beta-n}{\beta}\right) \tag{21}$$

$$M'_X(t) = \sum_{n=0}^{\infty} \frac{nt^{n-1}}{\alpha^n . n!} . \boldsymbol{B}\left(\frac{\beta+n}{\beta}, \frac{\beta-n}{\beta}\right) \tag{22}$$

$$M''_X(t) = \sum_{n=0}^{\infty} \frac{n(n-1)t^{n-2} . \alpha^n}{n!} . \boldsymbol{B}\left(\frac{\beta+n}{\beta}, \frac{\beta-n}{\beta}\right) \tag{23}$$

$$M^r{}_X(t) = \alpha^r . \boldsymbol{B}\left(\frac{\beta+r}{\beta}, \frac{\beta-r}{\beta}\right) \tag{24}$$

Where $\boldsymbol{B}$ is the type-II beta function.

Proof: We have that the Moment Generating Function mgf of T according to (Casella and Berger, 2002) is

$$M_X(t) = \int_{-\infty}^{\infty} e^{\rho t} f(x) dt = \int_{0}^{\infty} e^{tx} f(x) dx = \int_{0}^{\infty} e^{tx} \frac{\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}}{\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^2} dx$$

By using MacLaurin series the equation becomes:

$$M_X(t) = \int_{0}^{\infty} \left(1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \cdots\right) \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta}\left(\frac{x}{\alpha}\right)^{-1}\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-2} dx$$

$$M_X(t) = \int_{0}^{\infty} \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta}\left(\frac{x}{\alpha}\right)^{-1}\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-2} dx$$

$$+ \int_{0}^{\infty} tx \frac{\beta}{\alpha} \frac{(tx)^2}{2!}\left(\frac{x}{\alpha}\right)^{\beta}\left(\frac{t}{\alpha}\right)^{-1}\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-2} dx$$

$$+ \int_{0}^{\infty} \frac{(tx)^2}{2!} \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta}\left(\frac{x}{\alpha}\right)^{-1}\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-2} dx + \int_{0}^{\infty} \frac{(tx)^3}{3!} \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta}\left(\frac{x}{\alpha}\right)^{-1}\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-2} dx +$$

By substituting $y = \left(\frac{x}{\alpha}\right)^{\beta}$, we find $x = y^{\frac{1}{\beta}}\alpha$ with $dx = \frac{\alpha}{\beta} y^{\frac{1}{\beta}-1} dy$. This means that if boundary of $x = 0$ then boundary of $y = 0$ and for $x = \infty$, then $y = \infty$. Then, the moment generating function it can be written as follows:

$$M_X(t) = \int_{0}^{\infty} \frac{1}{(1+y)^2} dy + t\alpha \int_{0}^{\infty} \frac{y^{\frac{1}{\beta}}}{(1+y)^2} dy + (t\alpha)^2 \int_{0}^{\infty} \frac{y^{\frac{2}{\beta}}}{(1+y)^2} dy + (t\alpha)^3 \int_{0}^{\infty} \frac{y^{\frac{3}{\beta}}}{(1+y)^2} dy$$

By applying Beta Function:

$$M_X(t) = \boldsymbol{B}(1,1) + t\alpha \boldsymbol{B}\left(\frac{\beta+1}{\beta}, \frac{\beta-1}{\beta}\right) + \frac{(t\alpha)^2}{2!} \boldsymbol{B}\left(\frac{\beta+1}{\beta}, \frac{\beta-1}{\beta}\right) + \frac{(t\alpha)^3}{3!}\left(\frac{\beta+1}{\beta}, \frac{\beta-1}{\beta}\right) + \cdots$$

Therefore, the mgt of LL distribution is:

$$M_X(t) = \sum_{n=0}^{\infty} \frac{(t\alpha)^n}{n!} . \boldsymbol{B}\left(\frac{\beta+n}{\beta}, \frac{\beta-n}{\beta}\right) \quad \blacksquare$$

By differentiating the mgt that we have before then the 1st and 2nd moments of LL distribution are retrieved as follows:

1st moment of LL distribution:

$$M'_X(t) = \frac{d}{dt} M_X(t)$$

$$M'_X(t) = \frac{d}{dt} \sum_{n=0}^{\infty} \frac{(t\alpha)^n}{n!} . \boldsymbol{B}\left(\frac{\beta+n}{\beta}, \frac{\beta-n}{\beta}\right)$$

$$M'_X(t) = \sum_{n=0}^{\infty} \frac{nt^{n-1} . \alpha^n}{n!} . \boldsymbol{B}\left(\frac{\beta+n}{\beta}, \frac{\beta-n}{\beta}\right) \quad \blacksquare$$

$$M'_X(t=0) = \alpha . \boldsymbol{B}\left(\frac{\beta+1}{\beta}, \frac{\beta-1}{\beta}\right)$$

2^{nd} moment of LL distribution:

$$M''_X(t) = \frac{d^2}{dt^2} M_X(t)$$

$$M''_X(t) = \frac{d^2}{dt^2} \sum_{n=0}^{\infty} \frac{(t\alpha)^n}{n!} . \boldsymbol{B}\left(\frac{\beta+n}{\beta}, \frac{\beta-n}{\beta}\right)$$

$$M''_X(t) = \sum_{n=0}^{\infty} \frac{n(n-1)t^{n-2}.\alpha^n}{n!} . \boldsymbol{B}\left(\frac{\beta+n}{\beta}, \frac{\beta-n}{\beta}\right)$$

$$M''_X(t=0) = \alpha^2 . \boldsymbol{B}\left(\frac{\beta+2}{\beta}, \frac{\beta-2}{\beta}\right)$$ ■

Similarly, the r^{th} moment in general, is

$$M^r{}_X(t) = \frac{d^r}{dt^r} M_X(t)$$

$$M^r{}_X(t) = \frac{d^r}{dt^r} \sum_{n=0}^{\infty} \frac{(t\alpha)^n}{n!} . \boldsymbol{B}\left(\frac{\beta+n}{\beta}, \frac{\beta-n}{\beta}\right)$$

$$M^r{}_X(t) = \sum_{n=0}^{\infty} \frac{n(n-1)(n-2)\ldots\big(n-(r-1)\big)(n-r)t^{n-r}\alpha^n}{n!} . \boldsymbol{B}\left(\frac{\beta+n}{\beta}, \frac{\beta-n}{\beta}\right)$$

$$M^r{}_X(t) = \alpha^r . \boldsymbol{B}\left(\frac{\beta+r}{\beta}, \frac{\beta-r}{\beta}\right)$$ ■

In addition, the mean, variance, skewness and the kurtosis of the log-logistic distribution are given, respectively by The mean of the LL distribution is

$$\mu = E(T) = \alpha . \boldsymbol{B}\left(\frac{\beta+1}{\beta}, \frac{\beta-1}{\beta}\right) \tag{25}$$

The Variance of the LL distribution is

$$\sigma^2 = V(T) = E(T^2) - (E(T))^2$$

$$= \alpha^2 . \boldsymbol{B}\left(\frac{\beta+2}{\beta}, \frac{\beta-2}{\beta}\right) - \left(\alpha . \boldsymbol{B}\left(\frac{\beta+1}{\beta}, \frac{\beta-1}{\beta}\right)\right)^2 \tag{26}$$

$$Skewness[X] = \frac{\left[\boldsymbol{B}\left(\frac{\beta+3}{\beta}, \frac{\beta-3}{\beta}\right) - 3\boldsymbol{B}\left(\frac{\beta+2}{\beta}, \frac{\beta-2}{\beta}\right)\boldsymbol{B}\left(\frac{\beta+1}{\beta}, \frac{\beta-1}{\beta}\right) + 2\left(\boldsymbol{B}\left(\frac{\beta+1}{\beta}, \frac{\beta-1}{\beta}\right)\right)^3\right]}{\left(\boldsymbol{B}\left(\frac{\beta+2}{\beta}, \frac{\beta-2}{\beta}\right) - \left(\boldsymbol{B}\left(\frac{\beta+1}{\beta}, \frac{\beta-1}{\beta}\right)\right)^2\right)^{\frac{3}{2}}} \tag{27}$$

$$Kurtosis[X] == \frac{\left[\begin{array}{c}\boldsymbol{B}\left(\frac{\beta+4}{\beta}, \frac{\beta-4}{\beta}\right) - 4\boldsymbol{B}\left(\frac{\beta+1}{\beta}, \frac{\beta-1}{\beta}\right)\boldsymbol{B}\left(\frac{\beta+3}{\beta}, \frac{\beta-3}{\beta}\right) - 3\left(\boldsymbol{B}\left(\frac{\beta+2}{\beta}, \frac{\beta-2}{\beta}\right)\right)^2 \\ +12\boldsymbol{B}\left(\frac{\beta+2}{\beta}, \frac{\beta-2}{\beta}\right)\left(\boldsymbol{B}\left(\frac{\beta+1}{\beta}, \frac{\beta-1}{\beta}\right)\right)^2 - 6\left(\boldsymbol{B}\left(\frac{\beta+1}{\beta}, \frac{\beta-1}{\beta}\right)\right)^4\end{array}\right]}{\left(\boldsymbol{B}\left(\frac{\beta+2}{\beta}, \frac{\beta-2}{\beta}\right) - \left(\alpha . \boldsymbol{B}\left(\frac{\beta+1}{\beta}, \frac{\beta-1}{\beta}\right)\right)^2\right)^2} \tag{28}$$

2.5 Characteristic Function

Proposition 4.4.5: The characteristic function of the LL distribution $\varphi_X(t)$ is given by

$$\varphi_X(t) = E(e^{itx}) = \int_0^{\infty} e^{itx} f(x) dx \tag{29}$$

As can be seen from the above proposition, the characteristic function of the LL distribution cannot be computed analytically. However, applying the concept of complex analysis; we can express as:

First, since the $e^{itx} = \cos(tx) + i\sin(tx)\ \forall t \in \mathbb{R}$;

$$\varphi_X(t) = \int_0^\infty \cos tx + i\,sin\ \ tx \left\{\frac{\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}}{\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^2}\right\} dx$$

Second, evaluating the above equation into two parts to solve it easily, we get:

i)

$$\int_0^\infty \cos tx \left\{\frac{\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}}{\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^2}\right\} dx$$

ii)

$$\int_0^\infty i\,sin\ \ tx \left\{\frac{\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}}{\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^2}\right\} dx$$

Third, we can transform $sin\ (tx)\ and\ cos\ (tx)$ by applying MacLaurin Series, then we find the results of each part:

i)

$$\boldsymbol{B}(1,1) - \frac{(t\alpha)^2}{2!}\boldsymbol{B}\left(\frac{\beta+2}{\beta},\frac{\beta-2}{\beta}\right) + \frac{(t\alpha)^4}{4!}\boldsymbol{B}\left(\frac{\beta+4}{\beta},\frac{\beta-4}{\beta}\right) + \frac{(t\alpha)^6}{6!}\boldsymbol{B}\left(\frac{\beta+6}{\beta},\frac{\beta-6}{\beta}\right) + \cdots$$

ii)

$$it\alpha\boldsymbol{B}\left(\frac{\beta+1}{\beta},\frac{\beta-1}{\beta}\right) - \frac{(it\alpha)^3}{3!}\boldsymbol{B}\left(\frac{\beta+3}{\beta},\frac{\beta-3}{\beta}\right) + \frac{(it\alpha)^5}{5!}\boldsymbol{B}\left(\frac{\beta+5}{\beta},\frac{\beta-5}{\beta}\right) - \frac{(it\alpha)^7}{7!}\boldsymbol{B}\left(\frac{\beta+7}{\beta},\frac{\beta-7}{\beta}\right) + \cdots$$

Hence, the characteristic function is the sum of the two parts:

$$\varphi_X(t) = part\ i + part\ ii$$

$$\varphi\_X\ (t) = \boldsymbol{B}(1,1) + it\alpha\boldsymbol{B}\left(\frac{\beta+1}{\beta},\frac{\beta-1}{\beta}\right) - \frac{(t\alpha)^2}{2!}\boldsymbol{B}\left(\frac{\beta+2}{\beta},\frac{\beta-2}{\beta}\right) - \frac{(it\alpha)^3}{3!}\boldsymbol{B}\left(\frac{\beta+3}{\beta},\frac{\beta-3}{\beta}\right) + \frac{(t\alpha)^4}{4!}\boldsymbol{B}\left(\frac{\beta+4}{\beta},\frac{\beta-4}{\beta}\right) + \frac{(it\alpha)^5}{5!}\boldsymbol{B}\left(\frac{\beta+5}{\beta},\frac{\beta-5}{\beta}\right) - \cdots$$

Therefore, the characteristic function of the LL distribution is:

$$\varphi_X(t) = \sum_{n=0}^{\infty}\frac{(it\alpha)^n}{n!}\boldsymbol{B}\left(\frac{\beta+n}{\beta},\frac{\beta-n}{\beta}\right) \tag{30}$$

We can also derive the norm characteristic function of the LL distribution:

$$|\varphi_X(t)| = \boldsymbol{B}(1,1) = 1 \tag{31}$$

Since the value of the norm characteristic function of the LL distribution equals 1. It shows us that the characteristic function of the LL distribution is a finite function.

More details about the log-logistic distribution can be found in (Ekawati et al. 2015)

3. Extensions of Log-Logistic Distribution

In recent years, the ability to propose new probability models to deal with reliability and survival analysis has increased.

Many extensions (or generalizations) of the log-logistic distribution have been proposed in the last two decades. In terms of applications, the log-logistic distribution and its generalizations have become the most popular models for survival and reliability data. Some recent applications have included: modeling for AIDS and Melanoma data (de Santana, Ortega, Cordeiro, & Silva, 2012); used for minification process (Gui, 2013); modeling breast cancer data (Ramos et al. 2013); (Tahir et al. 2014); modeling on censored survival data (Lemonte, 2014); modeling time up to first calving of cows (Louzada & Granzotto, 2016); modeling, inference, and use to a polled Tabapua Race Time up to First Calving Data (Granzotto et al. 2017); modeling positive real data in many areas (Lima & Cordeiro, 2017); analysing a right-censored data (Shakhatreh, 2018); modeling lung cancer data (Alshangiti, et al. 2016); and modeling of breaking stress data (Aldahlan, 2020).

Because of the increasing interest in terms of applications and methodology, we feel it is timely that a survey is provided of the log-logistic distribution and its generalizations. In this study, we provide such a review. We review in the following sections nearly twenty generalizations. For each generalized one, we try to give expressions for the cdf, pdf, reliability (or survival) function, the failure (or hazard) rate function, the reversed hazard function, the quantile function and the cumulative hazard function. Sometimes not all of these functions are given if they are not stated in the original source or if nice closed form expressions are not known.

In the statistics and probability, the literature on statistical theory abounds in surveys of topics of current interest. Ghitany (1998) provided thorough reviews of the recent modifications of the gamma distribution. Pham and Lai (2007) provided through reviews of the recent generalizations of the Weibull distribution. Nadarajah (2013) provided an extensive survey of the exponentiated Weibull distribution. Li and Nadarajah (2020) provided an extensive review of the student's t distribution and its generalizations. Rahman et al. (2020) provided an expository review of the transmuted probability distributions. Tomy et al. 2020) provided an extensive review of the recent generalizations of the exponential distribution. Dey, et al. (2021) provided an extensive review of the transmuted distributions. We contribute to the literature by reviewing the log-logistic distribution and its generalizations. While it is common to come across reviews which deal with generalizations of an existing distributions, our approach is significantly different. Our focus is on the method. To be the best of our knowledge, there is no other work which attempts to bring together at one place nearly twenty generalizations (or extensions) of the LL distribution. In this work, we have reviewed only univariate log-logistic distribution and related distributions. A future work is to review bivariate, multivariate, matrix variate and complex variate log-logistic distribution.

4. Review of the Methods for Building New Log-Logistic Distributions

In this section, we present up-to-date review of the methods for building new families of continuous probability distributions. In probability and applied statisticians have shown great interest in building and generating new generalized probability models that extend well-known probability distributions and are more flexible for data modeling in many different disciplines of applications.in recent decades, some different extensions of continuous distributions have received great attention in the recent literature. Gupta and Kundu (2009) discussed six different techniques for the induction of shape/skewness parameter(s) in probability distributions namely: (1) method of proportional hazard model, (2) method of proportional reversed hazard model, (3) method of proportional cumulants model, (4) method of proportional odds model, (5) method of power transformed model, and (6) method of Azzalini's skewed model. On the other hand, Lee et al. (2013), reviewed the different methods of generating new probability distributions and they focused on the two main techniques; adding parameters and combining existing probability distributions. On the other hand, they discussed three methods developed before 1980s, and they are: (1) method of differential equation, (2) method of transformation, and (3) method of quantile. Then they discussed five generating techniques developed since 1980s, those are : (1) method of adding parameters to an existing distribution, (2) composite method, (3) the beta-generated method, (4) the transformed-transformer (T-X) method, and (5) methods of generating skewed distributions. Tahir and Nadarajah (2015) wrote a review of methods for generating probability distributions with more than 300 reference papers, most of these distributions were introduced in the recent years, (Nadarajah et al. 2013); (Tahir and Cordeiro (2015), (Ahmad et al. 2019) and (De Brito et al. 2019) gives an excellent review on the recent developments of univariate continuous distributions.

This study work offers a survey of recently different methods for developing families of probability distributions. The technique of obtaining a new generalized distribution is of different forms; generally speaking, the five methods developed since 1980s, can be named them as a combination methods because of the reason because of the reason that these techniques attempt to add an extra parameters to an existing probability distribution or combining existing distributions into new distributions (Lee et al., 2013). In this work, we will categorize into two headings; (1) generator method or Parameter Induction method, and (2) Compound method.

4.1 Generator Method

There are several different methods described in the literature used to extend well-known probability distributions. Probably, one of the most popular method is to consider distribution generators and is called generator method (also known as parameter induction method). The method deals with an induction a shape parameter(s) to a baseline (or parent) distribution to improve goodness-of-fits and to explore tail properties. In the generator method, we can categorize into; exponentiated-G family, beta-G family, gamma-G family, Marshall-Olkin class, Kumaraswamy class, transmuted family, cubic transmuted family, a general transmuted family, alpha-power transformation, T-X family method, Zubair-G family, and the Cordeiro-Tahir's family.

4.1.1 The Exponentiated-G family of Distributions

The exponentiated-G family of distributions can be traced back to Gompertz (1825), (Verhulst 1838;1845;1847) and Lehmann (1953). It is one of the simplest methods for parameter induction techniques. The method adds one shape parameter to an existing distribution. If $G(x)$ is the cumulative distribution function (cdf) of the baseline model, then

$$F(x) = G(x)^{\gamma}, \gamma > 0 \tag{32}$$

$F(x)$ is also the cdf of the new distribution and it is called the exponentiated-G distribution (Exp-G distribution) with exponent parameter γ.

The distribution G is the baseline distribution and γ is a positive real parameter. The variable x can take any of the form

$$x = x - \mu \; or \; x = \frac{x-\mu}{\sigma} \; or \; x = k\left(\frac{x-\mu}{\sigma}\right) \; or \; x = k\left(\frac{x-\mu}{\sigma}\right)^{\frac{1}{\sigma}} .$$

The probability density function (pdf) corresponding to (32) is

$$f(x) = \gamma(x)G(x)^{\gamma-1}, \tag{33}$$

These family of distributions became famous after the papers by (Mudholkar & Srivastava, 1993) exponentiated-Weibull distribution, Gupta et al. (1998) named the family to the proportional reversed hazard rate (PHR) model, and (Gupta and Kundu 1999, 2001; 2002) exponentiated -exponential distribution. This family of distributions is considered in many papers in the literature such as (Mudholkar et al. 1995); (Mudholkar & Hutson, 1996), (Choudhury, 2005), (Singh et al. 2005), (Nadarajah and Kotz 2006), (Barrios & Dios, 2012), (Shakil & Ahsanullah, 2012) and (Gholam, 2013), among many others.

Exponentiated log-logistic distribution

This new extended distribution was developed by (Rosaiah et al. 2006), (Aslam & Jun, 2010), (Rao et al. 2012) and (Chaudhary & Kumar, 2014). The cdf of Exponentiated log-logistic distribution (ELL) is given as

$$F_{ELL}(x) = P_{ELL}(\{y|y \le x\}) = \left[\frac{\left(\frac{x}{\alpha}\right)^{\beta}}{1+\left(\frac{x}{\alpha}\right)^{\beta}}\right]^{\gamma} \; for\, x \ge 0 \tag{34}$$

where $\alpha, \beta, and\, \gamma > 0$ are the unknown parameters of the model.

The ELL is obtained from the CDF of the log-logistic by $F(x) = G(x)^{\gamma}$. The pdf of ELL is as follows:

$$f_{ELL}(x) = \frac{\beta\gamma\left(\frac{x}{\alpha}\right)^{\beta\gamma-1}}{\alpha\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{\gamma+1}} \; for\, x \ge 0 \tag{35}$$

The ELL distribution may be taken as a parametric model for survival analysis, if the lifetimes show a large variability. For any lifetime random variable t, the reliability (survival) functions $S(t)$, the hazard rate function $h(t)$, the reversed hazard rate function $r(t)$ and the cumulative hazard rate function $H(t)$, associated with (34) and (35) are

$$S_{ELL}(t) = 1 - F_{ELL}(t) \tag{36}$$

$$h_{ELL}(t) = \frac{f_{ELL}(x)}{1 - F_{ELL}(t)} \tag{37}$$

$$r_{ELL}(t) = \frac{f_{ELL}(x)}{F_{ELL}(x)} \tag{38}$$

$$H_{ELL}(t) = -\log[1 - F_{ELL}(t)] \tag{39}$$

Two-parameter Exponentiated log-logistic distribution:

Chaudhary (2007, 2019) studied the exponentiated log-logistic distribution regarding to the standard log-logistic distribution (when $\alpha = 1$) and called two-parameter exponentiated log-logistic distribution.

If the cdf of the standard LL distribution is

$$F(x;\ \beta) = \frac{x^{\beta}}{1 + x^{\beta}}, x < 0, \qquad \beta > 0 \tag{40}$$

Then the cdf of the two-parameter exponentiated LL distribution is given by

$$F(x;\ \beta) = \left\{\frac{x^{\beta}}{1 + x^{\beta}}\right\}^{\gamma}, x < 0, \qquad \beta > 0 \tag{41}$$

the pdf corresponding to (41) is given by

$$f(x) = \frac{\beta\gamma x^{\beta\gamma}}{x\ [1 + x^{\beta}]^{\gamma+1}}\ for\ x > 0 \tag{42}$$

The survivor function, hazard rate function and the quantile function are given by respectively

$$h(x) = \frac{\beta\gamma x^{\beta\gamma}}{\left[1 - \left\{\frac{x^{\beta}}{1 + x^{\beta}}\right\}^{\gamma}\right]}\ for\ x > 0 \tag{43}$$

$$S(x) = 1 - \left\{\frac{x^{\beta}}{1 + x^{\beta}}\right\}^{\gamma}\ for\ x > 0 \tag{44}$$

$$x_q = \left(q^{-\frac{1}{\gamma}} - 1\right)^{-\frac{1}{\beta}} \qquad for\ 0 < q > 1. \tag{45}$$

4.1.2 The Beta-G Family of Distributions

Beta distribution is a continuous probability distribution with two positive shape parameters, $a\ and\ b.$ It is the natural extension of the uniform distribution and the prior of the binomial distribution. It can rescale and shift to create a new probability distribution with a wide range of shapes and apply for a different application.

Eugene et al. (2002) and Jones (2004)were introduced the Beta-G family of distributions based on the parameter induction technique. The generator method that Eugene et al. (2002) proposed is as follows. For any continuous baseline cumulative distribution function cdf

$$G(x) = G(x, \theta)$$

where θ is the parameter vector, the cdf of the beta-G, $F(x)$ say, is given by

$$F(x) = I_{G(x)}(a, b) = \frac{1}{\boldsymbol{B}(a, b)}\int_0^{G(x)} x^{a-1}(1 - x)^{b-1}dx, \tag{46}$$

Where $a, b > 0$ are additional shape parameters to those in θ that aim to provide greater flexibility of its tails and to introduce skewness. $x \in (0,1)$, $\boldsymbol{B}(a, b) = \int_0^{\infty} x^{a-1}(1 - x)^{b-1}dx$ represents the beta function and $I_{G(x)}(a, b) = \frac{1}{\boldsymbol{B}(a,b)}\int_0^{x} x^{a-1}(1 - x)^{b-1}dx$ represents the incomplete beta function ratio, $\boldsymbol{B}(a, b) = \int_0^{1} x^{a-1}(1 - x)^{b-1}dx = \frac{\Gamma(a)+\Gamma(b)}{\Gamma(a+b)}$ is the beta function, $\boldsymbol{\Gamma}(.)$ is the gamma function, and $\boldsymbol{B}(a, b) = \int_0^{x} x^{a-1}(1 - x)^{b-1}dx$ is the incomplete beta function.

The pdf of the beta-G family of distribution takes the form

$$f(x) = \frac{1}{B(a, b)}g(x)G(x)^{a-1}[1 - G(x)]^{b-1} \tag{47}$$

The beta-G family is also called the beta logit family. These family of distributions become much more popular after Eugene et al. (2002) .This class is studied and discussed the estimation methods and the characterization by maximum entropy by (Zografos & Balakrishnan, 2009) and the moments from beta-G family are studied by (Cordeiro & Nadarajah, 2011)

Beta log-logistic distribution

Lemonte, (2014) proposed and studied the beta log-logistic distribution. The new distribution is quite flexible to model and analyze positive real data. The cdf of the beta log-logistic (BLL) distribution is given by

$$F_{BLL}(t) = I_{\frac{x^{\beta}}{\alpha^{\beta}+x^{\beta}}}(a,b) \tag{48}$$

The pdf corresponding to (48) is

$$f_{BLL}(x) = \frac{\left(\frac{\beta}{\alpha}\right)}{B(a,b)} \frac{\left(\frac{x}{\alpha}\right)^{a\beta-1}}{\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{a+b}}, \qquad x > 0. \tag{49}$$

For any lifetime random variable t, the reliability (survival) functions $S(t)$, the hazard rate function $h(t)$, the reversed hazard rate function $r(t)$ and the cumulative hazard rate function $H(t)$, associated with (48) and (49) are

$$S(t) = 1 - I_{G(x)}(a,b) = 1 - I_{\frac{x^{\beta}}{\alpha^{\beta}+x^{\beta}}}(a,b) \tag{50}$$

$$h(t) = \frac{f_{BLL}(x)}{1 - F_{BLL}(t)} \tag{51}$$

$$r(t) = \frac{f_{BLL}(x)}{F_{BLL}(x)} \tag{52}$$

$$H(t) = -\log[1 - F_{BLL}(t)] \tag{53}$$

This extension distribution can be used in many fields, like economics, reliability analysis in engineering, survival analysis, hydrology, and other disciplines as the LL distribution.

4.1.3 The Gamma-G Family of Distributions

Zografos and Balakrishnan (2009)proposed a simple generator approach (or parameter induction technique). If $G(x,\theta)$, $g(x,\theta)$, and $S(x,\theta)$ are the cdf, pdf and survival function of the baseline distribution respectively. Then the cdf of the gamma-G family is given by

$$F(x,\theta) = \frac{1}{\Gamma(\delta)} \int_0^{-logS(x,\theta)} t^{\delta-1} e^{-t} dt, \quad \delta > 0 \tag{54}$$

And the pdf of the gamma-G family is given by

$$f(x,\theta) = \frac{1}{\Gamma(\delta)} [-logS(x,\theta)]^{\delta-1} g(x,\theta). \tag{55}$$

The corresponding hazard (or failure) rate function is given by

$$h(x,\theta) = \frac{1}{\Gamma\big(\delta, -logS(x,\theta)\big)} [-logS(x,\theta)]^{\delta-1} g(x,\theta). \tag{56}$$

Another gamma-G family was proposed by (Ristic and Balakrishnan, 2012) which is slightly different from the above generator. The cdf and pdf of their generator was defined by

$$F(x,\theta) = 1 - \frac{1}{\Gamma(\delta)} \int_0^{-logS(x,\theta)} t^{\delta-1} e^{-t} dt, \quad \delta > 0 \tag{57}$$

And the pdf of the gamma-G family is given by

$$f(x,\theta) = \frac{1}{\Gamma(\delta)} [-logF(x,\theta)]^{\delta-1} g(x,\theta). \tag{58}$$

The corresponding hazard function is given by

$$h(x,\theta) = \frac{1}{\Gamma(\delta, -logF(x,\theta))}[-logF(x,\theta)]^{\delta-1} g(x,\theta). \quad (59)$$

Nadarajah et al. (2015b) studied the mathematical and statistical properties of the Zografos-Balakrishnan-G family of distributions.

Zografos-Balakrishnan log-logistic distribution

Ramos et al. (2013) proposed a gamma log-logistic distribution by using the Zografos-Balakrishnan -G technique and they called the Zografos-Balakrishnan Log-logistic distribution (ZB-G). The ZB-G family is a LL distribution plus an extra shape parameter $\delta > 0.$ The pdf and cdf of the Zografos-Balakrishnan Log-logistic distribution is given by using the equations of (1) and (2):

$$f(x,\theta) = \frac{\beta}{\alpha^{\beta}\Gamma(\delta)} x^{\beta-1}\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-2}\left\{\log\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]\right\}^{\delta-1}. \quad (60)$$

And the corresponding cdf is given by

$$F(x,\theta) = \frac{1}{\Gamma(\delta)}\int_0^{log\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]} t^{\delta-1}e^{-t}dt, \quad (61)$$

The method for Ristic and Balakrishnan (2012) is an alternative method that can be used to extend the LL distribution and it can be an open research question.

4.1.4 The Marshall-Olkin Family of Distributions

Marshall and Olkin, (1997) introduced a simple generator approach of adding a single parameter to a family of well-known distributions and several authors applied their technique to generalize the well-known probability distributions in the last two decades.

If $G(x,\theta)$ and $S(x,\theta)$ are the cdf and survival function of the baseline distribution depending on the vector parameter , and $\gamma > 0$ is an additional parameter known as tilt parameter. Then the survival function of Marshall and Olkin (MO) family is given by

$$S_{MO}(x;\,\gamma,\theta) = \frac{\gamma.S(x,\theta)}{1-\bar{\gamma}\,S(x,\theta)}, \qquad \theta,\gamma > 0, x\in\mathbb{R}. \quad (62)$$

Where $\bar{\gamma} = 1-\gamma$. Note that, if $\gamma = 1$, then $S_{MO}(x,\gamma,\theta) = S(x,\theta)$ which means that we obtain the baseline distribution. Applying (62), the generalized versions of the well-known distribution have been proposed.

The pdf corresponding to (62) is given by

$$f(x;\,\gamma,\theta) = \frac{\gamma g(x;\,\theta)}{\{1-\bar{\gamma}\,S(x,\theta)\}^2}, \quad (63)$$

And the hazard rate function h(t) is given by

$$h(x;\,\gamma,\theta) = \frac{1}{1-\bar{\gamma}\,S(x,\theta)} h(x;v=\theta), \qquad x\in\mathbb{R} \quad (64)$$

Marshall-Olkin log-logistic distribution

Gui (2013) introduced the Marshall-Olkin Log-logistic distribution (MOLL) and studied the mathematical and statistical properties of the proposed model and used it to models of time series. The survival function, pdf, and the hazard rate function of the proposed distribution are given by:

$$S_{MOLL}(x;\alpha,\beta,\gamma) = \frac{\gamma.\alpha^{\beta}}{x^{\beta}+\gamma.\alpha^{\beta}}, \qquad \alpha,\beta,\gamma,x > 0, \quad (65)$$

$$f_{MOLL}(x;\alpha,\beta,\gamma) = \frac{\alpha^{\beta}\beta\gamma x^{\beta-1}}{(x^{\beta}+\gamma.\alpha^{\beta})^2}, \qquad \alpha,\beta,\gamma,x > 0, \quad (66)$$

$$h_{MOLL}(x;\alpha,\beta,\gamma) = \frac{\beta x^{\beta-1}}{x^{\beta}+\gamma.\alpha^{\beta}}, \qquad \alpha,\beta,\gamma,x > 0, \quad (67)$$

Other authors who studied the further results involving reliability analysis, the estimation of the parameters and the uses of the Marshall-Olkin log-logistic distribution are (Alshangiti et al. 2016), and (Shakhatreh, 2018), (Nasiru et al.

2019)

4.1.5 The Alpha Power Transformation

Mahdavi and Kundu (2017) proposed a new generator technique that many authors applied to introduce for new statistical distributions to increase flexibility of the given family. The technique adds a new parameter to the baseline distribution. The cdf of Alpha power transformation (AP) is defined as

$$F_{APT}(x;\ \theta,\gamma) = \frac{\gamma^{G(x;\theta)} - 1}{\gamma - 1}{}' \quad \theta,\gamma > 0\ ,\gamma \neq 1, x \in \mathbb{R}. \tag{68}$$

where $G(x;\theta)$ is the cdf of the baseline distribution and θ is the vector parameter.

The pdf corresponding to (68) is given as

$$F_{APT}(x;\ \theta,\gamma) = \frac{\log(\gamma)\,\gamma^{G(x;\theta)}}{\gamma - 1} g(x;\theta); \qquad \theta,\gamma > 0\ ,\gamma \neq 1, x \in \mathbb{R}. \tag{69}$$

Two-parameter Alpha Power Transformed Log-logistic distribution

Several researchers have applied the alpha power technique to extend the log-logistic distribution.

Malik and Ahmad (2020) proposed the two-parameter alpha power log-logistic distribution (APLL) of two unknown parameters θ scale parameters and γ shape parameter. Note that, Malik and Ahmad (2020) extended the standard log-logistic distribution (where the shape parameter equals 1).

If the cdf of the standard LL distribution takes the form:

$$F(x;\theta) = \frac{x^{\theta}}{1 + x^{\theta}}, \tag{70}$$

Then the cdf of the APLL is given by

$$F_{APLL}(x;\ \theta,\gamma) = \frac{\gamma\left(\frac{x^{\theta}}{1 + x^{\theta}}\right) - 1}{\gamma - 1} \qquad \theta,\gamma > 0\ ,\gamma \neq 1, x \in \mathbb{R}. \tag{71}$$

The pdf corresponding to (71) is given by

$$f_{APLL}(x;\ \theta,\gamma) = \frac{\log(\gamma)}{\gamma - 1}\frac{\theta x^{\theta-1}}{(1 + x^{\theta})^2}\gamma^{\left(\frac{x^{\theta}}{1+x^{\theta}}\right)}; \qquad \theta,\gamma > 0\ ,\gamma \neq 1, x \in \mathbb{R}. \tag{72}$$

The survival function of the APLL is defined as

$$S_{APLL}(x;\ \theta,\gamma) = \gamma^{\left(\frac{x^{\theta}}{1+x^{\theta}}\right)\left[\frac{\gamma\left(\frac{1}{1+x^{\theta}}\right)-1}{\gamma-1}\right]}; \qquad \theta,\gamma > 0\ ,\gamma \neq 1, x \in \mathbb{R}. \tag{73}$$

The hazard rate function of the APLL is given by

$$h_{APLL}(x;\ \theta,\gamma) = \log(\gamma)\frac{\theta x^{\theta-1}}{(1 + x^{\theta})^2}\frac{1}{\gamma\left(\frac{1}{1 + x^{\theta}}\right) - 1}; \qquad \theta,\gamma > 0\ ,\gamma \neq 1, x \in \mathbb{R}. \tag{74}$$

The reverse hazard rate function of the APLL is given by

$$r_{APLL}(x;\ \theta,\gamma) = \log(\gamma)\frac{\theta x^{\theta-1}}{(1 + x^{\theta})^2}\frac{\gamma\left(\frac{1}{1 + x^{\theta}}\right)}{\gamma\left(\frac{1}{1 + x^{\theta}}\right) - 1}; \qquad \theta,\gamma > 0\ ,\gamma \neq 1, x \in \mathbb{R}. \tag{75}$$

The Alpha Power Transformed Log-logistic distribution

Aldahlan (2020) proposed an Alpha Power Transformed Log-logistic distribution (APTLL) and studied the mathematical and statistical properties of the new distribution. Aldahlan (2020) extended the two-parameter log-logistic distribution. If the cdf of the two-parameter baseline log-logistic distribution takes the form:

$$F(x;\alpha,\beta) = \frac{1}{1 + \left(\frac{x}{\alpha}\right)^{-\beta}}, \tag{76}$$

Then the cdf of the APTLL is given by

$$F_{APTLL}(x;\ \alpha,\beta,\gamma) = \frac{\gamma^{\left(\frac{1}{1+\left(\frac{x}{\alpha}\right)^{-\beta}}\right)} - 1}{\gamma - 1} \qquad \alpha,\beta,\gamma > 0\ , \gamma \neq 1, x > 0. \tag{77}$$

The pdf corresponding to (77) is given by

$$f_{APTLL}(x;\ \alpha,\beta,\gamma) = \frac{\log(\gamma)}{\alpha^{\beta}(\gamma-1)} \frac{\beta x^{\beta-1}}{\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{2}} \gamma^{\left(\frac{1}{1+\left(\frac{x}{\alpha}\right)^{-\beta}}\right)}; \qquad \alpha,\beta,\gamma > 0\ , \gamma \neq 1, x > 0. \tag{78}$$

The survival function of the APTLL is defined as

$$S_{APTLL}(x;\alpha,\beta,\gamma) = \frac{\gamma - \gamma^{\left(\frac{1}{1+\left(\frac{x}{\alpha}\right)^{-\beta}}\right)}}{\gamma - 1}; \qquad \alpha,\beta,\gamma > 0\ , \gamma \neq 1, x > 0. \tag{79}$$

The hazard rate function of the APTLL is given by

$$h_{APLL}(x;\alpha,\beta,\gamma) = \frac{\frac{\alpha^{-\beta}\log(\gamma)\beta x^{\beta-1}}{\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{2}} \gamma^{\left(\frac{1}{1+\left(\frac{x}{\alpha}\right)^{-\beta}}\right)}}{\gamma - \gamma^{\left(\frac{1}{1+\left(\frac{x}{\alpha}\right)^{-\beta}}\right)}}; \qquad \alpha,\beta,\gamma > 0\ , \gamma \neq 1, \qquad x > 0. \tag{80}$$

The reverse hazard rate function of the APTLL is given by

$$r_{APTLL}(x;\ \alpha,\beta,\gamma) = \frac{\frac{\alpha^{-\beta}\log(\gamma)\beta x^{\beta-1}}{\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{2}} \gamma^{\left(\frac{1}{1+\left(\frac{x}{\alpha}\right)^{-\beta}}\right)}}{\gamma^{\left(\frac{1}{1+\left(\frac{x}{\alpha}\right)^{-\beta}}\right)} - 1}; \qquad \alpha,\beta,\gamma > 0\ , \gamma \neq 1, x > 0. \tag{81}$$

The cumulative hazard rate function of the APTLL is given by

$$H_{APTLL}(x;\ \alpha,\beta,\gamma) = -\log\left[\frac{\gamma - \gamma^{\left(\frac{1}{1+\left(\frac{x}{\alpha}\right)^{-\beta}}\right)}}{\gamma - 1}\right]; \qquad \alpha,\beta,\gamma > 0\ , \gamma \neq 1, x > 0. \tag{82}$$

4.1.6 The Kumaraswamy-G Family of Distributions

Kumaraswamy Distribution

Kumaraswamy (1980) introduced a new two-parameter continuous distribution on (0,1) named Kumaraswamy distribution. Kumaraswamy distribution has the cdf and pdf of the form

$$G(x) = 1 - (1 - x^{a})^{b}, \tag{83}$$

where $x \in (0,1)$ and $a\ and\ b$ are both shape parameters.

$$g(x) = a\, b\, x^{a-1}(1 - x^{a})^{b-1}, \tag{84}$$

The Kumaraswamy distribution has the same basic shape properties to the beta distribution; where (1) $a > 1\ and\ b > 1$ unimodal; (2) $a < 1\ and\ b < 1$ bathtub; (3) $a \leq 1\ and\ b > 1$ decreasing; (4) $a > 1\ and\ b \leq 1$ increasing, and (4) $a = b = 1$ constant.

Kumaraswamy-G family of distributions

Jones (2009) and Cordeiro and de Castro (2011) introduced the Kumaraswamy-G family of distributions by extending the beta-G family of distributions by applying Kumaraswamy distribution as a generator instead of the beta generator. That is; the cdf of the Kumaraswamy-G of distributions is derived by replacing the cdf in (46) by the Kumaraswamy distribution as the following:

The cdf of the Kumaraswamy-G family of distributions

$$F_{KU-G}(x) = 1 - [1 - G(x)^a]^b, \tag{85}$$

The pdf corresponding to (85) is given by

$$f_{KU-G}(x) = a\, b\, g(x) G(x)^{a-1}[1 - G(x)^a]^{b-1}, \tag{86}$$

Where $x > 0,\ a, b > 0$ are the two extra shape parameters in addition to those in the baseline model whose role are to govern tail weights and skewness. The pdf of the Kumaraswamy-G family has many similar properties to the beta-G family, but has some advantages in terms of mathematical tractability, since it doesn't involve any specific function such as the beta function.

Kumaraswamy log-logistic distribution

de Santana et al. (2012) and Muthulakshmi, (2013) proposed and studied the mathematical and statistical properties of the Kumaraswamy log-logistic distribution. de Santana et al. (2012) proposed a new distribution that contain several essential distributions as sub-models to the extended distribution.

The cdf and pdf of the Kumaraswamy Log-logistic distribution (KULL) are defined as

$$F_{KULL}(x) = \left[\left(1 - \frac{1}{1 + \left(\frac{x}{\alpha}\right)^{\beta}}\right)^{a}\right]^{b}, \tag{87}$$

$$f_{KULL}(x) = \frac{a\, b\, \beta}{\alpha^{a\beta}} x^{a\beta - 1}\left[1 + \left(\frac{x}{\alpha}\right)^{\beta}\right]^{-(a+1)}\left\{1 - \left[1 - \frac{1}{1 + \left(\frac{x}{\alpha}\right)^{\beta}}\right]^{a}\right\}^{b-1}, \qquad x > 0 \tag{88}$$

The survival function corresponding to (87) is

$$S_{KULL}(x) = 1 - F_{KULL}(x) = \left[1 - \left(1 - \frac{1}{1 + \left(\frac{x}{\alpha}\right)^{\beta}}\right)^{a}\right]^{b}, \tag{89}$$

The hazard function corresponding to (87) is

$$h_{KULL}(x) = \frac{a\, b\, \beta}{\alpha^{a\beta}} x^{a\beta - 1}\left[1 + \left(\frac{x}{\alpha}\right)^{\beta}\right]^{-(a+1)}\left\{1 - \left[1 - \frac{1}{1 + \left(\frac{x}{\alpha}\right)^{\beta}}\right]^{a}\right\}^{-1}, \qquad x > 0 \tag{90}$$

where α is the scale parameter, and the shape parameters $a, b, and\ \beta$ govern the skewness of (87).

4.1.7 The McDonald-G family of Distributions

McDonald Distribution

McDonald, (2008) introduced a new distribution called McDonald distribution with cdf of the form

$$F(x;\ a, b, c) = I_{x^c}(ac^{-1}, b) = \frac{B_{x^c}(ac^{-1}, b)}{B(ac^{-1}, b)} = \frac{1}{B(ac^{-1}, b)}\int_0^{x^c} x^{ac^{-1}-1}(1 - x)^{b-1} dx \tag{91}$$

where $a, b, c > 0$ are the three shape parameters. Some of the special cases of the Mc distribution includes the beta type 1 distribution (c=1) and the Kumaraswamy distribution (a=1).

The pdf corresponding to (91) is

$$f(x;\ a,b,c) = \frac{c}{B(ac^{-1},b)} x^{a-1}(1-x^c)^{b-1}, 0 < x < 1 \tag{92}$$

McDonald-G family of distributions

Alexander et al. (2012) introduced the McDonald-G family of distributions by replacing the upper limit x^c of the integral in equation (91) with $G(x^c)$.

For any baseline cumulative distribution function (cdf) $G(x)$, the resulting cdf $F(x)$ of the Mc-generalized family of distribution Mc-G is

$$F_{McG}(x,\theta) = I_{G(x)^c}(ac^{-1},b) = \frac{1}{B(ac^{-1},b)} \int_0^{G(x)^c} x^{ac^{-1}-1}(1-x)^{b-1}dx \tag{93}$$

where the $I_{G(x)^c}(a,b)$ is the incomplete beta function ratio and $I_{G(x)^c}(a,b) = \frac{B_{G(x)^c}(ac^{-1},b)}{B(ac^{-1},b)}$

The pdf corresponding to (93) is

$$f_{McG}(x,\theta) = \frac{c}{B(ac^{-1},b)} g(x)\{G(x)\}^{a-1}\{1-G(x)^c\}^{b-1} \tag{94}$$

Lemonte and Cordeiro (2013) stated that this method of parameter induction facilitates the computation of several statistical and mathematical properties of the G family of probability distributions.

McDonald Log-logistic distribution

Tahir et al. (2014) introduced and studied the McDonald log-logistic distribution and they considered three of the above extended models of log-logistic distribution, (1) beta log-logistic distribution; (2) Kumaraswamy log-logistic distribution; and (3) gamma log-logistic or Zografos-Balakrishnan log-logistic distribution.

Using the equations of the cdf and pdf of the McDonald-G family, they obtained the CDF and pdf of the McDonald log-logistic distribution.

The cdf of the McDonald is given by

$$F_{McLL}(x,\theta) = I_w(ac^{-1},b) = \frac{1}{B(a,b)} \int_0^{w^c} w^{ac^{-1}-1}(1-w)^{b-1}dw \tag{95}$$

Where $w = CDF\ of\ LL\ distribution = \left[1+\left(\frac{x}{\alpha}\right)^{-\beta}\right]^{-1}$ and $a,b,c,\alpha,\beta > 0$, where $a,b,c, and\ \beta$ are shape parameters while α is a scale parameter.

The corresponding pdf of the (95) is given by

$$f_{McLL}(x,\theta) = \frac{c}{B(ac^{-1},b)}\left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{a\beta-1}\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-(a+1)}\left[1-\left\{1-\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-1}\right\}^{c}\right]^{b-1}, \tag{96}$$

For a lifetime random variable t, the survivor function, the hazard (failure) rate function, the reversed hazard rate function, the cumulative hazard rate function and the quantile function of the McDLL distribution are given by

$$S(t,\theta) = 1 - F(t,\theta) = 1 - I_w(ac^{-1},b) \tag{97}$$

$$h(t,\theta) = \frac{c\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{a\beta-1}\left[1-\left\{1-\left[1+\left(\frac{t}{\alpha}\right)^{\beta}\right]^{-1}\right\}^{c}\right]^{b-1}}{B(ac^{-1},b)\left[1+\left(\frac{t}{\alpha}\right)^{\beta}\right]^{(a+1)}[I_w(ac^{-1},b)\,]} \tag{98}$$

$$r(t,\theta)=\frac{c\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\alpha\beta-1}\left[1-\left\{1-\left[1+\left(\frac{t}{\alpha}\right)^{\beta}\right]^{-1}\right\}^{c}\right]^{b-1}}{B(ac^{-1},b)\left[1+\left(\frac{t}{\alpha}\right)^{\beta}\right]^{(\alpha+1)}\left[1-I_w(ac^{-1},b)\right]} \tag{99}$$

$$H(t,\theta)=\int_0^t h(t,\theta)dt=-\log S(t,\theta)=-\log[1-I_w(ac^{-1},b)] \tag{100}$$

And

$$X_q=\left\{\left[\left(I_q^{-1}(ac^{-1},b)\right)^{-\frac{1}{c}}-1\right]\alpha\right\}^{-\frac{1}{\beta}} \tag{101}$$

4.1.8 The Transmuted Family of Distributions

Shaw and Buckley (2009) introduced a new method of parameter induction (generator technique) and several researchers applied their method to extend and generalize the well-known probability distributions in the last decade. They named the family as quadratic transmuted family of distributions and they used it to financial mathematics and other applied fields.

For any baseline distribution with cdf $G(x;\theta)$ depending on the vector parameter θ, then the CDF of the transmuted family is given by

$$F(x;\gamma,\theta)=(1+\gamma)G(x;\theta)-\gamma G(x;\theta)^2, \tag{102}$$

where $\theta>0, |\gamma|\geq 1, x\in\mathbb{R}$. θ is the vector parameter, γ is the extra shape parameter. When $\gamma=0$, we obtain the baseline distribution, i.e., $F(x;\gamma,\theta)=G(x;\theta)$.

The pdf corresponding to (102) is given by using differentiation

$$f(x;\gamma,\theta)=g(x;\gamma,\theta)[1+\gamma-2\gamma G(x;\theta)], \tag{103}$$

Aryal and Tsokos (2009, 2011) first highlight the method in (114) and proposed a couple of transmuted probability distributions that would offer more distributional flexibility in reliability and environmental analysis. For detail about this family we refer to (Bourguignon et al. 2016)and (Alizadeh et al. 2017) who studied the general properties of this family, while (Tahir and Cordeiro, 2015) have introduced a list of quadratic transmuted family of distributions. Rahman et al. (2020) have provided an up-to-date list of popular transmuted -G classes of distributions.

Transmuted Log-logistic distribution

Aryal (2013) proposed and studied some of the statistical and mathematical properties of the transmuted log-logistic distribution. using the equations (102) and (103); they derived the cdf and pdf of the transmuted LL distribution as follows:

$$F_{TLL}(x;\gamma,\theta)=\frac{(1+\gamma)\alpha^{\beta}x^{\beta}+x^{2\beta}}{(\alpha^{\beta}+x^{\beta})^2}, x>0 \tag{104}$$

The pdf of the transmuted log-logistic distribution is given by

$$f_{TLL}(x;\gamma,\theta)=\frac{\beta\alpha^{\beta}x^{\beta-1}\left[(1+\gamma)(\alpha^{\beta}+x^{\beta})+2\gamma x^{\beta}\right]}{(\alpha^{\beta}+x^{\beta})^3}, x>0 \tag{105}$$

More details about the transmuted LL distribution and its applications to real data sets can be found in (Granzotto and Louzada, 2015); (Louzada and Granzotto, 2016); and (Adeyinka, 2019).

4.1.9 The Cubic Transmuted Family of Distributions

Granzotto et al. (2017) introduced a new parameter induction technique of generating probability distributions called Cubic Transmutation technique. The reason behind developing of cubic transmuted family was that the quadratic transmuted distribution captures the complexity of unimodal data but the real-life data become more complex to use them, and sometimes cannot be fitted by applying the quadratic transmuted family.

For any baseline distribution with cdf $G(x;\theta)$ depending on the vector parameter θ, then the CDF of the cubic transmuted family is given by

$$F(x;\gamma,\theta)=\gamma_1 G(x;\theta)+(\gamma_2-\gamma_1)G(x;\theta)^2+(1-\gamma_2)G(x;\theta)^3, \qquad x\in\mathbb{R}, \tag{106}$$

where $\gamma_1\in[0,1], and\ \gamma_2\in[-1,1]$.

The pdf corresponding to (106) is given by

$$f(x;\gamma,\theta) = g(x;\theta)[\gamma_1 + 2\,(\gamma_2 - \gamma_1)G(x;\theta) + 3(1-\gamma_2)G(x;\theta)^2], \qquad x \in \mathbb{R}, \tag{107}$$

Rahman et al. (2019, 2020) proposed two new cubic transmuted families of distributions. On the other hand, Aslam et al. (2018)introduced another cubic transmuted-G family of distributions using the T-X idea of (Alzaatreh et al, 2013).

Rahman et al. (2020) have provided an up-to-date list of popular cubic transmuted -G classes of distributions. In general, the cubic transmuted distributions show better flexibility to handle multi-modal data than the quadratic transmuted distributions.

Cubic transmuted Log-logistic distribution

Granzotto et al. (2017) proposed the cubic transmuted log-logistic distribution. they used the transmuted log-logistic distribution to derive the pdf of the cubic transmuted log-logistic distribution.

The pdf of the cubic transmuted log-logistic distribution is given by

$$f_{CTLL}(x;\gamma,\theta) = \frac{\beta x^{\beta-1}e^{\alpha}}{(1+e^{\alpha}x^{\beta})^4}\left[\gamma_1\left(1-e^{2\alpha}x^{2\beta}\right) + \gamma_2 e^{\alpha}x^{\beta}\left(2-e^{\alpha}x^{\beta}\right) + 3e^{2\alpha}x^{2\beta}\right], x > 0 \tag{108}$$

The cdf corresponding to (108) is given by

$$F_{CTLL}(x;\gamma,\theta) = \frac{x^{\beta}e^{\alpha}}{(1+e^{\alpha}x^{\beta})}\left[\gamma_1 + (\gamma_2 - \gamma_1)\frac{x^{\beta}e^{\alpha}}{(1+e^{\alpha}x^{\beta})} + (1-\gamma_2)\frac{x^{2\beta}e^{2\alpha}}{(1+e^{\alpha}x^{\beta})^2}\right], x > 0 \tag{109}$$

4.1.10 The General Transmuted Family of Distributions

Merovci et al. (2016) proposed and studied the statistical and mathematical properties of the generalized transmuted family of distributions. (Alizadeh et al. 2017) introduced a new generalized transmuted family of distributions and have described it as a linear combination of exponentiated densities in terms of the same parent distribution. Recently, Rahman et al. (2018) introduced a couple of new general transmuted family of distributions and they named k-transmuted families and they defined by

$$F(x;\theta) = G(x;\theta) + [1 - G(x;\theta)]\sum_{i=1}^{k}\gamma_i[G(x;\theta)]^i, x\ x \in \mathbb{R}, \tag{110}$$

where $\gamma_i \in [-1,1]$ for $i = 1,2,\ldots,k$ and $-k \le \sum_{i=1}^{k}\gamma_i \le 1$, and

$$F(x;\theta) = G(x;\theta) + G(x;\theta)\sum_{i=1}^{k}\gamma_i[G(x;\theta)]^i, x\ x \in \mathbb{R}, \tag{111}$$

where $\gamma_1 \in [-1,1]$ and $\gamma_i \in [-1,1]$for $i = 1,2,\ldots,k$.

AL-Kadim (2018) introduced a generalized family of transmuted distribution which turned out to be a special case of family (110).

4.1.11 The Weibull-G Family of Distributions

Alzaatreh et al. (2013) and Bourguignon et al. (2014) proposed the Weibull-G family of continuous probability distributions. Weibull-G family is an interesting technique of inducting an extra shape parameter(s) to an existing G distribution. Considering the cdf of the Weibull distribution which is given by

$$F(x) = 1 - e^{-\alpha x^{\beta}}, x > 0 \tag{112}$$

where $\alpha, \beta > 0$ are the unknown parameters of the Weibull distribution.

They defined the cdf of the Weibull-G family by replacing $x\ with\ G(x;\theta)/S(x;\theta)$ $[S(x;\theta) = 1 - G(x;\theta),]$, the cdf is given by

$$F(x;\alpha,\beta,\theta) = \int_0^{\frac{G(x;\theta)}{1-G(x;\theta)}}\alpha\beta t^{\beta-1}e^{-\alpha t^{\beta}}dt \ = 1 - \exp\left\{-\alpha\left[\frac{G(x;\theta)}{S(x;\theta)}\right]^{\beta}\right\}, x \in D \subseteq \mathbb{R};\ \alpha,\beta > 0, \tag{113}$$

Where $G(x;\theta)$ is the baseline cdf, which depends on a parameter vector $\boldsymbol{\theta}' = (\alpha,\beta)$.

The pdf corresponding to (113) is given by

$$f(x;\alpha,\beta,\theta)=\alpha\beta g(x;\theta)\frac{G(x;\theta)^{\beta-1}}{S(x;\theta)^{\beta+1}}exp\left\{-\alpha\left[\frac{G(x;\theta)}{S(x;\theta)}\right]^{\beta}\right\}, \tag{114}$$

The hazard rate function of the Weibull-G family is given by

$$h(x;\alpha,\beta,\theta)=\frac{\alpha\beta g(x;\theta)G(x;\theta)^{\beta-1}}{S(x;\theta)^{\beta+1}}=\frac{\alpha\beta G(x;\theta)^{\beta-1}}{S(x;\theta)^{\beta}}h(x;\theta), \tag{115}$$

where$h(x;\theta)=g(x;\theta)/S(x;\theta)$. $the\ multiplying\ quantity\ \alpha\beta G(x;\theta)^{\beta-1}/S(x;\theta)^{\beta+1}$ works as a corrected factor for the failure rate function of the parent (or baseline) distribution.

The Weibull-G family can deal with general situations in modeling and analysing time-to-event data with different shapes of the hazard (or failure) rate function. For example, if the baseline distribution is a log-logistic distribution the

$$\frac{G(x;\theta)^{\beta-1}}{S(x;\theta)^{\beta+1}}=\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-1},x>0 \tag{116}$$

Weibull-Log-logistic Distribution

Oluyede et al. (2016)proposed the log-logistic Weibull distribution and they applied it into a lifetime data. Considering a series system and assuming that the lifetime of the component follow the log-logistic and Weibull distribution with survival functions $S_1(t;\theta)=\left[1+\left(\frac{t}{s}\right)^{c}\right]^{-1}$, $and\ S_2(t;\theta)=e^{-\alpha t^{\beta}}$, respectively. The survival $S(t)=P(T>t)$ of the system is given by

$$S(t)=\prod_{i=1}^{2}S_i(t) \tag{117}$$

The cdf of the log-logistic-Weibull distribution is given by

$$F(x)=1-\left(1+\left(\frac{x}{s}\right)^{c}\right)^{-1}e^{-\alpha x^{\beta}}, \tag{118}$$

The corresponding pdf is given by

$$f(x)=e^{-\alpha x^{\beta}}\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-1}\left\{\alpha\beta x^{\beta-1}+\frac{cx^{c-1}}{(s^{c}+x^{c})}\right\}, \tag{119}$$

Where $s,c,\alpha,\beta>0\ and\ x\geq 0.$

4.1.12 The Exponentiated Generalized Family of Distributions

Cordeiro et al. (2013) proposed a new family of distributions, called the exponentiated generalized (EG) class of distributions. This method belongs to the parameter induction method, where the purpose is to add two new extra shape parameters to the baseline (or parameter) distribution.

The cdf of the EG class of distribution is given by

$$F_{EG}(x;\theta)=\{1-[1-G(x)]^{a}\}^{b},x\in\mathbb{R}, \tag{120}$$

where $a>0, and\ b>0$ are two extra parameters whose role is to govern the skewness and create distributions with heavier/lighter tails.

The pdf corresponding to (120) is given by

$$f_{EG}(x;\theta)=ab[1-G(x)]^{a-1}\{1-[1-G(x)]^{a}\}^{b-1}g(x),x\in\mathbb{R}, \tag{121}$$

where $g(x)=\frac{dG(x)}{dx}$ is the pdf of the baseline (or parent) distribution. the two parameters in (121) can add entropy to the center of the Exponentiated Generalized density or possible control both tail weights.

Exponentiated generalized log-logistic distribution

Lima and Cordeiro (2017) studied the mathematical and statistical properties of a new four-parameter survival model applying the exponentiated generalized (EG) class.

The cdf of the new distribution is given by

$$F_{EGLL}(x;\theta)=\left\{1-\left[1-\frac{x^{\beta}}{\alpha^{\beta}+x^{\beta}}\right]^{a}\right\}^{b},x\in\mathbb{R}, \tag{122}$$

The pdf corresponding to (122) is given by

$$f_{EGLL}(x;\theta)=\frac{ab\beta\left(\frac{x}{\alpha}\right)^{\beta-1}}{\alpha\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{2}}\left[1-\frac{x^{\beta}}{\alpha^{\beta}+x^{\beta}}\right]^{a-1}\left\{1-\left[1-\frac{x^{\beta}}{\alpha^{\beta}+x^{\beta}}\right]^{a}\right\}^{b-1},x\in\mathbb{R}, \tag{123}$$

The hazard rate function of the EGLL is given by

$$h_{EGLL}(x;\theta)=\frac{ab\beta\left(\frac{x}{\alpha}\right)^{\beta-1}\left[1-\frac{x^{\beta}}{\alpha^{\beta}+x^{\beta}}\right]^{a-1}\left\{1-\left[1-\frac{x^{\beta}}{\alpha^{\beta}+x^{\beta}}\right]^{a}\right\}^{b-1}}{\alpha\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{2}\left\{1-\left[1-\frac{x^{\beta}}{\alpha^{\beta}+x^{\beta}}\right]^{a}\right\}^{b}},x\in\mathbb{R}, \tag{124}$$

The quantile function of EGLL is given by inverting (122)

$$x_q=\frac{\alpha^{\beta}\left\{1-\left[1-q^{\frac{1}{\beta}}\right]^{\frac{1}{\alpha}}\right\}}{\left[1-q^{\frac{1}{\beta}}\right]^{\frac{1}{\alpha}}},x\in\mathbb{R}, \tag{125}$$

4.1.13 The T-X Family of Distributions

Alzaatreh et al. (2013) proposed a new family of distributions, called the T-X family of distributions as an extension to the beta-G family of distributions introduced by (Eugene et al., 2002).

If the cdf of the baseline distribution is $G(x;\theta)$ then the cdf of the T-X family is given as

$$F(x)=\int_{a}^{WF(x)}r(t)dt=\{RW(F(x))\}, \tag{126}$$

where R is the cdf of the baseline random variable, and the function $W(F(x)$ must satisfy the following conditions:

1. $W(F(x)\in[a,b]$,
2. W is differentiable and monotonically non-decreasing
3. $W(F(x)\to a$ as $x\to-\infty$ and $W(F(x)\to b$ as $x\to\infty$

Where $[a,b]$ is the support of the baseline random variable for $-\infty\le a<b\le\infty$.

The pdf corresponding to (126) if it exists is given by

$$f(x)\to\left\{\frac{d}{dx}WF(x)\right\}r\{W(F(x))\}. \tag{127}$$

Alzaatreh and Ghosh (2015) introduced three sub-families of the T-X family which are beta-exponential-X family, gamma-X family, and the Weibull-X family. These three sub-families demonstrate that the T-X family consists of many sub-classes of distributions within each sub-family, one can introduce many new distributions as well as relateits members to many existing probability distributions. More details about the T-X family of distributions and their modifications we can refer to (Nasiru, 2018).

4.1.14 The Weibull-X Family of Distributions

Alzaatreh and Ghosh (2015) proposed the Weibull-X family of distributions. Using the concept of T-X family of distributions introduced by Alzaatreh et al. (2013).

The cdf and the pdf of the T-X family of distributions are given by;

$$F(x) = \int_{a}^{-\log[1-G(x)]} r(t)dt \quad (128)$$

$$f(x) \rightarrow \frac{g(x)}{1-G(x)} r\{-\log[1-G(x)] = h_g(x) r H_f(x). \quad (129)$$

Where $h_g(x)$ and $H_f(x)$ are the hazard rate and the cumulative hazard rate functions associated with $g(x)$.

If a random variable T follows the Weibull distribution with a pdf of

$$r(t) = \left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1} e^{-\left(\frac{t}{\alpha}\right)^{\beta}}, t > 0 \quad (130)$$

The pdf and cdf of the Weibull-X family of distributions is given by

$$f(x) \rightarrow \frac{\beta g(x)}{\alpha(1-G(x))}\left(\frac{-\log[1-G(x)]}{\alpha}\right)^{\beta-1} exp\left\{-\left(\frac{-\log[1-G(x)]}{\alpha}\right)^{\beta}\right\} \quad (131)$$

$$F(x) = 1 - exp\left\{-\left(\frac{-\log[1-G(x)]}{\alpha}\right)^{\beta}\right\}, x \in \mathbb{R} \quad (132)$$

5.1.15 The Exponentiated Kumaraswamy-G family

Lemonte et al. (2013)proposed the exponentiated Kumaraswamy-G family of distributions. The cdf and the pdf of the EK family are given by

$$F(x;\theta) = \left[1-(1-x^{\alpha})^{\beta}\right]^{\gamma}, x \in (0,1) \quad (133)$$

$$f(x;\theta) = \alpha\beta\gamma x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\left[1-(1-x^{\alpha})^{\beta-1}\right]^{\gamma-1}, x \in (0,1) \quad (134)$$

where $\alpha\beta$ and γ are extra positive shape parameters. The distribution (134) provides more options for analysing data restricted to the interval (0,1).

5.1.16 The Generalized Weibull-G Family of Probability Distributions

Cordeiro et al. (2015)proposed a new generalized Weibul-G family of probability distributions and they studied the mathematical and statistical properties of the family and some special models.

The cdf of the generalized Weibull family is given by

$$F(x;\alpha,\beta,\theta) = \alpha\beta \int_{0}^{-\log[1-G(x;\theta)]} t^{\alpha-1} e^{-\alpha t^{\beta}} dt = 1 - \exp\left\{-\alpha(-\log[1-G(x;\theta)])^{\beta}\right\} \quad (135)$$

The pdf corresponding to (130) is given by

$$f(x;\alpha,\beta,\theta) = \frac{\alpha\beta g(x;\theta)}{[1-G(x;\theta)]}\{-\log[1-G(x;\theta)]\}^{\beta-1} exp\left\{-\alpha(-\log[1-G(x;\theta)])^{\beta}\right\} \quad (136)$$

The generalized Weibull-Log-logistic Distribution

Cordeiro et al. (2015) introduced the generalized Weibull log-logistic distribution (GWLL).

the pdf of the GWLL with a log-logistic distribution having a scale parameter of a and a shape parameter of b is given by;

$$f(x) = \frac{\alpha\beta a x^{b-1}}{b^{a}}\left(1+\left(\frac{x}{b}\right)^{a}\right)^{-1}\left\{\log\left(1+\left(\frac{x}{b}\right)^{a}\right)\right\}^{\beta-1} exp\left\{-\alpha\left(\log\left[\left(1+\left(\frac{x}{b}\right)^{a}\right)\right]\right)^{\beta}\right\} \quad (137)$$

The literature on the generalized log-logistic distribution using generator (or parameter induction) methods is enrich enough and is also rapidly improving. We now describe the generalization of the log-logistic distribution using compounding and other methods in the following.

5.1.17 The Gamma Uniform Family of Probability Distributions

Torabi and Hedesh (2016) proposed a new generator approach called the gamma-uniform family of distributions. the cdf of the family is given by

$$F(x) = \alpha\beta \int_{0}^{\frac{x-a}{b-x}} \frac{e^{-\frac{w}{\beta}} w^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} dw = 1 - \mathrm{Q}\left\{\alpha \frac{x-a}{\beta(b-x)}\right\}, a < x < b. \quad (138)$$

The pdf and hazard rate function corresponding to (138) are given by

$$f(x) = \frac{(b-a)e^{-\frac{x-a}{\beta(b-x)}}\left(\frac{x-a}{b-x}\right)^{\alpha-1}}{(\mathrm{b}-\mathrm{x})^2\Gamma(\alpha)\beta^\alpha}, a < x < b \tag{139}$$

$$h(x) = \frac{(b-a)e^{-\frac{x-a}{\beta(b-x)}}\left(\frac{x-a}{b-x}\right)^{\alpha-1}}{(\mathrm{b}-\mathrm{x})^2\Gamma(\alpha,\frac{x-a}{\beta(b-x)})\beta^\alpha}, a < x < b \tag{140}$$

This distribution can be applied for modeling any data set with changing the extra parameters.

4.2 Compounding Methods

The compounding method is a technique that combine two or more existing distributions

4.2.1 The Exponentiated G-Geometric Class

Nadarajah et al. (2015)introduced the exponentiated G-geometric also known as the generalized G-geometric class. If $T[G(x;\ \tau);\ \gamma] = G(x;\tau)^\gamma$ and N be a geometric random variable with failure probability parameter $p \in (0,1)$ and probability mass function $\mathbb{P}(N=n) = qp^{n-1}, n = 1,2,\dots,p.$ where p is the success probability.

If we define $X = min(Y_1,\dots,Y_N)$, the unconditional of cdf of the exponentiated G-geometric (EGG) class can follow as

$$F_{EGG}(x;\theta) = \sum_{n=1}^{\infty}(1-p)p^{n-1}[1-(1-G(x;\tau)^\gamma)^n] = \frac{G(x;\tau)^\gamma}{\{1-p[1-G(x;\tau)^\gamma]\}} \tag{141}$$

Where $\gamma > 0, 0 < p < 1$ are two extra shape parameters. This new family can also be obtained by modification or an extension of Marshall-Olkin family of distribution by replacing the baseline CDF with its exponentiation and adding two more extra shape parameters to the original distribution.

The pdf corresponding to (141) is given by

$$f_{EGG}(x;\theta) = \frac{\gamma(1-p)G^{\gamma-1}(x;\tau)g(x;\tau)}{\{1-p[1-G(x;\tau)^\gamma]\}^2} \tag{142}$$

The exponentiated log-logistic geometric distribution

Mendoza et al. (2016)proposed and studied the exponentiated log-logistic geometric distribution. they derived some of the mathematical and statistical properties of the EEG Log-logistic distribution.

If $\{Y_i\}_{i=1}^{Z}$ are an i.i.d random variables having the exponentiated log-logistic distribution with cdf and pdf

$$F_{ELL}(x) = P_{ELL}(\{y|y \le x\}) = \left[\frac{\left(\frac{x}{\alpha}\right)^\beta}{1+\left(\frac{x}{\alpha}\right)^\beta}\right]^\gamma \ for\ x \ge 0 \tag{143}$$

$$f_{ELL}(x) = \frac{\beta\gamma\left(\frac{x}{\alpha}\right)^{\beta\gamma-1}}{\alpha\ \left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^{\gamma+1}} \ for\ x \ge 0 \tag{144}$$

where $\alpha, \beta, and\ \gamma > 0$ are the unknown parameters of the model. For $\gamma = 1$, we obtain as a special case the LL distribution.

Then,

- The conditional density function of EGG log-logistic function for $X_1 = \min(\{Y_i\}_{i=1}^{Z})$: is

$$f(x|z;\gamma,\alpha,\beta) = \frac{z\beta\gamma}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta\gamma-1}\left[1+\left(\frac{x}{\alpha}\right)^\beta\right]^{-(\gamma+1)}\left\{1-\ \left[1+\left(\frac{x}{\alpha}\right)^{-\beta}\right]^{-\gamma}\right\}^{z-1} \tag{145}$$

The pdf of the exponentiated log-logistic geometric type I (ELLGI) distribution reduces to

$$f_{ELLGI}(x;\ \theta) = \frac{(1-p)\beta\gamma}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta\gamma-1}\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-(\gamma+1)}\left\{1-p\left[1-\left\{1+\left(\frac{x}{\alpha}\right)^{-\beta}\right\}^{-\gamma}\right]\right\}^{-2} \tag{146}$$

Where $x > 0$, θ is the vector of parameters $((\alpha, \beta, \gamma > 0, and\ p \in (0,1)$..

The cdf corresponding to (146) is given by

$$F_{ELLGI}(x;\ \theta) = \left[1+\left(\frac{x}{\alpha}\right)^{-\beta}\right]^{-\gamma}\left\{1-p\left[1-\left\{1+\left(\frac{x}{\alpha}\right)^{-\beta}\right\}^{-\gamma}\right]\right\}^{-1}, x > 0 \tag{147}$$

The hazard rate function corresponding to (146) is given by

$$h_{ELLGI}(x;\ \theta) = \frac{(1-p)\beta\gamma}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta\gamma-1}\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-(\gamma+1)}\left\{1-p\left[1-\left\{1+\left(\frac{x}{\alpha}\right)^{-\beta}\right\}^{-\gamma}\right]\right\}^{-1}$$
$$\times\left\{1-p\left[1-\left\{1+\left(\frac{x}{\alpha}\right)^{-\beta}\right\}^{-\gamma}\right]-\left\{1+\left(\frac{x}{\alpha}\right)^{-\beta}\right\}^{-\gamma}\right\}^{-1}, x > 0 \tag{148}$$

- The conditional density function of EGG log-logistic function for $X_2 = \max(\{Y_i\}_{i=1}^{Z})$: is

$$f(x|z; \gamma, \alpha, \beta) = \frac{z\beta\gamma}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta\gamma-1}\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-(\gamma+1)}\left[1+\left(\frac{x}{\alpha}\right)^{-\beta}\right]^{-\gamma(z-1)} \tag{149}$$

The pdf of the exponentiated log-logistic geometric type II (ELLGII) distribution reduces to

$$f_{ELLGII}(x;\ \theta) = \frac{(1-p)\beta\gamma}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta\gamma-1}\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-(\gamma+1)}\left\{1-p\left[1+\left(\frac{x}{\alpha}\right)^{-\beta}\right]^{-\gamma}\right\}^{-2} \tag{150}$$

Where $x > 0$, θ is the vector of parameters $((\alpha, \beta, \gamma > 0, and\ p \in (0,1)$..

The cdf corresponding to (150) is given by

$$F_{ELLGII}(x;\ \theta) = (1-p)\left[1+\left(\frac{x}{\alpha}\right)^{-\beta}\right]^{-\gamma}\left\{1-p\left[1+\left(\frac{x}{\alpha}\right)^{-\beta}\right]^{-\gamma}\right\}^{-1}, x > 0 \tag{151}$$

The hazard rate function corresponding to (150) is given by

$$h_{ELLGII}(x;\ \theta) = \frac{\frac{(1-p)\beta\gamma}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta\gamma-1}\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-(\gamma+1)}\left\{1-p\left[1+\left(\frac{x}{\alpha}\right)^{-\beta}\right]^{-\gamma}\right\}^{-2}}{1-(1-p)\left[1+\left(\frac{x}{\alpha}\right)^{-\beta}\right]^{-\gamma}\left\{1-p\left[1+\left(\frac{x}{\alpha}\right)^{-\beta}\right]^{-\gamma}\right\}^{-1}} \tag{152}$$

4.3 Other Methods

In this section, we present some new techniques for generating new probability distributions.

4.3.1 Khan and Khosa's Generalized Log-Logistic Distribution

Khan and Khosa (2015) proposed a generalized log-logistic distribution that belongs to the PH family and they described that it has properties identical to those of log-logistic, and tend to the Weibull in the limit, and they defined that these features enable the model to handle all kinds of hazard functions.

The pdf, survivor function, hazard rate function, and the cumulative hazard rate function of the Generalized log-logistic distribution is given by respectively;

$$f(t;\theta) = \frac{k\rho(\rho t)^{k-1}}{(1+(\gamma t)^k)^2}, \mathrm{t} > 0, \tag{153}$$

$$S(t;\theta) = [\,1+(\gamma t)^k]^{-\frac{p^k}{\gamma^k}}, \tag{154}$$

$$h(t;\theta) = \frac{k\rho(\rho t)^{k-1}}{1+(\gamma x)^k}, \tag{155}$$

$$H(t;\theta) = \frac{p^k}{\gamma^k}\log[\,1+(\gamma t)^k], \tag{156}$$

5. Estimation of the Parameters

In the statistics literature, due to the importance of the LL distribution and its generalizations, the estimation of the unknown parameters has been widely studied from the two main inferential statistics of thought: frequentists and Bayesians. From the frequentist approach, there are several different techniques that were proposed for parameter estimation but the maximum likelihood estimators (MLEs) are commonly used in most of the studies because of their appealing properties and can be applied when building confidence regions and intervals and also in test statistics. Although MLE method has been proven to be consistent, asymptotically efficient under very general conditions, it was found that it breaks down as one parameter tending to cause the likelihood to be infinite, rendering the other parameters inconsistent and is not applicable to J-shape distributions (Cheng and Amin, 1983). Ranneby, (1984) mentioned that the MLE is unbound and inefficient in the estimation of the mixtures of continuous distributions and heavy-tailed distributions. Therefore, to overcome the drawbacks in the MLE method (Cheng and Amin, 1979) proposed a maximum product of spacings (MPS) estimator to deal with those problems as it will return valid results over a much wider range of distributions and their generalizations. The method was also developed independently by (Ranneby, 1984).

In this study, we applied the maximum product spacings (MPS) method to estimate the unknown parameters for most of the generalized distributions discussed in this paper. For more information about the MPS estimator we can refer to (Cheng and Amin, 1979, 1983; Kawanishi, 2020; Ranneby, 1984; Thongkairat et al. 2018). The MPS estimator is a general technique used for estimating unknown parameters from observations with continuous univariate distributions and is an alternative to the MLE method. The log-likelihood for the distribution parameters can be maximized by using nonlinear likelihood equations obtained by differentiating the log-likelihood or by using software programs and packages. In this study, we applied the MPS (Maximum Product Spacing) package (version 2.3.1) (Teimouri, 2018) available in the R programming language to estimate the unknown parameters of the generalized distributions. The package has been continuously updated and more information can be obtained from https://cran.rstudio.com/web/packages/MPS/index.html. Currently, Bayesian inference of the log-logistic parameters and some of its generalizations has also received attention in the literature and some of them are still needs to be studied.

6. Real-life Data Application

In this section, we compare the performances of some of the generalizations of the log-logistic distribution in Section 5 using a real-life data set. The data represent the survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929 to 1938 (Lee and Wang, 2003). The data are:

(154.0, 139.0, 129.0, 129.0, 127.0, 126.0, 125.0, 117.0, 115.0, 111.0, 109.0, 109.0, 105.0, 103.0, 96.0, 93.0, 90.0,

89.0, 88.0, 83.0, 80.0, 78.0, 69.0, 68.0, 67.0, 67.0, 65.0, 65.0, 62.0, 61.0, 60.0, 60.0, 60.0, 59.0, 58.0, 57.0, 56.0,

55.0, 54.0, 52.0, 51.0, 51.0, 51.0, 49.0, 48.0, 47.0, 46.0, 46.0, 45.0, 45.0, 44.0, 43.0, 43.0, 43.0, 42.0, 41.0, 41.0,

41.0, 40.0, 40.0, 40.0, 39.0, 39.0, 38.0, 38.0, 38.0, 37.0, 37.0, 37.0, 35.0, 35.0, 32.0, 31.0, 31.0, 30.0, 29.1, 28.2,

27.9, 24.0, 24.0, 23.6, 23.4, 23.0, 21.1, 21.0, 21.0, 20.9, 20.4, 19.8, 17.9, 17.5, 17.3, 17.2, 16.8, 16.5, 16.3, 16.2,

15.7, 15.5, 14.8, 14.4, 14.4, 13.5, 12.3, 12.2, 11.8, 11.0, 10.3, 8.4, 8.4, 7.5, 7.4, 6.8, 6.6, 6.3, 6.2, 5.6, 5.0, 4.0,

0.3, 0.3)

Table 1. Descriptive statistics of the data

Mean	Median	Variance	Skewness	Kurtosis	Minimum	Maximum
46.33	40	1244.464	1.03	0.35	0.3	154

We fitted the following scale-shape variations of some of the surveyed distributions: The beta log-logistic distribution (BLL), the Kumaraswamy log-logistic distribution (KWLL), the exponentiated log-logistic distribution (ELL), the exponentiated generalized log-logistic distribution (EGLL), the Zografos-Balakrishnan log-logistic distribution (ZBLL), the Marshall-Olkin log-logistic distribution (MO-LL), the Weibull-log-logistic distribution (WLL), the Weibull-X (T-X) log-logistic distribution (WXLL) , the exponentiated Kumaraswamy log-logistic distribution (EKWLL), the gamma-uniform log-logistic distribution (GLL), and the Log-logistic (LL) distribution.

Each distribution was fitted by the method of maximum product spacings (MPS). Table 2 gives the values of the MPS estimates of the model parameters for the BLL, KWLL, ELL, ZBLL, EGLL, MO-LL, WLL, WXLL, EKWLL, GLL, and LL models fitted to the exceedances of breast cancer data. We estimate the unknown parameters of each model by maximum product spacings (MPS). There exist many maximization methods in R packages like NM (Nelder-Mead), BFGS (Broyden-Fletcher Goldfarb-Shanno), NR (Newton-Raphson), BHHH (Berndt-Hall-Hall-Hausman), and SANN (Simulated-Annealing) methods. In this study, the maximum product spacing estimators (MPS) are computed using Nelder-Mead optimization (NM) and the measures of goodness of fit AIC, BIC, CAIC, HQIC, Anderson-Darling (A*) and Cramer-von Misses (W*) are used to compare the ten selected models.

Table 2. MPS estimators of the model parameters and the maximum of the log-likelihood function

Distributions	MPS estimators of the parameters					
	a	**b**	**c**	**α**	**β**	$\boldsymbol{\ell(\hat{\theta})}$
GLL (a, α, β)	0.765			1.495	63.618	**-579.011**
WXLL (a, b, α, β)	0.878	0.865		1.439	44.828	-579.143
KWLL (a, b, α, β)	0.366	4.311		3.113	207.018	-579.211
EGLL (a, b, α, β)	6.870	0.691		1.683	187.670	-579.301
WLL (a, b, α, β)	14.442	0.610		0.115	222.676	-579.411
BLL (a, b, α, β)	0.696	6.885		1.661	203.623	-579.412
ZBLL (a, α, β)	0.351			3.011	79.117	-580.609
ELL (a, α, β)	0.319			3.305	71.682	-580.748
EKWLL (a, b, c, α, β)	3.603	5.835	0.181	1.700	82.094	-579.330
MO-LL (a, α, β)	1.137			1.800	32.683	**-587.677**
LL (α, β)				1.856	35.177	-587.599

Table 3 lists the values of the following statistics: Akaike Information Criterion (AIC), values of Bayesian Information Criterion (BIC), values of Consistent Akaike Information Criterion (CAIC), values of Hannan-Quinn information Criterion (HQIC), Cramer-von Misses statistic (W*), and Anderson-Darlin Statistic (A*). The smaller the values of these criteria the better the fit. For more information about these criteria, we refer to (Anderson & Burnham, 2004; Bierens, 2004; Burnham & Anderson, 2004)and (Fang, 2011).

Table 3. Goodness-of-fit tests for the generalizations of the 2-parameter log-logistic distribution

Distribution	Goodness of fit (G-O-F) criteria					
	BIC	**AIC**	**CAIC**	**HQIC**	**A***	**W***
GLL	**1172.411**	**1164.023**	**1164.229**	**1167.430**	0.463	0.063
WXLL	1177.469	1166.286	1166.631	1170.828	0.407	0.055
KWLL	1177.607	1166.424	1166.769	1170.966	0.538	0.079
EGLL	1177.786	1166.602	1166.947	1171.144	0.422	0.057
WLL	1178.006	1166.823	1167.168	1171.365	0.417	0.057
BLL	1178.008	1166.825	1167.170	1171.367	0.431	0.058
ZBLL	1175.606	1167.218	1167.423	1170.625	0.442	0.052
ELL	1175.885	1167.498	1167.703	1170.904	0.439	0.051
EKWLL	1182.639	1168.660	1169.182	1174.338	0.476	0.062
LL	1184.791	1179.199	1179.301	1181.47	**1.257**	**0.2096**
MO-LL	**1189.742**	**1181.355**	**1181.560**	**1184.761**	**1.084**	**0.131**

We can see that the 3-parameter gamma-uniform log-logistic distribution gives the smallest AIC, CAIC, BIC, and HQIC values. The GLL distribution provides significantly better fits than all of the other distributions, including the WELL and other distributions. MO-LL and LL distributions give the largest values for all criterion and tests, also the EKWLL distribution, these distributions may be thought to give worst fits. The maximum likelihood estimates of the best fitting GLL distribution are $\hat{\alpha} = 0.7652255$, $\hat{\beta} = 1.4947180, \hat{\gamma} = 63.6183351.$ The fitting was performed using the R package MPS (Teimouri, 2018).

Shifted log-logistic distribution or generalized log-logistic distribution or simply a three-parameter log-logistic distribution is an extension of the 2-parameter log-logistic distribution by an adding a shift parameter or location parameter. If we apply the generalization of the shifted log-logistic distribution by adding an extra parameter(s) and then apply the above breast cancer data. The comparison of the selected models plus the Shifted (3-parameter) Log-logistic distribution (SLL) are listed in Table 4.

The pdf of the shifted log-logistic distribution is given by

$$f(x)=\frac{\frac{\beta}{\alpha}\left(\frac{x-\mu}{\alpha}\right)^{\beta-1}}{(1+(\frac{x-\mu}{\alpha})^{\beta})^{2}}, \quad (157)$$

Table 4. Goodness-of-fit tests for the generalizations of the shifted log-logistic distribution

Distribution	Goodness of fit (G-O-F) criteria					
	BIC	**AIC**	**CAIC**	**HQIC**	**A***	**W***
GLL	**1178.489**	**1167.306**	**1167.650**	**1171.848**	**0.352**	**0.051**
WXLL	1182.851	1168.872	1169.394	1174.549	0.411	0.059
WLL	1183.550	1169.572	1170.093	1175.249	0.365	0.052
KWLL	1183.807	1169.828	1170.350	1175.506	0.372	0.051
ZBLL	1181.69	1170.506	1170.851	1175.048	0.410	0.051
ELL	1181.954	1170.771	1171.116	1175.313	0.412	0.051
BLL	1185.310	1171.331	1171.331	1177.009	0.395	0.062
EGLL	1185.337	1171.358	1171.880	1177.035	0.385	0.052
EKWLL	1188.942	1172.168	1172.904	1178.941	0.437	0.054
SLL	1183.224	1174.836	1175.042	1178.243	0.795	0.130
MO-LL	**1188.349**	**1177.165**	**1177.510**	**1181.707**	**0.606**	**0.085**
LL	**1184.791**	**1179.199**	**1179.301**	**1181.470**	**1.257**	**0.2096**

We can see that the 4-parameter gamma-uniform log-logistic distribution gives the smallest AIC, CAIC, BIC, HQIC, A*, and W* values. The GLL distribution provides significantly better fits than all of the other distributions, including the WELL and other distributions. MO-LL and LL distributions give the largest values for all criterion and tests, also the EKWLL distribution, these distributions may be thought to give worst fits. The maximum likelihood estimates of the best fitting GLL distribution are $\hat{\alpha}=1.7621529$, $\hat{\beta}=0.9813133$, $\hat{\gamma}=27.1799543$, $\hat{\lambda}$ = -1.8892325.

TTT Plot

In the survival and reliability analysis, there is a qualitative information about the failure rate shape, which can help in selecting a specified model. The Total time on test (TTT) plot or TTT transform is a device used for assessing the empirical behavior of the hazard (or failure) rate function. The hazard rate may be constant, decrease, increase, be an upside-down bathtub shaped, bathtub shaped or indicate a more complicated process. The TTT plot for the above survival data is displayed in Fig 1, which reveals an increasing hazard rate function. This plot reveals that the distributions with increasing hazard rate function could be good candidates for modeling the above cancer data. In our case, the GLL is the one that best fits the data.

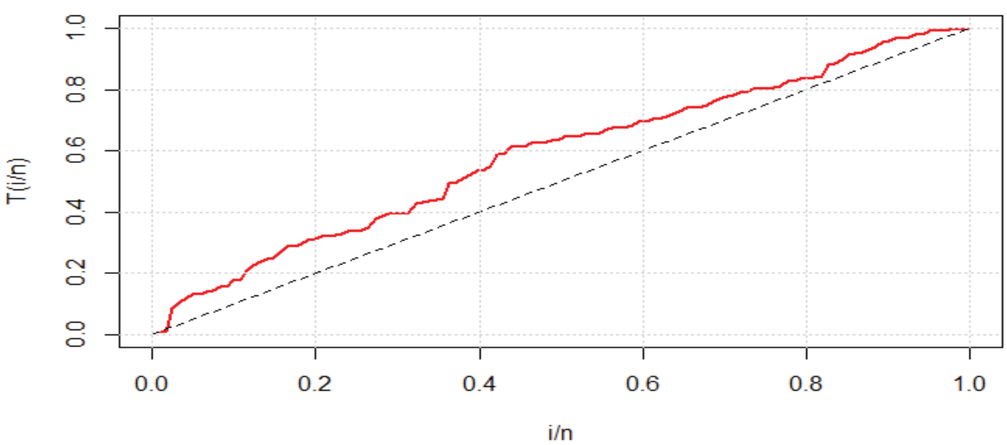


Figure 1. The TTT-plot for Cancer data

7. Censored Data and G-Classes of the Log-Logistic Distribution

The generating and extension of the existing probability distributions is an essential topic in survival and reliability analysis and has been applied in many applications in fields like social sciences, biological sciences, engineering, economics, physics, chemistry, medicine among others. Extension and generating of an existing probability distribution might allow to the resulting distribution to accommodate non-monotone forms for the failure rate function. Lai et al (2013) stated that the time of failure or life can have different interpretations depending on the area applications. Therefore, we can obtain more flexible distributions for modeling this kind of random variables. On the other hand, effectiveness and tractability for modeling censored data require, among other things, closed form expressions for the cumulative distribution function.

The LL distribution has been found to be very useful for modeling and analysing the incomplete (censored and truncated) data in the area of survival and reliability analysis. The LL distribution is an effective model for censored data, especially where the hazard rate is non-monotonic (i.e. incidence of an event increases after some finite time and then slowly decreases). When it comes to the extension of the log-logistic distribution using the G-classes of distributions, the Kumaraswamy-G family of distributions can be effective and tractable models for incomplete (or censored) data. The Marshall-Olkin family and the Exponentiated-G family of distributions can also be effective and tractable models for censored data, provided G is in closed form. However, beta-G and Mc-G distributions may not be effective or tractable models for censored data since their cumulative distribution functions involve the incomplete beta function.

8. Concluding Remarks

The LL distribution is one of the most commonly used distributions in survival and reliability analysis, particularly for events where the hazard rate is non-monotonic. It has also appeared in the literature under other names, such as Fisk distribution. Various LL extensions and generalizations have been introduced in recent years. In this paper, we have listed twenty distributions obtained from different generated families and compounding methods on the LL distribution. We review some of the statistical and mathematical properties of the LL distribution. We expect that these extensions or generalizations of the LL distributions will be an addition to the art of constructing useful models and lifetime distributions in general. One can discover easier the type of G-Classes that still is not applied to the LL distribution. The generalization of log-logistic distribution through generator, compounding and other methods was first applied in the area of survival and reliability analysis. After that time, several researchers have successfully applied these methods of generalizations to model lifetime and survival data. At present, these methods are being applied in the areas of engineering, economics, environmental, medical, hydrology, social science among others to handle more complex data.

9. Future Projects after the Survey

We hope our work will be of value to the statistical and probability community. As for the scope of future work, the possible future projects are: (1) to propose more new extensions of the LL distribution which have not been attempted; (2) to review and extend some of the generalized LL distributions, (3) to prepare a review and new developments on parametric survival models, (4) to derive some mathematical and statistical properties of the new extended distributions, (5) to estimate the parameters of the new extensions using both the classical and Bayesian approaches; (6) to review the multi variate, matrix variate and complex variate of the generalized LL distribution; and (6) to applied the new extensions of the LL distribution into a real-life data sets.

Acknowledgements

We would like to express our heartfelt appreciation to Pan African University, Institute for Basic Sciences, Technology and Innovation (PAUSTI), and African Union (AU) Commission for their financial support during the research study. The first author would also like to thank the supervisors for their tireless effort during the study. Also, the authors would like to thank the editor and reviewers for their careful checking of the details and for the insightful comments and suggestions that made this manuscript better.

References

Ahmad, Z., Hamedani, G. G., & Butt, N. S. (2019). Recent developments in distribution theory: A brief survey and some new generalized classes of distributions. *Pakistan Journal of Statistics and Operation Research*, *15*(1), 87–110. https://doi.org/10.18187/pjsor.v15i1.2803

AL-Kadim, K. A. (2018). {P}roposed {G}eneralized {F}ormula for {T}ransmuted {D}istribution. *Journal of University of Babylon*, *26*(4), 66–74.

Aldahlan, M. A. (2020). Alpha Power Transformed Log-Logistic Distribution with Application to Breaking Stress Data. *Advances in Mathematical Physics*, *2020*. https://doi.org/10.1155/2020/2193787

Alexander, C., Cordeiro, G. M., Ortega, E. M. M., & Sarabia, J. M. (2012). Generalized beta-generated distributions. *Computational Statistics and Data Analysis*, *56*(6), 1880–1897. https://doi.org/10.1016/j.csda.2011.11.015

Alizadeh, M., Merovci, F., & Hamedani, G. G. (2017). Generalized transmuted family of distributions: Properties and applications. *Hacettepe Journal of Mathematics and Statistics*, *46*(4). https://doi.org/10.15672/HJMS.201610915478

Alshangiti, A. M., Kayid, M., & Alarfaj, B. (2016). Further results involving Marshall–Olkin log-logistic distribution: reliability analysis, estimation of the parameter, and applications. *SpringerPlus*, *5*(1). https://doi.org/10.1186/s40064-016-2007-x

Alzaatreh, A., & Ghosh, I. (2015). On the Weibull-X family of distributions. *Journal of Statistical Theory and Applications*, *14*(2), 169. https://doi.org/10.2991/jsta.2015.14.2.5

Alzaatreh, A., Lee, C., & Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron*, *71*(1), 63–79. https://doi.org/10.1007/s40300-013-0007-y

Anderson, D., & Burnham, K. (2004). Model selection and multi-model inference. *Second. NY: Springer-Verlag*, *63*(2020), 10.

Aryal, G. R. (2013). Transmuted log-logistic distribution. *Journal of Statistics Applications & Probability*, *2*(1), 11.

Aryal, G. R., & Tsokos, C. P. (2009). On the transmuted extreme value distribution with application. *Nonlinear Analysis, Theory, Methods and Applications*, *71*(12), e1401–e1407. https://doi.org/10.1016/j.na.2009.01.168

Aryal, G. R., & Tsokos, C. P. (2011). Transmuted Weibull Distribution: A Generalization of the Weibull Probability Distribution. *European Journal of Pure and Applied Mathematics*, *4*(2), 89–102.

Ashkar, F., & Mahdi, S. (2003). Comparison of two fitting methods for the log-logistic distribution. *Water Resources Research*, *39*(8).

Aslam, M., Hussain, Z., & Asghar, Z. (2018). Cubic transmuted-g family of distributions and its properties. *Stochastics and Quality Control*, *33*(2), 103–112.

Aslam, M., & Jun, C.-H. (2010). A double acceptance sampling plan for generalized log-logistic distributions with known shape parameters. *Journal of Applied Statistics*, *37*(3), 405–414.

Bacon, R. W. (1993). PRACTITIONERS CORNER: A Note on the Use of the Log-Logistic Functional Form for Modelling Saturation Effects. *Oxford Bulletin of Economics and Statistics*, *55*(3), 355–361. https://doi.org/10.1111/j.1468-0084.1993.mp55003006.x

Barrios, R., & Dios, F. (2012). Exponentiated Weibull distribution family under aperture averaging for Gaussian beam waves. *Optics Express*, *20*(12), 13055–13064.

Bennett, S. (1983). Log-logistic regression models for survival data. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, *32*(2), 165–171.

Bierens, H. J. (2004). Information criteria and model selection. *Manuscript, Penn State University*.

Bourguignon, M., Ghosh, I., & Cordeiro, G. M. (2016). General Results for the Transmuted Family of Distributions and New Models. *Journal of Probability and Statistics*, *2016*. https://doi.org/10.1155/2016/7208425

Bourguignon, M., Silva, R. B., & Cordeiro, G. M. (2014). The Weibull-G Family of Probability Distributions. *Journal of Data Science*, *12*(1), 53–68. https://doi.org/10.6339/JDS.2014.12(1).1210

BRÜEDERL, J., & Diekmann, A. (1995). The log-logistic rate model: Two generalizations with an application to demographic data. *Sociological Methods & Research*, *24*(2), 158–186.

Burnham, K. P., & Anderson, D. R. (2004). Multimodel inference: understanding AIC and BIC in model selection. *Sociological Methods & Research*, *33*(2), 261–304.

Casella, G., & Berger, R. L. (2002). *Statistical inference* (Vol. 2). Duxbury Pacific Grove, CA.

Chaudhary, A. K. (2019). Frequentist Parameter Estimation of Two-Parameter Exponentiated Log-logistic Distribution BB. *NCC Journal*, *4*(1), 1–8. https://doi.org/10.3126/nccj.v4i1.24727

Chaudhary, A. K., & Kumar, V. (2014). Bayesian Estimation of Three-Parameter Exponentiated Log-logistic Distribution, *9*(2), 66–81.

Chaudhary, A. K., & Ph, D. (2007). Bayesian Analysis of Two-Parameter Exponentiated Log-logistic Distribution, *25*(1), 1–12.

Cheng, R. C. H., & Amin, N. A. K. (1979). Maximum product of spacings estimation with applications to the lognormal distribution. *Math Report*, *791*.

Cheng, R. C. H., & Amin, N. A. K. (1983). Estimating parameters in continuous univariate distributions with a shifted origin. *Journal of the Royal Statistical Society: Series B (Methodological)*, *45*(3), 394–403.

Choudhury, A. (2005). A simple derivation of moments of the exponentiated Weibull distribution. *Metrika*, *62*(1), 17–22.

Collatt, D. (2003). David Collett Modelling Survival Data in Medical Research - Copy.pdf.

Cordeiro, G. M., & de Castro, M. (2011). A new family of generalized distributions. *Journal of Statistical Computation and Simulation*, *81*(7), 883–898. https://doi.org/10.1080/00949650903530745

Cordeiro, G. M., Edwin, E. M., & Ramires, T. G. (2015). A new generalized Weibull family of distributions: mathematical properties and applications. *Journal of Statistical Distributions and Applications*, *2*(1). https://doi.org/10.1186/s40488-015-0036-6

Cordeiro, G. M., & Nadarajah, S. (2011). Closed-form expressions for moments of a class of beta generalized distributions. *Brazilian Journal of Probability and Statistics*, *25*(1), 14–33. https://doi.org/10.1214/09-BJPS109

Cordeiro, G. M., Ortega, E. M. M., & Cunha, D. C. C. (2013). The Exponentiated Generalized Class of Distributions. *Journal of Data Science*, *11*(1), 109–142. https://doi.org/10.6339/JDS.2013.11(1).1086

De Brito, C. R., Rêgo, L. C., De Oliveira, W. R., & Gomes-Silva, F. (2019). Method for generating distributions and classes of probability distributions: The univariate case. *Hacettepe Journal of Mathematics and Statistics*, *48*(3), 897–930. https://doi.org/10.15672/HJMS.2018.619

de Santana, T. V. F., Ortega, E. M. M., Cordeiro, G. M., & Silva, G. O. (2012). {T}he {K}umaraswamy-{L}og-{L}ogistic {D}istribution. *{J}ournal of {S}tatistical {T}heory and {A}pplications*, *11*(3), 265–291.

Dey, S., Kumar, D., Anis, M. Z., Nadarajah, S., & Okorie, I. (2021). A Review of Transmuted Distributions. *Journal of the Indian Society for Probability and Statistics*. https://doi.org/10.1007/s41096-021-00096-0

Diekmann, A. (1992). The log-logistic distribution as a model for social diffusion processes. *Journal of Scientific and Industrial Research*, *51*, 285–290.

Ekawati, D., Warsono, W., & Kurniasari, D. (2015). On the Moments, Cumulants, and Characteristic Function of the Log-Logistic Distribution. *IPTEK The Journal for Technology and Science*, *25*(3).

Eugene, N., Lee, C., & Famoye, F. (2002). BETA-NORMAL DISTRIBUTION AND ITS APPLICATIONS. *Communications in Statistics - Theory and Methods*, *31*(4), 497–512. https://doi.org/10.1081/STA-120003130

Fang, Y. (2011). Asymptotic equivalence between cross-validations and Akaike information criteria in mixed-effects models. *Journal of Data Science*, *9*(1), 15–21.

Fisk, P. R. (1961). The graduation of income distributions. *Econometrica: Journal of the Econometric Society*, 171–185.

Ghitany, M. E. (1998). On a recent generalization of gamma distribution. *Communications in Statistics - Theory and Methods*, *27*(1), 223–233. https://doi.org/10.1080/03610929808832662

Gholam, H. H. (2013). Characterizations of exponentiated distributions. *Pakistan Journal of Statistics and Operation Research*, *9*(1), 17–24. https://doi.org/10.18187/pjsor.v9i1.435

Gompertz, B. (1825). XXIV. On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies. In a letter to Francis Baily, Esq. FRS &c. *Philosophical Transactions of the Royal Society of London*, (115), 513–583.

Granzotto, D. C. T., Louzada, F., & Balakrishnan, N. (2017). Cubic rank transmuted distributions: inferential issues and applications. *Journal of Statistical Computation and Simulation*, *87*(14), 2760–2778. https://doi.org/10.1080/00949655.2017.1344239

Granzotto, D. C. T., & Louzada, F. (2015). The Transmuted Log-Logistic Distribution: Modeling, Inference, and an Application to a Polled Tabapua Race Time up to First Calving Data. *Communications in Statistics - Theory and Methods*, *44*(16), 3387–3402. https://doi.org/10.1080/03610926.2013.775307

Gui, W. (2013). Marshall-olkin extended log-logistic distribution and its application in minification processes. *Applied Mathematical Sciences*, *7*(77–80), 3947–3961. https://doi.org/10.12988/ams.2013.35268

Gupta, R. C., & Akman, H. O. (1995). On the reliability studies of a weighted inverse Gaussian model. *Journal of*

Statistical Planning and Inference, *48*(1), 69–83.

Gupta, R. C., Akman, O., & Lvin, S. (1999). A study of log-logistic model in survival analysis. *Biometrical Journal: Journal of Mathematical Methods in Biosciences*, *41*(4), 431–443.

Gupta, R. C., Gupta, P. L., & Gupta, R. D. (1998). Modeling failure time data by Lehman alternatives. *Communications in Statistics-Theory and Methods*, *27*(4), 887–904.

Gupta, R. D., & Kundu, D. (1999). Theory & methods: Generalized exponential distributions. *Australian & New Zealand Journal of Statistics*, *41*(2), 173–188.

Gupta, R. D., & Kundu, D. (2001). Exponentiated exponential family: an alternative to gamma and Weibull distributions. *Biometrical Journal: Journal of Mathematical Methods in Biosciences*, *43*(1), 117–130.

Gupta, R. D., & Kundu, D. (2009). Introduction of shape/skewness parameter (s) in a probability distribution. *J Probab Stat Sci*, *7*, 153–171.

Gupta, R. D., & Kundu, D. (2002). Generalized exponential distributions: statistical inferences. *Journal of Statistical Theory and Applications*, *1*(1), 101–118.

Jones, M. C. (2009). Kumaraswamy's distribution: A beta-type distribution with some tractability advantages. *Statistical Methodology*, *6*(1), 70–81. https://doi.org/10.1016/j.stamet.2008.04.001

Jones, M. C. (2004). Families of distributions arising from distributions of order statistics. *Test*, *13*(1), 1–43.

Kawanishi, T. (2020). Maximum likelihood and the maximum product of spacings from the viewpoint of the method of weighted residuals. *Computational and Applied Mathematics*, *39*(3). https://doi.org/10.1007/s40314-020-01179-7

Khan, S. A., & Khosa, S. K. (2015). Generalized log-logistic proportional hazard model with applications in survival analysis. *Journal of Statistical Distributions and Applications*, *3*(1). https://doi.org/10.1186/s40488-016-0054-z

Kleiber, C., & Kotz, S. (2003). *Statistical size distributions in economics and actuarial sciences* (Vol. 470). John Wiley & Sons.

Kumaraswamy, P. (1980). A generalized probability density function for double-bounded random processes. *Journal of Hydrology*, *46*(1–2), 79–88. https://doi.org/10.1016/0022-1694(80)90036-0

Lee, C., Famoye, F., & Alzaatreh, A. Y. (2013). Methods for generating families of univariate continuous distributions in the recent decades. *Wiley Interdisciplinary Reviews: Computational Statistics*, *5*(3), 219–238. https://doi.org/10.1002/wics.1255

Lee, E. T., & Wang, J. (2003). *Statistical methods for survival data analysis* (Vol. 476). John Wiley & Sons.

Lehmann, E. L. (1953). The power of rank tests. *The Annals of Mathematical Statistics*, 23–43.

Lemonte, A. J. (2014). The beta log-logistic distribution. *Brazilian Journal of Probability and Statistics*, *28*(3), 313–332. https://doi.org/10.1214/12-BJPS209

Lemonte, A. J., Barreto-Souza, W., & Cordeiro, G. M. (2013). The exponentiated Kumaraswamy distribution and its log-transform. *Brazilian Journal of Probability and Statistics*, *27*(1), 31–53. https://doi.org/10.1214/11-BJPS149

Lemonte, A. J., & Cordeiro, G. M. (2013). An extended Lomax distribution. *Statistics*, *47*(4), 800–816. https://doi.org/10.1080/02331888.2011.568119

Li, R., & Nadarajah, S. (2020). A review of Student's t distribution and its generalizations. *Empirical Economics*, *58*(3), 1461–1490. https://doi.org/10.1007/s00181-018-1570-0

Lima, S. R., & Cordeiro, G. M. (2017). The extended log-logistic distribution: Properties and application. *Anais Da Academia Brasileira de Ciencias*, *89*(1), 3–17. https://doi.org/10.1590/0001-3765201720150579

Little, C. L., Adams, M. R., Anderson, W. A., & Cole, M. B. (1994). Application of a log-logistic model to describe the survival of Yersinia enterocolitica at sub-optima pH and temperature. *International Journal of Food Microbiology*, *22*(1), 63–71.

Louzada, F., & Granzotto, D. C. T. (2016). The transmuted log-logistic regression model: a new model for time up to first calving of cows. *Statistical Papers*, *57*(3), 623–640. https://doi.org/10.1007/s00362-015-0671-5

Mahdavi, A., & Kundu, D. (2017). A new method for generating distributions with an application to exponential distribution. *Communications in Statistics - Theory and Methods*, *46*(13), 6543–6557. https://doi.org/10.1080/03610926.2015.1130839

Malik, A. S., & Ahmad, S. P. (2020). An Extension of Log-Logistic Distribution for Analyzing Survival Data. *Pakistan*

Journal of Statistics and Operation Research, *16*(4), 789–801. https://doi.org/10.18187/PJSOR.V16I4.2961

Marshall, A. W., & Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, *84*(3), 641–652.

McDonald, J. B. (2008). Some generalized functions for the size distribution of income. In *Modeling income distributions and Lorenz curves* (pp. 37–55). Springer.

Mendoza, N. V. R., Ortega, E. M. M., & Cordeiro, G. M. (2016). The exponentiated-log-logistic geometric distribution: Dual activation. *Communications in Statistics - Theory and Methods*, *45*(13), 3838–3859. https://doi.org/10.1080/03610926.2014.909937

Merovci, F., Alizadeh, M., & Hamedani, G. G. (2016). Another generalized transmuted family of distributions: Properties and applications. *Austrian Journal of Statistics*, *45*(3), 71–93. https://doi.org/10.17713/ajs.v45i3.109

Midhu, N. N., Sankaran, P. G., & Nair, N. U. (2013). A class of distributions with the linear mean residual quantile function and it's generalizations. *Statistical Methodology*, *15*, 1–24.

Mudholkar, G. S., & Hutson, A. D. (1996). The exponentiated Weibull family: some properties and a flood data application. *Communications in Statistics--Theory and Methods*, *25*(12), 3059–3083.

Mudholkar, G. S., & Srivastava, D. K. (1993). Exponentiated Weibull family for analyzing bathtub failure-rate data. *IEEE Transactions on Reliability*, *42*(2), 299–302.

Mudholkar, G. S., Srivastava, D. K., & Freimer, M. (1995). The exponentiated Weibull family: A reanalysis of the bus-motor-failure data. *Technometrics*, *37*(4), 436–445.

Muthulakshmi, S. (2013). Double sampling plan for Truncated Life test based on Kumaraswamy-Log-Logistic Distribution. *IOSR Journal of Mathematics*, *7*(4), 29–37. https://doi.org/10.9790/5728-0742937

Nadarajah, S., Cordeiro, G. M., & Ortega, E. M. M. (2013). The exponentiated Weibull distribution: A survey. *Statistical Papers*, *54*(3), 839–877. https://doi.org/10.1007/s00362-012-0466-x

Nadarajah, S., Cordeiro, G. M., & Ortega, E. M. M. (2015a). The exponentiated G geometric family of distributions. *Journal of Statistical Computation and Simulation*, *85*(8), 1634–1650. https://doi.org/10.1080/00949655.2014.885977

Nadarajah, S., Cordeiro, G. M., & Ortega, E. M. M. (2015b). The zografos-balakrishnan-G family of distributions: Mathematical properties and applications. *Communications in Statistics - Theory and Methods*, *44*(1), 186–215. https://doi.org/10.1080/03610926.2012.740127

Nadarajah, S., & Kotz, S. (2006). The exponentiated type distributions. *Acta Applicandae Mathematica*, *92*(2), 97–111.

Nandram, B. (1989). Discrimination between the complementary log-log and logistic model for ordinal data. *Communications in Statistics-Theory and Methods*, *18*(6), 2155–2164.

Nasiru, S. (2018). A New Generalization of Transformed-Transformer Family of Distributions. Retrieved from http://ir.jkuat.ac.ke:8080/bitstream/handle/123456789/4674/MS400-0002-16%28THESIS FINAL%29.pdf?sequence=1&isAllowed=y

Nasiru, S., Mwita, P. N., & Ngesa, O. (2019). Alpha power transformed Fréchet distribution. *Applied Mathematics and Information Sciences*, *13*(1), 129–141. https://doi.org/10.18576/AMIS/130117

Oluyede, B. O., Foya, S., Warahena-Liyanage, G., & Huang, S. (2016). The log-logistic weibull distribution with applications to lifetime data. *Austrian Journal of Statistics*, *45*(3), 43–69. https://doi.org/10.17713/ajs.v45i3.107

Pham, H., & Lai, C. D. (2007). On recent generalizations of the Weibull distribution. *IEEE Transactions on Reliability*, *56*(3), 454–458. https://doi.org/10.1109/TR.2007.903352

Prentice, R. L. (1976). A Generalization of the Probit and Logit Methods for Dose Response Curves. *Biometrics*, *32*(4), 761–768. https://doi.org/10.2307/2529262

Prentice, R. L., & Kalbfleisch, J. D. (1979). Hazard rate models with covariates. *Biometrics*, 25–39.

Rahman, M. M., Al-Zahrani, B., & Shahbaz, M. Q. (2018). A general transmuted family of distributions. *Pakistan Journal of Statistics and Operation Research*, *14*(2), 463–481. https://doi.org/10.18187/pjsor.v14i2.2334

Rahman, M. M., Al-Zahrani, B., & Shahbaz, M. Q. (2019). Cubic transmuted Weibull distribution: properties and applications. *Annals of Data Science*, *6*(1), 83–102.

Rahman, M. M., Al-Zahrani, B., & Shahbaz, M. Q. (2020). Cubic transmuted pareto distribution. *Annals of Data Science*, *7*(1), 91–108.

Rahman, M. M., Al-Zahrani, B., Shahbaz, S. H., & Shahbaz, M. Q. (2020). Transmuted probability distributions: A review. *Pakistan Journal of Statistics and Operation Research*, *16*(1), 83–94. https://doi.org/10.18187/PJSOR.V16I1.3217

Ramos, M. W. A. (2013). The Zografos-Balakrishnan Log-Logistic Distribution. *Properties and Application, Journal of Statistical Theory and Applications*, *12*(3), 244–255.

Ranneby, B. (1984). The maximum spacing method. An estimation method related to the maximum likelihood method. *Scandinavian Journal of Statistics*, 93–112.

Rao, G. S., Kantam, R. R. L., Rosaiah, K., & Prasad, S. V. (2012). Reliability test plans for type-II exponentiated log-logistic distribution. *Journal of Reliability and Statistical Studies*, *5*(1), 55–64.

Reath, J., Dong, J., & Wang, M. (2018). Improved parameter estimation of the log-logistic distribution with applications. *Computational Statistics*, *33*(1), 339–356. https://doi.org/10.1007/s00180-017-0738-y

Rosaiah, K., Kantam, R. R. L., & Kumar, S. (2006). Reliability test plans for exponentiated log-logistic distribution. *Stochastics and Quality Control*, *21*(2), 279–289.

Rowinski, P. M., Strupczewski, W. G., & Singh, V. P. (2002). A note on the applicability of log-Gumbel and log-logistic probability distributions in hydrological analyses: I. Known pdf. *Hydrological Sciences Journal*, *47*(1), 107–122.

Samuel, A. F. (2019). On Transmuted Four Parameters Generalized Log-Logistic Distribution. *International Journal of Statistical Distributions and Applications*, *5*(2), 32. https://doi.org/10.11648/j.ijsd.20190502.12

Shakhatreh, M. K. (2018). A new three-parameter extension of the log-logistic distribution with applications to survival data. *Communications in Statistics - Theory and Methods*, *47*(21), 5205–5226. https://doi.org/10.1080/03610926.2017.1388399

Shakil, M., & Ahsanullah, M. (2012). Review on Order Statistics and Record Values from F^{α} Distributions. *Pakistan Journal of Statistics and Operation Research*, 101–120.

Shaw, W. T., & Buckley, I. R. C. (2009). The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map, 1–8. Retrieved from http://arxiv.org/abs/0901.0434

Singh, K. P., Lee, C. M. -., & George, E. O. (1988). On Generalized Log-Logistic Model for Censored Survival Data. *Biometrical Journal*, *30*(7), 843–850. https://doi.org/10.1002/bimj.4710300714

Singh, U., Gupta, P. K., & Upadhyay, S. K. (2005). Estimation of three-parameter exponentiated-Weibull distribution under type-II censoring. *Journal of Statistical Planning and Inference*, *134*(2), 350–372.

Tahir, M.H., Mansoor, M., Zubair, M., & Hamedani, G. G. (2014). McDonald log-logistic distribution with an application to breast cancer data. *Journal of Statistical Theory and Applications*, *13*(1), 65. https://doi.org/10.2991/jsta.2014.13.1.6

Tahir, M. H., & Cordeiro, G. M. (2015). *Compounding of distributions: a survey and new generalized classes*. *Journal of Statistical Distributions and Applications* (Vol. 3). Journal of Statistical Distributions and Applications. https://doi.org/10.1186/s40488-016-0052-1

Tahir, Muhammad H., & Nadarajah, S. (2015). Parameter induction in continuous univariate distributions: Well-established G families. *Anais Da Academia Brasileira de Ciencias*, *87*(2), 539–568. https://doi.org/10.1590/0001-3765201520140299

Teimouri, M. (2018). MPS: An R package for modelling new families of distributions. *ArXiv Preprint ArXiv:1809.02959*.

Thongkairat, S., Yamaka, W., & Sriboonchitta, S. (2018). Maximum product spacings method for the estimation of parameters of linear regression. *Journal of Physics: Conference Series*, *1053*(1). https://doi.org/10.1088/1742-6596/1053/1/012110

Tomy, L., Jose, M., & Veena, G. (2020). A review on recent generalizations of exponential distribution, (October). https://doi.org/10.15406/bbij.2020.09.00313

Torabi, H., & Hedesh, N. M. (2016). The gamma-uniform distribution and its applications Kybernetika Hamzeh Torabi ; Narges Montazeri Hedesh, (January 2012).

Verhulst, P.-F. (1838). Notice sur la loi que la population suit dans son accroissement. *Corresp. Math. Phys.*, *10*, 113–126.

Verhulst, P. (1847). Deuxième mémoire sur la loi d'accroissement de la population. *Mémoires de l'académie Royale Des Sciences, Des Lettres et Des Beaux-Arts de Belgique*, *20*, 1–32.

Verhulst, P. F. (1845). Recherches Mathématiques sur La Loi D'Accroissement de la Population, Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Bruxelles, 18, Art. 1, 1-45. N.

Zografos, K., & Balakrishnan, N. (2009). On families of beta- and generalized gamma-generated distributions and associated inference. *Statistical Methodology*, *6*(4), 344–362. https://doi.org/10.1016/j.stamet.2008.12.003

Copyrights

Copyright for this article is retained by the author(s), with first publication rights granted to the journal.

This is an open-access article distributed under the terms and conditions of the Creative Commons Attribution license (http://creativecommons.org/licenses/by/4.0/).

International Journal of Statistics and Probability; Vol. 10, No. 3; May 2021
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

Compound Archimedean Copulas

Moshe Kelner[1], Zinoviy Landsman[2] & Udi E. Makov[2]

[1] Department of Statistics, Haifa University, Israel

[2] Actuarial Research Center, Haifa University, Israel

Correspondence: Moshe Kelner, Department of Statistics, Haifa University, Israel

Received: March 9, 2021 Accepted: April 21, 2021 Online Published: April 24, 2021

doi:10.5539/ijsp.v10n3p126 URL: https://doi.org/10.5539/ijsp.v10n3p126

Abstract

The copula function is an effective and elegant tool useful for modeling dependence between random variables. Among the many families of this function, one of the most prominent family of copula is the Archimedean family, which has its unique structure and features. Most of the copula functions in this family have only a single dependence parameter which limits the scope of the dependence structure. In this paper we modify the generator of Archimedean copulas in a way which maintains membership in the family while increasing the number of dependence parameters and, consequently, creating new copulas having more flexible dependence structure.

Keywords: compound Archimedean copula, dependence structure, association measures, finite tail dependence

1. Introduction

Dependence between variables has raised much research interest, where the challenge has always been to find a suitable multivariate distribution with which to model it. One promising direction is based on copula functions which provide a powerful tool geared to building multidimensional distributions with given marginals. By Sklar's Theorem (Sklar's, 1959), every multivariate cumulative distribution function $F(X_1,...,X_d) = P(X_1 \le x_1,...,X_d \le x_d)$ of a random vector $(X_1,...,X_d)$ can be uniquely written in term of the separate parts of its marginals $F_i(x_i) = P(X_i \le x_i)$ which are set of univariate distributions, and copula C, which holds the dependence structure between them, such as

$$\begin{aligned} F(\mathbf{x}) &= P(X_1 \le x_1,...,X_d \le x_d) = C(F_1(x_1),...,F_d(x_d)), \\ x_i &\in (-\infty,\infty), \ i = 1,...,d. \end{aligned} \tag{1}$$

By Nelsen (2006), the copula function must meet three properties. 1. For $u_i = F_i(x_i)$ when at least one of the marginals has zero value then $C(u_1,...,0,...,u_d) = 0$, 2. if all marginals except for u_i are equal to one, then $C(1,...,u_i,...,1) = u_i$, and 3. C is a d-dimensional non-decreasing function, i.e. $\sum_{i_1=1}^{2} ... \sum_{i_d=1}^{2} (-1)^{i_1+...+i_d} C(u_{1,i_1},...,u_{d,i_d}) \ge 0$, for any $[a,b] \in (0,1)^d$ ordered $a_i < b_i$ and $u_{j,1} = a_j$, $u_{j,2} = b_j$ for $j = 1,...,d$. This function has been comprehensively researched by Joe (1997), Nelsen (2006), and Durante & Sempi (2015) to mention only a few. See also references by Druet & Kotz (2001), and Genest & MacKay (1986). There are several families of copulas, the most common are the elliptical, which developed from an elliptically distributed random variables, the extreme value copula, which enables a suitable dependence structure for rare events, and the Archimedean family. The advantage of the Archimedean family is the unique structure that is expressed in its generator function. Different choices of generator functions yield different copulas with their particular expression of dependence. Many interesting parametric functions belong to this family, which contains a wealth of dependence structures (Embrechts et al., 2001). Among the most common are the Clayton copula (1978) in which the tails of the distribution are more dependent on the negative tail than on the positive, Frank (1979) which is symmetric Archimedean copula and Gumbel (1960) in which the tails of the distribution are more dependent on the positive tail than on the negative.

Table 1. Examples of families of Archimedean copulas

Family	$C_\theta(u,v)$	$\varphi_\theta(t)$	Range of θ
Clayton	$\left[\max\left(u^{-\theta}+v^{-\theta}-1,0\right)\right]^{-\frac{1}{\theta}}$	$\frac{1}{\theta}\left(t^{-\theta}-1\right)$	$\theta \in [-1,\infty)\setminus\{0\}$
Gumble	$\exp\left(-\left[(-\ln u)^{\theta}+(-\ln v)^{\theta}\right]^{-\frac{1}{\theta}}\right)$	$(-ln(t))^{\theta}$	$\theta \in [1,\infty)$
Frank	$-\frac{1}{\theta}\ln\left[1+\frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\right]$	$-ln\left(\frac{e^{-\theta t}-1}{e^{-\theta}-1}\right)$	$\theta \in (-\infty,\infty)$

A two-dimensional Archimedean copula is denoted by:

$$C_\theta(u,v) = \varphi_\theta^{-1}(\varphi_\theta(u) + \varphi_\theta(v)), \tag{2}$$

where $\varphi_\theta(t)$ is the Archimedean generator and θ is the dependence parameter. The generator φ is a continuous, strictly decreasing convex function $\varphi : [0,1] \to [0,+\infty]$ such that $\varphi_\theta(1) = 0$. Kimberling (1974) proved that $\varphi_\theta(t)$ is a completely monotone function. Schweizer and Sklar (1983) showed that $\varphi_\theta^{-1}(t)$ induces a bivariate copula if it is convex. McNeil & Neslehova (2009) showed that the necessary and sufficient condition for d-dimensional copula is for $\varphi_\theta^{-1}(t)$ to be d-monotone on $[0,\infty)$, i.e., to satisfy $(-1)^k \frac{d^k\varphi_\theta^{-1}(t)}{dt^k} \geq 0$ on $[0,\infty)$ and $k \in [1, d-2]$. Many properties of the Archimedean copulas and their generators are introduced and have been proven in Nelsen (2006). Some of them listed as follows:

- The *pseudo-inverse* of the generator φ is the function $\varphi_\theta^{[-1]}(t) = \begin{cases} \varphi_\theta^{-1}(t), & 0 \leq t \leq \varphi_\theta(0), \\ 0, & \varphi_\theta(0) \leq t \leq \infty. \end{cases}$, Nelsen (2006,4.1.2).
- The distribution function of Archimedean copula C with generator $\varphi_\theta(t)$ is denoted by $K_C(t) = t - \frac{\varphi_\theta(t)}{\varphi_\theta'(t^+)}$, Nelsen (2006,4.3.4).
- The density of Archimedean copula C is given by $c_\theta(u,v)$ =-$\frac{\varphi''(C(u,v))\varphi'(u)\varphi'(v)}{[\varphi'(C(u,v))]^3}$, Nelsen (2006,4.3.6).

Archimedean copulas have been used in different fields, such as actuarial science (Albrecher et al., 2011, Thilini et al., 2020), finance risk models (McNeil et al., 2005), portfolio allocations (Hennessy & Harvey, 2002), and hydrology (Chen and Guo, 2019). Several researchers have been involved in generating new Archimedean copulas, using its generator. Joh and Hu (1996) introduced families of multivariate copulas with tractable dependence structure, which was obtained by a mixture of a distributions called max-infinitely divisible. Genest et al., (1993) showed five different ways of generating alternative models having an Archimedean generator. The methods were right and left composition, scaling, composition via exponentiation, and linear combination. As an example, they generated a new generator which is a combination of Clayton's Frank's and Gummble's bivariate copulas, given by

$$\varphi_{\alpha,\beta,\gamma}(t) = \log\left(\frac{1-(1-\gamma)^\beta}{1-(1-\gamma t^\alpha)^\beta}\right), \quad \alpha > 0, \beta > 1, 0 < \gamma < 1,\ 0 < t \leq 1. \tag{3}$$

Morillas (2005) introduced a method designed to produce new copulas such that $C_\varphi(x_1,...,x_n) = \varphi^{[-1]}(C(\varphi(x_1),...,\varphi(x_n)))$. She showed sufficient conditions for the new copula but didn't investigate its behavior as compared with that of the original. Spreeuw (2010) presented a flexible family of Archimedean copula where the inverse of an Archimedean generator was generated from ψ, a utility function which is nondecreasing and concave. He assumed that ψ defined on $[0,1]$ so $-\psi$ is strictly decreasing and convex and could therefore serve as a generator. He transformed ψ in order to get $\psi(0) = -1$ and defined an Archimedean generator of the form $\varphi(s) = \max[1 + \beta(\psi(0) - \psi(s)), 0]$, $s \geq 0, \beta > 0$. Bernardino & Rulliere (2017) proposed conversion of the generator that allowed choosing an upper tail dependence without changes in the shape of the copula. They changed only part of a given generator and called it an Upper-Patched generator because the transformation is local and affects only the upper tail dependence. The new generator is given by $\varphi(t) = P_{d-1}(t) + (1-P_{d-1}(t))\varphi_D(t)$, where $t < t_0$, $\varphi_D(t)$ is a non-strict generator with endpoint $d_0 \leq t_0$ and $u_2 := \max(u_1,u_2)$. Xie et al., (2017) extended the Durante copula to a multivariate case by applying Marshall-Olkin distribution ideas (Marshall & Olkin, 1967). In our work we choose a different approach for enriching the Archimedean family and to apply it for two dimensions. We intend to replace θ, the generator parameter by new parameters, and propose a methodology for generating new copulas characterized by enhanced structures and improved properties. In section 2, we introduce a compounding method and the notion of compound generator. A short introduction of dependence measures is given in section 3. An example of generating a compound copula and a comparison of the original and the resulting compound copula is given in section 4. Conclusions are given in section 5.

2. Compound Archimedean Copula

In this paper we present a tool for generating new Archimedean copulas and we provide an extension to this family by creating new generators. This is achieved by using a compound of an existing generator with respect to $g_\eta(\theta)$, a probability density of the dependence parameter θ,

$$\varphi_M(t) = \int_\Theta \varphi_\theta(t)\, g_\eta(\theta)\, d\theta, \tag{4}$$

where M denotes compound. We now give sufficient conditions on the new generator guaranteeing that the resulting copula belongs to the Archimedean family.

Theorem 1 *Let* $\varphi_M(t) = \int_\Theta \varphi_\theta(t)\, g_\eta(\theta)\, d\theta$ *be a compound of* $\varphi_\theta(t)$, *a generator of an Archimedean copula, with respect to* $g_\eta(\theta)$, *a density function of* θ. *Then for any* $\varphi_\theta(t)$ *and density function* $g_\eta(\theta)$ *the compound* $\varphi_M(t)$ *is also an Archimedean copula generator.*
Proof. We need to show that the inverse of the Archimedean generator $\varphi_M^{-1}(t)$ is strictly decreasing and convex and that the generator satisfying $\varphi_M(1) = 0$.

Using the fact that for Archimedean generator complies $\varphi_\theta(1) = 0$ we get

$$\varphi_M(1) = \int_\Theta \varphi_\theta(1)\, g_\eta(\theta)\, d\theta = 0. \tag{5}$$

It is obvious that

$$\varphi_\theta^{-1}(\varphi_\theta(t)) = t. \tag{6}$$

By differentiating both sides of Eq. (6) by t, we get

$$\varphi_\theta^{-1\prime}(\varphi_\theta(t))\, \varphi_\theta'(t) = 1, \tag{7}$$

$$\varphi_\theta'(t) = \frac{1}{\varphi_\theta^{-1\prime}(\varphi_\theta(t))}. \tag{8}$$

Taking into account that $\varphi_\theta^{-1}(t)$ is strictly decreasing, we conclude that $\varphi_\theta^{-1'}(t) < 0$ and $\varphi_\theta^{-1\prime}(\varphi_\theta(t)) < 0$. Finally, by Eq.(8), we obtain that $\varphi_\theta'(t) < 0$. Similarly we get

$$\varphi_M'(t) = \frac{1}{\varphi_M^{-1\prime}(\varphi_M(t))}. \tag{9}$$

Let $\varphi_M'(t)$ be the first derivative by t of the compound generator $\varphi_M(t)$ such as

$$\varphi_M'(t) = \int_\Theta \varphi_\theta'(t)\, g_\eta(\theta)\, d\theta. \tag{10}$$

Taking into account that $\varphi_\theta'(t) < 0$ we conclude that $\varphi_M'(t) < 0$ and $\varphi_M^{-1'}(\varphi_M(t)) < 0$ and that leads to $\varphi_M^{-1\prime}(t) < 0$. Similarly, we get

$$\frac{d}{dt}\varphi_M^{-1\prime}(t) \;=\; \varphi_M^{-1''}(t) = -\frac{\varphi_M''\left(\varphi_M^{-1}(t)\right)}{\left(\varphi_M'\left(\varphi_M^{-1}(t)\right)\right)^2}\varphi_M^{-1\prime}(t) \tag{11}$$

where $\varphi_M''(t)$ is the second derivative by t, i.e.

$$\varphi_M''(t) = \int_\Theta \varphi_\theta''(t)\, g_\eta(\theta)\, d\theta. \tag{12}$$

Due to the fact that the generator $\varphi_\theta(t)$ is convex (Nelsen, 2006. Theorem 4.1.4), we get that $\varphi_\theta''(t) > 0$ and taking into account (12) we get $\varphi_M''(t) > 0$. We can, therefore, conclude that $\varphi_M''\left(\varphi_M^{-1}(t)\right) > 0$. Using $\varphi_M^{-1\prime}(t) < 0$, from Eq.(11) the desired result $\varphi_M^{-1''}(t) > 0$ is obtained.

corollary 2 *For any* $\varphi_\theta(0) < \infty$, *the copula* $C_\theta(u, v)$ *is defined as a non-strict copula (Nelsen, 2006), and* $C_M(u, v)$, *the compound copula, is also non-strict , and the compound generator holds the same end value,* $\varphi_M(0)$ *as that of the original generator,* $\varphi_\theta(0)$.

Proof. Let $\varphi_\theta(t)$ be a non-strict generator, then $\varphi_\theta(0)$ is a real number smaller than infinity. Let $\varphi_M(t)$ be a compound generator defined in Eq.(4) then

$$\begin{aligned} \varphi_M(0) &= \int_\theta \varphi_\theta(0)\, g_\eta(\theta) d\theta \\ &= \varphi_\theta(0) \int_\theta g_\eta(\theta) d\theta = \varphi_\theta(0) < \infty. \end{aligned} \tag{13}$$

3. Measures of Association and Compound Copula

The most common approach for characterizing a copula is measuring the strength of dependence which the data hold and its asymptotic properties. In this paper, we focus on two key measures of association, Kendall's tau, which is known as a bivariate concordance and discordance measure, and cross-ratio, which describes local dependence. We also examine the finite tail dependence of the compound copula (Sweeting and Fotiou, 2011).

Kendall's tau. Genest and Rivest (1993) showed that for Archimedean copulas this measure is given by:

$$\tau_\theta = 1 + 4\int_0^1 \frac{\varphi_\theta(t)}{\varphi'_\theta(t)}dt. \tag{14}$$

Let $\varphi_\theta(t)$ be an Archimedean generator resulting in Kendall's tau τ_θ as introduced in Eq.(14). Let $g_\eta(\theta)$ be a compounding distribution of θ used to create a compound generator shown by (4). By substituting $\varphi_\theta(t)$ into $\varphi_M(t)$ the Kendall's tau for a compound copula is defined as

$$\tau_M = 1 + 4\int_0^1 \left[\frac{\int_\theta \varphi_\theta(t)\, g_\eta(\theta)\, d\theta}{\int_\theta \varphi'_\theta(t)\, g_\eta(\theta)\, d\theta}\right]dt. \tag{15}$$

τ_M can be expressed as an expectation of τ_θ with respect to a specified distribution g^*_η, as seen in the following theorem.

Theorem 3 *Let the compounding distribution defined in (4), then* $\tau_M = E_{g^*_\eta}[\tau_\theta]$, *where* $g^*_\eta = \frac{\varphi'_\theta(t)g_\eta(\theta)}{\int_\theta \varphi'_\theta(t)g_\eta(\theta)d\theta}$.

Proof.

$$\begin{aligned} E_{g^*_\eta}[\tau_\theta] &= 1 + 4\int_\theta g^*_\eta\left[\int_{t=0}^1 \frac{\varphi_\theta(t)}{\varphi'_\theta(t)}dt\right]d\theta. \\ &= 1 + 4\int_{t=0}^1\int_\theta \frac{\varphi_\theta(t)}{\varphi'_\theta(t)}\frac{\varphi'_\theta(t)\, g_\eta(\theta)}{\int_\theta \varphi'_\theta(t)\, g_\eta(\theta)\, d\theta}d\theta dt \\ &= 1 + 4\int_{t=0}^1\left[\frac{\int_\theta \varphi_\theta(t)\, g_\eta(\theta)\, d\theta}{\int_\theta \varphi'_\theta(t)\, g_\eta(\theta)\, d\theta}\right]dt = \tau_M. \end{aligned} \tag{16}$$

Cross-ratio. The cross-ratio function is a commonly used tool to describe local dependence between two correlated variables. It can detect characteristics of association that cannot be captured by any other global dependence measures as Kendall's tau (Abrams et al., 2020). Oakes (1989) defined the measure as:

$$R_\theta(u,v) = \frac{C_\theta(u,v)\frac{d^2}{dudv}C_\theta(u,v)}{\frac{d}{dv}C_\theta(u,v)\frac{d}{du}C_\theta(u,v)}. \tag{17}$$

Positive or negative local dependence and independence at a location (u, v) are obtained for $R_\theta(u, v) > 1$, $0 < R_\theta(u, v) < 1$ and $R_\theta(u, v) = 1$, respectively. Using basic derivative rules he gave a simplified measure for the Archimedean copula, $R_\theta(u, v) = r_\theta\{C_\theta(u, v)\} = r_\theta(s) = \frac{-s\varphi''_\theta(s)}{\varphi'_\theta(s)}|_{s=C_\theta(u,v)}$. For the compound copula r_θ is replaced by r_M and $C_\theta(u, v)$ is replaced by $C_M(u, v)$.

Finite tail dependence. The coefficient of tail dependence measures the amount of dependence in the upper and the lower tail of distribution at the limit. For a copula function this boundary does not always exist. An alternative which we adopt here is to calculate the measure at a finite value k, (Sweeting and Fotiou, 2011). The finite upper tail dependence (FUTD) for the compound copula at k is defined as

$$\lambda_U^{(k)} = \frac{1 - 2k + C_M(k,k)}{1-k}, \tag{18}$$

and the finite lower tail dependence (FLTD) as

$$\lambda_L^{(k)} = \frac{C_M(k,k)}{k}. \tag{19}$$

4. Generating a Compound Copula: An Example

In this section, we introduce an example of generating a compound copula. We show the benefits gained by using the compound copula, as compared to the standard Archimedean copula, by comparing the values of the dependence measures discussed above.

Example 1.

Let $C_\theta(u, v)$ be a copula introduced by Nelsen (2006, 4.2.8)

$$C_\theta (u, v) = \max\left[\frac{\theta^2 uv - (1-u)(1-v)}{\theta^2 - (\theta-1)^2 (1-u)(1-v)}, 0\right]. \tag{20}$$

This copula is an Archimedean copula with generator function

$$\varphi_\theta (t) = \frac{1-t}{1+(\theta-1)t}, \theta > 1. \tag{21}$$

Note that this a non-strict copula with $\varphi_\theta (0) = 1$.

Let us assume that θ, the dependence parameter, is distributed

$$g_\eta(\theta) = \frac{1}{\ln(b) - \ln(a)} \frac{1}{\theta},\ a < \theta < b. \tag{22}$$

Then using Eq.(4) we get a compound generator of the form

$$\begin{aligned}
\varphi_M(t) &= \frac{1}{\ln(b)-\ln(a)} \int_a^b \frac{1}{\theta} \frac{1-t}{1+(\theta-1)t} d\theta && (23)\\
&= \frac{1}{\ln(b)-\ln(a)} \left(\ln(\theta) - \ln((\theta-1)t+1)\right)_a^b \\
&= \frac{1}{\ln(b)-\ln(a)} \begin{pmatrix} \log(b) - \ln((b-1)t+1) \\ -\log(a) + \ln((a-1)t+1) \end{pmatrix} \\
&= 1 + \frac{\ln\left(\frac{((a-1)t+1)}{((b-1)t+1)}\right)}{\ln\left(\frac{b}{a}\right)},\quad \forall\, b > a, && (24)
\end{aligned}$$

which provides $\varphi_M(1) = 1 + \frac{\ln\left(\frac{a}{b}\right)}{\ln\left(\frac{b}{a}\right)} = 1 - 1 = 0$, and $\varphi_M(0) = 1 + \frac{\ln(1)}{\ln\left(\frac{b}{a}\right)} = 1$, with an inverse

$$\varphi_M^{-1}(t) = \frac{ba^t - ab^t}{ba^t - ba^{t+1} + ab^{t+1} - ab^t}. \tag{25}$$

By placing Eq.(25) and Eq.(23) in Eq.(2) , a new compound copula is obtained:

$$C_M (u, v) = \max\left[\frac{ba^k - ab^k}{ba^k - ba^{k+1} + ab^{k+1} - ab^k}, 0\right], \tag{26}$$

for $k = (\varphi_M(u) + \varphi_M(v)) = \left(1 + \frac{\ln\left(\frac{((a-1)u+1)}{((b-1)u+1)}\right)}{\ln\left(\frac{b}{a}\right)}\right) + \left(1 + \frac{\ln\left(\frac{((a-1)v+1)}{((b-1)v+1)}\right)}{\ln\left(\frac{b}{a}\right)}\right)$,

$\forall\, b > a$.

Kendall's tau: Using Eq.(14), for the original copula we get

$$\tau_\theta = 1 - \frac{2(\theta+2)}{3\theta}, \tag{27}$$

and for the compound generator in Eq.(23) we get

$$\begin{aligned}\tau_M &= 1+4(\\ &\int_{t=0}^{1} \frac{(((a-1)t+1)((b-1)t+1))\left(\ln\left(\frac{b}{a}\right)+\ln\left(\frac{((a-1)t+1)}{((b-1)t+1)}\right)\right)}{a-b})dt \\ &= 1+\frac{4}{6(a-b)}\left[\begin{array}{c} 2(a-1)(b-1)\left(\ln\left(\frac{b}{a}\right)\right) \\ +3(a+b-2)\ln(\frac{b}{a})+b-a \\ +(2(a-1)(b-1)+3(a+b))\ln(\frac{a}{b}) \\ -\frac{2(b-1)}{(a-1)}+\frac{3(a+b)}{(a-1)}-\frac{3(a+b)}{(b-1)}+\frac{2(a-1)}{(b-1)} \\ +6\ln(\frac{b}{a})+\frac{(3a-b-2)\ln(a)}{(a-1)^2}+\frac{(a+2-3b)\ln(b)}{(b-1)^2} \\ -\frac{6a}{(a-1)}+\frac{6b}{(b-1)} \end{array}\right].\end{aligned} \quad (28)$$

The **cross ratio** for the original copula,is given by:

$$r_\theta(s)=\frac{-s\left(\frac{2(\theta-1)^2(1-s)}{(1+(\theta-1)s)^3}+\frac{2(\theta-1)}{(1+(\theta-1)s)^2}\right)}{-\frac{\theta}{(1+(\theta-1)t)^2}}, \quad (29)$$

where $s=C_\theta(u,v)=\max\left[\frac{\theta^2uv-(1-u)(1-v)}{\theta^2-(\theta-1)^2(1-u)(1-v)},0\right], \quad r_\theta(s)\in[0,2]$.

For the compound copula we get

$$\begin{aligned} r_M(s) &= \frac{-s\left[\frac{(b-a)(2(a-1)(b-1)s+a+b-2)}{((a-1)s+1)^2((b-1)s+1)^2\log(\frac{b}{a})}\right]}{-\left(\frac{(b-a)}{((a-1)s+1)((b-1)s+1)\log(\frac{b}{a})}\right)} \\ &= \frac{s(2(a-1)(b-1)s+a+b-2)}{((a-1)s+1)((b-1)s+1)}, \end{aligned} \quad (30)$$

where $s=C_M(u,v)$.

The upper and lower tail dependence are given by (18)-(19) with (23) substituted in $C_M(u,v)$. We now explore graphically the three measures of dependence discussed above with respect to the original and compound copulas. Figures 1,2 and 3 are related to Kendall's tau, Cross-ratio and FLTD, respectively.

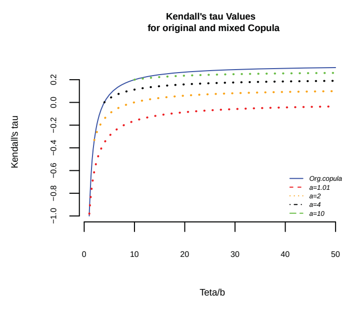


Figure 1. Kendall's tau for mixed and original copula

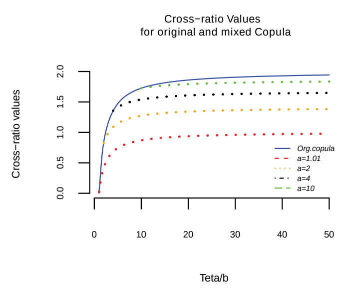


Figure 2. Cross ratio for mixed and original copula

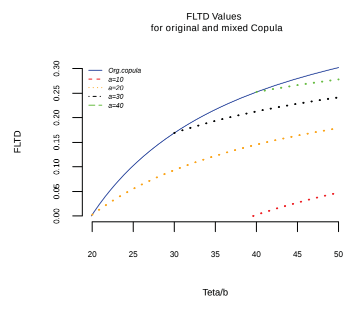


Figure 3. FLTD for mixed and original copula

The blue lines represent the values of the measures as a function of θ for the original copulas. The other four lines represent the values for the compound copula, where each line corresponds to different fixed values of a and varying values of b. From the graphs, it can be seen that the compound copula offers a richer choice of dependence structures.

Figure 4. presents Kendall's tau (x-axis) vs. Cross-ratio values (y-axis) for the original and the compound copulas. The top left represents the original and for the bottom left the compound copula. The right pair is similar to that on the left

except that Kendall's tau is limited to (0 – 0.1). We note that while for each value of Kendall's tau there is only one corresponding value of the cross-ratio, the range of such values in the compound copula is much wider. This clearly offers extended modeling possibilities.

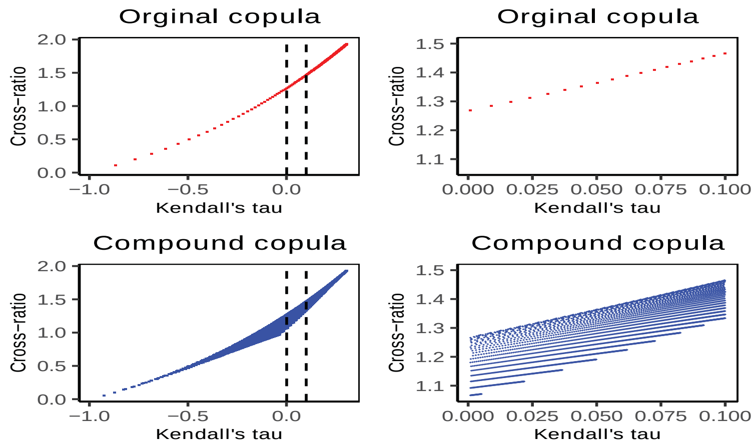


Figure 4. Kendall's tau vs. Cross-ratio for mixed and original copula

Example 2.

Let $C_\theta(u, v)$ be a copula introduced by Nelsen (2006, 4.2.16)

$$C_\theta(u,v) = \frac{1}{2}\left[\left(u + v - 1 - \theta\left(\frac{1}{u} + \frac{1}{v} - 1\right)\right) + \sqrt{\left(u + v - 1 - \theta\left(\frac{1}{u} + \frac{1}{v} - 1\right)\right)^2 + 4\theta}\right]. \tag{31}$$

With generator function

$$\varphi_\theta(t) = \left(\frac{\theta}{t} + 1\right)(1 - t), \quad \theta > 0 \tag{32}$$

note that this is a strict copula with $\varphi_\theta(0) = \infty$. Let us assume that θ, the dependence parameter, is distributed Gamma i.e. $g_{\alpha,\beta}(\theta) = \frac{e^{-\theta\beta}\beta^\alpha\theta^{\alpha-1}}{\Gamma(\alpha)}$. Then using Eq.(4) the compound generator equal to

$$\begin{aligned}\varphi_M(t) &= \int_{\theta=0}^{\infty} g_{\alpha,\beta}(\theta)\varphi_\theta(t)\,d\theta \\ &= \int_{\theta=0}^{\infty} \frac{e^{-\theta\beta}\beta^\alpha\theta^{\alpha-1}}{\Gamma(\alpha)}\left(\left(\frac{\theta}{t} + 1\right)(1 - t)\right)d\theta \\ &= \frac{(\alpha + \beta t)(1 - t)}{\beta t},\end{aligned} \tag{33}$$

with an inverse

$$\varphi_M^{[-1]} = \frac{1}{2\beta}\left((-\alpha - \beta t + \beta) + \sqrt{(\alpha\beta - \beta - t)^2 - 4\beta t}\right). \tag{34}$$

And using Eq.(2) a new compound copula is obtained

$$\begin{aligned}C_M(u,v) &= \frac{1}{2\beta}\left(\left(-\alpha - \beta\cdot\left(\left(\frac{(\alpha+\beta u)(1-u)}{\beta u}\right) + \left(\frac{(\alpha+\beta v)(1-v)}{\beta v}\right)\right) + \beta\right)\right) + \\ &\quad \frac{1}{2\beta}\left(\sqrt{\begin{array}{c}\left(\alpha + \beta\cdot\left(\left(\frac{(\alpha+\beta u)(1-u)}{\beta u}\right) + \left(\frac{(\alpha+\beta v)(1-v)}{\beta v}\right)\right) - \beta\right)^2 + \\ 4\beta\left(\left(\frac{(\alpha+\beta u)(1-u)}{\beta u}\right) + \left(\frac{(\alpha+\beta v)(1-v)}{\beta v}\right)\right)\end{array}}\right),\end{aligned} \tag{35}$$

which it is a strict copula with $\varphi_M(0) = \infty$. We now explore this copula function using two dependence measures. Kendall's tau and Blomqvit's β, which describe the position of pairs of observations relative to their quadrants and obtained by

$$\beta_l = 4 \cdot C\left(\frac{1}{2}, \frac{1}{2}\right) - 1 \tag{36}$$

For this purpose we substitute (33) and (32) into Kendall's tau Eq.(14) , and calculate Blomqvit's β by substituting (31) and (35) into Eq.(36) for $u = \frac{1}{2}$ and $v = \frac{1}{2}$. We will explore this two measures numerically. Figure 5 relates to Kendall's tau and Blomqvit's β.

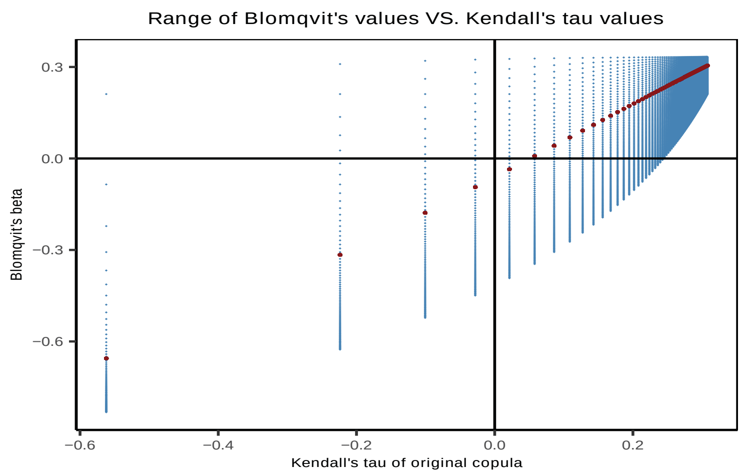


Figure 5. Kendall's tau vs. Blomqvit's β for the original and the compound copulas

The red dots represent the values of Blomqvit's β corresponding to Kendall's tau in the original copula. We note that for each value of τ_θ there is a single corresponding Blomqvit's β. The blue lines represent the values of Blomqvit's β corresponding to Kendall's tau in the compound copula. For each τ_θ there are multiple values of Blomqvit's β. While for the original copula, positive values of τ_θ resulted in only positive values of Blomqvit's β and negative values of τ_θ resulted in only negative values of Blomqvit's β, this restriction is removed when it comes to the compound copula, e.g. in the second quadrant there are positive values of Blomqvit's β corresponding to negative values of τ_θ.

5. Conclusions

In this paper, we introduce a novel method for generating new members of Archimedean copulas. We use a compound distribution approach by which we compound the generator function of a copula with a density function of its dependence parameter. We therefore create a new compound generator function which subsequently generates a compound Archimedean copula. We demonstrate this process with particular Archimedean copulas and show that the compound copulas offer a higher degree of flexibility in terms of dependence measures.

References

Abrams, S., Janssen, P., Swanepoel, J., & Veraverbeke, N. (2020). Non-parametric estimation of the cross-ratio function. *Annals of the Institute of Statistical Mathematics, 72*, 771–801. https://doi.org/10.1007/s10463-019-00709-3

Albrecher, H., Constantinescu, C., & Loisel, S. (2011). Explicit ruin formulas for models with dependence among risks. *Insurance Math. Econom., 48*, 265-270. https://doi.org/10.1016/j.insmatheco.2010.11.007

Bernardino, E. D., & Rullire, D. (2017). A note on upper-patched generators for Archimedean copulas. *Probability and statistics. 21*,183-200. https://doi.org/10.1051/ps/2017003

Chen, L., & Guo, S. (2019). *Copulas and Its Application in Hydrology and Water Resources*. Springer, New York.

Clayton, D. G. (1978). A Model for Association in Bivariate Life Tables and Its Application in Epidemiological Studies of Familial Tendency in Chronic Disease Incidence. *Biometrika., 65*, 141C151. https://doi.org/ 10.1093/biomet/65.1.141

Drouet, M., & Kotz, S. (2001).*Correlation and dependence*. Imperial College Press, London.

Durante, F., Quesada-Molina, J. J., & Sempi, C. (2007a). A generalization of the Archimedean class of bivariate copulas. *Ann. Inst. Statist. Math., 59*, 487–498. https://doi.org/ 10.1007/s10463-006-0061-9

Durante, F., & Sempi, C. (2015). *Principles of copula theory*. Chapman and Hall, New York.

Embrechts, P., Lindskog, F., & McNeil, A. (2001). Modelling Dependence with Copulas and Applications to Risk Management. *Handbook of Heavy Tailed Distributions in Finance, ed. S. Rachev, Elsevier. 8*, 329-384.

Frank, M. J. (1979). On the Simultaneous Associativity of F(x,y) and x+y-F(x,y). *Aequationes Math. 19*, 194C226. https://doi.org/10.1007/BF02189866

Genest, C., & MacKay, R. J. (1986b). The joy of copulas: Bivariate distributions with uniform marginal. *Amer. Statist. 40*, 280–283. https://doi.org/10.2307/2684602

Genest, C., & Rivest, L. P. (1993). Statistical Inference Procedures for Bivariate Archimedean Copulas. *Journal of the American Statistical Association, 88*, 1034–1043. https://doi.org/10.2307/2290796

Gumbel, E. J. (1960). Distributions des valeurs extrimes en plusieurs dimensions, *Publ. Inst. Statist. Univ. Paris, 9*, 171-173.

Hennessy, D., & Harvey, L. (2002). The use of Archimedean copula to model portfolio allocations. *Matematical Finance, 12*, 143-154. https://doi.org/10.1111/1467-9965.00136

Joe, H. (1997). *Multivariate models and dependence concepts*. Chapman and Hall, New York.

Joh, H., & Hu, T. (1996). Multivariate Distributions from Mixtures of Max-Infinitely Divisible Distributions, *Journal of Multivariate Analysis, 57*, 240-265. https://doi.org/10.1006/jmva.1996.0032

Kimberling, C. H. (1974). A probabilistic interpretation of complete monotonicity. *Equations Math., 10*, 152-164. https://doi.org/10.1007/BF01832852

Marshall, A. W., & Olkin, I. (1967). A multivariate exponential distribution. *Amer. Statist. Assoc., 62*, 30–44. https://doi/abs/10.1080/01621459.1967.10482885

McNeil, A. J., Rdiger F., & Embrechts, P. (2005). *Quantitative Risk Management: Concepts, Techniques and Tools*. Revised edition. Economics Books, Princeton University Press, edition 2.

McNeil, A. J., & Nešlehová, J. (2009). Multivariate Archimedean copulas, d-monotone functions and L1-norm symmetric distributions. *The Annals of Statistics., 37*(5B), 3059-3097. https://doi.org/10.1214/07-AOS556

Morillas, M. P. (2005). A method to obtain new copulas from a given one. *Metrika: International Journal for Theoretical and Applied Statistics. 61*, 169-184. https://doi.org/10.1007/s001840400330

Nelsen, R. B. (2006). *An Introduction to Copulas*. Second ed. Springer, New York.

Oakes, D. (1989). Bivariate Survival Models Induced by Frailties. *Journal of the American Statistical Association, 84*, 487–93. https://doi.org/10.2307/2289934

Schweizer, B., & Sklar, A. (1983). *Probabilistic Metric Spaces*. North-Holland, New York.

Sklar, A. (1959). Fonctions de rpartition n dimensions et leurs marges. *Publications de l'Institut de Statistique de l'Universit de Paris, 8*, 229–231.

Spreeuw, J. (2010). Relationships between Archimedean copulas and Morgenstern utility functions. *Copula Theory and its Applications, 17*, 309-320. https://doi.org/ 10.1007/$978-3-642-12465-5\_17$

Sweeting, P. J., & Fotiou, F. (2011). *Calculating and communicating tail association and the risk of extreme loss A discussion paper*. Presented to The Institute and Faculty of Actuaries, London.

Thilini, D. K., Jackie, L., & David, P. (2020). On the use of Archimedean copulas for insurance modelling. *Annals of Actuarial Science*, 1-25. https://doi.org/ 10.1017/S1748499520000147

Xie, J., Lin, F., & Yang, J. (2017). On a generalization of Archimedean copula family. *Statistics & Probability Letters, 125*, 121-129. https://doi.org/10.1007/s00500-005-0523-7

Copyrights

Copyright for this article is retained by the author(s), with first publication rights granted to the journal.

This is an open-access article distributed under the terms and conditions of the Creative Commons Attribution license (http://creativecommons.org/licenses/by/4.0/).

Mechanical Proof of the Maxwell-Boltzmann Speed Distribution With Analytical Integration

Hejie Lin[1] & Tsung-Wu Lin[2]

[1] Department of Mechanical Engineering, Oakland University, Rochester Hills, Michigan, USA

[2] Department of Civil Engineering, National Taiwan University, Taipei, Taiwan

Correspondence: Hejie Lin, Department of Mechanical Engineering, Oakland University, Rochester Hills, Michigan, USA. E-mail: hejielin@oakland.edu

Received: March 4, 2021 Accepted: April 22, 2021 Online Published: April 24, 2021

doi:10.5539/ijsp.v10n3p135 URL: https://doi.org/10.5539/ijsp.v10n3p135

Abstract

The Maxwell-Boltzmann speed distribution is the probability distribution that describes the speeds of the particles of ideal gases. The Maxwell-Boltzmann speed distribution is valid for both un-mixed particles (one type of particle) and mixed particles (two types of particles). For mixed particles, both types of particles follow the Maxwell-Boltzmann speed distribution. Also, the most probable speed is inversely proportional to the square root of the mass.

The Maxwell-Boltzmann speed distribution of mixed particles is based on kinetic theory; however, it has never been derived from a mechanical point of view. This paper proves the Maxwell-Boltzmann speed distribution and the speed ratio of mixed particles based on probability analysis and Newton's law of motion. This paper requires the probability density function (PDF) $\psi^{ab}(u_a; v_a, v_b)$ of the speed u_a of the particle with mass M_a after the collision of two particles with mass M_a in speed v_a and mass M_b in speed v_b. The PDF $\psi^{ab}(u_a; v_a, v_b)$ in integral form has been obtained before. This paper further performs the exact integration from the integral form to obtain the PDF $\psi^{ab}(u_a; v_a, v_b)$ in an evaluated form, which is used in the following equation to get new distribution $\mathrm{P}^a_{new}(u_a)$ from old distributions $\mathrm{P}^a_{old}(v_a)$ and $\mathrm{P}^b_{old}(v_b)$. When $\mathrm{P}^a_{old}(v_a)$ and $\mathrm{P}^b_{old}(v_b)$ are the Maxwell-Boltzmann speed distributions, the integration $\mathrm{P}^a_{new}(u_a)$ obtained analytically is exactly the Maxwell-Boltzmann speed distribution.

$$\mathrm{P}^a_{new}(u_a) = \int_0^\infty \int_0^\infty \psi^{ab}(u_a; v_a, v_b)\mathrm{P}^a_{old}(v_a)\mathrm{P}^b_{old}(v_b)\mathrm{d}v_a\mathrm{d}v_b\,, \quad a, b = 1 \text{ or } 2$$

The mechanical proof of the Maxwell-Boltzmann speed distribution presented in this paper reveals the unsolved mechanical mystery of the Maxwell-Boltzmann speed distribution since it was proposed by Maxwell in 1860. Also, since the validation is carried out in an analytical approach, it proves that there is no theoretical limitation of mass ratio to the Maxwell-Boltzmann speed distribution. This provides a foundation and methodology for analyzing the interaction between particles with an extreme mass ratio, such as gases and neutrinos.

Keywords: Maxwell speed distribution, Maxwell-Boltzmann speed distribution, Maxwell-Boltzmann distribution, Avogadro's law, kinetic theory of gases, kinetic theory, thermodynamics, statistical mechanics, subatomic particles

1. Overview

James C. Maxwell (1860a,b) first provided the Maxwell speed distribution in 1860 on a statistical heuristic basis. Maxwell (1867) and Boltzmann (1872) carried out more investigations into the physical meaning of the distribution. Boltzmann (1877) derived the distribution again based on statistical thermodynamics. Nevertheless, none of their approaches were based on Newton's law of motion.

The simplest way to prove the Maxwell-Boltzmann speed distribution is from a statistical view, beginning from the normal distribution of the velocity v_x in x-direction as follows.

$$P(v_x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{v_x}{\sigma}\right)^2} \tag{1}$$

or

$$P(v_x) = \frac{h}{\sqrt{\pi}} e^{-h^2 v_x^2} \tag{2}$$

Where $\sigma = \frac{1}{\sqrt{2}h}$, $h = \sqrt{\frac{M}{2kT}}$, k is the Boltzmann constant, T is the equilibrium temperature, and M is the particle mass. Extending from the velocity v_x to three independent velocities (v_x, v_y, v_z) in three directions, and transferring it to spherical coordinates (v, θ, φ) using $v_x = v\sin\theta\cos\varphi$, $v_y = v\sin\theta\sin\varphi$, and $v_z = v\cos\theta$ gives

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(v_x)P(v_y)P(v_z)dv_x dv_y dv_z$$

$$= \int_0^{\infty}\frac{4h^3}{\sqrt{\pi}}v^2 e^{-h^2v^2}dv = \int_0^{\infty}P(v)dv \tag{3}$$

Where $P(v)$ is the Maxwell-Boltzmann speed distribution shown in Equation (4) (Brush, 1966, Landau et al., 1969, McQuarrie, 1976, Garrod, 1995, Maudlin, 2013).

$$P(v) = \frac{4h^3}{\sqrt{\pi}}v^2 e^{-h^2v^2} \tag{4}$$

In the Maxwell-Boltzmann speed distribution, the most probable speed, $\boldsymbol{v_{mp}}$, is inversely proportional to the square root of the mass for fixed temperatures as follows

$$v_{mp} = \frac{1}{h} = \sqrt{\frac{2kT}{M}} \tag{5}$$

Therefore, when two types of particles with mass M_1 and M_2 are mixed at the same temperature, the above equation gives the following mass-speed relationship

$$\frac{v_{1,mp}}{v_{2,mp}} = \sqrt{\frac{M_2}{M_1}} \tag{6}$$

An example of the theoretical Maxwell-Boltzmann speed distribution curves and their corresponding most probable speeds $v_{1,mp}$ and $v_{2,mp}$ of two types of particles with a mass ratio of nine are shown below.

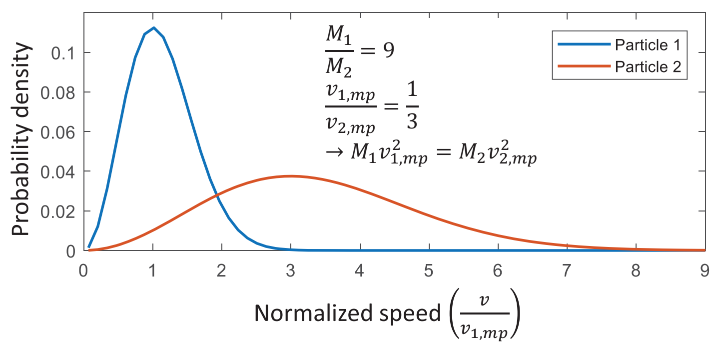


Figure 1. Maxwell-Boltzmann speed distributions of two types of particles

Boltzmann (1872) tried to provide mechanical proof of the Maxwell-Boltzmann speed distribution in 1872 by formulating the following equation.

$$dn = f(x,t)dx \cdot f(x',t)dx' \cdot \psi(\xi; x, x')d\xi \tag{7}$$

where $f(x,t)dx$ is the number of particles with speeds between x and x+dx, and similarly for $f(x',t)dx'$, dn is the number of particles with speeds between ξ and $\xi + d\xi$. In addition, the symbol $\psi(\xi; x, x')$ represents the probability density function (PDF) of the resulting speed after a collision between two particles. The definition is excellent and meaningful, but the method used to calculate this PDF ψ has yet to be created. Following Boltzmann's work in 1872, we derived the PDF ψ based on Newton's law of motion in this paper. The PDF ψ for equal mass particles had been provided by Lin et al. (2019).

To consider the collisions of unequal mass particles, we need to have four PDFs: $\psi^{11}(u_1; v_1, v_1')$, $\psi^{12}(u_1; v_1, v_2)$, $\psi^{21}(u_2; v_2, v_1)$, and $\psi^{22}(u_2; v_2, v_2')$. Where $\psi^{ab}(u_a; v_a, v_b)$ is the PDF of post-collision speed u_a of a particle with a mass M_a after the collision of two particles with mass M_a in a pre-collision speed v_a and mass M_b in a pre-collision speed v_b. For $a = b$, the PDF is identical to the collision of two equal mass particles (Lin et al., 2019). For $a \neq b$, the PDF will be given in this paper.

After the PDF $\psi^{ab}(u_a; v_a, v_b)$ was derived, a numerical iteration method (Lin et al., 2019) can be used to get a new distribution $\mathrm{P}^a_{new}(u_a)$ from the old distribution $\mathrm{P}^a_{old}(v_a)$, and set $\mathrm{P}^a_{old}(v_a) = \mathrm{P}^a_{new}(v_a)$ for the next iteration using the following equations.

$$\mathrm{P}^a_{new}(u_a) = \sum_{b=1}^{N} \mathrm{n}_b \int_0^\infty \int_0^\infty \psi^{ab}(u_a; v_a, v_b)\mathrm{P}^a_{old}(v_a)\mathrm{P}^b_{old}(v_b)\, dv_a dv_b, \quad a=1,2,\ldots N \tag{8}$$

Where n_a is the fraction of the number of particles with mass M_a, N is the total number of particle types, and $\sum_{b=1}^{N} \mathrm{n}_b = 1$.

Due to finite precision, the limit of computer memory, and computation time, the numerical iterations method can only apply to mixtures of gasses with a mass ratio between 0.01 and 100. In the cases of mixtures of molecules and subatomic particles, the extreme mass ratio is between 10^{-12} and 10^{12}. This paper provides an analytical integration method to show that the Maxwell-Boltzmann speed distribution is valid for even these extreme cases. When $\mathrm{P}^a_{old}(v_a)$ and $\mathrm{P}^b_{old}(v_b)$ are the Maxwell-Boltzmann speed distributions, the analytical integration $\mathrm{P}^a_{new}(u_a)$ obtained by the following equation will also be the Maxwell-Boltzmann speed distribution.

$$\mathrm{P}^a_{new}(u_a) = \int_0^\infty \int_0^\infty \psi^{ab}(u_a; v_a, v_b)\mathrm{P}^a_{old}(v_a)\mathrm{P}^b_{old}(v_b)\mathrm{d}v_a \mathrm{d}v_b \tag{9}$$

The integration is tedious, but the final result is exactly the Maxwell-Boltzmann speed distribution. Moreover, the RMS speed square is inversely proportional to the particle masses as predicted by Avogadro's law (Avogadro).

2. Velocity Diagram for a Collision of Two Particles

A velocity diagram for a collision of two particles is used to derive the PDF of two types of particles' post-collision speed. Two concentric circles in the 2D plane can be constructed as a velocity diagram for the collision of two particles in 3D space. The concentric circles velocity diagram provides a geometric relationship between the pre-collision and post-collision speeds. The concentric circles velocity diagram was used by Maxwell in his study of the Maxwell-Boltzmann speed distribution and is explained and proved in this section.

Concentric circles velocity diagrams are based on two reference frames: the fixed reference frame (O) and a center-of-mass (CM) reference frame (C). Speeds can be transferred between the fixed frame (O) and the CM frame (C).

2.1 The Fixed Reference Frame

In the fixed reference frame, before a collision, two particles with mass M_1 and M_2 are moving at pre-collision velocities $\vec{v}_1$ $(or\ \overrightarrow{OA})$ and $\vec{v}_2$ $(or\ \overrightarrow{OB})$. After the collision, the post-collision velocities of the two particles change to $\vec{u}_1$ $(or\ \overrightarrow{OP})$ and $\vec{u}_2$ $(or\ \overrightarrow{OQ})$, as shown in the figure below. It is important to note that the variables in the PDF $\psi^{12}(u_1; v_1, v_2)$ are speeds, which are the magnitudes of the velocities. Note that the vector $\vec{v}$ without an arrow-hat v indicates the length of the vector. For example, $u_1 = |\vec{u}_1|$.

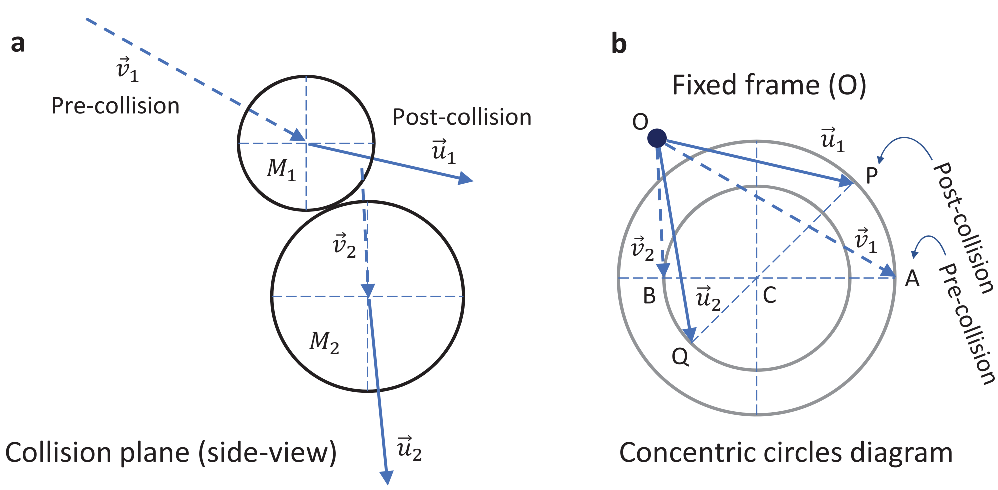


Figure 2. Velocities of two particles before and after a collision

2.2 The Center-of-mass Reference Frame

The center-of-mass (CM) reference frame uses the center of mass of two particles as its origin point (C). The velocity of the center-of-mass of two particles is related to the two masses M_1 and M_2 and their corresponding velocities $\vec{v}_1$ and

$\vec{v}_2$ as

$$\vec{v}_c \equiv \frac{M_1\vec{v}_1+M_2\vec{v}_2}{M_1+M_2} \equiv m_1\vec{v}_1 + m_2\vec{v}_2 \tag{10a}$$

where m_1 and m_2 are mass ratio and are defined as: $m_1 \equiv \frac{M_1}{M_1+M_2}$; $m_2 \equiv \frac{M_2}{M_1+M_2}$.

Before the collision, the pre-collision velocities of particles 1 and 2 in the CM frame are

$$\vec{v}_{1c} \equiv \vec{v}_1 - \vec{v}_c \tag{10b}$$

$$\vec{v}_{2c} \equiv \vec{v}_2 - \vec{v}_c \tag{10c}$$

It can be shown that after the collision, the post-collision velocities of particle 1 and particle 2 change to $\vec{u}_{1c}$ $(or\ \overrightarrow{CP})$ and $\vec{u}_{2c}$ $(or\ \overrightarrow{CQ})$ according to the following first three rules, which satisfy the last two conservations of momentum and energy.

1. Point P will be on Circle C-A. $\therefore\ u_{1c} = v_{1c}$
2. Point Q will be on Circle C-B. $\therefore\ u_{2c} = v_{2c}\ \ and\ \ \frac{u_{1c}}{u_{2c}} = \frac{|\overline{CP}|}{|\overline{CQ}|} = \frac{|\overline{CA}|}{|\overline{CB}|} = \frac{m_2}{m_1}$
3. Velocity $\overrightarrow{CP}$ and $\overrightarrow{CQ}$ have opposite directions. $\therefore\ m_1\vec{u}_{1c} + m_2\vec{u}_{2c} = 0$
4. Conservation of momentum.

$$\mathrm{m}_1(\vec{v}_c + \vec{u}_{1c}) + \mathrm{m}_2(\vec{v}_c + \vec{u}_{2c}) = \vec{v}_c = \mathrm{m}_1(\vec{v}_c + \vec{v}_{1c}) + \mathrm{m}_2(\vec{v}_c + \vec{v}_{2c})$$

5. Conservation of energy (by $u_{1c} = v_{1c}$ and $u_{2c} = v_{2c}$).

$$\mathrm{m}_1(\vec{v}_c + \vec{u}_{1c})\cdot(\vec{v}_c + \vec{u}_{1c}) + \mathrm{m}_2(\vec{v}_c + \vec{u}_{2c})\cdot(\vec{v}_c + \vec{u}_{2c}) = v_c^2 + \mathrm{m}_1 u_{1c}^2 + \mathrm{m}_2 u_{2c}^2$$

$$\mathrm{m}_1(\vec{v}_c + \vec{v}_{1c})\cdot(\vec{v}_c + \vec{v}_{1c}) + \mathrm{m}_2(\vec{v}_c + \vec{v}_{2c})\cdot(\vec{v}_c + \vec{v}_{2c}) = v_c^2 + \mathrm{m}_1 v_{1c}^2 + \mathrm{m}_2 v_{2c}^2$$

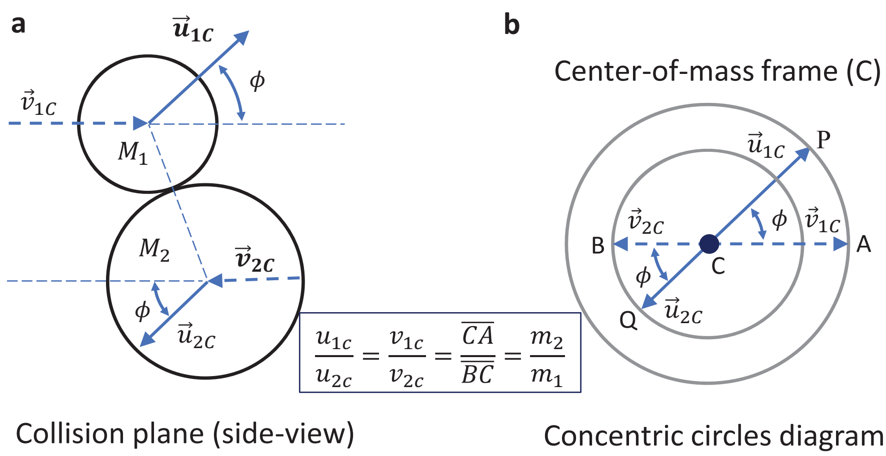


Figure 3. Post-collision velocities of two particles relative to the center of mass

3. Probability Density Function of the Post-Collision Speed in Integral Form

After any collision between two particles, the resulting speeds depend on two random factors: (1) the random directions of the pre-collision velocities $\vec{v}_1$ relative to $\vec{v}_2$ represented by a random angle α (see Figure 4), and (2) the random direction of the post-collision velocity $\vec{u}_1$ represented by a random angle β (see Figure 4). These two random factors are discussed below to prepare for the derivation of the PDF $\psi^{12}(u_1; v_1, v_2)$.

$$\Psi^{12}(u_1; v_1, v_2) = \int_0^{\pi} \mathrm{P}_{u_1|\alpha}(u_1)\mathrm{P}_\alpha(\alpha)d\alpha = \int_0^{\pi} \mathrm{P}_{\beta|\alpha}(\beta)\left|\frac{d\beta}{du_1}\right|\mathrm{P}_\alpha(\alpha)d\alpha \tag{11}$$

The right-hand side of the above equation will be derived in the following sections. And, the final PDF Ψ^{12} will be obtained at the end of this section.

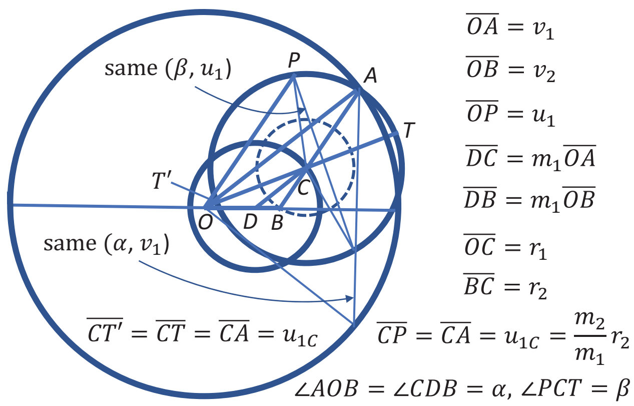


Figure 4. Three sphere surfaces *O-A, D-C, C-P* with centers at *O, D, C,* and radius $v_1, m_1 v_1, \frac{m_2}{m_1} r_2$

3.1 The Randomness of the Directions in 3D Before the Collision

For two given pre-collision speeds v_1 and v_2, the post-collision speeds depend on the directions of the two pre-collision velocities $\vec{v}_1$ and $\vec{v}_2$. The two pre-collision velocities both have random directions. The directions of these two pre-collision velocities determine the radius of circle *C-A* and circle *C-B*. The randomness of the two velocities can be reduced to one random angle α which considers only the relative direction between $\vec{v}_1$ and $\vec{v}_2$ and is defined as the angle between the two pre-collision velocities $\vec{v}_1$ and $\vec{v}_2$ (in the fixed frame) as shown in Figure 4.

For fixed magnitudes of v_1 and v_2, if $\vec{v}_2 = \overline{OB}$ is also fixed in the direction, but the direction of $\vec{v}_1 = \overline{OA}$ is changed, then point *A* will be located on a spherical surface, as shown in Figure 4. And the probability of point *A* on the surface is uniformly distributed since $\vec{v}_1$ has equal opportunity in any direction. Therefore the probability density of point A located on the sphere surface *O-A* at angle α is

$$\mathrm{P}_\alpha(\alpha) = \frac{1}{2}\sin\alpha \tag{12}$$

3.2 The Randomness of the Directions in 3D after the Collision

Similar to the pre-collision directions defined by α (representing the relative moving direction before the collision), the location of the point P will be defined by β, as the angle between $\vec{v}_C = \overline{OC}$ and $\vec{u}_{1C} = \overline{CP}$, as shown in Figure 4, such that the same angle β will result in the same magnitude u_1 of the different velocity $\vec{u}_1$.

The band area between β and $\beta + \mathrm{d}\beta$ on the sphere surface *C-P* is $2\pi(u_{1C}\sin\beta)(u_{1C}\mathrm{d}\beta)$, the total area of the sphere surface is $4\pi u_{1C}^2$, and the ratio is $\frac{1}{2}\sin\beta\, d\beta$. Therefore the probability density of point P located on the sphere surface *C-P* at angle β is

$$\mathrm{P}_{\beta|\alpha}(\beta) = \frac{1}{2}\sin\beta \tag{13}$$

3.3 Considering all the Possible Directions in 3D Before and after Collisions

For a fixed α, the r_1 and r_2 as shown in Figure 4, can be computed as

$$\begin{aligned} r_1(\alpha; v_1, v_2) = \overline{OC} &= \sqrt{(m_1 v_1\cos\alpha + m_2 v_2)^2 + (m_1 v_1 \sin\alpha)^2} \\ &= m_1\sqrt{v_1^2 + \left(\frac{m_2}{m_1}v_2\right)^2 + 2v_1\left(\frac{m_2}{m_1}v_2\right)\cos\alpha} \end{aligned} \tag{14a}$$

$$\begin{aligned} r_2(\alpha; v_1, v_2) = \overline{BC} &= \sqrt{(m_1 v_1\cos\alpha - m_1 v_2)^2 + (m_1 v_1 \sin\alpha)^2} \\ &= m_1\sqrt{v_1^2 + v_2^2 - 2v_1 v_2\cos\alpha} \end{aligned} \tag{14b}$$

So the relation between u_1 and β for fixed r_1 and r_2 is given by

$$u_1(\beta;r_1,r_2)=\overline{OP}=\sqrt{\left(\frac{m_2}{m_1}r_2\cos\beta+r_1\right)^2+\left(\frac{m_2}{m_1}r_2\sin\beta\right)^2}$$

$$=\sqrt{r_1^2+\left(\frac{m_2}{m_1}r_2\right)^2+2r_1\left(\frac{m_2}{m_1}r_2\right)\cos\beta} \tag{15}$$

Hence

$$\frac{du_1}{d\beta}=\frac{-r_1\left(\frac{m_2}{m_1}r_2\right)\sin\beta}{\sqrt{r_1^2+\left(\frac{m_2}{m_1}r_2\right)^2+2r_1\left(\frac{m_2}{m_1}r_2\right)\cos\beta}}=\frac{-r_1\left(\frac{m_2}{m}r_2\right)\sin\beta}{u_1} \tag{16}$$

3.4 Probability Density Function ψ^{12} in Integral Form

The PDF $\psi^{12}(u_1;v_1,v_2)$ of the post-collision speed u_1 for two given pre-collision speeds v_1 and v_2 can be obtained by summating all densities for all possible directions, i.e., α between 0 and π, yields

$$\Psi^{12}(u_1;v_1,v_2)=\int_0^{\pi}\mathrm{P}_{\beta|\alpha}(\beta)\left|\frac{d\beta}{du_1}\right|\mathrm{P}_{\alpha}(\alpha)d\alpha=\int_0^{\pi}\frac{u_1}{4r_1\left(\frac{m_2}{m_1}r_2\right)}\sin\alpha\,d\alpha$$

$$=\int_0^{\pi}\frac{u_1\sin\alpha d\alpha}{4m_1m_2\sqrt{v_1^2+\left(\frac{m_2}{m_1}v_2\right)^2+2v_1\left(\frac{m_2}{m_1}v_2\right)\cos\alpha}\sqrt{v_1^2+v_2^2-2v_1v_2\cos\alpha}}$$

$$=\frac{u_1}{4m_1m_2}\int_0^{\pi}\frac{\sin\alpha d\alpha}{\sqrt{C+2B\cos\alpha-A\cos^2\alpha}} \tag{17}$$

Where

$$A=4v_1^2\hat{v}_2^2=r^4\sin^2(2\theta) \tag{18a}$$

$$B=v_1\left(\frac{m_2}{m_1}v_2\right)(v_1^2+v_2^2)-v_1v_2\left[v_1^2+\left(\frac{m_2}{m_1}v_2\right)^2\right]$$

$$=2v_1\hat{v}_2(v_1^2-\hat{v}_2^2)\left(\frac{m_2-m_1}{\sqrt{4m_1m_2}}\right)=r^4\sin(2\theta)\cos(2\theta)\tan\varphi \tag{18b}$$

$$C=\left[v_1^2+\left(\frac{m_2}{m_1}v_2\right)^2\right](v_1^2+v_2^2) \tag{18c}$$

$$\sqrt{B^2+AC}=v_1\left(\frac{m_2}{m_1}v_2\right)(v_1^2+v_2^2)+v_1v_2\left[v_1^2+\left(\frac{m_2}{m_1}v_2\right)^2\right]$$

$$=2v_1\hat{v}_2(v_1^2+\hat{v}_2^2)\left(\frac{1}{\sqrt{4m_1m_2}}\right)=r^4\sin(2\theta)\,/\cos\varphi \tag{18d}$$

$$\hat{v}_2=\sqrt{\frac{m_2}{m_1}}v_2,\quad r=\sqrt{v_1^2+\hat{v}_2^2},\quad \theta=\tan^{-1}\left(\frac{\hat{v}_2}{v_1}\right) \tag{18e:g}$$

$$\varphi=\tan^{-1}\left(\frac{m_2-m_1}{\sqrt{4m_1m_2}}\right),\quad m_1=\frac{M_1}{M_1+M_2},\quad m_2=\frac{M_2}{M_1+M_2} \tag{18h:j}$$

4. Probability Density Function of the Post-Collision Speed in Evaluated Form

The PDF ψ^{12}, shown in Equation (17), is in integral form and can be used to compute numerically the probability of post-collision speed of particles 1 after a collision. Because the PDF in integral form can be computed numerically, it can be used in Equation (8) to mechanical proof the Maxwell-Boltzmann speed distribution using numerical iteration. The limitation of computer-aided proof of the Maxwell-Boltzmann speed distribution is that two particles' mass ratio cannot be near infinite.

In order for the PDF ψ^{12} to be used for analytical proof of the Maxwell-Boltzmann speed distribution, the PDF ψ^{12}must be evaluated and formulated for all the possible pre-collision speeds of the two particles as detailed in the following sections.

4.1 Integrate the Probability Density Function

The PDF of the post-collision speed in the previous section is in integral form with an interval from 0 to π. Since some α values in the interval near the lower bound $\alpha = 0$ and the upper bound $\alpha = \pi$ cannot have a valid real number but some imaginary number, it is required to use the proper interval $(\alpha_{lb}, \alpha_{ub})$ of the integration and integrate the PDF ψ^{12} of Equation (17) as

$$\Psi^{12}(u_1; v_1, v_2) = \frac{u_1}{4m_1m_2}\int_{\alpha_{lb}}^{\alpha_{ub}} \frac{\sin\alpha\, d\alpha}{\sqrt{C + 2B\cos\alpha - A\cos^2\alpha}}$$

$$= \frac{u_1}{4m_1m_2\sqrt{A}}\left[\sin^{-1}\frac{B - A\cos\alpha}{\sqrt{B^2 + AC}}\right]_{\alpha_{lb}}^{\alpha_{ub}}$$

$$= \frac{u_1}{4m_1m_2r^2\sin(2\theta)}\left[\sin^{-1}(\sin\varphi\cos(2\theta) - \cos\varphi\sin(2\theta)\cos\alpha)\right]_{\alpha_{lb}}^{\alpha_{ub}} \tag{19a}$$

$$= \frac{u_1(\gamma_{ub} - \gamma_{lb})}{4m_1m_2r^2\sin(2\theta)} \tag{19b}$$

where

$$\gamma_{lb,ub} = \sin^{-1}\left(\sin\varphi\cos(2\theta) - \cos\varphi\sin(2\theta)\cos\alpha_{lb,ub}\right) \tag{20a}$$

$$r = \sqrt{v_1^2 + \hat{v}_2^2},\ \theta = \tan^{-1}\left(\frac{\hat{v}_2}{v_1}\right),\ \varphi = \tan^{-1}\left(\frac{m_2 - m_1}{\sqrt{4m_1m_2}}\right) \tag{20b:d}$$

4.2 Determine the Bounds of Interval

We will determine the lower bound (α_{lb}) and upper bound (α_{ub}) of $\angle AOB$ (α) for given pre-collision speed (v_1), pre-collision speed (v_2) and a target post-collision speed (u_1) in the above PDF $\psi^{12}(u_1; v_1, v_2)$. The $\angle AOB$ (α) is the angle between the pre-collision speed (v_1) and pre-collision speed (v_2), as shown in Figure 4.

The possible range of $\angle AOB$ (α) is from 0 to π. However, some ranges of α $(\angle AOB)$, near 0 or/and π, are impossible to reach the target post-collision speed (u_1) because u_1 $(\overline{OP})$ is bound by $\overline{OT'}$ and $\overline{OT}$ as following

$$\overline{OT'} \le \overline{OP} \le \overline{OT} \tag{21}$$

Where

$$\overline{OP} = u_1 \tag{22a}$$

$$\overline{OT'} = \left| r_1(\alpha; v_1, v_2) - \frac{m_2}{m_1} r_2(\alpha; v_1, v_2) \right| \tag{22b}$$

$$\overline{OT} = r_1(\alpha; v_1, v_2) + \frac{m_2}{m_1} r_2(\alpha; v_1, v_2) \tag{22c}$$

Where, r_1 and r_2 as shown in Figure 4, can be computed from α by Equation (14).

The inequation of Equation (21) can be formulated in terms of v_1, v_2, u_1, α and separated into two inequations as

$$\left| m_1\sqrt{v_1^2 + \left(\frac{m_2}{m_1}v_2\right)^2 + 2v_1\left(\frac{m_2}{m_1}v_2\right)\cos\alpha} - m_2\sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos\alpha} \right| \le u_1 \tag{23a}$$

$$u_1 \le m_1\sqrt{v_1^2 + \left(\frac{m_2}{m_1}v_2\right)^2 + 2v_1\left(\frac{m_2}{m_1}v_2\right)\cos\alpha} + m_2\sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos\alpha} \tag{23b}$$

Let the α in lower bound and upper bound of u_1 be α_{ub} and α_{lb} respectively and express the boundary values of the above equation using a plus-minus sign in conjunction with $\alpha_{lb,ub}$ for a more compact formulation as

$$u_1 = \left| m_1\sqrt{v_1^2 + \left(\frac{m_2}{m_1}v_2\right)^2 + 2v_1\left(\frac{m_2}{m_1}v_2\right)\cos\alpha_{lb,ub}} \pm m_2\sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos\alpha_{lb,ub}} \right| \tag{24}$$

The $\cos\alpha_{lb}$ and $\cos\alpha_{ub}$ in the above equation can be solved by taking square twice (Note 1), as shown in Equation (25). For convenience, let us assign a new variable $c_{lb,ub}$ to the solutions of $\cos\alpha_{lb,ub}$ as

$$\cos\alpha_{lb,ub} = \frac{(m_2-m_1)(v_1^2-u_1^2)\pm\sqrt{4m_1m_2}u_1\sqrt{v_1^2+\hat{v}_2^2-u_1^2}}{\sqrt{4m_1m_2}\,v_1\hat{v}_2} \equiv c_{lb,ub} \quad (25a)$$

Or in polar coordinate as follows, where $\theta_o = \cos^{-1}(u_1/r)$.

$$\cos\alpha_{lb,ub} = \frac{\sin\varphi\cos(2\theta)\pm\sin(2\theta_o\mp\varphi)}{\cos\varphi\ \sin(2\theta)} \equiv c_{lb,ub} \quad (25b)$$

Because the value of any cosine function cannot be smaller than -1 or larger than $+1$, we must limit the values of $\cos\alpha_{lb,ub}$ between -1 and +1 as

$$\cos\alpha_{lb} = \min(c_{lb},+1) \quad \text{and} \quad \cos\alpha_{ub} = \max(c_{ub},-1) \quad (26)$$

In the table below, $\gamma_{lb,ub}$ for each case is calculated by Equation (20a) as follows.

For Case 1 and Case 3, $|c_{lb,ub}| > 1$: $\cos\alpha_{lb,ub} = \pm 1 \quad (\alpha_{lb} = 0,\ \alpha_{ub} = \pi)$

$$\begin{aligned}\gamma_{lb,ub} &= \sin^{-1}(\sin\varphi\cos(2\theta)\mp\cos\varphi\sin(2\theta)) = \mp\sin^{-1}(\sin(2\theta\mp\varphi)) \\ &= \mp\min\big((2\theta\mp\varphi),(\pi-2\theta\pm\varphi)\big)\ \epsilon\,[-\pi/2,\pi/2]\end{aligned} \quad (27)$$

For Case 2 and Case 4, $|c_{lb,ub}| \le 1$: $\cos\alpha_{lb,ub} = c_{lb,ub} \quad (\alpha_{lb,ub} = \cos^{-1}c_{lb,ub})$

$$\begin{aligned}\gamma_{lb,ub} &= \sin^{-1}(\sin\varphi\cos(2\theta)-\sin\varphi\cos(2\theta)\mp\sin(2\theta_o\mp\varphi)) \\ &= \mp\sin^{-1}(\sin(2\theta_o\mp\varphi)) \\ &= \mp\min\big((2\theta_0\mp\varphi),(\pi-2\theta_0\pm\varphi)\big)\ \epsilon\,[-\pi/2,\pi/2]\end{aligned} \quad (28)$$

Equation (27) and Equation (28) can be combined to

$$\gamma_{lb,ub} = \mp\min\big((2\theta\mp\varphi),(\pi-2\theta\pm\varphi),(2\theta_0\mp\varphi),(\pi-2\theta_0\pm\varphi)\big) \quad (29)$$

Table 1. The functions γ_{lb} and γ_{ub} for all cases

	IF	Use	α	γ	
Case 1	$c_{lb} > 1$	$c_{lb} = \cos(\alpha_{lb}) = 1$	$\alpha_{lb} = 0$	$\gamma_{lb} = -\min[(2\theta-\varphi),(\pi-2\theta+\varphi)]$*	4 Possible L bounds
Case 2	$c_{lb} < 1$	$\alpha_{lb} = \cos^{-1}(c_{lb})$		$\gamma_{lb} = -\min[(2\theta_0-\varphi),(\pi-2\theta_0+\varphi)]$*	
Case 3	$c_{ub} < -1$	$c_{ub} = \cos(\alpha_{ub}) = -1$	$\alpha_{ub} = \pi$	$\gamma_{ub} = \min[(2\theta+\varphi),(\pi-2\theta-\varphi)]$ **	4 Possible U bounds
Case 4	$c_{ub} > -1$	$\alpha_{ub} = \cos^{-1}(c_{ub})$		$\gamma_{ub} = \min[(2\theta_0+\varphi),(\pi-2\theta_0-\varphi)]$**	

* $\gamma_{lb} = -\min[(2\theta-\varphi),(\pi-2\theta+\varphi),(2\theta_0-\varphi),(\pi-2\theta_0+\varphi)]$

** $\gamma_{ub} = \min[(2\theta+\varphi),(\pi-2\theta-\varphi),(2\theta_0+\varphi),(\pi-2\theta_0-\varphi)]$

Now we have determined the bounds $\alpha_{lb,ub}$ of the integral and the values $\gamma_{lb,ub}$. Let's summarize the PDF again as following

$$\Psi^{12}(u_1;v_1,v_2) = \frac{u_1(\gamma_{ub}-\gamma_{lb})}{8m_1m_2v_1\hat{v}_2} \quad (30)$$

where

$$\gamma_{lb,ub} = \mp\min\big((2\theta\mp\varphi),(\pi-2\theta\pm\varphi),(2\theta_0\mp\varphi),(\pi-2\theta_0\pm\varphi)\big) \quad (31a)$$

$$\hat{v}_2 = \sqrt{\frac{m_2}{m_1}}v_2,\quad \theta = \tan^{-1}\left(\frac{\hat{v}_2}{v_1}\right),\quad \varphi = \tan^{-1}\left(\frac{m_2-m_1}{\sqrt{4m_1m_2}}\right) \quad (31b{:}d)$$

$$r = \sqrt{v_1^2+\hat{v}_2^2},\quad \theta_o = \cos^{-1}\left(\frac{u_1}{r}\right) \quad (31e{:}f)$$

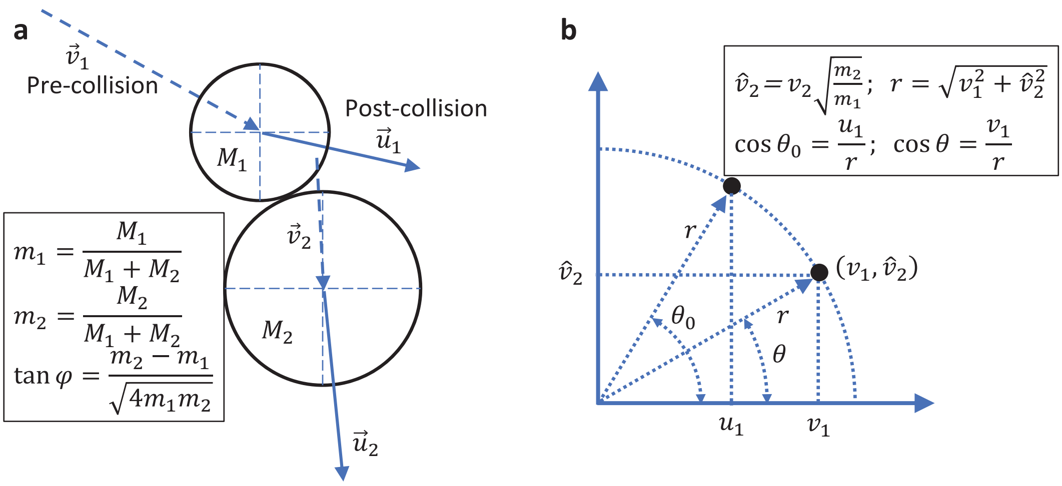


Figure 5. Relations between (r, θ, θ_0) and $(u_1, v_1, \hat{v}_2)$

4.3 Regions of $(\gamma_{ub} - \gamma_{lb})$ by the Lower and Upper Bounds

The PDF Ψ^{12} in the equation above cannot be formulated with one formula but 16 formulas. Because γ_{lb} has four possible formulas of $(2\theta - \varphi)$, $(\pi - 2\theta + \varphi)$, $(2\theta_0 - \varphi)$, $(\pi - 2\theta_0 + \varphi)$, and γ_{ub} have four possible formulas of $(2\theta + \varphi)$, $(\pi - 2\theta - \varphi)$, $(2\theta_0 + \varphi)$, $(\pi - 2\theta_0 - \varphi)$, the total combination of $(\gamma_{ub} - \gamma_{lb})$ is sixteen (4×4), as shown in the table below.

Table 2. Sixteen formulas of $(\gamma_{ub} - \gamma_{lb})$ for computing $\psi^{12}(u_1; v_1, v_2)$

	Lower bound	Case 1: $c_{lb} > 1$ $\rightarrow \alpha_{lb} = 0$		Case 2: $c_{lb} < 1$ $\rightarrow \alpha_{lb} = \cos^{-1}(c_{lb})$	
Upper bound	$\gamma_{ub} - \gamma_{lb}$ / $-\gamma_{lb}$ / γ_{ub}	$2\theta - \varphi$	$\pi - 2\theta + \varphi$	$2\theta_0 - \varphi$	$\pi - 2\theta_0 + \varphi$
Case 3: $c_{ub} < -1$ ↓ $\alpha_{ub} = \pi$	$2\theta + \varphi$	**A** 4θ	**B** (D-J Boundary) $\pi + 2\varphi$	**C** (A-K Boundary) $2\theta + 2\theta_0$	**D** $(m_1 > m_2)$ $\pi + 2\varphi + 2\theta - 2\theta_0$
	$\pi - 2\theta - \varphi$	**E** (G-M Boundary) $\pi - 2\varphi$	**F** $2\pi - 4\theta$	**G** $(m_1 < m_2)$ $\pi - 2\varphi - 2\theta + 2\theta_0$	**H** (F-P Boundary) $2\pi - 2\theta - 2\theta_0$
Case 4: $c_{ub} > -1$ ↓ $\alpha_{ub} = \cos^{-1}(c_{ub})$	$2\theta_0 + \varphi$	**I** (A-K Boundary) $2\theta + 2\theta_0$	**J** $(m_1 > m_2)$ $\pi + 2\varphi - 2\theta + 2\theta_0$	**K** $4\theta_0$	**L** (D-J Boundary) $\pi + 2\varphi$
	$\pi - 2\theta_0 - \varphi$	**M** $(m_1 < m_2)$ $\pi - 2\varphi + 2\theta - 2\theta_0$	**N** (F-P Boundary) $2\pi - 2\theta - 2\theta_0$	**O** (G-M Boundary) $\pi - 2\varphi$	**P** $2\pi - 4\theta_0$

The formulas of $(\gamma_{ub} - \gamma_{lb})$ in the table above can be visualized as eight regions (ADFGJKMP) with coordinates of v_1 and $\hat{v}_2$ as shown in Figure 6. Each region represents a formula of $(\gamma_{ub} - \gamma_{lb})$. Both γ_{ub} and γ_{lb} have four possible formulas, as shown in Equation (29), which in terms depends on the pre-collision speeds of two particles before a collision. The formula of $(\gamma_{ub} - \gamma_{lb})$ is the addition of γ_{ub} (Case 3 and Case 4) and $-\gamma_{lb}$ (Case 1 and Case 2) as shown in the table above.

In Figure 6, eight formulas (BCEHILNO) are not shown as regions. It can easily be proved that these eight formulas (BCEHILNO) are the boundary between two regions. For example, Formula E or O $(\pi - 2\varphi)$ is the boundary (part of BL_2) between Formula G and Formula M and can be obtained by equating the Formula G and Formula M as

$$\pi - 2\varphi - 2\theta + 2\theta_0 = \pi - 2\varphi + 2\theta - 2\theta_0 \;\rightarrow\; \theta = \theta_0 \;\rightarrow\; BL_2$$

Substituting the above equation of the boundary line (BL_2) into Formulas G and M shows that Formula G and Formula M merge to Formula E or O as

$$\text{Formula G:}\;\; \pi - 2\varphi - 2\theta + 2\theta_0 \;\rightarrow\; \pi - 2\varphi \;\rightarrow\; \text{Formula E or O}$$

$$\text{Formula M:}\;\; \pi - 2\varphi + 2\theta - 2\theta_0 \;\rightarrow\; \pi - 2\varphi \;\rightarrow\; \text{Formula E or O}$$

Other examples for the boundary lines BL_6 and BL_7 between Formula M-A and Formula M-0 can be obtained by

Formula M = Formula A and Formula M = 0 as

$$\pi - 2\varphi + 2\theta - 2\theta_0 = 4\theta \quad \rightarrow \quad \theta = \pi/2 - \varphi - \theta_0 \quad \rightarrow \ BL_6$$
$$\pi - 2\varphi + 2\theta - 2\theta_0 = 0 \quad \rightarrow \quad \theta = -(\pi/2 - \varphi - \theta_0) \quad \rightarrow \ BL_7$$

All the boundary lines in Figure 6 can be found from the formulas of two adjacent regions as described above and are listed in the table below.

Table 3. Equations of the boundary lines

Boundary line (BL)	Adjacent regions	Equation of boundary line
BL_0	K-0	$\theta_0 = 0;\ r = u_1$
BL_1	F-0	$\theta = \pi/2$
BL_2	F-P; G-M; K-A	$\theta = \theta_0;\ v_1 = u_1$
BL_3	A-0	$\theta = 0$
$BL_{4,8}$	P-M; F-G, P-D; F-J	$\theta = \pi/2 \pm \varphi - \theta_0$
$BL_{5,9}$	G-0, J-0	$\theta = \pi/2 \mp \varphi + \theta_0$
$BL_{6,10}$	G–K; M–A, J–K; D–A	$\theta = \pi/2 \mp \varphi - \theta_0$
$BL_{7,11}$	M-0, D-0	$\theta = -(\pi/2 \mp \varphi - \theta_0)$

For the case of $m_1 \leq m_2$, the angle φ is greater than zero ($\varphi = \tan^{-1}\left(\frac{m_2 - m_1}{\sqrt{4m_1m_2}}\right) \geq 0$), and there are six regions (AFGKMP) of $(\gamma_{ub} - \gamma_{lb})$, as shown in Figure 6(a). For the case of $m_1 \geq m_2$, the angle φ is less than zero ($\varphi \leq 0$), and region G becomes Region J, and region M becomes Region D, as shown in Figure 6(b).

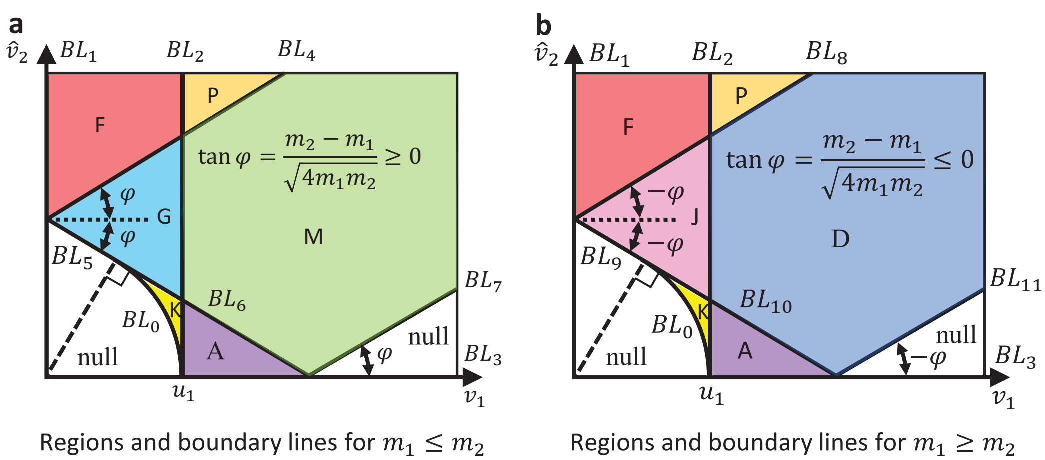


Figure 6. Eight regions of $(\gamma_{ub} - \gamma_{lb})$ and boundary lines

As it can be observed in Figure 6, the shapes of regions of $(\gamma_{ub} - \gamma_{lb})$ is related to the angle φ, which is defined as $\tan^{-1}\left(\frac{m_2 - m_1}{\sqrt{4m_1m_2}}\right)$, as shown in Equation (18h). Four combinations of m_1 and m_2 are used as examples to visualize the regions of $(\gamma_{ub} - \gamma_{lb})$ for some typical mass ratios. The four mass ratios are (a) $m_1 = 0.5, m_2 = 0.5$, (b) $m_1 = 0.3, m_2 = 0.7$, (c) $m_1 = 0.1, m_2 = 0.9$, (d) $m_1 = 0.01$, $m_2 = 0.99$. Where (a) is a special case representing equal-mass particles. The angle $\varphi = \tan^{-1}\left(\frac{m_2 - m_1}{\sqrt{4m_1m_2}}\right)$ determines the slope of the boundary lines BL_4 to BL_{11} as shown in Figure 6 and Figure 7.

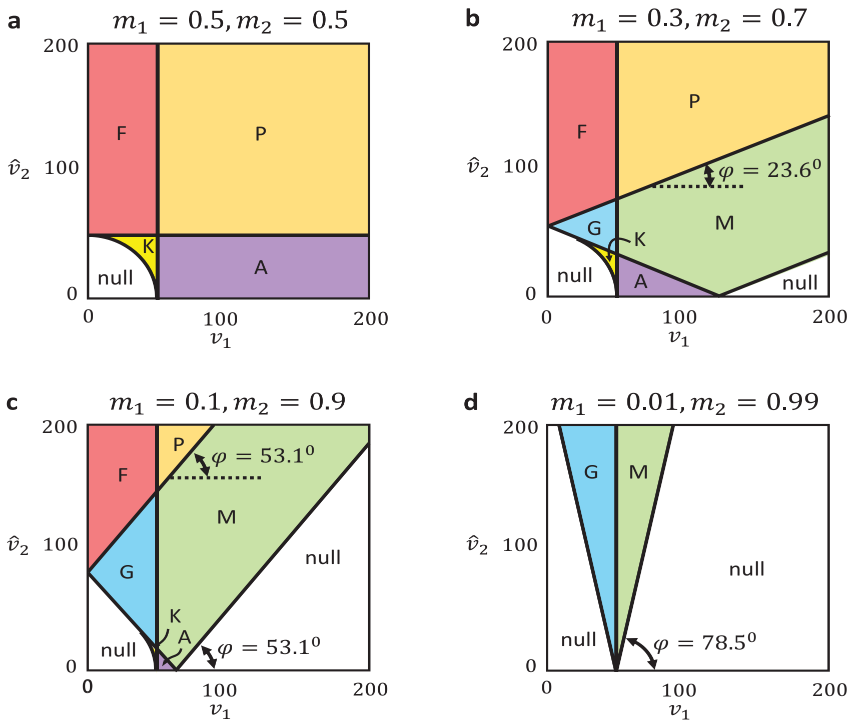


Figure 7. Regions A to P for the formulas of $(\gamma_{ub} - \gamma_{lb})$

for computing $\psi^{12}(u_1; v_1, v_2)$ with $u_1 = 50$

5. Analytical Proof by Double Integrals

The Maxwell-Boltzmann speed distribution is proved by examining speed distributions before and after a random collision of two particles in a steady-state system. We assumed the speed distributions of both particle 1 and particle 2 before a collision are the Maxwell-Boltzmann speed distributions. Suppose the integrated speed distributions of both particle 1 and particle 2 after the collision are also the Maxwell-Boltzmann speed distribution. In that case, it proves that the Maxwell-Boltzmann speed distribution is the right speed distribution of the steady-state system.

Before a random collision, we let the speed distributions $\mathrm{P}^1_{old}(v_1), \mathrm{P}^2_{old}(v_2)$ of particle 1 and particle 2 respectively be the Maxwell-Boltzmann speed distribution as

$$\mathrm{P}^1_{old}(v_1) = \frac{4h_1^3}{\sqrt{\pi}} v_1^2 e^{-h_1^2 v_1^2} \tag{32a}$$

$$\mathrm{P}^2_{old}(v_2) = \frac{4h_2^3}{\sqrt{\pi}} v_2^2 e^{-h_2^2 v_2^2} \tag{32b}$$

Where $h_1 = \sqrt{\frac{M_1}{2kT}}$, $h_2 = \sqrt{\frac{M_2}{2kT}}$ and k is the Boltzmann constant, T is the temperature.

The speed distributions $\mathrm{P}^1_{new}(u_1)$ of particle 1 after the collision can be calculated from the following equation with the PDF $\psi^{12}(u_1; v_1, v_2)$ computed by Equation (30) as

$$\mathrm{P}^1_{new}(u_1) = \int_0^\infty \int_0^\infty \psi^{12}(u_1; v_1, v_2)\, \mathrm{P}^1_{old}(v_1) \mathrm{P}^2_{old}(v_2)\, dv_1\, dv_2 \tag{33a}$$

$$= \int_0^\infty \int_0^\infty \psi^{12}(u_1; v_1, v_2)\, \frac{4h_1^3}{\sqrt{\pi}} v_1^2 e^{-h_1^2 v_1^2} \frac{4h_2^3}{\sqrt{\pi}} v_2^2 e^{-h_2^2 v_2^2}\, dv_1\, dv_2 \tag{33b}$$

Where $\psi^{12}(u_1; v_1, v_2)$ is the PDF of post-collision speed u_1 of particle 1 after a random collision of particle 1 with pre-collision speed v_1 and particle 2 with pre-collision speed v_2 as

$$\psi^{12}(u_1; v_1, v_2) = \frac{(\gamma_{ub} - \gamma_{lb}) u_1}{8 m_1 m_2 v_1 \hat{v}_2} \tag{34a}$$

with $$(\gamma_{ub} - \gamma_{lb}) = 0, \quad \text{for } r \le u_1 \quad \text{or} \quad \gamma_{ub} \le \gamma_{lb} \tag{34b}$$

5.1 Simplify the Distribution Parameters

To further evaluate the speed distribution $\mathrm{P}^1_{new}(u_1)$ as in Equation (33), the speed distribution can be first simplified by eliminating h_2 by the definition of the Boltzmann constant k as shown below

$$\frac{h_2}{h_1} = \frac{\sqrt{\frac{M_2}{2kT}}}{\sqrt{\frac{M_1}{2kT}}} = \sqrt{\frac{M_2}{M_1}} = \sqrt{\frac{m_2}{m_1}} \tag{35}$$

Since $v_2 = \sqrt{\frac{m_1}{m_2}}\hat{v}_2$ (as defined in Equation (18e)) and simplified symbol h_1 as h will yield

$$\mathrm{P}^1_{new}(u_1) = \int_0^\infty \int_0^\infty \psi^{12}(u_1; v_1, v_2)\, \frac{4h_1^3}{\sqrt{\pi}} v_1^2 e^{-h_1^2 v_1^2} \frac{4h_2^3}{\sqrt{\pi}} v_2^2 e^{-h_2^2 v_2^2}\, dv_1\, dv_2$$

$$= \int_0^\infty \int_0^\infty \psi^{12}(u_1; v_1, v_2) \frac{16h^6}{\pi}\, v_1^2 \hat{v}_2^2 e^{-h^2(v_1^2+\hat{v}_2^2)}\, dv_1\, d\hat{v}_2$$

$$= \frac{16h^6}{\pi} \int_0^\infty \int_0^\infty \frac{(\gamma_{ub}-\gamma_{lb})u_1 v_1 \hat{v}_2}{8m_1 m_2} e^{-h^2(v_1^2+\hat{v}_2^2)}\, dv_1\, d\hat{v}_2 \tag{36}$$

5.2 Convert Coordinate to Polar Coordinate

The speed distribution, as shown in the equation above, has a double integral for v_1 and $\hat{v}_2$. By converting Cartesian coordinates to polar coordinates: $v_1 = r\cos\theta, \hat{v}_2 = r\sin\theta,\ dv_1 d\hat{v}_2 = r d\theta dr.$ We can evaluate the first integral for θ then evaluate the second integral for r as following.

$$\mathrm{P}^1_{new}(u_1) = \frac{2h^6}{m_1 m_2 \pi} \int_{u_1}^\infty \int_0^{\pi/2} (\gamma_{ub} - \gamma_{lb}) u_1\, r\cos\theta\, r\sin\theta\, e^{-h^2 r^2} r d\theta\, dr$$

$$= \frac{h^6 u_1}{m_1 m_2 \pi} \int_{u_1}^\infty \left[\int_0^{\pi/2} (\gamma_{ub} - \gamma_{lb}) \sin(2\theta)\, d\theta\right] e^{-h^2 r^2} r^3\, dr \tag{37}$$

In the flowing section, we will evaluate the integral for θ.

5.3 Evaluate the Integral for θ

In this section, we will evaluate the inner integral $\int_0^{\pi/2}(\gamma_{ub} - \gamma_{lb})\sin(2\theta)\, d\theta$ in Equation (37). Where $(\gamma_{ub} - \gamma_{lb})$ could be any of the eight formulas depends on the pre-collision speeds v_1 and $\hat{v}_2$. Because $(\gamma_{ub} - \gamma_{lb})$ has a different formula for different v_1 and $\hat{v}_2$, the above integral needs to be separated into different integrals for different regions. For example, the integral path across regions MGF, $\int_{MGF}$, need to be evaluated by three integrals of corresponding formulas $(\gamma_{ub} - \gamma_{lb})$ from Table 2 and bounded by the relevant boundary lines from Table 3 as follows.

$$\int_{MGF} f\, d\theta = \int_{BL_7}^{BL_2} f_M\, d\theta + \int_{BL_2}^{BL_4} f_G\, d\theta + \int_{BL_4}^{BL_1} f_F\, d\theta$$

$$= \int_{-(\pi/2-\varphi-\theta_0)}^{\theta_0} f_M\, d\theta + \int_{\theta_0}^{\pi/2+\varphi-\theta_0} f_G\, d\theta + \int_{\pi/2+\varphi-\theta_0}^{\pi/2} f_F\, d\theta$$

Where

$$f_R = (\gamma_{ub} - \gamma_{lb})_R \sin(2\theta)$$

Substituting $(\gamma_{ub} - \gamma_{lb})_R$ for each region from Table 2 into the above three integrals and evaluating each integral will yield

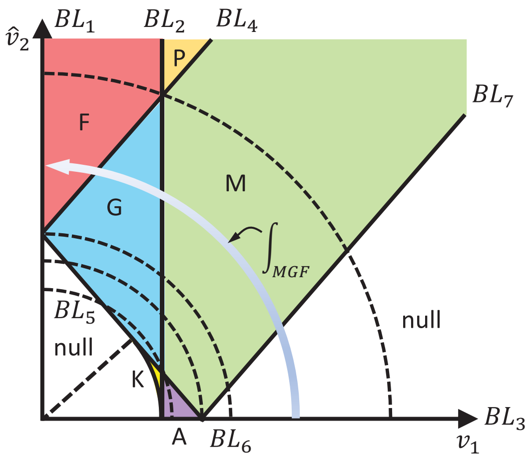
$$\int_{MGF} f\, d\theta = \int_{MGF} (\gamma_{ub} - \gamma_{lb}) \sin(2\theta)\, d\theta = 8m_1 m_2 \sin(2\theta_0)$$

It can be shown that the integral for θ integrating through different paths for different r results in the same function of θ_0 as

$$\int_{MPF} f\, d\theta = \int_{MGF} f\, d\theta = \int_{MG} f\, d\theta = \int_{AMG} f\, d\theta = \int_{AKG} f\, d\theta$$

$$= \int_0^{\pi/2} (\gamma_{ub} - \gamma_{lb}) \sin(2\theta)\, d\theta = 8m_1m_2 \sin(2\theta_0) \quad (38)$$

The detailed integrations are shown in Note 2.



$$\int_{MPF} = \int_{BL_7}^{BL_4} f_M\, d\theta + \int_{BL_4}^{BL_2} f_P\, d\theta + \int_{BL_2}^{BL_1} f_F\, d\theta$$

$$\int_{MGF} = \int_{BL_7}^{BL_2} f_M\, d\theta + \int_{BL_2}^{BL_4} f_G\, d\theta + \int_{BL_4}^{BL_1} f_F\, d\theta$$

$$\int_{MG} = \int_{BL_7}^{BL_2} f_M\, d\theta + \int_{BL_2}^{BL_5} f_G\, d\theta$$

$$\int_{AMG} = \int_{BL_3}^{BL_6} f_A\, d\theta + \int_{BL_6}^{BL_2} f_M\, d\theta + \int_{BL_2}^{BL_5} f_G\, d\theta$$

$$\int_{AKG} = \int_{BL_3}^{BL_2} f_A\, d\theta + \int_{BL_2}^{BL_6} f_K\, d\theta + \int_{BL_6}^{BL_5} f_G\, d\theta$$

Where $f_R = (\gamma_{ub} - \gamma_{lb})_R \sin(2\theta)$ and $(\gamma_{ub} - \gamma_{lb})_R$ for the six regions R as (AFGKMP) are listed in Table 2.

Figure 8. Integration paths

5.4 Evaluate the Integral for r

Substituting Equation (38) into Equation (37) yields

$$P_{new}^1(u_1) = \frac{8h^6u_1}{\pi} \int_{u_1}^{\infty} \sin(2\theta_0)\, e^{-h^2r^2} r^3\, dr$$

$$= \frac{8h^6u_1}{\pi} \int_{u_1}^{\infty} \frac{2u_1}{r} \sqrt{1 - \left(\frac{u_1}{r}\right)^2}\, e^{-h^2r^2} r^3\, dr$$

$$= \frac{16h^6u_1^2}{\pi} \int_{u_1}^{\infty} \sqrt{r^2 - u_1^2}\, e^{-h^2r^2} r dr \quad (39)$$

5.5 Cange Variable from r *to* v

Changing the variable from r to v by $v^2 = r^2 - u_1^2$, $2vdv = 2rdr$, and using $\int_0^{\infty} P_v(v)dv = \frac{4h^3}{\sqrt{\pi}} \int_0^{\infty} v^2\, e^{-h^2v^2} dv = 1$, yields

$$P_{new}^1(u_1) = \frac{16h^6u_1^2}{\pi} \int_0^{\infty} v\, e^{-h^2(u_1^2+v^2)} v dv$$

$$= \frac{4h^3}{\sqrt{\pi}} u_1^2 e^{-h^2u_1^2} \left[\frac{4h^3}{\sqrt{\pi}} \int_0^{\infty} v^2\, e^{-h^2v^2} dv\right]$$

$$= \frac{4h^3}{\sqrt{\pi}} u_1^2 e^{-h^2u_1^2} \quad (40)$$

This concludes the derivation of the Maxwell-Boltzmann speed distribution $P_{new}^1(u_1)$ of the post-collision speed of particle 1 as

$$P_{new}^1(u_1) = \frac{4h_1^3}{\sqrt{\pi}} u_1^2 e^{-h_1^2u_1^2} \quad (41)$$

The PDF $P_{new}^2(u_2)$ of the post-collisiton speed of particle 2 will also be exactly the Maxwell-Boltzmann speed distribution as

$$\mathrm{P}_{new}^{2}(u_2) = \frac{4h_2^3}{\sqrt{\pi}} u_2^2 e^{-h_2^2 u_2^2} \tag{42}$$

The above PDF $\mathrm{P}_{new}^{2}(u_2)$ can be concluded by simply treat particle 2 as particle 1 and following the same procedure for getting $\mathrm{P}_{new}^{1}(u_1)$.

6. Conclusion and Outlook

This paper analytically proved the Maxwell-Boltzmann speed distribution and the speed ratio of mixed particles based on particles' collision mechanics. The proof is based on a probability density function of post-collision velocities developed in Sections 3 and 4. The probability density function of the post-collision speed reveals the microscopic mechanics behind the macroscopic phenomenon. It can be used as a mathematical tool in the fields of statistical thermodynamics and kinetic theory.

The derivation of the Maxwell-Boltzmann speed distribution results in another significant outcome: the Maxwell-Boltzmann speed distribution is valid for interactions with extreme mass ratios between molecules and subatomic particles, where the mass ratio is between 10^{-12} and 10^{12}.

This article gives mechanical proof of the Maxwell-Boltzmann speed distribution and the speed ratio for monatomic particles only. The same procedures can be applied to polyatomic particles. The PDF of the post-collision speed must be extended to including the rotation of the molecules unless the rotation is small and its effect can be neglected. Moreover, the procedures may also be applied to charged particles. The chemical characteristic of molecules may be revealed from the speed and spin distributions induced by the mutual interaction of two different molecules. Our method provided in this paper may have a significant impact on this kind of research.

Acknowledgments

We appreciate the words "Everything is mechanics" said by Chao-Chung Yu (1915~2014), late president of the National Taiwan University. Indeed, the mechanics are the foundation of the universe. We also appreciate Prof. Yu for his encouragement and support.

References

Avogadro, A. (1822). Essai d'une maniere de determiner les masses relatives des molecules elementaires des corps, et les proportions selon lesquelles elles entrent dans ces combinations. *Journal de Physique, 73*, 58-76.

Boltzmann, L. (1872). Weitere studien uber das Warmegleichgewicht unter Gasmolekulen, Sitzungsbenchte der Kaiserlichen Akademie der Wissenschaften in Wien. *Mathematisch-Naturwissenschaftliche Classe, 66*, 275-370.

Boltzmann, L. (1877). Über die Beziehung zwischen dem zweitenHauptsatz der mechanischen Wärmetheorie und der Wahrscheinlichkeitsrechnung respektive den Sätzen über das Wärmegleichgewicht." Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften in Wien, Mathematisch-Naturwissenschaftliche Classe. Abt. II, *76*, 1877, 373–435. Reprinted in Wissenschaftliche Abhandlungen, Vol. II, pp. 164–223, Leipzig: Barth, 1909. https://doi.org/10.1017/CBO9781139381437.011

Brush, S. G. (1966). Kinetic theory Vol.2. Irreversible processes. Pergamon Press, ISBN 9781483155937 1483155935.Selection.2:Ludwig Boltzmann:Further Studies on the Thermal Equilibrium of Gas Molecules (from Sitzungsbenchte der Kaiserlichen Akademie der Wissenschaften,Vienna, 1872), 88-175. https://doi.org/10.1016/B978-0-08-011869-7.50009-6

Garrod, C. (1995). *Statistical mechanics and thermodynamics*. University of California, Davis, New York Oxford: Oxford University Press, ISBN 0-19-508523-X (IBM software).

Landau L. D., & Lifshitz, E. M. (1969). *Statistical physics*. Institute of Physical Problems, U.S.S.R. Academy of Science, Vol.5 of Course of Theoretical Physic, Translated from the Russian by Peierls, E. and Peierls, R.F., London, Paris: Pergamon Press, ISBN 0-201-0-04167-7.

Lin, T.-W., & Lin, H. (2019). Mechanical Proof of the Maxwell Speed Distribution, International Journal of Statistics and Probability, *8*(2), 90-98. https://doi.org/10.5539/ijsp.v8n2p90

Maudlin, T. (2013). How to derive the equilibrium velocity distribution two ways (Neither of which is what you are expecting). Cosmology Summer School, Santa Cruz, July 26, 2013, NYU. Retrieved from http://hipacc.ucsc.edu/IPC2013/slides/130626_TimMaudlin.pdf

Maxwell, J. C. (1860a). V. Illustrations of the dynamical theory of gases, Part I. On the motions and collisions of perfectly elastic spheres, The London, Edinburgh, and Dubin Philosophical Magazine and Journal of Science, 4th Series, *19*, 19-32. https://doi.org/10.1080/14786446808639951

Maxwell, J. C. (1860b). II. Illustrations of the dynamical theory of gases, Part II. On the process of diffusion of two or more kinds of moving particles among one another, The London. Edinburgh, and Dubin Philosophical Magazine and Journal of Science, 4th Series, *20*, 21-37, https://doi.org/10.1080/14786446008642902

Maxwell, J. C. (1867). XV. On the dynamical theory of gases, The London, Edinburgh, and Dubin Philosophical Magazine and Journal of Science, 4th Series, *35*, 129-145, & *36*, 185-217, https://doi.org/10.1080/14786446808639951, https://doi.org/10.1080/14786446808639963

McQuarrie, D. A. (1976). *Statistical mechanics.* Indiana University, Harper Collins Publishers, ISBN 06-044366-9.

Notes

Note 1.

To solve for $\cos\alpha_{lb,ub}$ from the following equation.

$$u_1 = \left| m_1\sqrt{v_1^2 + \left(\frac{m_2}{m_1}v_2\right)^2 + 2v_1\left(\frac{m_2}{m_1}v_2\right)\cos\alpha_{lb,ub}} \pm m_2\sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos\alpha_{lb,ub}} \right|$$

Let $c_{lb,ub} \equiv \cos\alpha_{lb,ub}$, and square to get

$$u_1^2 = \left(m_1^2v_1^2 + m_2^2v_2^2 + 2m_1m_2v_1v_2c_{lb,ub}\right) + \left(m_2^2v_1^2 + m_2^2v_2^2 - 2m_2^2v_1v_2c_{lb,ub}\right)$$
$$\pm 2\sqrt{\left(m_1^2v_1^2 + m_2^2v_2^2 + 2m_1m_2v_1v_2c_{lb,ub}\right)\left(m_2^2v_1^2 + m_2^2v_2^2 - 2m_2^2v_1v_2c_{lb,ub}\right)}$$

Rearrange and square again to get

$$\left[\left(m_1^2v_1^2 + m_2^2v_2^2 + 2m_1m_2v_1v_2c_{lb,ub}\right) + \left(m_2^2v_1^2 + m_2^2v_2^2 - 2m_2^2v_1v_2c_{lb,ub}\right) - u_1^2\right]^2$$
$$-4\left(m_1^2v_1^2 + m_2^2v_2^2 + 2m_1m_2v_1v_2c_{lb,ub}\right)\left(m_2^2v_1^2 + m_2^2v_2^2 - 2m_2^2v_1v_2c_{lb,ub}\right) = 0$$

or

$$\left(m_1^2v_1^2 + m_2^2v_2^2 + 2m_1m_2v_1v_2c_{lb,ub}\right)^2 + \left(m_2^2v_1^2 + m_2^2v_2^2 - 2m_2^2v_1v_2c_{lb,ub}\right)^2 + u_1^4$$
$$-2u_1^2\left(m_1^2v_1^2 + m_2^2v_2^2 + 2m_1m_2v_1v_2c_{lb,ub} + m_2^2v_1^2 + m_2^2v_2^2 - 2m_2^2v_1v_2c_{lb,ub}\right)$$
$$-2\left(m_1^2v_1^2 + m_2^2v_2^2 + 2m_1m_2v_1v_2c_{lb,ub}\right)\left(m_2^2v_1^2 + m_2^2v_2^2 - 2m_2^2v_1v_2c_{lb,ub}\right) = 0$$

or

$$A'\,c_{lb,ub}^2 + 2B'c_{lb,ub} + C' = 0 \tag{N1}$$

Where

$$A' = (2m_1m_2v_1v_2)^2 + (2m_2^2v_1v_2)^2 + 2(2m_1m_2v_1v_2)(2m_2^2v_1v_2)$$
$$= 4(m_1m_2 + m_2^2)^2v_1^2v_2^2 = 4m_2^2v_1^2v_2^2 = 4m_1m_2v_1^2\hat{v}_2^2$$
$$C' = u_1^4 - 2u_1^2(m_1^2v_1^2 + m_2^2v_2^2 + m_2^2v_1^2 + m_2^2v_2^2)$$
$$+(m_1^2v_1^2 + m_2^2v_2^2)^2 + (m_2^2v_1^2 + m_2^2v_2^2)^2 - 2(m_1^2v_1^2 + m_2^2v_2^2)(m_2^2v_1^2 + m_2^2v_2^2)$$
$$= u_1^4 - 2u_1^2\left((m_1^2 + m_2^2)v_1^2 + 2m_2^2v_2^2\right) + (m_1^2 - m_2^2)^2v_1^4$$
$$= u_1^4 - 2u_1^2\left((m_1^2 + m_2^2)v_1^2 + 2m_1m_2\hat{v}_2^2\right) + (m_1 - m_2)^2v_1^4$$
$$B' = 2m_1m_2v_1v_2(m_1^2v_1^2 + m_2^2v_2^2) - 2m_2^2v_1v_2(m_2^2v_1^2 + m_2^2v_2^2)$$
$$+2m_2^2v_1v_2(m_1^2v_1^2 + m_2^2v_2^2) - 2m_1m_2v_1v_2(m_2^2v_1^2 + m_2^2v_2^2)$$
$$+2u_1^2(m_2^2 - m_1m_2)v_1v_2$$
$$= 2m_1m_2v_1v_2(m_1^2v_1^2 - m_2^2v_1^2) + 2m_2^2v_1v_2(m_1^2v_1^2 - m_2^2v_1^2)$$

$$
\begin{aligned}
&+2u_1^2(m_2^2 - m_1m_2)v_1v_2 \\
&= 2m_2v_1v_2(m_1^2 - m_2^2)v_1^2 + 2u_1^2(m_2^2 - m_1m_2)v_1v_2 \\
&= 2m_2v_1v_2(m_1 - m_2)(v_1^2 - u_1^2) \\
&= \sqrt{4m_1m_2}v_1\hat{v}_2(m_1 - m_2)(v_1^2 - u_1^2)
\end{aligned}
$$

$$
\begin{aligned}
B'^2 - A'C' &= 4m_1m_2v_1^2\hat{v}_2^2 \\
&\quad \left((m_1 - m_2)^2(v_1^2 - u_1^2)^2 - u_1^4 + 2u_1^2\left((m_1^2 + m_2^2)v_1^2 + 2m_1m_2\hat{v}_2^2\right) - (m_1 - m_2)^2v_1^4\right) \\
&= 4m_1m_2v_1^2\hat{v}_2^2u_1^2\left(4m_1m_2(v_1^2 + \hat{v}_2^2) - u_1^2(1 - (m_1 - m_2)^2)\right) \\
&= (4m_1m_2v_1\hat{v}_2u_1)^2(v_1^2 + \hat{v}_2^2 - u_1^2)
\end{aligned}
$$

Therefore, the $c_{lb,ub}$ in Equation (N1) are given by

$$
\begin{aligned}
c_{lb,ub} &= \frac{-\sqrt{4m_1m_2}v_1\hat{v}_2(m_1 - m_2)(v_1^2 - u_1^2) \pm 4m_1m_2v_1\hat{v}_2u_1\sqrt{v_1^2 + \hat{v}_2^2 - u_1^2}}{4m_1m_2v_1^2\hat{v}_2^2} \\
&= \frac{(m_2 - m_1)(v_1^2 - u_1^2) \pm \sqrt{4m_1m_2}u_1\sqrt{v_1^2 + \hat{v}_2^2 - u_1^2}}{\sqrt{4m_1m_2}\, v_1\hat{v}_2}
\end{aligned}
\quad \text{(N2)}
$$

By using $r = \sqrt{v_1^2 + \hat{v}_2^2},\ \theta = \tan^{-1}\left(\frac{\hat{v}_2}{v_1}\right),\ \theta_o = \cos^{-1}\left(\frac{u_1}{r}\right),\ \varphi = \tan^{-1}\left(\frac{m_2 - m_1}{\sqrt{4m_1m_2}}\right),$

the equation above becomes

$$
\begin{aligned}
c_{lb,ub} &= \frac{\sin\varphi(\cos^2\theta - \cos^2\theta_o) \pm \cos\varphi\cos\theta_o\sin\theta_o}{\cos\varphi\cos\theta\sin\theta} = \frac{\sin\varphi(\cos(2\theta) - \cos(2\theta_o)) \pm \cos\varphi\sin(2\theta_o)}{\cos\varphi\ \sin(2\theta)} \\
&= \frac{\sin\varphi\cos(2\theta) \pm \sin(2\theta_o \mp \varphi)}{\cos\varphi\ \sin(2\theta)}
\end{aligned}
\quad \text{(N3)}
$$

Because the value of any cosine function cannot be smaller than -1 or larger than +1, we must limit the values of $\cos\alpha_{lb,ub}$ between -1 and +1 as

$$\cos\alpha_{lb} = \min(c_{lb}, +1) = \min\left(\frac{\sin\varphi\cos(2\theta) + \sin(2\theta_o - \varphi)}{\cos\varphi\ \sin(2\theta)}, +1\right) \quad \text{(N4)}$$

$$\cos\alpha_{ub} = \max(c_{ub}, -1) = \max\left(\frac{\sin\varphi\cos(2\theta) - \sin(2\theta_o + \varphi)}{\cos\varphi\ \sin(2\theta)}, -1\right) \quad \text{(N5)}$$

Note 2.

a) Integration paths across the regions (AKGF) and (AKG'):

A: $\int_{BL_3}^{BL_2} f_A\, d\theta = \int_0^{\theta_0} (4\theta)\sin(2\theta)\, d\theta = [-2\theta\cos(2\theta)]_0^{\theta_0} + [\sin(2\theta)]_0^{\theta_0}$

$= -2\theta_0\cos(2\theta_0) + \sin(2\theta_0)$

K: $\int_{BL_2}^{BL_6} f_K\, d\theta = \int_{\theta_0}^{\frac{\pi}{2} - \varphi - \theta_0} (4\theta_0)\sin(2\theta)\, d\theta = [-2\theta_0\cos(2\theta)]_{\theta_0}^{\frac{\pi}{2} - \varphi - \theta_0}$

$= -2\theta_0\cos(\pi - 2\varphi - 2\theta_0) + 2\theta_0\cos(2\theta_0)$

G': $\int_{BL_6}^{BL_5} f_G\, d\theta = \int_{\frac{\pi}{2}-\varphi-\theta_0}^{\frac{\pi}{2}-\varphi+\theta_0} (\pi - 2\varphi - 2\theta + 2\theta_0)\sin(2\theta)\, d\theta$

$= \frac{1}{2}[-(\pi - 2\varphi - 2\theta + 2\theta_0)\cos(2\theta)]_{\frac{\pi}{2}-\varphi-\theta_0}^{\frac{\pi}{2}-\varphi+\theta_0} - \frac{1}{2}[\sin(2\theta)]_{\frac{\pi}{2}-\varphi-\theta_0}^{\frac{\pi}{2}-\varphi+\theta_0}$

$= 2\theta_0\cos(\pi - 2\varphi - 2\theta_0) + \frac{1}{2}\sin(\pi + 2\varphi - 2\theta_0) + \frac{1}{2}\sin(\pi - 2\varphi - 2\theta_0)$

G: $\int_{BL_6}^{BL_4} f_G\, d\theta = \int_{\frac{\pi}{2}-\varphi-\theta_0}^{\frac{\pi}{2}+\varphi-\theta_0} (\pi - 2\varphi - 2\theta + 2\theta_0)\sin(2\theta)\, d\theta$

$= \frac{1}{2}[-(\pi - 2\varphi - 2\theta + 2\theta_0)\cos(2\theta)]_{\frac{\pi}{2}-\varphi-\theta_0}^{\frac{\pi}{2}+\varphi-\theta_0} - \frac{1}{2}[\sin(2\theta)]_{\frac{\pi}{2}-\varphi-\theta_0}^{\frac{\pi}{2}+\varphi-\theta_0}$

$= (2\varphi - 2\theta_0)\cos(\pi + 2\varphi - 2\theta_0) + 2\theta_0\cos(\pi - 2\varphi - 2\theta_0)$

$-\frac{1}{2}\sin(\pi + 2\varphi - 2\theta_0) + \frac{1}{2}\sin(\pi - 2\varphi - 2\theta_0)$

F: $\int_{BL_4}^{BL_1} f_F\, d\theta = \int_{\frac{\pi}{2}+\varphi-\theta_0}^{\frac{\pi}{2}} (2\pi - 4\theta)\sin(2\theta)\, d\theta$

$= [-(\pi - 2\theta)\cos(2\theta)]_{\frac{\pi}{2}+\varphi-\theta_0}^{\frac{\pi}{2}} - [\sin(2\theta)]_{\frac{\pi}{2}+\varphi-\theta_0}^{\frac{\pi}{2}}$

$= -(2\varphi - 2\theta_0)\cos(\pi + 2\varphi - 2\theta_0) + \sin(\pi + 2\varphi - 2\theta_0)$

Combine the results to get

$\int_{AKGF} (\gamma_2 - \gamma_1)\sin(2\theta)\, d\theta = \int_{AKG'} (\gamma_2 - \gamma_1)\sin(2\theta)\, d\theta$

$= \sin(2\theta_0) + \frac{1}{2}\sin(\pi - 2\varphi - 2\theta_0) + \frac{1}{2}\sin(\pi + 2\varphi - 2\theta_0)$

$= \sin(2\theta_0) + \sin(\pi - 2\theta_0)\cos(2\varphi) = (1 + \cos(2\varphi))\sin(2\theta_0)$

$= 2\cos^2\varphi\ \sin(2\theta_0) = 8m_1m_2\sin(2\theta_0)$

It is interesting to note that all the terms with a cosine function are canceled.

For $m_1 = m_2 = 1/2$, $\varphi = 0$, only two integration paths are needed: (AKF) and (APF).

$\int_{AKF} (\gamma_2 - \gamma_1)\sin(2\theta)\, d\theta = \int_{APK} (\gamma_2 - \gamma_1)\sin(2\theta)\, d\theta$

$= \sin(2\theta_0) + \sin(\pi - 2\theta_0) = 2\sin(2\theta_0) = 8m_1m_2\sin(2\theta_0)$

b) Integration paths across the regions (AMGF), (AMG'**)**, and (M'G'**)**:

A: $\int_{BL_3}^{BL_6} f_A\, d\theta = \int_0^{\frac{\pi}{2}-\varphi-\theta_0} (4\theta)\sin(2\theta)\, d\theta = [-2\theta\cos(2\theta)]_0^{\frac{\pi}{2}-\varphi-\theta_0} + [\sin(2\theta)]_0^{\frac{\pi}{2}-\varphi-\theta_0}$

$= -(\pi - 2\varphi - 2\theta_0)\cos(\pi - 2\varphi - 2\theta_0) + \sin(\pi - 2\varphi - 2\theta_0)$

M: $\int_{BL_6}^{BL_2} f_M \, d\theta = \int_{\frac{\pi}{2}-\varphi-\theta_0}^{\theta_0} (\pi - 2\varphi + 2\theta - 2\theta_0)\sin(2\theta)\, d\theta$

$$= \frac{1}{2}[-(\pi - 2\varphi + 2\theta - 2\theta_0)\cos(2\theta)]_{\frac{\pi}{2}-\varphi-\theta_0}^{\theta_0} + \frac{1}{2}[\sin(2\theta)]_{\frac{\pi}{2}-\varphi-\theta_0}^{\theta_0}$$

$$= (\pi - 2\varphi - 2\theta_0)\cos(\pi - 2\varphi - 2\theta_0) - \frac{1}{2}(\pi - 2\varphi)\cos(2\theta_0)$$

$$-\frac{1}{2}\sin(\pi - 2\varphi - 2\theta_0) + \frac{1}{2}\sin(2\theta_0)$$

M': $\int_{BL_7}^{BL_2} f_M \, d\theta = \int_{-\left(\frac{\pi}{2}-\varphi-\theta_0\right)}^{\theta_0} (\pi - 2\varphi + 2\theta - 2\theta_0)\sin(2\theta)\, d\theta$

$$= \frac{1}{2}[-(\pi - 2\varphi + 2\theta - 2\theta_0)\cos(2\theta)]_{-\left(\frac{\pi}{2}-\varphi-\theta_0\right)}^{\theta_0} + \frac{1}{2}[\sin(2\theta)]_{-\left(\frac{\pi}{2}-\varphi-\theta_0\right)}^{\theta_0}$$

$$= -\frac{1}{2}(\pi - 2\varphi)\cos(2\theta_0) + \frac{1}{2}\sin(\pi - 2\varphi - 2\theta_0) + \frac{1}{2}\sin(2\theta_0)$$

G': $\int_{BL_2}^{BL_5} f_G \, d\theta = \int_{\theta_0}^{\frac{\pi}{2}-\varphi+\theta_0} (\pi - 2\varphi - 2\theta + 2\theta_0)\sin(2\theta)\, d\theta$

$$= \frac{1}{2}[-(\pi - 2\varphi - 2\theta + 2\theta_0)\cos(2\theta)]_{\theta_0}^{\frac{\pi}{2}-\varphi+\theta_0} - \frac{1}{2}[\sin(2\theta)]_{\theta_0}^{\frac{\pi}{2}-\varphi+\theta_0}$$

$$= \frac{1}{2}(\pi - 2\varphi)\cos(2\theta_0) + \frac{1}{2}\sin(\pi + 2\varphi - 2\theta_0) + \frac{1}{2}\sin(2\theta_0)$$

G: $\int_{BL_2}^{BL_4} f_G \, d\theta = \int_{\theta_0}^{\frac{\pi}{2}+\varphi-\theta_0} (\pi - 2\varphi - 2\theta + 2\theta_0)\sin(2\theta)\, d\theta$

$$= \frac{1}{2}[-(\pi - 2\varphi - 2\theta + 2\theta_0)\cos(2\theta)]_{\theta_0}^{\frac{\pi}{2}+\varphi-\theta_0} - \frac{1}{2}[\sin(2\theta)]_{\theta_0}^{\frac{\pi}{2}+\varphi-\theta_0}$$

$$= (2\varphi - 2\theta_0)\cos(\pi + 2\varphi - 2\theta_0) + \frac{1}{2}(\pi - 2\varphi)\cos(2\theta_0)$$

$$-\frac{1}{2}\sin(\pi + 2\varphi - 2\theta_0) + \frac{1}{2}\sin(2\theta_0)$$

F: $\int_{BL_4}^{BL_1} f_F \, d\theta = \int_{\frac{\pi}{2}+\varphi-\theta_0}^{\frac{\pi}{2}} (2\pi - 4\theta)\sin(2\theta)\, d\theta$

$$= -(2\varphi - 2\theta_0)\cos(\pi + 2\varphi - 2\theta_0) + \sin(\pi + 2\varphi - 2\theta_0)$$

Combine the results to get

$$\int_{AMGF} (\gamma_2 - \gamma_1)\sin(2\theta)\, d\theta = \int_{AMG'} (\gamma_2 - \gamma_1)\sin(2\theta)\, d\theta = \int_{M'G'} (\gamma_2 - \gamma_1)\sin(2\theta)\, d\theta$$

$$= \frac{1}{2}\sin(\pi - 2\varphi - 2\theta_0) + \frac{1}{2}\sin(2\theta_0) + \frac{1}{2}\sin(\pi + 2\varphi - 2\theta_0) + \frac{1}{2}\sin(2\theta_0)$$

$$= \sin(2\theta_0) + \sin(\pi - 2\theta_0)\cos(2\varphi) = 8m_1 m_2 \sin(2\theta_0)$$

The integration path (M'G') is the only path for very small m_1.

c) Integration paths across the regions (AMPF) and (M'PF):

A: $\int_{BL_3}^{BL_6} f_A \, d\theta = \int_0^{\frac{\pi}{2}-\varphi-\theta_0} (4\theta)\sin(2\theta)\, d\theta$

$= -(\pi - 2\varphi - 2\theta_0)\cos(\pi - 2\varphi - 2\theta_0) + \sin(\pi - 2\varphi - 2\theta_0)$

M: $\int_{BL_6}^{BL_4} f_M \, d\theta = \int_{\frac{\pi}{2}-\varphi-\theta_0}^{\frac{\pi}{2}+\varphi-\theta_0} (\pi - 2\varphi + 2\theta - 2\theta_0)\sin(2\theta)\, d\theta$

$= \frac{1}{2}[-(\pi - 2\varphi + 2\theta - 2\theta_0)\cos(2\theta)]_{\frac{\pi}{2}-\varphi-\theta_0}^{\frac{\pi}{2}+\varphi-\theta_0} + \frac{1}{2}[\sin(2\theta)]_{\frac{\pi}{2}-\varphi-\theta_0}^{\frac{\pi}{2}+\varphi-\theta_0}$

$= -(\pi - 2\theta_0)\cos(\pi + 2\varphi - 2\theta_0) + (\pi - 2\varphi - 2\theta_0)\cos(\pi - 2\varphi - 2\theta_0)$

$+ \frac{1}{2}\sin(\pi + 2\varphi - 2\theta_0) - \frac{1}{2}\sin(\pi - 2\varphi - 2\theta_0)$

M': $\int_{BL_7}^{BL_4} f_M \, d\theta = \int_{-\left(\frac{\pi}{2}-\varphi-\theta_0\right)}^{\frac{\pi}{2}+\varphi-\theta_0} (\pi - 2\varphi + 2\theta - 2\theta_0)\sin(2\theta)\, d\theta$

$= \frac{1}{2}[-(\pi - 2\varphi + 2\theta - 2\theta_0)\cos(2\theta)]_{-\left(\frac{\pi}{2}-\varphi-\theta_0\right)}^{\frac{\pi}{2}+\varphi-\theta_0} + \frac{1}{2}[\sin(2\theta)]_{-\left(\frac{\pi}{2}-\varphi-\theta_0\right)}^{\frac{\pi}{2}+\varphi-\theta_0}$

$= -(\pi - 2\theta_0)\cos(\pi + 2\varphi - 2\theta_0) + \frac{1}{2}\sin(\pi + 2\varphi - 2\theta_0) + \frac{1}{2}\sin(\pi - 2\varphi - 2\theta_0)$

P: $\int_{BL_4}^{BL_2} f_P \, d\theta = \int_{\frac{\pi}{2}+\varphi-\theta_0}^{\theta_0} (2\pi - 4\theta_0)\sin(2\theta)\, d\theta = [-(\pi - 2\theta_0)\cos(2\theta)]_{\frac{\pi}{2}+\varphi-\theta_0}^{\theta_0}$

$= (\pi - 2\theta_0)\cos(\pi + 2\varphi - 2\theta_0) - (\pi - 2\theta_0)\cos(2\theta_0)$

F: $\int_{BL_2}^{BL_1} f_F \, d\theta = \int_{\theta_0}^{\frac{\pi}{2}} (2\pi - 4\theta)\sin(2\theta)\, d\theta = [-(\pi - 2\theta)\cos(2\theta)]_{\theta_0}^{\frac{\pi}{2}} - [\sin(2\theta)]_{\theta_0}^{\frac{\pi}{2}}$

$= (\pi - 2\theta_0)\cos(2\theta_0) + \sin(2\theta_0)$

Combine the results to get

$$\int_{AMPF} (\gamma_2 - \gamma_1)\sin(2\theta)\, d\theta = \int_{M'PF} (\gamma_2 - \gamma_1)\sin(2\theta)\, d\theta$$

$$= \frac{1}{2}\sin(\pi - 2\varphi - 2\theta_0) + \frac{1}{2}\sin(\pi + 2\varphi - 2\theta_0) + \sin(2\theta_0)$$

$$= \sin(\pi - 2\theta_0)\cos(2\varphi) + \sin(2\theta_0) = 8m_1 m_2 \sin(2\theta_0)$$

Copyrights

Copyright for this article is retained by the author(s), with first publication rights granted to the journal.

This is an open-access article distributed under the terms and conditions of the Creative Commons Attribution license (http://creativecommons.org/licenses/by/4.0/).

International Journal of Statistics and Probability; Vol. 10, No. 3; May 2021
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

Effect of Education on Attitude Towards Domestic Violence in Nigeria: An Exploration Using Propensity Score Methodology

Olanrewaju Davies Eniade[1], Joshua Odunayo Akinyemi[1,2], Oyindamola Bidemi Yusuf[1], Rotimi Felix. Afolabi[1], Olufunmilayo I. Fawole[1]

[1] Department of Epidemiology and Medical Statistics, College of Medicine, University of Ibadan, Ibadan, Nigeria

[2] Demography and Population Studies, Schools of Social Sciences and Public Health, University of the Witwatersrand, Johannesburg, South Africa

Correspondence: Olanrewaju Davies Eniade, Department of Epidemiology and Medical Statistics, College of Medicine, University of Ibadan, Ibadan, Nigeria

Received: March 15, 2021 Accepted: April 20, 2021 Online Published: April 30, 2021

doi:10.5539/ijsp.v10n3p154 URL: https://doi.org/10.5539/ijsp.v10n3p154

Abstract

Propensity Score Methodology (PSM) was used to investigate the effect of education on attitude towards domestic violence (ATDV) among men and women in Nigeria.

A total of 14,495 and 33,419 records were extracted for men and women respectively from the 2016-2017 Multiple Indicator Cluster Survey (MICS) in Nigeria. The outcome variable was ATDV. The study framework described the role of education on ATDV in the light of demographic characteristics, socioeconomic profile, and lifestyle. Selection bias was checked among the levels of education using the multinomial logit regression. Propensity scores (PS) and PS weights were generated for the treatment variable and average treatment effects (ATE) of ATDV were estimated using logistic regression that combined regression adjustment and inverse-probability weight. Descriptive statistics, odds ratios and 95%CI were presented.

The mean age of men and women were 30.8 ±10.2 years and 29 ±9.4 years respectively. About 22% men and 35% women justified domestic violence (DV) respectively. Selection bias was found between the covariates and level of education ($p<0.05$). PSM effectively corrected the selection bias (SD diff ≈ 0, Variance ratio ≈ 1). Men (AOR = 0.84, 95% CI: 0.78, 0.92) and women (AOR=0.94, 95%CI: 0.80, 2.22) who have attained tertiary level of education were less likely to justify DV in comparison to their uneducated counterparts.

Tertiary education was protective for ATDV among men and women. The use of PSM effectively controlled for selection bias in estimating the effect of education on ATDV. PSM will enable researchers make causal inference from non-experimental/cross-sectional studies in situations where randomized control trials are not feasible.

Keywords: propensity score, attitude towards domestic violence, treatment effect, selection bias

1. Background

1.1 Introduction

Experimental studies remain the gold standard when the measurement of a causal relationship is of interest. Scholars solely rely on Randomized Control Trials (RCT) to make causal inference in various fields of research. However, randomization, manipulation, and intervention are impossible in some research especially in evaluating the effect of programs (Oliver et al., 2002). For instance, it will be unethical for a researcher to deny some set of people access to education programs because of research. Similarly, it will be unacceptable for a researcher to expose some women to violence and watch if their access to maternal health care will be poorer than those who were not exposed to violence (Kean, Lock, & Howard-Lock, 1991; Sayar et al., 2019). However, analysis of observational data is an option, but the generalizability and the reliability of such findings are questionable especially in studies where a causal factor is of interest. The major problem of a non-experimental study is "selection bias" which is known as the systematic difference between the treatment (exposed) and control (non-exposed) group based on any number of covariates This systematic difference (selection bias) was corroborated by Shadish in a study where study participants who self-selected themselves into the training group performed better than those who were randomly assigned to the same training group (Shadish, Luellen, & Clark, 2006). Findings from Shadish study confirmed the claim of Rosenbaum and Rubin that participants who were not randomly assigned to treatment will tend to give a better report on the treatment or the exposure.

All efforts to adjust and correct for selection bias such as structural equation modeling (SEM) and adjusted regression showed no improvement (Cepeda, Boston, Farrar, & Strom, 2003). Only the use of PSM can effectively control for the selection bias (Arikan, van de Vijver, & Yagmur, 2018). Many studies have used PSM to address the problem of selection bias in quasi-experimental and cross-sectional study designs (Feng, Zhou, Zou, Fan, & Li, 2012; Rubin, 1997; Shadish et al., 2006; Yang, Imbens, Cui, Faries, & Kadziola, 2016; Yaya, Gunawardena, & Bishwajit, 2019). PSM is a statistical method that has proven useful for evaluating treatment effects when using non-experimental or observational data (Guo & Fraser, 2015). PSM is used when researchers need to assess the effect of covariates on the outcome variable using survey data, census, administration data, and other observational data without any intervention by random assignment rules (Rubin, 1997).

Domestic violence (DV) was defined as "any use of physical, sexual, psychological or economic violence of one family member, irrespective of person's age, gender or any other personal circumstance of the victim or the perpetrator of violence" (DHS, 2018). Attitude towards domestic violence (ATDV) has been identified as an indicator of the degree of social acceptance of DV and a known predictor of victimization and perpetration of DV. People's ATDV determines whether such violent acts will be reported or not (NBS, 2017; Okenwa-Emegwa, Lawoko, & Jansson, 2016). A higher proportion of men (25%) and women (35%) justified DV for reasons like, "wife burns the food", "argues with him", "goes out without telling him", "neglects the children", or "refuses sexual intercourse with him" (NDHS, 2018; Okenwa-Emegwa et al., 2016). The magnitude, extent, and predictors of ATDV against women have been examined among men and women (Fawole, Aderonmu, & Fawole, 2005; Okenwa-Emegwa et al., 2016). Factors such as Islamic religion, residency in the northern region, the South-South region, low levels of education, and low household wealth index have been reported to influence the justification of DV. Of the reported associated factors of DV and ATDV, studies have implicated education, but majority of the evidence was been based on observational studies which have limitations when it comes to "causal inference". It is on this premise that the present study is aimed at employing PSM to investigate the effect of education on ATDV.

Further, the use of PSM to estimate the effect of drug use on violent behaviors while adjusting for selection bias among students in South-West Nigeria showed that drug use was associated with the likelihood of violent behavior. (Yusuf, Akinyemi, Adedokun, & Omigbodun, 2014). Also, IPV has been linked as a risk factor for maternal health care utilization and poor pregnancy outcome using PSM (Yaya et al., 2019).

Studies have shown that a higher level of education was protective against the risks of DV among men and women (Bates, Schuler, Islam, & Islam, 2004; Koenig, Ahmed, Hossain, & Mozumder, 2003; Okenwa-Emegwa et al., 2016; Wang, 2016). Since ATDV is an indicator of the degree of social acceptance of DV and a known predictor of victimization and perpetration of DV (NBS, 2017), Since ATDV is an indicator of the degree of social acceptance of DV and a known predictor of victimization and perpetration of DV, it is important to investigate whether education will also be a protective factor for ATDV among the general population to be able to make policies that will protect current and potential victims of domestic violence and enhance a protective ATDV among the perpetrators.

We aimed to examine the effect of education on ATDV among men and women in Nigeria using PSM. We hypothesized that PSM will improve the estimation of the effect of educational level on ATDV among men and women.

2. Materials and Methods

2.1 Study Design and Setting

We used the 2016-2017 Multiple Indicator Cluster Survey (MICS5), a cross-sectional study carried out among adults (men and women) of age 15 to 49 years in Nigeria. Nigeria is the most populous African country with an estimated population of about 206 million inhabitants consisting of 99.1 million females (Thomas & Crow, 2020; Worldometer, 2020). Nigeria has 36 states and a Federal Capital Territory (political divisions). Nigeria has more than 50 ethnic groups among which Yoruba, Hausa/Fulani, and the Igbo are the dominants. Also, Islam and Christianity are the predominant religions practiced.

2.2 Study Population and Sampling Procedures

The study population included men and women who are between the ages of 15 and 49. The survey used the sampling frame to determine the enumeration areas (EAs), local government areas (LGAs), states, and zones in Nigeria as prepared in the 2006 Population Census of the Federal Republic of Nigeria. Details of the sampling procedure were provided in the MICS5 report. (NBS, 2017). For this analysis, records of men and women who responded to the questions on ATDV were sorted, resulting to a total of 14,495 and 33,419 records of men and women respectively.

2.3 Study Variables

We used ATDV as the outcome variable. ATDV was categorized as "DV justified" and "DV not justified". ATDV was measured by asking the respondents the following question. In your opinion, is a husband justified for hitting or beating his wife in the following situations: If she goes out without telling him, if she neglects the children, If she argues with him, If she refuses to have sex with him, If she burns the food.

Any respondent who said yes to any of the five questions was said to have justified DV. Also, whosoever said no to all the five questions does not justify DV. The treatment variable was Educational level, while the covariates were age, religion, occupation type, residential type, geopolitical region, marital status, wealth index, ethnicity, number of children, age at first sex, alcohol use, tobacco use, and media use.

2.4 Data Analysis

We described the demographic characteristics, socio-economic, and lifestyle using frequency tables and percentages. Association between the treatment variable (educational level) and all the categorical variables were tested using the chi-square test. The PSM was thereafter used to estimate the effect of level of education on ATDV.

2.5 Techniques Used in Propensity Score Methods

The approach was in three stages. First, we checked for imbalance (selection bias) between the treatment variable and the covariates using multinomial regression. Each of the study covariates were used as the outcome variable in the model and the treatment variable (Educational level) as the explanatory variable in the model,

Multinomial equation

$$\log\left(\frac{\pi_{ij}}{\pi_{iJ}}\right) = \alpha_j + \chi_i\beta j$$

Where πij is the probability of a response of the dependent that is greater or equal to a given category (i=2…4), πiJ is the probability of the response less than the given category (i=1), αj is a constant and βj is a vector of regression coefficients, for j=1,2,…,J−1. Xi is a vector of the covariates. At the second stage, we estimated generalized PS expressed as which is the generalized PS of receiving treatment dose *d* for participants *k* with observed covariate X. The inverse of the PSW was obtained for participants. The inverse PSW is expressed as $\frac{1}{e\ (X_{k,d})}$ (Bergstra et al., 2019).

Stage three was achieved by using the "tebalance summary" on "stata MP 14" to check if the standardized difference of the weighted scores is close to zero and the variance ratio for the weighted scores are close to one for all the covariates (SD diff ≈ 0, Variance ratio ≈ 1). If the result obtained satisfied the above criterion (i.e SD diff ≈ 0, Variance ratio ≈ 1), then selection bias has been corrected (i.e covariates are balanced) otherwise the selection bias has not been corrected. Lastly, we used the "teffect ipw" command on stata MP 14 to estimate the effect of the treatment (level of education). The "teffect ipw" command conducted a logistic regression that combined regression adjustment and inverse-probability weights between the study outcome variable ATDV and the propensity weight of the treatment variable. This provided the average treatment effect (ATE) which measures the effect of the PSW of educational level on ATDV.

Also, the potential outcome means (PO mean) which measures the effect of education on ATDV without the use of PS (Feng et al., 2012; Lu, Guo, & Li, 2020). Data were weighted to reflect educational level differentials in the population of men and women. Descriptive statistics, odds ratios, and 95%CI were presented. All analyses were conducted at 5% level of significance using stata MP 14 (StataCorp, 2015).

3. Results

3.1 Respondents Profile

Information about the socio-economic, demographic characteristics of men and women were presented in Tables 1 and 2. Men had a mean age of 29 years (SD=10 years). Of the 14,495 men who participated in this study, 22% justified DV. About 10.7% had no education while 17.3% had tertiary education. Close to half (48.2%) of the respondents were married. More than half (53.1%) of the respondents had no children, and about 32.7% used alcohol. Also, 97.3% do not smoke and most of them (56.6%) had media exposure. Also, 32.6% were residents of urban areas and 13.4% were from the South-West region. There was a preponderance (38.8%) of Hausa men in this study, 13.4% were Yorubas, and majority (44.8%) of the respondents were rich.

The mean age of female participants was 29 ± 9.4 years. Also, 34.5% justified DV and 10.8% of the women have attained the tertiary level of education. There was a preponderance of married women (70.7%) in this study and 28.1% had no children. Further, 18.6% used alcohol while almost all (99.6%) don't engage in cigarette smoking. More than a half (59.8%) were exposed to media and 32.0% were urban dwellers.

Table 1. Demographic Characteristics of men and women

	Men		**Women**	
Variables	**Frequency (n=14495)**	**Percentage (%)**	**Frequency**	**Percentage (%)**
Age				
15-19	3283	22.6	6312	18.9
20-24	2257	15.6	5569	16.7
25-29	2070	14.3	5835	17.5
30-34	2018	13.9	5211	15.6
35-39	1883	13.0	4343	13.0
40-44	1684	11.6	3564	10.7
45-49	1300	9.0	2585	7.7
Age Mean(SD)	29.1(10.0)		29(9.4)	
Education				
None	1552	10.7	4687	14
Primary	3443	23.8	12125	36.3
Secondary	6995	48.3	13006	38.9
Tertiary	2505	17.3	3601	10.8
Ethnicity				
Hausa	5555	38.3	13093	39.2
Igbo	1856	12.8	4715	14.1
Yoruba	1886	13	4234	12.7
Other ethnic group	5198	35.9	11377	34
Geopolitical Zones				
North central	2978	20.5	6767	20.2
North east	2338	16.1	4942	14.8
North west	3753	25.9	9124	27.3
South east	1381	9.5	3595	10.8
South-South	2109	14.5	4642	13.9
South west	1936	13.4	4349	13
Residence				
Urban	4722	32.6	10703	32
Rural	9773	67.4	22716	68
Marital status				
Married	6983	48.2	23569	70.7
Divorced/ widowed	225	1.6	8356	25.1
Single	7279	50.2	1400	4.2

Table 2. Respondents' profile

	Men		Women	
Variables	**Frequency (n=14495)**	**Percentage (%)**	**Frequency (n=33419)**	**Percentage (%)**
Parity				
None	7703	53.1	9395	28.1
1 – 2	2186	15.1	7327	21.9
3 – 4	2004	13.8	7376	22.1
more than 4	2602	18	9321	27.9
Total	14495	100	33419	100
Wealth index				
Poor	5138	35.4	12080	36.1
Average	2858	19.7	6612	19.8
Rich	6499	44.8	14727	44.1
Total	14495	100	33419	100
Alcohol				
Yes	4738	32.7	6229	18.6
No	9757	67.3	27189	81.4
Total	14495	100	33418	100
Tobacco use				
Yes	398	2.7	119	0.4
No	14097	97.3	33299	99.6
Total	14495	100	33418	100
Media Exposure				
No	6294	43.4	19978	59.8
Yes	8201	56.6	13441	40.2
Total	14495	100	33419	100

3.2 Selection Bias

The result from the multinomial logit model that was fitted to check for selection bias in the data was presented in the supplementary tables. The results revealed that selection bias was present in the data ($p < 0.05$).

3.3 Weighted Propensity Scores

Table 3 shows the standardized difference and variance ratio of the weighted PS for men and women. The standardized difference of the weighted scores is close to zero and the variance ratio is close to one. This implied that selection bias has been addressed with the use of PSM. Also, the similarities in the trends for each level of education presented in Figures 1 and 2 implied a good overlap in the estimated PS for educational level among men.

Table 3. Weighted propensity scores for men

	Men						Women					
	Primary		Secondary		Tertiary		Primary		Secondary		Tertiary	
	SD	VR	SD	VR	SD	VR	SD	VR	SD	VR	SD	VR
Variable												
Residence												
Rural	-0.37	1.58	-0.39	1.59	-0.37	1.58	-0.18	1.14	-0.13	1.11	-0.22	1.17
Marital status												
Divorced/widowed	-0.13	0.42	-0.13	0.42	-0.16	0.34	0.12	1.13	0.03	1.04	0.06	1.06
Single	0.58	1.48	0.63	1.49	0.59	1.48	0.06	1.68	0.1	2.13	0.1	2.23
Wealth index												
Poor	-0.36	1.24	-0.31	1.21	-0.39	1.28	-0.08	0.89	-0.07	0.89	-0.19	0.71
Middle	0.06	0.99	0.03	0.1	1.3	1.1	0.16	1.31	0.18	1.35	0.21	1.4
Rich	0.07	0.97	0.57	0.7	0.04	0.4	-0.85	1.16	0.04	1.06	0.08	1.1
Parity												
1-2	-0.53	0.55	-0.53	0.55	-0.59	0.49	0	0.99	0.03	1.04	-0.07	0.9
3-4	-0.38	0.6	-0.42	0.55	-0.45	0.52	0	1	0.01	1.02	-0.07	0.9
>4	0.31	1.96	0.27	1.84	0.41	2.24	-0.09	0.91	-0.06	0.95	0.09	1.07
Alcohol												
No	0.56	1	0.62	0.97	0.66	0.94	-0.85	1.12	0.04	1	0.14	1.03
Smoke												
No	-0.15	3.81	-0.17	4.33	-0.1	2.74	0	0.92	-0.02	1.45	-0.02	1.51
Media use												
No	-0.36	0.75	-0.37	0.75	-0.35	0.76	0.13	1.3	0.09	1.21	0.1	1.24

SD = Standard difference, VR= Variance Ratio

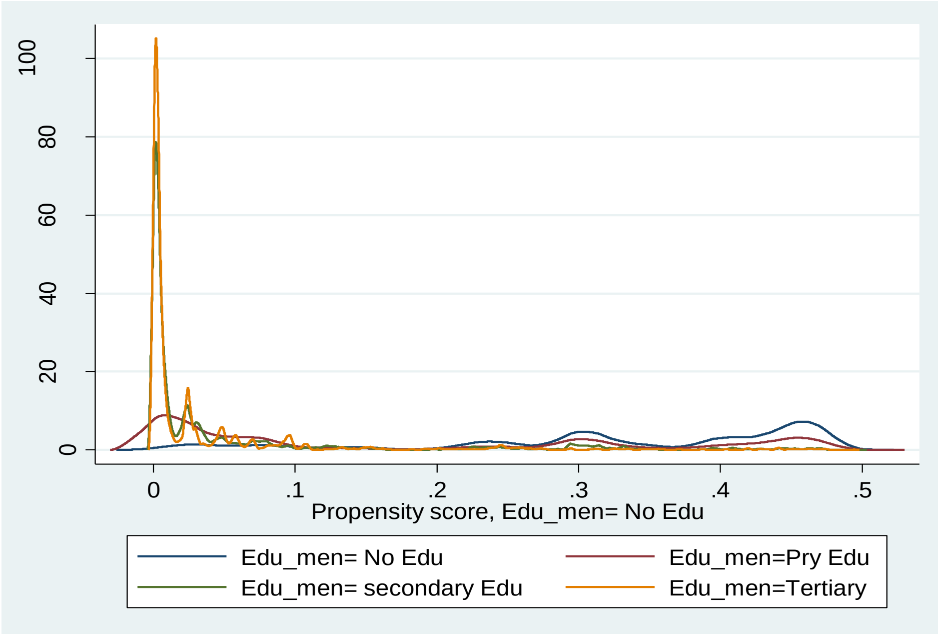


Figure 1. Overlap plot for the propensity score of level of education (Men)

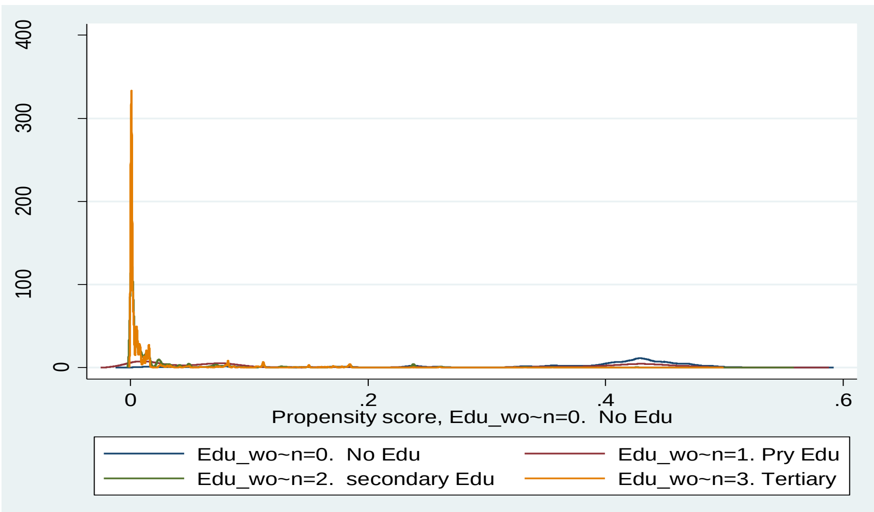


Figure 2. Overlap plot for the propensity score of level of education (Women)

No Edu (blue line): This represents the uneducated

Pry Edu (pink line): Those who have attained primary level of education

Secondary Edu (Green line): Those who have attained secondary level of education

Tertiary Edu (Green line): Those who have attained tertiary level of education

3.4 Treatment Effect for Attitude Towards Domestic Violence Among Men

The result from the multivariate analysis was presented in table 4. In comparison with uneducated men, those who have attained a tertiary level of education (AOR = 0.84, 95% CI: 0.78, 0.92) were less likely to justify DV. Similarly, the Yorubas (AOR = 1.12, 95% CI: 0.96, 1.31) were more likely to justify DV relative to Hausa men. The same pattern was observed for men from the rich wealth quintile (AOR = 1.07, 95% CI: 0.99, 1.17) compared to poor men. Also, men who were exposed to media (AOR = 1.02, 95% CI: 1.00, 1.03) were more likely to justify DV relative to their unexposed counterparts.

Women who have attained a tertiary level of education (AOR = 0.94, 95% CI: 0.80, 2.22) were less likely to justify DV compared to their uneducated counterparts. Similarly, Yoruba women (AOR = 0.99, 95% CI: 0.84, 1.17) were less likely to justify DV relative to Hausa women and rich women (AOR = 0.91, 95% CI: 0.81, 1.02) were less likely to justify DV compared to poor women.

Table 4. Treatment effect for attitude towards domestic violence among men and women

Logistic regression that combined regression adjustment and inverse-probability weighting						
	Men			**Women**		
	AOR	**95% CI**		**AOR**	**95% CI**	
Variables		**Lower**	**Upper**		**Lower**	**Upper**
Education						
None	ref					
Primary	0.94	0.87	1.03	1.03	0.88	2.41
Secondary	0.92	0.85	1	1.04	0.88	2.42
Tertiary	0.84	0.78	0.92	0.94	0.8	2.22
Residence						
Urban	ref					
Rural	1.05	1.03	1.07	1.04	1.02	1.06
Ethnicity						
Hausa	ref					
Igbo	1.04	0.9	1.2	1.07	0.96	1.18
Yoruba	1.12	0.96	1.31	0.99	0.84	1.17
Others	1.07	1.00	1.15	0.99	0.92	1.07
Marital Status						
Married	ref					
Single	1.06	0.97	1.16	0.95	0.83	1.08
Widowed/divorced	1.02	0.95	1.09	0.99	0.94	1.03
Parity						
None	ref					
1-2	1	0.96	1.05	1.04	1.02	1.07
3-4	1.12	0.97	1.28	1.06	1.02	1.1
>4	1.03	0.94	1.13	1.05	1.01	1.09
Media use						
Yes	1.02	1.00	1.03	1.02	1.01	1.04
No	ref					
Wealth index						
Poor	ref					
Average	1.07	0.99	1.17	0.96	0.86	1.08
Rich	1.06	0.97	1.16	0.91	0.81	1.02

AOR= Adjusted odds ratio for the treatment effects

4. Discussion

The exigency of policies that will enhance protective ATDV among the general population necessitated the investigation of the role of education on ATDV. A powerful statistical technique that is capable of providing better estimates was explored in this study.

The effect of education on ATDV was assessed among men and women in Nigeria using nationally representative data. We used ATDV as the main outcome variable, education as the treatment variable, while the explanatory variables were demographic characteristics, Socioeconomic profile, lifestyle, and others. Selection bias was detected in the data which led to the use of PSM since it's capable of minimizing selection bias in the data. The effectiveness of PSM has been established in previous studies (Yang et al., 2016; Yaya et al., 2019; Yusuf et al., 2014). This study showed that a lower proportion of men justified DV compared to women. Although, the prevalence of ATDV was higher than that of Ukraine and Ghana, but almost similar to that of Moldova and Namibia (Sardinha & Catalan, 2018). The disparity in the descriptive findings could be a consequence of the differences in the attributes of the countries, such as cultural beliefs and level of campaign against DV in the different countries. Arisi and Oromareghake reported that some cultures in Nigeria considered women as inferior beings, only beneficial in the kitchen, for pleasure and temptation (Arisi & Oromareghake, 2011). Also, it was known as common practice among men that women must kneel to beg their husbands when they are mistreated by their husbands (Arisi & Oromareghake, 2011). Krause also corroborated the findings by further explaining that some cultures considered those acts of wife-beating as a legitimate requital for a wife's defiance rather than seeing it as violence (Krause, Gordon-Roberts, VanderEnde, Schuler, & Yount, 2016). A higher proportion of women justified DV in this study. These findings were similar to that of a report in Palestine. The study buttressed that victims of DV are usually restrained from justifying DV to avoid marital separation as it could affect the children and their sustenance (Haj-Yahia, 2005).

Our results showed that only men who had primary education, secondary education, and tertiary education were less likely to justify DV, this is contrary to the previous finding where men who had primary and secondary education justified DV (Okenwa-Emegwa et al., 2016). This study and the previous study used a nationally representative data and the definitions of ATDV were similar, but the disparity could be as a result of the differences in the methods of analysis i.e the PSM that was used for this study has addressed the selection bias in the data thereby providing a better estimate (Cepeda et al., 2003). This paper has its limitations. The PSM is only capable of adjusting for selection bias. This method may not be capable of addressing other forms of bias, such as measurement bias. However, this limitation does not erode the strength of this study as it added to knowledge about statistical methodology and alternatives to improve findings from non-experimental studies.

Education played a crucial role in ATDV among men and women in Nigeria. Tertiary education was protective for ATDV among men and women. The use of PSM effectively controlled for selection bias in estimating the effect of education on ATDV. PSM will enable researchers to make causal inferences from non-experimental/ cross-sectional studies in situations where randomized control trials are not feasible.

List of abbreviations

Attitude towards domestic violence (ATDV)

PSM: Propensity Score Methodology

MICS: Multiple Indicator Cluster Survey

PS: Propensity scores

ATE: Average treatment effects

DV: Domestic violence

RCT: Randomized Control Trials

SEM: Structural equation modeling

NDHS: Nigerian Demographic and Health Survey

EAs: Enumeration areas

PO mean: Potential outcome means

UNICEF: United Nations International Children's Emergency Fund

Competing interests

We do not have any competing interests.

Ethical approval and consent to participate

Secondary data was used for this study. Informed consent and ethical approval were obtained for the primary data collection by United Nations International Children's Emergency Fund (UNICEF). Every confidential information and personal identifier has been excluded from the dataset before it was made available for this study. As a result, the confidentiality and anonymity of the respondents are guaranteed. Also, permission to use the MICS 2016/2017 dataset was requested and granted by UNICEF.

Acknowledgements

The authors would like to appreciate the UNICEF MICS team for making the dataset available for this study.

Funding

There was no funding for this study.

References

Arikan, S., van de Vijver, F. J., & Yagmur, K. (2018). Propensity Score Matching Helps to Understand Sources of DIF and Mathematics Performance Differences of Indonesian, Turkish, Australian, and Dutch Students in PISA. *International Journal of Research in Education and Science*, *4*(1), 69-81. https://doi.org/10.21890/ijres.382936

Arisi, R. O., & Oromareghake, P. (2011). Cultural violence and the Nigerian woman. *African Research Review, 5*(4). https://doi.org/10.4314/afrrev.v5i4.69290

Bates, L. M., Schuler, S. R., Islam, F., & Islam, M. K. (2004). Socioeconomic factors and processes associated with domestic violence in rural Bangladesh. *International family planning perspectives*, 190-199. https://doi.org/10.1363/3019004

Bergstra, S. A., Winchow, L.-L., Murphy, E., Chopra, A., Salomon-Escoto, K., Fonseca, J. E., . . . Landewé, R. B. (2019). How to treat patients with rheumatoid arthritis when methotrexate has failed? The use of a multiple propensity score to adjust for confounding by indication in observational studies. *Annals of the rheumatic diseases*, *78*(1), 25-30. https://doi.org/10.1136/annrheumdis-2018-213731

Cepeda, M. S., Boston, R., Farrar, J. T., & Strom, B. L. (2003). Comparison of logistic regression versus propensity score when the number of events is low and there are multiple confounders. *American journal of epidemiology, 158*(3), 280-287. https://doi.org/10.1093/aje/kwg115

DHS, M. (2018). Demographic and health surveys. *Calverton: Measure DHS*.

Fawole, O. I., Aderonmu, A. L., & Fawole, A. O. (2005). Intimate partner abuse: wife beating among civil servants in Ibadan, Nigeria. *African journal of reproductive health*, 54-64. https://doi.org/10.2307/3583462

Feng, P., Zhou, X. H., Zou, Q. M., Fan, M. Y., & Li, X. S. (2012). Generalized propensity score for estimating the average treatment effect of multiple treatments. *Statistics in medicine, 31*(7), 681-697. https://doi.org/10.1002/sim.4168

Guo, S., & Fraser, M. W. (2015). *Propensity score analysis*: Sage.

Haj-Yahia, M. M. (2005). On the characteristics of patriarchal societies, gender inequality, and wife abuse: The case of Palestinian society. *Adalah's Newsletter, 20*, 1-6.

Kean, W., Lock, C., & Howard-Lock, H. (1991). Gold complex research in medical science. Difficulties with experimental design. *Inflammopharmacology, 1*(2), 103-114. https://doi.org/10.1007/BF02735392

Koenig, M. A., Ahmed, S., Hossain, M. B., & Mozumder, A. K. A. (2003). Women's status and domestic violence in rural Bangladesh: individual-and community-level effects. *Demography, 40*(2), 269-288. https://doi.org/10.1353/dem.2003.0014

Krause, K. H., Gordon-Roberts, R., VanderEnde, K., Schuler, S. R., & Yount, K. M. (2016). Why do women justify violence against wives more often than do men in Vietnam? *Journal of interpersonal violence, 31*(19), 3150-3173. https://doi.org/10.1177/0886260515584343

Lu, D., Guo, F., & Li, F. (2020). Evaluating the causal effects of cellphone distraction on crash risk using propensity score methods. *Accident Analysis & Prevention*, *143*, 105579. https://doi.org/10.1016/j.aap.2020.105579

NBS, U. (2017). Multiple Indicator Cluster Survey 2016-17, Survey Findings Report: Nigeria National Bureau of Statistics and United Nations Children's Fund Abuja.

NDHS. (2018). Nigeria Demographic and Health Survey Key Indicators Report: Abuja.

Okenwa-Emegwa, L., Lawoko, S., & Jansson, B. (2016). Attitudes toward physical intimate partner violence against women in Nigeria. *Sage Open*, *6*(4), 2158244016667993. https://doi.org/10.1177/2158244016667993

Oliver, P. C., Piachaud, J., Done, J., Regan, A., Cooray, S., & Tyrer, P. (2002). Difficulties in conducting a randomized controlled trial of health service interventions in intellectual disability: implications for evidence-based practice. *Journal of Intellectual Disability Research*, *46*(4), 340-345. https://doi.org/10.1046/j.1365-2788.2002.00408.x

Rosenbaum, P. R., & Rubin, D. B. (1984). Reducing bias in observational studies using subclassification on the propensity score. *Journal of the American statistical Association, 79*(387), 516-524. https://doi.org/10.1080/01621459.1984.10478078

Rubin, D. B. (1997). Estimating causal effects from large data sets using propensity scores. *Annals of internal medicine, 127*(8_Part_2), 757-763. https://doi.org/10.7326/0003-4819-127-8_Part_2-199710151-00064

Sardinha, L., & Catalan, H. E. N. (2018). Attitudes towards domestic violence in 49 low-and middle-income countries: A gendered analysis of prevalence and country-level correlates. *PloS one, 13*(10), e0206101. https://doi.org/10.1371/journal.pone.0206101

Sayar, Z., Czuprynska, J., Patel, J. P., Benjamin, R., Roberts, L. N., Patel, R. K., . . . Arya, R. (2019). What are the difficulties in conducting randomised controlled trials of thromboprophylaxis in myeloma patients and how can we address these? Lessons from apixaban versus LMWH or aspirin as thromboprophylaxis in newly diagnosed multiple myeloma (TiMM) feasibility clinical trial. *Journal of thrombosis and thrombolysis, 48*(2), 315-322. https://doi.org/10.1007/s11239-019-01891-0

Shadish, W. R., Luellen, J. K., & Clark, M. (2006). Propensity Scores and Quasi-Experiments: A Testimony to the Practical Side of Lee Sechrest.

StataCorp, L. (2015). Stata statistical software: release 14.[computer program]. *StataCorp LP*.

Thomas, A., & Crow, B. (2020). *Third world atlas*: Taylor & Francis. https://doi.org/10.4324/9781003075394

Wang, L. (2016). Factors influencing attitude toward intimate partner violence. *Aggression and Violent Behavior, 29*, 72-78. https://doi.org/10.1016/j.avb.2016.06.005

Worldometer. (2020) 1–22. https://doi.org/10.1055/a-1024-0773

Yang, S., Imbens, G. W., Cui, Z., Faries, D. E., & Kadziola, Z. (2016). Propensity score matching and subclassification in observational studies with multi-level treatments. *Biometrics, 72*(4), 1055-1065. https://doi.org/10.1111/biom.12505

Yaya, S., Gunawardena, N., & Bishwajit, G. (2019). Association between intimate partner violence and utilization of facility delivery services in Nigeria: a propensity score matching analysis. *BMC public health, 19*(1), 1131. https://doi.org/10.1186/s12889-019-7470-1

Yusuf, O., Akinyemi, J. O., Adedokun, & Omigbodun. (2014). Propensity score methodology in estimating the effect of drug use on violent behaviour among students in south-west Nigeria. *Nigerian Journal of Psychiatry, 12*(100.0), 120.

Copyrights

Copyright for this article is retained by the author(s), with first publication rights granted to the journal.

This is an open-access article distributed under the terms and conditions of the Creative Commons Attribution license (http://creativecommons.org/licenses/by/4.0/).

ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

Reviewer Acknowledgements

International Journal of Statistics and Probability wishes to acknowledge the following individuals for their assistance with peer review of manuscripts for this issue. Their help and contributions in maintaining the quality of the journal is greatly appreciated.

Many authors, regardless of whether *International Journal of Statistics and Probability* publishes their work, appreciate the helpful feedback provided by the reviewers.

Reviewers for Volume 10, Number 3

Abouzar Bazyari, Persian Gulf University, Iran

Adekola Lanrewaju Olumide, Bells University of Technology, Nigeria

Carla J. Thompson, University of West Florida, USA

Carla Santos, New University of Lisbon, Portugal

Daoudi Hamza, Ibn Khaldoun University, Algeria

Felix Almendra-Arao, UPIITA del Instituto Politécnico Nacional, México

Kartlos Kachiashvili, Georgian Technical University, Georgia

Mohamed HSSIKOU, Ibn Zohr University, Morocco

Philip Westgate, University of Kentucky, USA

Pourab Roy, US Food and Drug Administration, USA

Renisson Neponuceno de Araujo Filho, Universidade Federal do Tocantins, Brazil

Robert Montgomery, University of Kansas Medical Center, USA

Sohair F. Higazi, University of Tanta, Egypt

Soukaina Douissi, National School of Applied Sciences (ENSA) Cadi Ayyad University, Morocco

Wei Zhang, The George Washington University, USA

Wojciech Gamrot, University of Economics, Poland

Wendy Smith

On behalf of,

The Editorial Board of *International Journal of Statistics and Probability*

Canadian Center of Science and Education

ACADEMIC AND OPEN ACCESS

Canadian Center of Science and Education

CCSENET.ORG

➢ CALL FOR MANUSCRIPTS

International Journal of Statistics and Probability is a peer-reviewed journal, published by Canadian Center of Science and Education. The journal publishes research papers in all aspects of statistics and probability. The journal is available in electronic form in conjunction with its print edition. All articles and issues are available for free download online.

We are seeking submissions for forthcoming issues. All manuscripts should be written in English. Manuscripts from 3000–8000 words in length are preferred. All manuscripts should be prepared in LaTeX or MS-Word format, and submitted online, or sent to: ijsp@ccsenet.org

Paper Selection and Publishing Process

a) Submission acknowledgement. If you submit manuscript online, you will receive a submission acknowledgement letter sent by the online system automatically. For email submission, the editor or editorial assistant sends an e-mail of confirmation to the submission's author within one to three working days. If you fail to receive this confirmation, please check your bulk email box or contact the editorial assistant.

b) Basic review. The editor or editorial assistant determines whether the manuscript fits the journal's focus and scope. And then check the similarity rate (CrossCheck, powered by iThenticate). Any manuscripts out of the journal's scope or containing plagiarism, including self-plagiarism are rejected.

c) Peer Review. We use a double-blind system for peer review; both reviewers' and authors' identities remain anonymous. The submitted manuscript will be reviewed by at least two experts: one editorial staff member as well as one to three external reviewers. The review process may take four to ten weeks.

d) Make the decision. The decision to accept or reject an article is based on the suggestions of reviewers. If differences of opinion occur between reviewers, the editor-in-chief will weigh all comments and arrive at a balanced decision based on all comments, or a second round of peer review may be initiated.

e) Notification of the result of review. The result of review will be sent to the corresponding author and forwarded to other authors and reviewers.

f) Pay the article processing charge. If the submission is accepted, the authors revise paper and pay the article processing charge (formatting and hosting).

g) E-journal is available. E-journal in PDF is available on the journal's webpage, free of charge for download. If you need the printed journals by post, please order at http://www.ccsenet.org/journal/index.php/ijsp/store/hardCopies.

h) Publication notice. The authors and readers will be notified and invited to visit our website for the newly published articles.

More Information

E-mail: ijsp@ccsenet.org

Website: http://ijsp.ccsenet.org

Paper Submission Guide: http://ijsp-author.ccsenet.org

Recruitment for Reviewers: http://www.ccsenet.org/journal/index.php/ijsp/editor/recruitment

➢ JOURNAL STORE

To order back issues, please contact the journal editor and ask about the availability of journals. You may pay by credit card, PayPal, and bank transfer. If you have any questions regarding payment, please do not hesitate to contact the journal editor or editorial assistant.

Price: $40.00 USD/copy   Shipping fee: $20.00 USD/copy

ABOUT CCSE

The Canadian Center of Science and Education (CCSE) is a private for-profit organization delivering support and services to educators and researchers in Canada and around the world.

The Canadian Center of Science and Education was established in 2006. In partnership with research institutions, community organizations, enterprises, and foundations, CCSE provides a variety of programs to support and promote education and research development, including educational programs for students, financial support for researchers, international education projects, and scientific publications.

CCSE publishes scholarly journals in a wide range of academic fields, including the social sciences, the humanities, the natural sciences, the biological and medical sciences, education, economics, and management. These journals deliver original, peer-reviewed research from international scholars to a worldwide audience. All our journals are available in electronic form in conjunction with their print editions. All journals are available for free download online.

Mission

To work for future generations

Values

Scientific integrity and excellence

Respect and equity in the workplace

CONTACT US

1595 Sixteenth Ave, Suite 301,
Richmond Hill, Ontario, L4B 3N9,
Canada
Tel: 1-416-642-2606
Fax: 1-416-642-2608
E-mail: info@ccsenet.org
Website: www.ccsenet.org

The journal is peer-reviewed
The journal is open-access to the full text
The journal is included in:

- Aerospace Database
- BASE (Bielefeld Academic Search Engine)
- EZB (Elektronische Zeitschriftenbibliothek)
- Google Scholar
- JournalTOCs
- Library and Archives Canada
- LOCKSS
- MIAR
- PKP Open Archives Harvester
- SHERPA/RoMEO
- Standard Periodical Directory
- Ulrich's

International Journal of Statistics and Probability

Bimonthly

Publisher Canadian Center of Science and Education
Address 1595 Sixteenth Ave, Suite 301, Richmond Hill, Ontario, L4B 3N9, Canada
Telephone 1-416-642-2606
Fax 1-416-642-2608
E-mail ijsp@ccsenet.org
Website http://ijsp.ccsenet.org

ISSN 1927-7032

9 771927 703213 03>