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Demystifying the Concept of Probability

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Abstract

This paper attempts to demystify the concepts underlying the study of probability. It also attempts to simplify the processes that govern the pragmatic use of probability. The paper employs practical examples in order to enable the use of probability in numerous fields, including sports, medicine, engineering, education, business, gambling, weather patterns, etc. At some point or another, we all employ probability without being aware that we are doing so. The conscious recognition that we are using probability makes our choices clearer and our decisions more informed.

Keywords: experiment, sample space, outcome, event, impossibility, certainty, axiomatic

Probability Theory

According to Lightner (1991), Probability theory had its root in the 16th century when Gerolamo Cardano, an Italian Mathematician and Physician, addressed the first work on the topic, ***The Book on Games of Chance (Liber de Ludo Aleae)***. Cardano's book contains the foundations of Mathematical Probability Theory about one hundred years before Pascal and Fermat. After its inception, the knowledge of probability was brought to the attention of great Mathematicians. Probability is the branch of Mathematics concerning numerical descriptions of how likely an event is to occur. The probability of an event is a number between **0** and **1**, where **0** indicates impossibility of the event and **1** indicates certainty of the event. All possible outcomes, ***a***, of an experiment is referred to as Sample Space, ***U***. That is, a is a subset of **U,** $\boldsymbol{A \subset U}$**.**

Equally Likely Events

When the outcomes of a random experiment have an equal likelihood of occurrence, they are called *equally likely events*. Like during a coin toss, the probability of getting a head or a tail is equally likely. Therefore, equally likely events have the same theoretical probability of occurring.

If the outcomes of an event are equally likely then we can calculate the probability using the formula:

$$\boldsymbol{Probability\ of\ an\ event = \frac{Number\ of\ successful\ outcomes}{Total\ number\ of\ possible\ outcomes}}$$

$$\boldsymbol{P(A) = \frac{n(A)}{n(U)}}$$

Example:

A bag contains 1 red, 3 green, 4 yellow, and 2 black marbles. What is the probability of pulling a green marble from the bag without looking?

Solution:

$$P(\text{green}) = \frac{Number\ of\ successful\ outcomes}{Total\ number\ of\ possible\ outcomes}$$

$$= \frac{3}{10}$$

$$= 0.3\ (30\%)$$

The Impossible Event

If P(A) = 0, then the event is an absolute impossibility, that is, the event will never occur, for example, the

probability of a person walking on the sun.

That is, if A = Ø (empty set).

Then, $P(A) = \frac{n(A)}{n(U)} = \frac{n(\emptyset)}{n(U)} = \frac{0}{n(U)} = 0 = P(\emptyset)$

Hence, P (Ø) = 0

The Certain Event

If P (A) = 1, then the event is an absolute certainty, that is, the event will occur, for example, the probability of a person walking in the park.

That is, if A = U (universal set).

Then, $P(A) = \frac{n(A)}{n(U)} = \frac{n(U)}{n(U)} = \frac{1}{n(U)} = 0 = P(U)$

Hence, P (U) = 1

Types of Probability

There are three major types of probabilities:

1. Experimental Probability
2. Theoretical Probability
3. Axiomatic Probability

Experimental Probability

Experimental Probability (relative frequency) is found by repeating an experiment and observing the outcomes. Therefore, experimental probability is the result of an experiment. The experimental probability can be calculated based on the number of possible outcomes by the total number of trials.

$$\boldsymbol{P(event) = \frac{number\ of\ times\ event\ occurs}{total\ number\ of\ trials}}$$

Example:

A coin is tossed 10 times. A head is recorded 7 times and tail 3 times.

Solution:

P(head)$= \frac{number\ of\ times\ event\ occurs}{total\ number\ of\ trials}$

$= \frac{7}{10}$

P(tail)$= \frac{number\ of\ times\ event\ occurs}{total\ number\ of\ trials}$

$= \frac{3}{10}$

Theoretical Probability

Theoretical probability is what is expected to happen after an event. It is mainly based on the reasoning behind probability.If the number of favorable outcomes and the number of possible outcomes can be determined, the probability can be calculated using the following formula:

$$\boldsymbol{Theoretical\ Probability = \frac{number\ of\ favourable\ outcomes}{total\ size\ of\ sample\ space}}$$

Example:

What is the probability of rolling a 3 on a number cube?

Solution:

$$P(3) = \frac{number\ of\ favourable\ outcomes}{total\ size\ of\ sample\ space}$$

$$= \frac{1}{6}$$

$$= 0.16$$

What is the probability of rolling a number less than 3 on a number cube?

Solution:

$$P(\text{less than } 3) = \frac{number\ of\ favourable\ outcomes}{total\ size\ of\ sample\ space}$$

$$= \frac{2}{6}$$

$$= \frac{1}{3}$$

$$= 0.33$$

Axiomatic Probability

In *axiomatic probability*, a set of rules or axioms are set which applies to all types of probabilities. Shafer and Vovk (2012) posit that these axioms are set by Andrey Nikolaevich Kolmogorov and are known as Kolmogorov's three axioms. With the axiomatic approach to probability, the chances of occurrence or non-occurrence of the events can be quantified.

Let ***S*** be the sample space of a random experiment. If a number ***P (A)*** assigned to each event $A \in S$ satisfies the following axioms, then ***P (A)*** is called the probability of ***A.***

Axiom 1: $P(A) \geq 0$

Axiom 2: $P(S) = 1$

Axiom 3: If $\{A_1, A_2, \ldots\}$ is a sequence of mutually exclusive events i.e., $A_i \cap A_j = \phi$

When, $i \neq j$

Then,

$$P\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Finding Probability by using the Complement

The complement **A'** of the event A consists of all the elements in the sample space that are not in **A**. The complement rule states that the sum of the probabilities of an event and its complement must be equal to **1**.

Given an event A,

$$P(U) = P(A) + P(A')$$

So, $1 = P(A) + P(A')$

That is $P(A) = 1 - P(A')$

Since $P(U) = 1$

$$P(A) = \frac{n(A)}{n(U)}$$

And $(PA') = \frac{n(A')}{n(U)}$

Example:

1. A coin is tossed once and the results are observed and noted.

Calculate the probability that:

I. a head appears

II. a tail appears

Solution

Sample space U = {H, T}

n (U) = 2

Let A = {head} = {H}

n (A) = 1

Let A'= {tail} = {T}

n (A') = 1

I. $P(A) = \frac{n(A)}{n(U)} = \frac{1}{2} = 0.5$

The probability of a head appearing is 0.5.

II. $P(A') = \frac{n(A')}{n(U)} = \frac{1}{2} = 0.5$

The probability of a tail appearing is 0.5.

2. A bag contains only red, yellow, and green marbles. The probability of choosing a red marble is $\frac{1}{4}$, the probability of choosing a yellow marble is $\frac{1}{2}$. What is the probability of choosing a green marble?

P (red) + P (yellow) + P (green) = 100%

25% + 50% + P (green) = 100%

75% + P (green) = 100%

P (green) = 100% - 75%

P (green) = 25%

Mutually Exclusive Events

Two events are ***mutually exclusive*** or ***disjoint*** events, if they both cannot occur in the same trial of an experiment. For example, rolling a 4 and an odd number on a number cube are mutually exclusive events because they both cannot happen at the same time. Suppose both ***A*** and ***B*** is two mutually exclusive events:

$A \cap B = \emptyset$ (disjoint set)

So $P(A \cup B) = P(A) + P(B)$

Where A and B are mutually exclusive events.

That is, P (both ***A*** and ***B*** will occur) = 0

P (either ***A*** or ***B*** will occur) = P (***A***) + P (***B***)

Example:

A fair number cube is rolled once and the result observed. What is the probability that a 2 or a 3 appears?

Solution:

The probability of a 2 appearing is $P(E_1) = \frac{1}{6}$

The probability of a 3 appearing is $P(E_2) = \frac{1}{6}$

The probability that a 2 or a 3 appearing is $P(E_1 \cup E_2) = P(E_1) + P(E_2)$

$$= \frac{1}{6} + \frac{1}{6}$$

$$= \frac{2}{6}$$

$$= \frac{1}{3}$$

The Addition Law of Probability

The general rule for mutually exclusive events is called the addition law of probability. The addition law of probability states that if E_1, E_2, E_3,…E_n are mutually exclusive events, then the probability of any one of the events occurring is given by:

$$\mathbf{P(E_1 \text{ or } E_2 \text{ or } E_3, \ldots \text{ or } E_n) = P(E_1 \cup E_2 \cup E_3 \ldots \cup E_n)}$$

$$\mathbf{= P(E_1) + P(E_2) + P(E_3) + \ldots P(E_n)}$$

That is, the probability of any one of the several mutually exclusive events occurring is equal to the sum of their individual probabilities.

Independent Events

An event ***B*** is said to be independent of another event ***A,*** if the probability of ***B*** occurring is not influenced by whether ***A*** has or has not occurred.

$$\mathbf{P(A \text{ and } B) = P(A \cap B) = P(A) \times P(B)}$$

Where ***A*** and ***B*** are independent events.

Example:

A coin and a number cube are tossed at the same time. Determine the probability that a tail and a 2 will result.

Solution:

The probability of a tail appearing is: $P(T) = \frac{1}{2}$

The probability of a 2 appearing is: $P(2) = \frac{1}{6}$

Therefore, the probability of a tail and a 2 appearing is: $\mathbf{P(T \cap 2) = P(T) \times P(2)}$

$$= \frac{1}{2} \times \frac{1}{6}$$

$$= \frac{1}{12}$$

Hence, the probability of a tail and a 2 appearing is $\frac{1}{12}$

The Multiplication Law of Probability

The general rule for independent events is called the multiplication law of probability. The multiplication law of probability states that if E_1, E_2, E_3, … E_n are independent events, then the probability of all the events occurring simultaneously is given by:

$$\mathbf{P(E_1 \text{ and } E_2 \text{ and } E_3 \text{ and } \ldots E_n)}$$

$$\mathbf{= P(E_1 \cap E_2 \cap E_3 \ldots \cap E_n)}$$

$$\mathbf{= P(E_1) \times P(E_2) \times P(E_3) \times \ldots P(E_n)}$$

That is, the probability of all the events occurring simultaneously is equal to the product of their individual probabilities.

Dependent Events

An event ***A*** and ***B*** is said to be dependent on another event ***A***, if the probability of ***B*** occurring is influenced by whether A has or has not occurred. The conditional probability of ***B*** given ***A*** is:

$$P\left(\frac{B}{A}\right) = \frac{P(A \text{ and } B)}{P(A)} = \frac{P(A \cap B)}{P(A)}$$

$$P(A \cap B) = P(A)\, X\, P(\frac{B}{A})$$

Example:

A box contains 10 similar balls. Four balls are green. Calculate the probability that a ball drawn at random is green. If the ball is green and not replaced, calculate the probability that a second ball drawn at random is also green. Hence, determine the probability of drawing two green balls.

Solution:

The number of green balls, n(G) = 4

The total number of balls, n(U) = 10

Therefore, $P(G) = \frac{n(G)}{n(U)} = \frac{4}{10} = \frac{2}{5}$

Hence, the probability of drawing a green ball is $\frac{2}{5}$.

The total number of green balls remaining, n(G) = 4 – 1 = 3

The total number of balls remaining, n(U) = 10 – 1 = 9

Therefore, P(G/G) = $\frac{n(G)}{n(U)} = \frac{3}{9} = \frac{1}{3}$

Hence, the probability of drawing a second green ball is $\frac{1}{3}$.

Thus, P (G and G) = P(G) X P(G/G)

$\frac{2}{5}$ X $\frac{1}{3}$

$\frac{2}{15}$

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Alpha Power Poisson-G Distribution With an Application to Bur XII Distribution Lifetime Data

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Abstract

We propose a new method of adding two shape parameters to a family of distributions for more flexibility and wider scope of applications called Alpha power Poisson-g distribution. A special case has been considered in details namely; one parameter exponential distribution. Various properties of the proposed distribution, including explicit expressions for the moments, quantiles, moment generating function, mean and median deviation, Bonferroni and Lorenz curve, order statistics and expression of the Renyi entropies are derived. The maximum likelihood estimators of unknown parameters cannot be obtained in explicit forms, and they have to be obtained by solving non-linear equations only. Further we consider an extension of the two-parameter Bur XII distribution also, mainly for data analysis purposes. Three data sets have been analyzed to show how the proposed models work in practice. We also carried out Monte Carlo simulation to further investigate the properties of the proposed method of estimation.

Keywords: Bur XII distribution, moments, Bonferroni and Lorenz curve, and maximum likelihood estimation.8

1. Introduction

Several methods have been proposed and developed to generate a new generalized family of distributions. This is to address the monotone failure rate exhibited by classical distributions which often makes it not suitable to model real lie data. Generated family of continuous distributions is a new improvement for creating and extending the usual classical distributions. The newly generated families have been broadly studied in several areas as well as yield more flexibility in applications. Eugene et al. (2002) proposed the beta generated method that uses the beta distribution with parameters α and β as the generator to develop the beta generated distributions. The CDF of a beta-generated random variable X is defined as

$$G(x) = \int_0^{F(x)} q(t)dt \tag{1}$$

where $q(t)$ is the PDF of a beta random variable and $F(x)$ is the CDF of any random variable X. Alzaatreh et al. (2013) introduced a new method for generating families of continuous distributions called $T-X$ family by replacing the beta PDF with a PDF, $r(t)$, of a continuous random variable and applying a function $W(F(x))$ that satisfies some specific conditions.

Zografos and Balakrishnan (2009) suggested a generated family using gamma distribution which is defined as follows

$$G_1(x) = \frac{1}{\Gamma(v)} \int_0^{-log[1-J(x;\xi)]} t^{v-1}e^{-t}dt, \tag{2}$$

Kumaraswamy generalized family provided by (Cordeiro and de Castro, 2011). Ristic and Balakrishnan, 2011) proposed an alternative gamma generator for any continuous distribution $J(x)$ which is defined as

$$G_1(x) = 1 - \frac{1}{\Gamma(v)} \int_0^{-log[1-J(x;\xi)]} t^{v-1}e^{-t}dt, \tag{3}$$

Where, $\Gamma(v) = G_1(x) = \frac{1}{\Gamma(v)}\int_0^\infty t^{v-1}e^{-t}dt,$ is the gamma function.

Further, some generated families were studied by several authors, for example, the Cordeiro and de Castro (2011) developed the Kumaraswamy-G, kummer beta by Pescim et al. (2012), exponentiated generalized class by Cordeiro et al. (2013),), Weibull-G by Bourguignon et al. (2014), exponentiated half-logistic by Cordeiro et al. (2014), transmuted exponentiated generalized-G by Yousof et al. (2015), the type I half-logistic by (Cordeiro et al., 2015), the Kumaraswamy Weibull by Hassan and Elgarhy (2016), transmuted geometric-G by Afify et al. (2016a) , Kumaraswamy transmuted-G family of distribution was studied by Afify et al. (2016b). Nofal et al. (2017) developed the transmuted-G family Alizadeh et al. (2017) developed the generalized odd generalized exponential, exponentiated Weibull-H by Cordeiro et al. (2017), alpha power transformation Mahdavi and Kundu (2017), exponentiated generalized-G Poisson by Aryal and Yousof (2017), Alizadeh et al. (2018) proposed and studied transmuted Weibull-G, Marshall-Olkin generalized-G Poisson by Korkmaz et al. (2018). Oluyede, et al. (2018) introduced the gamma Weibull-G and odd Lomax-G family by Cordeiro et al. (2019).

The aim of this paper is to introduce two extra parameters to a family of distributions functions to bring more flexibility and enhance the scope of applications to the given family. We call this new method as Alpha Power Poisson-g family of (APP-G) distribution. The proposed APP-G distribution is tractable and very easy to use; hence it can be used quite effectively to model data exhibiting different shapes of the hazard function. We have use the APPG method to a two-parameter Bur XII distribution, and generated a four-parameter Alpha power Bur XII Poisson ($APBXIIP$) distribution with more modeling potentials. It is observed that the four-parameter $APBXIIP$ distribution has several desirable properties. The PDF and the hazard functions of $APBXII$P distribution can take similar shapes as the Weibull, Gamma or logistic distribution. The PDF of $APBXIIP$ distribution can be expressed in explicit form; hence it can be used quite conveniently for analyzing censored data exhibiting non-monotone failure rate.

This paper is organized as follows. In section 2, we introduced the Alpha power Poisson-g family of distribution are examined. The Alpha Power Bur XII Poisson distribution, survival hazard rate, hazard function, and mixture representation of $APBXIIP$ distribution are given in section 3. Section 4 contains the ordinary and incomplete moments, mean and median deviation, moment generating function, Bonferroni and Lorenz curves. Renyi entropy and Order statistics are considered in section 5. Monte Carlo simulation study is conducted to examine the Absolute bias and mean square error of the maximum likelihood estimators. Also, the results on the estimation of the parameters of the $APBXIIP$ model via the maximum likelihood estimation technique is given in section 6 which also includes real data applications. Concluding remarks is given in section 7.

2. Alpha Power Poisson–g Method

Suppose that a system has N subsystems functioning independently at a given time where N has zero truncated Poisson (ZTP) distribution with parameter θ. It is the conditional probability distribution of a Poisson-distributed random variable, given that the value of the random variable is not zero. The probability mass function (pmf) of N is given by

$$P(N = n) = \frac{1}{(1 - e^{-\theta})}\frac{e^{-\theta}\theta^n}{n!}, \qquad n = 1,2, \ldots. \tag{5}$$

the expected value and variance are respectively given by

$$E(N) = \frac{\theta}{1 - e^{-\theta}}$$

And

$$Var(N) = \frac{\theta + \theta^2}{1 - e^{-\theta}} - \frac{\theta^2}{(1 - e^{-\theta})^2}$$

Suppose that the failure time of each subsystem has the Alpha power transformed-G (APT-G) distribution defined by the cumulative distribution function (CDF) and probability density function (PDF) are respectively given by

$$H_{APT}(x; \alpha, \xi) = \begin{cases} \dfrac{\alpha^{H(x)} - 1}{\alpha - 1}, & if\ \alpha > 0, \alpha \neq 1 \\ H(x), & \alpha = 0 \end{cases} \tag{6}$$

and

$$h_{APT}(x;\alpha,\xi)=\begin{cases}\dfrac{log\alpha}{\alpha-1}h(x)\alpha^{H(x)}, & if\ \alpha>0,\alpha\neq 1\\ H(x), & if\ \alpha=0\end{cases} \tag{7}$$

where α is the additional shape parameter. Let Z_i denote the failure time of the i^{th} subsystem and let $X = min\{Z_1,\ldots,Z_N\}$. Then the conditional CDF of X given N is

$$F(x/N)=1-P(X>x/N)=1-\left(1-H_{APT}(x;\alpha,\xi)\right)^N \tag{8}$$

Therefore, the unconditional cdf of X, as described in Ramos et al. (2015), can be expressed as

$$F(x;\alpha,\theta,\xi)=\left[\frac{1}{1-e^{-\theta}}\right]\left[1-e^{-\theta\left(\frac{\alpha^{H(x)}-1}{\alpha-1}\right)}\right] \tag{9}$$

The cdf in (9) is called the Alpha power G Poisson (APGP) family of distributions. The corresponding pdf is

$$f(x;\alpha,\theta,\xi)=\left[\frac{\theta\log(\alpha)}{(1-e^{-\theta})(\alpha-1)}\right]h(x)\alpha^{\bar{H}(x)}e^{-\theta\left(\frac{\alpha^{\bar{H}(x)}-1}{\alpha-1}\right)} \tag{10}$$

An expression for the survival and the hazard function for APGP family of distribution is respectively given by

$$S(x;\alpha,\theta,\xi)=1-\left[\frac{1}{1-e^{-\theta}}\right]\left[1-e^{-\theta\left(\frac{\alpha^{\bar{H}(x)}-1}{\alpha-1}\right)}\right] \tag{11}$$

And

$$\bar{h}(x;\alpha,\theta,\xi)=\frac{\theta log(\alpha)\bar{h}(x)\alpha^{\bar{H}(x)}e^{-\theta\left(\frac{\alpha^{\bar{H}(x)}-1}{\alpha-1}\right)}}{(1-e^{-\theta})(\alpha-1)\left\{1-\left[\frac{1}{1-e^{-\theta}}\right]\left[1-e^{-\theta\left(\frac{\alpha^{\bar{H}(x)}-1}{\alpha-1}\right)}\right]\right\}} \tag{12}$$

Using the power series expansion

$$e^x=\sum_{v=0}^{\infty}\frac{x^v}{v!} \tag{13}$$

we express the PDF in (12) as

$$f(x;\alpha,\theta,\xi)=\left[\frac{\theta log\alpha}{(1-e^{-\theta})}\right]\bar{h}(x)\alpha^{\bar{H}(x)}\sum_{i=0}^{\infty}\frac{(-\theta)^i(-1)^i}{i!\,(\alpha-1)^{i+1}}\left(1-\alpha^{\bar{H}(x)}\right)^i \tag{14}$$

Also, applying series expansion given by

$$(1-p)^v=\sum_{j=0}^{\infty}(-1)^j\binom{v}{j}p^v \tag{15}$$

We have,

$$f(x;\alpha,\theta,\xi)=\left[\frac{\theta log\alpha}{(1-e^{-\theta})}\right]\bar{h}(x)\sum_{i,j=0}^{\infty}\frac{(-\theta)^i(-1)^{i+j}}{i!\,(\alpha-1)^{i+1}}\binom{i}{j}\alpha^{(j+1)\bar{H}(x)} \tag{16}$$

Since,

$$\alpha^u = \sum_{t=0}^{\infty} \frac{(log\alpha)^t}{t!} u^t \tag{17}$$

Finally, we have

$$f(x;\alpha,\theta,\xi) = \left[\frac{\theta}{(1-e^{-\theta})}\right] \bar{h}(x) \sum_{i,j,k=0}^{\infty} \frac{(-\theta)^i (-1)^{i+j} (log\alpha)^{k+1} (j+1)^k}{i!\, k!\, (\alpha-1)^{i+1} (k+1)} \binom{i}{j} (k+1)[\bar{H}(x)]^k \tag{18}$$

Where $H(x)$ and $h(x)$ are the DF and the PDF of the baseline distribution respectively. Then (18) can be expressed as

$$f(x;\alpha,\theta,\xi) = \sum_{k=0}^{\infty} \varphi_k \pi_{k+1}(x) \tag{19}$$

where

$$\varphi_k = \left[\frac{\theta}{(1-e^{-\theta})}\right] \sum_{i,j=0}^{\infty} \frac{(-\theta)^i (-1)^{i+j} (log\alpha)^{k+1} (j+1)^k}{i!\, k!\, (\alpha-1)^{i+1} (k+1)}$$

And

$$\pi_{k+1} = (k+1) h(x) [H(x)]^k \tag{20}$$

This is the Exp-G PDF with power parameter $(k + 1)$. By integrating (19), we obtain the mixture representation of $F(x)$ as

$$F(x;\alpha,\theta,\xi) = \sum_{k=0}^{\infty} \varphi_k \Pi_{k+1}(x) \tag{21}$$

where Π_{k+1} is the cdf of the Exp-G family with power parameter (k + 1). Equation (20) reveals that the EGGP density function is a linear combination of Exp-G densities. Thus, some structural properties of the new family such as the ordinary and incomplete moments and the generating function can be immediately obtained from well-established properties of the Exp-G distributions. The properties of Exp-G distributions have been studied by many authors in recent years, see Mudholkar and Srivastava (1993) and Mudholkar, Srivastava and Freimer (1995) for exponentiated Weibull (EW) distributions, R. C. Gupta, P. L. Gupta and R. D. Gupta (1998) for exponentiated Pareto distributions, Gupta and Kundu (1999) for exponentiated exponential distributions, Nadarajah and Kotz (2006) for the exponentiated-type distributions, Nadarajah (2005) for exponentiated Gumbel distributions, Shirke and Kakade (2006) for exponentiated log-normal distributions and Nadarajah and Gupta (2007) for exponentiated gamma distributions (EGa), among others.

3. Alpha Power Bur XII Poisson Distribution and Properties

The Bur XII (BXII) distribution was proposed by Burr (1942) has several applications in in many areas including lifetime testing, reliability, failure time modeling and acceptance sampling plans. This distribution is a very popular distribution for modeling lifetime data exhibiting monotone failure rate. Tadikamalla (1980) investigated the properties of Bur XII distribution and its related models, namely: logistic, compound Weibull gamma, Weibull exponential, Pareto type II (Lomax) distributions. In recent time several modifications have been made to BXII distribution to allow for wider applications and this includes: Zimmer et al. (2008) proposed and studied a three-parameter Burr XII distribution; Silva et al. proposed and studied the properties of the log-Burr XII regression models with censored data. Afify et al. (2016) investigated the properties of Weibull BXII; Zografos-Balakrishnan BXII distribution was studied by Altun et al. (2018) distribution and many others**.** The CDF of Bur XII distribution is given by

$$\bar{H}(x;\lambda,\eta) = 1 - \left(1 + x^{\lambda}\right)^{-\eta}, \qquad x > 0 \tag{22}$$

And the corresponding CDF is given by

$$\bar{h}(x;\lambda,\eta) = \lambda\eta\left(1+x^{\lambda}\right)^{-\eta}, \qquad x > 0 \tag{23}$$

Where λ *and* η are positive shape parameters. Putting (22) in (9), we obtain a CDF of a more flexible distribution named Alpha Power Bur XII Poisson (APBXIIP) distribution given by

$$F(x;\alpha,\theta,\lambda,\eta) = \left[\frac{1}{1-e^{-\theta}}\right]\left[1-e^{-\theta\left(\frac{\alpha^{\left(1-\left(1+x^{\lambda}\right)^{-\eta}\right)}-1}{\alpha-1}\right)}\right] \tag{24}$$

And the associated PDF is given by

$$f(x;\alpha,\theta,\lambda,\eta) = \left[\frac{\lambda\eta\theta log\alpha}{(1-e^{-\theta})(\alpha-1)}\right]x^{\lambda-1}\left(1+x^{\lambda}\right)^{-\eta-1}\alpha^{\left(1-\left(1+x^{\lambda}\right)^{-\eta}\right)}e^{-\theta\left(\frac{\alpha^{\left(1-\left(1+x^{\lambda}\right)^{-\eta}\right)}-1}{\alpha-1}\right)} \tag{25}$$

Plots of the cdf and PDF of the $APBXIIP$ distribution are displayed in Figure 1 for some parameter values.

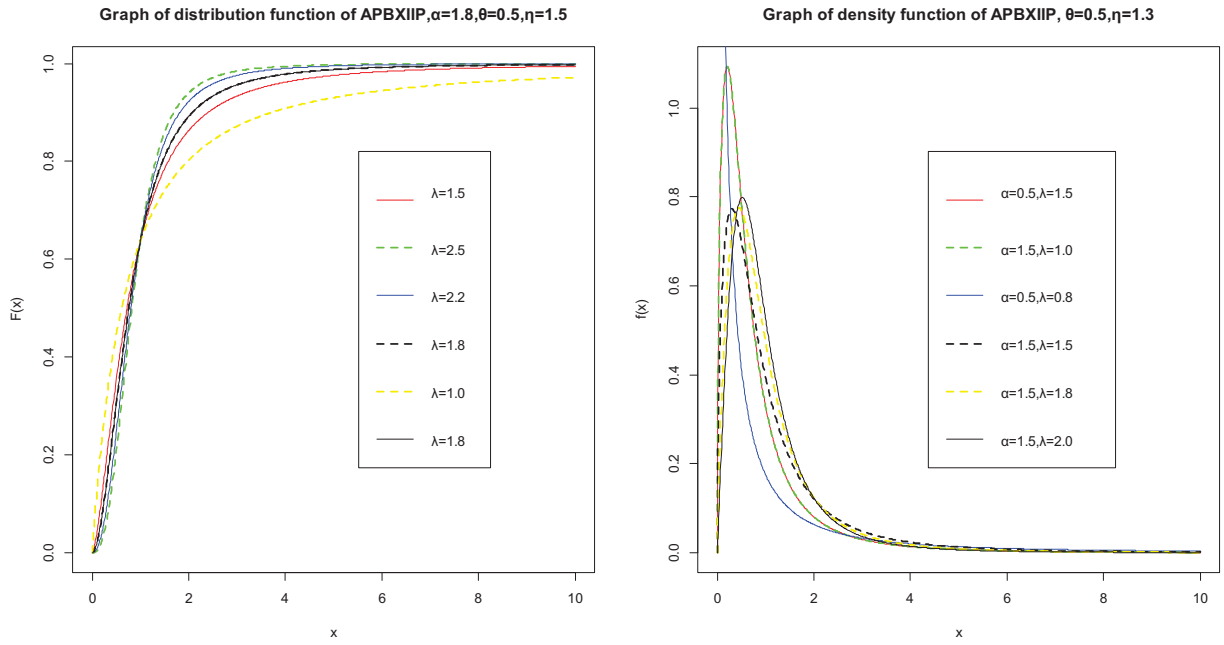

Figure 1. Plot of $APBXIIP$ Distribution and Density Function

An expression for its Survival function $(S(x))$ and hazard function $(h(x))$ is respectively given by

$$\boldsymbol{S(x) = 1 - \left[\frac{1}{1-e^{-\theta}}\right]\left[1-e^{-\theta\left(\frac{\alpha^{\left(1-\left(1+x^{\lambda}\right)^{-\eta}\right)}-1}{\alpha-1}\right)}\right]} \tag{26}$$

and

$$h(x)=\frac{\left[\frac{\lambda\eta\theta log\alpha}{(1-e^{-\theta})(\alpha-1)}\right]x^{\lambda-1}\left(1+x^{\lambda}\right)^{-\eta-1}\alpha^{\left(1-\left(1+x^{\lambda}\right)^{-\eta}\right)}e^{-\theta\left(\frac{\alpha^{\left(1-\left(1+x^{\lambda}\right)^{-\eta}\right)}-1}{\alpha-1}\right)}}{1-\left[\frac{1}{1-e^{-\theta}}\right]\left[1-e^{-\theta\left(\frac{\alpha^{\left(1-\left(1+x^{\lambda}\right)^{-\eta}\right)}-1}{\alpha-1}\right)}\right]} \quad (27)$$

Figure (2) and (3) are the graphs of the hazard function of $APBXIIP$ distribution for various values of the parameters.

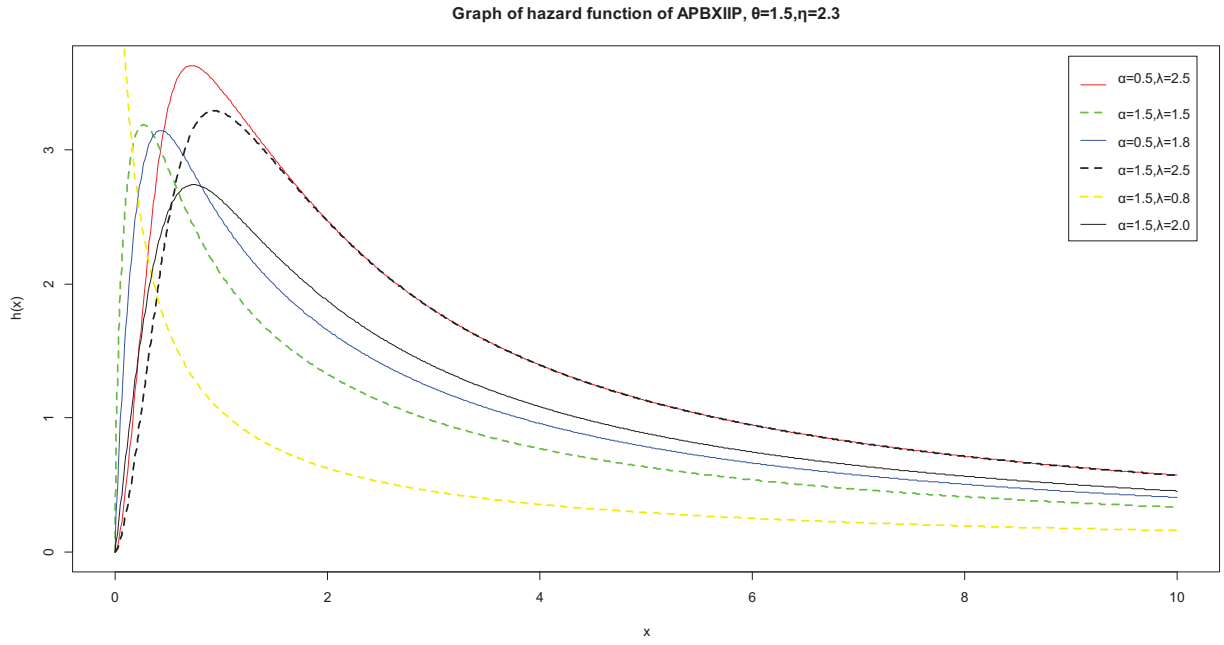

Figure 2. Plot of $APBXIIP$ Hazard Function

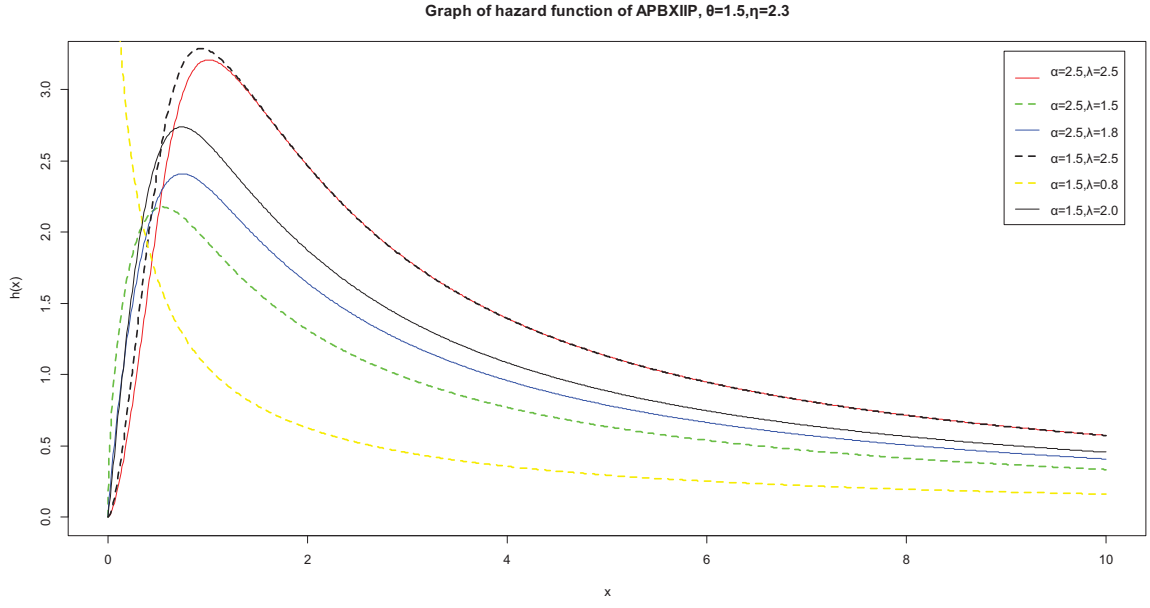

Figure 3. Plot of $APBXIIP$ Hazard Function

Plot of hazard APBXIIP hazard rate function shows different shapes including decreasing, increasing, as well as inverted (upside down) bathtub shapes as shown in figure (2) and (3).

3.1 Quantile Function of APBXIIP Model

The $APBXIIP$ quantile, say $Q(u) = F^{-1}(u)$, can easily be obtained by inverting The CDF given in (28) as follows:

$$x_u = \left[1 - \left(1/\log(\alpha)\left\{1 - log\left(\frac{\{1-\alpha\}}{\theta}\left[1 - u\left(1 - e^{-\theta}\right)\right]\right)\right\}\right)^{1/\lambda}\right]^{1/\eta}, \tag{28}$$

the u^{th} quantile for $u \in (0,1)$

for $u = 0.25, 0.5, 0.75$, we have the lower quartile, middle quartile (median) and the upper quartile of the $APBXIIP$ distribution respectively, given by

$$x_{0.25} = \left[1 - \left(1/\log(\alpha)\left\{1 - log\left(\frac{\{1-\alpha\}}{\theta}\left[1 - 0.25\left(1 - e^{-\theta}\right)\right]\right)\right\}\right)^{1/\lambda}\right]^{1/\eta}, \tag{29}$$

$$x_{0.5} = \left[1 - \left(1/\log(\alpha)\left\{1 - log\left(\frac{\{1-\alpha\}}{\theta}\left[1 - 0.5\left(1 - e^{-\theta}\right)\right]\right)\right\}\right)^{1/\lambda}\right]^{1/\eta}, \tag{30}$$

And

$$x_{0.75} = \left[1 - \left(1/\log(\alpha)\left\{1 - log\left(\frac{\{1-\alpha\}}{\theta}\left[1 - 0.75\left(1 - e^{-\theta}\right)\right]\right)\right\}\right)^{1/\lambda}\right]^{1/\eta}, \tag{31}$$

We use the quantile function of X given in (28) to obtain a numerical value for the Bowley's skewness B_S, Kenny and Keeping (1962) and Moor's kurtosis M_k, Moors (1988). These measures are given by

$$B_S = \frac{Q(3/4) + Q(1/4) - 2Q(1/2)}{Q(3/4) - Q(1/4)} \; and \; M_k = \frac{Q(3/8) - Q(1/8) + Q(7/8) - Q(5/8)}{Q(6/8) - Q(2/8)}$$

Table 1 drawn below gives numerical values of Bowley's skewness and Moor's kurtosis of APBXIIP model for a fixed value of $\alpha = 0.1, \theta = 0.2$ and varying the values of λ and η.

Table 1. Table of Bowley's skewness and Moor's kurtosis for APBXIIP model

$Quantiles$	$Parameters$			
	$\lambda = 1.5, \eta = 4.7$	$\lambda = 2.5, \eta = 217$	$\lambda = 1.5, \eta = 0.7$	$\lambda = 0.5, \eta = 0.2$
$Q(1/4)$	0.9192	0.6462	0.5516	0.8175
$Q(1/2)$	0.9224	0.6644	0.5812	0.8518
$Q(3/4)$	0.9300	0.6846	0.6142	0.8849
$Q(1/8)$	0.9117	0.6377	0.5379	0.8000
$Q(3/8)$	0.9187	0.6551	0.5661	0.8348
$Q(5/8)$	0.9261	0.6742	0.5972	0.8686
$Q(7/8)$	0.9340	0.6956	0.6321	0.9006
B_S	0.0270	0.0520	0.0543	−0.0178
M_k	1.0068	1.0104	1.0080	0.9911

3.2 Mixture Representation of ***APBXIIP*** *Distribution*

The PDF of $APBXIIP$ distribution given in (25) can be presented in a mixture form using the expansion series given in (13), (15) and (17) to obtain

$$f(x) = \frac{\lambda\eta\theta}{(1 - e^{-\theta})} \sum_{i,j,k,l}^{\infty} \frac{(-1)^{i+j+l}(-\theta)^i}{i!\,k!\,(\alpha - 1)^{i+1}} \binom{i}{j} \binom{k}{l} (1 + j)^k x^{\lambda - 1}\left(1 + x^{\lambda}\right)^{l - \eta - 1} \tag{32}$$

4. Ordinary and Incomplete Moments of APBXIIP Model

Several properties of a distribution can be examined via their moments. The r^{th} moment about the origin of X has a APBXIIP distribution is obtained as follows:

Using the mixture representation of APBXIIP model given in (32), we have

$$E(X)^r = \mu'_r = \frac{\lambda\eta\theta}{(1-e^{-\theta})}\sum_{i,j,k,l}^{\infty}\frac{(-1)^{i+j+l}(-\theta)^i}{i!\,k!\,(\alpha-1)^{i+1}}\binom{i}{j}\binom{k}{l}(1+j)^k\int_{-\infty}^{\infty}x^{\lambda-1}\left(1+x^{\lambda}\right)^{l-\eta-1}dx \quad (33)$$

By letting $p = x^{\lambda}, x = p^{1/\lambda}, dx = {}^{1}\!/_{\lambda}\, p^{1/\lambda-1}dp$ and putting it in (33), we have

$$\mu'_r = \frac{\eta\theta}{(1-e^{-\theta})}\sum_{i,j,k,l}^{\infty}\frac{(-1)^{i+j+l}(-\theta)^i}{i!\,k!\,(\alpha-1)^{i+1}}\binom{i}{j}\binom{k}{l}(1+j)^k\int_{-\infty}^{\infty}p^{r/\lambda}(1+p)^{l-\eta-1}dp \quad (34)$$

Also, taking $p = {}^{1}\!/_{(1-u)}, dp = (1-u)^{-2}du$ and substitute in (34), we have

$$\mu'_r = \frac{\eta\theta}{(1-e^{-\theta})}\sum_{i,j,k,l}^{\infty}\frac{(-1)^{i+j+l}(-\theta)^i}{i!\,k!\,(\alpha-1)^{i+1}}\binom{i}{j}\binom{k}{l}(1+j)^k\int_{-\infty}^{\infty}u^{r/\lambda}(1-u)^{\eta-1-l-\frac{r}{\lambda}}dp \quad (35)$$

Finally, we have

$$\mu'_r = \frac{\eta\theta}{(1-e^{-\theta})}\sum_{i,j,k,l}^{\infty}\frac{(-1)^{i+j+l}(-\theta)^i}{i!\,k!\,(\alpha-1)^{i+1}}\binom{i}{j}\binom{k}{l}(1+j)^k B\left[\left(\frac{r}{\lambda}+1\right),\left(\eta-l-\frac{r}{\lambda}\right)\right] \quad (36)$$

$where\ B(q,n) = \int_0^1 z^{q-1}(1-z)^{n-1}dz$, is the standard beta function with $q>0$ and $n>0$.

The mean of APBXIIP distribution can be estimated by taking $r=1$ in equation (), we have

$$\mu'_1 = \mu = \frac{\eta\theta}{(1-e^{-\theta})}\sum_{i,j,k,l}^{\infty}\frac{(-1)^{i+j+l}(-\theta)^i}{i!\,k!\,(\alpha-1)^{i+1}}\binom{i}{j}\binom{k}{l}(1+j)^k B\left[\left(\frac{1}{\lambda}+1\right),\left(\eta-l-\frac{1}{\lambda}\right)\right] \quad (37)$$

Table 2 drawn below gives the first six moments, variance (σ^2) and the coefficient of variation (CV) taking the values of parameters $\lambda = 1.5,\ \eta = 4.1$ and varying the values of α and θ.

The coefficient of variation (CV), variance (σ^2), and standard deviation $(SD = \sigma)$ can be easily obtained and are given by

$$\sigma^2 = \mu'_2 - \mu^2,\ \ and\ \ CV = \frac{\sigma}{\mu} = \frac{(\mu'_2-\mu^2)^{1/2}}{\mu} = \left(\frac{\mu'_2}{\mu^2}-1\right)^{1/2}$$

Table 2. Table of moments of $APBXIIP$ Distribution

Moments	$\alpha=0.1,\theta=0.5$	$\alpha=1.1,\theta=1.5$	$\alpha=2.1,\theta=2.5$	$\alpha=5.0,\theta=5.0$
μ'_1	0.2181	0.3003	0.2802	0.2242
μ'_2	0.0955	0.1674	0.1420	0.0814
μ'_3	0.0769	0.1569	0.1218	0.0472
μ'_4	0.1122	0.2468	0.1803	0.0500
μ'_5	0.3296	0.7479	0.5300	0.1218
μ'_6	6.6154	15.1467	10.6245	2.2960
σ^2	0.0479	0.0772	0.0635	0.0311
CV	1.0035	0.9252	0.8993	0.7865

Further, one can determine the r^{th} central moment and r^{th} cumulant of X defined respectively by,

$$\mu_r = E\{(X-\mu)^r\} = \sum_{q=0}^{r}\binom{r}{q}\mu'_{r-q}(-1)^q\mu^q, \quad \kappa_r = \mu'_r - \sum_{q=1}^{r-1}\binom{r-1}{q-1}\kappa_q\mu'_{r-q},$$

With $\kappa_1 = \mu$. One can express several measure of skewness and kurtosis based cumulants (central moments)
Consequently, the r^{th} incomplete moment of a distribution is given by

$$E(X^r) = \int_t^{\infty} x^r f(x)dx \tag{38}$$

Putting equation (32) in (38), we obtain

$$\mu'_s(t) = \frac{\lambda\eta\theta}{(1-e^{-\theta})}\sum_{i,j,k,l}^{\infty}\frac{(-1)^{i+j+l}(-\theta)^i}{i!\,k!\,(\alpha-1)^{i+1}}\binom{i}{j}\binom{k}{l}(1+j)^k\int_0^{\infty} x^{\lambda-1}\left(1+x^{\lambda}\right)^{l-\eta-1}dx \tag{39}$$

By letting $p = x^{\lambda}, x = p^{1/\lambda}, dx = {}^{1}/_{\lambda}\,p^{1/\lambda-1}dp$ and putting it in (39), we have

$$\mu'_r(t) = \frac{\eta\theta}{(1-e^{-\theta})}\sum_{i,j,k,l}^{\infty}\frac{(-1)^{i+j+l}(-\theta)^i}{i!\,k!\,(\alpha-1)^{i+1}}\binom{i}{j}\binom{k}{l}(1+j)^k\int_0^{t^{\lambda}} p^{r/\lambda}(1+p)^{l-\eta-1}dp \tag{40}$$

Also, taking $p = {}^{1}/_{(1-u)}, dp = (1-u)^{-2}du$ and substitute in (40), we have

$$\mu'_r = \frac{\eta\theta}{(1-e^{-\theta})}\sum_{i,j,k,l}^{\infty}\frac{(-1)^{i+j+l}(-\theta)^i}{i!\,k!\,(\alpha-1)^{i+1}}\binom{i}{j}\binom{k}{l}(1+j)^k\int_0^{\frac{t^{\lambda}}{(1+t^{\lambda})}} u^{r/\lambda}(1-u)^{\eta-1-l-\frac{r}{\lambda}}dp \tag{41}$$

Finally, we have

$$\mu'_r = \frac{\eta\theta}{(1-e^{-\theta})}\sum_{i,j,k,l}^{\infty}\frac{(-1)^{i+j+l}(-\theta)^i}{i!\,k!\,(\alpha-1)^{i+1}}\binom{i}{j}\binom{k}{l}(1+j)^k B\left[\frac{t^{\lambda}}{(1+t^{\lambda})},\left(\frac{r}{\lambda}+1\right),\left(\eta-l-\frac{r}{\lambda}\right)\right] \tag{42}$$

$where\ B(z;q,n) = \int_0^z y^{z-1}(1-y)^{n-1}dy\,, is\ the\ beta\ function$

By taking $r = 1,$ we obtain an expression for the first incomplete moment as

$$\mu'_1 = \frac{\eta\theta}{(1-e^{-\theta})}\sum_{i,j,k,l}^{\infty}\frac{(-1)^{i+j+l}(-\theta)^i}{i!\,k!\,(\alpha-1)^{i+1}}\binom{i}{j}\binom{k}{l}(1+j)^k B\left[\frac{t^{\lambda}}{(1+t^{\lambda})},\left(\frac{1}{\lambda}+1\right),\left(\eta-l-\frac{1}{\lambda}\right)\right] \tag{43}$$

It should be noted that $\mu'_r(t)$ always exists.

4.1 Moment Generating Function

The moment generating function of a random variable X is defined by

$$E(e^{tX}) = \int_{-\infty}^{\infty} e^{tX}f(x)dx \tag{44}$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r f(x) dx$$

$$= \sum_{i,j,k,l,r}^{\infty} \frac{t^r}{r!} \frac{\eta\theta}{(1-e^{-\theta})} \frac{(-\theta)^i (-1)^{i+j+l}}{i!\, k!\, (\alpha-1)^{i+1}} \binom{i}{j} \binom{k}{l} (1+j)^k B\left[\left(\frac{r}{\lambda}+1\right), \left(\eta - l - \frac{r}{\lambda}\right)\right]$$

4.2 Mean Deviation

The mean deviation, about the mean and the median, are used to determine the degree of spread in a population. Let μ and M be the mean and the median of the $APBXIIP$ distribution given by (43) and (30) respectively.

The mean deviation of $APBXIIP$ model about the mean can be obtained as

$$\Gamma_1(X) = E|X-\mu| = \int_0^{\infty} |X-\mu| f(x;\alpha,\theta,\lambda,\eta) dx, \tag{45}$$

$$= 2\mu F(\mu;\alpha,\theta,\lambda,\eta) - 2\mu + 2\int_{\mu}^{\infty} x f(x;\alpha,\theta,\lambda,\eta) dx$$

$$= 2\mu \left[\frac{1}{1-e^{-\theta}}\right] \left[1 - e^{-\theta\left(\frac{\alpha^{\left(1-\left(1+x^{\lambda}\right)^{-\eta}\right)}-1}{\alpha-1}\right)}\right] - 2\mu + \frac{2\eta\theta}{(1-e^{-\theta})}$$

$$\times \sum_{i,j,k,l}^{\infty} \frac{(-1)^{i+j+l}}{i!\, k!\, (\alpha-1)^{i+1}} \binom{i}{j} \binom{k}{l} (1+j)^k B\left[\frac{\mu^{\lambda}}{(1+\mu^{\lambda})}, \left(\frac{r}{\lambda}+1\right), \left(\eta - l - \frac{r}{\lambda}\right)\right] \tag{46}$$

The mean deviation of $APBXIIP$ about the median can also be obtained as

$$\Gamma_2(X) = E|X-M| = \int_0^{\infty} |X-M| f(x) dx, \tag{47}$$

$$= -\mu + 2\int_{m}^{\infty} x f(x;v,w,\lambda) dx$$

$$= -\boldsymbol{\mu} + \frac{2\eta\theta}{(1-e^{-\theta})} \sum_{i,j,k,l}^{\infty} \frac{(-\theta)^i (-1)^{i+j+l}}{i!\, k!\, (\alpha-1)^{i+1}} \binom{i}{j} \binom{k}{l} (1+j)^k B\left[\frac{m^{\lambda}}{(1+m^{\lambda})}, \left(\frac{r}{\lambda}+1\right), \left(\eta - l - \frac{r}{\lambda}\right)\right]$$

4.3 Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves have been found suitable to study income and poverty analysis in the field of economics and also in other field like insurance, reliability, and demography. The Bonferroni and Lorenz curves are defined by

$$B(f) = \frac{1}{p\mu} \int_0^{q} x f(x;\alpha,\theta,\lambda,\eta) dx \tag{48}$$

and

$$L(f) = \frac{1}{\mu} \int_0^{q} x f(x;\alpha,\theta,\lambda,\eta) dx \tag{49}$$

Respectively, where $\mu = E(X)$ and $q = F^{-1}(p)$. In the case of $APBXIIP$ distribution, we obtain

$$B(f) = \frac{\theta\eta}{f\mu(1-e^{-\theta})}\sum_{i,j,k,l}^{\infty}\frac{(-\theta)^i(-1)^{i+j+l}}{i!\,k!\,(\alpha-1)^{i+1}}\binom{i}{j}\binom{k}{l}(1+j)^k B\left[\frac{t^\lambda}{(1+t^\lambda)};\left(\frac{r}{\lambda}+1\right),\left(\eta-l-\frac{r}{\lambda}\right)\right] \quad (50)$$

and

$$L(p) = \frac{\eta\theta}{\mu(1-e^{-\theta})}\sum_{i,j,k,l}^{\infty}\frac{(-\theta)^i(-1)^{i+j+l}}{i!\,k!\,(\alpha-1)^{i+1}}\binom{i}{j}\binom{k}{l}(1+j)^k B\left[\frac{t^\lambda}{(1+t^\lambda)};\left(\frac{r}{\lambda}+1\right),\left(\eta-l-\frac{r}{\lambda}\right)\right] \quad (51)$$

$$where\ B(l;m,n) = \int_0^l y^{m-1}(1-y)^{n-1}dy\,,\ is\ the\ beta\ function$$

5. Renyi Entropy

Renyi (1961), gave a useful mathematical expression that can be used to measure the entropy of a $APBXIIP$ distribution given by

$$I_R^{(v)} = \frac{1}{1-v}log\left[\int_0^\infty f_{\text{APBXIIP}}(x;\zeta)^v dx\right], \quad v>0, v\neq 1 \quad (52)$$

Putting equation (25) in (52), we have

$$I_R^{(v)} = \frac{1}{1-v}log(W^v)$$

Where,

$$W^v = \left[\int_0^\infty\left\{\left[\frac{\lambda\eta\theta log\alpha}{(1-e^{-\theta})(\alpha-1)}\right]x^{\lambda-1}\left(1+x^\lambda\right)^{-\eta-1}\alpha^{\left(1-\left(1+x^\lambda\right)^{-\eta}\right)}e^{-\theta\left(\frac{\alpha^{\left(1-\left(1+x^\lambda\right)^{-\eta}\right)}-1}{\alpha-1}\right)}\right\}^v dx\right] \quad (53)$$

Using series expansion given in (13), (15), and (17) in (53), we have

$$W^v = \frac{\lambda^{v-1}\eta^v\theta^v}{(1-e^{-\theta})^v(\alpha-1)^v}\sum_{i,j,k,l}^{\infty}\frac{(-v\theta)^i}{i!\,k!}\left(log(\alpha)\right)^{k+v}(-1)^{i+j+l}\binom{i}{j}\binom{k}{l}(v+j)^k$$

$$\times B\left[\frac{V(\lambda-1)+1}{\lambda},\frac{v(\lambda\eta-2\lambda+1)+l\lambda\eta-1}{\lambda}\right] \quad (54)$$

6. Order Statistics

The concept of order statistics is generally applied in modeling some certain random system. Most especially, for $r=1,\dots,n$, the first order statistics of a statistical sample is obtained when $r=1$ and for the largest order statistics is when $r=n$.

Suppose $X_{(1)},X_{(2)},\dots,X_{(n)}$ be an ordered sample that follows the $APBXIIP$ distribution, the pdf of $X_{(r)}$ is computed as

$$f_r(x;\xi) = \frac{1}{B(r,n-r+1)}F_{APBXIIP}(x;\xi)^{r-1}[1-F_{APBXIIP}(x;\xi)]^{n-r}f_{APBXIIP}(x;\zeta) \quad (55)$$

Then by applying the series expansion given in (15) to (55), we have

$$f_r(x;\xi) = \frac{f_{APBXIIP}(x;\xi)}{B(r,n-r+1)}\sum_{i=1}^{n-r}(-1)^i\binom{n-r}{i}F_{APBXIIP}(x;\xi)^{r+i-1} \tag{56}$$

Now, by substituting equation (9) and (10) in $f_r(x;\xi)$, we have

$$f_r(x;\xi) = \frac{\left[\frac{\lambda\eta\theta log\alpha}{(1-e^{-\theta})(\alpha-1)}\right]x^{\lambda-1}\left(1+x^{\lambda}\right)^{-\eta-1}\alpha^{\left(1-\left(1+x^{\lambda}\right)^{-\eta}\right)}e^{-\theta\left(\frac{\alpha^{\left(1-\left(1+x^{\lambda}\right)^{-\eta}\right)}-1}{\alpha-1}\right)}}{B(r,n-r+1)}$$

$$\times\sum_{i=1}^{n-r}(-1)^i\binom{n-r}{i}\left\{\left[\frac{1}{1-e^{-\theta}}\right]\left[1-e^{-\theta\left(\frac{\alpha^{\left(1-\left(1+x^{\lambda}\right)^{-\eta}\right)}-1}{\alpha-1}\right)}\right]\right\}^{r+i-1} \tag{57}$$

Using the series expansion given in (15) in (57)

$$\left\{\left[1-e^{-\theta\left(\frac{\alpha^{\left(1-\left(1+x^{\lambda}\right)^{-\eta}\right)}-1}{\alpha-1}\right)}\right]\right\}^{r+i-1} = \sum_{j=0}^{r+i-1}(-1)^j\binom{r+i-1}{j}e^{-\theta j\left(\frac{\alpha^{\left(1-\left(1+x^{\lambda}\right)^{-\eta}\right)}-1}{\alpha-1}\right)}$$

It then follows that,

$$f_r(x;\xi) = \frac{\lambda\eta\theta log\alpha}{(\alpha-1)B(r,n-r+1)}x^{\lambda-1}\left(1+x^{\lambda}\right)^{-\eta-1}\alpha^{\left(1-\left(1+x^{\lambda}\right)^{-\eta}\right)}$$

$$\times\sum_{i=0}^{n-r}\sum_{j=0}^{r+i-1}(-1)^{i+j}\binom{n-r}{i}\binom{r+i-1}{j}\left(\frac{1}{1-e^{-\theta}}\right)^{r+i}e^{-\theta(j+1)\left(\frac{\alpha^{\left(1-\left(1+x^{\lambda}\right)^{-\eta}\right)}-1}{\alpha-1}\right)} \tag{58}$$

Also, applying (13) and (17) in (58), we have an expression for the r^{th} order statistics of $APBXIIP$ distribution given by

$$f_r(x;\xi) = \frac{\lambda\eta\theta}{B(r,n-r+1)}\sum_{i=0}^{n-r}\sum_{j=0}^{n+r-1}\sum_{k,l,p}^{\infty}\frac{(-1)^{i+j+k+p}}{(\alpha-1)^{k+1}m!}\binom{n-r}{i}\binom{r+i-1}{j}\binom{k}{l}\binom{m}{p}$$

$$\times[\log(\alpha)]^{m+1}\left(\frac{1}{1-e^{-\theta}}\right)^{r+i}[-\theta(j+1)]^k(m+1)^k x^{\lambda-1}\left(1+x^{\lambda}\right)^{-[\eta(1+p)+1]} \tag{59}$$

It should be noted that Renyi entropy is an extension of Shannon entropy that Renyi entropy tends to Shannon entropy as $v \longrightarrow 1$

7. Maximum Likelihood Estimates of the Parameters

Estimators are obtained for the APBXIIP parameters depending on the maximum likelihood estimates are derived. Suppose $X_1, X_2, \ldots, X_n$ be a random sample from $APBXIIP$ distribution with observed values $x_1, x_2, \ldots, x_n$. The log likelihood function of $APBXIIP$ model, denoted by l, is obtained as follows

$$l = nlog\left[\frac{\lambda\eta\theta log\alpha}{(1-e^{-\theta})(\alpha-1)}\right] + (\lambda-1)\sum_{i=1}^{n}x_i - (\eta+1)\sum_{i=1}^{n}\left(1+{x_i}^{\lambda}\right)$$

$$+\log(\alpha)\sum_{i=1}^{n}\left(1-\left(1+{x_i}^{\lambda}\right)^{-\eta}\right) - \frac{\theta}{(\alpha-1)}\sum_{i=1}^{n}\left[\alpha^{\left(1-\left(1+x^{\lambda}\right)^{-\eta}\right)}-1\right] \tag{60}$$

The partial derivative of the log-likelihood function with respect to the unknown parameters are given by

$$\frac{\partial l}{\partial \alpha} = \frac{n(\alpha - 1 - \alpha\log(\alpha))}{\alpha(\alpha-1)\log(\alpha)} + \frac{1}{\alpha}\sum_{i=1}^{n}\left(1-\left(1+{x_i}^{\lambda}\right)^{-\eta}\right) + \frac{1}{(\alpha-1)^2}\sum_{i=1}^{n}\left[\alpha^{\left(1-\left(1+{x_i}^{\lambda}\right)^{-\eta}\right)} - 1\right] \quad (61)$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} - \frac{ne^{-\theta}}{1-e^{-\theta}} - \frac{1}{(\alpha-1)}\sum_{i=1}^{n}\left[\alpha^{\left(1-\left(1+x^{\lambda}\right)^{-\eta}\right)} - 1\right] \quad (62)$$

$$-\frac{1}{\alpha(\alpha-1)}\sum_{i=1}^{n}\alpha^{\left(1-\left(1+{x_i}^{\lambda}\right)^{-\eta}\right)}\left(1-\left(1+{x_i}^{\lambda}\right)^{-\eta}\right) \quad (63)$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} x_i - (\eta+1)\sum_{i=1}^{n}{x_i}^{\lambda} log(x) + \eta\log(\alpha)\sum_{i=1}^{n}{x_i}^{\lambda} log(x_i)\left(1+{x_i}^{\lambda}\right)^{-\eta-1}$$

$$-\frac{\theta\,\eta\log(\alpha)}{(\alpha-1)}\sum_{i=1}^{n}{x_i}^{\lambda}\left(1+{x_i}^{\lambda}\right)^{-\eta-1} log(x_i) \quad (64)$$

$$\frac{\partial l}{\partial \eta} = \frac{n}{\eta} - \sum_{i=1}^{n}\left(1+{x_i}^{\lambda}\right) - \log(\alpha)\sum_{i=1}^{n}\left(1+{x_i}^{\lambda}\right)^{-\eta} log\left(1+{x_i}^{\lambda}\right) \quad (65)$$

The maximum likelihood of the model parameters are obtained by solving the non-linear equations $\frac{\partial l}{\partial \alpha} = \frac{\partial l}{\partial \theta} = \frac{\partial l}{\partial \lambda} = \frac{\partial l}{\partial \eta} = 0$. The solutions to these equations can be obtained by solving simultaneously, numerically using iterative method such as Newton-Raphson iteration technique. For interval estimation of the parameters, the 4×4 observed information matrix $I(\xi) = \{I_{qp}\}$ for $(\alpha, \theta, \lambda, \eta)$. U7nder certain regularity conditions, the asymptotic properties of the maximum likelihood method shows that: $\sqrt{n}\left(\hat{\xi} - \xi\right) \xrightarrow{d}$ indicates the convergence in distribution, with mean 0. Then the $100(1-w)\%$ confidence intervals for $\alpha, \theta, \lambda \; and \; \eta$ are given respectively as follows:

$$\hat{\alpha} \pm Z_w SE(\hat{\alpha}), \hat{\theta} \pm Z_w SE\left(\hat{\theta}\right), \hat{\lambda} \pm Z_w SE\left(\hat{\lambda}\right) and \; \hat{\eta} \pm Z_w SE(\hat{\eta})$$

It should be noted that the variances of $\alpha, \theta, \lambda, \eta$ are the diagonal elements of $I^{-1}(\xi)$ corresponding to the model parameters.

8. Simulation Study

A simulation study is carried to examine the performance of MLE for APBXIIP model in terms of their absolute bias (AB), mean square error (MSE). In this context, we employ of the most used simulation techniques to evaluate the performance of estimators is by Monte Carlo simulation, see, for example Lemonte (2013), Cordeiro and Lemonte (2014) and De Andrade et al. (2019) Simulated procedures are carried out as follows:

A sample sizes of $n = 50, 100, 150 \; and \; 200$ are generated from APBXIIP distribution with selected values for $\alpha, \theta, \lambda \; and \; \eta$. We consider 2000 Monte Carlo replications. The simulation process is performed in R software using Broyden-Fletcher-Goldfarb-Shannon (BFGS) maximization method in the optimum script. To ensure that the experiment is reproducible, we use the seed for the random number generator: set.seed (103). The results of the simulation are presented in Table 3, including the Absolute mean (AB), standard error (SE) and the mean square error. The results obtained shows that the $APBXIIP$ estimates exhibits desirable properties even for small sample sizes. In general, the MSE approaches zero as the sample size increases, as expected.

Table 3. Means, Absolute Biases and MSE of $\hat{\alpha}, \hat{\theta}, \hat{\lambda}, and\ \hat{\eta}$ for APBXII model

Parameter	n	AB	SE	MSE
$\alpha = 0.1$	50	3.6699	3.8566	28.3349
	100	5.1857	5.4407	56.4927
	150	4.0680	2.4804	22.7010
	200	3.1195	1.8514	13.1590
	250	2.9146	1.3805	10.4007
$\theta = 0.5$	50	1.7670	0.9618	4.0474
	100	1.5682	0.7460	3.0158
	150	1.5337	0.5967	2.7083
	200	1.6821	0.5355	3.1162
	250	1.5481	0.4613	3.1162
$\lambda = 1.2$	50	0.4173	0.1312	2.6094
	100	0.3852	0.1072	0.1599
	150	0.4536	0.0749	0.2114
	200	0.3969	0.0715	0.1626
	250	0.3606	0.0646	0.1342
$\eta = 0.5$	50	1.2080	1.0521	2.5662
	100	0.8161	0.8212	1.3404
	150	1.3779	0.6358	2.3028
	200	0.5556	0.5860	0.6521
	250	0.5149	0.5407	0.5575

8.1 Real Data Applications

To demonstrate the flexibility proposed family of distributions, -2*log-likelihood statistic $(-2l)$, Akaike information criterion $(AIC = 2k - 2l)$, Bayesian information criterion $(BIC = kln(n) - 2l)$, Consistent Akaike information criterion $(CAIC = AIC + 2\frac{k(k+1)}{n-k-1})$ and Hannan–Quinn information criterion (HQIC) are calculated for APBXIIP model and its sub-models, where n is the number of observations, and k is the number of estimated parameters. The goodness-of-fit statistic, Anderson–Darling (A^*) and Cramer–von Mises ($W*$) are also presented in the Table. The best model correspond among the class considered is the model having minimum value of these statistics as the best model. In this study, numerical results (of maximum likelihood estimates and goodness of fit criteria) are calculated by using the goodness.fit (.) command in the Model Adequacy package available in R language. The AIC, CAIC, BIC, HQIC,A^*, and W^* are given for the sub-models Alpha Power $BXII$ $(APBXII)$ model, Bur XII Poisson $(BXIIP)$ model and the BXII model. Tables 6, 8 and 10 respectively. The PDF of APBXIIP sub-models are given as

$$f_{APBXII}(x;\alpha,\theta,\lambda,\eta) = \frac{\lambda\eta log\alpha}{(\alpha-1)} x^{\lambda-1}\left(1+x^{\lambda}\right)^{-\eta-1}\alpha^{\left(1-\left(1+x^{\lambda}\right)^{-\eta}\right)},\ \ x>0;\lambda,\eta,\alpha>0$$

$$f_{BXIIP}(x;\alpha,\theta,\lambda,\eta) = \frac{\lambda\eta\theta}{(1-e^{-\theta})} x^{\lambda-1}\left(1+x^{\lambda}\right)^{-\eta-1} e^{-\theta\left(1-\left(1+x^{\lambda}\right)^{-\eta}\right)}, x>0;\theta,\eta,\alpha>0$$

The following are the data sets which we have used in this study. Data set 1 represents the lifetime of 50 devices and was used by Aarset (1987).

Data set 1: 0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6,7, 11, 12, 18, 18, 18, 18, 18, 21, 32,36, 40, 45, 46, 47, 50, 55, 60, 63, 63,67, 67, 67, 67, 72, 75, 79, 82, 82, 83,84, 84, 84, 85, 85, 85, 85, 85, 86, 86.

Data set 2 contains intervals in days between 109 successive coal-mining disasters in Great Britain, for the period 1875–1951, published by Mahdavi and Kundu (2017). The data are given as:

Data set 2: 1, 4, 4, 7, 11, 13, 15, 15, 17, 18, 19, 19, 20, 20, 22, 23, 28, 29, 31, 32, 36,37, 47, 48, 49, 50, 54, 54, 55, 59, 59, 61, 61, 66, 72, 72, 75, 78, 78, 81, 93, 96, 99, 108, 113, 114, 120, 120, 120,123, 124, 129, 131, 137, 145, 151, 156, 171, 176, 182, 188, 189, 195, 203, 208, 215, 217, 217, 217, 224, 228, 233, 255, 271, 275, 275, 275, 286, 291, 312, 312, 312, 315, 326,326, 329, 330, 336, 338, 345, 348, 354, 361, 364, 369, 378, 390, 457, 467, 498, 517, 566, 644, 745, 871,

1312, 1357, 1613, 1630.

Data set 3 contains the remission times (in months) of a random sample of 128 bladder cancer patients. The data have been obtained from Lee and Wang (2003).

The data are given as: 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26,3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70,5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32,7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41,7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66,11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10,1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53,12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

Some descriptive statistics for the three data sets considered are presented in Table 4, including the range, mean, median, upper and lower quartile, and variance, among others. From the Table it can deduce that the three data sets are over-dispersed. The graph of Total Test Time (TTT curves) to this data sets are presented in Figure 5, which shows that data set 1 and 3 exhibits non-monotone failure rate and data set 2 exhibits a decreasing failure rate. The graph of empirical density is drawn in figure 4 which shows that data set 1 is moderately skewed to the right and data set 2 and 3 are highly skewed to the right.

Table 4. Descriptive statistics for the data sets

Statistic	*Data set I*	Data set 2	Data set 3
n	50	109	128
Lower quartile	13.50	54.0	3.35
Upper quartile	81.25	312.0	11.84
Median	48.50	145.0	6.40
Mean	45.69	233.3	9.37
minimum	0.1	1.0	0.08
Maximum	86.00	1630.0	79.05
Variance	1078.16	87873.33	110.43
range	86.1	1630.1	79.13

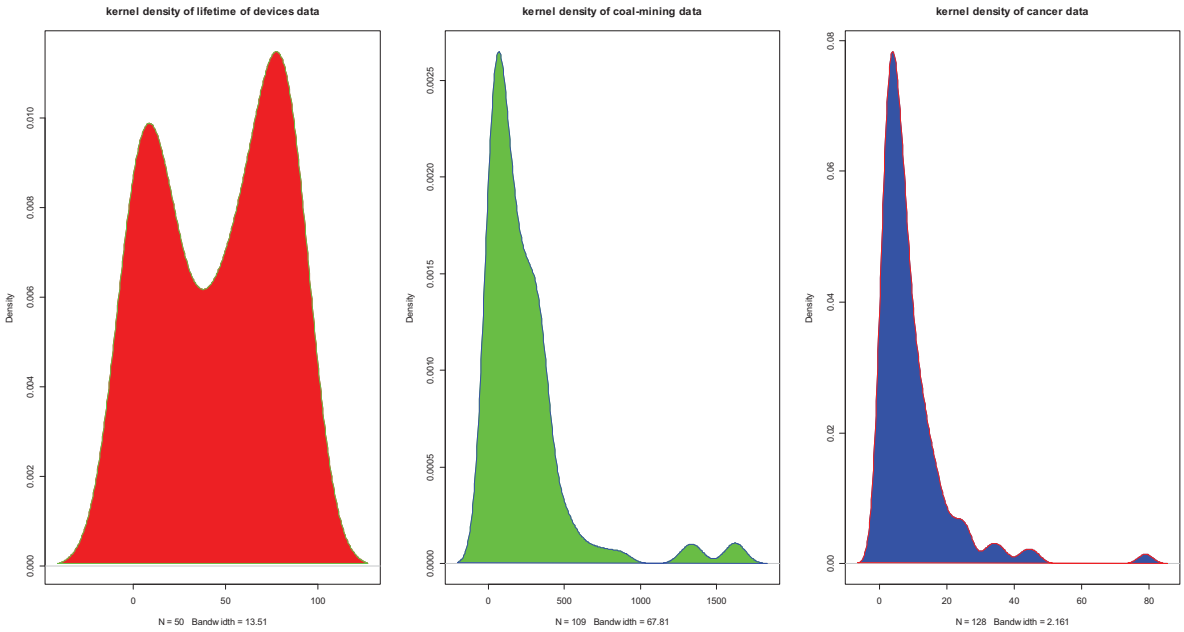

Diagram 1 Diagram 2 Diagram 3

Figure 4. Kernel density Plot for the three failure data

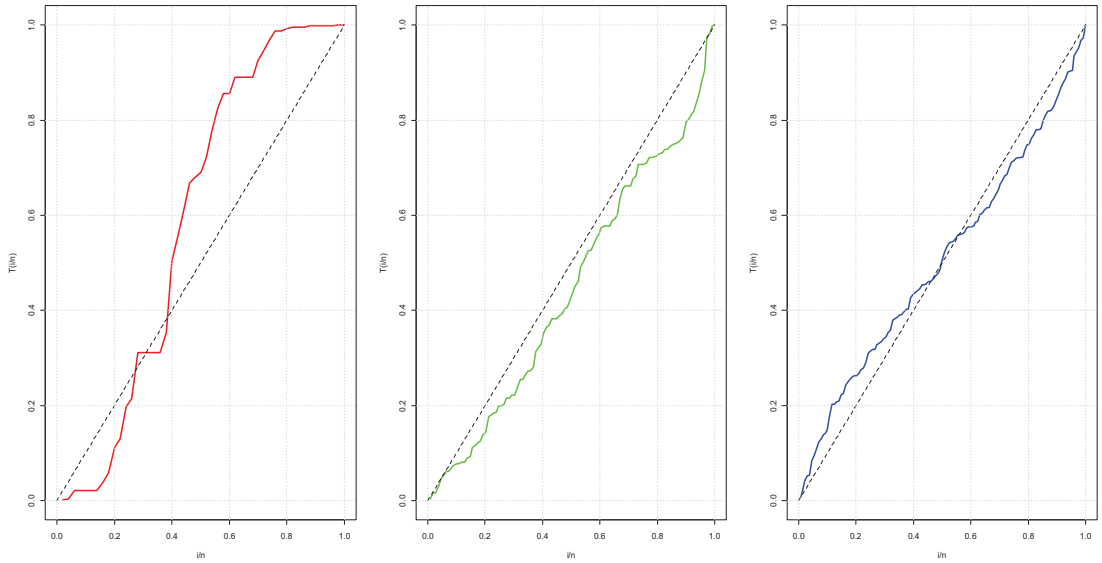

Diagram I Diagram II Diagram III

Figure 5. TTT Plot of the three failure data

Table 5. MLEs and SEs (in Parenthesis), confidence interval (in curly bracket) for the device lifetime data

Distribution	α	θ	λ	η
APBXIIP	10.28(1.96) {6.44,14.12}	−4.45(1.77) {−7.92, −0.98}	0.58(0.17) {0.25}	1.38(0.46) {0.48,2.28}
APBXII	21.61(12.21) {−2.32,45.54}	−(−) {−}	0.98 (0.29) {0.42,1.55}	0.56(0.19) {0.19,0.93}
PBXII	−(−) {−}	−8.71(4.76) {−18.04,0.62}	0.48(0.20) {0.09,0.87}	1.53(0.79) {−0.02,3.08}
BXII	−(−) {−}	−(−) {−}	1.26(0.32) {0.63,1.89}	0.25(0.07) {0.11,0.39}

Table 6. The AIC, BIC, CAIC, and A^*, W^* statistics for device lifetime data

Distribution	$-2l$	*AIC*	*BIC*	*CAIC*	HQIC	A^*	W^*
APBXIIP	510.019	518.019	525.668	518.908	520.932	4.5035	0.7858
APBXII	521.757	527.757	533.493	528.279	529.942	5.0513	0.9090
PBXII	514.864	520.864	526.599	521.386	523.048	4.8438	0.8645
BXII	544.728	548.728	552.553	548.983	550.184	5.8572	1.0947

Table 7. MLEs and SEs (in Parenthesis), confidence interval (in curly bracket) for the coal-mining data

Distribution	α	θ	λ	η
APBXIIP	12.25(10.40) {−8.13,32.63}	−10.60(2.85) {−13.45, −7.75}	0.65(0.13) {0.40,0.91}	1.19(0.27) {0.66,1.72}
APBXII	13.92(3.70) {6.67,21.17}	−(−) {−}	3.07 (3.36) {−3.52,9.66}	0.12(0.13) {−0.14,0.38}
PBXII	−(−) {−}	−21.99(6.73) {−35.18, −8.80}	0.58(0.17) {0.25,0.91}	1.25(0.42) {0.43,2.07}
BXII	−(−) {−}	−(−) {−}	4.39(1.48) {1.49,7.29}	0.5(0.07) {0.36,0.64}

Table 8. The AIC, BIC, CAIC, A^*,and W^* statistic for coal-mining data

$Distribution$	$-2l$	AIC	BIC	$CAIC$	HQIC	A^*	W^*
$APBXIIP$	1432.242	1440.242	1451.007	1440.626	1444.608	2.6117	0.4583
$APBXII$	1526.755	1532.755	1540.829	1532.984	1536.030	2.8623	0.4925
$PBXII$	1438.772	1444.772	1452.846	1445.0	1448.046	2.9399	0.5136
$BXII$	1598.631	1602.631	1608.013	1602.744	1604.813	3.9302	0.6655

Table 9. MLEs and SEs (in Parenthesis), confidence interval (in curly bracket) for the cancer data

$Distribution$	α	θ	λ	η
$APBXIIP$	10.53(9.14) {−7.29,28.35}	−4.51(1.39) {−7.23, −1.86}	1.09(0.32) {0.46,1.72}	1.36(0.46) {0.45,2.26}
$APBXII$	17.31(5.70) {6.14,28.48}	−(−) {−}	1.85(0.31) {1.24,2.46}	0.52(0.10) {0.32,0.72}
$PBXII$	−(−) {−}	1.13(1.77) {−2.34,4.60}	−7.04(0.29) {−7.61, −6.47}	1.12(0.34) {0.45,1.79}
$BXII$	−(−) {−}	−(−) {−}	0.24(0.36) {−0.47,0.95}	2.33(0.04) {2.25,2.41}

Table 10. The AIC, BIC, CAIC, A^*, and W^* statistics for cancer data

$Distribution$	$-2l$	AIC	BIC	$CAIC$	HQIC	A^*	W^*
$APBXIIP$	826.422	834.4218	845.829	834.747	839.057	0.6264	0.0944
$APBXII$	851.982	857.981	866.538	858.175	861.458	1.8386	0.2865
$PBXII$	834.792	840.792	849.348	840.985	844.268	1.2370	0.1882
$BXII$	907.034	911.035	916.740	911.132	913.353	4.5426	0.7475

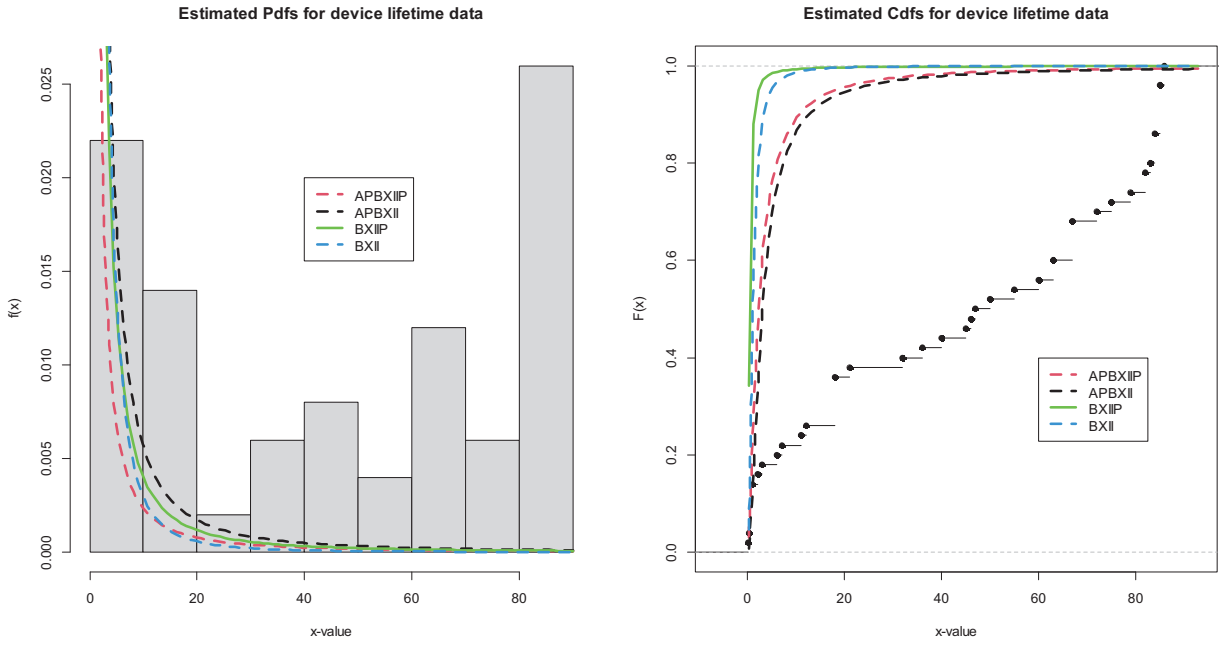

Figure 6. Estimated PDF and CDF function and other competing models for device lifetime data

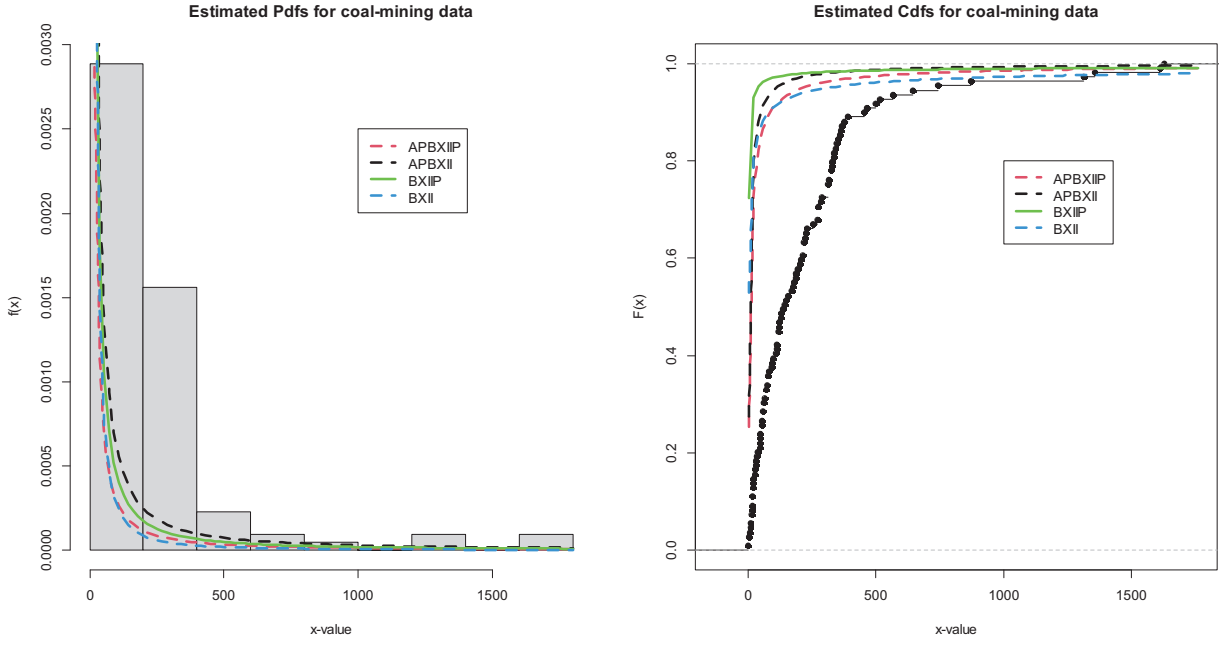

Figure 7. Estimated PDF and CDF function and other competing models for coal-mining data

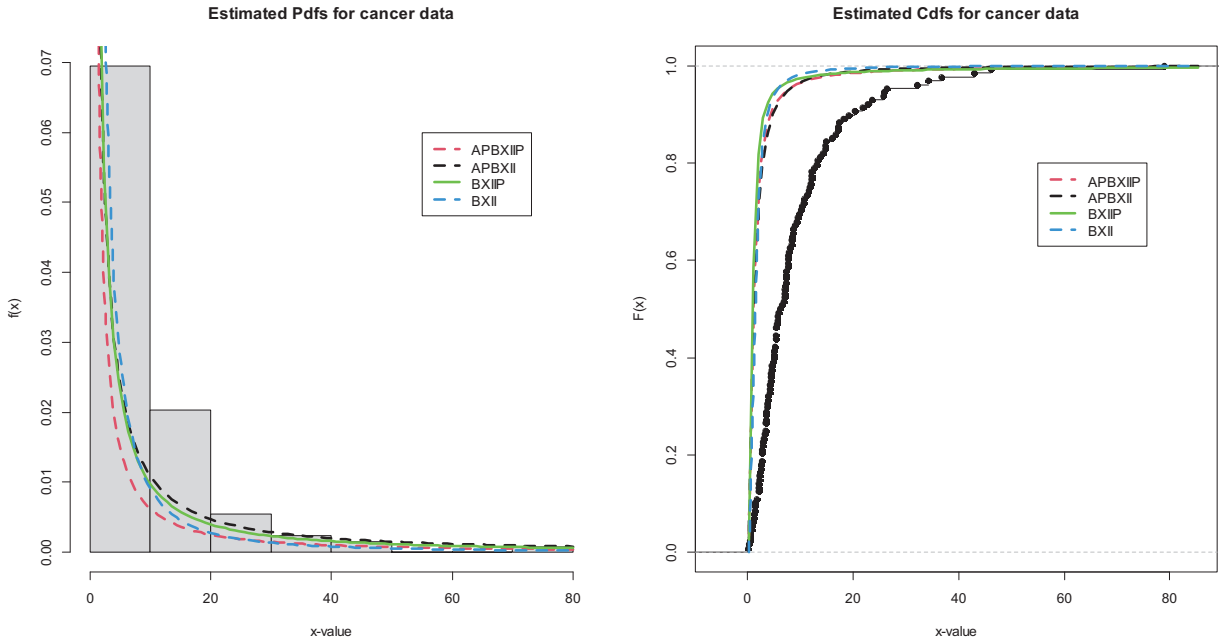

Figure 8. Estimated PDF and CDF function and other competing models for Yarn specimen data

Based on Tables 6, 8 and 10, it is evident that $APBXIIP$ model provides the best fit and can therefore be taken as the best model based on the data considered. Figures 6, 7, and 8 provide more information on the flexibility of the $APBXIIP$ model.

7. Conclusion

A new $APPG$ family of distribution has been introduced to incorporate skewness to a classical distribution functions. We have used that method to the Bur XII distribution functions, and a new four-parameter $APBXIIP$ distribution has been introduced and studied. The proposed distribution has several desirable properties, which enable it use for modeling data that exhibits different shape of the hazard function. Maximum likelihood estimation procedure is used to estimate the values of the unknown parameters. Three data analysis has been performed based on four-parameter $APBXIIP$ distribution. It is observed that the four-parameter $APBXIIP$ distribution provides a good fit to the data sets. Monte Carlo simulation is carried out to validate the use of maximum likelihood estimation.

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A New Odd Fréchet Lehmann Type II–G Family of Distributions: A Power Function Distribution With Theory and Applications

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Abstract

Modeling complex random phenomena frequently observed in reliability engineering and medical science once thought to be an enigma. Scientists and practitioners agree that an appropriate but simple model is the best choice for this investigation. We contribute a new family referred to as an odd Fréchet Lehmann type-II (OFrLII) G family of distributions to address these issues. This new family has involved a shape parameter that modulated the tails of new models. Furthermore, we develop a list of eight new sub-models for a new family and a power function distribution (OFrLII–PF) nominated for detailed discussion. We derive several complementary mathematical properties and explicit expressions for the moments, quantile function, and order statistics. We plot possible shapes of the density and the hazard rate functions over the particular choices of the model parameters. We follow a technique known as maximum likelihood estimation to estimate unknown model parameters and a simulation study established to assess the asymptotic behavior of these MLEs. The applicability of the OFrLII–G family, is evaluated via OFrLII –PF distribution. For this, we fit two engineering and one COVID–19 pandemic dataset. Supportive results of OFrLII–PF distribution declare it as a better fit model against the well-established competitor's ones. A modified odd Fréchet Lehmann Type II–G Family of Distributions: A Power Function Distribution with Theory and Applications

Keywords: Lehmann type distribution, Fréchet distribution, power function distribution, COVID–19, failure rate function; moments, Entropy, maximum likelihood estimation

Mathematics Subject Classification: 60E05, 62P12, 62P30

1. Introduction

Over a long time, modeling complex random phenomena predominantly in reliability engineering and medical sciences consider an enigma for researchers. For this exploration, an appropriate but simple model is the first choice of scientists and practitioners. Several bounded and unbounded but simple to complex lifetime models have been developed to overcome these challenges, but a revolutionary change in the research world is attributed to [1]. The study by [1] developed one of the most spartan families known as a Lehmann type–I (L–I) with (cumulative distribution function CDF $[P^a(z)]$. L – I was the simple exponentiated version of any arbitrary baseline model. Lehmann's work was further discussed by [2] for the exponential distribution. In the meantime, [3] proposed a new technique to generate models with CDF $\left[P(z)/\left(P(z)+a\bar{P}(z)\right)\right]$. Study by [4] proposed a beta generated–*P* family with CDF $F(x)=\int_0^{Z(x)} b(t)\,dt$, where $b(t)$ is the PDF of beta distribution with $G(x;\zeta)\,\epsilon\,(0,1)$ is a CDF of any arbitrary baseline model. [5] proposed an odd log-logistic–*P* family with CDF $[P(z)/\bar{P}(z)]$. [6] proposed a quadratic rank transmutation map with CDF $[(1+c)P(z)-cP^2(z)]$. [7] proposed a Kumaraswamy generalized–*P* family with CDF $[1-[1-P^a(z)]^b]$. [8] proposed a gamma–*P* family with CDF $[-logP(z)]$. [9] developed a dual transformation and established a Lehmann type–II (L–II) *P* family with CDF $Z(x)=1-\left(1-P(z)\right)^a$. [10] proposed a *T–X* family with CDF $[1-R(W[G(z)])]$. [11] proposed a Weibull–*P* family with CDF $\left[1-e^{\left(-a[P(z)/\bar{P}(z)]^b\right)}\right]$. [12] proposed a beta Marshall–Olkin–*P* family with CDF $\left[I_{P-MO(z)}(a,b)\right]$. [13] proposed a DUS transformation to generate new models with CDF $\left[\left(e^{P(z)}-1\right)/(e-1)\right]$.[14] proposed a Logistic–X family with CDF $\left[\left[1+\left[-log[\bar{P}(z)]\right]^{-a}\right]^{-1}\right]$.[15] proposed an alpha transformation with CDF $\left[\left(\alpha^{P(z)}-1\right)/(\alpha-1)\right]$. [16] developed an odd Fréchet (OFré)-G family with CDF is given by

$Z(x) = \mathrm{e}^{-\left[\frac{1-G(z)}{G(z)}\right]^a}$. [17] proposed a new alpha power transformation with CDF $\left[P(z)\alpha^{P(z)}/\alpha\right]$. Study by [18] proposed another technique with CDF $\left[\left(a^{P(z)} - e^{P(z)}\right)/(a-e)\right]$ to generate new models. [19] proposed a Gull alpha power Weibull–*P* family with CDF $\left[\alpha P(z)/\alpha^{P(z)}\right]$. [20] proposed a new Kumaraswamy–*P* family with CDF $\left[1-\left[1-\left(1-\bar{P}(z)^{P(z)}\right)^a\right]^b\right]$. [21] proposed a new logarithmic–*P* family with CDF $\left[1 - Log\left[\left(2-\lambda P(z)\right)/log(2)\right]\right]$ and many others. Attracted features of L–I,–II compelled the researchers to explore new areas for modeling and discuss the hidden characteristics of classical and modified models. For recent examples, see the latest work of the references. [22] generalized a new model via L–II–*P* family. [23] discussed exponentiated PF distribution with L–I. [24] developed a generalized version of L–II. [25] developed the *P* family of a generalized version of L–II with CDF $\left[1-\left(\left(1-P(z)\right)/\left(1-aP(z)\right)\right)^b\right]$, and [26] discussed a beta version of L–II with CDF $\left[I_{1-\left(1-P(z)\right)^a}(a,b)\right]$.

1.1 Definition

We have proposed a new family, known as odd Fréchet Lehmann type-II (O*Fr*LII) G family of distributions with CDF

$$F_{\mathrm{OFrLII}-G}(x;\phi) = 1-\left(1-e^{-\left(\frac{1-G(x;\phi)}{G(x;\phi)}\right)}\right)^a \; ; x \in \mathbb{R}, a, \phi > 0, \tag{1}$$

where $G(x;\phi)\,\epsilon\,(0,1)$ is a CDF of any arbitrary baseline model based on the parametric vector ϕ depends on (r x 1) with $a > 0$ as a shape parameter. O*Fr*LII – G family is obtained by replacing the CDF of L–II with the CDF of the OFré – G family withholding a power parameter of the OFré – G family, which equals one.

In Table 1 we present eight new sub-models survival functions $S(x;\phi)$ corresponding to classical baseline models $G(x;\phi)$.

Table 1. List of New Sub-models $S(x;\phi)$ corresponding to $G(x;\phi)$ functions

Model	Support	Baseline model	Survival models	$\boldsymbol{\phi}$
Rayleigh	$(0,\infty)$	$1-e^{-bx^2}$	$\left(1-e^{1-\left(1-e^{-bx^2}\right)^{-1}}\right)^a$	a,b
Gompertz	$(0,\infty)$	$1-e^{-b(e^{cx}-1)}$	$\left(1-e^{1-\left(1-e^{-b(e^{cx}-1)}\right)^{-1}}\right)^a$	a,b,c
Pareto	(m,∞)	$1-\left(\frac{m}{x}\right)^b$	$\left(1-e^{1-\left(1-\left(\frac{m}{x}\right)^b\right)^{-1}}\right)^a$	a,b
Fréchet	$(0,\infty)$	$e^{-bx^{-c}}$	$\left(1-e^{1-\left(e^{-bx^{-c}}\right)^{-1}}\right)^a$	a,b,c
Burr-X	$(0,\infty)$	$\left(1-e^{-(bx)^2}\right)^c$	$\left(1-e^{1-\left(1-e^{-(bx)^2}\right)^{-c}}\right)^a$	a,b,c
Weibull	$(0,\infty)$	$1-e^{-bx^c}$	$\left(1-e^{1-\left(1-e^{-bx^c}\right)^{-1}}\right)^a$	a,b,c
Lomax	$(0,\infty)$	$1-(1+xb^{-1})^{-c}$	$\left(1-e^{1-\left(1-\left(1+xb^{-1}\right)^{-c}\right)^{-1}}\right)^a$	a,b,c
Power Function	$(0,\mathrm{g}_0)$	$\left(\frac{x}{\mathrm{g}_0}\right)^b$	$\left(1-e^{1-\left(\frac{\mathrm{g}_0}{x}\right)^b}\right)^a$	a,b

Let $\mathrm{g}(x;\phi) = dG(x;\phi)/dx$ is the probability density function (PDF) of any baseline model. The associated PDF $\left(f_{\mathrm{OFrLII}-G}(x;\phi)\right)$, hazard rate function HRF $\left(h_{\mathrm{OFrLII}-G}(x;\phi)\right)$, and quantile function $\left(Q_{\mathrm{OFrLII}-G}(q;\phi)\right)$corresponding to O*Fr*LII–G family are, given by, respectively

$$f_{\mathrm{OFrLII}-G}(x;\phi) = \frac{a\mathrm{g}(x;\phi)}{G^2(x;\phi)}e^{-\left(\frac{1-G(x;\phi)}{G(x;\phi)}\right)}\left(1-e^{-\left(\frac{1-G(x;\phi)}{G(x;\phi)}\right)}\right)^{a-1}, \tag{2}$$

$$h_{\mathrm{OFrLII}-G}(x;\phi)=\frac{ag(x;\phi)e^{-\left(\frac{1-G(x;\phi)}{G(x;\phi)}\right)}}{G^2(x;\phi)\left(1-e^{-\left(\frac{1-G(x;\phi)}{G(x;\phi)}\right)}\right)}, \tag{3}$$

and

$$Q_{\mathrm{OFrLII}-G}(q;\phi)=G^{-1}\left(1-log\left(1-(1-q)^{\frac{1}{a}}\right)\right)^{-1}, q\in(0,1). \tag{4}$$

Now and onward, an odd Fréchet Lehmann type-II (O*Fr*LII) G family random variable X corresponding to $f_{\mathrm{OFrLII}-G}(x;\phi)$ will be denoted by $X\sim$ O*Fr*LII $-G(x;\phi)$ and to the best of our knowledge, no study has been done in the past that relates to our new family. This study has the following motivations:

(i) To propose a new family that generates flexibility and improves the features of baseline models.

(ii) Closed-form features of CDF, PDF and HRF of new models are simple to interpret.

(iii) New models offer greater distributional flexibility in terms of high kurtosis.

(iv) It offers a better fit over the asymmetric, and bathtub-shaped random phenomena particularly associated with the engineering, and medical sciences events.

This paper is assembled on the following steps. The construction of a new family is discussed in Section 1. General characteristics of a new family are developed in Section 2. A detailed discussion of O*Fr*LII–PF distribution (sub-model) is done in Section 3. A technique to estimate the model parameters named maximum likelihood estimation and a simulation study are discussed in Section 4. Real-life data sets are analyzed in Section 6 and finally, the conclusion is reported in Section 7.

2. General Characteristics

2.1 Useful Representation

Linear representation of CDF and PDF has a significant role in providing more ease for complex mathematical measures. For O*Fr*LII–G family we utilize binomial and exponential series expansions and it is given by

$$(1-z)^{\beta}=\sum_{i=0}^{\infty}(-1)^i\binom{\beta}{i}z^i,\ |z|<1;\ \ e^z=\sum_{j=0}^{\infty}\frac{z^j}{j!}.$$

Infinite linear combinations of CDF

$$F_{\mathrm{OFrLII}-G}(x;\phi)=1-\sum_{i,j,k=0}^{\infty}\binom{a}{i}\binom{j}{k}\frac{(-1)^{i+j+k}i^j}{j!}G^{k-j}(x;\phi),$$

$$F_{\mathrm{OFrLII}-G}(x;\phi)=1-\sum_{i,j,k=0}^{\infty}\Delta_{i,j,k}G^c(x;\phi), \tag{5}$$

and PDF for O*Fr*LII–G family are given by

$$f_{\mathrm{OFrLII}-G}(x;\phi)=a\sum_{i,j,k,l,m=0}^{\infty}\binom{i}{j}\binom{a-1}{k}\binom{l}{m}\frac{(-1)^{i+j+k+l+m}}{i!\,l!}\mathrm{g}(x;\phi)\,G^{j-i-l+m-2}(x;\phi),$$

$$f_{\mathrm{OFrLII}-G}(x;\phi)=a\sum_{i,j,k=0}^{\infty}\nabla_{i,j,k,l,m}\,\mathrm{g}(x;\phi)G^d(x;\phi), \tag{6}$$

respectively, where $\Delta_{i,j,k}=\binom{a}{i}\binom{j}{k}\frac{(-1)^{i+j+k}i^j}{j!}, c=k-j, \nabla_{i,j,k,l,m}=\binom{i}{j}\binom{a-1}{k}\binom{l}{m}\frac{(-1)^{i+j+k+l+m}}{i!l!}, d=j-i-l+$

$m-2$. The expansions in (5) and (6) provide us the exponentiated-G (Exp-G) family which is quite useful for the generalization of models.

2.2 Moments

The *r*-th ordinary moment (sayμ'_r) of *X* is given by

$$\mu'_r = \int_{-\infty}^{+\infty} x^r f(x)dx\,.$$

By following (6), we obtain

$$\mu'_{r-\mathrm{OF}r\mathrm{LII}-G} = a\sum_{i,j,k,l,m=0}^{\infty} \nabla_{i,j,k,l,m}\, I^r{}_d(x;\phi), \tag{7}$$

where coefficient $\nabla_{i,j,k,l,m}=\binom{i}{j}\binom{a-1}{k}\binom{l}{m}\frac{(-1)^{i+j+k+l+m}}{i!l!}, d=j-i-l+m-2$ and $I^r{}_d(x;\phi)=\int_{-\infty}^{+\infty} x^r \mathrm{g}(x;\phi)G^d(x;\phi)dx.$

2.3 Incomplete Moments

The first incomplete moment has a significant role in the discussion of Bonferroni and Lorenz curves. The r – th incomplete moments $\varphi_r(t)=\int_{-\infty}^{t} x^r f(x)dx$ directly followed by (7) are given by

$$\varphi_{r-\mathrm{OF}r\mathrm{LII}-\mathrm{G}}(t;\phi) = a\sum_{i,j,k,l,m=0}^{\infty} \nabla_{i,j,k,l,m}\, I^{r,t}{}_d(x;\phi),$$

where $I^{r,t}{}_d(x;\phi)=\int_{-\infty}^{t} x^r \mathrm{g}(x;\phi)G^d(x;\phi)dx$.For parent distributions, integrals $I^r{}_d(x;\phi)$ and $I^{r,t}{}_d(x;\phi)$ can be solved numerically.

2.4 Residual and Reversed Residual Life Functions

The residual life function is defined by $R_t(x)=\frac{S(x+t)}{S(t)}$. The residual life function of *X* is given by

$$R_{t-\mathrm{OF}r\mathrm{LII}-\mathrm{G}}(x) = \frac{\left(1-e^{-\left(\frac{1-G(x+t;\phi)}{G(x+t;\phi)}\right)}\right)^a}{\left(1-e^{-\left(\frac{1-G(t;\phi)}{G(t;\phi)}\right)}\right)^a}\,.$$

Furthermore, reversed residual life function is defined by $\bar{R}_t(x)=\frac{S(x-t)}{S(t)}$. The reversed residual life function of *X* is given by

$$\bar{R}_{t-\mathrm{OF}r\mathrm{LII}-\mathrm{G}}(x) = \frac{\left(1-e^{-\left(\frac{1-G(x-t;\phi)}{G(x-t;\phi)}\right)}\right)^a}{\left(1-e^{-\left(\frac{1-G(t;\phi)}{G(t;\phi)}\right)}\right)^a}\,.$$

2.5 Moment Generating Function

Moment generating function $M_X(t)$ is defined as $M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\mu'_r$ and it is given by

$$M_{X-\mathrm{OF}r\mathrm{LII}-G}(x;\phi) = a\sum_{r=0}^{\infty}\frac{t^r}{r!}\sum_{i,j,k=0}^{\infty}\nabla_{i,j,k,l,m}\, I^r{}_d(x;\phi), \tag{8}$$

where coefficient $\nabla_{i,j,k,l,m} = \binom{i}{j}\binom{a-1}{k}\binom{l}{m}\frac{(-1)^{i+j+k+l+m}}{i!l!}, d = j-i-l+m-2$

and $I^r{}_d(x;\phi) = \int_{-\infty}^{+\infty} x^r \mathrm{g}(x;\phi)G^d(x;\phi)dx.$

2.6 Entropy

When a system is quantified by randomness in general, it is known as entropy. [27] entropy of X is given by

$$H_\delta(X) = \frac{1}{1-\delta}log\int_{-\infty}^{\infty} f^\delta(x)dx\ , \qquad \delta > 0\ and\ \delta \neq 1. \tag{9}$$

By following (2), we simplify $f(x;\phi)$ in terms of $f^\delta(x;\phi)$, we get

$$f^\delta{}_{\mathrm{OF}r\mathrm{LII}-G}(x;\phi) = \frac{a^\delta \mathrm{g}^\delta(x;\phi)}{G^{2\delta}(x;\phi)}e^{-\delta\left(\frac{1-G(x;\phi)}{G(x;\phi)}\right)}\left(1-e^{-\left(\frac{1-G(x;\phi)}{G(x;\phi)}\right)}\right)^{\delta(a-1)}.$$

The expansion of $e^{-\delta\left(\frac{1-G(x;\phi)}{G(x;\phi)}\right)}$ and $\left(1-e^{-\left(\frac{1-G(x;\phi)}{G(x;\phi)}\right)}\right)^{\delta(a-1)}$ provide us Exp-G and the last expression can be written as follows

$$f^\delta{}_{\mathrm{OF}r\mathrm{LII}-G}(x;\phi) = \left(\begin{array}{c}\frac{a^\delta \mathrm{g}^\delta(x;\phi)}{G^{2\delta}(x;\phi)}\sum_{i=0}^{\infty}\frac{(-1)^i\delta^i}{i!}\times\frac{1}{G^i(x;\phi)}\sum_{j=0}^{\infty}\binom{i}{j}(-1)^j G^j(x;\phi)\times \\ \sum_{k=0}^{\infty}\binom{\delta(a-1)}{k}(-1)^k\sum_{l=0}^{\infty}\frac{(-1)^l k^l}{l!}\times\frac{1}{G^l(x;\phi)}\sum_{m=0}^{\infty}\binom{l}{m}(-1)^m G^m(x;\phi)\end{array}\right).$$

Now place the last information in (9) which provides us a reduced form of Rényi entropy for X and it is given as follows

$$H_{\delta-\mathrm{OF}r\mathrm{LII}-G}(X) = \frac{1}{1-\delta}loga^\delta\sum_{i,j,k,l,m=0}^{\infty}\nabla^*_{i,j,k,l,m}\, I^\delta{}_d(x;\phi),$$

where

$\nabla^*_{i,j,k,l,m} = \binom{i}{j}\binom{\delta(a-1)}{k}\binom{l}{m}\frac{(-1)^{i+j+k+l+m}\delta^i k^l}{i!l!}, d = j-i-l+m-2\delta, I^\delta{}_d(x;\phi) = \int_{-\infty}^{+\infty}\mathrm{g}^\delta(x;\phi)G^d(x;\phi)dx.$

2.7 Distribution of Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n follows to the $\mathrm{OF}r\mathrm{LII}$–G family and $X_{(1:n)} < X_{(2:n)} < \ldots < X_{(n:n)}$ be the corresponding order statistics. The PDF of $X_{(i)}$ is given by

$$f_{(i:n)}(x;\phi) = \frac{1}{B(i,n-i+1)!}\big(F(x;\phi)\big)^{i-1}\big(1-F(x;\phi)\big)^{n-i}f(x;\phi), i = 1,2,3,\ldots n.$$

Using the fact that

$$\big(1-F(x;\phi)\big)^{n-i} = \sum_{m=0}^{n-i}(-1)^m\binom{n-i}{m}F(x;\phi)^m,$$

and place the last information in $f_{(i:n)}(x;\phi)$, we obtain the most refined form of OS PDF and expression may be written as follows

$$f_{(i:n)}(x;\phi) = \frac{f(x;\phi)}{B(i,n-i+1)!}\sum_{m=0}^{n-i}(-1)^m\binom{n-i}{m}F(x;\phi)^{i+m-1}. \tag{10}$$

$F(x;\phi)$, and $f(x;\phi)$ are the associated CDF with the corresponding PDF of the O–L–II–G family and

$$F(x;\phi)^{i+m-1} = \sum_{i,j,k,l,o=0}^{\infty} \binom{n-i}{m}\binom{i+m-1}{j}\binom{j}{k}\binom{l}{o}\frac{(-1)^{j+k+l+m+o}}{l!}k^l G^{o-l}(x;\phi).$$

Hence one may obtain the straightforward expression of OS PDF by inserting the last information in (10).

2.8 Bivariate Extension

In this sub-section, we present a simple bivariate extension of the O*Fr*LII–G family. A joint CDF of the O*Fr*LII family is given by

$$F_{O-L-II-G}(x,y;\phi) = 1-\left(1-e^{-\left(\frac{1-G(x,y;\phi)}{G(x,y;\phi)}\right)}\right)^a, x,y\in\mathbb{R}, a,\phi>0,$$

where $G(x,y;\phi)$ is a bivariate continuous distribution function along with marginal CDF's $G_1(x;\phi)$ and $G_2(y;\phi)$. We refer to it as a bivariate O*Fr*LII–G family of distributions. The marginal CDF's of *X* and *Y* is given by respectively are given by

$$F_{X-\mathrm{OFr\,LII}-G}(x;\phi) = 1-\left(1-e^{-\left(\frac{1-G_1(x;\phi)}{G_1(x;\phi)}\right)}\right)^a,$$

$$F_{Y-\mathrm{OFr\,LII}-G}(y;\phi) = 1-\left(1-e^{-\left(\frac{1-G_2(y;\phi)}{G_2(y;\phi)}\right)}\right)^a.$$

The joint PDF of (X,Y) can be determined easily by following $f_{X,Y}(x,y;\phi) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x\partial y}$. Furthermore, the marginal PDFs of X and Y are given by, respectively

$$f_{O-L-II-G-X}(x;\phi) = \frac{ag_1(x;\phi)}{G^2{}_1(x;\phi)}e^{-\left(\frac{1-G_1(x;\phi)}{G_1(x;\phi)}\right)}\left(1-e^{-\left(\frac{1-G_1(x;\phi)}{G_1(x;\phi)}\right)}\right)^{a-1},$$

$$f_{O-L-II-G-Y}(y;\phi) = \frac{ag_2(y;\phi)}{G^2{}_2(y;\phi)}e^{-\left(\frac{1-G_2(y;\phi)}{G_2(y;\phi)}\right)}\left(1-e^{-\left(\frac{1-G_2(y;\phi)}{G_2(;\phi)}\right)}\right)^{a-1}.$$

The conditional CDFs of *X* and *Y* are given by, respectively

$$F_{(X/Y)-\mathrm{OFr\,LII}-\mathrm{G}}(x/y;\phi) = \frac{1-\left(1-e^{-\left(\frac{1-G(x,y;\phi)}{G(x,y;\phi)}\right)}\right)^a}{1-\left(1-e^{-\left(\frac{1-G_2(y;\phi)}{G_2(y;\phi)}\right)}\right)^a},$$

$$F_{(Y/X)-\mathrm{OFr\,LII}-\mathrm{G}}(y/x;\phi) = \frac{1-\left(1-e^{-\left(\frac{1-G(x,y;\phi)}{G(x,y;\phi)}\right)}\right)^a}{1-\left(1-e^{-\left(\frac{1-G_1(x;\phi)}{G_1(x;\phi)}\right)}\right)^a}.$$

2.9 Inference

In this sub-section, we estimate unknown parameters of the O*Fr*LII–G family with the assistance of maximum likelihood estimation and the ordinary least square method.

2.9.1 Maximum Likelihood Estimation (MLE)

Let $x_1, x_2, x_3, \ldots, x_n$ be a random sample of size *n* from the O*Fr*LII–G family, then the log-likelihood function Log *L* $= Log\, L(\phi)$ is given by

$$Log\, L_{OFrLII-G} = nloga + \sum_{i=1}^{n} \log\big(g(x_i;\phi)\big) - 2\sum_{i=1}^{n} log\, G(x_i;\phi) + \sum_{i=1}^{n}\left(\frac{1-G(x_i;\phi)}{G(x_i;\phi)}\right)$$

$$+ (a-1)\sum_{i=1}^{n} log\left(1 - e^{-\left(\frac{1-G(x_i;\phi)}{G(x_i;\phi)}\right)}\right).$$

The partial derivatives of $Log\, L_{OFrLII-G} = l$ for a and ϕ are

$$\frac{\partial l}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} log\left(1 - e^{-\left(\frac{1-G(x_i;\phi)}{G(x_i;\phi)}\right)}\right),$$

$$\frac{\partial l}{\partial \phi} = \sum_{i=1}^{n} \frac{g'_{\phi}(x_i;\phi)}{g(x_i;\phi)} - 2\sum_{i=1}^{n} \frac{G'_{\phi}(x_i;\phi)}{G(x_i;\phi)} + \sum_{i=1}^{n} \frac{G'_{\phi}(x_i;\phi)}{G^2(x_i;\phi)} + (a-1)\sum_{i=1}^{n} \frac{G'_{\phi}(x_i;\phi)e^{-\left(\frac{1-G(x;\phi)}{G(x;\phi)}\right)}}{G^2(x_i;\phi)\left(1 - e^{-\left(\frac{1-G(x;\phi)}{G(x;\phi)}\right)}\right)},$$

respectively, where $g'_{\phi}(x_i;\phi) = \partial g(x_i;\phi)/\partial\phi$ and $G'_{\phi}(x_i;\phi) = \partial G(x_i;\phi)/\partial\phi$. By substituting $\partial l/\partial a$ and $\partial l/\partial\phi$ equal to zero and to obtain ML estimators $\hat{\zeta} = \big(\hat{a}, \hat{\phi}\big)^T$ of $\zeta = (a, \phi)$, we solve these equations simultaneously. As per the prior expressions are not in closed form. Hence, R software will be a better choice to find out its numerical study by using any iterative methods.

2.9.2 Ordinary Least Square (OLS)

Let $x_1, x_2, x_3, \ldots, x_n$ be a random sample of size n from the O*Fr*LII–G family, then the expectation of the empirical CDF is known as OLS estimates and it is given by $E\big(F_{OFrLII-G}(x;\phi)\big) = \frac{j}{n+1}; j = 1,2,3, \ldots, n.$ OLS estimates of unknown parameters (a, ϕ) can be obtained by maximizing the $A(x;\phi) = \left(E\big(F_{OFrLII-G}(x;\phi)\big) - \frac{j}{n+1}\right)^2$ by taking the partial derivatives $\partial A(x;\alpha)/\partial\alpha$ and $\partial A(x;\phi)/\partial\phi$.

3. A New Odd Lehmann type-II Power Function (O*Fr*LII–PF) Distribution

In this section, we derive several explicit expressions for a sub-model of the O*Fr*LII–G family, known as an odd Lehmann type-II power function (O*Fr*LII–PF) distribution. For this, we have the CDF and PDF of power function as

$$G_{PF}(x;b) = \left(\frac{x}{g_0}\right)^b,$$

and

$$g_{PF}(x;b) = \frac{b}{(g_0)^b} x^{b-1};\ b > 0, 0 < x < g_0,$$

respectively. Henceforth, the analytical expressions for CDF, PDF, and HRF of O*Fr*LII–PF distribution are given by respectively

$$F_{OFrLII-PF}(x;a,b) = 1 - \left(1 - e^{1-\left(\frac{g_0}{x}\right)^b}\right)^a, \tag{11}$$

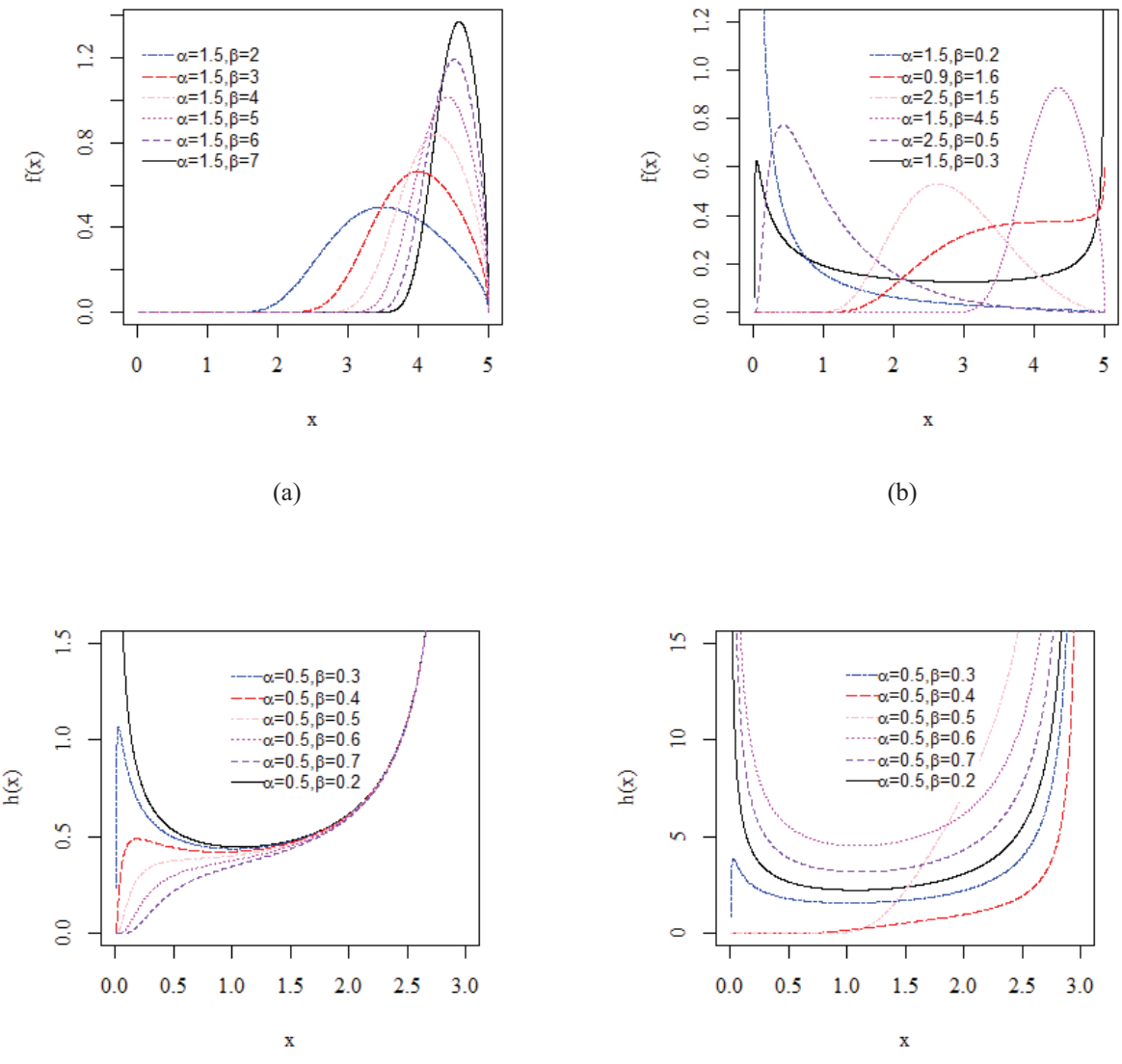
$$f_{OFrLII-PF}(x;a,b) = \frac{ab(g_0)^b}{x^{1+b}} e^{1-\left(\frac{g_0}{x}\right)^b}\left(1 - e^{1-\left(\frac{g_0}{x}\right)^b}\right)^{a-1}, \tag{12}$$

$$h_{OFrLII-PF}(x;a,b) = \frac{ab(g_0)^b e^{1-\left(\frac{g_0}{x}\right)^b}}{(x^{1+b})\left(1-e^{1-\left(\frac{g_0}{x}\right)^b}\right)}, \tag{13}$$

where $0 < x < g_0$, and $a, b > 0$ are two shape parameters.

3.1 Shapes of Density and Hazard Rate Functions

In this sub-section, several curves of PDF and HRF for *X at* different choices of model parameters are sketched out in Figure 1. Note that PDFs (*a and b*) curves have increasing, decreasing, symmetric, upside down, and bathtub shapes. However, HRF's (*c and d*) possess upside-down increasing, U-shaped, bathtub-shaped, and increasing curves.



(a) (b)

(c) (d)

Figure 1. Different curves of density and hazard rate functions

3.2 Useful Expansions

Infinite linear combinations of CDF

$$F_{\mathrm{OFrLII}-PF}(x;a,b) = 1 - \sum_{i,j=0}^{\infty} \binom{a}{i} \frac{(-1)^{i+j} e^{j} i^{j} (\mathrm{g}_0)^{bj}}{j!} x^{-bj}. \tag{14}$$

$$F_{O-L-II-PF}(x;a,b) = 1 - \sum_{i,j=0}^{\infty} A_{i,j} x^{-bj},$$

and PDF for X is given by

$$f_{\mathrm{OFrLII}-PF}(x;a,b) = ab \sum_{i,j,k=0}^{\infty} \binom{a-1}{j} \frac{(-1)^{i+j+k} e^{1+j} (\mathrm{g}_0)^{b(i+k+1)}}{i!\,k!} x^{-b(i+k+1)-1}, \tag{15}$$

$$f_{\mathrm{OFrLII}-PF}(x;a,b) = ab \sum_{i,j,k=0}^{\infty} B_{i,j,k}\, x^{-b(i+k+1)-1},$$

where $A_{i,j} = \binom{a}{i} \frac{(-1)^{i+j} e^{j} i^{j} (\mathrm{g}_0)^{bj}}{j!}$, $B_{i,j,k} = \binom{a-1}{j} \frac{(-1)^{i+j+k} e^{1+j} (\mathrm{g}_0)^{b(i+k+1)}}{i!k!}$.

3.3 Moments

The *r-th* ordinary moments for X is defined as

$$\mu'_{r-\mathrm{OFrLII}-PF} = ab \int_0^{\mathrm{g}_0} x^{r} \frac{(\mathrm{g}_0)^{b}}{x^{1+b}} e^{1-\left(\frac{\mathrm{g}_0}{x}\right)^{b}} \left(1 - e^{1-\left(\frac{\mathrm{g}_0}{x}\right)^{b}}\right)^{a-1} dx,$$

and after few simplifications we obtain

$$\mu'_{r-\mathrm{OFrLII}-PF} = ab \sum_{i,j,k=0}^{\infty} B_{i,j,k} \int_0^{\mathrm{g}_0} x^{r-b(i+k+1)-1}\, dx.$$

Hence, the *r*-th ordinary moments are obtained by solving the last integral and it is given by

$$\mu'_{r-\mathrm{OFrLII}-PF} = ab \sum_{i,j,k=0}^{\infty} B_{i,j,k} \frac{((\mathrm{g}_0)^{r-\chi_{b,i,k}})}{r-\chi_{b,i,k}}, \tag{16}$$

where $B_{i,j,k} = \binom{a-1}{j} \frac{(-1)^{i+j+k} e^{1+j} (\mathrm{g}_0)^{b(i+k+1)}}{i!k!}$, $\chi_{b,i,k} = b(i+k+1)$.

The derived expression in (16) is quite useful in the development of several statistical measures. For instance: to deduce the mean and negative moments of X, substitute $r = 1$ and $r = -w$ with (16), respectively, and it is given by

$$\mu'_{1-\mathrm{OFrLII}-PF} = ab \sum_{i,j,k=0}^{\infty} B_{i,j,k} \frac{((\mathrm{g}_0)^{1-\chi_{b,i,k}})}{1-\chi_{b,i,k}}, \tag{17}$$

and

$$\mu'_{-w-\mathrm{OFrLII}-PF} = ab \sum_{i,j,k=0}^{\infty} B_{i,j,k} \frac{\left((\mathrm{g}_0)^{-(w+\chi_{b,i,k})}\right)}{1-\chi_{b,i,k}}.$$

Furthermore, for fractional positive and fractional negative moments for X, substitute $r = (m/n)$ and $r = -(m/n)$ with (16), respectively, and the expressions are, respectively, given by

$$\mu'_{(m/n)-\mathrm{OFrLII}-PF} = ab \sum_{i,j,k=0}^{\infty} B_{i,j,k} \frac{\left((\mathrm{g}_0)^{(m/n)-\chi_{b,i,k}}\right)}{1-\chi_{b,i,k}},$$

and

$$\mu'_{-(m/n)-\mathrm{OF}r\mathrm{LII}-PF} = ab\sum_{i,j,k=0}^{\infty} B_{i,j,k}\frac{\left((\mathrm{g}_0)^{-(m/n)-\chi_{b,i,k}}\right)}{1-\chi_{b,i,k}}.$$

The moment generating function $M_X(t)$ is defined as $M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu'_r$. It is obtained for X as

$$M_{X-\mathrm{OF}r\mathrm{LII}-PF}(t) = ab\sum_{r=0}^{\infty}\frac{t^r}{r!}\sum_{i,j,k=0}^{\infty} B_{i,j,k}\frac{((\mathrm{g}_0)^{r-\chi_{b,i,k}})}{r-\chi_{b,i,k}}.$$

The characteristic function of X is defined as $\emptyset_X(t) = \sum_{r=0}^{\infty}\frac{(it)^r}{r!}\mu'_r$. It is obtained for X as

$$\emptyset_{X-\mathrm{OF}r\mathrm{LII}-PF}(t) = ab\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\sum_{i,j,k=0}^{\infty} B_{i,j,k}\frac{((\mathrm{g}_0)^{r-\chi_{b,i,k}})}{r-\chi_{b,i,k}}.$$

The factorial generating function is defined as $F_x(t) = E(1+t)^x = E\left(e^{xln(1+t)}\right) = \sum_{r=0}^{\infty}\frac{(ln(1+t))^r}{r!}\mu'_r$. It is obtained for X as

$$F_{x-\mathrm{OF}r\mathrm{LII}-PF}(t) = ab\sum_{r=0}^{\infty}\frac{\left(ln(1+t)\right)^r}{r!}\sum_{i,j,k=0}^{\infty} B_{i,j,k}\frac{((\mathrm{g}_0)^{r-\chi_{b,i,k}})}{r-\chi_{b,i,k}}.$$

The Mellin transformation is defined as$M_x(m) = \int_0^{\infty} x^{m-1}f(x)dx$. It is obtained for X as

$$M_{x-\mathrm{OF}r\mathrm{LII}-PF}(m) = ab\sum_{i,j,k=0}^{\infty} B_{i,j,k}\frac{\left((\mathrm{g}_0)^{(m-1)-\chi_{b,i,k}}\right)}{(m-1)-\chi_{b,i,k}}.$$

The central moments $\mu_s = \sum_{k=0}^{s}\binom{S}{k}(-1)^k\left(\mu'_1\right)^s\mu'_{s-k}$ and first four cumulants$K_1 = \mu'_1$, $K_2 = \mu'_2 - \mu'^2_1$, $K_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1$, $K_4 = \mu'_4 - 4\mu'_3\mu'_1 - 3\mu'^2_2 + 12\mu'_2\mu'^2_1 - 6\mu'^4_1$for X may easily be defined by ordinary moments. To study the tail and peak behavior for X, a measure of skewness ($\beta_1 = \mu_3^2/\mu_2^3$) and measure of kurtosis ($\beta_2 = \mu_4/\mu_2^2$), play a significant role, respectively. **S**ome numerical results of the first four ordinary moments ($\mu'_1, \mu'_2, \mu'_3, \mu'_4$), σ^2 = variance, β_1 = skewness, and β_2 = kurtosis for some choices of model parameters for $\mathrm{g}_0 = 1.3$ is presented in Table 2.

Table 2. Some numerical results of moments, variance, skewness, and kurtosis

Statistics			$\mathbf{a = 1.5}$		
μ'_r	$b = 0.4$	$b = 0.5$	$b = 0.6$	$b = 0.7$	$b = 0.8$
μ'_1	0.3230	0.4024	0.4724	0.5336	0.5873
μ'_2	0.1832	0.2443	0.3050	0.3637	0.4199
μ'_3	0.1363	0.1861	0.2381	0.2911	0.3441
μ'_4	0.1172	0.1617	0.2091	0.2587	0.3096
σ^2	0.0910	0.0992	0.0997	0.0932	0.0799
β_1	0.4279	0.1765	0.0791	0.0411	0.0274
β_2	1.3768	0.8019	0.4972	0.3126	0.1823
Statistics	$b = 0.9$	$b = 0.1$	$b = 0.3$	$b = 0.1$	$b = 0.5$
μ'_r	$a = 2.01$	$a = 2.1$	$a = 1.9$	$a = 1.9$	$a = 0.9$
μ'_1	0.5657	0.0216	0.1791	0.0287	0.5590
μ'_2	0.3761	0.0068	0.0771	0.0103	0.4416
μ'_3	0.2833	0.0037	0.0486	0.0062	0.4108
μ'_4	0.2344	0.0027	0.0372	0.0047	0.4168
σ^2	0.0154	0.0061	0.0427	0.0092	0.1636
β_1	0.1807	39.2183	1.9478	29.3150	0.0049
β_2	-0.1411	55.0211	3.7508	40.8929	0.2454

We observe that the results of moments are decreasing whereas variance, skewness, and kurtosis have flexible performances at different values of ***a*** and ***b***.

3.4 Incomplete Moments and Residual Life Function

The r – th lower incomplete moments is defined as $\Phi_r(x) = \int_0^t x^r f(x)dx$. It is obtained for X as

$$\Phi_{r-\mathrm{OF}r\mathrm{LII}-PF}(x) = ab \sum_{i,j,k=0}^{\infty} B_{i,j,k} \frac{(t^{r-\chi_{b,i,k}})}{r-\chi_{b,i,k}}. \tag{18}$$

The first incomplete moment is obtained by simply substituting $r = 1$ in (18) and it is given by

$$\Phi_{1-\mathrm{OF}r\mathrm{LII}-PF}(x) = ab \sum_{i,j,k=0}^{\infty} B_{i,j,k} \frac{(t^{1-\chi_{b,i,k}})}{1-\chi_{b,i,k}}. \tag{19}$$

The residual life function is defined by $R_t(x) = \frac{S(x+t)}{S(t)}$. The residual life function and associated CDF of X are given by

$$R_{t(x)-\mathrm{OF}r\mathrm{LII}-PF}\left({}^{t}/_{x}\right) = \frac{\left(1-e^{1-\left(\frac{\mathrm{g}_0}{x+t}\right)^b}\right)^a}{\left(1-e^{1-\left(\frac{\mathrm{g}_0}{t}\right)^b}\right)^a}, x > 0.$$

$$F_{R(t)-\mathrm{OF}r\mathrm{LII}-PF}\left({}^{t}/_{x}\right) = 1 - \frac{\left(1-e^{1-\left(\frac{\mathrm{g}_0}{x+t}\right)^b}\right)^a}{\left(1-e^{1-\left(\frac{\mathrm{g}_0}{t}\right)^b}\right)^a};\ x > 0.$$

Furthermore, the reversed residual life function is defined by $\bar{R}_t(x) = \frac{S(x-t)}{S(t)}$. The reversed residual life function and

associated CDF of *X* are given by

$$\bar{R}_{t(x)-\mathrm{OF}r\mathrm{LII}-PF}\left({}^{t}/_{x}\right)=\frac{\left(1-e^{1-\left(\frac{\mathrm{g}_0}{x-t}\right)^b}\right)^a}{\left(1-e^{1-\left(\frac{\mathrm{g}_0}{t}\right)^b}\right)^a},x>0.$$

The mean residual life function $E\left(R_{t(x)}\right)=\frac{1}{S(t)}\left(\mu_{1,t}^{/}-\int_0^t xf(x)dx\right)-t;\ t\geq 0,$ reversed residual life function $E\left(\bar{R}_{t(x)}\right)=t-\frac{1}{F(t)}\int_0^t xf(x)dx;t\geq 0,$ and strong mean inactivity time (SMIT) $M(t)=t^2-\frac{1}{F(t)}\int_0^t x^2f(x)dx,$ may easily be derived by following equations (17) and (19) and for SMIT substitute *r* = 2 with (17), respectively. Furthermore, the Lorenz $L(x)$ and Bonferroni $B(x)$ curves have a significant role not only in the study of economics, the distribution of income, poverty, or wealth, but it has a vital role in fields of insurance, demography, medicine, reliability engineering, and others. The first incomplete moment is very useful in the discussion of Lorenz and Bonferroni curves and it is obtained for *X* respectively, by

$$L(x)=\frac{\int_0^t xf(x)dx}{\mu_1^{/}},\qquad B(x)=\frac{L(x)}{F(x)},$$

, and

$$L(x)=\frac{\sum_{i,j,k=0}^{\infty}B_{i,j,k}\frac{(t^{1-\chi_{b,i,k}})}{1-\chi_{b,i,k}}}{\sum_{i,j,k=0}^{\infty}B_{i,j,k}\frac{((\mathrm{g}_0)^{1-\chi_{b,i,k}})}{1-\chi_{b,i,k}}},\qquad B(x)=\frac{\sum_{i,j,k=0}^{\infty}B_{i,j,k}\frac{(t^{1-\chi_{b,i,k}})}{1-\chi_{b,i,k}}}{\sum_{i,j,k=0}^{\infty}B_{i,j,k}\frac{((\mathrm{g}_0)^{1-\chi_{b,i,k}})}{1-\chi_{b,i,k}}\left(1-\left(1-e^{1-\left(\frac{\mathrm{g}_0}{x}\right)^b}\right)^a\right)}.$$

3.5 Distribution of Order Statistics

In reliability analysis and life testing of a component in quality control, order statistics (OS) has a noteworthy contribution. Let $X_1, X_2, X_3, ..., X_n$ be a random sample of size *n* follows to the O–L–II–PF distribution and $\{X_{(1)}<X_{(2)}<X_{(3)}<...<X_{(n)}\}$ be the corresponding order statistics. The PDF of *i*-th OS is given by

$$f_{(i:n)}(x)=\frac{1}{B(i,n-i+1)!}\left(F(x)\right)^{i-1}\left(1-F(x)\right)^{n-i}f(x),\quad i=1, 2, 3,..., n.$$

By incorporating (11) and (12), *i*-th OS PDF for *X* is given by

$$f_{(i:n)-\mathrm{OF}r\mathrm{LII}-PF}(x;a,b)=\left(\begin{array}{c}\frac{1}{B(i,n-i+1)!}\left(1-\left(1-e^{1-\left(\frac{\mathrm{g}_0}{x}\right)^b}\right)^a\right)^{i-1}\left(\left(1-e^{1-\left(\frac{\mathrm{g}_0}{x}\right)^b}\right)^a\right)^{n-i}\times\\ \frac{ab(\mathrm{g}_0)^b}{x^{1+b}}e^{1-\left(\frac{\mathrm{g}_0}{x}\right)^b}\left(1-e^{1-\left(\frac{\mathrm{g}_0}{x}\right)^b}\right)^{a-1}\end{array}\right).$$

Minimum OS PDF

$$f_{(1:n)-\mathrm{OF}r\mathrm{LII}-PF}(x;a,b)=\left(\begin{array}{c}\frac{1}{B(i,n-i+1)!}\left(\left(1-e^{1-\left(\frac{\mathrm{g}_0}{x}\right)^b}\right)^a\right)^{n-1}\times\\ \frac{ab(\mathrm{g}_0)^b}{x^{1+b}}e^{1-\left(\frac{\mathrm{g}_0}{x}\right)^b}\left(1-e^{1-\left(\frac{\mathrm{g}_0}{x}\right)^b}\right)^{a-1}\end{array}\right),$$

and maximum OS PDF for *X* is given by

$$f_{(n:n)-\mathrm{O}Fr\mathrm{LII}-PF}(x;a,b)=\frac{1}{B(i,n-i+1)!}\left(\begin{array}{c}\left(1-\left(1-e^{1-\left(\frac{\mathrm{g}_0}{x}\right)^b}\right)^a\right)^{n-1}\times\\ \frac{ab(\mathrm{g}_0)^b}{x^{1+b}}e^{1-\left(\frac{\mathrm{g}_0}{x}\right)^b}\left(1-e^{1-\left(\frac{\mathrm{g}_0}{x}\right)^b}\right)^{a-1}\end{array}\right).$$

The *i*-th OS CDF is defined by

$$F_{(i:n)}(x)=\sum_{r=i}^{n}\binom{n}{r}\big(F(x)\big)^r\big(1-F(x)\big)^{n-r}.$$

By incorporating (11), we obtain the *i*-th OS CDF for *X* and it is given by

$$F_{(i:n)-\mathrm{O}Fr\mathrm{LII}-PF}(x;a,b)=\sum_{r=1}^{n}\binom{n}{r}\left(1-\left(1-e^{1-\left(\frac{\mathrm{g}_0}{x}\right)^b}\right)^a\right)^r\left(\left(1-e^{1-\left(\frac{\mathrm{g}_0}{x}\right)^b}\right)^a\right)^{n-r}.$$

3.6 Quantile Function

The q^{th} quantile function of the O*Fr*LII–PF distribution is obtained by inverting the CDF. It is defined as $q=F\big(x_q\big)=P\big(X\le x_q\big),\ q\in(0,1)$. Then; the quantile function for *X* is given by

$$x_{q-\mathrm{O}Fr\mathrm{LII}-PF}=\frac{\mathrm{g}_0}{(1-log(1-(1-q)^{1/a})\,)^{1/b}}. \tag{20}$$

To derive the 1st quartile, median and 3rd quartile of *X*, one may place *q* = 0.25, 0.5, and 0.75 respectively in (20). Henceforth, to generate random numbers, one may assume that the CDF for *X* follows to uniform distribution *u*= *U* (0, 1).

3.7 Bivariate and Multivariate Extensions

In this sub-section, we develop the bivariate and multivariate extensions for the O*Fr*LII–PF distribution by following the Morgenstern family and the Clayton family.

The CDF of the Bi– O*Fr*LII–PF distribution followed by the Morgenstern family for the random vector (V_1,V_2) is

$$F_{\phi-\mathrm{O}Fr\mathrm{LII}-PF}(V_1,V_2)=\Big(1+\phi\big(1-F_1(v_1)\big)\big(1-F_2(v_2)\big)\Big)F_1(v_1)F_2(v_2),$$

where $|\phi|\le 1,\ F_1(v_1)=1-\left(1-e^{1-\left(\frac{\mathrm{g}_{01}}{v_1}\right)^{b_1}}\right)^{a_1}$, and $F_2(v_2)=1-\left(1-e^{1-\left(\frac{\mathrm{g}_{02}}{v_2}\right)^{b_2}}\right)^{a_2}$.

The CDF of the Bi– O*Fr*LII–PF distribution followed by the Clayton family for the random vector (X,Y) is

$$C(x,y)=\big(x^{-(\zeta_1+\zeta_2)}+y^{-(\zeta_1+\zeta_2)}-1\big)^{-\frac{1}{(\zeta_1+\zeta_2)}};\ \zeta_1+\zeta_2\ge 0.$$

Let $v_1\sim$ O– O*Fr*LII–PF (α_1,β_1), and $v_2\sim$ O– O*Fr*LII–PF (α_2,β_2). Then setting

$$x=F_1(v_1)=1-\left(1-e^{1-\left(\frac{\mathrm{g}_{01}}{v_1}\right)^{b_1}}\right)^{a_1}\text{ and }\ y=F_2(v_2)=1-\left(1-e^{1-\left(\frac{\mathrm{g}_{02}}{v_2}\right)^{b_2}}\right)^{a_2}.$$

The CDF of the Bi– O*Fr*LII–PF distribution followed by the Clayton family for the random vector (V_1,V_2) is

$$G_{Bi-\mathrm{O}Fr\mathrm{LII}-PF}(v_1,v_2)=\left(\begin{array}{c}\left(1-\left(1-e^{1-\left(\frac{\mathrm{g}_{01}}{v_1}\right)^{b_1}}\right)^{a_1}\right)^{(\zeta_1+\zeta_2)}+\\ \left(1-\left(1-e^{1-\left(\frac{\mathrm{g}_{02}}{v_2}\right)^{b_2}}\right)^{a_2}\right)^{(\zeta_1+\zeta_2)}-1\end{array}\right)^{-\frac{1}{(\zeta_1+\zeta_2)}}.$$

A simple *n*-dimensional extension of the last version will be

$$H(x_1, x_2, x_3, \dots, x_n) = \left(\sum_{i=1}^{n} \left(\left(1 - \left(1 - e^{1-\left(\frac{g_{0i}}{x_i}\right)^{b_i}} \right)^{a_i} \right)^{(\zeta_1+\zeta_2)} \right) + 1 - n \right)^{-\frac{1}{(\zeta_1+\zeta_2)}}.$$

4. Inference

In this section, we discuss an estimation technique for O*Fr*LII–PF distribution known as the method of maximum likelihood estimation.

Let $X_1, X_2, \dots, X_n$ be a random sample of size *n* from *X*, then the likelihood function $L(\phi) = \prod_{i=1}^{n} f_{OFrLII-PF}(x_i; a, b)$ of *X* is given by

$$L_{OFrLII-PF}(\phi) = \frac{(ab(g_0)^b)^n}{\prod_{i=1}^{n} {x_i}^{1+b}} \prod_{i=1}^{n} e^{1-\left(\frac{g}{x_i}\right)^b} \prod_{i=1}^{n} \left(1 - e^{1-\left(\frac{g_0}{x_i}\right)^b} \right)^{a-1}.$$

The log-likelihood function, $l_{OFrLII-PF}(\phi)$ of *X* is given by

$$l_{O-L-II-PF}(\phi) = n\big(loga + logb + blog(g_0)\big) - (1+b)\sum_{i=1}^{n} logx_i + \sum_{i=1}^{n} \left(1 - \left(\frac{g_0}{x_i}\right)^b \right) + (a-1)\sum_{i=1}^{n} log\left(1 - e^{1-\left(\frac{g_0}{x_i}\right)^b} \right). \quad (21)$$

The partial derivatives *w.r.t* a and b of (21) yield

$$\frac{\partial l_{OFrLII-PF}(\phi)}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} log\left(1 - e^{1-\left(\frac{g_0}{x_i}\right)^b} \right),$$

and

$$\frac{\partial l_{OFrLII-PF}(\phi)}{\partial b} = \frac{n}{b} - \sum_{i=1}^{n} logx_i - \sum_{i=1}^{n} \left(\frac{g_0}{x_i}\right)^b log\left(\frac{g_0}{x_i}\right) - (a-1)\sum_{i=1}^{n} \left(\frac{\left(\frac{g_0}{x_i}\right)^b e^{1-\left(\frac{g_0}{x_i}\right)^b} log\left(\frac{g_0}{x_i}\right)}{\left(1 - e^{1-\left(\frac{g_0}{x_i}\right)^b} \right)} \right),$$

respectively. The maximum likelihood estimates $(\hat{\phi} = \hat{a}, \hat{b})$ for the O*Fr*LII–PF distribution can be obtained by maximizing (21) or by solving the prior non-linear equations simultaneously. These non-linear equations although do not provide an analytical solution for the MLEs and the optimum value of a, and b. Consequently, the Newton-Raphson type algorithm is an appropriate choice in the support of MLEs.

4.1 Simulation Experiment

In this sub-section, we perform a simulation experiment to observe the asymptotic performance of MLE's $\hat{\phi} = (\hat{a}, \hat{b})$. For this, we discuss the following algorithm.

Step -1. A random sample $x_1, x_2, x_3, \dots, x_n$ of sizes *n* = 25, 50, 100, 200, 300, 400, 500, and 1000 from (20).

Step -2. The required results are obtained based on the different combinations of the model parameters place in S-I (a =2.2, b = 1.9), S-II (a =3.9, b = 3.1), S-III (a =0.9, b = 0.5), and S-IV (a =0.5, b = 2.1).

Step -3. Results of mean, variance (short Var), Bias, and root mean square error (short RMSE) are calculated with the assist of statistical software R with its exclusive function *nlmib*. These results are presented in Tables 3 to 10.

Step -4. Each sample is replicated N = 1000 times.

Step -5. Gradual decrease with the increase in sample sizes is observed in mean, biases, RMSEs, and Var.

Furthermore, the following measures are defined in the development of average estimate (AE), variance, bias, and RMSE, and these measures are:

$$AE = \frac{1}{N}\sum_{i=1}^{N}\hat{\phi}_i\,, Var = \frac{1}{N}\sum_{i=1}^{N}(\phi - \bar{\phi}_i)^2\,, Bias = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\phi}_i - \phi\right),$$

$$RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\phi}_i - \phi\right)^2}.$$

Table 3. Mean, Variance, Bias, and Root Mean Square Error for S-I

Sample	Mean $_a$	Var $_a$	Bias $_a$	RMSE $_a$
25	2.1081	0.3401	-0.0918	0.5904
50	1.9898	0.1323	-0.2102	0.4201
100	1.9358	0.0592	-0.2641	0.3591
200	1.9067	0.0273	-0.2932	0.3366
300	1.8995	0.0172	-0.3004	0.3278
400	1.8943	0.0131	-0.3056	0.3263
500	1.8914	0.0103	-0.3085	0.3248
1000	1.8860	0.0051	-0.3139	0.3220

Table 4. Mean, Variance, Bias, and Root Mean Square Error for S-I

Sample	Mean $_b$	Var $_b$	Bias $_b$	RMSE $_b$
25	1.7632	0.0371	-0.1367	0.2362
50	1.7299	0.0176	-0.1700	0.2157
100	1.7162	0.0080	-0.1837	0.2045
200	1.7072	0.0043	-0.1927	0.2037
300	1.7061	0.0028	-0.1938	0.2010
400	1.7046	0.0021	-0.1953	0.2007
500	1.7034	0.0016	-0.1965	0.2007
1000	1.7021	0.0007	-0.1978	0.1999

Table 5. Mean, Variance, Bias, and Root Mean Square Error for S-II

Sample	Mean $_a$	Var $_a$	Bias $_a$	RMSE $_a$
25	3.3402	0.9733	-0.5597	1.1343
50	3.1289	0.3695	-0.7710	0.9818
100	3.0361	0.1628	-0.8638	0.9534
200	2.9858	0.0760	-0.9142	0.9548
300	2.9736	0.0476	-0.9263	0.9516
400	2.9647	0.0362	-0.9352	0.9544
500	2.9593	0.0283	-0.9406	0.9555
1000	2.9504	0.0140	-0.9495	0.9569

Table 6. Mean, Variance, Bias, and Root Mean Square Error for S-II

Sample	Mean $_b$	Var $_b$	Bias $_b$	RMSE $_b$
25	2.7772	0.0696	-0.3227	0.4169
50	2.7327	0.0333	-0.3672	0.4101
100	2.7150	0.0153	-0.3849	0.4043
200	2.7030	0.0083	-0.3969	0.4073
300	2.7017	0.0054	-0.3982	0.4050
400	2.6995	0.0041	-0.4004	0.4055
500	2.6979	0.0031	-0.4020	0.4060
1000	2.6962	0.0015	-0.4037	0.4056

Table 7. Mean, Variance, Bias, and Root Mean Square Error for S-III

Sample	Mean $_a$	Var $_a$	Bias $_a$	RMSE $_a$
25	0.9572	0.0586	0.0572	0.2487
50	0.9120	0.0235	0.0120	0.1540
100	0.8899	0.0107	-0.0101	0.1042
200	0.8782	0.0048	-0.0217	0.0731
300	0.8751	0.0031	-0.0248	0.0610
400	0.8730	0.0024	-0.0269	0.0556
500	0.8719	0.0018	-0.0280	0.0516
1000	0.8696	0.0009	-0.0303	0.0432

Table 8. Mean, Variance, Bias, and RMSE for S-III

Sample	Mean $_b$	Var $_b$	Bias $_b$	RMSE $_b$
25	0.5037	0.0057	0.0037	0.0760
50	0.4893	0.0026	-0.0106	0.0526
100	0.4827	0.0011	-0.0172	0.0382
200	0.4789	0.0006	-0.0210	0.0325
300	0.4783	0.0004	-0.0216	0.0295
400	0.4777	0.0003	-0.0222	0.0282
500	0.4772	0.0002	-0.0227	0.0274
1000	0.4765	0.0001	-0.0234	0.0257

Table 9. Mean, Variance, Bias, and Root Mean Square Error for S-IV

Sample	Mean $_a$	Var $_a$	Bias $_a$	RMSE $_a$
25	0.4514	0.0109	-0.0485	0.1155
50	0.4365	0.0045	-0.0634	0.0928
100	0.4288	0.0021	-0.0711	0.0847
200	0.4244	0.0009	-0.0755	0.0815
300	0.4232	0.0006	-0.0767	0.0805
400	0.4225	0.0004	-0.0775	0.0803
500	0.4221	0.0003	-0.0778	0.0801
1000	0.4214	0.0001	-0.0785	0.0796

Table 10. Mean, Variance, Bias, and Root Mean Square Error for S-IV

Sample	Mean $_b$	Var $_b$	Bias $_b$	RMSE $_b$
25	1.4678	0.0478	-0.6321	0.6688
50	1.4323	0.0222	-0.6676	0.6841
100	1.4147	0.0105	-0.6852	0.6928
200	1.4031	0.0054	-0.6968	0.7007
300	1.4014	0.0035	-0.6985	0.7010
400	1.3996	0.0026	-0.7003	0.7022
500	1.3985	0.0021	-0.7015	0.7029
1000	1.3966	0.0010	-0.7033	0.7040

5. Analysis of Engineering and COVID-19 Events

In this section, we analyze three real-life data sets. These data sets are related to the engineering sector and the COVID-19 pandemic particularly outbreaks in the United Kingdom. The first data set illustrates the failure times of 50 devices put on life test at time zero discussed by [28] and explicitly, data set is: 0.1, 0.2, 1.0, 1.0, 1.0, 1.0, 1.0, 2.0, 3.0, 6.0, 7.0, 11.0, 12.0, 18.0, 18.0, 18.0, 18.0, 18.0, 21.0, 32.0, 36.0, 40.0, 45.0, 45.0, 47.0, 50.0, 55.0, 60.0, 63.0, 63.0, 67.0, 67.0, 67.0, 67.0, 72.0, 75.0, 79.0, 82.0, 82.0, 83.0, 84.0, 84.0, 84.0, 85.0, 85.0, 85.0, 85.0, 85.0, 86.0, 86.0. The second data set illustrates the lifetimes (in days) of 30 electronic devices discussed by [29] and the data set is: 0.020, 0.029, 0.034, 0.044, 0.057, 0.096, 0.106, 0.139, 0.156, 0.164, 0.167, 0.177, 0.250, 0.326, 0.406, 0.607, 0.650, 0.672, 0.676, 0.736, 0.817, 0.838, 0.910, 0.931, 0.946, 0.953, 0.961, 0.981, 0.982 , 0.990. The third data set represents mortality rate under COVID-19 pandemic outbreaks in United Kingdom (UK) from 1 December 2020 to 29 January 2021 [30]. The data set is: 0.1292, 0.3805,0.4049, 0.2564, 0.3091, 0.2413, 0.1390, 0.1127, 0.3547, 0.3126, 0.2991, 0.2428, 0.2942,0.0807, 0.1285, 0.2775, 0.3311, 0.2825, 0.2559, 0.2756, 0.1652, 0.1072, 0.3383, 0.3575, 0.2708, 0.2649, 0.0961, 0.1565, 0.1580, 0.1981, 0.4154, 0.3990, 0.2483, 0.1762, 0.1760, 0.1543, 0.3238, 0.3771, 0.4132, 0.4602, 0.3523, 0.1882, 0.1742, 0.4033, 0.4999, 0.3930, 0.3963, 0.3960, 0.2029, 0.1791, 0.4768, 0.5331, 0.3739, 0.4015, 0.3828, 0.1718, 0.1657, 0.4542, 0.4772, 0.3402.

The O*Fr*LII–PF distribution is compared with its competitors (CDFs are presented in Table 11) based on some criteria called, -Log-likelihood (-LL), Akaike information criterion (AIC), along with the goodness of fit statistics Cramer-Von Mises (CM), Anderson-Darling (AD), and Kolmogorov Smirnov (KS) with its *p*-value. Some choices of descriptive statistics are presented in Table 12. Tables 13 to 15 illustrate the estimates of the parameters, standard errors (in parenthesis), and goodness of fit statistics as well. Conventionally the minimum value of goodness of fit statistics is the criteria for a better fit model that O*Fr*LII–PF distribution eventually satisfies. Hence; we support that O*Fr*LII–PF distribution is a better fit model among all of its well-established competitors over the engineering and COVID-19 events.

Furthermore, the empirically fitted density (a) and distribution function plots (b) Probability-Probability (c) and Kaplan-Meier survival plots (d), along with the total time on test transform (e) and box plots (f), are presented in Figures 2 to 4, respectively. These plots provide sufficient information about the closest fit to subject data. All the numerical results are calculated with the assistance of statistical software R with its exclusive package AdequacyModel (https://www.r-project.org/).

Table 11. List of competitive models CDFs

Model	Model	Parameter / variable Range	Reference
PF	$P(x)=\left(\frac{x}{\mathrm{g}_0}\right)^a$	$a>0,\ 0<x<\mathrm{g}_0$	[31]
Gen-PF	$P(x)=1-(\mathrm{g}_0-x)^a(\mathrm{g}_0-m)^{-a}$	$a>0,$ $m<x<\mathrm{g}_0$	[32]
W-PF	$P(x)=1-e^{-a\left(\frac{x^b}{\mathrm{g}_0{}^b-x^b}\right)^c}$	$a,b,c>0$ $0<x<\mathrm{g}_0$	[33]
MO-PF	$P(x)=1-\frac{a\left(1-\left(\frac{x}{\mathrm{g}_0}\right)^b\right)}{\left(\frac{x}{\mathrm{g}_0}\right)^b+\alpha\left(1-\left(\frac{x}{\mathrm{g}_0}\right)^b\right)}$	$a,b>0$ $0<x<\mathrm{g}_0$	[34]
Kum-PF	$P(x)=1-\left(1-\left(\frac{x}{\mathrm{g}_0}\right)^{ab}\right)^c$	$a,b,c>0$ $0<x<\mathrm{g}_0$	[35]
Tr-PF	$P(x)=(1+a)\left(\frac{x}{\mathrm{g}_0}\right)^b-a\left(\frac{x}{\mathrm{g}_0}\right)^{2b}$	$\lvert a\rvert\le 1,b>0$ $0<x<\mathrm{g}_0$	[36]
PF-Poi	$P(x)=\frac{e^{b\left(\frac{x}{\mathrm{g}_0}\right)^a}-1}{e^b-1}$	$a,b>0$ $0<x<\mathrm{g}_0$	[37]

Table 12. Descriptive statistics

Data set	Min	Q_1	Median	Mean	Q_3	Max	Sk	Kur
50 devices	0.100	13.50	48.50	45.67	81.25	86.00	-0.14	1.410
30 devices	0.020	0.143	0.506	0.494	0.892	0.990	0.060	1.310
COVID-19	0.0807	0.176	0.288	0.288	0.385	0.533	0.047	1.961

Table 13. Parameter estimates, standard errors (in parenthesis), and goodness of fit statistics for failure times of 50 devices data

Model	$\hat{a}$	$\hat{b}$	$\hat{c}$	-LL	AIC	CM	AD	K-S (p-value)
O*Fr*LII–PF	0.3585 (0.0536)	0.2183 (0.0327)	-	200.4441	404.8882	0.0480	0.3715	0.07779 (0.9227)
MO–PF	7.6657 (5.7076)	0.2558 (0.1544)	-	212.5529	429.1057	0.1179	0.8264	0.1739 (0.0969)
PF–Poi	2.1129 (0.9889)	0.4589 (0.1468)	-	216.0639	436.1277	0.0661	0.5192	0.2091 (0.0259)
Tr–PF	-0.4479 (0.2411)	0.6009 (0.1233)	-	218.0597	440.1195	0.0522	0.4295	0.2194 (0.0162)

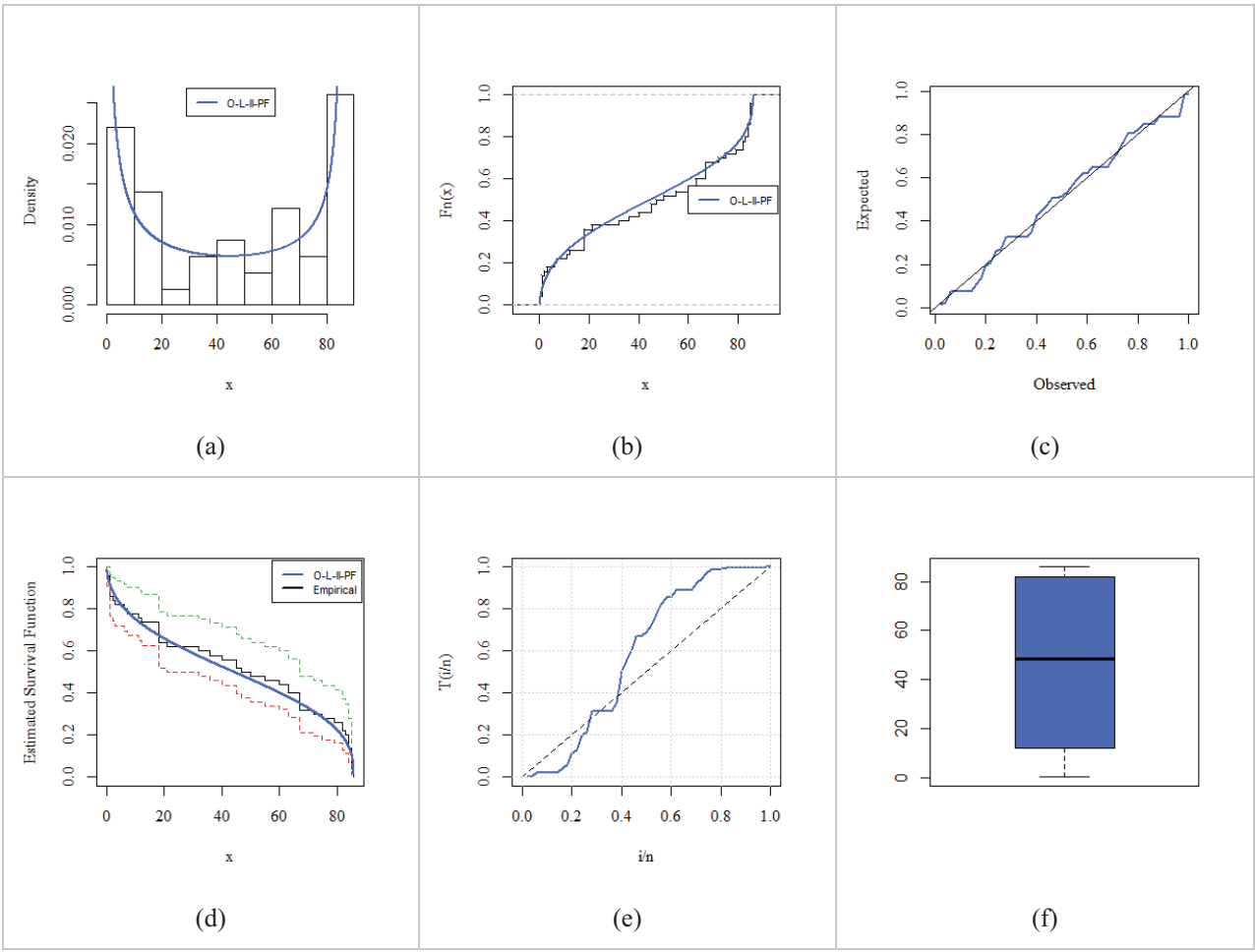


Figure 2. Fitted Plots for failure times of 50 devices data

Table 14. Parameter estimates, standard errors (parenthesis), and goodness of fit statistics for lifetimes (in days) of 30 electronic devices data

Model	$\hat{a}$	$\hat{b}$	$\hat{c}$	-LL	AIC	CM	AD	K-S (p-value)
O*Fr*LII–PF	0.4968 (0.0978)	0.3334 (0.0568)	-	-6.1325	-8.2650	0.0713	0.4338	0.1368 (0.5802)
Kum–PF	7.9804 (126.37)	0.0713 (1.1292)	0.5807 (0.1262)	-4.3578	-2.7157	0.1002	0.6206	0.1616 (0.3731)
Gen–PF	0.7525 (0.1373)	-	-	-1.9424	-1.8849	0.0783	0.4680	0.2728 (0.0183)
MO–PF	2.5015 (1.8981)	0.5305 (0.2620)	-	-1.6694	0.6611	0.1212	0.7603	0.1894 (0.2038)
PF–Poi	0.9523 (1.1370)	0.6597 (0.2397)	-	-1.2320	1.5359	0.1118	0.6983	0.1803 (0.2515)
Tr–PF	-0.2277 (0.3731)	0.7396 (0.2037)	-	-1.0026	1.9947	0.1068	0.6643	0.1891 (0.2055)
PF–I	0.8198 (0.1496)	-	-	-0.7829	0.4340	0.1021	0.6332	0.1960 (0.1741)

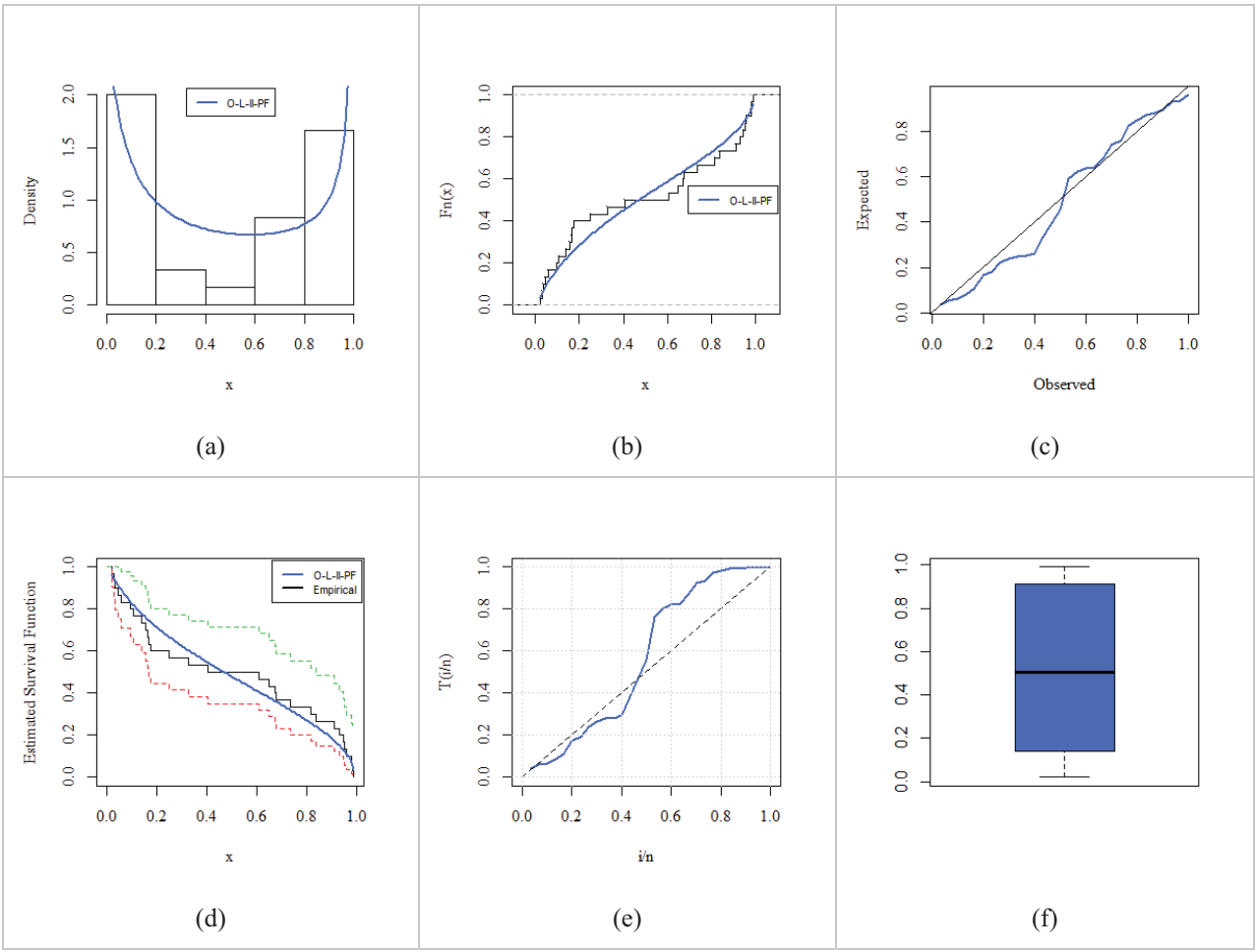


Figure 3. Fitted plots for lifetimes (in days) of 30 electronic devices data

Table 15. Parameter estimates, standard errors (parenthesis), and goodness of fit statistics for mortality rate data under COVID-19 in UK

Model	$\hat{a}$	$\hat{b}$	$\hat{c}$	-LL	AIC	CM	AD	K-S (p-value)
O*Fr*LII–PF	1.2260 (0.1850)	0.9217 (0.0766)	-	-48.1722	-92.3444	0.0743	0.4286	0.0760 (0.8524)
MO–PF	0.1860 (0.0797)	2.8671 (0.4567)	-	-47.0925	-90.1851	0.0871	0.5176	0.0901 (0.6813)
W–PF	10.9575 (14.695)	1.5807 (1.2693)	1.4814 (0.8148)	-46.2305	-86.4610	0.0830	0.5201	0.1027 (0.5175)

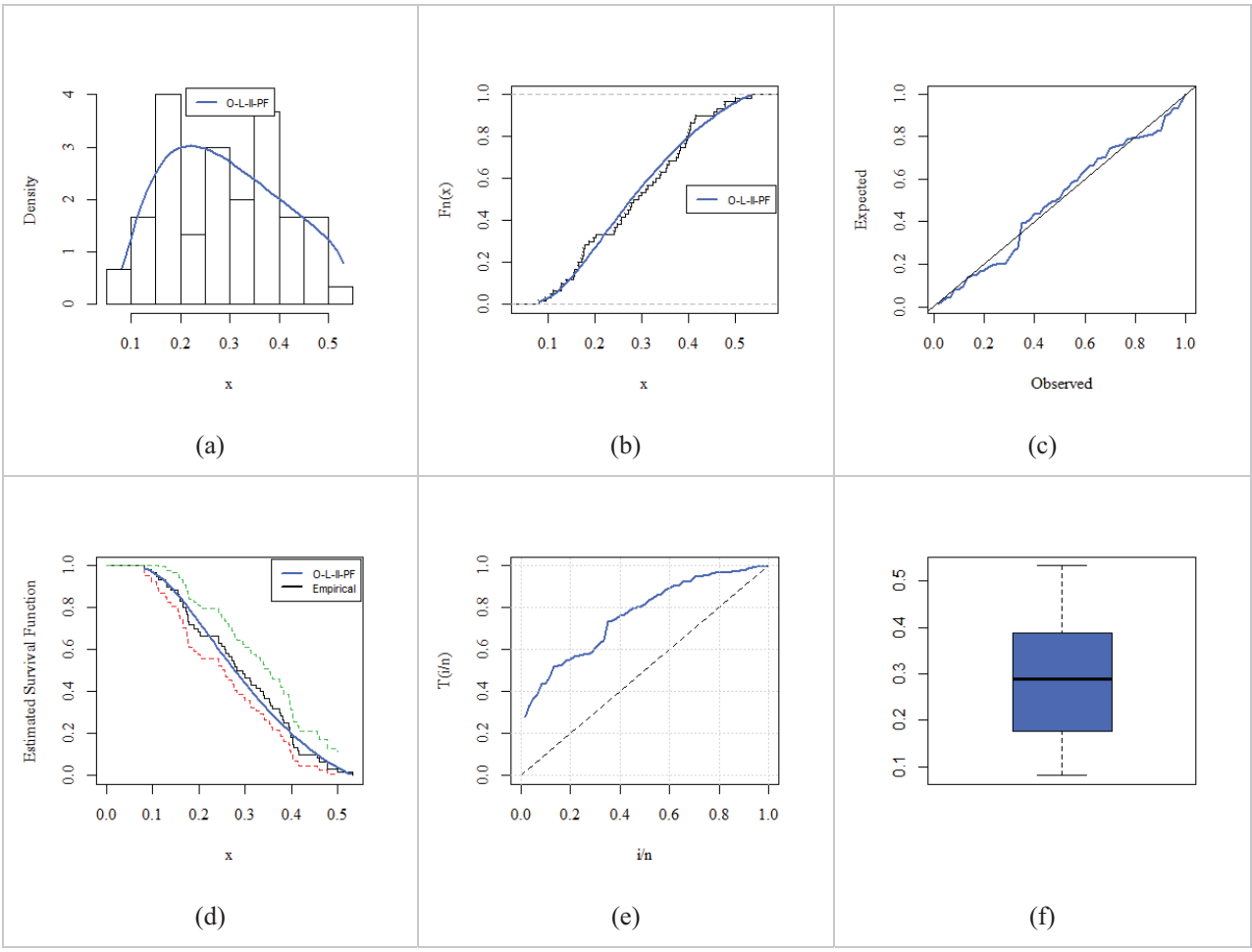


Figure 4. Fitted plots for the United Kingdom COVID-19 dat

6. Conclusion

This paper proposed a new family that generates flexible models in terms of PDF and HRF. It is referred to as odd Fréchet Lehmann type-II (OFrLII) G family of distributions. Several general characteristics of the proposed family and its sub-model (OFrLII–PF) are discussed in detail. Furthermore. OFrLII–PF distribution explored flexible shapes of PDF, including left-skewed, right-skewed, symmetric, or bathtub shaped, and HRF possessed U-shaped, increasing, or bathtub shaped. Applicability of OFrLII–PF distribution was explored over the engineering and COVID-19 pandemic events. Finally, closed-form PDF, CDF, and HRF of OFrLII–PF distribution attract researchers to opt for the model for forecasting and prediction resolution. Furthermore, it has outperformed estimates, and closest fit to datasets of interest expect to consider it as a better alternative than the PF distribution .

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Inferences About a Quantile Shift Measure of Effect Size When There Is a Covariate

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Abstract

When comparing two independent groups, a possible appeal of the quantile shift measure of effect size is that its magnitude takes into account situations where one or both distributions are skewed. Extant results indicate that a percentile bootstrap method performs reasonably well given the goal of making inferences about this measure of effect size. The goal here is to suggest a method for making inferences about this measure of effect size when there is a covariate. The method is illustrated with data dealing with the wellbeing of older adults.

Keywords: linear model, quantile regression estimator, bootstrap, robust effect size

1. Introduction

Consider two independent groups having unknown distributions. Here, the first group is viewed as a control group and the other group is an experimental group. Let δ denote some parameter that characterizes how the distributions differ. There is now a wide range of choices for δ with each providing a different perspective on how the groups compare (e.g., Huberty, 2002; Grissom & Kim, 2012; Wilcox, 2022b).

Note that the median of the experimental group corresponds to the Qth quantile of the control group. That is, Q reflects the extent the median of the experimental group is unusual relative to the control group and is generally known as a quantile shift measure of effect size. A possible appeal of this measure of effect size is that its relative magnitude takes into account whether one or both distributions are skewed. Extant results indicate that a reasonably accurate confidence interval for Q can be computed via a percentile bootstrap method (e.g., Wilcox, 2022b). However, when there is a covariate, there are no results on how to proceed. The goal here is to suggest a method for making inferences about Q, given a value for some covariate, followed by a simulation study that deals with how well the proposed method performs.

To review the motivation for Q as well as some of its properties, first consider the situation where there is no covariate. To begin, let θ_j and τ_j denote some measure of location and scale, respectively, associated with the jth group ($j = 1, 2$). Certainly the most common approach to comparing two distributions is to take the measure of location θ_j to be the population mean or median and to view the measure of scale, τ_j, as a nuisance parameter. More formally use $\delta = \theta_1 - \theta_2$ to characterize how the groups differ and test

$$H_0 : \theta_1 = \theta_2 \tag{1}$$

or compute a confidence interval for $\theta_1 - \theta_2$.

Another general approach is to use a measure of effect size that takes into account both measures of location and some measure of variation. Broadly, this approach uses

$$\delta = \frac{\theta_1 - \theta_2}{f(\tau_1, \tau_2)}, \tag{2}$$

where $f(\tau_1, \tau_2)$ is some function of τ_1 and τ_2 to be determined. Seemingly, the best-known version of (2) is where $\theta_j = \mu_j$, the population mean, $\tau_j = \sigma_j$, the population standard deviation, and by assumption $\sigma_1 = \sigma_2 = \sigma$ (homoscedasticity), in which case (2) becomes

$$\Delta = \frac{\mu_1 - \mu_2}{\sigma}. \tag{3}$$

A common practice (e.g., Cohen, 1988) is to view $\Delta = 0.2$, 0.5 and 0.8 as being small, medium and large, respectively. Presumably, what constitutes a large effect size can depend on the situation. However, for illustrative purposes, Cohen's suggestion is assumed henceforth.

There are two basic concerns with Δ. First, it assumes homoscedasticity. Kulinskaya et al. (2008) derived a heteroscedastic measure of effect size given by

$$\delta_{\text{kms}} = \frac{\mu_1 - \mu_2}{s}, \tag{4}$$

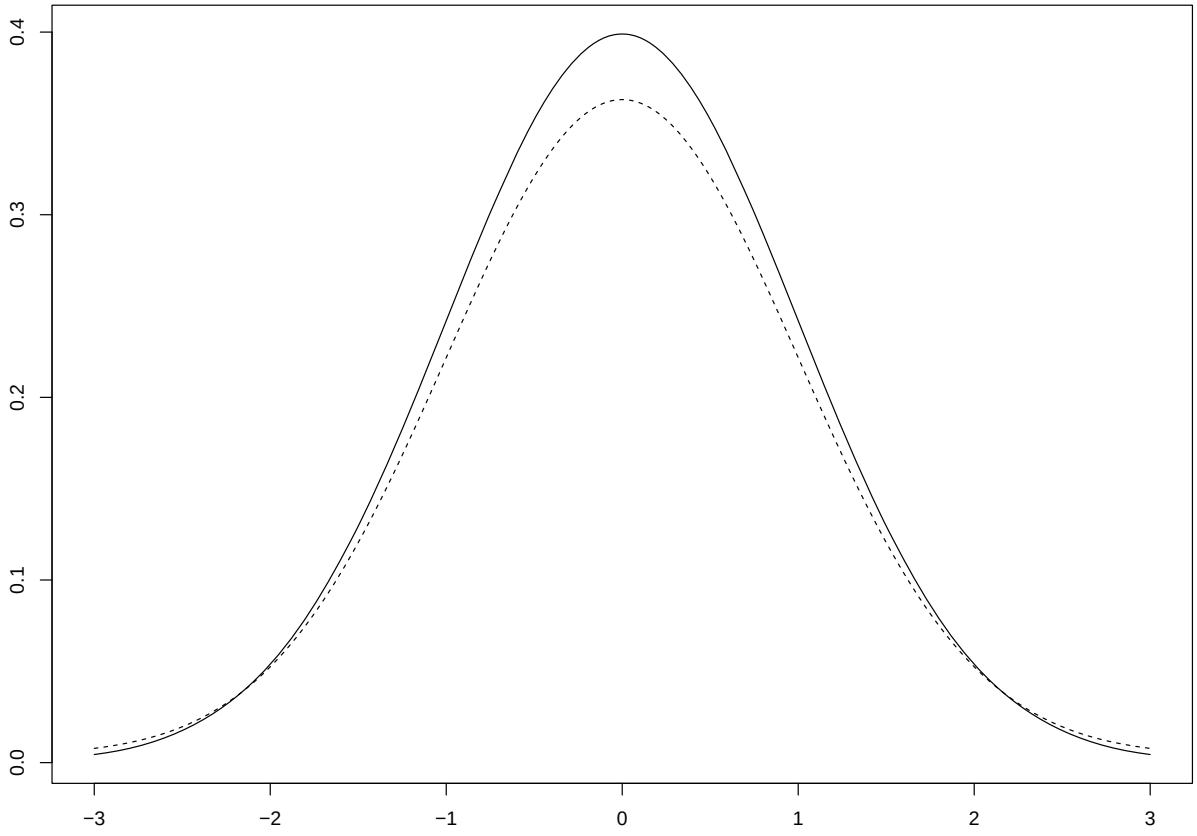

Figure 1. The solid line is a standard normal distribution, $\sigma^2 = 1$. The dashed line is a mixed normal distribution, $\sigma^2 = 10.9$

where

$$s^2 = \frac{(1-q)\sigma_1^2 + q\sigma_2^2}{q(1-q)}.$$

$q = n_1/N$, $N = n_1 + n_2$ and n_j are the sample sizes. Wilcox (2022a) reports results using this measure of effect size when dealing with an interaction in a two-way design.

The second concern is that Δ is not robust (e.g., Algina et al., 2005), roughly meaning that even a small departure from normality can alter its value substantially. To be a bit more precise, the standard deviation is not robust (e.g., Hampel et al, 1986; Huber & Ronchetti, 1990; Staudte & Shearer, 1986). It is highly sensitive to the tails of a distribution, the result being that even a slight departure from a normal distribution has the potential of lowering Δ substantially. In particular, a large effect among the bulk of the participants can appear to be small when using Δ.

Following Algina et al. (2005), this issue is illustrated with the mixed normal distribution discussed by Tukey (1960). Its cumulative distribution function (cdf) is given by

$$H(x) = 0.9\Phi(x) + 0.1\Phi(x/10), \tag{5}$$

where $\Phi(x)$ is the cdf of a standard normal distribution. Figure 1 shows a plot of the standard normal and this mixed normal distribution. As is evident, the two distributions appear to be very similar. However, while the standard normal has variance one, the variance of the mixed normal is 10.9.

Now look at Figure 2. In the left panel, are two normal distributions with variance one. The means are 0 and 0.8, so $\Delta = 0.8$, which Cohen characterizes as large. In the right panel are two mixed normals again with means 0 and 0.8.

Now $\Delta = 0.8/\sqrt{10.9} = 0.24$, which is relatively small. Algina et al. (2005) deal with this issue by replacing the mean and variance in (3) with a 20% trimmed mean and Winsorized variance, which is rescaled to estimate the variance when dealing with a normal distribution. A similar modification of δ_{kms} is straightforward. These methods help deal with heavy-tailed distributions such as the mixed normal, but there is an inherent assumption that the distributions are symmetric.

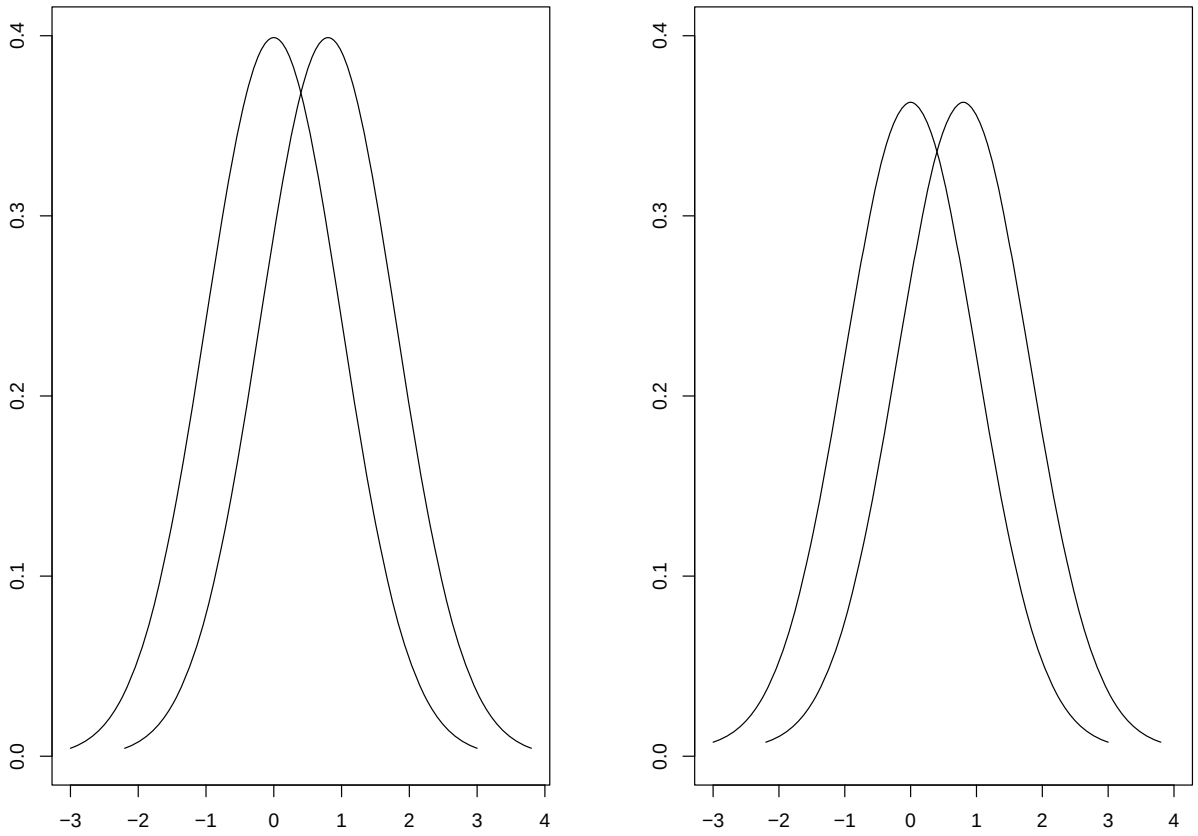

Figure 2. A slight departure from a normal distribution can substantially lower Δ, masking a large effect among the bulk of the participants. The left panel shows two normal distributions where $\Delta = 0.8$. The right panel shows to mixed normals where $\Delta = 0.24$

To underscore some concerns when dealing with skewed distributions, it helps to first note that under normality, $\Delta = 0.2$ indicates that the mean of the experimental group corresponds to the 0.42 quantile of the control group. That is, the experimental group shifts the mean of the control from the $q_1 = 0.5$ quantile to the $q_2 = 0.42$ quantile. Let $\delta_q = q_1 - q_2$, which captures the spirit of a standardized difference, Δ, without imposing any parametric family of distributions. Given that $\Delta = 0.2$ is viewed as a small effect size when dealing with normal distributions, it follows that $\delta_q = 0.08$ is considered small as well. In a similar manner, if $\Delta = 0.5$ and 0.8 are considered medium and large effect size under normality, respectively, this means that $\delta_q = 0.19$ and $\delta_q = 0.29$ are considered medium and large effect size as well.

Wilcox (2022b, section 5.3.4) describes possible concerns about skewed distributions when using Δ or some robust, heteroscedastic version of Δ. Note, for example, that in terms of magnitude, there is no distinction between $\Delta = 0.5$ and $\Delta = -0.5$. Both would be viewed as a median effect size. But consider the situation where the control group has a lognormal distribution, which has mean equal to 1.65, which is the $q_1 = 0.69$ quantile. Suppose the experimental group has a lognormal distribution that has been shifted to have mean θ_2 which is the q_2 quantile associated with the control group. Of course, when the means are equal, $\Delta = \delta_q = 0$. But consider the case where $\Delta = 0.5$. This corresponds to shifting the mean from about the 0.69 quantile to the 0.29 quantile. So $\delta_q = 0.4$, suggesting a very large effect size rather than a medium effect size as suggested by Δ. It is readily verified that the reverse can happen where δ_q suggests a small effect size in contrast to Δ. This same concern occurs for any measure of effect size that implicitly assumes that the

distributions are symmetric.

One way of dealing with this concern in a robust, non-parametric manner is to first take θ_1 and θ_2 to be the population medians of the control group and the experimental group, respectively. Let Y_j denote some random variable of interest associated with the jth group and let

$$Q = P(Y_1 \leq \theta_2). \tag{6}$$

That is, θ_2, the median of the experimental group, is the Qth quantile of the control group. Following Wilcox (2022b), Q is taken to be a measure of effect size. The further Q is from 0.5 the larger the effect. Under normality and homoscedasticity, $\Delta = 0.2$, 0.5 and 0.8 correspond to $Q = 0.58$, 0.69 and 0.79, respectively.

Now consider the situation where there is a covariate X and let $Q(x)$ denote the value of Q given that $X = x$. Section 2 of this paper suggests a method for estimating $Q(x)$. Included is a proposed method for testing

$$H_0 : Q(x) = 0.5, \tag{7}$$

no effect, as well as a method for computing a $1 - \alpha$ confidence interval for $Q(x)$. Section 3 reports the results of a simulation study. Finally, the method is illustrated with data dealing with the physical and emotional wellbeing of older adults.

It is noted that testing (7) is open to the criticism that surely $Q(x)$ differs from 0.5 at some decimal place (Tukey, 1991). Assuming this view is reasonable, the goal is not to test (7), but rather determine the extent it is reasonable to make a decision about whether $Q(x)$ is less than or greater than 0.5 (Jones & Tukey, 2000). From this point of view, a p-value quantifies the strength of the empirical evidence that a decision can be made. But of course a p-value does not indicate the probability of a correct decision.

2. The Proposed Method

Let η_{jqx} denote the qth quantile of Y_j given that $X_j = x$. Here it is assumed that

$$\eta_{jqx} = \beta_{0jq} + \beta_{1jq}x. \tag{8}$$

The unknown slope, β_{1jq} and intercept, β_{0jq}, can be estimated via the well-known Koenker and Bassett (1978) quantile regression estimator yielding say b_{1jq} and b_{0jg}, respectively. Assuming (8) is true provides a straightforward method for estimating $Q(x)$. Let $\hat{\theta}_2 = b_{1,2,0.5}x + b_{0,2,0.5}$ denote the estimate of the conditional median of the experimental group given that $X = x$. As is evident, $\hat{\theta}_2$ corresponds to some quantile of the conditional distribution associated with the control group, given that $X = x$, which is $Q(x)$. An estimate of $Q(x)$, $\hat{Q}(x)$, is the value of q such that

$$b_{1,1,q}x + b_{0,1,q} = \hat{\theta}_2. \tag{9}$$

Here, (9) is solved with the Nelder and Mead (1965) algorithm.

Now consider the goal of testing (7) as well as computing a confidence interval for $Q(x)$. Here, a percentile bootstrap method is used. For theoretical results that motivate the use of this method, see Liu and Singh (1997). Consideration of this approach stems from past studies indicating that it frequently performs well when dealing with robust estimators (Wilcox, 2022b). Briefly, let (X_{ij}, Y_{ij}), $(i = 1, \ldots n_j;\ j = 1,2)$ denote a random sample of size n_j from the jth group. Generate a bootstrap sample from each group by sampling with replacement n_j pairs of values from group j. Based on these bootstrap values, compute the estimate of $Q(x)$ yielding $\hat{Q}^*(x)$. Repeat this process B times and label the results $\hat{Q}^*_b(x)$ $(b = 1, \ldots, B)$.

Let

$$P^* = \sum I(\hat{Q}^*_b(x) < 0.5), \tag{10}$$

where the indicator function $I(\hat{Q}^*_b(x) < 0.5) = 1$ if $\hat{Q}^*_b(x) < 0.5$, otherwise $I(\hat{Q}^*_b(x) < 0.5) = 0$. Then a (generalized) p-value for testing (7) is $2\min(P^*, 1 - P^*)$. To compute a $1 - \alpha$ confidence interval, first put the bootstrap estimates in ascending order and label the results $\hat{Q}^*_{(1)}(x) \leq \cdots \leq \hat{Q}^*_{(B)}(x)$. Let $\ell = \alpha B/2$ and $u = B = \ell$. Then a $1 - \alpha$ confidence interval for $Q(x)$ is

$$(\hat{Q}^*_{(\ell+1)}(x), \hat{Q}^*_{(u)}(x)). \tag{11}$$

This is called method Q henceforth. The choice for B is discussed in the next section of this paper.

3. Simulation Results

Simulations were used to get some sense of how well the percentile bootstrap performs when making inferences about $Q(x)$. First, some comments about choosing B are required. Racine and MacKinnon (2007) discuss this issue at length and proposed a method for choosing the number of bootstrap samples. Davidson and MacKinnon (2000) proposed a pretest procedure for choosing B. Typically $B \geq 500$ is used. However, a practical problem was that execution time using the Nelder-Mead method to solve (9) was much higher than expected. Even with $B = 100$ and $n_1 = n_2 = 20$, execution time was over 18 seconds on a MacBook Pro using a 2.9 GHz processor. The problem is that running a simulation with 1000 replications and $B = 500$ would require over 52 hours. Switching to alternative minimization functions in the R package optim did not improves matters. Here, the execution time was reduced by taking advantage of a quad core processor via the R package parallel. Now with $B = 200$, execution time for a single replication was a little over 26 seconds. That is, for 1000 replications, the execution time is a little over seven hours. Consequently, $B = 200$ was used in the simulations with 1000 replications.

Data were generated from four distributions: normal, symmetric and heavy tailed, skewed and relatively light-tailed, and skewed with heavy tails. Roughly, heavy-tailed distributions are characterized by outliers. More precisely, data were generated from four g-and-h distributions. Let Z denote a random variable having a standard normal distribution. Then

$$V = \begin{cases} \frac{exp(gZ)-1}{g} exp(hZ^2/2), & \text{if } g > 0 \\ Zexp(hZ^2/2), & \text{if } g = 0 \end{cases} \tag{12}$$

has a g-and-h distribution (Hoaglin, 1985), where g and h are parameters that determine the first four moments. The four distributions considered here are the standard normal distribution ($g = h = 0$), a symmetric heavy-tailed distribution ($g = 0$, h= 0.2), an asymmetric distribution with relatively light tails ($g = 1$, $h = 0$), and an asymmetric distribution with heavy tails ($g = h = 0.2$). The g-and-h distribution with $g = 1$ and $h = 0$ corresponds to a lognormal distribution that has been shifted to have a median of zero. Figure 3 shows plots of the four distributions used here. A review of five papers aimed at characterizing the extent distributions are non-normal (Wilcox, 2022b, section 4.2) suggests that the g-and-h distributions used here span what typically encountered in practice.

Inferences about $Q(x)$ were made based on two choices for x. Let $U_j = \hat{x}_{j,0.8}$ denote an estimate of the 0.8 quantile associated with the jth group. And let $L_j = \hat{x}_{j,0.2}$. Let $L = \max(L_1, L_2)$ and $U = \min(U_1, U_2)$. The first choice for x was $(L + U)/2$ and the second choice was U.

Estimates of the actual Type I error probability are reported in Table 1. Bradley (1978) suggests that as a general guide, when testing at the 0.05 level, the actual level should be between 0.025 and 0.075. As can be seen, the highest estimate is 0.053. Bradley's criterion is satisfied for the point $(L+U)/2$ with one exception, which occurred when $(n_1, n_2) = (20, 20)$, $g = 1$ and $h = 0.2$. The estimate is 0.020. For U there are situations where the estimate drops below 0.025 when one or both sample sizes are less than or equal to 50. The lowest estimate is 0.017. For $(n_1, n_2) = (100, 100)$, the estimated Type I error probability satisfies Bradley's criterion in all of the situations considered.

There is the issue of how the power of the proposed method compares to situations where the covariate is ignored or not available. Power can be higher or lower depending on the nature of the association. Consider, for example, $n = 50$, $g = h = 0$ and suppose the groups are compared for the covariate value corresponding U. If there is no association, $\beta_{01} = 0.5$ and $\beta_{02} = 0$, the power of the proposed method is 0.31. But if $H_0 : Q = 0.5$ is tested ignoring the covariate, power is 0.66. However, if $\beta_{01} = \beta_{02} = 0$, $\beta_{11} = 1$ and $\beta_{12} = 0$, the proposed method has power 0.568. In contrast, ignoring the covariate, the power is only 0.050 because in effect the hypothesis $H_0 : Q = 0.5$ is true.

4. Illustration

The proposed method is illustrated with data from the Well Elderly 2 study (Clark et al., 2011). Generally, this study dealt with an intervention program aimed at improving the physical and emotional well being of older adults. The focus here is on a measure of meaningful activities (MAPA). For each participant, cortisol was measured upon awakening and again 30-45 minutes later. The change in cortisol, generally known as the cortisol awakening response (CAR) has been found to be associated with measures of stress (e.g., Clow et al., 2004; Chida & Steptoe, 2009). Consequently, the goal is to compare MAPA measures with CAR taken as the covariate. The sample sizes are 232 for the control group and 141 for the intervention group.

Figure 4 shows a plot of the data and the 0.5 quantile regression lines. For the control group, the data points are indicated by a + and the solid line is the regression line. Table 2 summarizes the results for CAR=-0.2, -0.1 and 0.1. As can be seen, the first two p-values are less than or equal to 0.02. At CAR=-0.2, the estimate of Q is 0.711, which is moderately large.

To provide perspective, the groups were compared again based on the conditional median of the MAPA scores given a

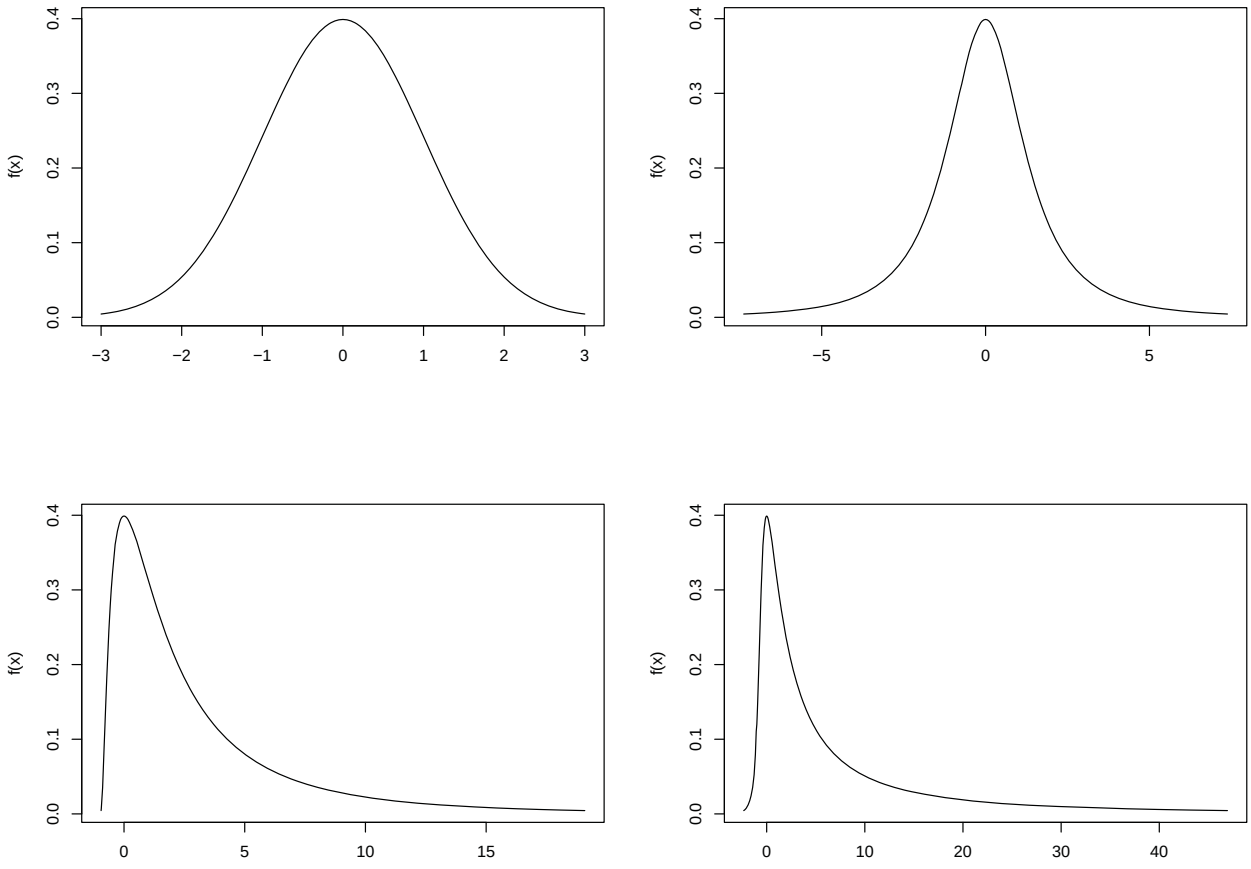

Figure 3. Distributions used in the simulations. Upper left, a standard normal distribution; upper right $g = 0, h = 0.2$; lower left, $g = 1, h = 0$; lower right, $g = 1, h = 0.2$

Table 1. Estimated Type I error probabilities, $\alpha = 0.05$

(n_1, n_2)	g	h	$(L + U)/2$	U
(20, 20)	0.0	0.0	0.030	0.019
	0.0	0.2	0.035	0.026
	1.0	0.0	0.030	0.021
	1.0	0.2	0.020	0.021
(20, 50)	0.0	0.0	0.028	0.031
	0.0	0.2	0.026	0.023
	1.0	0.0	0.029	0.026
	1.0	0.2	0.029	0.021
(50, 50)	0.0	0.0	0.031	0.034
	0.0	0.2.	0.035	0.017
	1.0	0. 0	0.038	0.021
	1.0	0.2	0.040	0.020
(100, 100)	0.0	0.0	0.049	0.034
	0.0	0.2	0.035	0.037
	1.0	0.0	0.039	0.030
	1.0	0.2	0.058	0.036

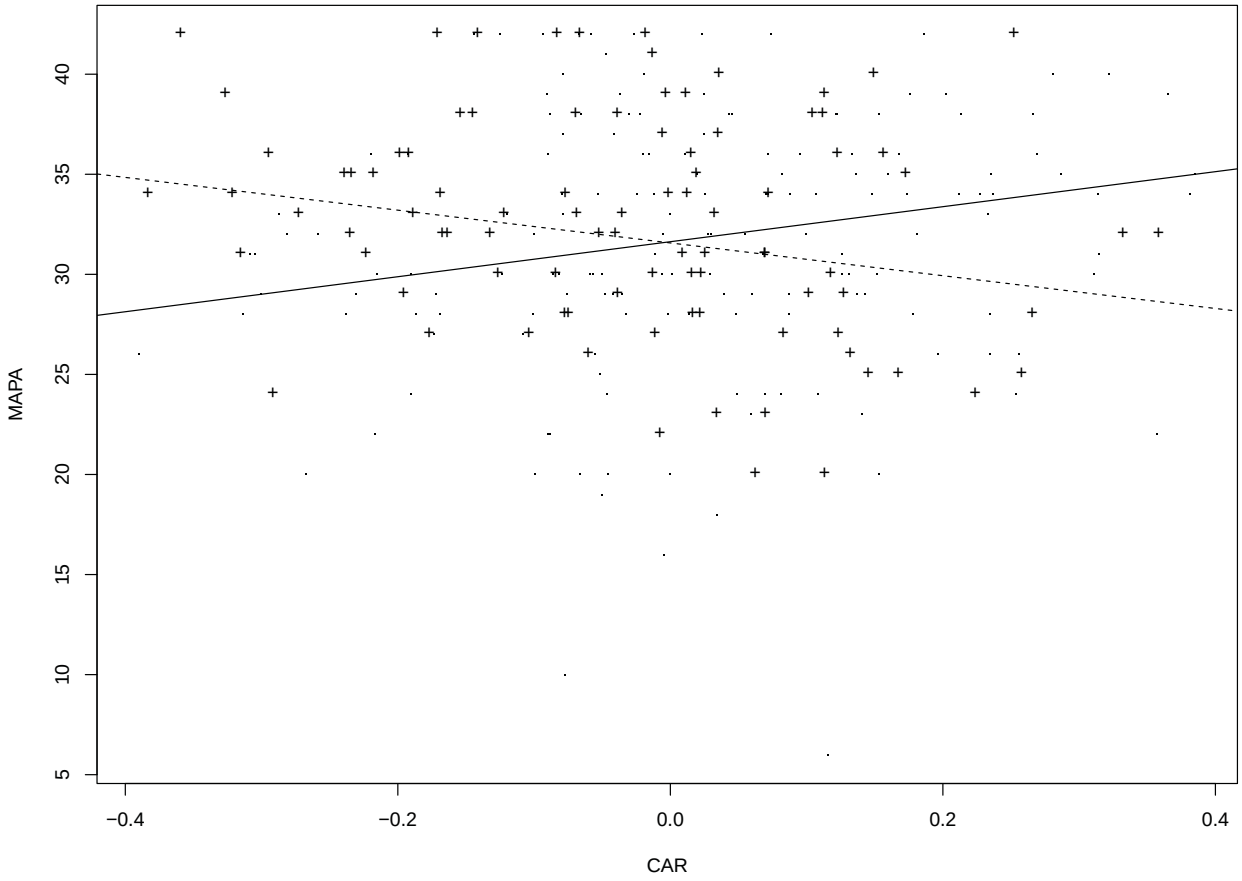

Figure 4. Solid line is the regression line for the control group

Table 2. Results for the Well Elderly data using method Q

CAR	p.value	Conf.Inter	$\hat{Q}$
-0.2	0.00	(0.588, 0.814)	0.711
-0.1	0.02	(0.509, 0.726)	0.586
0.1	0.36	(0.211, 0.632)	0.374

Table 3. Results when comparing the conditional medians

CAR	Est.1	Est.2	DIF	Conf.Inter	p.value
−0.2	30.8	33.5	−2.7	(−5.76, −0.28)	0.030
−0.1	31.3	32.6	−1.3	(−3.90, 1.21)	0.208
0.1	32.3	30.9	1.4	(−1.63, 4.51)	0.261

value for CAR. This was done via the method in Wilcox (2022b, section 12.1, method S1). The R function ancJN in the R package WRS was used. The results are reported in Table 3. As can be seen, the p-values differ substantially from those reported in Table 2, especially for CAR=−0.1 and 0.1, illustrating that the choice of method can make a practical difference. Of course, this is not surprising because the two methods used here are sensitive to different features of the data.

5. Concluding Remarks

An alternative to Q that reflects the approach given by (2) can be outlined as follows. Let $\theta_j(x)$ denote the conditional median of Y_j given that $X = x$. Let $\tau_j(x)$ denote the interquartile range of Y_j given that $X = x$, rescaled to estimate the standard deviation when the conditional distribution of the Y_j is normal. Using $\tau_j(x)$ as a robust measure of scale is convenient because it is readily estimated by the Koenker–Bassett regression estimator. Then an analog of (4) is readily derived, which is labeled ξ. However, when dealing with skewed distributions, this approach might be deemed unsatisfactory for reasons previously described.

A possible appeal of ξ is that it provides a measure of effect size without having to specify one of the groups as a control group. But perhaps this is not a serious concern when using Q. Imagine, for example, males and females are compared. One could use females as the control group, estimate Q, and then use males as the control group, which in general would yield a different estimate of Q.

It is not being suggested that Q should be used to the exclusion of other measures of effect size. The suggestion is that multiple perspective can be useful and that Q supplements other measures that might be deemed reasonable. A possible appeal of Q is that it provides a flexible way of characterizing the extent an experimental group improves upon a control group regardless of the shape of the distribution of the control group.

Finally, the R function anclin.QS.CIpb performs method Q. It is contained in the file Rallfun-v39, which can be downloaded from https://osf.io/dashboard. Simply source the file to gain access to anclin.QS.CIpb. By default, the covariate values are taken to be L, $(L+U)/2$ and U. The covariate values can be specified via the argument pts. Setting the argument MC=TRUE, the function will take advantage of a multicore processor if one is available provided the R package parallel has been installed. It is noted that the Nelder-Mead method was applied via the R function nelderv2, which is in the R package WRS as well as the Rallfun-v39 file. When using the R function optim instead, situations were found where for x sufficiently large, nonsensical estimates of Q were obtained in some instances. The reason for this is unknown.

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Reliability of a Meta-analysis of Air Quality−Asthma Cohort Studies

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Abstract

What may be a contributing cause of the replication problem in science – multiple testing bias – was examined in this study. Independent analysis was performed on a meta-analysis of cohort studies associating ambient exposure to nitrogen dioxide (NO2) and fine particulate matter (PM2.5) with development of asthma. Statistical tests used in 19 base papers from the meta-analysis were counted. Test statistics and confidence intervals from the base papers used for meta-analysis were converted to p-values. A combined p-value plot for NO2 and PM2.5 was constructed to evaluate the effect heterogeneity of the p-values. Large numbers of statistical tests were estimated in the 19 base papers – median 13,824 (interquartile range 1,536−221,184). Given these numbers, there is little assurance that test statistics used from the base papers for meta-analysis are unbiased. The p-value plot of test statistics showed a two-component mixture. The shape of the p-value plot for NO2 suggests the use of questionable research practices related to small p-values in some of the cohort studies. All p-values for PM2.5 fall on a 45-degree line in the p-value plot indicating randomness. The claim that ambient exposure to NO2 and PM2.5 is associated with development of asthma is not supported by our analysis.

Keywords: cohort studies, air quality, asthma, meta-analysis, multiple testing bias

1. Introduction

1.1 Irreproducible Science

Scholarly publishing in the science, technology and biomedicine fields produced about 2.5 million articles in over 28,000 peer-reviewed journals in 2015 (Ware & Mabe, 2015). Further, Ware & Mabe (2015) indicated continued growth in volumes of these articles at a rate of 3–3.5% per year. Yet research claims in observational studies, randomized trials and, in general, studies across multiple scientific disciplines often do not replicate (Chambers, 2015; Hubbard, 2015; Atmanspacher & Maasen, 2016; NASEM, 2016 & 2019; Harris, 2017; Randall & Welser, 2018; Ritchie, 2020).

The majority of irreproducible studies report positive associations between causative factors (e.g., a behavior or risk factor) and an outcome. Negative (null) studies – those with findings of no associations – are often not reported by researchers (Franco et al., 2014). If negative studies are submitted for publication, editors may reject them out of hand, so a false positive (irreproducible) study can mistakenly be presumed as established fact (Franco et al., 2014; Nissen et al.; 2016).

Published estimates of irreproducible studies or reports range from 51–100% in the biomedical field (Young et al., 2022):

- 41 of 80 studies (51%) examined in the primary care and general medicine field (Glasziou et al., 2008).
- 131 of 257 studies (51%) examined in the clinical psychology, cognitive psychology, cognitive neuroscience, developmental psychology, social psychology, school psychology and various inter-subdisciplinary fields (Hartshorne & Schachner, 2012).
- 129 of 238 studies (54%) examined in the fields of neuroscience, developmental biology, immunology, cell and molecular biology, general biology (Vasilevsky et al., 2013).
- 25 of 45 (56%) clinical studies published in high-impact-factor specialty medical journals in 1990-2003 (Ioannidis,

2005).

- 52 of 63 studies (78%) examined mostly from the oncology field; but several studies were from the fields of women's health and cardiovascular health (Prinz et al., 2011).
- 47 of 53 studies (89%) examined in the fields of haematology, oncology (Begley & Ellis, 2012).
- 52 of 52 studies (100%) examined in the field of nutrition (Young & Karr, 2011).

1.2 Cohort Studies

The beginnings of the cohort studies can be traced to the interest on life behavior and health status information; information that is important to public health (Samet & Munoz, 1988). In a cohort study researchers identify subjects at a point in time when they do not have an outcome of interest (e.g., a disease) and then later to compare the incidence of the disease among groups of exposed and unexposed subjects (Grimes & Schulz, 2002). Periodic follow-up with the subjects can be frequent to record their behaviors and changes in health status (Song & Chung, 2010).

A cohort study can take on a life of its own. The cohort Life Project in England, which examined children born within a narrow period in the 1950s, has become a decades-long study that has provided data for many published articles in a range of social science and health disciplines (Pearson, 2016). Researchers have published 2,500 papers on the 1958 cohort.

1.3 Meta-analysis

A meta-analysis is intended to offer a window into the reliability of a research finding. A meta-analysis examines a research finding by using test statistics from multiple individual studies found in literature (Glass et al., 1981). Two key assumptions of meta-analysis are that the test statistics drawn into the analysis are an unbiased estimate of the effect of interest (Boos & Stefanski, 2013) and that meta-analysis of multiple studies offers a pooled estimate with improved precision (Cleophas & Zwinderman, 2015).

Meta-analyses based on limited evidence, biased studies and/or poor-quality trials are prone to unreliable results (Pereira & Ioannidis, 2011; Packer, 2017). As researchers can often ask a lot of questions and compute many models in an observational study, any statistics coming from such a study may not be unbiased (Young & Kindzierski, 2019). Observational studies that have many hundreds of possible questions at issue may yield extreme findings due to chance. Also, modeling is typically used to reduce variability and aggressive modeling may lead to an underestimate of variability (Schisterman et al., 2009).

Any kind of variability across studies in a meta-analysis may be termed 'heterogeneity' (Higgens & Green, 2011). Heterogeneity occurs because the effects of interest in the subjects studied may not be the same. This can be examined by looking at the 'across study' variability versus the 'within study' variability (Cochran, 1952 & 1954). Very often there is more heterogeneity in meta-analysis than one would expect by chance. One way to deal with heterogeneity is to assume that summary statistics come from a consistent (normal) distribution with extra variability (DerSimonian & Laird, 1986); in which case the meta-analysis process can give a combined (weighted) estimate of an effect. However, the heterogeneity may be more complex and the assumption of selecting values from a normal distribution with extra variability may not be valid for meta-analysis.

1.4 Objective of Study

An independent examination of a meta-analysis drawing statistics from cohort studies was undertaken. The meta-analysis was published by Anderson et al. (2013a,b) and it explored whether associations with ambient air quality early in life lead to development of asthma later in life. As of December 11, 2021, this meta-analysis had 238 *Google Scholar* citations and 135 *Web of Science* citations.

An often-reported cause of the irreproducibility problem is related to researcher statistical methods (Colling & Szucs, 2018). In relation to this, we examined whether asking a lot of questions and computing many models can bias meta-analysis of cohort studies. Asking many questions and computing many models has been referred to as multiple testing and multiple modeling (MTMM) or more generally as multiple testing bias (Westfall & Young, 1993; Young & Karr, 2011). We used analysis search space and p-value plots to independently examine two aspects of the Anderson et al. (2013a,b) meta-analysis:

- Whether research findings in the base papers used for meta-analysis are susceptible to the multiple testing bias.
- Whether heterogeneity in test statistics used for meta-analysis is more complex than simple sampling from a single normal process.

2. Method

Pekkanen & Pearce (1999) note that there are two classes of causes of asthma – primary (related to the increase in risk

of developing the disorder) and secondary (related to asthma attacks or exacerbations). The Anderson et al. (2013a,b) meta-analysis focused on cohort studies of the association between ambient air quality components and development of asthma later in life, and hence on the primary causes of asthma.

A public standard operating procedure (SOP) of the test methods used here was initially filed with the Center for Open Science 'open science framework' (Young, 2019). Anderson et al. conducted a systematic review and meta-analysis of cohort studies of the association between two air quality components – particulate matter with aerodynamic equivalent diameter ⩽2.5 micron (PM2.5) and nitrogen dioxide (NO2) – and incidence of asthma. Incidence was defined as: i) incidence of diagnosed asthma or of new wheeze symptom between two assessments or, ii) in birth cohorts followed up to 10 years of age, a lifetime prevalence estimate of asthma or wheeze symptom.

To increase the number of test statistics for each air quality parameter (PM2.5 & NO2)–outcome pair, Anderson et al. scaled results for studies of particulate matter with aerodynamic diameter <10 μm (PM10) to PM2.5 using a factor of 0.65 and of oxides of nitrogen (NOx) to nitrogen dioxide (NO2) using a factor of 0.44. They indicated that most cohort studies they used for meta-analysis reported test statistics as odds ratios (ORs), but some reported them as relative risks (RRs) or hazard ratios (HRs).

Anderson et al. also indicated that all three quantitative health outcome estimates were combined for their meta-analysis because the outcome of interest (asthma) is quite common, but the effect size is relatively small. A small effect size can be interpreted as a weak relationship between two variables. Their outcomes are referred to here as effect estimates (EEs) with 95% confidence intervals (CIs). These EEs were standardized by Anderson et al. to a 10 $\mu g/m^3$ increment for PM2.5 and NO2.

Anderson et al. identified 17 cohorts in their review. This included eight birth and nine child/adult cohorts of relationships between air quality and incidence of asthma or wheeze symptom with a total of 99 EEs from 24 published studies. Most cohort studies were based on inferred 'within community exposure' contrasts dominated by traffic pollution. Twelve of the 17 cohorts reported at least one positive statistically significant association ($p<.05$) between an air quality component and a measure of asthma incidence. Of the 99 EEs identified, 29 were positive associations and statistically significant (i.e., $p<.05$) and the remaining 70 were null associations.

Thirteen of their cohorts reported results for oxides of nitrogen (NOx), mostly as nitrogen dioxide (NO2), and were used for their meta-analysis of NO2. Of these 13 cohorts, two had multiple publications. Anderson et al. did not state which of the publications they drew upon for their EEs and CIs of the two cohort populations. Also, five cohorts were used for their meta-analysis of PM2.5. Of the five cohorts used, four had multiple publications. Again, Anderson et al. did not state which of the publications they drew upon for their EEs and CIs of the four cohort populations.

It is important to note that epidemiologic studies with null findings more likely remain unpublished compared to studies with positive findings (Chavalarias et al., 2016). Egger et al. (2001) and Sterne et al. (2001) note that this creates a distortion of the literature. This represents a potential problem because a meta-analysis drawing upon test statistics from the literature may only be using misleading, positive findings (Ioannidis, 2008; NASEM, 2019).

Anderson et al. reported the following combined results of their meta-analysis: (i) for the 13 cohort studies with NO2 estimates, the EE was 1.15 (95% CI 1.06 to 1.26) per 10 $\mu g/m^3$, and (ii) for the five cohort studies with estimates for PM2.5, the EE was 1.16 (95% CI 0.98 to 1.37) per 10 $\mu g/m^3$. Finally, Anderson et al. stated in their Abstract… "*The results are consistent with an effect of outdoor air pollution on asthma incidence*."

2.1 Analysis Search Space

Search space counting is introduced as a test of whether studies used in the meta-analysis are susceptible to multiple testing bias. We refer to the cohort studies used for meta-analysis as 'base papers'. Analysis search space (search space counts) represents an estimate of the number of statistical tests performed in a base paper.

Why might this be relevant? There is flexibility available to researchers to undertake a range of statistical tests and use different statistical models in an observational study before selecting, using and reporting only a portion of the test and model results (Young & Kindzierski, 2019). Wicherts et al. (2016) refers to this flexibility as 'researcher degrees of freedom' in the psychological sciences. Base papers with large search space counts suggest the use of a large number of statistical tests and statistical models and the potential for researchers to search through and only report a portion of their results (i.e., positive, statistically significant results).

Analysis search space was estimated for 19 of the Anderson et al. 24 base papers (80%). A listing of the 19 base papers is provided in Appendix A. Electronic copies of these base papers and any corresponding electronic supplementary information files were obtained and read. The number of outcomes, predictors, time lags and covariates reported in each base paper was separately counted as follows (Young & Kindzierski, 2019; Kindzierski et al., 2021):

• The product of outcomes, predictors, and time lags = number of questions at issue (i.e., Questions = outcomes × predictors × lags).

• A covariate may or may not be a confounder to a predictor variable. The only way to test for this is to include/exclude the covariate from a model. As it can be included or excluded, one way to approximate the 'modeling options' is to raise 2 to the power of the number of covariates (i.e., Models = 2^k, where k = number of covariates). Identifying covariates in a published article can be difficult as they might be stated anywhere in the article.

• Questions × Models = an approximation of analysis search space (Search Space).

Three examples of how to estimate analysis search space in observational cohort studies are provided in Appendix B. Estimates of analysis search space are considered to be lower bound approximations (Young & Kindzierski, 2019). What is presented here is based on information that is reported in each base paper evaluated. Finally, we specifically reviewed the 19 base papers focusing our attention on identifying whether a paper: i) discussed/mentioned the multiple testing bias issue in various forms (multiple testing, multiple comparisons, multiplicity) and/or, ii) made any mention of correcting for this issue.

2.2 p-value Plot

Epidemiologic research results have long been required to be statistically significant (NASEM, 1991). Further, environmental epidemiology traditionally uses confidence intervals instead of p-values from a hypothesis test to show statistical significance. As confidence intervals and p-values are derived from the same data set, they are interchangeable, and one can be estimated from the other (Altman & Bland, 2011a,b).

A positive association between two variables in an environmental epidemiology study can be considered statistically significant where the confidence interval for a test statistic excludes the null hypothesis or the p-value is less than .05. The p-value is a random variable derived from a distribution of the test statistic used to analyze data and to test a null hypothesis (Kindzierski et al., 2021). The p-value can be defined as the probability, if nothing is going on, of obtaining a result equal to or more extreme than what was observed.

Hung et al. (1997) indicate that under the null hypothesis, the p-value is distributed uniformly over the interval 0 to 1 regardless of sample size. A distribution of true null hypothesis points in a p-value plot should form a straight line (Schweder & Spjøtvoll, 1982). A plot of rank-ordered p-values related to true null hypothesis points should conform to a near 45-degree line (Westfall & Young, 1993). The plot can be used to assess the validity of a false finding being taken as true and can be used to test the reliability of the findings made in base papers used for meta-analysis.

A p-value plot was constructed using 18 Anderson et al. (2013a,b) p-value estimates – 13 for NO2 and five for PM2.5 – and interpreted after Schweder & Spjøtvoll (1982) and Young & Kindzierski (2019):

• The p-values were computed from the EEs and CIs assuming symmetrical CIs using JMP statistical software (SAS Institute, Cary, NC).

• The p-values were ordered from smallest to largest and plotted against the integers, 1, 2, 3, …

• If p-value results are random (i.e., a true null relationship), the p-value plot should roughly follow a 45-degree line indicating a uniform distribution.

• Alternatively, p-values should be on a roughly straight line with a slope considerably less than 45 degrees if a true relationship exists.

• If analysis search space counts are high and the corresponding plotted p-values exhibit a two-component – bilinear shape – then the p-values used for meta-analysis comprise a mixture and a general (over-all) finding is not supported. In addition, the p-value reported for the combined statistic of the meta-analysis is not valid. This is elaborated further in the study.

To assist in interpretation of the visual behavior of p-value plots, plots for 'plausible true null' and 'plausible true alternative' hypothesis outcomes based on meta-analysis of observational datasets were constructed (Appendix C). Hung et al. (1997) note the distribution of the p-value under the alternative hypothesis – where p-values are a measure of evidence against the null hypothesis – is a function of both sample size and the true value or range of true values of the tested parameter. The p-value plots presented in Appendix C represent examples of distinct (single) sample distributions for each condition – i.e., for true null associations and true effects between two variables. Evidence for p-value plots exhibiting behaviors outside of that shown in Appendix C should be treated as questionable particularly where analysis search space counts are high.

3. Results

3.1 Analysis Search Space

Estimated analysis search spaces for 19 base papers we examined from Anderson et al. (2013a,b) are presented in Table 1. These 19 papers represented 14 of the 17 cohort studies used by Anderson et al. for their meta-analysis. From Table 1, investigating multiple – 2 or more – asthma outcomes (i.e., Outcomes) in the cohort studies were as common as single outcome investigations. In addition, use of multiple Predictors and Lags was common. So was adjusting for multiple possible Covariate confounders. Examining multiple factors (i.e., outcomes, predictors, lags and covariates) seemingly represents a reasonable attempt to simulate/model possible exposure−disease combinations, however these multiple combinations can inflate the overall number of statistical tests performed in a single study.

Table 1. Counts and analysis search spaces for 19 base papers considered by Anderson et al. (2013a,b) in their meta-analysis

RowID	Study cohort	Outcomes	Predictors	Lags	Covariates	Questions	Models	Search space
1	BAMSE	7	3	4	6	84	64	5,376
2	British Columbia	1	8	4	7	32	128	4,096
3	CHS	1	2	8	15	16	32,768	524,288
4	CHS	1	6	5	10	30	1,024	30,720
5	CHS 2003	1	5	3	15	15	32,768	491,520
6	CHIBA	3	1	3	6	9	64	576
7	CHIBA	1	3	6	6	18	64	1,152
8	CHIBA	5	4	4	8	80	256	20,480
9	ECHRS	1	1	6	11	6	2,048	12,288
10	GINIplus+LISAplus	4	4	6	12	96	4,096	393,216
11	MISSEB	1	2	7	6	14	64	896
12	OLIN	1	3	4	5	3	32	96
13	OSLO	4	2	3	11	24	2,048	49,152
14	PIAMA	8	4	4	18	128	262,144	33,000,000
15	PIAMA	5	4	8	18	160	262,144	42,000,000
16	RHINE	1	2	1	8	2	256	512
17	TRAPCA	6	3	6	7	108	128	13,824
18	TRAPCA	7	3	4	9	84	512	43,008
19	AHSMOG	1	3	3	7	15	128	1,920

Notes. Refer to Appendix A for a listing of the 19 study cohort names; Questions = Outcomes × Predictors × Lags; Models = 2^k where k = number of Covariates; Search space = approximation of analysis search space = Questions × Models; none of the papers made any adjustments/corrections for multiple testing bias.

None the 19 base papers we reviewed made any adjustments/corrections for multiple testing bias in their analysis. Seventeen of the 19 papers made no mention of the issue. One paper (Morgenstern et al., 2007) stated they… *did not adjust for multiple testing* in their discussion. Another paper (McDonnell et al., 1999) used Bonferroni's correction but only to explain similarities in characteristics of four different groups making up their study cohort. They did not make any adjustment/correction for multiple testing bias in their subsequent analysis.

Summary statistics of possible numbers of statistical tests performed in the 19 base papers are presented in Table 2. The median number (interquartile range, IQR) of Questions and Models was 24 (IQR 15−84) and 256 (IQR 96–3,072), respectively. The median number (IQR) of possible statistical tests (Search space) of the 19 base papers was 13,824 (IQR 1,536−221,184).

Given the large numbers of possible tests, the statistics drawn from the cohort studies are unlikely to offer unbiased statistics for meta-analysis. Although not shown, covariates in each of the cohort studies vary considerably from study to study (the reader is referred to the original Anderson et al. 2013a supplemental files). For comparison purposes,

search space counts of air quality component−heart attack observational studies are also large – i.e., median (IQR) = 6,784 (2,600−94,208), n=14 (Young et al., 2019), and = 12,288 (2,496−58,368), n=34 (Young & Kindzierski, 2019).

Table 2. Summary statistics for counts estimated for 19 base papers considered by Anderson et al. (2013a,b) in their meta-analysis

Statistic	Outcomes	Predictors	Covariates	Lags	Questions	Models	Search space
Minimum	1	1	1	5	2	32	96
Lower quartile	1	2	4	7	15	96	1,536
Median	1	3	4	8	24	256	13,824
Upper quartile	5	4	6	12	84	3,072	221,184
Maximum	8	8	8	18	160	262,144	42,000,000

Notes. Questions = Outcomes × Predictors × Lags; Models = 2^k where k = number of Covariates; Search space = approximation of analysis search space = Questions × Models.

3.2 p-value Plot

Table 3 presents EEs, CIs and calculated p-values for 18 cohort studies used in their meta-analysis calculation. A plot of the sorted p-values versus the integers is given in Figure 1. Both p-values for NO2 (indicated by solid circles, ●) and PM2.5 (indicted by open circles, o) are combined in Figure 1. This (combining results for NO2 and PM2.5) is the same as what Anderson et al. (2013a) did to compute their meta-analytic statistic.

Table 3. Effect estimate (EE), lower confidence level (CL_{low}) and upper confidence level (CL_{high}) values and corresponding p-values estimated for cohort studies used by Anderson et al. (2013a,b) in their meta-analysis

Air Component	Study cohort, outcome	EE	CL_{low}	CL_{high}	p-value
NO2	BAMSE, wheeze	1.01	0.98	1.04	0.5135
	British Columbia, asthma	1.13	1.04	1.23	0.0073
	CHS 2003, asthma	1.03	1.01	1.05	0.0033
	CHS, asthma	1.24	1.06	1.46	0.0187
	CHIBA, asthma	1.32	1.02	1.71	0.0691
	ECRHS, asthma	1.43	1.02	2.00	0.0854
	KRAMER, asthma	1.19	0.85	1.68	0.3695
	MISSEB, asthma	1.32	0.73	2.41	0.4553
	OLIN, asthma	1.00	0.35	2.87	1.0000
	Oslo Birth Cohort, asthma	0.93	0.85	1.00	0.0673
	PIAMA, asthma	1.16	0.96	1.41	0.1634
	RHINE, asthma	1.46	1.07	1.99	0.0500
	TRACPA, asthma	0.71	0.14	3.48	0.7336
PM2.5	AHSMOG, asthma	1.08	0.85	1.38	0.5541
	British Columbia, asthma	1.10	0.90	1.35	0.3837
	CHIBA, asthma	1.86	0.90	3.86	0.2547
	PIAMA, asthma	2.06	0.91	4.66	0.2678
	TRACAP, asthma	1.60	0.45	5.70	0.6542

Notes. Study cohorts presented here are in the same order as presented in the Anderson et al. (2013a) Fig. 1 for NO2 study cohorts and Fig. 2 for PM2.5 study cohorts.

The Figure 1 relationship presents as bi-linear – six p-values are near or below nominal significance (.05) and the remaining p-values >.05 fall on an approximately 45-degree line. This is different from behavior of both plausible true null and true alternative hypothesis outcomes (Appendix C Figs. C−1 and C−2). A plausible true null hypothesis outcome presents as a sloped line from left to right at approximately 45 degrees, and a plausible true alternative hypothesis outcome presents as line with a majority of p-values below .05 in p-value plots.

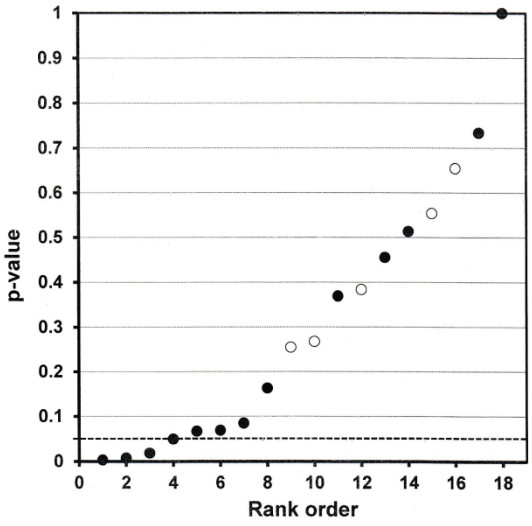

Figure 1. P-value plot for the Anderson et al. (2013a,b) meta-analysis (note: solid circles (●) are NOx p-values; open circles (o) are PMx p-values)

The p-value plots are basic technology used by others (e.g., Selwyn, 1989; Westfall & Young, 1993; Cao et al., 2007; Young et al. 2009; Ryan et al., 2013; Young & Kindzierski, 2019; Kindzierski et al., 2021). The two-component mixture of p-values in Figure 1 may be a combination of studies showing an association and no association, but both outcomes cannot be true. Questionable research practices (QRP) involve approaches used by researchers during data collection, analysis and reporting that can increase false-positive findings in published literature (de Vrieze, 2021; Ravn & Sørensen, 2021). QRP cannot be ruled out as an explanation for small p-values in several of the cohort studies in Table 3 and Figure 1. A p-value (mixture) relationship like this does not support a finding that exposure to ambient NOx and PM2.5 early in life is associated with development of asthma later in life.

Higgens and Green (2011) assert that heterogeneity will always exist in meta-analysis whether or not one can detect it using a statistical test. Statistical heterogeneity (I^2) quantifies the proportion of the variation in point estimates due to among−study differences. I^2 is a standard measure for heterogeneity and Anderson et al. (2013a,b) reported an I^2=64.1% (p<.001) for NO2 based on 13 study cohorts and I^2=7.4% (p=.364) for PM2.5 based on 5 study cohorts.

4. Discussion

Two statistical methods – analysis search space and p-value plots – were used to independently test the reliability of a meta-analysis of cohort studies published by Anderson et al. (2013a,b). Large search space counts without evidence of corrections for multiple testing bias in base papers is one measure for identifying limitations of a meta-analysis. Estimated analysis search spaces of the base studies used by Anderson et al. (2013a,b) – Tables 1 and 2 – indicates that there were large numbers of possible statistical tests performed in the base studies. Results taken from these studies are unlikely to offer unbiased measures for meta-analysis.

The p-value plot constructed for statistics taken from the base studies (Figure 1) shows a two-component mixture. This bi-linear pattern/shape is clearly different with p-value plot behavior of either plausible true null and true alternative hypothesis outcomes (Figs. C−1 and C−2). Taken together, this evidence does not support the meta-analysis as being a reliable study for other researchers to depend upon. This is discussed in more detail.

4.1 Interpretation of Anderson et al. Meta-analysis

The overall approach and EEs and CIs in the Anderson et al. (2013a,b) meta-analysis is taken at face value and interpretations of our independent tests are made from there. As to an air quality−asthma development relationship, there may also be other possible explanations for statistical associations in observational studies. Some of these explanations relate to study methodology and include (Clyde, 2000; Pocock et al., 2004; Ioannidis, 2008; Sarewitz, 2012; Chambers, 2015; Simonsohn et al., 2014; Hubbard, 2015; Harris, 2017; Young & Kindzierski 2019):

- Improper selection of datasets for analysis.
- Improper selection of statistical models.
- Flexible choices in methods to compute statistical results, including undertaking multiple testing and multiple modeling without statistical correction.

- Inadequate treatment of confounders and other latent variables.
- Selective reporting of results.
- Publication bias, non-reporting of null results.

There are many aspects of choice involved in modeling air quality−health effect relationships in observational studies (Young & Kindzierski, 2019). Some of these choices involve which parameters and confounding variables (covariates) to include in a model, what type of lag structure for covariates to use, which interactions need to be considered, and how to model nonlinear trends (Clyde, 2000). Because of the many potential parameters and confounders that may be included in a study, some aspect of model selection is often used. Even if models are selected in an unbiased manner, different model selection strategies may lead to very different models and outcomes for the same set of data. On the other hand, inherent bias may lead researchers to choose models that provide selective outcomes (Young & Kindzierski, 2019).

The p-value plot test and the resulting mixture relationship of p-values from the cohort studies (Figure 1) does not support a general air quality−asthma incidence relationship. Although Anderson et al. follow a typical statistical approach for meta-analysis, their approach will not be meaningful if EEs & CIs drawn from base studies are not unbiased and/or if the test statistics drawn from the base studies as a whole form a two-component mixture.

We consider the Anderson et al. study as a standard meta-analysis… a research question is selected, a computer search is undertaken for relevant published papers involving study cohorts, papers are identified, filtered, and a final set of base papers is selected. The etiology they examined is… *whether ambient air quality early in life leads to development of asthma later in life*. Each cohort study (i.e., base paper) they selected looked at air quality, including one or more of the following air components – carbon monoxide, nitrogen dioxide, sulfur dioxide, ozone, particulate matter (PM).

However, each cohort study varied in terms of the specific air components studied. They identified 18 study cohorts with a total of 99 EEs that examined air quality and asthma, but they only ended up doing a formal meta-analysis on NOx (NO or NO2) and PMx (PM10 and/or PM2.5). Three outcome estimates (ORs, RRs or HRs) with upper and lower confidence limits were extracted from the base papers and a random effects analysis assuming the statistics were normally distributed was used after DerSimonian & Laird (1986).

The Anderson et al. initial computer search identified 4,165 possibly relevant papers. From this, 266 papers were examined in detail and 13 cohorts studies were selected that reported on NOx and five cohort studies were selected that report on PM. A numerical meta-analysis was computed on the two datasets, NOx and PMx, separately and computed p-values for these datasets and combined the p-values into one figure (Figure 1).

In their search, Anderson et al. also identified asthma-related effect studies for other air quality components – e.g., carbon monoxide, ozone and sulfur dioxide. They did not explain reasons for excluding these components in the meta-analysis. As for NO2 and PM, the search space counts – numbers of possible statistical tests conducted – in the selected base studies (Tables 1 and 2) are considered large – median search space 13,824 (IQR 1,536−221,184). Given such large search spaces, there is little assurance that test statistics drawn from the base papers into their meta-analysis are unbiased.

4.2 Heterogeneity

Regarding the interpretation of quantitative measures of heterogeneity (I^2), Higgens et al. (2003) assign low, moderate and high I^2 values of 25%, 50%, and 75% for meta-analysis. Higgens and Green (2011) provide other guidance for interpretation: 0−40% may not be important, 30−60% may represent moderate heterogeneity, 50−90% may represent substantial heterogeneity and 75−100% represents considerable heterogeneity. The Higgens and Green (2011) and Higgens et al. (2003) guidance suggests that the Anderson et al. (2013a,b) meta-analysis of 13 NO2 study cohorts is associated with moderate to substantial heterogeneity.

Forstmeier et al. (2017) note that a key source of heterogeneity in meta-analysis is publication bias favoring positive effects, often due to researcher degrees of freedom (flexibility) to find a statistically significance effect more often than expected by chance As for the Anderson et al. meta-analysis, it is noted that heterogeneity in their NO2 dataset is not simply due to an increase in across-study (study-to-study) variability – but is a much more problematic two-component mixture (see Figure 1). Specifically, some NO2 studies have very small p-values that may suggest real causal relationships or QRP, whereas other p-values fall on a 45-degree line indicating randomness (i.e., no relationship at all). As for their meta-analysis of 5 PM2.5 study cohorts, all corresponding p-values fall on a 45-degree line indicating complete randomness (Figure 1).

There are two standard approaches to heterogeneity: 1) Find the covariates that give rise to the heterogeneity. Covariates are typically examined and usually they do not remove the heterogeneity. 2) Use the random effects analysis model of

DerSimonian & Laird (1986). This approach assumes that there is one normal distribution and a contribution of study-to-study variability, and that bias is not a big issue. This approach was specifically developed for randomized control trials (RCTs) where these assumptions are more reasonable. One normal distribution means one etiology with superimposed study-to-study variability.

However, the Anderson et al. meta-analysis is not of RCTs; it combines observational studies, and a mixture is observed (Figure 1) – some studies positive and some studies null. There is a difference between sampling from a normal distribution and a two-component mixture. Publication bias and multiple testing bias may, in part, explain the nature of the heterogeneity – i.e., two-component mixture (Figure 1) – for NO2. These factors cannot be dismissed as possible explanations for their findings.

A further possible hidden problem is that meta-analysis assumes that heterogeneity among statistics from relevant base papers is randomly distributed around the true value (Charlton, 1996). This holds true if errors across base papers used for meta-analysis are balanced – i.e., errors in some base studies in one direction are cancelled or balanced out by errors in other base studies in the other direction. Therefore, statistical pooling and averaging in meta-analysis in theory produces an error-reduced estimate of the underlying, unbiased, 'true' value.

However, pooling statistics from base papers in meta-analysis unavoidably includes hidden biases of the individual studies. Given pervasiveness of bias in science today (Sarewitz, 2012; Forstmeier et al., 2017), a more likely situation is that most researchers tend to make the same errors in the same direction – i.e., their test statistics have similar biases related to seeking out positive associations (which is more publishable). Such a condition violates a key assumption of meta-analysis and any statistic under this situation is not meaningful.

4.3 Real Versus Random (Chance) Associations

The p-value plot (Figure 1) exhibits a bi-linear appearance and is clearly different from p-value plots of both plausible true null and true alternative hypothesis outcomes (Appendix C Figs. C−1 and C−2). Six p-values are below or near .05, a value often taken as 'statistically significant', and twelve p-values appear completely random – a two-component mixture. Firstly, any sort of statistical averaging – weighted or not – for a mixture of this type in inappropriate. Secondly, both findings cannot be true. Evidence for real versus random statistical associations is considered further here (refer to Table 4). The factors presented in Table 4 are intended as a checklist related to methodology of the base papers for researchers to assist in the interpretation of real versus random statistical associations.

Table 4. Factors to consider when evaluating meta-analysis results presenting as bi-linear in a p-value plot

Possibility 1	**Possibility 2**
Small p-values true	**Small p-values false**
Most of the base papers show a small p-value	Evidence of QRP in base papers
There will be supporting literature	Covariates correlated with outcome, bias
There will be a reasonable etiology	Very large sample size elevates small bias to cause
No evidence of QRP in base papers	A number of Bradford Hill criterion not met
Most Bradford Hill criterion met	
Large p-values false	**Large p-values true**
Poor research technique in base papers	Distribution of large p-values is uniform
Underpowered studies	Good negative effect studies
Masking covariates hide real effect	No clear etiology
Role of chance	

Notes. QRP = questionable research practices.

4.3.1 Small p-values True & Large p-values False

First, suppose the small p-values represent true associations, i.e., there is a real air quality−asthma association. In this study, there are two small p-values – .00328 and .00732 (Table 3). P-values this small are often interpreted by researchers as being real. These two p-values are close to a .005−action level proposed by Johnson (2013), but larger than a .001−action level proposed by Boos and Stefanski (2011) for making such an interpretation. Here, the term 'action level' means that if the study is replicated, the replication will give a p-value less than .05.

These rules of thumb – the traditional .05 and more-recently proposed .005 and .001 p-value decision criteria – all presume only one statistical test and one p-value result, i.e., no multiple testing bias issues, and that the result is from a well-conducted study (i.e., with randomization, blinding and blocking). This is not true for the Anderson et al. (2013a,b) meta-analysis and its base studies. The median number of possible p-values over the base studies is 13,824. A small p-value can easily arise by chance given this many tests (Bock, 2016). When researchers have many different hypotheses and carry out many statistical tests on the same set of data, they run the risk of concluding that there are real differences or real associations when in fact there are none (Kavvoura et al., 2007).

Yet there is an abundance of published literature suggesting an overall statistical air quality component−adverse health relationship. There are large numbers of papers reporting positive statistical associations between some air quality variable and a health effect. For example, a Google Scholar search of the exact phrase "air pollution" and term "asthma" in a title over the years 1990−2020 returned 1,330 hits (search done 14 December 2021).

If small p-values are true, plausible explanations are needed for large p-values in Figure 1 being false. Here, one can speculate it is possible that some of the papers have a large p-value due to poor data, methods, small sample size or just chance. However, seven of 13 NO2 p-values and all five PM2.5 p-values are greater than .05 (Table 3 and Figure 1). This requires further rationalization given the presumed careful procedure used by Anderson et al. researchers to screen and select their study cohorts and base papers for meta-analysis.

4.3.2 Small p-values False & Large p-values True

On the other side of the coin is a possibility that the small p-values are false. How might this be the case? In the presence of large numbers of statistical tests performed in the base studies, two plausible ways related to methodology are offered which may contribute to a meta-analysis failing:

• P-hacking in base studies (Streiner, 2018). P-hacking is multiple testing and multiple modeling without statistical correction (Chambers, 2015; Hubbard, 2015; Harris, 2017). Search space counts of base studies aids understanding of this issue. Specifically, p-hacking cannot be ruled out as an explanation for small p-values coming from base studies where no statistical corrections are made for large numbers of test performed.

• Not properly controlling for covariates in base studies (Brenner, 1998; Wang & Yin 2013) such that controlling for them may make the small p-values disappear.

While both are important, p-hacking may be serious in published biomedical studies. For example, Hayat et al. (2017) randomly sampled and reviewed 216 of 1,023 published articles from seven top tier general public health journals for the year 2014 with an objective quantifying basic and advanced statistical methods used in public health research. These journals included: *Epidemiology*, *American Journal of Epidemiology*, *American Journal of Public Health*, *Bulletin of World Health Organization*, *European Journal of Epidemiology*, *American Journal of Preventive Medicine* and *International Journal of Epidemiology*. They reported that statistical corrections for multiple testing bias were only made in 5.1% of the 216 studies they reviewed (i.e., ~1−in−20 published studies).

We reviewed the Hayat et al. (2017) Supplemental Information and we emailed the corresponding author – Hayat – in attempts to identify which articles indicated adjustments for multiple testing bias. Both the Supplemental Information and email response provided by Hayat indicated 10 (not 11) or 4.6% of the 216 randomly sampled articles made adjustments for multiple testing bias. It is further speculated by us that the articles making adjustments may be genetic rather than traditional epidemiology articles, and that published traditional epidemiology articles making adjustments for multiple testing bias may be much less than 4.6%.

Kavvoura et al (2007) noted that there is an apparent tendency among epidemiology researchers to avoid making statistical corrections for multiple testing bias, highlighting statistically significant findings, and avoiding highlighting nonsignificant findings in their research papers. This behavior may be a problem, because many of these significant findings could in future turn out to be false positives.

4.4 Testing of Meta-analysis Claims

It has been stated previously that the body of literature available for meta-analysis may be distorted with positive−association studies and a systematic review or meta-analysis of these studies may be biased (Egger et al., 2001; Sterne et al., 2001) because researchers are summarizing information and data from a misleading, selected body of evidence (Ioannidis, 2008; NASEM, 2019). We believe this applies to the epidemiolocal literature, and this alone supports a need for testing of claims made in meta-analysis of this literature. Mayo (2018) endorses Karl Popper's approach of developing scientific knowledge by identifying and correcting errors through strong (severe) tests of scientific claims. We also support this approach.

It makes sense to step back from a detailed consideration of the air quality component−asthma claim made by Anderson

et al. (2013a,b). A scientist is expected to make a good case for a research claim, and it ought to survive a battery of severe but passable tests. Several examples exist in epidemiolocal literature where air quality component−chronic disease claims – both positive and null associations – have been independently tested.

While there is observational evidence that long-term exposure to particulate matter (PM) is associated with premature death in urban populations, confounding by unmeasured variables remains a valid concern in observational studies. Greven et al. (2011) attempted to independently replicate a long-term particulate matter exposure−acute mortality claim using the US Medicare Cohort Air Pollution Study (MCAPS) dataset. This dataset included individual−level information on time of death and age on a population of 18.2 million for the period 2000–2006.

Greven et al. suggested two ways to make a good case for a positive air quality component−health effect claim is to test for 'within location' or 'across location' effects. Positive 'across location' effects might be due to confounding whereas positive 'within location' effects would be less likely biased by confounding. Using this within location approach, Greven et al. was unable to replicate the long-term particulate matter exposure−acute mortality claim.

Another example is with the Young et al. (2017) analysis of an air quality−acute death claim for the eight most populous California air basins. This analysis included over 2,000,000 deaths and over 37,000 exposure days over a 13-year period. Young et al. examined each air basin individually (i.e., 'within location' analysis) and observed null effects like Greven et al (2011). Here one must keep in mind that as sample size goes to infinity, the standard error (SE) goes to zero. So, any small but statistically significant 'across location' effect observed between two air basin populations, each with large sample sizes, has a good chance of being due to bias.

This bias can largely be controlled using a method referred to as Local Control (Obenchain & Young, 2017). One clusters objects into many small clusters and does an analysis within each cluster. One can then observe how the analysis result changes (or does not change) across clusters. Obenchain & Young applied Local Control to a historical air quality (total suspended particulate−mortality) dataset describing a 'natural experiment' initiated by the federal Clean Air Act Amendments of 1970 (specifically, the Chay et al., 2003 dataset).

Chay et al. (2003) used a comprehensive county-level (US) dataset available compiled on population, mortality, total suspended particulate matter (TSP) levels and economic conditions for the period 1969–1974. Obenchain & Young replicated the Chay et al. finding of a no TSP−mortality association. Thus, the control of confounding is important. Variables can be put into a model or an analysis and be restricted to limited geographic regions (e.g., clusters) thereby reducing the influence of confounding factors.

Two statistical methods are demonstrated here as a form of testing of scientific claims made in meta-analysis of observational cohort studies:

- Search space counting and identifying whether corrections for multiple testing bias are made in base papers is one measure. Search space counting allows one to obtain a clearer picture of the numbers of statistical tests that may have been performed in base studies. Test statistics drawn from base studies with large search space counts are unlikely to offer unbiased measures for meta-analysis where no corrections are made for the number of statistical tests performed.
- Examining the behavior of p-values in a ranked plot (p-value plot) for test statistics drawn from base studies into a meta-analysis is another measure. P-value plotting provides a test of results where the underlying data itself remains hidden.

These tests enable a user to independently diagnose specific meta-analysis claims to judge the potential for use of QRP such as multiple testing bias, p-hacking, publication bias (Banks et al., 2016). The Anderson et al. (2013a,b) meta-analyses associating ambient air quality early in life with development of asthma later in life failed these tests. The p-value plot constructed for statistics used from the base studies by Anderson et al. (2013a,b) showed a two-component mixture. This bi-linear pattern/shape is clearly different with p-value plot behavior of both plausible true null and plausible true alternative hypothesis outcomes.

5. Findings

Estimation of analysis search spaces of 19 base papers used in the Anderson et al. meta-analysis indicated that the numbers of statistical tests possible were large – median 13,824 (interquartile range 1,536−221,184; range 96−42M) in comparison to actual statistical test results presented. Given such large search spaces, there is little assurance that test statistics drawn from the base papers into the meta-analysis are unbiased.

A p-value plot showed that heterogeneity of the NO2 results across studies is consistent with a two-component mixture. Meta-analytic averaging across a mixture is inappropriate. The shape of the p-value plot for NO2 appears consistent with use of questionable research practices to obtain small p-values in several of the cohort studies. As for PM2.5 results, all corresponding p-values fall on a 45-degree line in the p-value plot indicating complete randomness rather

than a true association.

Anderson et al. claim an association of air quality and development of asthma. Our analysis does not support their claim. Because of multiple testing bias, it cannot be ruled out that test statistics drawn from the base papers and used for meta-analysis by Anderson et al. are unbiased. Also, heterogeneity of test statistics across base papers used for the meta-analysis is more complex than simple sampling from a normal process.

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Appendix A

Appendix A.doc

Base papers used for search space counting are presented in the file “Appendix A.doc”.

Appendix B

Appendix B.doc

Three search space analysis examples are presented in the file “Appendix B.doc”.

Appendix C

Appendix C.doc

Summary statistics of datasets and p-value plots for true null associations and true effects between two variables in observational studies are presented in the file “Appendix C.doc”.

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