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Contents

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A Family of Left Lie Bol Loops

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Abstract

In this paper the left Bol split extension method is used to build left Bol Lie loops from the Lie groups H and K such that H is a Lie subgroup of $Aut(K)$. Furthermore, we investigated some of the properties of those loops constructed in this way. Examples are given for finite and infinite dimensional left Bol Lie loops. Moreover, we showed that the twisted semidirect product of Lie algebras is an Akivis algebra.

Keywords: loops and quasigroups, semidirect product, twisted semidirect product, Lie loops.

1. Introduction

Non-associative semidirect product of groups are investigated in the last decades intensively in (Johnson & Sharma,1980; Kinyon & Jones, 2000; Nagy & Strambach, 2008; Johnson & Smith, 2010; Greer & Raney, 2014). The twisted semidirect product of groups is also a known object in loop theory, its first appearance is in (Johnson & Sharma, 1980), where they called the method *the left Bol split extension*. The construction method was further surveyed and generalized by Johnson and Smith (Johnson & Smith, 2010). In this paper the left Bol Lie loops that are formed by the twisted semidirect product of Lie groups are investigated. We also explore examples of infinite dimensional left Bol Lie loops that are raised by the action of Lie subgroups of $GL(\mathcal{H})$ on the Hilbert space $\mathcal{H}$ over $\mathbb{C}$.

It is well-known that semidirect product of Lie algebras is a Lie algebra, so we naturally asked this question for the twisted semidirect product of Lie algebras of the Lie groups. We showed that twisted semidirect product of Lie algebras of the Lie groups is a Lie algebra which turned out to an Akivis algebra.

2. Preliminaries

We use the function evaluation in the backwards. If $\alpha : X \to Y$ is a function, then the function evaluation of α at the point $x \in X$ is denoted by $(x)\alpha$ or x^α. Let $\beta : Y \to Z$ be another function, then the composition of α and β is the function $\gamma := \alpha\beta$ such that $(x)^{\alpha\beta} := (x^\alpha)^\beta$ for all x in the domain of α. Let G be a group. The elements a, b of G is said to conjugate if there exists a $g \in G$ such that $g^{-1}ag = b$ where $g^{-1}ag := a^g$.

The nonempty set L with a binary operation, $\oplus$, is called a loop if there exists $e \in L$ such that for all $a \in L$ $a \oplus e = e \oplus a = a$, and the equations $a \oplus x = b$ and $y \oplus a = b$ have always unique solutions $x := a\backslash b$ and $y := b/a$ in L whenever a and b are given in L. The uniqueness of x and y lead us to define two new maps that are called the left division $\backslash : L \times L \to L$ $(a, b) \mapsto a\backslash b$, and the right division $/ : L \times L \to L$ $(a, b) \mapsto b/a$ such that $a \oplus (a\backslash b) = b$ and $(b/a) \oplus a = b$.

Let $(L, \oplus)$ be a loop. Given any $x \in L$, let $L_x : L \to L$ and $R_x : L \to L$ be two maps defined by $(a)R_x := a \oplus x$, $(b)L_x := x \oplus b$ where $a, b \in L$. The maps L_x and R_x are called the left and the right translation maps respectively for x. It is well known that if $(L, \oplus)$ is a loop, then the left and the right translation maps are bijective. The loop $(L, \oplus)$ is called a left Bol loop if the left Bol identity given in (1) is valid for all a, b, and c in L:

$$a \oplus (b \oplus (a \oplus c)) = (a \oplus (b \oplus a)) \oplus c. \quad (1)$$

Further readings on Bol loops can be found in (Robinson, 1966; Pflugfelder, 1990; Kiechle 2002). Next we define some groups acting on L, namely right multiplication group and left multiplication group. Right multiplication group, $Rmlt(L)$, of L is the permutation group generated by all right translations of L. The left multiplication group, $Lmlt(L)$, is defined similarly. The multiplication group of L, $Mlt(L)$, is the permutation group generated by all right and left translations of L. Hence, $Mlt(L) = \langle L_a, R_b : a, b \in L\rangle$.

G is called a Lie group if G is a group and G is a smooth manifold such that multiplication and inversion maps are smooth (Knapp, 2016). A Lie loop L is a loop and a smooth manifold such that multiplication, right and left division maps are all smooth (Nagy & Strambach, 2008). In this paper we mainly focus on the examples of Lie loops that are obtained from the twisted semidirect product of matrix Lie groups.

Let $M(n,\mathbb{C})$ be the set of matrices of size n by n with complex entries and let $GL(n,\mathbb{C})$ be the general linear group. A matrix Lie group is a closed subgroup of $GL(n,\mathbb{C})$. The list of matrix Lie groups can be found in (Wallach, 1988; Hein, 1990). The left, the right and the middle nuclei of $(L,\oplus)$ can be defined respectively as follows:

$$N_l = \{a \in L | (a \oplus x) \oplus y = a \oplus (x \oplus y);\ \forall x, y \in L\}. \tag{2}$$
$$N_r = \{a \in L | (x \oplus y) \oplus a = x \oplus (y \oplus a);\ \forall x, y \in L\}. \tag{3}$$
$$N_m = \{a \in L | (x \oplus a) \oplus y = x \oplus (a \oplus y);\ \forall x, y \in L\}. \tag{4}$$

Note that N_l, N_r, and N_m are all subgroups of L. The nucleus, $N(L)$, and the centrum, $C(L)$, of $(L,\oplus)$ are defined as follow:

$$N(L) := N_l \cap N_r \cap N_m. \tag{5}$$
$$C(L) := \{x \in L|\ x \oplus y = y \oplus x\ \forall y \in L\}. \tag{6}$$

The center of L is denoted by $Z(L)$ such that $Z(L) := C(L) \cap N(L)$. It is well known that the nucleus and the center of L are subgroups of L, see (Pflugfelder, 1990).

Let G be a group and A be a set, and suppose G acts on A from the right. We use $Aff(A,G)$ to denote the set of maps, $f_{(a,g)}$, such that $(b)f_{(a,g)} = a + b.g$ where $a, b \in A$ and $g \in G$. If G acts on A by function evaluation, then $(b)f_{(a,g)} = a + b^g$.

2.1 Semidirect and Twisted Semidirect Products

Let H and K be groups such that $H \leq Aut(K)$ and consider $G := K \times H$ as a set and define the multiplication, $\odot$, on G as follow:

$$(k_1, h_1) \odot (k_2, h_2) = (k_1 k_2^{h_1^{-1}}, h_1 h_2). \tag{7}$$

Note that we used juxtapositions for the product in K and H. It is well known that $(G,\odot)$ is a new group with the identity element (e_K, e_H). The product given in (7) is called *semidirect product* of K by H and denoted by $G = K \rtimes H$, see (Hall, 1999). Note that if H is acting on K trivially, then the semidirect product is the usual direct product. The sets $K_G = \{(k, e_H) : k \in K\}$ and $H_G = \{(e_K, h) : h \in H\}$ have a trivial intersection, and $K_G \cong K$ and $H_G \cong H$ such that K_G is normal subgroup of G.

If we replace h_1^{-1} with h_1 in (7), then we obtain a new binary operation $\overline{\odot}$ on $K \times H$ as given in (8). Johnson and Sharma (1980) named the $K \times H$ with $\overline{\odot}$ *the left Bol split extension* and they showed that if H is a non-abelian group of $Aut(K)$, then $(K \times H, \overline{\odot})$ is a left Bol loop. In the current paper we use the term *twisted semidirect product* for the binary operation $\overline{\odot}$.

$$(k_1, h_1)\overline{\odot}(k_2, h_2) = (k_1 k_2^{h_1}, h_1 h_2). \tag{8}$$

Each element of H is an automorphism from K to K and the notation $k_2^{h_1}$ stands for the image of k_2 under h_1. In general the twisted semidirect product is not necessarily associative.

2.2 Lie Algebra and Akivis Algebra

A Lie algebra, see (Humphreys, 1972; Knapp, 2016), is a vector space $\mathfrak{g}$ over a field $\mathbb{F}$ that endowed with bracket operation $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(x, y) \mapsto [x, y]$ that satisfies the following axioms:

1. The bracket operation is bilinear.
2. $[x, x] = 0$ for all $x \in \mathfrak{g}$.
3. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$.

The last axiom is called the *Jacobi* identity. Note that combining the first two axioms yields that $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$. Therefore, the bracket operation is skew-symmetric in a Lie algebra. A homomorphism, ϕ, of Lie algebras from $\mathfrak{g}_1$ to $\mathfrak{g}_2$ is a linear map that preserves the brackets that is $([x, y])\phi = [(x)\phi, (y)\phi]$ for $x, y \in \mathfrak{g}_1$. A derivation of a Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ is a linear map $f : \mathfrak{g} \to \mathfrak{g}$ such that $([x, y])f = [(x)f, y] + [x, (y)f]$ for all $x, y \in \mathfrak{g}$ (Hein, 1990). The set of all derivation of $\mathfrak{g}$ over $\mathbb{F}$ is denoted by $Der_{\mathbb{F}}(\mathfrak{g})$. The derivation is a Lie algebra if the bracket operation is defined as $[f, g] = gf - fg$ for $f, g \in Der_{\mathbb{F}}(\mathfrak{g})$. A Lie subalgebra $\mathfrak{h}$ is a vector subspace of $\mathfrak{g}$ such that $\mathfrak{h}$ is closed under bracket operation. A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called an ideal of $\mathfrak{g}$ if $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$.

An Akivis algebra $(\mathcal{A}, [.,.], \langle .,.,. \rangle)$ is a real vector space with a bilinear skew-symmetric map $(x, y) \mapsto [x, y] : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, called the commutator map, and a trilinear map $(x, y, z) \mapsto \langle x, y, z \rangle : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, called the associator map, such that the following identity (called the Akivis identity) holds (Figula & Strambach, 2007).

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = \langle x, y, z\rangle + \langle y, z, x\rangle + \langle z, x, y\rangle - (\langle x, z, y\rangle + \langle y, x, z\rangle + \langle z, y, x\rangle) \tag{9}$$

If $\mathcal{A}$ is a Lie algebra, then the left hand-side of the equality is zero by the Jacobi identity.

Let $\mathfrak{k}$, $\mathfrak{h}$ be Lie algebras over the same field $\mathbb{F}$, and suppose $\rho : \mathfrak{h} \to Der_{\mathbb{F}}(\mathfrak{k})$, $h \mapsto (h)\rho$ and $k \mapsto (k)^{(h)\rho}$, be a Lie algebra homomorphism. The set $\mathfrak{k} \times \mathfrak{h}$ is a Lie algebra endowed with the bracket operation given in (10).

$$[(k_1, h_1), (k_2, h_2)] = ([k_1, k_2] + (k_1)^{(h_2)\rho} - (k_2)^{(h_1)\rho}, [h_1, h_2]) \tag{10}$$

The new Lie algebra with the bracket defined in (10) is called a semidirect product of $\mathfrak{k}$ and $\mathfrak{h}$, and it is denoted by $\mathfrak{k}\rtimes_\rho\mathfrak{l}$. The following result is well known; see for example (Hein, 1990).

Theorem 2.1. *Let $\mathfrak{h}$ and $\mathfrak{k}$ be two Lie algebras over the field $\mathbb{F}$, and let $\rho : \mathfrak{h} \to Der_{\mathbb{F}}(\mathfrak{k})$ be a Lie algebra homomorphism. Then,*

1. *$\mathfrak{l} = \mathfrak{k}\rtimes_\rho\mathfrak{h}$ is a Lie algebra.*
2. *$\bar{\mathfrak{k}} = \{(k, 0) : k \in \mathfrak{k}\} \cong \mathfrak{k}$ such that $\mathfrak{k}$ is an ideal of $\mathfrak{l}$, i.e., $[\bar{\mathfrak{k}}, \mathfrak{l}] \subseteq \bar{\mathfrak{k}}$.*
3. *$\bar{\mathfrak{h}} = \{(0, h) : h \in \mathfrak{h}\} \cong \mathfrak{h}$ such that $\mathfrak{h}$ is a subalgebra of $\mathfrak{l}$, i.e., $[\bar{\mathfrak{h}}, \bar{\mathfrak{h}}] \subseteq \bar{\mathfrak{h}}$.*

We obtain a new bracket operation from (10) by interchanging ${k_1}^{h_2}$ and ${k_2}^{h_1}$ in (10). We call this new bracket operation, given in (11), the *twisted semidirect product* of $\mathfrak{k}$ and $\mathfrak{h}$. The set $\mathfrak{k} \times \mathfrak{h}$ with twisted semidirect product is denoted by $\mathfrak{k}\overline{\rtimes}_\rho\mathfrak{h}$.

$$[(k_1, h_1), (k_2, h_2)] = ([k_1, k_2] + (k_2)^{(h_1)\rho} - (k_1)^{(h_2)\rho}, [h_1, h_2]) \tag{11}$$

3. Main Results

Theorem 3.1. *Let H and K be Lie groups with $H \leq Aut(K)$ such that the evaluation map $ev : K \times H \to K$, $(k, h) \mapsto k^h$ is smooth. If $\mathcal{L} := (K \times H, \overline{\odot})$, then*

1. *$\mathcal{L}$ is a Lie group if and only if H is an abelian Lie group.*
2. *If H is not abelian, then $\mathcal{L}$ is a left Bol Lie loop, not a Lie group.*

Proof. We first prove the first argument. If $\mathcal{L}$ is a Lie group, then its group product $\overline{\odot}$ is associative. That is the equation (12) holds for all $(k_1, h_1), (k_2, h_2)$, and (k_3, h_3) in $\mathcal{L}$

$$[(k_1, h_1)\overline{\odot}(k_2, h_2)]\overline{\odot}(k_3, h_3) = (k_1, h_1)\overline{\odot}[(k_2, h_2)\overline{\odot}(k_3, h_3)] \tag{12}$$

The left hand side of the equation (12) is equal to $(k_1k_2^{h_1}{k_3}^{h_1h_2}, (h_1h_2)h_3)$ and the right hand side is equal to $(k_1k_2^{h_1}{k_3}^{h_2h_1}, h_1(h_2h_3))$. Two sides are equal if and only if ${k_3}^{h_1h_2} = {k_3}^{h_2h_1}$, but this forces that H is an abelian Lie group. Conversely, suppose that H is an abelian Lie group, then $\mathcal{L}$ is a smooth manifold as a cartesian product of smooth manifolds $K \times H$. Let e_K and e_H be the identity elements of K and H respectively, then it can be verified that (e_K, e_H) is the identity element of $\mathcal{L}$. Moreover, for any arbitrary element (k, h) of $\mathcal{L}$ the following equation is satisfied, hence the two sided inverse of (k, h) exists.

$$(k, h)\overline{\odot}((k^{-1})^{h^{-1}}, h^{-1}) = ((k^{-1})^{h^{-1}}, h^{-1})\overline{\odot}(k, h) = (e_K, e_H). \tag{13}$$

The group product $\overline{\odot}$ of $\mathcal{L}$ is associative as shown below:

$$\begin{aligned}
[(k_1, h_1)\overline{\odot}(k_2, h_2)]\overline{\odot}(k_3, h_3) &= (k_1k_2^{h_1}, h_1h_2)\overline{\odot}(k_3, h_3) && (14)\\
&= ((k_1k_2^{h_1}){k_3}^{h_1h_2}, (h_1h_2)h_3) && (15)\\
&= ((k_1k_2^{h_1}){k_3}^{h_2h_1}, (h_1h_2)h_3) && (16)\\
&= (k_1(k_2^{h_1}{k_3}^{h_2h_1}), h_1(h_2h_3)) && (17)\\
&= (k_1(k_2^{h_1}({k_3}^{h_2})^{h_1}), h_1(h_2h_3)) && (18)\\
&= (k_1(k_2k_3^{h_2})^{h_1}, h_1(h_2h_3)) && (19)\\
&= (k_1, h_1)\overline{\odot}(k_2k_3^{h_2}, h_2h_3) && (20)\\
&= (k_1, h_1)\overline{\odot}[(k_2, h_2)\overline{\odot}(k_3, h_3)]. && (21)
\end{aligned}$$

Note that we used the assumption that H is abelian in (16) to write $h_1h_2 = h_2h_1$, and we used the fact that h_1 is an automorphism over K to write $k_2^{h_1}k_3^{h_2h_1} = (k_2k_3^{h_2})^{h_1}$ in (19). We conclude that $\mathcal{L}$ satisfies all group axioms, so it is a group besides its smooth manifold structure. To show it is actually a Lie group requires to show that the group product and the inversion maps are smooth.

Let $\mu_{\mathcal{L}} : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ such that $((k_1, h_1), (k_2, h_2))\mu_{\mathcal{L}} = (k_1k_2^{h_1}, h_1h_2)$ and $i : \mathcal{L} \to \mathcal{L}$ such that $(k, h)i = ((k^{-1})^{h^{-1}}, h^{-1})$. Let μ_K and μ_H be the multiplication maps of K and H respectively and let i_K and i_H be the inversion maps of K and H. By assumption μ_K, μ_H and i_K, i_H are all smooth maps. Observe that

$$\begin{aligned}((k_1, h_1), (k_2, h_2))\mu_{\mathcal{L}} &= ((k_1, k_2^{h_1})\mu_K, (h_1, h_2)\mu_H) && (22)\\ &= (((k_1)id, (k_2, h_1)ev)\mu_K, (h_1, h_2)\mu_H) && (23)\\ &= (((k_1, (k_2, h_1)(id \times ev))\mu_K, (h_1, h_2)\mu_H) && (24)\\ &= ((k_1, (k_2, h_1))((id \times ev) \circ \mu_K), (h_1, h_2)\mu_H) && (25)\\ &= ((k_1, (k_2, h_1)), (h_1, h_2))((id \times ev) \circ \mu_K) \times \mu_H. && (26)\end{aligned}$$

$\mu_{\mathcal{L}}$ is smooth since the direct product of smooth maps and composition of smooth maps are smooth. Similar to multiplication the inversion map is also smooth that can be shown below.

$$\begin{aligned}(k, h)i_{\mathcal{L}} &= ((k^{-1})^{h^{-1}}, h^{-1}) && (27)\\ &= (((k)i_K, (h)i_H)ev, (h)i_H) && (28)\\ &= ((k, h)(i_K \times i_H) \circ ev), (h)i_H) && (29)\\ &= ((k, h), h))((i_K \times i_H) \circ ev) \times i_H. && (30)\end{aligned}$$

Therefore, $\mathcal{L}$ is a Lie group.

For the proof of the second argument let H be a non-abelian subgroup of $Aut(K)$, then $(K \times H, \overline{\odot})$ is a left Bol loop which has been shown in (Johnson & Sharma, 1980). For the convenience of readers we prefer to provide the proof. Suppose that H is non-abelian, then there exists $h_1, h_2 \in H$ such that $h_1h_2 \neq h_2h_1$. That means there exists a $k_3 \in K$ such that ${k_3}^{h_1h_2} \neq {k_3}^{h_2h_1}$, then for nonzero $k_1, k_2 \in K$, $(k_1{k_2}^{h_1})k_3^{h_1h_2} \neq (k_1{k_2}^{h_1})k_3^{h_2h_1}$ which is equivalent to:

$$[(k_1, h_1)\overline{\odot}(k_2, h_2)]\overline{\odot}(k_3, h_3) \neq (k_1, h_1)\overline{\odot}[(k_2, h_2)\overline{\odot}(k_3, h_3)]. \qquad (31)$$

Therefore, if H is not abelian, then the product on $\mathcal{L}$ is not associative, so $\mathcal{L}$ is not a Lie group. On the other hand, $\mathcal{L}$ is a smooth manifold as the cartesian product of smooth manifolds K and H. Moreover, for all $(k, h) \in \mathcal{L}$

$$(k, h)\overline{\odot}(e_K, e_H) = (e_K, e_H)\overline{\odot}(k, h) = (k, h) \qquad (32)$$

hence (e_K, e_H) is the neutral element of $\mathcal{L}$. We can always find unique (x_k, x_h) and (y_k, y_h) in $\mathcal{L}$ that satisfy the given equations in (33) and (34).

$$(k_1, h_1)\overline{\odot}(x_k, x_h) = (k_2, h_2) \quad \text{where} \quad (x_k, x_h) := (k_1, h_1)\backslash(k_2, h_2). \qquad (33)$$

$$(y_k, y_h)\overline{\odot}(k_2, h_2) = (k_1, h_1) \quad \text{where} \quad (y_k, y_h) := (k_1, h_1)/(k_2, h_2). \qquad (34)$$

The solutions $(x_k, x_h) = ((k_1^{-1}k_2)^{h_1^{-1}}, h_1^{-1}h_2)$ and $(y_k, y_h) = ((k_1(k_2^{-1})^{h_1h_2^{-1}}, h_1h_2^{-1})$ can be derived easily. We conclude that $\mathcal{L}$ is a loop. To show it is a Lie loop we also need to show that the twisted semidirect product, the left and the right division maps are all smooth. Based on (33) and (34) the right and the left division maps are derived as follow:

$$\begin{aligned}\backslash \;&:\; \mathcal{L} \times \mathcal{L} \to \mathcal{L} \quad \text{such that} \quad ((k_1, h_1), (k_2, h_2)) \mapsto ((k_1^{-1}k_2)^{h_1^{-1}}, h_1^{-1}h_2) && (35)\\ / \;&:\; \mathcal{L} \times \mathcal{L} \to \mathcal{L} \quad \text{such that} \quad ((k_1, h_1), (k_2, h_2)) \mapsto ((k_1(k_2^{-1})^{h_1h_2^{-1}}, h_1h_2^{-1}) && (36)\end{aligned}$$

We have already showed in the proof of first argument that the twisted semidirect product is smooth. On the other hand, the left and the right division maps can be written as direct products of smooth maps, thus they are also smooth as given below:

$$\begin{aligned}(k_1, h_1)\backslash(k_2, h_2) &= (((k_1, k_2), h_1), (h_1, h_2))(((i_K \times id_K)\mu_K) \times id_H)ev \times (i_H \times id_H)\mu_H && (37)\\ (k_1, h_1)/(k_2, h_2) &= (k_1, (k_2, (h_1, h_2)))(id_K \times ((i_K \times (id_H \times i_H)\mu_H)ev)\mu_K \times (id_H \times i_H)\mu_H && (38)\end{aligned}$$

We conclude that if H is non-abelian, then $\mathcal{L}$ is a left Bol Lie loop that is not a Lie group. □

Notice that if we set $h_1 = h_2$ in (35) and $k_1 = k_2$ in (36), then the following corollary is obtained.

Corollary 3.1.1. *Let H and K be groups such that $H \leq Aut(K)$ and let $\mathcal{L} := (K \times H, \overline{\odot})$. Then*

1. $(k_1, h_1)\backslash(k_2, h_2) = (e_K, h_1^{-1}h_2)$ *if* $k_1 = k_2$.
2. $(k_1, h_1)/(k_2, h_2) = (k_1k_2^{-1}, e_H)$ *if* $h_1 = h_2$.

Example 3.2. *Let $\mathbb{C}$ be the field of complex numbers and let n be a positive integer. It is well known that $\mathbb{C}^n$ is an additive Lie group and $GL(n, \mathbb{C}) \cong Aut(\mathbb{C}^n)$ after fixing a basis of $\mathbb{C}^n$. Suppose $\phi : GL(n, \mathbb{C}) \to Aut(\mathbb{C}^n)$ is the isomorphism and $i : G \to GL(n, \mathbb{C})$ be the inclusion map, where G is a non-abelian closed subgroup of $GL(n, \mathbb{C})$. A closed subgroup of $GL(n, \mathbb{C})$ is a Lie group, hence G is a Lie group. The map $\Phi := i \circ \phi$ is a homomorphism of Lie groups from G to $Aut(\mathbb{C}^n)$. On the other hand, the evaluation map is smooth since matrix multiplication is smooth. Therefore, $(\mathbb{C}^n \times G, \overline{\odot})$ is a left Bol Lie loop by Theorem 3.1.*

Example 3.3. *For any matrix $A \in M(n, \mathbb{C})$ we use $A^\top$ and A^* to denote the transpose of A and conjugate transpose of A respectively. Let $p, q \in \mathbb{N}$ and $p + q \geq 1$ and let $\alpha := \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$ and $\beta := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, where I_p, I_q, and I_n are the identity matrices. The pseudo-unitary group $U(p, q)$ and the symplectic group $Sp(n, \mathbb{C})$ are well-known non-abelian classical Lie groups given below:*

$$U(p, q) = \left\{A \in GL(p + q, \mathbb{C}) : A\alpha A^\top = \alpha\right\}. \tag{39}$$

$$Sp(n, \mathbb{C}) = \{A \in GL(2n, \mathbb{C}) : A\beta A^* = \beta\}. \tag{40}$$

Similar to example 3.2, $(\mathbb{C}^{p+q} \times U(p, q), \overline{\odot})$ and $(\mathbb{C}^{2n} \times Sp(n, \mathbb{C}), \overline{\odot})$ are both left Bol Lie loops.

Example 3.4. *Let a, b, and c be arbitrary real numbers. The Heisenberg group, H, consists of 3 by 3 matrices in form of*

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}. \tag{41}$$

Heisengerg group is a closed subgroup of $GL(3, \mathbb{R})$, so it is a matrix Lie group. The evaluation map, $ev : \mathbb{R}^3 \times H \to \mathbb{R}^3$, $(v, A) \mapsto v^\top A$ is smooth. It can be checked that the matrix multiplication in H is not commutative. Therefore, $(\mathbb{R}^3 \times H, \overline{\odot})$ is a left Bol Lie loop.

Corollary 3.4.1. *Let V be either finite or infinite dimensional linear space over a field $\mathbb{F}$ and let G be any non-abelian subgroup of $Aut(V)$, then $\mathcal{L} := (V \times G, \overline{\odot})$ is a left Bol loop such that:*

1. $Lmlt(\mathcal{L}) \subseteq Aff(V, G) \times G$.
2. $N(\mathcal{L}) = \{0\} \times Z(G)$.
3. $Z(\mathcal{L}) = \{(0, id)\}$.

Proof. $\mathcal{L}$ is a left Bol loop is immediate by Theorem 3.1 since G is a non-abelian subgroup of $Aut(V)$ and V is a linear space that means V is an additive group. The twisted semidirect product over $V \times G$ is written as below:

$$(v_1, g_1)\overline{\odot}(v_2, g_2) = (v_1 + (v_2{}^{g_1}), g_1g_2)$$

We first prove (1). Let $L_{(v,g)}$ be any left translation of $Lmlt(\mathcal{L})$, and let (w, h) be any element of $\mathcal{L}$. Then

$$(w, h)L_{(v,g)} = (v, g)\overline{\odot}(w, h) \tag{42}$$

$$= (v + w^g, gh) \tag{43}$$

$$= ((w)\phi_{(v,g)}, (h)L_g) \tag{44}$$

$$= (w, h)(\phi_{(v,g)} \times L_g). \tag{45}$$

For each $(w, h) \in \mathcal{L}$, $(w, h)L_{(v,g)} = (w, h)(\phi_{(v,g)} \times L_g)$, thus $L_{(v,g)} = \phi_{(v,g)} \times L_g$, and this implies $Lmlt(\mathcal{L}) \subseteq Aff(V, G) \times Lmlt(G)$. Note that $Lmlt(G) = G$ since G is a group, so $Lmlt(\mathcal{L}) \subseteq Aff(V, G) \times G$.

To see (2), we will determine the left, the middle and the right nuclei of $\mathcal{L}$. Let (v_1, g_1) and (v_2, g_2) be arbitrary elements of $\mathcal{L}$.

$$\begin{aligned} N_l(\mathcal{L}) &= \left\{(w,h) \in \mathcal{L} : [(w,h)\overline{\odot}(v_1,g_1)]\overline{\odot}(v_2,g_2) = (w,h)\overline{\odot}[(v_1,g_1)\overline{\odot}(v_2,g_2)]\right\} \\ &= \left\{(w,h) \in \mathcal{L} : (w + {v_1}^h, hg_1)\overline{\odot}(v_2,g_2) = (w,h)\overline{\odot}(v_1 + {v_2}^{g_1}, g_1g_2)\right\} \\ &= \left\{(w,h) \in \mathcal{L} : (w + {v_1}^h + {v_2}^{hg_1}, (hg_1)g_2) = (w + {v_1}^h + {v_2}^{g_1h}, h(g_1g_2))\right\} \\ &= \left\{(w,h) \in \mathcal{L} : {v_2}^{hg_1} = {v_2}^{g_1h}\right\}. \end{aligned}$$

The condition ${v_2}^{hg_1} = {v_2}^{g_1h}$ is independent from w, so w can be anything in V. Moreover, if ${v_2}^{hg_1} = {v_2}^{g_1h}$ for all v_2 in V, then $hg_1 = g_1h$ for all $g_1 \in G$, thus $h \in Z(G)$. Therefore, $N_l(\mathcal{L}) = V \times Z(G)$. In left Bol loops the left and the right nuclei are same (Robinson, 1966), hence we only need to find $N_r(\mathcal{L})$.

$$\begin{aligned} N_r(\mathcal{L}) &= \left\{(w,h) \in \mathcal{L} : [(v_1,g_1)\overline{\odot}(v_2,g_2)]\overline{\odot}(w,h) = (v_1,g_1)\overline{\odot}[(v_2,g_2)\overline{\odot}(w,h)]\right\} \\ &= \left\{(w,h) \in \mathcal{L} : (v_1 + {v_2}^{g_1}, g_1g_2)\overline{\odot}(w,h) = (v_1,g_1)\overline{\odot}[(v_2 + w^{g_2}, g_2h)\right\} \\ &= \{(w,h) \in \mathcal{L} : (v_1 + {v_2}^{g_1} + w^{g_1g_2}, (g_1g_2)h) = (v_1 + {v_2}^{g_1} + w^{g_2g_1}, g_1(g_2h))\} \\ &= \{(w,h) \in \mathcal{L} : w^{g_1g_2} = w^{g_2g_1}\}. \end{aligned}$$

$w^{g_1g_2} = w^{g_2g_1}$ for all $g_1, g_2 \in G$, so $w = 0$. On the other hand, $w^{g_1g_2} = w^{g_2g_1}$ is independent from h, so h can be anything in G, thus $N_r(\mathcal{L}) = \{0\} \times G$.

The nucleus of $\mathcal{L}$ is the intersection of left, right and middle nuclei. Therefore, $N(\mathcal{L}) = \{0\} \times Z(G)$.

Finally, let (w,h) be in the center of $\mathcal{L}$, then (w,h) is in the nucleus of $\mathcal{L}$. Therefore $w = 0$ and $h \in Z(G)$, but $(0,h)$ is also in the centrum, hence $(0,h)\overline{\odot}(v,g) = (v,g)\overline{\odot}(0,h)$ for all $(v,g) \in \mathcal{L}$. That is $(v^h, hg) = (v, gh)$ if and only if $v^h = v$ for all $v \in V$ if and only if h is the identity operator in G. Therefore, $Z(\mathcal{L}) = \{(0, id_V)\}$ where id_V is the identity map from V to V. □

Example 3.5. *Let $GL(\mathcal{H})$ be the group of invertible operators inside the space of bounded linear operators $L(\mathcal{H})$, where $\mathcal{H}$ is an infinite dimensional Hilbert space over $\mathbb{C}$. The infinite dimensional Hilbert space $\mathcal{H}$ over $\mathbb{C}$ is an additive group with the neutral element 0. It is an infinite dimensional manifold since it is locally homeomorphic to itself, and the addition and inversion maps are smooth. On the other hand, the group of invertible operators $GL(\mathcal{H})$ is open in $L(\mathcal{H})$ with respect to operator norm, so it is a Banach-Lie group. Therefore, $\mathcal{H} \times GL(\mathcal{H})$ is a smooth manifold as a cartesian product of smooth manifolds. Furthermore, if the evaluation map $ev : \mathcal{H} \times GL(\mathcal{H}) \to \mathcal{H}; (h,T) \mapsto h^T$ is smooth, then the twisted semidirect product, the left and the right division maps are smooth, hence $(\mathcal{H} \times GL(\mathcal{H}), \overline{\odot})$ is an infinite dimensional left Bol Lie loop by corollary 3.4.1.*

Lemma 3.6. *Any Lie algebra $\mathfrak{l}$, is an Akivis algebra with the trilinear operation defined by $\langle x,y,z\rangle : \mathfrak{l} \times \mathfrak{l} \times \mathfrak{l} \to \mathfrak{l}$; $(x,y,z) \mapsto [[x,y],z] - [x,[y,z]]$.*

Proof. Let $\mathfrak{l}$ be a Lie algebra, then there exists a bilinear skew-symmetric operation, $[.,.] : \mathfrak{l} \times \mathfrak{l} \to \mathfrak{l}$; $(x,y) \mapsto [x,y]$. To see that $\mathfrak{l}$ is indeed an Akivis algebra, we need to verify the Akivis identity: $[[x,y],z] + [[y,z],x] + [[z,x],y] = \alpha - \beta$ where α and β given below.

$$\alpha = \langle x,y,z\rangle + \langle y,z,x\rangle + \langle z,x,y\rangle \quad (46)$$

$$\beta = \langle x,z,y\rangle + \langle y,x,z\rangle + \langle z,y,x\rangle \quad (47)$$

Let $\gamma = [[x,y],z] + [[y,z],x] + [[z,x],y]$, then we want to show that $\gamma = \alpha - \beta$. Since $\mathfrak{l}$ is a Lie algebra it satisfies the Jacobi identity and this forces that $\gamma = 0$, hence we only need to show that $\alpha = \beta$

$$\begin{aligned} \alpha &= [[x,y],z] - [x,[y,z]] + [[y,z],x] - [y,[z,x]] + [[z,x],y] - [z,[x,y]] \\ &= ([[x,y],z] + [[y,z],x] + [[z,x],y]) - ([x,[y,z]] + [y,[z,x]] + [z,[x,y]]) \\ &= 2([[x,y],z] + [[y,z],x] + [[z,x],y]) \\ &= 2(0) = 0 \end{aligned}$$

We can similarly show that $\beta = 0$, so $\alpha = \beta = 0$. Therefore, any Lie algebra $\mathfrak{l}$ is an Akisvis algebra if the trilinear operation defined as in Lemma 3.6. □

Theorem 3.7. *Let $\mathfrak{h}$ and $\mathfrak{k}$ be two Lie algebras over the field $\mathbb{F}$, and let $\rho : \mathfrak{h} \to Der_{\mathbb{F}}(\mathfrak{k})$ be a Lie algebra homomorphism. Then,*

1. *$\mathfrak{l} = \mathfrak{k}\overline{\rtimes}_\rho\mathfrak{h}$ is an Akivis algebra with bracket and trilinear operations given in (48) and (49) below respectively.*

$$[(k_1, h_1), (k_2, h_2)] = ([k_1, k_2] + (k_2)^{(h_1)\rho} - (k_1)^{(h_2)\rho}, [h_1, h_2]) \text{ for all } k_1, k_2 \in \mathfrak{k} \text{ and } h_1, h_2 \in \mathfrak{h}. \quad (48)$$

$$\langle x, y, z\rangle = [[x, y], z] - [x, [y, z]] \text{ for all } x, y, z \in \mathfrak{l}. \quad (49)$$

2. *$\overline{\mathfrak{k}} = \{(k, 0) : k \in \mathfrak{k}\} \cong \mathfrak{k}$ is an ideal of $\mathfrak{l}$, i.e., $[\overline{\mathfrak{k}}, \mathfrak{l}] \subseteq \overline{\mathfrak{k}}$.*
3. *$\overline{\mathfrak{h}} = \{(0, h) : h \in \mathfrak{h}\} \cong \mathfrak{h}$ is a subalgebra of $\mathfrak{l}$, i.e., $[\overline{\mathfrak{h}}, \overline{\mathfrak{h}}] \subseteq \overline{\mathfrak{h}}$.*

Proof. 1. The bracket on $\mathfrak{l}$ is skew symmetric as follows:

$$\begin{aligned}[(k_1, h_1), (k_2, h_2)] &= ([k_1, k_2] + (k_2)^{(h_1)\rho} - (k_1)^{(h_2)\rho}, [h_1, h_2]) && (50)\\ &= (-[k_2, k_1] - ((k_1)^{(h_2)\rho} - (k_2)^{(h_1})\rho), -[h_2, h_1]) && (51)\\ &= -([k_2, k_1] + (k_1)^{(h_2)\rho} - (k_2)^{(h_1)\rho}, [h_2, h_1]) && (52)\\ &= -[(k_2, h_2), (k_1, h_1)] && (53)\end{aligned}$$

The bracket on $\mathfrak{l}$ is bilinear since the bracket operations on $\mathfrak{k}$ and $\mathfrak{h}$ are bilinear. On the other hand $(h)\rho$ is linear for each $h \in \mathfrak{h}$. Therefore, we only need to verify the Jacobi identity on $\mathfrak{l}$, and this can be seen as follow: Let $x = [[(k_1, h_1), (k_2, h_2)], (k_3, h_3)]$, $y = [[(k_2, h_2), (k_3, h_3)], (k_1, h_1)]$, and $z = [[(k_3, h_3), (k_1, h_1)], (k_2, h_2)]$. We want to show that $x + y + z = (0, 0)$. Notice that:

$$\begin{aligned}x &= [[(k_1, h_1), (k_2, h_2)], (k_3, h_3)] && (54)\\ &= [([k_1, k_2] + (k_2)^{(h_1)\rho} - (k_1)^{(h_2)\rho}, [h_1, h_2]), (k_3, h_3)] && (55)\\ &= ([[k_1, k_2] + (k_2)^{(h_1)\rho} - (k_1)^{(h_2)\rho}, k_3] + (k_3)^{([h_1,h_2])\rho} - ([k_1, k_2] + (k_2)^{(h_1)\rho} - (k_1)^{(h_2)\rho})^{(h_3)\rho}, [[h_1, h_2], h_3]) && (56)\\ &= ([[k_1, k_2], k_3] + [{k_2}^{(h_1)\rho}, k_3] - [{k_1}^{(h_2)\rho}, k_3] + {k_3}^{([h_1,h_2])\rho} - [k_1, k_2]^{(h_3)\rho} - {k_2}^{(h_1)\rho(h_3)\rho} + {k_1}^{(h_2)\rho(h_3)\rho}, [[h_1, h_2], h_3]). && (57)\end{aligned}$$

The map ρ is a Lie algebra homomorphism, so $([h_1, h_2])\rho = [(h_1)\rho, (h_2)\rho]$, and $[(h_1)\rho, (h_2)\rho] = (h_2)\rho(h_1)\rho - (h_1)\rho(h_2)\rho$ since $Der_{\mathbb{F}}(\mathfrak{k})$ is a Lie algebra with $[f, g] = gf - fg$ for each $f, g \in Der_{\mathbb{F}}(\mathfrak{k})$. Therefore,

$${k_3}^{([h_1,h_2])\rho} = {k_3}^{(h_2)\rho(h_1)\rho} - {k_3}^{(h_1)\rho(h_2)\rho} \quad (58)$$

On the other hand, $(h)\rho$ is a derivation on $\mathfrak{k}$, so $(h)\rho$ preserves the Leibniz rule, and this gives that:

$$[k_1, k_2]^{(h_3)\rho} = [{k_1}^{(h_3)\rho}, k_2] + [k_1, {k_2}^{(h_3)\rho}] \quad (59)$$

If we let $x = (x_1, x_2)$, then x_1 and x_2 coordinates are written as follow.

$$x_1 = [[k_1, k_2], k_3]+[{k_2}^{(h_1)\rho}, k_3]-[{k_1}^{(h_2)\rho}, k_3]+{k_3}^{(h_2)\rho(h_1)\rho}-{k_3}^{(h_1)\rho(h_2)\rho}-[{k_1}^{(h_3)\rho}, k_2]-[k_1, {k_2}^{(h_3)\rho}]-{k_2}^{(h_1)\rho(h_3)\rho}+{k_1}^{(h_2)\rho(h_3)\rho} \quad (60)$$

$$x_2 = [[h_1, h_2], h_3] \quad (61)$$

We can similarly find $y = (y_1, y_2)$ and $z = (z_1, z_2)$ such that

$$y_1 = [[k_2, k_3], k_1]+[{k_3}^{(h_2)\rho}, k_1]-[{k_2}^{(h_3)\rho}, k_1]+{k_1}^{(h_3)\rho(h_2)\rho}-{k_1}^{(h_2)\rho(h_3)\rho}-[{k_2}^{(h_1)\rho}, k_3]-[k_2, {k_3}^{(h_1)\rho}]-{k_3}^{(h_2)\rho(h_1)\rho}+{k_2}^{(h_3)\rho(h_1)\rho} \quad (62)$$

$$y_2 = [[h_2, h_3], h_1]) \quad (63)$$

$$z_1 = [[k_3, k_1], k_2]+[{k_1}^{(h_3)\rho}, k_2]-[{k_3}^{(h_1)\rho}, k_2]+{k_2}^{(h_1)\rho(h_3)\rho}-{k_2}^{(h_3)\rho(h_1)\rho}-[{k_3}^{(h_2)\rho}, k_1]-[k_3, {k_1}^{(h_2)\rho}]-{k_1}^{(h_3)\rho(h_2)\rho}+{k_3}^{(h_1)\rho(h_2)\rho} \quad (64)$$

$$z_2 = [[h_3, h_1], h_2] \quad (65)$$

The second coordinate of $x + y + z$ is $x_2 + y_2 + z_2 = [[h_1, h_2], h_3] + [[h_2, h_3], h_1] + [[h_3, h_1], h_2] = 0$ since $\mathfrak{h}$ is a Lie algebra. Moreover, a part of the first coordinate of $x + y + z$ is $[[k_2, k_3], k_1] + [[k_2, k_3], k_1] + [[k_3, k_1], k_2] = 0$ since $\mathfrak{k}$ is a Lie algebra. The rest of the first coordinate of $x + y + z$ is rewritten as:

$([k_2{}^{(h_1)\rho}, k_3] - [k_2{}^{(h_1)\rho}, k_3]) + (-[k_1{}^{(h_2)\rho}, k_3] - [k_3, k_1{}^{(h_2)\rho}]) + (k_3{}^{(h_2)\rho(h_1)\rho} - k_3{}^{(h_2)\rho(h_1)\rho}) + (-k_3{}^{(h_1)\rho(h_2)\rho} + k_3{}^{(h_1)\rho(h_2)\rho})$ $+ (-[k_1{}^{(h_3)\rho}, k_2] + [k_1{}^{(h_3)\rho}, k_2]) + (-[k_1, k_2{}^{(h_3)\rho}] - [k_2{}^{(h_3)\rho}, k_1]) + (-k_2{}^{(h_1)\rho(h_3)\rho} + k_2{}^{(h_1)\rho(h_3)\rho}) + (k_1{}^{(h_2)\rho(h_3)\rho} - k_1{}^{(h_2)\rho(h_3)\rho})$ $+ ([k_3{}^{(h_2)\rho}, k_1] - [k_3{}^{(h_2)\rho}, k_1]) + (k_1{}^{(h_3)\rho(h_2)\rho} - k_1{}^{(h_3)\rho(h_2)\rho}) + (k_2{}^{(h_3)\rho(h_1)\rho} - k_2{}^{(h_3)\rho(h_1)\rho})$, and this sum is zero since the sum in each parentheses is zero. Therefore, $x + y + z = (0, 0)$, and we conclude that the twisted semidirect product of Lie algebras is a Lie algebra. Therefore, $\mathfrak{l}$ is an Akivis Algebra by Lemma 3.6.

2. The first claim, $\mathfrak{k} \cong \overline{\mathfrak{k}}$, is clear. We will show that $[\overline{\mathfrak{k}}, \mathfrak{l}] \subseteq \overline{\mathfrak{k}}$. Let $(k, 0) \in \overline{\mathfrak{k}}$, and let $(k^*, h) \in \mathfrak{l}$. Then, $[(k, 0), (k^*, h)] = ([k, k^*] + (k^*)^{(0)\rho} - (k)^{(h)\rho}, [0, h])$ where $(k^*)^{(0)\rho} = (k^*)^{id} = k^*$ and $[0, h] = 0$, so $[(k, 0), (k^*, h)] = ([k, k^*] - k^* + ((k)^{(h)\rho}, 0) = (k^{**}, 0) \in \overline{\mathfrak{k}}$, where $k^{**} = [k, k^*] - k^* + (k)^{(h)\rho}$. Therefore, $\mathfrak{k}$ is an ideal of $\mathfrak{l}$.

3. $\mathfrak{h} \cong \overline{\mathfrak{h}}$ is clear. Let $(0, h), (0, h^*) \in \overline{\mathfrak{h}}$, then $[(0, h), (0, h^*)] = ([0, 0] + (0)^{(h)\rho} - (0)^{(h^*)\rho}, [h, h^*]) = (0, h^{**}) \in \overline{\mathfrak{h}}$. Therefore, $[\overline{\mathfrak{h}}, \overline{\mathfrak{h}}] \subseteq \overline{\mathfrak{h}}$, so $\mathfrak{h}$ is a subalgebra of $\mathfrak{l}$. □

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On Commutativity of Semiprime Rings with Multiplicative (generalized)-derivations

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Abstract

The aim of this paper is to explore the commutativity of semiprime rings admitting multiplicative (generalized)-derivations and satisfy certain hypotheses on appropriate subsets.

Keywords: semiprime ring, ideals, derivation, multiplicative (generalized)-derivation.

1. Introduction

Throughout this paper R denotes an associative ring with center $Z(R)$. Recall, a ring R is said to be prime ring if for any $a, b \in R$, $aRb = (0)$ implies either $a = 0$ or $b = 0$ and is semiprime ring if $aRa = (0)$ implies $a = 0$. For any $x, y \in R$, we shall denote the commutator and anti-commutator by the symbols $[x, y] = xy - yx$ and $(x \circ y) = xy + yx$ respectively. We shall frequently use the basic commutator and anti-commutator identities : $[xy, z] = x[y, z]+[x, z]y$, $[x, yz] = y[x, z]+[x, y]z$ and $(x \circ yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x.y]z$, $(xy \circ z) = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$. An additive map $f : R \to R$ is called a derivation of R if $f(xy) = f(x)y + xf(y)$ holds for all $x, y \in R$. Let $F : R \to R$ be a map together with another map $f : R \to R$ so that $F(xy) = F(x)y + xf(y)$ for all $x, y \in R$. If F is additive and f a derivation of R, then F is called generalized derivation of R and if $f = 0$, then F is called left multiplier of R. The notion of generalized derivation was introduced by Brešar (Brešar, 1991) . In (Havala, 1998), author gave an algebraic study of these mappings in prime rings. Obviously, every derivation is a generalized derivation. In this way generalized derivation covers both concepts of derivation and left multiplier of R. Let K be a nonempty subset of R, a map $f : K \to R$ is said to be centralizing on K, if $[f(x), x] \in Z(R)$ for all $x \in K$. In particular, if $[f(x), x] = 0$ for all $x \in K$, then f is called commuting on K.

In the literature, a number of authors have discussed the commutativity of prime rings and semiprime rings admitting derivations and generalized derivations satisfying certain algebraic identities, see (Ali, Kumar & Miyan, 2011), (Ali, Dhara & Fošner, 2011), (Andima & Pajoohesh, 2010), (Ashraf et al, 2007, 2001), (Daif & Bell, 1992), (Dhara & Pattanayak, 2011), (Hongan, 1997), where further references can be found.

Let us swing to the foundation examination of multiplicative (generalized)-derivations of associative rings. Inspired by the work of Martindale III (Martindale, 1969), Daif (Daif, 1991) introduced the concept of multiplicative derivations. Accordingly, a map $f : R \to R$ is called multiplicative derivation of R if $f(xy) = f(x)y + xf(y)$ holds for all $x, y \in R$. Of course, these maps are not necessarily additive. Goldmann and Sĕmrl (Goldmann & Sĕmrl, 1996) presented complete description of these maps. Further, Daif and Tammam-El-Sayiad (Daif & Tammam-El-Sayiad, 1997) extended the notion of multiplicative derivation to multiplicative generalized derivation as follows: A map $F : R \to R$ is called multiplicative generalized derivation of R if $F(xy) = F(x)y + xf(y)$ holds for all $x, y \in R$, where f is a derivation of R. Recently, Dhara and Ali (Dhara & Ali, 2013) made a slight generalization in above definition of multiplicative generalized derivation by relaxing the conditions on f. A map $F : R \to R$ (not necessarily additive) is said to be a multiplicative (generalized)-derivation if $F(xy) = F(x)y + xf(y)$ holds for all $x, y \in R$, where f can be any map (not necessarily additive nor a derivation). For convenience we denote it by a pair (F, f). In the previous couple of years many outcomes has been gotten in prime and semi-prime rings involving multiplicative (generalized)-derivations, see (Ali et al, 2015), (Ali et al, 2014), (Dhara & Ali, 2013), (Dhara et al, 2014) and (Khan, 2016). As multiplicative (generalized)-derivation is an extended notion of generalized derivation, so it is noteworthy to demonstrate the consequences of generalized derivations for multiplicative (generalized)-derivations.

The main objective of this paper is to take care of the issue raised by author in (Khan, 2016) and investigate the commutativity of R. Precisely, we concentrate on the following central-valued conditions: $f(x)F(y) \pm yx \in Z(R)$, $f(x)F(y) \pm xy \in Z(R)$, $f(x)F(y) \pm (x \circ y) \in Z(R)$, $f(x)F(y) \pm [x, y] \in Z(R)$, $F(xy) \pm F(x)F(y) \in Z(R)$, $F[x, y] \pm (x \circ y) \in Z(R)$, $F(x \circ y) \pm [x, y] \in Z(R)$, $F[x, y] \pm xy \in Z(R)$, $F(x \circ y) \pm xy \in Z(R)$, $F[x, y] \pm f(x) \circ y \in Z(R)$, $F(x \circ y) \pm [f(x), y] \in Z(R)$ where x and y are from an appropriate subset of R.

2. Main Results

Theorem 1. *Let R be a semiprime ring and I a nonzero ideal of R. Suppose that (F, f) is a multiplicative (generalized)-derivation of R. If $f(x)F(y) \pm yx \in Z(R)$for all $x, y \in I$, then f is commuting on I and I is commutative.*

Proof. We consider

$$f(x)F(y) \pm yx \in Z(R) \text{ for all } x, y \in I. \tag{1}$$

Replace y by yz in (1) to get $(f(x)F(y) \pm yx)z + f(x)yf(z) \pm y[z, x] \in Z(R)$ for all $x, y, z \in I$. On commuting with z we obtain

$$[f(x)yf(z), z] \pm [y[z, x], z] = 0 \text{ for all } x, y, z \in I. \tag{2}$$

In particular, putting $x = z$ to obtain

$$[f(z)yf(z), z] = 0 \text{ for all } y, z \in I. \tag{3}$$

Which implies that

$$f(z)yf(z)z = zf(z)yf(z) \text{ for all } x, y, z \in I \tag{4}$$

Substituting $yf(z)w$ for y in (4), we have

$$f(z)yf(z)wf(z)z = zf(z)yf(z)wf(z) \text{ for all } x, y, z, w \in I. \tag{5}$$

Using (4) in (5), we obtain $f(z)yzf(z)wf(z) = f(z)yf(z)zwf(z)$ for all $x, y, z, w \in I$. That is $xf(z)y[f(z), z]wf(z) = 0$ for all $x, y, z, w \in I$. It implies that $x[f(z), z]y[f(z), z]w[f(z), z] = 0$ for all $x, y, z, w \in I$. Therefore, $(I[f(z), z])^3 = (0)$ for all $z \in I$. But R has no nonzero nilpotent ideal, we conclude that $I[f(z), z] = (0)$ for all $z \in I$. Thus, $[f(z), z] = 0$ for all $z \in I$(See, (Herstein, 1976)).

Now, Replace y by yz in (2) and we get

$$[f(x)yzf(z), z] \pm [yz[z, x], z] = 0 \text{ for all } x, y, z \in I. \tag{6}$$

Right multiply (2) by z and subtract (6) from it, we obtain $[f(x)y[f(z), z], z] \pm [y[[z, x], z], z] = 0$ for all $x, y, z \in I$. Using the fact that $I[f(z), z] = (0)$ for all $z \in I$, we get

$$[y[[z, x], z], z] = 0 \text{ for all } x, y, z \in I. \tag{7}$$

Replace y by xy in (7), we obtain

$$x[y[[z, x], z], z] + [x, z]y[[z, x], z] = 0 \text{ for all } x, y, z \in I. \tag{8}$$

Using (7), it reduces to

$$[x, z]y[[z, x], z] = 0 \text{ for all } x, y, z \in I. \tag{9}$$

Replace y by zy in (9), we get

$$[x, z]zy[[x, z], z] = 0 \text{ for all } x, y, z \in I \tag{10}$$

Left multiply (9) by z and subtract from (10), we get $[[x, z], z]y[[x, z], z] = 0$ for all $x, y, z \in I$. That is $[[x, z], z]I[[x, z], z] = (0)$ for all $x, z \in I$. Semiprimeness of I yields that

$$[[x, z], z] = 0 \text{ for all } x, z \in I. \tag{11}$$

Linearizing (11) with respect to z and using (11), we have

$$[[x, z], t] + [[x, t], z] = 0 \text{ for all } x, t, z \in I. \tag{12}$$

Replace z by zt in (12), we get $z[[x, t], t] + [z, t][x, t] + ([[x, z], t] + [[x, t], z])t + z[[x, t], t] = 0$ for all $x, t, z \in I$. Using (11) and (12), we obtain

$$[z, t][x, t] = 0 \text{ for all } x, t, z \in I. \tag{13}$$

Replace x by xy in (13) to get $[z, t]x[y, t] + [z, t][x, t]y = 0$ for all $x, y, t, z \in I$. Using (13), we obtain $[z, t]x[y, t] = 0$ for all $x, y, t, z \in I$. In particular, $[y, t]I[y, t] = (0)$ for all $y, t \in I$. It implies that $[y, t] = 0$ for all $y, t \in I$. Hence, $[I, I] = (0)$ as desired.

Theorem 2. *Let R be a semiprime ring and I a nonzero ideal of R. Suppose that (F, f) is a multiplicative (generalized)-derivation of R. If $f(x)F(y) \pm xy \in Z(R)$ for all $x, y \in I$, then f is commuting on I.*

Proof. We consider

$$f(x)F(y) \pm xy \in Z(R) \text{ for all } x, y \in I. \tag{14}$$

Replace y by yz in (14), we get

$$(f(x)F(y) \pm xy)z + f(x)yf(z) \in Z(R) \text{ for all } x, y, z \in I. \tag{15}$$

On commuting with z in (15), we obtain $[f(x)yf(z), z] = 0$ for all $x, y, z \in I$. In particular, put $x = z$, we get $[f(z)yf(z), z]$ for all $y, z \in I$. It coincides with (3), hence Theorem 1. insures the conclusion.

Theorem 3. *Let R be a semiprime ring and I a nonzero ideal of R. Suppose that (F, f) is a multiplicative (generalized)-derivation of R. If $f(x)F(y) \pm (x \circ y) \in Z(R)$for all $x, y \in I$, then f is commuting on I and I is commutative.*

Proof. We consider

$$f(x)F(y) \pm (x \circ y) \in Z(R) \text{ for all } x, y \in I \tag{16}$$

Replace y by yz in (16) to obtain $(f(x)F(y) \pm (x \circ y))z + f(x)yf(z) \mp y[x, z] \in Z(R)$ for all $x, y, z \in I$. On commuting both sides by z, we get $[f(x)yf(z), z] \mp [y[z, x], z] = 0$ for all $x, y, z \in I$. It coincides with (2), hence Theorem 1. insure the conclusions.

Theorem 4. *Let R be a semiprime ring and I a nonzero ideal of R. Suppose that (F, f) is a multiplicative (generalized)-derivation of R. If $f(x)F(y) \pm [x, y] \in Z(R)$for all $x, y \in I$, then f is commuting on I and I is commutative.*

Proof. We consider

$$f(x)F(y) \pm [x, y] \in Z(R) \text{ for all } x, y \in I \tag{17}$$

Replace y by yz in (17) to obtain $(f(x)F(y) \pm [x, y])z + f(x)yf(z) \pm y[x, z] \in Z(R)$ for all $x, y, z \in I$. On commuting both sides by z, we have

$$[f(x)yf(z), z] \pm [y[x, z], z] = 0 \text{ for all } x, y, z \in I \tag{18}$$

Substituting $x = z$ and we get $[f(z)yf(z), z] = 0$ this is same as (3) so by theorem 1, we obtain $[f(z), z] = 0$ for all $z \in I$. Replace y by yz in (18), we get

$$[f(x)yzf(z), z] \pm [yz[x, z], z] = 0 \text{ for all } x, y, z \in I \tag{19}$$

Right multiply (18) by z and subtract (19) from it and we get $[f(x)y[f(z), z], z] \pm [y[[x, z], z], z] = 0$ for all $x, y, z \in I$. Using the fact that $I[f(z), z] = 0$ for all $z \in I$, we obtain $[y[[x, z], z], z] = 0$ for all $x, y, z \in I$. It coincides with (7), hence Theorem 1. insures the conclusion.

Corollary 5. *Let R be a semiprime ring. If (F, f) is a multiplicative (generalized) -derivation of R such that any one of the following*

i. $f(x)F(y) \pm [x, y] \in Z(R)$

ii. $f(x)F(y) \pm (x \circ y) \in Z(R)$

iii. $f(x)F(y) \pm yx \in Z(R)$

holds for all $x, y \in R$, then R is commutative.

Theorem 6. *Let R be a semiprime ring and I a nonzero left ideal of R. Suppose that (F, f) is a multiplicative (generalized)-derivation of R. If $F(xy) \pm F(x)F(y) \in Z(R)$ holds for all $x, y \in I$, then $I[f(z), z] = (0)$ for all $z \in I$.*

Proof. We consider

$$F(xy) \pm F(x)F(y) \in Z(R) \text{ for all } x, y, z \in I. \tag{20}$$

Replace y by yz in (20), we get $(F(xy) \pm F(x)F(y))z + xyf(z) \pm F(x)yf(z) \in Z(R)$ for all $x, y, z \in I$. On commuting with z and using (20), we obtain

$$[xyf(z), z] \pm [F(x)yf(z), z] = 0 \text{ for all } x, y, z \in I. \tag{21}$$

Replace x by xz in (21) to get

$$[xzyf(z), z] \pm [F(x)zyf(z), z] \pm [xf(z)yf(z), z] = 0 \text{ for all } x, y, z \in I. \tag{22}$$

Replace y by zy in (21) and subtract it from (22), we have

$$[xf(z)yf(z), z] = 0 \text{ for all } x, y, z \in I. \tag{23}$$

Substitute $f(z)x$ for x in (23), we get $f(z)[xf(z)yf(z), z] + [f(z), z]xf(z)yf(z) = 0$ for all $x, y, z \in I$. Relation (23) reduce it to

$$[f(z), z]xf(z)yf(z) = 0 \text{ for all } x, y, z \in I. \tag{24}$$

Replace x by xz in (24) and we get

$$[f(z), z]xzf(z)yf(z) = 0 \text{ for all } x, y, z \in I. \tag{25}$$

Replace y by yz in (24), we have

$$[f(z), z]xf(z)zyf(z) = 0 \text{ for all } x, y, z \in I \tag{26}$$

Subtract (25) from (26)to obtain $[f(z), z]x[f(z), z]yf(z) = 0$ for all $x, y, z \in I$. It implies that $(I[f(z), z])^3 = (0)$ for all $z \in I$. Hence, we conclude that $I[f(z), z] = (0)$ for all $z \in I$.

Corollary 7. Let R be a semiprime ring and (F, f) a multiplicative (generalized)-derivation of R. If $F(xy) \pm F(x)F(y) \in Z(R)$ holds for all $x, y \in R$, then f is a commuting map.

Theorem 8. *Let R be a semiprime ring and I a nonzero left ideal of R. Suppose that (F, f) is a multiplicative (generalized)-derivation of R. If $F[x, y] \pm (x \circ y) \in Z(R)$ for all $x, y \in I$, then $I[x, f(x)] = (0)$ or $I[x, f(Z(R))] = (0)$ for all $x \in I$.*

Proof. We consider

$$F[x, y] \pm (x \circ y) \in Z(R) \text{ for all } x, y \in I. \tag{27}$$

If $Z(R) = (0)$ then

$$F[x, y] \pm (x \circ y) = 0 \text{ for all } x, y \in I. \tag{28}$$

Replace y by yx in (28) and we get $(F[x, y] \pm (x \circ y))x + [x, y]f(x) = 0$ for all $x, y \in I$. It reduces to

$$[x, y]f(x) = 0 \text{ for all } x, y \in I \tag{29}$$

Replace y by $f(x)y$ in (29), we have $f(x)[x, y]f(x) + [x, f(x)]yf(x) = 0$ for all $x, y \in I$. Using (29), we obtain

$$[x, f(x)]yf(x) = 0 \text{ for all } x, y \in I. \tag{30}$$

Replace y by yx in (30) and we get

$$[x, f(x)]yxf(x) = 0 \text{ for all } x, y \in I. \tag{31}$$

Right multiply (30) by x and subtract from (31), to obtain $[x, f(x)]y[x, f(x)] = 0$ for all $x, y \in I$. Since I is a left ideal of R, so we have $y[x, f(x)]Ry[x, f(x)] = (0)$ for all $x, y \in I$. Semiprimeness of R yields that $y[x, f(x)] = 0$ for all $x, y \in I$. Hence, we conclude that $I[x, f(x)] = (0)$ for all $x \in I$.

If $Z(R) \neq (0)$ then there exist $0 \neq t \in Z(R)$. Replace y by yt in (27), we get $(F[x, y] \pm (x \circ y))t + [x, y]f(t) \in Z(R)$ for all $x, y \in I$. Using (27), we get $[x, y]f(t) \in Z(R)$ for all $x, y \in I$. On commuting with $r \in R$, we have

$$[[x, y]f(t), r] = 0 \text{ for all } x, y \in I \text{ and } r \in R. \tag{32}$$

Replace x by yx in (32), we get $[y[x, y]f(t), r] = y[[x, y]f(t), r] + [y, r][x, y]f(t) = 0$ for all $x, y \in I$ and $r \in R$. Using (32), we obtain

$$[y, r][x, y]f(t) = 0 \text{ for all } x, y \in I \text{ and } r \in R. \tag{33}$$

Replace r by pr in (33) where $p \in R$, we get $p[y, r][x, y]f(t) + [y, p]r[x, y]f(t) = 0$ for all $x, y \in I$ and $r, p \in R$. Using (33), we get $[y, p]r[x, y]f(t) = 0$ for all $x, y \in I$ and $r, p \in R$. Substitute $f(t)r$ for r and in particular, we get $[x, y]f(t)R[x, y]f(t) = (0)$ for all $x, y \in I$. Semiprimeness of R implies that

$$[x, y]f(t) = 0 \text{ for all } x, y \in I. \tag{34}$$

Replace y by $f(t)y$ in (34), we get $f(t)[x, y]f(t)+[x, f(t)]yf(t) = 0$ for all $x, y \in I$. Equation (34) forces that $[x, f(t)]yf(t) = 0$ for all $x, y \in I$. It implies $[x, f(t)]y[x, f(t)] = 0$ for all $x, y \in I$. Since I is a left ideal of R so we have $y[x, f(t)]Ry[x, f(t)] = (0)$ for all $x, y \in I$. Semiprimeness of R yields that $y[x, f(t)] = 0$ for all $x, y \in I$ and $t \in Z(R)$. Hence, we conclude that $I[x, f(Z(R))] = (0)$ for all $x \in I$.

Theorem 9. *Let R be a semiprime ring and I a nonzero left ideal of R. Suppose that (F, f) is a multiplicative (generalized)-derivation of R. If $F(x \circ y) \pm [x, y] \in Z(R)$ for all $x, y \in I$, then $I[x, f(x)] = (0)$ or $I[x, f(Z(R))] = (0)$ for all $x \in I$.*

Proof. We consider

$$F(x \circ y) \pm [x, y] \in Z(R) \text{ for all } x, y \in I. \tag{35}$$

If $Z(R) = (0)$ then

$$F(x \circ y) \pm [x, y] = 0 \text{ for all } x, y \in I. \tag{36}$$

Replace y by yx in (36), we get $(F(x \circ y) \pm [x, y])x + (x \circ y)f(x) = 0$ for all $x, y \in I$. Using (36) to obtain

$$(x \circ y)f(x) = 0 \text{ for all } x, y \in I \tag{37}$$

Replace y by $f(x)y$ in (37) and we get $f(x)(x \circ y)f(x) + [x, f(x)]yf(x) = 0$ for all $x, y \in I$. Relation (37) implies that

$$[x, f(x)]yf(x) = 0 \text{ for all } x, y \in I. \tag{38}$$

Replace y by yx in (38), we obtain

$$[x, f(x)]yxf(x) = 0 \text{ for all } x, y \in I. \tag{39}$$

Right multiply (38) by x and subtract from (39), we get $[x, f(x)]y[x, f(x)] = 0$ for all $x, y \in I$. Since I is a left ideal of R, so we have $y[x, f(x)]Ry[x, f(x)] = (0)$ for all $x, y \in I$. Semiprimeness of R yields that $y[x, f(x)] = 0$ for all $x, y \in I$. Hence, we conclude that $I[x, f(x)] = (0)$ for all $x \in I$.

If $Z(R) \neq (0)$ then there exist $0 \neq t \in Z(R)$. Replace y by yt in (27) to get $(F[x, y] \pm (x \circ y))t + (x \circ y)f(t) \in Z(R)$ for all $x, y \in I$. Using (27), we left with $(x \circ y)f(t) \in Z(R)$ for all $x, y \in I$. On commuting with $r \in R$, we obtain

$$[(x \circ y)f(t), r] = 0 \text{ for all } x, y \in I \text{ and } r \in R. \tag{40}$$

Replace y by xy in (40), we get $x[(x \circ y)f(t), r] + [x, r](x \circ y)f(t) = 0$ for all $x, y \in I$ and $r \in R$. Equation (40) reduce it to

$$[x, r](x \circ y)f(t) = 0 \text{ for all } x, y \in I \text{ and } r \in R. \tag{41}$$

Replace y by py in (41) where $p \in R$, we have $[x, r]p(x \circ y)f(t) + [x, r][x, p]yf(t) = 0$ for all $x, y \in I$ and $r, p \in R$. Using the fact that $(x \circ y)f(t) \in Z(R)$ for all $x, y \in I$, we get $[x, r](x \circ y)f(t)p + [x, r][x, p]yf(t) = 0$ for all $x, y \in I$ and $r, p \in R$. Using (41) to obtain

$$[x, r][x, p]yf(t) = 0 \text{ for all } x, y \in I \text{ and } r, p \in R. \tag{42}$$

Replacing r by sr where $s \in R$ in (42) and we have $s[x, r][x, p]yf(t) + [x, s]r[x, p]yf(t) = 0$ for all $x, y \in I$ and $p, r, s \in R$. Using (42) to obtain

$$[x, s]r[x, p]yf(t) = 0 \text{ for all } x, y \in I \text{ and } p, r, s \in R. \tag{43}$$

Replace y by yx in (43), we get

$$[x, s]r[x, p]yxf(t) = 0 \text{ for all } x, y \in I \text{ and } p, r, s \in R. \tag{44}$$

Right multiply (43) by x and subtract from (44) to get $[x, s]r[x, p]y[x, f(t)] = 0$ for all $x, y \in I$ and $p, r, s \in I$. Replace r by ry and y by ry, we obtain $[x, s]ry[x, p]ry[x, f(t)] = 0$ for all $x, y \in I$ and $p, r, s \in I$. In particular, $[x, f(t)]ry[x, f(t)]ry[x, f(t)] = 0$ for all $x, y \in I$, $r \in I$ and $t \in Z(R)$. It implies $(Ry[x, f(Z(R))])^3 = (0)$ for all $x, y \in I$. But R has no nonzero nilpotent ideal, so we have $Ry[x, f(Z(R))] = (0)$ for all $x, y \in I$. Hence, we conclude that $I[x, f(Z(R))] = (0)$ for all $x \in I$.

Theorem 10. *Let R be a semiprime ring and I a nonzero left ideal of R. Suppose that (F, f) is a multiplicative (generalized)-derivation of R. If $F[x, y] \pm xy \in Z(R)$ holds for all $x, y \in I$, then $I[x, f(x)] = (0)$ or $I[x, f(Z(R))] = (0)$ for all $x \in I$.*

Proof. We consider

$$F[x, y] \pm xy \in Z(R) \text{ for all } x, y \in I. \tag{45}$$

If $Z(R) = (0)$ then it is easy to prove that $I[x, f(x)] = (0)$ for all $x \in I$.

If $Z(R) \neq (0)$ then there exist $0 \neq t \in Z(R)$. Replace y by yt in (45) to obtain $(F[x, y] \pm xy)t + [x, y]f(t) \in Z(R)$ for all $x, y \in I$. Using (45), we get $[x, y]f(t) \in Z(R)$ for all $x, y \in I$. On commuting with $r \in R$, we have $[[x, y]f(t), r] = 0$ for all $x, y \in I$ and $r \in R$. It coincides with (32), hence Theorem 9. insure the conclusions.

Theorem 11. *Let R be a semiprime ring and I a nonzero left ideal of R. Suppose that* (F, f) *is a multiplicative (generalized)-derivation of R. If* $F(x \circ y) \pm xy \in Z(R)$ *holds for all* $x, y \in I$*, then* $I[x, f(x)] = (0)$ *or* $I[x, f(Z(R))] = (0)$ *for all* $x \in I$.

Proof. We consider

$$F(x \circ y) \pm xy \in Z(R) \text{ for all } x, y \in I. \tag{46}$$

If $Z(R) = (0)$ then it is easy to prove that $I[x, f(x)] = (0)$ for all $x \in I$.

If $Z(R) \neq (0)$ then there exist $0 \neq t \in Z(R)$. Replace y by yt in (46) and we get $(F[x, y] \pm xy)t + (x \circ y)f(t) \in Z(R)$ for all $x, y \in I$. Using (46), we get $(x \circ y)f(t) \in Z(R)$ for all $x, y \in I$. On commuting with $r \in R$, we obtain $[(x \circ y)f(t), r] = 0$ for all $x, y \in I$ and $r \in R$. It coincides with (40), hence Theorem 10. insure the conclusions.

Theorem 12. *Let R be a semiprime ring and I a nonzero left ideal of R. Suppose that* (F, f) *is a multiplicative (generalized)-derivation of R. If* $F[x, y] \pm f(x) \circ y \in Z(R)$ *holds for all* $x, y \in I$*, then* $I[x, f(x)] = (0)$ *or* $I[x, f(Z(R))]$ *for all* $x \in I$.

Proof. We consider

$$F[x, y] \pm f(x) \circ y \in Z(R) \text{ for all } x, y \in I. \tag{47}$$

If $Z(R) = (0)$ then we have

$$F[x, y] \pm f(x) \circ y = 0 \text{ for all } x, y \in I. \tag{48}$$

Substitute yx for y in (48) to get $(F[x, y] \pm f(x) \circ y)x + [x, y]f(x) \mp y[f(x), x] = 0$ for all $x, y \in I$. By (48), it reduces to

$$[x, y]f(x) \mp y[f(x), x] = 0 \text{ for all } x, y \in I. \tag{49}$$

Replace y by $f(x)y$ in (49), we get

$$f(x)[x, y]f(x) + [x, f(x)]yf(x) \mp f(x)y[f(x), x] = 0 \text{ for all } x, y \in I. \tag{50}$$

Left multiply (49) by $f(x)$ and subtract from (50), we obtain $[x, f(x)]yf(x) = 0$ for all $x, y \in I$.. Since I is a left ideal in R, it implies that $y[x, f(x)]Ry[x, f(x)] = (0)$ for all $x, y \in I$. Semiprimeness of R yields that $y[x, f(x)] = 0$ for all $x, y \in I$. We conclude that $I[x, f(x)] = (0)$ for all $x \in I$.

If $Z(R) \neq (0)$ then there exist some $0 \neq t \in Z(R)$. Replace y by yt in (47), we get $(F[x, y] + f(x) \circ y)t + [x, y]f(t) \in Z(R)$ for all $x, y \in I$. Using (47) to obtain $[x, y]f(t) \in Z(R)$ for all $x, y \in I$. That is $[[x, y]f(t), r] = 0$ for all $x, y \in I$ and $r \in R$. It coincides with (32), hence Theorem 9. yields that $I[x, f(Z(R))] = (0)$ for all $x \in I$.

Theorem 13. *Let R be a semiprime ring and I a nonzero left ideal of R. Suppose that* (F, f) *is a multiplicative (generalized)-derivation of R. If* $F(x \circ y) \pm [f(x), y] \in Z(R)$ *holds for all* $x, y \in I$*, then* $I[x, f(x)] = (0)$ *or* $I[x, f(Z(R))] = (0)$*for all* $x \in I$.

Proof. We consider

$$F(x \circ y) \pm [f(x), y] \in Z(R) \text{ for all } x, y \in I. \tag{51}$$

If $Z(R) = (0)$ then we have

$$F(x \circ y) \pm [f(x), y] = 0 \text{ for all } x, y \in I. \tag{52}$$

Replace y by yx in (52) and we obtain $F(x \circ y)x + (x \circ y)f(x) \pm [f(x), y]x \pm y[f(x), x] = 0$ for all $x, y \in I$. Using (52), we left with

$$(x \circ y)f(x) \pm y[f(x), x] = 0 \text{ for all } x, y \in I. \tag{53}$$

Replace y by $f(x)y$ in (53) and we get

$$f(x)(x \circ y)f(x) + [x, f(x)]yf(x) \pm f(x)y[f(x), x] = 0 \text{ for all } x, y \in I. \tag{54}$$

Left multiply (53) by $f(x)$ and subtract it from (54), we obtain $[x, f(x)]yf(x) = 0$ for all $x, y \in I$. It implies that $[x, f(x)]y[x, f(x)] = (0)$ for all $x, y \in I$. Semiprimeness of R yields that $y[x, f(x)] = (0)$ for all $x, y \in I$. We conclude that $I[x, f(x)] = (0)$ for all $x \in I$.

If If $Z(R) \neq (0)$ then there exist some $0 \neq t \in Z(R)$. Replace y by yt in (51), we get $(F(x \circ y) + [f(x), y])t + (x \circ y)f(t) \in Z(R)$ for all $x, y \in I$. Using (51), we obtain $(x \circ y)f(t) \in Z(R)$ for all $x, y \in I$. That is $[(x \circ y)f(t), r] = 0$ for all $x, y \in I$ and $r \in R$. It coincides with (40), hence Theorem 10. yields that $I[x, f(Z(R))] = (0)$ for all $x \in I$.

Corollary 14. Let R be a semi-prime ring. Suppose that (F, f) is a multiplicative (generalized)-derivation of R. If any one of the following

i. $F[x, y] \pm (x \circ y) \in Z(R)$

ii. $F(x \circ y) \pm [x, y] \in Z(R)$

iii. $F[x, y] \pm xy \in Z(R)$

iv. $F(x \circ y) \pm xy \in Z(R)$

v. $F[x, y] \pm (f(x) \circ y) \in Z(R)$

vi. $F(x \circ y) \pm [f(x), y] \in Z(R)$

holds for all $x, y \in R$, then either f is commuting map or $f(Z(R)) \subseteq Z(R)$.

3. Examples

In this section, we build a few examples to show that the condition of semiprimeness in our results is not superfluous. **Example 1.** Consider

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in S \right\},$$

where S is any arbitrary ring.

We define maps $F, f : R \to R$ by

$$F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & bc \\ 0 & 0 & 0 \end{pmatrix}, f\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & c^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

it is verified that F is a multiplicative (generalized)-derivations associated with the maps f and it is easy to see that the identities $f(x)F(y) \pm [x, y] \in Z(R)$,$f(x)F(y) \pm (x \circ y) \in Z(R)$ and $f(x)F(y) \pm yx \in Z(R)$ are satisfied for all $x, y \in R$. Here R is not a semiprime ring because

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (0).$$

Note that R is not commutative. Hence, the condition of semi-primeness in Corollary 5. can not be omitted.

Example 2. Consider $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{Z}_2 \right\}$ be a ring over integers modulo 2 and let $I = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{Z}_2 \right\}$, be a left ideal in R. We define maps $F, f : R \to R$ by

$$F\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & nb \\ 0 & 0 \end{pmatrix}, f\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & (n-1)b \\ 0 & 0 \end{pmatrix},$$

where n is any positive integer. Then it is verified that F is a multiplicative (generalized)-derivations associated with the maps f and it is easy to see that the identities $F(xy) \pm F(x)F(y) \in Z(R)$ are satisfied for all $x, y \in I$. Here R is not a semiprime ring, but observe that $I[f(x), x] \neq (0)$ for all $x \in I$. Hence, the condition of semiprimeness in Theorem 6. is essential.

Example 3. Consider $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ stands for the ring of integers. We define maps $F, f : R \to R$ by

$$F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & bc \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, f\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b & a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then it is verified that F is a multiplicative (generalized)-derivations associated with the maps f and it is easy to see that the identities $F[x, y]\pm(x\circ y) \in Z(R)$,$F(x\circ y)\pm[x, y] \in Z(R)$,$F[x, y]\pm xy \in Z(R)$, $F(x\circ y)\pm xy \in Z(R)$, $F[x, y]\pm(f(x)\circ y) \in Z(R)$ and $F(x \circ y) \pm [f(x), y] \in Z(R)$ are satisfied for all $x, y \in R$. Clearly, R is not a semiprime ring. Note that f is neither commuting on R nor maps $Z(R)$ into $Z(R)$. Hence, the condition of semiprimeness in Corollary 14. can not be removed.

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Products of Reflections and Triangularization of Bilinear Forms

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Abstract

The present article is motivated by the theorem of Cartan-Dieudonné which states that every orthogonal transformation is a product of reflections. Its purpose is to determine, for each orthogonal transformation, the minimal number of factors in a decomposition into a product of reflections, and to propose an effective algorithm giving such a decomposition. With the orthogonal transformations g of a quadratic space (V, q), it associates couples (S, ϕ) where S is a subspace of V, and ϕ an non-degenerate bilinear form on S such that $\phi(y, y) = q(y)$ for every y in S. In general, the minimal decompositions of g into a product of reflections correspond to the bases of S in which the matrix of ϕ is lower triangular. Therefore, we need an algorithm of triangularization of bilinear forms. Affine isometries are also taken into consideration.

Keywords: orthogonal transformations, bilinear forms.

Let V be a vector space of finite dimension n over a field K, q a quadratic form on V which is momentarily assumed to be non-degenerate, and $\mathrm{O}(V, q)$ the group of its orthogonal transformations. Since the characteristic of K may be 2, the associated bilinear form b_q is defined in this way:

$$\forall x,\ y \in V, \qquad \mathrm{b}_q(x, y) = q(x + y) - q(x) - q(y)\,;$$

thus $\mathrm{b}_q(x, x) = 2q(x)$ for all x. Every non-isotropic vector $v \in V$ determines a *reflection* $\mathrm{R}(v)$:

$$\forall x \in V, \qquad \mathrm{R}(v)(x) = x - \frac{\mathrm{b}_q(x, v)}{q(v)}\, v\,.$$

The theorem of Cartan-Dieudonné (see (Dieudonné, 1958)) states that every $g \in \mathrm{O}(V, q)$ is a product of reflections, where the number of reflections is $\leq n$. Nevertheless, there are exceptions when the field K is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. When q is anisotropic (for instance when $K = \mathbb{R}$ and q is euclidean), it is easy to prove that the minimal number of reflections for a particular g is the dimension of $\mathrm{im}(g - \mathbf{1})$, the image of $g - \mathbf{1}_V$ (where $\mathbf{1}_V$ is the identity mapping of V, also denoted by $\mathbf{1}$ if this short notation is clear enough). The determination of this minimal number is much more difficult when there are non-zero isotropic vectors x (such that $q(x) = 0$). Here this minimal number proves to be the dimension of $\mathrm{im}(g-\mathbf{1})$ when it is not totally isotropic, and $\dim(\mathrm{im}(g - \mathbf{1})) + 2$ when it is totally isotropic; because of the above mentioned exceptions, K is assumed not to be isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

I first tackled this problem with the Clifford algebra $\mathrm{Cl}(V, q)$ (the associative and unital algebra generated by the elements x of V with the relations $x^2 = q(x)$); but in this article, contrary to (Helmstetter 2017), I present only the part of my research that can be explained without mentioning Clifford algebras. Nevertheless, the Clifford algebras suggested new points of view and new definitions that I shall explain at once. Firstly, the hypothesis that q is non-degenerate has been removed, because it causes a dreadful loss of effectiveness in the treatment of Clifford algebras. We must pay attention to $\ker(\mathrm{b}_q)$, the subspace of all $x \in V$ such that $\mathrm{b}_q(x, y) = 0$ for all $y \in V$, and to $\ker(q)$, the subspace of all $x \in \ker(\mathrm{b}_q)$ such that $q(x) = 0$; since $\mathrm{b}_q(x, x) = 2q(x)$, the equality $\ker(q) = \ker(\mathrm{b}_q)$ holds whenever the characteristic of K is $\neq 2$. When $\ker(q) \neq \ker(\mathrm{b}_q)$, q is said to be *defective*. Secondly, we must distinguish $\mathrm{Iso}(V, q)$, the group of *isometries* of (V, q), and its subgroup $\mathrm{O}(V, q)$, the group of *orthogonal transformations*; a linear transformation g of V is an isometry if (by definition) $q(g(x)) = q(x)$ for all $x \in V$; an isometry g is an orthogonal transformation if $\ker(g-\mathbf{1}) \supset \ker(\mathrm{b}_q)$. For instance, every reflection $\mathrm{R}(v)$ is an orthogonal transformation, and $\mathrm{im}(\mathrm{R}(v)-\mathbf{1})$ is the line spanned by v (except when q is defective and $v \in \ker(\mathrm{b}_q)$). A linear transformation g is an isometry if and only if it extends to an automorphism of $\mathrm{Cl}(V, q)$; it is an orthogonal transformation if and only if it extends to a twisted inner automorphism of $\mathrm{Cl}(V, q)$ according to this definition which involves the parity gradation of $\mathrm{Cl}(V, q)$: the twisted inner automorphism determined by an invertible, even or odd element $a \in \mathrm{Cl}(V, q)$ is $b \longmapsto aba^{-1}$ if a or b is even, $b \longmapsto -aba^{-1}$ if a and b are odd. Thirdly, every orthogonal transformation g can be determined by a couple (S, ϕ) where S is a subspace of V containing $\mathrm{im}(g - \mathbf{1})$, and ϕ is a non-degenerate bilinear form on S such that $\phi(y, y) = q(y)$ for all $y \in S$. Since we shall meet plenty of such couples

(S,ϕ), I propose to call them *transformers* of (V,q). When q is non-degenerate (in other words, $\ker(\mathrm{b}_q) = 0$), then g admits only one transformer (S,ϕ), and $S = \mathrm{im}(g-\mathbf{1})$. But in other cases, there may be plenty of transformers over each $g \in \mathrm{O}(V,q)$, sometimes of various dimensions; therefore, the determination of their minimal dimension is important:

$$\text{minimal } \dim(S) = \dim(\mathrm{im}(g-\mathbf{1})) + \dim(\mathrm{im}(g-\mathbf{1}) \cap \ker(q))\,.$$

This minimal dimension s gives the minimal number of factors in a decomposition of g into a product of reflections; it is s when q admits a minimal-dimensional transformer (S,ϕ) that is not totally isotropic; in the other cases, it is $s+2$ (only $s+1$ if q is defective).

The quadratic space (V,q) is said to be embedded in $(W,\tilde{q})$ if there is an injective linear mapping $f : V \to W$ such that $\tilde{q}(f(x)) = q(x)$ for all x; for convenience, V will be treated as a subspace of W, and $\tilde{q}$ as an extension of q. Such an embedding is especially interesting if $\tilde{q}$ is non-degenerate; indeed, we shall realize that an isometry g of (V,q) is an orthogonal transformation if and only if it extends to an orthogonal transformation $\tilde{g}$ of $(W,\tilde{q})$ such that $\mathrm{im}(\tilde{g}-\mathbf{1}_W) \subset V$; in other words, $\mathrm{O}(V,q)$ is the image of the subgroup of all $\tilde{g} \in \mathrm{O}(W,\tilde{q})$ such that $\mathrm{im}(\tilde{g}-\mathbf{1}_W) \subset V$; the image of each $\tilde{g}$ is its restriction to V; moreover, the suitable extensions $\tilde{g}$ of g are in bijection with the transformers (S,ϕ) over g.

Example. When q is the null quadratic form on V, then $\mathrm{Iso}(V,q)$ is the linear group $\mathrm{GL}(V)$ whereas $\mathrm{O}(V,q)$ is the trivial group $\{\mathbf{1}_V\}$. There is a non-degenerate embedding $(W,\tilde{q})$ where W is the direct sum of V and the dual space V^*, and where $\tilde{q}(x,\ell) = \ell(x)$ for all $x \in V$ and all $\ell \in V^*$. Every $g \in \mathrm{GL}(V)$ has extensions $\tilde{g}$ in $\mathrm{O}(W,\tilde{q})$, and there is a canonical extension $(x,\ell) \longmapsto (g(x),\ \ell\circ g^{-1})$; but $\mathrm{im}(\tilde{g}-\mathbf{1}_W)$ is not contained in V if $g \neq \mathbf{1}_V$; indeed, Lemma 1.2 (here below) shows that the conditions $\mathrm{im}(\tilde{g}-\mathbf{1}_W) \subset V$ is equivalent to $\ker(\tilde{g}-\mathbf{1}_W) \supset V$. When $g = \mathbf{1}_V$, the extensions $\tilde{g}$ are well known: see (Chevalley, 1954), section III.1.7; they are in bijection with the elements ω of $\bigwedge^2(V)$; if $\omega = \sum_{i=1}^r y_i \wedge z_i$, the associated orthogonal transformation $\mathrm{F}(\omega)$ maps each (x,ℓ) to $(x + \sum_i(\ell(y_i)\,z_i - \ell(z_i)\,y_i),\ \ell)$. Thus $\mathrm{F}(\omega)\circ\mathrm{F}(\omega') = \mathrm{F}(\omega+\omega')$. The calculation of the transformer (S,ϕ) associated with $\mathrm{F}(\omega)$ (according to Theorem 2.2 below) is easy when $(y_1,z_1,y_2,z_2,\ldots,y_r,z_r)$ is linearly independant: S is the subspace with basis $(y_1,z_1,\ldots,y_r,z_r)$, and ϕ is the alternate bilinear form on S such that $\phi(y_i,z_i) = 1$, $\phi(y_i,z_j) = 0$ if $i \neq j$, and $\phi(y_i,y_j) = \phi(z_i,z_j) = 0$ for all i and j. Thus we obtain a bijection between the elements of $\bigwedge^2(V)$ and the transformers (S,ϕ) of $(V,0)$.

Let us suppose that the orthogonal transformation g is a product of reflections $\mathrm{R}(v_1)\mathrm{R}(v_2)\cdots\mathrm{R}(v_s)$ involving s linearly independent vectors; then g admits the transformer (S,ϕ) where S is the subspace with basis $(v_1,\ldots,v_s)$, and where ϕ has a lower triangular matrix in this basis; in other words, $\phi(v_i,v_j) = 0$ whenever $i<j$; since $\phi(y,y) = q(y)$ for all $y \in S$, this property completely determines ϕ. Conversely, if (S,ϕ) is a transformer for g, and if the matrix of ϕ is lower triangular in some basis $(v_1,\ldots,v_s)$ of S, then $g = \mathrm{R}(v_1)\cdots\mathrm{R}(v_s)$. Thus we are led to the problem which shall be the subject of the second part of this article: if ϕ is a bilinear form on a vector space S (of finite dimension s), are there bases of S where the matrix of ϕ is lower triangular, and how can we calculate one of them?

Although every transformer (S,ϕ) involves a non-degenerate bilinear form ϕ, I will solve the problem of triangularization even when ϕ is degenerate; in the frame of Clifford algebras, there are at least two problems that require triangularisation even for degenerate bilinear forms. When ϕ is a non-zero alternate bilinear form, its matrix is alternate in every basis of S; therefore, it cannot be triangularized. All other bilinear forms can be triangularized, except when K is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Bilinear forms over $\mathbb{Z}/2\mathbb{Z}$ are outside the scope of this article; here, I do not more than showing (just below) a bilinear form over $\mathbb{Z}/2\mathbb{Z}$ that cannot be triangularized although it is not alternate. I shall present an algorithm of triangularization where every phase is almost trivial, except the "correction procedure"; this procedure is the only phase that requires K not to be isomorphic to $\mathbb{Z}/2\mathbb{Z}$; therefore, the presence of this unpleasant procedure is not the result of a clumsiness.

Example. Here, exceptionally, K is the field $\mathbb{Z}/2\mathbb{Z}$. Let us consider the following non-degenerate bilinear form ϕ on K^3:

$$\phi((\xi_1,\xi_2,\xi_3),(\zeta_1,\zeta_2,\zeta_3)) = (\xi_1\zeta_2 - \xi_2\zeta_1) + (\xi_2+\xi_3)\zeta_3\,.$$

If the matrix of ϕ is triangular in a basis (v_1,v_2,v_3), then $\phi(v_1,v_1)$, $\phi(v_2,v_2)$ and $\phi(v_3,v_3)$ are all $\neq 0$ because ϕ is non-degenerate. Unfortunately, only two vectors of K^3 are not isotropic for the quadratic form $v \longmapsto \phi(v,v)$: $(0,0,1)$ and $(1,0,1)$. Therefore, ϕ cannot be triangularized.

1. Preliminary Lemmas

The first lemma is useful only in characteristic 2.

Lemma 1.1. *For every* $g \in \mathrm{Iso}(V,q)$ *we have* $\mathrm{im}(g-\mathbf{1}) \cap \ker(\mathrm{b}_q) \subset \ker(q)$*; in other words,* $\mathrm{im}(g-\mathbf{1}) \cap \ker(\mathrm{b}_q) = \mathrm{im}(g-\mathbf{1}) \cap \ker(q)$.

Proof. If $g(x)-x$ is in $\ker(\mathrm{b}_q)$, then

$$q(x) = q(g(x)) = q(x) + q(g(x)-x) + \mathrm{b}_q(x,\ g(x)-x) = q(x) + q(g(x)-x),$$

whence $q(g(x) - x) = 0$. □

Lemma 1.1 implies that $\mathrm{O}(V, q) = \mathrm{Iso}(V, q)$ if and only if $\ker(q) = 0$.

For every subspace U of V, $U^\perp$ is the subspace of all $x \in V$ such that $\mathrm{b}_q(x, u) = 0$ for all $u \in U$.

Lemma 1.2. *For every* $g \in \mathrm{Iso}(V, q)$*, the subspaces* $\ker(g - \mathbf{1})$ *and* $\mathrm{im}(g - \mathbf{1})$ *are orthogonal. When* $\ker(q) = 0$*, then* $\ker(g - \mathbf{1}) = (\mathrm{im}(g - \mathbf{1}))^\perp$.

Proof. For all $x, y \in V$ we have

$$\mathrm{b}_q(x,\ g(y) - y) = -\mathrm{b}_q(g(x) - x,\ g(y))\,;$$

therefore, every x in $\ker(g - \mathbf{1})$ is orthogonal to every $g(y) - y$ in $\mathrm{im}(g - \mathbf{1})$. Conversely, if x is orthogonal to all $g(y) - y$, then $g(x) - x$ is in $\ker(\mathrm{b}_q)$, therefore in $\ker(q)$; and $x \in \ker(g - \mathbf{1})$ if $\ker(q) = 0$. □

When q is non-degenerate, the orthogonal group $\mathrm{O}(V, q)$ contains a normal subgroup $\mathrm{SO}(V, q)$ of index 2 which no reflection $\mathrm{R}(v)$ can belong to. The same holds true when q is degenerate but non-defective; indeed, q induces a non-degenerate quadratic form q'' on the quotient $V'' = V/\ker(q)$, every $g \in \mathrm{O}(V, q)$ gives a transformation $g'' \in \mathrm{O}(V'', q'')$, and $\mathrm{SO}(V, q)$ is the inverse image of $\mathrm{SO}(V'', q'')$ by the homomorphism $g \longmapsto g''$. If g is a product of reflections, the parity of the number of reflections depends on whether g is, or not, in the subgroup $\mathrm{SO}(V, q)$. All this is null and void when q is defective; in this case, $\ker(\mathrm{b}_q)$ contains vectors v such that $q(v) \neq 0$ and $\mathrm{R}(v) = \mathbf{1}_V$.

Now we consider a bilinear form ϕ on some vector space S, and we define the quadratic form q by $q(y) = \phi(y, y)$ for all $y \in S$. Consequently,

$$\forall x,\ y \in S, \quad \phi(x, y) + \phi(y, x) = \mathrm{b}_q(x, y)\,. \tag{1.1}$$

Let $\mathrm{RKer}(\phi)$ (resp. $\mathrm{LKer}(\phi)$) be the subspace of all $x \in S$ such that $\phi(v, x) = 0$ (resp. $\phi(x, v) = 0$) for all $v \in S$. If U is a subspace of S, we denote by $\mathrm{R}^\perp_\phi(U)$ (resp. $\mathrm{L}^\perp_\phi(U)$) the subspace of all $x \in S$ such that $\phi(u, x) = 0$ (resp. $\phi(x, u) = 0$) for all $u \in U$. When $U \subset \ker(\mathrm{b}_q)$, then $\mathrm{R}^\perp_\phi(U) = \mathrm{L}^\perp_\phi(U)$, and the notation $\mathrm{LR}^\perp_\phi(U)$ is allowed.

Lemma 1.3. *Let* U_1 *and* U_3 *be two subspaces of* S *such that* $\phi(U_1, U_3) = 0$ *and such that the restrictions of* ϕ *to* U_1 *and* U_3 *are non-degenerate. Then we have* $S = U_1 \oplus U_2 \oplus U_3$ *if* $U_2 = \mathrm{R}^\perp_\phi(U_1) \cap \mathrm{L}^\perp_\phi(U_3)$.

Proof. For every $x \in S$, there is a unique $x_1 \in U_1$ (resp. $x_3 \in U_3$) such that $\phi(u, x) = \phi(u, x_1)$ for all $u \in U_1$ (resp. $\phi(x, u) = \phi(x_3, u)$ for all $u \in U_3$). If we set $p_1(x) = x_1$ and $p_3(x) = x_3$, then p_1 and p_3 are projectors such that $\mathrm{im}(p_1) = U_1$, $\ker(p_1) = \mathrm{R}^\perp_\phi(U_1)$, $\mathrm{im}(p_3) = U_3$, $\ker(p_3) = \mathrm{L}^\perp_\phi(U_3)$. Since $\phi(U_1, U_3) = 0$, we have $p_1 p_3 = p_3 p_1 = 0$. Thus, if we set $p_2 = \mathbf{1} - p_1 - p_3$, we obtain a projector on $\ker(p_1) \cap \ker(p_3) = U_2$. □.

Lemma 1.3 can be applied when $U_1 = 0$ or $U_3 = 0$, because the unique bilinear form on $\{0\}$ is non-degenerate.

The next lemma, motivated by the frequent presence of $g - \mathbf{1}$, does not require V to be a vector space; it holds true already for an additive group.

Lemma 1.4. *Let* g_1 *and* g_2 *be homomorphisms from an additive group* V *into itself, and* $g = g_1 g_2$ *their product. Let us consider these four assertions:*

$$\begin{aligned}
(im): &\quad \mathrm{im}(g_1 - \mathbf{1}) \cap \mathrm{im}(g_2 - \mathbf{1}) = 0\,;\\
(Im): &\quad \mathrm{im}(g_1 - \mathbf{1}) + \mathrm{im}(g_2 - \mathbf{1}) = \mathrm{im}(g - \mathbf{1})\,;\\
(ker): &\quad \ker(g_1 - \mathbf{1}) + \ker(g_2 - \mathbf{1}) = V\,;\\
(Ker): &\quad \ker(g_1 - \mathbf{1}) \cap \ker(g_2 - \mathbf{1}) = \ker(g - \mathbf{1})\,.
\end{aligned}$$

The following four implications hold true:

$$(im) \Rightarrow (Ker)\,, \qquad (ker) \Rightarrow (Im)\,; \tag{1.2}$$

$$(im)\,\&\,(Im) \Longleftrightarrow (ker)\,\&\,(Ker)\,. \tag{1.3}$$

Proof. I will prove only (1.2) because we shall never use (1.3) which is mentioned here only because it would be a pity to mutilate Lemma 1.4; yet the proof of (1.3) is more difficult. The two inclusions

$$\mathrm{im}(g_1 - \mathbf{1}) + \mathrm{im}(g_2 - \mathbf{1}) \supset \mathrm{im}(g - \mathbf{1}) \qquad \text{and} \qquad \ker(g_1 - \mathbf{1}) \cap \ker(g_2 - \mathbf{1}) \subset \ker(g - \mathbf{1})$$

are obvious consequences of

$$g - \mathbf{1} = (g_1 - \mathbf{1})\, g_2 + (g_2 - \mathbf{1}) = g_1\, (g_2 - \mathbf{1}) + (g_1 - \mathbf{1})\,.$$

Let us prove $(im) \Rightarrow (Ker)$. If (im) is true and $g(x) = x$, then $(g_1 - \mathbf{1})g_2(x) = (g_2 - \mathbf{1})(x) = 0$, whence $g_2(x) = x = g_1(x)$; this means that (Ker) is true. Now let us prove that (ker) implies $\mathrm{im}(g_1 - \mathbf{1}) \subset \mathrm{im}(g - \mathbf{1})$; since $\mathrm{im}(g_2 - \mathbf{1}) \subset \mathrm{im}(g - \mathbf{1})$ for the same reasons, (Im) follows. Let us consider $y = (g_1 - \mathbf{1})(x)$ and let us write $x = x_1 + x_2$ where $g_1(x_1) = x_1$ and $g_2(x_2) = x_2$; thus $y = (g_1 - \mathbf{1})(x_2) = g_1(g_2 - \mathbf{1})(x_2) + (g_1 - \mathbf{1})(x_2) = (g - \mathbf{1})(x_2)$. □

Remark. When $\dim(V)$ is infinite, which properties of an isometry g ensure that it extends to a twisted inner automorphism of $\mathrm{Cl}(V, q)$? The necessary condition $\ker(g - \mathbf{1}) \supset \ker(\mathrm{b}_q)$ is no longer sufficient. Indeed, an isometry g extends to a twisted inner automorphism (and is called an orthogonal transformation) if and only if the codimension of $\ker(g - \mathbf{1})$ is finite, and if $\ker(g - \mathbf{1})$ is orthogonally closed according to this definition: a subspace U of V is orthogonally closed if $U^{\perp\perp} = U$. I recall that $U^{\perp\perp} \supset U$ and $U^{\perp\perp\perp} = U^{\perp}$ for every subspace U. When the codimension of $\ker(\mathrm{b}_q)$ is infinite, the property $\ker(g - \mathbf{1}) \supset \ker(\mathrm{b}_q)$ is much weaker. When $\ker(q) = 0$, then $\ker(g - \mathbf{1})$ is orthogonally closed for every isometry g because Lemma 1.2 is always valid. But if $\ker(q)$ contains a vector $u \neq 0$, then every $\ell \in V^*$ determines an isometry $g : x \longmapsto x + \ell(x)\,u$ such that $\ker(g - \mathbf{1}) = \ker(\ell)$; and g is an orthogonal transformation if and only if there is $v \in V$ such that $\ell(x) = \mathrm{b}_q(v, x)$ for all $x \in V$; even when $\ker(\ell) \supset \ker(\mathrm{b}_q)$, the existence of v is exceptional. Besides, for every orthogonal transformation g, there is an orthogonal decompostion $V = V_1 \oplus V_2$ such that $\dim(V_1)$ is finite, $\mathrm{im}(g - \mathbf{1}) \subset V_1$ and $\ker(g - \mathbf{1}) \supset V_2$; it reduces the study of g to the finite-dimensional case. Nothing interesting will occur as long as no other concept and no other hypothesis (for instance, the presence of a topology) is introduced.

2. The Main Theorems for Transformers

A transformer of (V, q) is a couple (S, ϕ) where ϕ is a non-degenerate bilinear form on a subspace S of V, and satisfies the condition $\phi(y, y) = q(y)$ for all $y \in S$. The following two theorems justify this definition.

Theorem 2.1. *Let (S, ϕ) be a transformer of (V, q). There is a unique linear endomorphism g of V such that* $\mathrm{im}(g - \mathbf{1}) \subset S$, *and such that*

$$\forall x \in V,\ \forall y \in S, \quad \phi(g(x) - x,\ y) = -\mathrm{b}_q(x, y)\,; \tag{2.1}$$

it is an orthogonal transformation of (V, q). Moreover,

$$\ker(g - \mathbf{1}) = S^{\perp}, \tag{2.2}$$

$$\mathrm{im}(g - \mathbf{1}) = \mathrm{LR}_{\phi}^{\perp}(S \cap \ker(\mathrm{b}_q))\,; \tag{2.3}$$

$$\dim(S) \ \geq\ \dim(\mathrm{im}(g - \mathbf{1})) + \dim(\mathrm{im}(g - \mathbf{1}) \cap \ker(q))\,; \tag{2.4}$$

$$\forall y,\, z \in S, \quad \phi(g(y), g(z)) = \phi(y, z)\,. \tag{2.5}$$

The reverse transformer $(S, \phi^{\dagger})$, where $\phi^{\dagger}$ is defined by $\phi^{\dagger}(x, y) = \phi(y, x)$, gives the inverse transformation g^{-1}.

Proof. Since ϕ is non-degenerate, it is clear that (2.1) determines an endomorphism g. Every $x \in \ker(g)$ must be in S, and $\phi(x, y) = \mathrm{b}_q(x, y)$ for all $y \in S$, whence $\phi(y, x) = 0$ because of (1.1), and $x = 0$ since ϕ is non-degenerate. Therefore, g is bijective. Let us prove that it is an isometry; for all $x \in V$, we have $g(x) = x + (g(x) - x)$, whence

$$\begin{aligned} q(g(x)) - q(x) &= q(g(x) - x) + \mathrm{b}_q(x,\ g(x) - x) \\ &= q(g(x) - x) - \phi(g(x) - x,\ g(x) - x) \ = q(y) - \phi(y, y) \quad \text{if } y = g(x) - x\,; \end{aligned}$$

thus $q(g(x)) = q(x)$ as expected. From (2.1) we deduce that $g(x) - x = 0$ if and only if $x \in S^{\perp}$; consequently, (2.2) holds true, and g is an orthogonal transformation. If ℓ is a linear form on S, there is $x \in V$ such that $\ell(y) = -\mathrm{b}_q(x, y)$ for all $y \in S$ if and only if ℓ vanishes on $S \cap \ker(\mathrm{b}_q)$. On another side, a vector z of S belongs to $\mathrm{im}(g - \mathbf{1})$ if and only if the linear form $y \longmapsto \phi(z, y)$ is equal to $y \longmapsto -\mathrm{b}_q(x, y)$ for some $x \in V$; this occurs if and only if $z \in \mathrm{L}_{\phi}^{\perp}(S \cap \ker(\mathrm{b}_q))$; this proves (2.3). Since ϕ is non-degenerate,

$$\begin{aligned} \dim(S) &= \dim(S \cap \ker(\mathrm{b}_q)) + \dim(\mathrm{L}_{\phi}^{\perp}(S \cap \ker(\mathrm{b}_q))) \\ &\geq \dim(\mathrm{im}(g - \mathbf{1}) \cap \ker(q)) + \dim(\mathrm{im}(g - \mathbf{1}))\,, \end{aligned}$$

in accordance with (2.4). The fact that g^{-1} can be derived from $(S, \phi^{\dagger})$ is equivalent to the following fact:

$$\forall y \in S,\ \forall x \in V, \quad \phi(y,\ g(x) - x) = \mathrm{b}_q(y,\ g(x))\,; \tag{2.6}$$

this formula (2.7) is a consequence of (1.1) and (2.1):

$$\phi(y,\ g(x) - x) = \mathrm{b}_q(g(x) - x,\ y) - \phi(g(x) - x,\ y) = \mathrm{b}_q(g(x) - x,\ y) + \mathrm{b}_q(x, y) \ = \mathrm{b}_q(g(x),\ y)\,.$$

Finally, we derive (2.5) from (2.1) and (2.6); for all $y,\ z \in S$,

$$\phi(g(y), g(z)) - \phi(y, z) = \phi(g(y) - y,\ g(z)) + \phi(y,\ g(z) - z) = -\mathrm{b}_q(y, g(z)) + \mathrm{b}_q(y, g(z)) = 0.$$

The proof of Theorem 2.1 is complete. □

When q is non-degenerate, the equality (2.3) means that $\mathrm{im}(g-\mathbf{1}) = S$. A transformer (S, ϕ) gives the transformation $\mathbf{1}$ if and only if $S \subset \ker(\mathrm{b}_q)$. The trivial transformer $(0, 0)$ (on the null subspace $\{0\}$) always gives $\mathbf{1}$. Now we come to the reciprocal theorem.

Theorem 2.2. *Every $g \in \mathrm{O}(V, q)$ admits a transformer (S, ϕ) such that*

$$\dim(S) = \dim(\mathrm{im}(g-\mathbf{1})) + \dim(\mathrm{im}(g-\mathbf{1}) \cap \ker(q)) . \tag{2.7}$$

We can require S not to be totally isotropic, except in these two cases:

if $\mathrm{im}(g-\mathbf{1}) \cap \ker(q) = 0$ *and* $\mathrm{im}(g-\mathbf{1})$ *is totally isotropic;*

if $\mathrm{im}(g-\mathbf{1}) \cap \ker(q) \neq 0$ *and* $(\ker(g-\mathbf{1}))^{\perp}$ *is totally isotropic.*

Proof. There is an easy case and a difficult case.

The easy case: $\mathrm{im}(g-\mathbf{1}) \cap \ker(q) = 0$. In this case, (2.7) means that $S = \mathrm{im}(g-\mathbf{1})$. Let us prove that the equation (2.1) determines a bilinear form ϕ; we must verify that every equality $g(x) - x = g(x') - x'$ implies $\mathrm{b}_q(x, y) = \mathrm{b}_q(x', y)$ for all $y \in S$; indeed, this equality means $x - x' \in \ker(g-\mathbf{1})$; therefore, $x - x'$ is orthogonal to $\mathrm{im}(g-\mathbf{1}) = S$ and $\mathrm{b}_q(x - x', y) = 0$. This bilinear form ϕ is non-degenerate; indeed, if $\phi(z, y) = 0$ for all $z \in S$, then $\mathrm{b}_q(x, y) = 0$ for all $x \in V$, therefore $y \in \ker(\mathrm{b}_q)$, whence $y \in S \cap \ker(\mathrm{b}_q) = \mathrm{im}(g-\mathbf{1}) \cap \ker(q) = 0$. When $y = g(x) - x$, we can prove that $q(g(x)) - q(x) = q(y) - \phi(y, y)$ as we did it in the proof of Theorem 2.1; and here, this equality implies $\phi(y, y) = q(y)$ for all $y \in S$.

The difficult case: $\mathrm{im}(g-\mathbf{1}) \cap \ker(q) \neq 0$. Let $(b_1, \ldots, b_t)$ be a basis of $S_0 = \mathrm{im}(g-\mathbf{1}) \cap \ker(q)$, and S' a subspace such that $\mathrm{im}(g-\mathbf{1}) = S_0 \oplus S'$. Moreover, let V' be a subspace such that $V = \ker(\mathrm{b}_q) \oplus V'$ and $V' \supset S'$. Since q is non-degenerate on V', there is an orthogonal transformation g' of V' and there is $(c_1, \ldots, c_t)$ in V' such that

$$\forall x \in V', \quad g(x) = g'(x) + \sum_{i=1}^{t} \mathrm{b}_q(x, c_i)\, b_i . \tag{2.8}$$

In V' we can find a linearly independent sequence $(a_1, \ldots, a_t)$ such that $g(a_i) - a_i = b_i$ for $i = 1, 2, \ldots, t$. Consequently, $g'(a_i) = a_i$ and $\mathrm{b}_q(a_i, c_i) = 1$ for $i = 1, 2, \ldots, t$, but $\mathrm{b}_q(a_i, c_j) = 0$ if $i \neq j$. This proves that $(c_1, \ldots, c_t)$ spans a subspace S_1 of dimension t which b_q puts in duality with the space spanned by $(a_1, \ldots, a_t)$. Moreover, $S_1 \cap (S_0 \oplus S') = 0$ because $S_0 \oplus S'$ (that is $\mathrm{im}(g-\mathbf{1})$) is orthogonal to the subspace spanned by $(a_1, \ldots, a_t)$; indeed, for all $x \in V$,

$$\mathrm{b}_q(a_i,\ g(x) - x) = -\mathrm{b}_q(g(a_i) - a_i,\ g(x)) = -\mathrm{b}_q(b_i,\ g(x)) = 0 .$$

Let us set $S = S_0 \oplus S' \oplus S_1$. This subspace S is orthogonal to $\ker(g-\mathbf{1})$; indeed, we already know that $S_0 \oplus S'$ (that is $\mathrm{im}(g-\mathbf{1})$) is orthogonal to $\ker(g-\mathbf{1})$; since $\ker(g-\mathbf{1}) \supset \ker(\mathrm{b}_q)$, it suffices to prove that S_1 is orthogonal to $V' \cap \ker(g-\mathbf{1})$; this follows from (l2.8), where the equality $g(x) = x$ implies implies $\mathrm{b}_q(x, c_i) = 0$ for $i = 1, 2, \ldots, t$.

Now we construct ϕ. The equation (2.1) involves only the restriction of ϕ to $(S_0 \oplus S') \times S$, and as in the previous easy case, it actually determines this restriction, because every equality $g(x) - x = g(x') - x'$ implies that $x - x'$ is in $\ker(g-\mathbf{1})$, therefore orthogonal to S. Since $S_0 \subset \ker(\mathrm{b}_q)$, it is clear that ϕ vanishes on $(S_0 \oplus S') \times S_0$. Since the vectors a_i are orthogonal to $S_0 \oplus S'$ (see above), ϕ vanishes on $S_0 \times (S_0 \oplus S')$ too:

$$\phi(b_i, y) = \phi(g(a_i) - a_i,\ y) = -\mathrm{b}_q(a_i, y) = 0 \quad \text{if } y \in S_0 \oplus S' .$$

Since $\phi(b_i, c_j) = \phi(g(a_i) - a_i,\ c_j) = -\mathrm{b}_q(a_i, c_j)$, we have $\phi(b_i, c_i) = -1$, but $\phi(b_i, c_j) = 0$ if $i \neq j$. On another side, the restriction of ϕ to S' is non-degenerate; indeed, if y is an element of S' such that $\phi(z, y) = 0$ for all $z \in S'$, then $\phi(z, y) = 0$ for all $z \in S_0 \oplus S'$; therefore, $\mathrm{b}_q(x, y) = -\phi(g(x) - x,\ y) = 0$ for all $x \in V$, whence $y \in S' \cap \ker(\mathrm{b}_q) = 0$. Since the equation (2.1) is now satisfied, we can deduce the equality $q(g(x)) - q(x) = q(y) - \phi(y, y)$ from $y = g(x) - x$ as above, and claim that $\phi(y, y) = q(y)$ for all $y \in S_0 \oplus S'$. To complete the construction of ϕ, we have only to worry about the equalities $\phi(y, y) = q(y)$ and $\phi(y, z) + \phi(z, y) = \mathrm{b}_q(y, z)$ when y is in S_1. Since S_0 and S_1 are orthogonal, we realize that $\phi(c_i, b_i) = 1$ for $i = 1, 2, \ldots, t$, but $\phi(c_i, b_j) = 0$ if $i \neq j$. Let us choose a basis $(d_1, \ldots, d_r)$ of S', and consider the matrix Φ of ϕ in the basis $(b_1, \ldots, b_t,\ d_1, \ldots, d_r,\ c_1, \ldots, c_t)$ of S:

$$\Phi = \begin{pmatrix} 0 & 0 & -\mathbf{1}_t \\ 0 & M & N \\ \mathbf{1}_t & N' & P \end{pmatrix} ;$$

the submatrix M is invertible since it gives the restriction of ϕ to S'; consequently the matrix Φ is invertible. The submatrix N' is determined by N and the restriction of b_q to $S_1 \times S'$; but when $t \geq 2$, the submatrix P is not completely determined by the condition $\phi(y, y) = q(y)$ for all $y \in S_1$.

It remains to prove that there are non totally isotropic choices of S if and only if $\ker(g - \mathbf{1})^\perp$ is not totally isotropic. When q is defective, there is $u \in \ker(\mathrm{b}_q)$ such that $q(u) \neq 0$; since $\ker(g - \mathbf{1})^\perp$ contains u, it is never totally isotropic, and we must prove that there is always a non totally isotropic choice of S; indeed, the equality (2.8) remains true if we replace c_1 with $c_1 + u$; since $q(c_1 + u) = q(c_1) + q(u) \neq q(c_1)$, we can choose c_1 in such a way that $q(c_1) \neq 0$. Now let us suppose that $\ker(q) = \ker(\mathrm{b}_q)$. Since (2.2) implies $S \subset \ker(g - \mathbf{1})^\perp$, every choice of S is totally isotropic if $\ker(g - \mathbf{1})^\perp$ is totally isotropic. Conversely, let the above constructed subspace S be totally isotropic, and let us prove that $V' \cap \ker(g - \mathbf{1})^\perp$ is totally isotropic (therefore, $\ker(g - \mathbf{1})^\perp$ too). From (2.8) we deduce that $V' \cap \ker(g - \mathbf{1})$ is the intersection of $V' \cap S_1^\perp$ and $\ker(g' - \mathbf{1}_{V'})$, and also that $\mathrm{im}(g' - \mathbf{1}'_{V'}) = S'$. Since q is non-degenerate on V', $\ker(g' - \mathbf{1}_{V'}) = V' \cap S'^\perp$. Thus $V' \cap \ker(g - \mathbf{1})$ is the intersection of $V' \cap S'^\perp$ and $V' \cap S_1^\perp$, whence $V' \cap \ker(g - \mathbf{1})^\perp = S' \oplus S_1$. If S is totally isotropic, the same is true for $S' \oplus S_1$ and $\ker(g - \mathbf{1})^\perp$. □

When q is non-degenerate, the correspondance between transformers and orthogonal transformations is bijective. In Section 4, it is explained that the same is true for a non-defective q such that $\dim(\ker(q)) = 1$. Whatever q may be, if $(V, q) \to (W, \tilde{q})$ is an embedding such that $V \subset W$, every transformer (S, ϕ) of (V, q) is also a transformer of $(W, \tilde{q})$; consequently, every $g \in \mathrm{O}(V, q)$ has an extension $\tilde{g} \in \mathrm{O}(W, \tilde{q})$ such that $\mathrm{im}(\tilde{g} - \mathbf{1}_W) \subset V$. Conversely, if $\tilde{q}$ is non-degenerate, every $\tilde{g} \in \mathrm{O}(W, \tilde{q})$ such that $\mathrm{im}(\tilde{g} - \mathbf{1}_W) \subset V$ admits a transformer (S, ϕ) such that $S \subset V$; thus there is a bijection between the transformers of (V, q) and the elements $\tilde{g} \in \mathrm{O}(W, \tilde{q})$ such that $\mathrm{im}(\tilde{g} - \mathbf{1}_W) \subset V$. This fact gives a structure of group on the set of transformers of (V, q). This structure does not depend on the choice of the embedding; indeed, if (V, q) is embedded in $(W, \tilde{q})$ and in $(W', \tilde{q}')$ (with non-degenerate $\tilde{q}$ and $\tilde{q}'$), then $(W, \tilde{q})$ and $(W', \tilde{q}')$ can be embedded in the same non-degenerate space $(W'', \tilde{q}'')$ in such a way that we get twice the same embedding $(V, q) \to (W'', \tilde{q}'')$; it is easy to construct $(W'', \tilde{q}'')$ (despite a little difficulty when q is defective).

When K is the field $\mathbb{R}$ of real numbers, the groups under consideration are Lie groups. The dimension of the group of transformers is always $n(n-1)/2$; indeed, there is canonical bijection from $\bigwedge^2(W)$ onto the Lie algebra of $\mathrm{O}(W, \tilde{q})$ which maps every $y \wedge z$ to the operator $x \longmapsto \mathrm{b}_{\tilde{q}}(x, y)\, z - \mathrm{b}_{\tilde{q}}(x, z)\, y$, and the image of $\bigwedge^2(V)$ is actually the Lie algebra of the subgroup determined by the condition $\mathrm{im}(\tilde{g} - \mathbf{1}_W) \subset V$. The dimension of $\mathrm{O}(V, q)$ depends on $k = \dim(\ker(q))$; it is $(n(n-1) - k(k-1))/2 = (n-k)(n+k-1)/2$. The group $\mathrm{Iso}(V, q)$ is isomorphic to a semi-direct product of $\mathrm{O}(V, q)$ and $\mathrm{GL}(\ker(q))$.

Theorem 2.3 gives an example of a product of transformers.

Theorem 2.3. *Let (S_1, ϕ_1) and (S_2, ϕ_2) be two transformers of (V, q) such that $S_1 \cap S_2 = 0$, and let g_1 and g_2 be the associated orthogonal transformations. Their product $g = g_1 g_2$ admits the following transformer (S, ϕ): $S = S_1 \oplus S_2$; ϕ coincides with ϕ_1 on S_1, with ϕ_2 on S_2, and for all $y_1 \in S_1$ and $y_2 \in S_2$ we have $\phi(y_1, y_2) = 0$ (whence $\phi(y_2, y_1) = \mathrm{b}_q(y_1, y_2)$).*

Proof. Since (V, q) can be embedded in a non-degenerate space $(W, \tilde{q})$, it suffices to prove Theorem 2.3 when q is non-degenerate. This hypothesis implies $\mathrm{im}(g_1 - \mathbf{1}) = S_1$ and $\ker(g_1 - \mathbf{1}) = S_1^\perp$, and similarly $\mathrm{im}(g_2 - \mathbf{1}) = S_2$ and $\ker(g_2 - \mathbf{1}) = S_2^\perp$. Since $S_1 \cap S_2 = 0$, we have $S_1^\perp + S_2^\perp = V$, consequently, $\ker(g_1 - \mathbf{1}) + \ker(g_2 - \mathbf{1}) = V$, and Lemma 1.4 implies that $\mathrm{im}(g - \mathbf{1}) = \mathrm{im}(g_1 - \mathbf{1}) + \mathrm{im}(g_2 - \mathbf{1})$. It follows that $S = S_1 \oplus S_2$.

Let us consider vectors x, y_1 and y_2 respectively in V, S_1 and S_2. Let us calculate $\phi(g(x) - x,\ y_2)$ when $g(x) - x$ is in S_2; from $g - \mathbf{1} = (g_1 - \mathbf{1})g_2 + (g_2 - \mathbf{1})$ and $S_1 \cap S_2 = 0$, we deduce $g(x) - x = g_2(x) - x$; consequently,

$$\phi(g(x) - x,\ y_2) = -\mathrm{b}_q(x, y_2) = \phi_2(g_2(x) - x,\ y_2) = \phi_2(g(x) - x,\ y_2)\,;$$

therefore, ϕ coincides with ϕ_2 on S_2. Now we suppose that $g(x) - x$ is in S_1; for the same reasons as above, this implies $g_2(x) = x$ and $g(x) - x = g_1(x) - x$; consequently,

$$\begin{aligned}
\phi(g(x) - x,\ y_1) &= -\mathrm{b}_q(x, y_1) = \phi_1(g_1(x) - x,\ y_1) = \phi_1(g(x) - x,\ y_1)\,,\\
\phi(g(x) - x,\ y_2) &= -\mathrm{b}_q(x, y_2) = \phi_2(g_2(x) - x,\ y_2) = 0\,;
\end{aligned}$$

therefore, ϕ coincides with ϕ_1 on S_1, and $\phi(S_1, S_2) = 0$. □

Corollary 2.4. *Let (S_1, ϕ_1) and (S_2, ϕ_2) be two transformers of (V, q) such that $S_1 \subset S_2$, and $\phi_1(y, z) = \phi_2(z, y)$ for all $y, z \in S_1$. Let g_1 and g_2 be the associated orthogonal transformations. Their product $g = g_1 g_2$ admits the following transformer (S, ϕ): $S = \mathrm{R}^\perp_{\phi_2}(S_1)$ and ϕ is the restriction of ϕ_2 to S. And their product $g' = g_2 g_1$ admits the following transformer (S', ϕ'): $S' = \mathrm{L}^\perp_{\phi_2}(S_1)$ and ϕ' is the restriction of ϕ_2 to S'.*

Proof. The equalities $g = g_1 g_2$ and $g' = g_2 g_1$ are equivalent to $g_2 = g_1^{-1} g$ and $g_2 = g' g_1^{-1}$, and g_1^{-1} is given by the reverse transformer $(S_1, \phi_1^\dagger)$ where $\phi_1^\dagger$ coincides with the restriction of ϕ_2 to S_1. Since ϕ_1 is non-degenerate, we have $S_2 = S_1 \oplus \mathrm{R}_{\phi_2}^\perp(S_1)$ and $S_2 = \mathrm{L}_{\phi_2}^\perp(S_1) \oplus S_1$ (see Lemma 1.3). With Theorem 2.3, it is easy to verify that $g_2 = g_1^{-1} g$ and $g_2 = g' g_1^{-1}$ if g and g' are determined by the transformers described in Corollary 2.4. □

3. Products of Reflections

Let (S, ϕ) be a transformer of (V, q) such that $\dim(S) = 1$; thus S is spanned by a non-zero vector v and $\phi(v, v) = q(v)$; since ϕ is non-degenerate, we have $q(v) \neq 0$ and v determines a reflection $\mathrm{R}(v)$; and since $\phi(\mathrm{R}(v)(x) - x,\ v) = -\mathrm{b}_q(x, v)$ for all $x \in V$, we realize that $\mathrm{R}(v)$ admits (S, ϕ) as a transformer. Thus the reflections are the orthogonal transformations determined by the one-dimensional transformers. The following theorem is an immediate consequence of Theorem 2.3 and Corollary 2.4.

Theorem 3.1. *Let us consider a reflection* $\mathrm{R}(v)$ *and the orthogonal transformation* h *determined by a transformer* (T, ψ)*. The products* $g = \mathrm{R}(v)\, h$ *and* $g' = h\, \mathrm{R}(v)$ *admit the following transformers* (S, ϕ) *and* (S', ϕ')*:*

if v *is outside* T*, then* $S = S' = T \oplus Kv$*, the restrictions of* ϕ *and* ϕ' *to* T *coincide with* ψ*, and* $\phi(v, y) = \phi'(y, v) = 0$ *for all* $y \in T$ *(whence* $\phi(y, v) = \phi'(v, y) = \mathrm{b}_q(v, y)$*; and of course,* $\phi(v, v) = \phi'(v, v) = q(v)$*);*

if v *belongs to* T*, then* $S = \mathrm{R}_\psi^\perp(v)$ *and* $S' = \mathrm{L}_\psi^\perp(v)$*, and* ϕ *and* ϕ' *are the restrictions of* ψ *to* S *and* S' *respectively.*

Corollary 3.2. *For every* $g \in \mathrm{O}(V, q)$ *and for every sequence* $(v_1, v_2, \ldots, v_s)$ *of linearly independent vectors in* V*, these two assertions are equivalent:*

$g = \mathrm{R}(v_1)\, \mathrm{R}(v_2) \cdots \mathrm{R}(v_s)$;

g admits the transformer (S, ϕ) *where* $(v_1, \ldots, v_s)$ *is a basis of* S*, and* ϕ *has a lower triangular matrix in this basis.*

Theorem 3.1 and its corollary provide an effective method to calculate the product (S, ϕ) of two transformers (S_1, ϕ_1) and (S_2, ϕ_2) when a triangularizing basis is known for one factor. Since $S \subset S_1 + S_2$, the product can be calculated in the subspace $S_1 + S_2$ without worrying about the non-degenerate embeddings that were previously necessary to prove that it is well defined. For instance, if (S, ϕ) is the transformer for a product of reflections $\mathrm{R}(w_1) \cdots \mathrm{R}(w_k)$, then S is contained in the subspace spanned by $(w_1, \ldots, w_k)$.

Section 5 shall be devoted to the proof of the next theorem, and to the construction of an effective algorithm of triangularization; this theorem requires the hypotheses that K is not isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Theorem 3.3. *If* ϕ *is a bilinear form on some space* S*, and if* ϕ *is not alternate, there are bases of* S *where the matrix of* ϕ *is lower triangular.*

In Theorem 3.3, it is clear that ϕ is alternate if and only if S is totally isotropic for the quadratic form $y \longmapsto \phi(y, y)$.

The previous statements enable us to prove that every $g \in \mathrm{O}(V, q)$ can be decomposed into a product of reflections, and to evaluate the minimal number of reflections in such a decomposition. The minimal dimension of a transformer for g is given by (2.7); as in the proof of Theorem 2.2, we consider two cases (and we suppose $g \neq \mathbf{1}_V$).

In the easy case $\mathrm{im}(g - \mathbf{1}) \cap \ker(q) = 0$, the unique minimal transformer involves $S = \mathrm{im}(g - \mathbf{1})$, and we set $s = \dim(S)$. If S is not totally isotropic, the minimal number of reflections is s. If S is totally isotropic, the minimal number of reflections is $> s$; if v is any non-isotropic vector (therefore, outside S), the transformer for $\mathrm{R}(v)\, g$ (or $g\, \mathrm{R}(v)$) involves the subspace $S \oplus Kv$ which is not totally isotropic; consequently, it is a product of $s + 1$ reflections, and g itself is a product of $s + 2$ reflections. If q is non-defective, g cannot be a product of $s + 1$ reflections, because the parity of the number of reflections is determined by g. On the contrary, if q is defective, we have $\mathrm{R}(w) = \mathbf{1}_V$ for every non-isotropic $w \in \ker(\mathrm{b}_q)$, and the equality $g = \mathrm{R}(w)\, g$ proves that g is a product of $s + 1$ reflections.

In the difficult case $\mathrm{im}(g - \mathbf{1}) \cap \ker(q) \neq 0$, the dimension s of a minimal transformer (S, ϕ) is $\dim(\mathrm{im}(g - \mathbf{1})) + \dim(\mathrm{im}(g - \mathbf{1}) \cap \ker(q))$, and we can require S not to be totally isotropic if and only if $\ker(g - \mathbf{1})^\perp$ is not totally isotropic; if it is not, the minimal number of reflections is s. On the contrary, if $\ker(g - \mathbf{1})^\perp$ is totally isotropic, the same is true for its subspace $\ker(\mathrm{b}_q)$; this means that q is non-defective; and the same argument (involving $\mathrm{R}(v)\, g$ or $g\, \mathrm{R}(v)$) proves that the minimal number of reflections is $s + 2$.

Remark. When the support S of a transformer (S, ϕ) is totally isotropic, the dimension s of S is even, because ϕ is a non-degenerate and alternate bilinear form on S. There is a basis $(y_1, z_1, \ldots, y_r, z_r)$ of S (where $r = s/2$) such that $\phi(y_i, z_i) = 1$ for $i = 1, 2, \ldots, r$, but $\phi(y_i, z_j) = 0$ whenever $i \neq j$, and $\phi(y_i, y_j) = \phi(z_i, z_j) = 0$ for all i and j; and it is convenient to consider $\omega = \sum_{i=1}^{r} y_i \wedge z_i$ in $\bigwedge^2(S)$ because the transformation determined by (S, ϕ) is the transformation $\mathrm{F}(\omega)$ such that

$$\forall x \in V, \qquad \mathrm{F}(\omega)(x) = x + \sum_{i=1}^{r} \left(\mathrm{b}_q(x, y_i)\, z_i - \mathrm{b}_q(x, z_i)\, y_i\right). \tag{3.1}$$

If q is non-degenerate, then $4r = 2s \le n$; therefore, a totally isotropic S (such that $S \ne 0$) can appear only when $n \ge 4$. This explains that $s+2 \le n$. Nevertheless, when q is degenerate, it may happen that $s+2 > n$, as in the following example.

Example. Let (V, q) be the space with basis (u_1, u_2, u_3) over $\mathbb{R}$, provided with the quadratic form q such that $q(\xi_1 u_1 + \xi_2 u_2 + \xi_3 u_3) = \xi_1 \xi_2$; thus $\ker(q)$ is the line $\mathbb{R}u_3$. Let g be the orthogonal transformation such that

$$g(\xi_1 u_1 + \xi_2 u_2 + \xi_3 u_3) = \xi_1 (u_1 + u_3) + \xi_2 u_2 + \xi_3 u_3\,. \tag{3.2}$$

It is determined by the transformer (S, ϕ) such that (u_2, u_3) is a basis of S, ϕ is alternate and $\phi(u_2, u_3) = 1$; this agrees with (3.1). Therefore, when g is expressed as a product of reflections, the minimal number of reflections is 4. Let us calculate the transformer (T, ψ) for $h = \mathrm{R}(u_1 + u_2)\, g$. Since $T = \mathbb{R}(u_1 + u_2) \oplus S$, we have $T = V$; since $\psi(u_1 + u_2,\, u_2) = \psi(u_1 + u_2,\, u_3) = 0$, we have $\psi(u_1, u_2) = 0$ and $\psi(u_1, u_3) = -1$; the matrix Ψ of ψ in the basis (u_1, u_2, u_3) is written below. In this example, it is easy to find a basis (v_1, v_2, v_3) where the matrix Ψ' of ψ is lower triangular; for instance,

$$\begin{cases} v_1 = u_1 + u_2 + u_3\,, \\ v_2 = u_1 + 2u_2\,, \\ v_3 = u_1 + 2u_2 - 2u_3\,, \end{cases} \qquad \Psi = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \qquad \Psi' = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 3 & 4 & 2 \end{pmatrix}.$$

The result of this calculation is

$$g = \mathrm{R}(u_1 + u_2)\,\mathrm{R}(u_1 + u_2 + u_3)\,\mathrm{R}(u_1 + 2u_2)\,\mathrm{R}(u_1 + 2u_2 - 2u_3)\,. \tag{3.3}$$

There is an non-degenerate embedding $(W, \tilde{q})$ with a basis $(u_1, \ldots, u_4)$ such that $\tilde{q}(\sum_{i=1}^{4} \xi_i u_i) = \xi_1\xi_2 + \xi_3\xi_4$. The extension $\tilde{g}$ maps u_4 to $u_4 - u_2$; and (3.3) gives a decomposition of $\tilde{g}$ if the reflections operate on W.

Remark. When $K = \mathbb{Z}/2\mathbb{Z}$, the group $\mathrm{O}(V, q)$ is different from the subgroup $\mathrm{O_R}(V, q)$ generated by the reflections in the following two exceptional cases (see (Helmstetter & Micali, 2008), section 5.7). Dieudonné's exceptional case occurs when V is the direct sum of $\ker(q)$ (perhaps reduced to 0) and a hyperbolic subspace of dimension 4 (with a basis $(u_1, \ldots, u_4)$ such that $q(\sum_i \xi_i u_i) = \xi_1\xi_2 + \xi_3\xi_4$); in this case, the quotient $\mathrm{O}(V, q)/\mathrm{O_R}(V, q)$ is a group of order 2. The other case occurs when V is the direct sum of $\ker(q)$ and a hyperbolic space of dimension 2; in this case, $\mathrm{O}(V, q)/\mathrm{O_R}(V, q)$ is isomorphic to the additive group $\ker(q)$; it is exceptional only if $\ker(q) \ne 0$ (an eventuality which Dieudonné did not accept in (Dieudonné, 1958)). If we use (3.2) to define an orthogonal transformation g over $\mathbb{Z}/2\mathbb{Z}$, then g is not a product of reflections; and neither is its extension $\tilde{g}$ to a hyperbolic space of dimension 4.

4. The Non-defective Case $\dim(\ker(q)) = 1$

It is sensible to ask whether an orthogonal transformation g of (V, q) may admit several transformers. By means of a non-degenerate embedding $(W, \tilde{q})$, this question is easily reduced to the following one: does $\mathbf{1}_V$ admit several transformers, in other words, are there non-trivial transformers (S, ϕ) such that $S \subset \ker(\mathrm{b}_q)$? When q is defective, the answer is obviously "yes" because the reflection associated with each non-isotropic $v \in \ker(\mathrm{b}_q)$ is equal to $\mathbf{1}_V$, and it admits the one-dimensional transformer spanned by v. When q is not defective, the condition $S \subset \ker(\mathrm{b}_q)$ implies that $\dim(S)$ is even, and it can be satisfied by a non-trivial transformer if and only if $\dim(\ker(\mathrm{b}_q)) \ge 2$. Thus we have proved the following theorem.

Theorem 4.1. *The correspondance between the orthogonal transformations and the transformers is bijective (only) in these two cases:*

when q is non-degenerate (in other words, $\ker(\mathrm{b}_q) = 0$*);*

when q is non-defective and $\dim(\ker(q)) = 1$.

The non-defective case $\dim(\ker(q)) = 1$ deserves some attention because it can be used in the study of the affine isometries of an affine space E provided with a non-degenerate quadratic form χ. An affine space E is a set on which a vector space $\vec{E}$ operates in a simply transitive way (by translations); the non-degenerate quadratic form χ is defined on $\vec{E}$; every affine transformation g of E has a linear part $\vec{g}$ in $\mathrm{GL}(\vec{E})$, and g is an affine isometry if and only if $\vec{g} \in \mathrm{O}(\vec{E}, \chi)$; the set of all affine isometries is the group $\mathrm{Af.Iso}(E, \chi)$. For convenience, we set $n = \dim(E) + 1$, and we suppose that $E = \vec{E}$; thus $\mathrm{O}(E, \chi)$ is the subgroup of all $g \in \mathrm{Af.Iso}(E, \chi)$ such that $g(0) = 0$. For every $a \in E$, let $a^\sharp$ be the linear form on E such that $a^\sharp(b) = \mathrm{b}_\chi(a, b)$ for all $b \in E$; the mapping $a \longmapsto a^\sharp$ is a linear bijection $E \to E^*$, and the inverse bijection is denoted by $\ell \longmapsto \ell^\flat$; moreover, we define a dual quadratic form χ^* on E^* by setting $\chi^*(\ell) = \chi(\ell^\flat)$. Let V be the space of all affine forms $x : E \to K$; thus E^* is the subspace of all $\ell \in V$ such that $\ell(0) = 0$, and every $x \in V$ has a linear part $\vec{x} \in E^*$ such that $\vec{x}(a) = x(a) - x(0)$. Let q be the quadratic form on V defined by $q(x) = \chi^*(\vec{x}) = \chi(\vec{x}^\flat)$. Thus V is a space of dimension n provided with a non-defective quadratic form q such that $\dim(\ker(q)) = 1$; indeed, $\ker(q)$ is the set of all constant functions $E \to K$. Every affine transformation g of E determines a linear transformation $g^\sharp$ of V which maps every $x \in V$ to the affine form $a \longmapsto x(g(a))$. From this definition, it follows that $(g_1g_2)^\sharp = g_2^\sharp g_1^\sharp$. Besides, $\ker(g^\sharp - \mathbf{1}) \supset \ker(q)$ because $g^\sharp$ leaves invariant every constant function $E \to K$. It is easy to prove that the mapping $g \longmapsto g^\sharp$ induces an anti-isomorphism from $\mathrm{Af.Iso}(E,\chi)$ onto $\mathrm{O}(V,q)$. The inverse anti-isomorphism is denoted by $h \longmapsto h^\flat$.

By this anti-isomorphism $\flat$, the reflections in (V,q) are in bijection with the affine reflections in (E,χ); if v is a non-isotropic element of V, the set of all $a \in E$ such that $v(a) = 0$ is an affine hyperplane of E, and $(\mathrm{R}(v))^\flat$ is the affine reflection determined by this affine hyperplane:

$$\forall a \in E, \quad (\mathrm{R}(v))^\flat(a) = a - \frac{v(a)}{q(v)}\,\vec{v}^\flat\,. \tag{4.1}$$

Thus the decomposition into products of affine reflections in $\mathrm{Af.Iso}(E,\chi)$ is reduced to the decomposition into products of reflections in $\mathrm{O}(V,q)$.

Let g be an element of $\mathrm{Af.Iso}(E,\chi)$ (other than $\mathbf{1}_E$). We must find out whether $\mathrm{im}(g^\sharp - \mathbf{1}) \cap \ker(q)$ is reduced to 0 or not. If it is, there is a hyperplane H of V that contains $\mathrm{im}(g^\sharp - \mathbf{1})$ but not $\ker(q)$; since H does not contain $\ker(q)$, there is a point $p \in E$ such that H is the subset of all $x \in V$ such that $x(p) = 0$; and since H contains $\mathrm{im}(g^\sharp - \mathbf{1})$, we have $g^\sharp(H) = H$ and $g(p) = p$. Conversely, if $g(p) = p$ for some $p \in E$, then $g^\sharp(x)(p) = x(g(p)) = x(p)$ for all $x \in V$, and $(g^\sharp - \mathbf{1})(x)$ cannot be a constant function $\neq 0$. Therefore, the easy case $\mathrm{im}(g^\sharp - \mathbf{1}) \cap \ker(q) = 0$ occurs if and only if $g(p) = p$ for some $p \in E$. If $g(p) = p$, then $g = T\,\vec{g}\,T^{-1}$ where T is the translation $a \longmapsto a + p$, and the decomposition of g into a product of affine reflections is reduced to the decomposition of $\vec{g}$ into a product of reflections in $\mathrm{O}(E,\chi)$.

Now we consider the difficult case $\mathrm{im}(g^\sharp - \mathbf{1}) \supset \ker(q)$. We have $g(a) = \vec{g}(a) + g(0)$ for all $a \in E$, and $g(0)$ is not in $\mathrm{im}(\vec{g} - \mathbf{1}_E)$ because the equality $g(0) = \vec{g}(b) - b$ is equivalent to $g(-b) = -b$, which is only possible in the above easy case. According to Theorem 2.2, we must find out whether $\ker(g^\sharp - \mathbf{1})^\perp$ is totally isotropic or not; since it contains $\ker(q)$, it is determined by its image by the mapping $x \longmapsto \vec{x}^\flat$. For all $x \in V$ and all $a \in E$, we have:

$$(g^\sharp - \mathbf{1})(x)(a) = \mathrm{b}_\chi(\vec{x}^\flat,\ (\vec{g} - \mathbf{1}_E)(a) + g(0));$$

therefore, x is in $\ker(g^\sharp - \mathbf{1})$ if and only if $\vec{x}^\flat$ is orthogonal to $\mathrm{im}(\vec{g} - \mathbf{1}_E)$ and $g(0)$; and y is in $\ker(g^\sharp - \mathbf{1})^\perp$ if and only if $\vec{y}^\flat$ is in the direct sum of $\mathrm{im}(\vec{g} - \mathbf{1}_E)$ and the line $Kg(0)$. Consequently, $\ker(g^\sharp - \mathbf{1})^\perp$ *is totally isotropic in* (V,q) *if and only if* $\mathrm{im}(\vec{g} - \mathbf{1}_E) \oplus Kg(0)$ *is totally isotropic in* (E,χ).

We must also know how to deduce $s = \dim(S)$ from $d = \dim(\mathrm{im}(\vec{g} - \mathbf{1}_E))$. The dimensions of $\mathrm{im}(\vec{g} - \mathbf{1}_E) \oplus Kg(0)$ and $\ker(g^\sharp - \mathbf{1})^\perp$ are $d + 1$ and $d + 2$. The dimension of $\mathrm{im}(g^\sharp - \mathbf{1})$ is $d + 1$ because of this fact: the sum of the dimensions of $\ker(g^\sharp - \mathbf{1})$ and $\mathrm{im}(g^\sharp - \mathbf{1})$ is n, but the sum of the dimensions of $\ker(g^\sharp - \mathbf{1})$ and $\ker(g^\sharp - \mathbf{1})^\perp$ is $n + 1$ because $\ker(g^\sharp - \mathbf{1}) \supset \ker(q)$. From (2.7) we deduce $s = d + 2$. Since $s \le n$, we have $d \le n - 2$, in agreement with $g(0) \notin \mathrm{im}(\vec{g} - \mathbf{1}_E)$.

When $\mathrm{im}(\vec{g} - \mathbf{1}_E) \oplus Kg(0)$ is totally isotropic, may it occur that $s + 2 > n$? The example below shows that it occurs when $n = 3$ and $d = 0$. But other occurences are only possible with $d > 0$. Since χ is non-degenerate, we have $2(d + 1) \le n - 1$ when $\mathrm{im}(\vec{g} - \mathbf{1}_E) \oplus Kg(0)$ is totally isotropic; moreover, d is even like s; consequently, $n \ge 7$ if $d > 0$; and it is easy to realize that $s + 2 < n$ when $n \ge 7$ and $2(d + 1) \le n - 1$.

Example. Let (E,χ) be the vector space with basis (e_1, e_2) over $\mathbb{R}$, where $\chi(\xi_1 e_1 + \xi_2 e_2) = \xi_1\xi_2$; and let g be the translation of vector e_1. In general, a translation is a product of two reflections; but here we shall need four reflections because e_1 is isotropic. With the notation used just above, we have $n = 3$, $d = 0$ because $\vec{g} = \mathbf{1}_E$, and $s = 2$; but since S will prove to be totally isotropic in (V,q), we need $s + 2$ reflections. Let u_1, u_2 and u_3 be the affine forms that map every $\xi_1 e_1 + \xi_2 e_2$ respectively to ξ_1, ξ_2 and 1; thus (u_1, u_2, u_3) is a basis of V. The mapping $x \longmapsto \vec{x}^\flat$ maps u_1, u_2, u_3 respectively to $e_2, e_1, 0$; consequently, $q(\xi_1 u_1 + \xi_2 u_2 + \xi_3 u_3) = \xi_1\xi_2$. An easy calculation shows that $g^\sharp$ maps u_1, u_2, u_3 respectively to $u_1 + u_3$, u_2, u_3; thus $g^\sharp$ coincides with the orthogonal transformation defined by (3.2). We already know that S is spanned by (u_2, u_3), and we translate (3.3) here in this way:

$$g = (\mathrm{R}(u_1 + 2u_2 - 2u_3))^\flat(\mathrm{R}(u_1 + 2u_2))^\flat(\mathrm{R}(u_1 + u_2 + u_3))^\flat(\mathrm{R}(u_1 + u_2))^\flat\ ;$$

$(\mathrm{R}(u_1 + 2u_2 - 2u_3))^\flat(\mathrm{R}(u_1 + 2u_2))^\flat$ is the translation of vector $2e_1 + e_2$, and $(\mathrm{R}(u_1 + u_2 + u_3))^\flat(\mathrm{R}(u_1 + u_2))^\flat$ is the translation of vector $-e_1 - e_2$.

5. An Algorithm of Triangularization

Theorem 3.3 states that there are bases $(v_1, \ldots, v_s)$ of S where the matrix of ϕ is lower triangular, provided that ϕ is not alternate; this must be proved when $s \geq 2$, and to prove it, I propose an algorithm of triangularization. There are two standard versions of this algorithm; the left side version calculates the vectors v_i in the increasing order of the indices i; as a by-product, it gives a basis of RKer(ϕ). When the dimension t of LKer(ϕ) and RKer(ϕ) is $\neq 0$, it gives a triangularizing basis $(v_1, \ldots, v_s)$ where $\phi(v_i, v_i) \neq 0$ for $i = 1, 2, \ldots, s-t$, and $(v_{s-t+1}, \ldots, v_s)$ is a basis of RKer(ϕ). The right side version calculates the vectors v_i in the decreasing order of the indices, and when $t \neq 0$, then $(v_1, \ldots, v_t)$ is a basis of LKer(ϕ). Each version requires $s-1$ steps if $t = 0$, and $s-t$ steps if $t \geq 1$.

The space (S, ϕ) is given by a basis $(u_1, \ldots, u_s)$ and the matrix of ϕ in this basis. When the k-th step of the left side algorithm begins, we know a sequence $(v_1, \ldots, v_{k-1}, \dot{v}_k)$ such that $\phi(v_i, v_i) \neq 0$ for $i = 1, 2, \ldots, k-1$, $\phi(\dot{v}_k, \dot{v}_k) \neq 0$, $\phi(v_i, v_j) = 0$ whenever $i < j$, and $\phi(v_i, \dot{v}_k) = 0$ for $i = 1, 2, \ldots, k-1$. In particular, the first step begins with a vector $\dot{v}_1$ such that $\phi(\dot{v}_1, \dot{v}_1) \neq 0$; such a vector $\dot{v}_1$ exists because ϕ is not alternate. In general, the instructions of this algorithm order to set $v_k = \dot{v}_k$; but sometimes, the vector $\dot{v}_k$ must be "corrected" (replaced by a suitable v_k); the "correction procedure" (the instruction ⟪8⟫ below) is the only phase that may fail when $K \cong \mathbb{Z}/2\mathbb{Z}$. The k-th step is performed according to the following eight instructions.

⟪1⟫ In the basis $(u_1, u_2, \ldots, u_s)$ we choose a subsequence $(x_1, x_2, \ldots, x_{s-k})$ such that $(v_1, \ldots, v_{k-1}, \dot{v}_k, x_1, \ldots, x_{s-k})$ is a basis of S.

⟪2⟫ For $j = 1, 2, \ldots, s-k$, and as long as the "stop rule" (written just below) does not interrupt the calculations, we calculate the scalars $\xi_1, \ldots, \xi_k$ that let the vector $y_j = \xi_1 v_1 + \cdots + \xi_{k-1} v_{k-1} + \xi_k \dot{v}_k + x_j$ satisfy the following conditions:

$$\phi(v_1, y_j) = \phi(v_2, y_j) = \cdots = \phi(v_{k-1}, y_j) = \phi(\dot{v}_k, y_j) = 0\,; \tag{5.1}$$

the properties of the sequence $(v_1, \ldots, \dot{v}_k)$ show that (5.1) is a regular system of k linear equations with a lower triangular matrix; therefore, the calculation of $\xi_1, \ldots, \xi_k$ is easy. When $k = s-1$, we have to calculate only one vector y_1, and then we go to ⟪3⟫. When $k \leq s-2$, the stop rule interrupts the calculations in these two cases:

when we find a vector y_j such that $\phi(y_j, y_j) \neq 0$, we go to ⟪4⟫;

when we find two vectors y_i and y_j such that $\phi(y_i, y_i) = \phi(y_j, y_j) = 0$ and $\phi(y_i, y_j) + \phi(y_j, y_i) \neq 0$, we go to ⟪5⟫.

When the stop rule never interrupts the calculations, we go to ⟪6⟫.

⟪3⟫ When $k = s-1$, we set $v_{s-1} = \dot{v}_{s-1}$ and $v_s = y_1$. Thus we have found a triangularizing basis $(v_1, \ldots, v_s)$. If $\phi(v_s, v_s) \neq 0$, then ϕ is non-degenerate. If $\phi(v_s, v_s) = 0$, then RKer(ϕ) is the line spanned by v_s.

In the next instructions, we have $k \leq s-2$.

⟪4⟫ When $\phi(y_j, y_j) \neq 0$, we set $v_k = \dot{v}_k$ and $\dot{v}_{k+1} = y_j$, and we start the $(k+1)$-th step (we return to ⟪1⟫ where we replace k with $k+1$).

⟪5⟫ When $\phi(y_i, y_i) = \phi(y_j, y_j) = 0$ and $\phi(y_i, y_j) + \phi(y_j, y_i) \neq 0$, we set $v_k = \dot{v}_k$ and $\dot{v}_{k+1} = y_i + y_j$, and we start the $(k+1)$-th step.

⟪6⟫ When the stop rule never interrupts the calculations, the restriction of ϕ to the subspace spanned by $(y_1, \ldots, y_{s-k})$ (that is $\mathrm{R}_\phi^\perp(v_1, \ldots, \dot{v}_k)$) is alternate. If there is a couple (i, j) such that $\phi(y_i, y_j) \neq 0$, we go to ⟪8⟫. If all $\phi(y_i, y_j)$ (with $i, j \in \{1, 2, \ldots, s-k\}$) vanish, we go to ⟪7⟫.

⟪7⟫ If all $\phi(y_i, y_j)$ vanish, then we set $v_k = \dot{v}_k$, $v_{k+1} = y_1$, $v_{k+2} = y_2$, $\ldots$, $v_s = y_{s-k}$. Thus we have found a triangularizing basis $(v_1, \ldots, v_s)$, where $(v_{k+1}, \ldots, v_s)$ is a basis of RKer(ϕ); therefore, $t = s-k$.

⟪8⟫ Let (i, j) be a couple (with $i \neq j$) such that

$$\phi(y_i, y_i) = \phi(y_j, y_j) = 0 \quad \text{and} \quad \phi(y_i, y_j) = -\phi(y_j, y_i) \neq 0. \tag{5.2}$$

We look for scalars κ, λ, μ that ensure the three properties required from the vectors $v_k = \dot{v}_k + \kappa y_i$ and $\dot{v}_{k+1} = \dot{v}_k + \lambda y_i + \mu y_j$. Here are these properties:

$$\phi(v_k, \dot{v}_{k+1}) = \phi(\dot{v}_k, \dot{v}_k) + \kappa\, \phi(y_i, \dot{v}_k) + \kappa\mu\, \phi(y_i, y_j) = 0\,, \tag{5.3}$$

$$\phi(v_k, v_k) = \phi(\dot{v}_k, \dot{v}_k) + \kappa\, \phi(y_i, \dot{v}_k) \neq 0\,, \tag{5.4}$$

$$\phi(\dot{v}_{k+1}, \dot{v}_{k+1}) = \phi(\dot{v}_k, \dot{v}_k) + \lambda\, \phi(y_i, \dot{v}_k) + \mu\, \phi(y_j, \dot{v}_k) \neq 0\,. \tag{5.5}$$

(8a) If $\phi(y_i, \dot{v}_k) = 0$, the condition (5.4) is void. We set $\lambda = 0$, we choose an invertible μ compatible with (5.5), and we calculate κ by means of (5.3). When v_k and $\dot{v}_{k+1}$ have been calculated, we start the $(k+1)$-th step.

(8b) If $\phi(y_i, \dot{v}_k) \neq 0$, we choose an invertible κ compatible with (5.4), we calculate μ by means of (5.3), and we choose λ compatible with (5.5); in general, the choice $\lambda = 0$ is correct. When v_k and $\dot{v}_{k+1}$ have been calculated, we start the $(k + 1)$-th step. If $\phi(y_i, \dot{v}_k) \neq 0$ and $\phi(y_j, \dot{v}_k) = 0$, it is preferable (but not indispensable) to permute i and j and to apply (8a) instead of (8b).

These instructions involve the correction procedure ⟪8⟫ as rarely as possible (it is involved only when the restriction of ϕ to $\mathrm{R}^{\perp}_{\phi}(v_1, \ldots, \dot{v}_k)$ is alternate and $\neq 0$); this choice is suggested by an algorithm elaborated for a similar problem which involves a very painful correction procedure. Since here the correction procedure is not so painful, it is acceptable to modify the stop rule in such a way that ⟪8⟫ is involved as frequently as possible. When $k \leq s - 2$, the new stop rule interrupts the calculations in ⟪2⟫ as soon as we meet a non-zero $\phi(y_i, y_j)$; when $i = j$, we go to ⟪4⟫; when $i \neq j$ and $\phi(y_i, y_i) = \phi(y_j, y_j) = 0$, we go to ⟪5⟫, except when (5.2) is true; when (5.2) is true, we go to ⟪8⟫. Thus the instruction ⟪6⟫ becomes superfluous; if the new stop rule never interrupts the calculation, the restriction of ϕ to $\mathrm{R}^{\perp}_{\phi}(v_1, \ldots, \dot{v}_k)$ is completely null, and we go directly to ⟪7⟫.

The right side algorithm requires symmetric instructions. The k-th step starts with a sequence $(\dot{v}_{s-k+1}, v_{s-k+2}, \ldots, v_s)$ satisfying obvious conditions. In the instruction ⟪2⟫, we set $y_j = x_j + \xi_1 \dot{v}_{s-k+1} + \xi_2 v_{s-k+2} + \cdots + \xi_k v_s$, and the unknown scalars $\xi_1, \ldots, \xi_k$ are determined by a system of k liner equations with an upper triangular matrix. In the correction procedure ⟪8⟫, we set $v_{s-k+1} = \kappa y_i + \dot{v}_{s-k+1}$ and $\dot{v}_{s-k} = \lambda y_i + \mu y_j + \dot{v}_{s-k+1}$; and the unknown scalars κ, λ, μ must satisfy

$$\begin{aligned} \phi(\dot{v}_{s-k}, v_{s-k+1}) &= \kappa\, \phi(\dot{v}_{s-k+1}, y_i) - \kappa\mu\, \phi(y_i, y_j) + \phi(\dot{v}_{s-k+1}, \dot{v}_{s-k+1}) = 0\,, \\ \phi(v_{s-k+1}, v_{s-k+1}) &= \kappa\, \phi(\dot{v}_{s-k+1}, y_i) + \phi(\dot{v}_{s-k+1}, \dot{v}_{s-k+1}) \neq 0\,, \\ \phi(\dot{v}_{s-k}, \dot{v}_{s-k}) &= \lambda\, \phi(\dot{v}_{s-k+1}, y_i) + \mu\, \phi(\dot{v}_{s-k+1}, y_j) + \phi(\dot{v}_{s-k+1}, \dot{v}_{s-k+1}) \neq 0\,. \end{aligned}$$

The left and right side versions are the ordered versions. But there are plenty of disordered versions where the vectors of a triangularizing basis are calculated in an arbitrary disorder; there is only one restriction in the choice of this disorder when $t \geq 2$: the last step produces simultaneously t isotropic vectors which give a connected subsequence in the resulting basis $(v_1, \ldots v_s)$ (not necessarily at the beginning or at the end). Lemma 1.3 (which involves two subspaces U_1 and U_2 of S on which ϕ is non-degenerate) is the foundation of all these versions; the left side version uses it when $U_2 = 0$, the right side version when $U_1 = 0$, and the disordered versions use it in its full generality. There is an example of disordered algorithm in Section 7.

6. Orthogonal Transformations Inside (S, ϕ)

The notation is the same as in Section 5; here we emphasize the quadratic form q on S such that $q(y) = \phi(y, y)$ for all $y \in S$. When T is a subspace of S, the notation (T, ϕ) means the subspace T provided with the restriction of ϕ to T. When this restriction is non-degenerate, (T, ϕ) is a transformer for (S, q), and induces an orthogonal transformation g on S such that $\mathrm{im}(g - \mathbf{1}_S) \subset T$. Besides, Lemma 1.3 implies $S = T \oplus \mathrm{R}^{\perp}_{\phi}(T) = \mathrm{L}^{\perp}_{\phi}(T) \oplus T$.

Theorem 6.1. *If the restriction of ϕ to T is non-degenerate, the orthogonal transformation g induced by (T, ϕ) maps $\mathrm{R}^{\perp}_{\phi}(T)$ onto $\mathrm{L}^{\perp}_{\phi}(T)$; moreover,*

$$\forall x,\ y \in \mathrm{R}^{\perp}_{\phi}(T), \quad \phi(g(x), g(y)) = \phi(x, y)\,. \tag{6.1}$$

Proof. When $\mathrm{b}_q(x, y) = \phi(x, y) + \phi(y, x)$, the equation (2.1) gives

$$\forall x \in S,\ \forall y \in T, \quad \phi(g(x),\ y) = -\phi(y, x)\,;$$

therefore, $g(x)$ is in $\mathrm{L}^{\perp}_{\phi}(T)$ if and only if x is in $\mathrm{R}^{\perp}_{\phi}(T)$. For all $x,\ y \in S$,

$$\phi(x,\ g(y)) - \phi(g^{-1}(x),\ y) = \phi(x,\ g(y) - y) - \phi(g^{-1}(x) - x,\ y)\,;$$

both $g(y) - y$ and $g^{-1}(x) - x$ belong to T; when x and y belong respectively to $\mathrm{L}^{\perp}_{\phi}(T)$ and $\mathrm{R}^{\perp}_{\phi}(T)$, then $\phi(x,\ g(y) - y)$ and $\phi(g^{-1}(x) - x,\ y)$ vanish, and $\phi(x, g(y)) = \phi(g^{-1}(x), y)$ in accordance with (6.1). □

The equality (6.1) is also true when x and y belong to T: see Theorem 2.1, formula (2.5); in general, it is false when x and y are arbitrary elements of S.

When ϕ is degenerate, Theorem 6.1 gives a property of $\mathrm{LKer}(\phi)$ and $\mathrm{RKer}(\phi)$; as in Section 5, their dimension is denoted by t. The restriction of ϕ to a subspace T of dimension $s-t$ is non-degenerate if and only if $\mathrm{LKer}(\phi) \cap T = T \cap \mathrm{RKer}(\phi) = 0$; when it is non-degenerate, then $\mathrm{LKer}(\phi) = \mathrm{L}^{\perp}_{\phi}(T)$ and $\mathrm{RKer}(\phi) = \mathrm{R}^{\perp}_{\phi}(T)$; therefore, the orthogonal transformation induced by (T, ϕ) maps $\mathrm{RKer}(\phi)$ bijectively onto $\mathrm{LKer}(\phi)$.

Theorem 6.1 also enables us to perform operations on a triangularizing basis $(v_1, \ldots, v_s)$ of (S, ϕ). Let us consider a subsequence $(v_{h+1}, v_{h+2}, \ldots, v_{h+c+d})$ where h, c, d are integers such that $c > 0$, $d > 0$ and $0 \leq h \leq s - c - d$. Let T_1 be the subspace spanned by $(v_{h+1}, \ldots, v_{h+c})$, T_2 the subspace spanned by $(v_{h+c+1}, \ldots, v_{h+c+d})$, and $S' = T_1 \oplus T_2$. When v_j is never isotropic for $h < j \leq h + c$, let g_1 be the orthogonal transformation of (S', q) induced by the transformer (T_1, ϕ); it is equal to the product of the reflections $\mathrm{R}(v_j)$ with $j = h + 1,\ h + 2,\ \ldots,\ h + c$. And when v_j is never isotropic for $h + c < j \leq h + c + d$, let g_2 be the orthogonal transformation of (S', q) induced by the reverse transformer $(T_2, \phi^\dagger)$; it is the product of the reflections $\mathrm{R}(v_j)$ with $j = h+c+d,\ h+c+d-1,\ \ldots,\ h+c+1$. We obtain another triangularizing basis if we replace the subsequence $(v_{h+1},\ \ldots,\ v_{h+c+d})$ with

$$(g_1(v_{h+c+1}),\ \ldots,\ g_1(v_{h+c+d}),\ v_{h+1},\ \ldots,\ v_{h+c}) \qquad \text{or} \quad (v_{h+c+1},\ \ldots,\ v_{h+c+d},\ g_2(v_{h+1}),\ \ldots,\ g_2(v_{h+c}))\,.$$

7. Examples

First example: a rotation in a euclidean plane

Let (V, q) be a euclidean plane over $\mathbb{R}$, provided with a basis (e_1, e_2) such that $q(\xi_1 e_1 + \xi_2 e_2) = \xi_1^2 + \xi_2^2$, whence $\mathrm{b}_q(\xi_1 e_1 + \xi_2 e_2,\ \zeta_1 e_1 + \zeta_2 e_2) = 2(\xi_1\zeta_1 + \xi_2\zeta_2)$. Let g be the rotation of angle 2θ such that $\sin(\theta) \neq 0$ (so that $g \neq \mathbf{1}$); its matrix G is written below. Since $g - \mathbf{1}$ is a bijection $V \to V$, the formula (2.1) gives $\phi(x, y) = -\mathrm{b}_q((g - \mathbf{1})^{-1}(x),\ y)$; therefore, the matix Φ of ϕ is obtained by transposition of $-2(G - \mathbf{1})^{-1}$:

$$G = \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix}, \qquad \Phi = \frac{1}{\sin(\theta)} \begin{pmatrix} \sin(\theta) & \cos(\theta) \\ -\cos(\theta) & \sin(\theta) \end{pmatrix}.$$

Let us consider $v_1 = \cos(\lambda)e_1 + \sin(\lambda)e_2$ and $v_2 = \cos(\mu)e_1 + \sin(\mu)e_2$; which are the couples (λ, μ) for which $g = \mathrm{R}(v_1)\,\mathrm{R}(v_2)$? According to Corollary 3.2, this is true if and only if $\phi(v_1, v_2) = 0$; let us verify that this equation agrees with the answer that has been known for already more than 2000 years:

$$\phi(v_1, v_2) = \begin{pmatrix} \cos(\lambda) & \sin(\lambda) \end{pmatrix} \Phi \begin{pmatrix} \cos(\mu) \\ \sin(\mu) \end{pmatrix} = \frac{\sin(\theta - \lambda + \mu)}{\sin(\theta)}\ ;$$

thus $g = \mathrm{R}(v_1)\,\mathrm{R}(v_2)$ if and only if $\lambda - \mu = \theta$ modulo π.

Second example with a correction procedure

Here (V, q) is given by the basis (e_1, e_2, e_3, e_4) over $\mathbb{R}$, and the quadratic form q such that $q(\sum_{i=1}^4 \xi_i e_i) = \xi_1\xi_2 + \xi_3\xi_4$. Let us apply the left and right side algorithms to the orthogonal transformation g of (V, q) described by the matrix G just below. This matrix G determines over the field $\mathbb{Z}/2\mathbb{Z}$ an orthogonal transformation that is not a product of reflections (it belongs to Dieudonné's exceptional case). The image of $g - \mathbf{1}$ is the subspace S spanned by (e_1, e_3, e_4); $g - \mathbf{1}$ maps $e_3 - e_4,\ e_2 - e_3,\ -e_2$ respectively to $e_1,\ e_3,\ e_4$, and the matrix Φ of ϕ in the basis (e_1, e_3, e_4) easily follows:

$$G = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{pmatrix}, \qquad \Phi = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let us begin the left side algorithm with $\dot v_1 = e_3 + e_4$. Since this choice of $\dot v_1$ is also acceptable for the field $\mathbb{Z}/2\mathbb{Z}$, we are sure to need a correction; indeed, the predictable failure of the algorithm over $\mathbb{Z}/2\mathbb{Z}$ can be explained only by its failure during a correction procedure. By means of the basis $(\dot v_1, e_1, e_3)$ of S, we start the calculation of a basis (y_1, y_2) of $\mathrm{R}_\phi^\perp(\dot v_1)$. For $y_1 = \xi_1\dot v_1 + e_1$, the condition $\phi(\dot v_1, y_1) = 0$ gives $\xi_1 = 0$, whence $y_1 = e_1$ and $\phi(y_1, y_1) = 0$. Therefore, we also calculate $y_2 = \xi_1\dot v_1 + e_3$; the condition $\phi(\dot v_1, y_2) = 0$ gives again $\xi_1 = 0$, whence $y_2 = e_3$, $\phi(y_2, y_2) = 0$, and $\phi(y_1, y_2) = -\phi(y_2, y_1) = 1$. Since this agrees with (5.2), a correction is necessary; since $\phi(y_1, \dot v_1) = 0$ and $\phi(y_2, \dot v_1) = 1$, we follow (8a) in the instruction ⟪8⟫. We set $v_1 = \dot v_1 + \kappa y_1$ (whence $\phi(v_1, v_1) = 1$) and $\dot v_2 = \dot v_1 + \mu y_2$; the condition $\phi(v_1, \dot v_2) = 0$ gives $1 + \kappa\mu = 0$, and the condition $\phi(\dot v_2, \dot v_2) \neq 0$ gives $1 + \mu \neq 0$. As it was predictable, these two conditions cannot be satisfied over the field $\mathbb{Z}/2\mathbb{Z}$. But over $\mathbb{R}$, they are satisfied with $\mu = 1$ and $\kappa = -1$. Consequently, we start the second step of the algorithm with $v_1 = -e_1 + e_3 + e_4$ and $\dot v_2 = 2e_3 + e_4$.

Since $(v_1, \dot v_2, e_4)$ is a basis of S, we set $y_1 = \xi_1 v_1 + \xi_2\dot v_2 + e_4$ and we calculate ξ_1 and ξ_2 with the equations $\phi(v_1, y_1) = \phi(\dot v_2, y_1) = 0$, which give $\xi_1 + 2 = 3\xi_1 + 2\xi_2 + 2 = 0$, whence $\xi_1 = -2$ and $\xi_2 = 2$. According to the instruction ⟪3⟫, we set $v_2 = \dot v_2$ and $v_3 = y_1 = -2v_1 + 2v_2 + e_4$. Here is the basis (v_1, v_2, v_3) and the matrix Φ' of ϕ in this basis:

$$\begin{cases} v_1 = -e_1 + e_3 + e_4\,, \\ v_2 = \qquad\ \ 2e_3 + e_4\,, \\ v_3 = 2e_1 + 2e_3 + e_4\,, \end{cases} \qquad \Phi' = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 3 & 4 & 2 \end{pmatrix}.$$

The conclusion of this calculation is $g = \mathrm{R}(v_1)\,\mathrm{R}(v_2)\,\mathrm{R}(v_3)$.

Now let us start the right side algorithm with $\dot v_3 = e_3 + e_4$ and the basis $(e_1, e_3, \dot v_3)$ of S. The calculation of $y_1 = e_1 + \xi_1 \dot v_3$ such that $\phi(y_1, \dot v_3) = 0$ gives $\xi_1 = 0$ and $y_1 = e_1$. Therefore, we also calculate $y_2 = e_3 + \xi_1 \dot v_3$ such that $\phi(y_2, \dot v_3) = 0$; we find $\xi_1 = -1$ and $y_2 = -e_4$. Thus $\phi(y_1, y_1) = \phi(y_2, y_2) = 0$ and $\phi(y_1, y_2) = -\phi(y_2, y_1) = 1$; and a correction is necessary. Since $\phi(\dot v_3, y_1) = 0$ and $\phi(\dot v_3, y_2) = -1$, we set $v_3 = \kappa y_1 + \dot v_3$ (whence $\phi(v_3, v_3) = 1$) and $\dot v_2 = \mu y_2 + \dot v_3$. The conditions $\phi(\dot v_2, v_3) = 0$ and $\phi(\dot v_2, \dot v_2) \neq 0$ give $-\kappa\mu + 1 = 0$ and $-\mu + 1 \neq 0$; they are satisfied with $\mu = \kappa = -1$. Thus we start the second step with $\dot v_2 = e_3 + 2e_4$ and $v_3 = -e_1 + e_3 + e_4$, and with the basis $(e_4, \dot v_2, v_3)$ of S. We must calculate $y_1 = e_4 + \xi_1 \dot v_2 + \xi_2 v_3$ with the conditions $\phi(y_1, \dot v_2) = \phi(y_1, v_3) = 0$; they give the equations $2\xi_1 + 3\xi_2 = -1 + \xi_2 = 0$, and determine $\xi_2 = 1$ and $\xi_1 = -3/2$. Here is the final result of this calculation:

$$\begin{cases} v_1 = -e_1 - \frac{1}{2}e_3 - e_4\,, \\ v_2 = \qquad e_3 + 2e_4\,, \\ v_3 = -e_1 + e_3 + e_4\,, \end{cases} \qquad \Phi' = \begin{pmatrix} 1/2 & 0 & 0 \\ -2 & 2 & 0 \\ -3/2 & 3 & 1 \end{pmatrix}.$$

As above, $g = \mathrm{R}(v_1)\,\mathrm{R}(v_2)\,\mathrm{R}(v_3)$.

Third example (an ordinary example)

Let (V, q) be the space over $\mathbb{R}$ determined by the orthogonal basis $(e_1, \dots, e_6)$ such that $q(e_i) = 1$ for $i = 1, 2, 3, 4$, and $q(e_i) = -1$ for $i = 5, 6$; and let g be the orthogonal transofrmation of (V, q) given by the following matrix:

$$G = \begin{pmatrix} 3/10 & -3/5 & -4/5 & 2/5 & 0 & -1/2 \\ -2/5 & -1/5 & 2/5 & 4/5 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & -1 \\ -1 & -2 & 0 & -1 & 2 & -1 \\ 1/5 & -2/5 & 4/5 & -2/5 & 1 & 1 \\ -11/10 & -9/5 & -2/5 & -4/5 & 2 & -3/2 \end{pmatrix}.$$

The kernel of $g - \mathbf{1}$ is spanned by $2e_1 - e_2 - e_3$ and $e_1 - e_2 - e_4 + 2e_5 - e_6$. There are well known algorithms to find a convenient basis $(u_1, \dots, u_4)$ of $S = \mathrm{im}(g - \mathbf{1})$; then the matrix Φ of ϕ in this basis is calculated with (2.1):

$$\begin{cases} u_1 = (g-\mathbf{1})(-2e_1 - e_4 - 2e_5) = e_1 + 2e_3 - e_6\,, \\ u_2 = \frac{1}{2}(g-\mathbf{1})(e_3 + 2e_4 + 2e_5) = e_2 - e_3 + e_6\,, \\ u_3 = \frac{1}{2}(g-\mathbf{1})(e_5) = e_4 + e_6\,, \\ u_4 = \frac{1}{2}(g-\mathbf{1})(-2e_2 - 3e_4 - 5e_5) = e_5 - 2e_6\,. \end{cases} \qquad \Phi = \begin{pmatrix} 4 & 0 & 2 & -4 \\ -2 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 3 & -5 \end{pmatrix}.$$

Let us first experiment with the left side algorithm. We begin with $\dot v_1 = u_1$, and the basis $(\dot v_1, u_2, u_3, u_4)$ of S. We calculate $y_1 = \xi_1 \dot v_1 + u_2$ with the condition $\phi(\dot v_1, y_1) = 0$; immediately, we obtain $y_1 = u_2$. We begin the second step with $v_1 = u_1$, $\dot v_2 = u_2$, and the basis $(v_1, \dot v_2, u_3, u_4)$. We calculate $y_1 = \xi_1 v_1 + \xi_2 \dot v_2 + u_3$ with the conditions $\phi(v_1, y_1) = \phi(\dot v_2, y_1) = 0$, which give the equations $4\xi_1 + 2 = -2\xi_1 + \xi_2 - 2 = 0$, whence $\xi_1 = -1/2$, $\xi_2 = 1$, and $y_1 = -\frac{1}{2}u_1 + u_2 + u_3$. Unfortunately, $\phi(y_1, y_1) = 0$ and we must calculate also $y_2 = \xi_1 v_1 + \xi_2 \dot v_2 + u_4$; the equations $4\xi_1 - 4 = -2\xi_1 + \xi_2 + 2 = 0$ give $\xi_1 = 1$, $\xi_2 = 0$ and $y_2 = u_1 + u_4$. Since $\phi(y_2, y_2) = -5$, we begin the third step with $v_1 = u_1$, $v_2 = u_2$ and $\dot v_3 = u_1 + u_4$. In this final step, we calculate $y_1 = \xi_1 v_1 + \xi_2 v_2 + \xi_3 \dot v_3 + u_3$; the wanted conditions give the equations

$$4\xi_1 + 2 = -2\xi_1 + \xi_2 - 2 = 4\xi_1 + 2\xi_2 - 5\xi_3 + 5 = 0\,; \tag{7.1}$$

consequently, $\xi_1 = -1/2$ and $\xi_2 = \xi_3 = 1$. Here is the resulting basis $(v_1, \dots, v_4)$ and the matrix Φ' of ϕ in this basis:

$$\begin{cases} v_1 = u_1\,, \\ v_2 = u_2\,, \\ v_3 = u_1 + u_4\,, \\ v_4 = \frac{1}{2}u_1 + u_2 + u_3 + u_4\,, \end{cases} \qquad \Phi' = \begin{pmatrix} 4 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 4 & 2 & -5 & 0 \\ 0 & 3 & 2 & 1 \end{pmatrix}.$$

We have $g = \mathrm{R}(v_1)\mathrm{R}(v_2)\mathrm{R}(v_3)\mathrm{R}(v_4)$ with $v_1 = e_1 + 2e_3 - e_6$, $v_2 = e_2 - e_3 + e_6$, $v_3 = e_1 + 2e_3 + e_5 - 3e_6$, $v_4 = \frac{1}{2}e_1 + e_2 + e_4 + e_5 - \frac{1}{2}e_6$.

Now let us experiment with the disordered algorithm that gives the vectors of a triangularizing basis in the disorder (v_1, v_4, v_2, v_3). To take advantage of the vanishing of $\phi(u_4, u_1)$, we begin with $\dot v_1 = u_4$, the basis $(\dot v_1, u_1, u_2, u_3)$ and $y_1 = \xi_1 \dot v_1 + u_1$; the condition $\phi(\dot v_1, y_1) = 0$ gives immediately $y_1 = u_1$. Therefore, we start the second step with $v_1 = u_4$,

$\dot{v}_4 = u_1$ and with the basis $(v_1, u_2, u_3, \dot{v}_4)$; we calculate $y_1 = \xi_1 v_1 + u_3 + \xi_2 \dot{v}_4$ with the conditions $\phi(v_1, y_1) = \phi(y_1, \dot{v}_4) = 0$. The resulting equations $-5\xi_1 + 3 = 4\xi_2 = 0$ give $\xi_1 = 3/5$, $\xi_2 = 0$ and $y_1 = u_3 + \frac{3}{5}u_4$, whence $\phi(y_1, y_1) = 3/5$. Therefore, we start the third (and last) step with $v_1 = u_4$, $\dot{v}_2 = u_3 + \frac{3}{5}u_4$, $v_4 = u_1$, and with the basis $(v_1, \dot{v}_2, u_2, v_4)$. We calculate $y_1 = \xi_1 v_1 + \xi_2 \dot{v}_2 + u_2 + \xi_3 v_4$ with the conditions $\phi(v_1, y_1) = \phi(\dot{v}_2, y_1) = \phi(y_1, v_4) = 0$, which give the equations

$$-5\xi_1 + 2 = -2\xi_1 + \frac{3}{5}\xi_2 + \frac{6}{5} = -2 + 4\xi_3 = 0\,; \tag{7.2}$$

consequently, $\xi_1 = 2/5$, $\xi_2 = -2/3$, $\xi_3 = 1/2$. Here is the resulting basis (v_1, v_2, v_3, v_4), and the matrix of ϕ in this basis:

$$\left\{\begin{array}{l} v_1 = u_4, \\ v_2 = u_3 + \frac{3}{5}u_4, \\ v_3 = \frac{1}{2}u_1 + u_2 - \frac{2}{3}u_3, \\ v_4 = u_1, \end{array}\right. \qquad \Phi' = \begin{pmatrix} -5 & 0 & 0 & 0 \\ -2 & 3/5 & 0 & 0 \\ -2/3 & -7/5 & 5/3 & 0 \\ -4 & -2/5 & 2/3 & 4 \end{pmatrix}.$$

Thus $g = \mathrm{R}(v_1)\mathrm{R}(v_2)\mathrm{R}(v_3)\mathrm{R}(v_4)$ with $v_1 = e_5 - 2e_6$, $v_2 = e_4 + \frac{3}{5}e_5 - \frac{1}{5}e_6$, $v_3 = \frac{1}{2}e_1 + e_2 - \frac{2}{3}e_4 - \frac{1}{6}e_6$, $v_4 = e_1 + 2e_3 - e_6$.

To compare these two versions, we compare the square matrices associated with the systems of equations (7.1) and (7.2):

$$\begin{pmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & 2 & -5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -5 & 0 & 0 \\ -2 & 3/5 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

The first matrix is just a lower triangular matrix, with 6 meaningful entries. Along the diagonal of the second matrix, there is a lower triangular submatrix of order 2, and a submatrix of order 1 which would appear to be upper triangular if it were larger; the main fact is that the second matrix contains only 4 meaningful entries. For a space S of arbitrary dimension s, the calculation is shorter if we calculate the vectors of a triangularizing basis $(v_1, \ldots, v_s)$ in this disorder: firstly v_1 and v_s (either (v_1, v_s) or (v_s, v_1)), secondly v_2 and v_{s-1} (either (v_2, v_{s-1}) or (v_{s-1}, v_2)), thirdly v_3 and v_{s-2}, and so forth…

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On Finding Geodesic Equation of Student T Distribution

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Abstract

Student t distribution has been widely applied in the course of statistics. In this paper, we focus on finding a geodesic equation of the two parameter student t distributions. To find this equation, we applied both the well-known Darboux Theorem and a triply of partial differential equations taken from Struik D. J. (Struik, D. J., 1961) or Grey A (Grey A., 1993), As expected, the two different approaches reach the same type of results. The solution proposed in this paper could be used as a general solution of the geodesic equation for the student t distribution.

Mathematical Subject Classification 62E99

Keywords: Darboux theorem, geodesic equation, small sample**,** size, student t distribution, triply partial differential equation.

1. Introduction

The student t distribution was first discovered by W.S. Gosset. Since the Irish brewery for which Gosset was working did not want the other breweries to know the statistical method they were using, Gosset published under the pseudonym of a student. Most statistical textbooks describe the t distribution in the following way: If $X_1, X_2, \ldots\ldots X_n$ are independent, identically distributed, random variables, each having the same normal distribution with the expected value u and standard deviation v, then $\sqrt{n}(\overline{X}-u)/v$ has a unit normal distribution. This statistic can be used in the construction of tests and confidence intervals relating to the value of u, provided that v is known. If v is not known, it is reasonable to replace it by the sample estimator "s", given the statistic $T=\sqrt{n}(\overline{X}-u)/s$. This process has been used for some time without allowing for differences between the distribution of $\sqrt{n}(\overline{X}-u)/v$ and $\sqrt{n}(\overline{X}-u)/s$. Statisticians realized that the two distributions are not identical, but the determination of the actual distribution had difficulties. Gosset obtained the distribution of $T'=T/\sqrt{n-1}$ and gave a short table of it's cumulative distribution function. We can show that T' is distributed as a ratio of a unit normal variable, z, and Chi, $\chi_{(n-1)}$, where the two variables are mutually independent. The divisor $\sqrt{n-1}$ was introduced by Fisher(1925a) who defined t with ν degree of freedom as the distribution of $t_\nu = z(\frac{\chi_\nu}{\nu})^{\frac{1}{2}}$. This quantity is usually called student t and the corresponding distribution is called the student t distribution. In this paper, we used two different algorithms to find the geodesic equation of the student t distribution.

2. List the Fundamental Tensor

The probability density function for the student t distribution is given by:

$$f(x)=\frac{1}{v}\frac{\Gamma(\frac{r+1}{2})}{\sqrt{\pi r}\,\Gamma(\frac{r}{2})}[1+\frac{1}{r}(\frac{x-u}{v})^2]^{-\frac{r+1}{2}} \qquad x\in \mathrm{R}, \quad (\mathrm{u,v})\in \mathrm{R}\times\mathrm{R}_+\} .$$

where u is a location parameter, v is a scale parameter and r is defined as the degree of freedom.

Define

$$a = \frac{1}{r}, \qquad b = \frac{r+1}{2}, \qquad c_r = \frac{\Gamma(\frac{r+1}{2})}{\sqrt{\pi r}\,\Gamma(\frac{r}{2})}.$$

Then,

$$\ln f(x) = \ln c_r - b\ln(1 + a(\frac{x-u}{v})^2) - \ln v. \tag{2.1}$$

From above equation (2.1), we derive the first and second partial derivatives:

$$\frac{\partial \ln f}{\partial u} = \frac{2ab(x-u)}{v^2 + a(x-u)^2}, \qquad \frac{\partial^2 \ln f}{\partial u^2} = \frac{2ab(a(x-u)^2 - v^2)}{(a(x-u)^2 + v^2)^2}. \tag{2.2}$$

$$\frac{\partial \ln f}{\partial v} = \frac{2ab(x-u)^2 v^{-3}}{1 + a(x-u)^2 v^{-2}} - v^{-1}, \qquad \frac{\partial^2 \ln f}{\partial v^2} = v^{-2}\left[1 + \frac{-6ab(x-u)^2 v^{-2}(1 + a(x-u)^2 v^{-2}) + 4a^2 b(x-u)^4 v^{-4}}{(1 + a(x-u)^2 v^{-2})^2}\right] \tag{2.3}$$

$$\frac{\partial^2 \ln f}{\partial v \partial u} = \frac{-4abv(x-u)}{(a(x-u)^2 + v^2)^2} \tag{2.4}$$

Then we take the expected values of (2.2),(2.3)and (2.4) to derive the metric tensor components for the student t distribution:

$$E = -E(\frac{\partial^2 \ln f}{\partial u^2}) = -2ab(\frac{-r}{v^2(r+3)}) = \frac{r+1}{v^2(r+3)}, \tag{2.5}$$

$$F = -E(\frac{\partial^2 \ln f}{\partial v \partial u}) = 0, \tag{2.6}$$

$$G = -E(\frac{\partial^2 \ln f}{\partial v^2}) = \frac{-1}{v^2}(1 - 3\frac{r+1}{r+3}). = \frac{1}{v^2}(\frac{2r}{r+3}) \tag{2.7}$$

More detailed proof for equation (2.5), (2.6) and (2.7) can be found in Chen W.W.S.[3]. Using the above results we can further derive their derivatives and six well known Christoffel Symbols as follows:

$$E_u = 0; \quad E_v = \frac{-2(r+1)}{v^3(r+3)}; \quad \mathrm{G_u} = 0; \quad \mathrm{G_v} = \frac{-4r}{v^3(r+3)}$$

$$EG - F^2 = EG = \frac{2r(r+1)}{v^4(r+3)^2}; \quad \mathrm{F} = 0; \quad \mathrm{F_u} = 0; \quad \mathrm{F_v} = 0 \tag{2.8}$$

$$\Gamma_{11}^{1} = \frac{E_u}{2E} = 0; \qquad \Gamma_{12}^{2} = \frac{G_u}{2G} = 0;$$
$$\Gamma_{11}^{2} = \frac{-E_v}{2G} = \frac{2(r+1)}{v^3(r+3)} \frac{v^2(r+3)}{4r} = \frac{r+1}{2rv} \tag{2.9}$$

$$\Gamma_{22}^{1} = \frac{-G_u}{2E} = 0; \quad \Gamma_{12}^{1} = \frac{E_v}{2E} = \frac{-2(r+1)}{\mathrm{v}^3(r+3)} \frac{\mathrm{v}^2(r+3)}{2(\mathrm{r}+1)} = \frac{\text{-1}}{\mathrm{v}}$$
$$\Gamma_{22}^{2} = \frac{G_v}{2G} = \frac{-4r}{v^3(r+3)} \frac{v^2(r+3)}{4r} = \frac{-1}{v} \tag{2.10}$$

3. The Geodesic Equation

To find the geodesic equation of the student t distribution, we solve a triply of partial differential equations, given in the appendix I. We seek its solution in the following section.

$$\frac{\mathrm{d}^2 u}{ds^2} - \frac{2}{v} \frac{dudv}{dsds} = 0 \tag{3.1}$$

$$\frac{\mathrm{d}^2 v}{ds^2} + \frac{r+1}{2rv} (\frac{du}{ds})^2 - \frac{1}{v} (\frac{dv}{ds})^2 = 0 \tag{3.2}$$

$$ds^2 = \frac{r+1}{v^2(r+3)} du^2 + \frac{2r}{v^2(r+3)} dv^2 \tag{3.3}$$

We only need two out of above three equations to find the student t model geodesic equation. We will choose the first (3.1) and the third (3.3) equations. To simplify the notation, we let

$$p = \frac{\mathrm{du}}{\mathrm{ds}} \qquad \text{then} \quad \frac{\mathrm{dp}}{\mathrm{ds}} - \frac{2}{v}(p\frac{dv}{ds}) = 0 \tag{3.4}$$

$$\frac{\frac{\mathrm{dp}}{\mathrm{ds}}}{\mathrm{p}} - \frac{2}{v} \frac{dv}{ds} = 0 \tag{3.5}$$

Integrate (3.5) on both sides with respect to p, to get

$$\ln p - 2\ln v = C_1 \qquad \text{or} \qquad \ln \mathrm{pv}^{-2} = C_1$$
$$pv^{-2} = e^{C_1} = A_t \qquad \frac{\mathrm{du}}{\mathrm{ds}} = A_t v^2$$

Where C_1 is an arbitrary constant and A_t is a temperary constant. We will define its value later. Finally, we derive:

$$ds^2 = \frac{du^2}{A_t^2 v^4} \tag{3.6}$$

Substitute equation (3.6) into equation (3.3)

$$\frac{du^2}{A_t^2 v^4} = \frac{r+1}{v^2(r+3)} du^2 + \frac{2r}{v^2(r+3)} dv^2$$

$$[(r+3)-A_t^2v^2(r+1)]du^2 = 2rA_t^2v^2dv^2$$
$$\mathrm{du}^2 = \frac{2rA_t^2v^2dv^2}{(r+3)-A_t^2v^2(r+1)} \tag{3.7}$$

Then take the square root of equation (3.7), to get

$$\mathrm{du} = \frac{\pm\sqrt{2r}A_t v dv}{\sqrt{(r+3)-A_t^2v^2(r+1)}}$$

Integrate the equation on both sides to derive the geodesic equation of the student t distribution as follows:

$$\pm u \pm \int \frac{\sqrt{2r}A_t v dv}{\sqrt{(r+3)-A_t^2v^2(r+1)}} = B \tag{3.8}$$

Where A_t and B are an arbitrary constant.

Alternatively, we can find the geodesic equation of the student t distribution by solving one partial differential equation. This idea originated from the French mathematician Darboux and is now known as Darboux's theory.

$$\nabla Z = 1; \quad \text{i.e.} \quad \frac{EZ_v^2 - 2FZ_uZ_v + GZ_u^2}{EG-F^2} = 1$$

where

$$E = -E(\frac{\partial^2 \ln f}{\partial u^2}) = \frac{r+1}{v^2(r+3)}, \quad F = -E(\frac{\partial^2 \ln f}{\partial v \partial u}) = 0,$$
$$G = -E(\frac{\partial^2 \ln f}{\partial v^2}) = \frac{-1}{v^2}(1-3\frac{r+1}{r+3}). = \frac{1}{v^2}(\frac{2r}{r+3})$$

$$\nabla Z = 1; \text{ equivalent to : } \frac{r+1}{v^2(r+3)}Z_v^2 + \frac{2r}{v^2(r+3)}Z_u^2 = \frac{2r(r+1)}{v^4(r+3)^2}$$
$$or \quad (\mathrm{r}+1)Z_v^2 + 2rZ_u^2 = \frac{2r(r+1)}{v^2(r+3)} \tag{3.9}$$

From (3.9), we derive

$$Z_u^2 = \frac{r+1}{v^2(r+3)} - \frac{\mathrm{r}+1}{2r}Z_v^2 = A^2 \tag{3.10}$$

Then (3.10) separated into two parts as follows;

Part 1:

$$Z_u^2 = A^2, \quad Z_u = \pm A \quad \text{or} \quad Z = \pm Au \tag{3.11}$$

Part 2:

$$\frac{r+1}{v^2(r+3)} - \frac{\mathrm{r}+1}{2r}Z_v^2 = A^2$$
$$Z_v = \pm[\frac{2r}{r+1}(\frac{r+1}{v^2(r+3)} - A^2)]^{\frac{1}{2}}$$

$$Z = \pm\int [\frac{2r}{r+1}(\frac{r+1}{v^2(r+3)} - A^2)]^{\frac{1}{2}} dv \tag{3.12}$$

We put (3.11) and (3.12) together to find one general solution for equation (3.9):

$$Z = \pm Au \pm \int [\frac{2r}{r+1}(\frac{r+1}{v^2(r+3)} - A^2)]^{\frac{1}{2}} dv \tag{3.13}$$

Now, to find the Geodesic equation of the student t distribution we only need to differentiate equation (3.13) by the constant A.

$$i.e.\ \frac{\partial Z}{\partial A} = B.$$

$$\pm u \pm \int \frac{\sqrt{2r}Avdv}{\sqrt{\frac{(r+1)^2}{r+3} - A^2v^2(r+1)}} = B \tag{3.14}$$

We found that equations (3.8) and (3.14) are of the same type. The difference is only by a constant. The difference may be adjusted by using the constant A_t.

4. Concluding Remarks

Rao(1945) presented a "geodesic distance" (or "Rao distance"), which has outstanding theoretical properties.

However,it was based on a demanding differential geometrical approach. This "geodesic distance" concept, a generalization of the well-known Mahalanobis distance, had to wait until more interest in differential geometry was raised by Efron. This paper uses a simple econometric problem to demonstrate the reason the student t geodesic equation is useful. Let A be a stock represented by its yield $y \sim N(\mu,\sigma_0^2)$, with the unknown expected yield μ and the known risk σ_0^2. Assume we

want to test $H_0\ \mu = \mu_0$ *versus* $H_a\ \mu \neq \mu_0$ where μ_0 is some specified value with a sample size of one. The optimal test in this situation has a critical region $H = \langle \bar{x} : / |\bar{x} - \mu_0| > t_{1-\alpha/2}\ \sigma_0 \rangle$. The test seeks to answer the question: Is the distance between the two normal populations $N(\mu_0,\sigma_0^2)$ and $N(\bar{x},\sigma_0^2)$, big enough to reject H_0? The answer depends on σ and on the distributional assumption. If we let σ tend to infinitely large,then the distance between $N(\mu_0,\sigma^2)$ and $N(\bar{x},\sigma^2)$ should converge to zero. For σ to tend to zero,then the distance will become infinitely large. For this reason, the family of t distribution should not be identified with a flat plane but with a curved surface. This is why the geodesic equation should be used instead of the t distribution.

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Appendix I

We list the six well known Christoffel Symbols as follows. For detail derivation see Struik or Grey.

$$\Gamma_{11}^{1} = \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}, \qquad \Gamma_{12}^{2} = \frac{EG_u - FE_v}{2(EG - F^2)}$$

$$\Gamma_{11}^{2} = \frac{2EF_u - EE_v - FE_u}{2(EG - F^{2)})}, \qquad \Gamma_{22}^{1} = \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)}$$

$$\Gamma_{12}^{1} = \frac{GE_v - FG_u}{2(EG - F^2)}, \qquad \Gamma_{22}^{2} = \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}$$

In general, the solution of the geodesic equation depends upon a pair of partial differential equations as below.

$$\frac{d^2u}{ds^2} + \Gamma_{11}^{1}\left(\frac{du}{ds}\right)^2 + 2\,\Gamma_{12}^{1}\left(\frac{du}{ds}\frac{dv}{ds}\right) + \Gamma_{22}^{1}\left(\frac{dv}{ds}\right)^2 = 0$$

$$\frac{d^2v}{ds^2} + \Gamma_{11}^{2}\left(\frac{du}{ds}\right)^2 + 2\,\Gamma_{12}^{2}\left(\frac{du}{ds}\frac{dv}{ds}\right) + \Gamma_{22}^{2}\left(\frac{dv}{ds}\right)^2 = 0$$

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Resistance to Noise of Non-linear Observers in Canonical Form Application to a Sludge Activation Model

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Abstract

The aim of this study was to increase the resistance to noise of an observer of a non-linear MISO system transformed into canonical regulation form of order n. For this, the principle idea was to add n observers on the output equations of the main observer. By adjusting the time scale of the output observers, the resistance to noise of the final estimates is considerably increased. The proposed method is illustrated by model simulations based on a non-linear Sludge Activation Model (SAM)

Keywords: non-linear systems, state observers, continuous time

1. Introduction

State observers have been intensely exploited since (Luenberger, 1966), to model, control or identify linear and non-linear systems, including the studies of (Krener & Isidori, 1983; Zheng, Boutat, & Barbot, 2009) relating to non-linear systems transformable into a canonical form. The key idea in such approaches is to produce approximate measures of non-linearity of order 1, as in Extended *Luenberger* Observers (ELO) (Ciccarella, Mora, & Germani, 1993). Approximations of non-linearities in the canonical form (which results in ELO) have already been suggested (Bestle & Zeitz, 1983), and this approach can be extended to higher order approximations (Röbenack & Lynch, 2004). An observer using a partial non-linear observer canonical form (POCF) (Röbenack & Lynch, 2006) has weaker observability and integrability existence conditions than the well-established non-linear observer canonical form (OCF). Non-linear sliding mode observers use a quasi-Newtonian approach, applied after pseudo-derivations of the output signal (Efimov & Fridman, 2011). State observers using Extended Kalman Filters (EKF) provide another method of transforming non-linear systems (Boker & Khalil, 2013), (Rauh, Butt, & Aschemann, 2013). Finding an appropriate method for parameter synthesis remains one of the major difficulties with state observers for non-linear systems. (Tornambè, 1992), (Mobki, Sadeghia, & Rezazadehb, 2015) proposed high-gain state observers to deal with this problem. High-gain state observers reduce observation errors for a range of predetermined amplitudes or fluctuations by making the observations independent of parameters. The weak point of this method is its sensitivity to noise and uncertainty.

In network identification and encryption, observers with delays are used to synchronize chaotic oscillators, as shown in several studies (Ibrir, 2009; Martínez-Guerra, et al., 2011). Noise and uncertainty are not critical factors in such a context. This can be very different in the case of industrial processes, as shown in a recent study (Bodizs, 2011), where the performances of observers using ELO, EKF or Integrated Kalman Filters (IKF) are compared. The influence of noise and uncertainty on these observer types was emphasized, with more reliable results produced by ELO observers, which permit the exact state reconstruction of highly perturbed systems. For PI and ELO observer classes, (Söffker, et al., 2002) demonstrated a compensation effect on measurement errors ; (Khalifa & Mabrouk, 2015) addressed the problem of uncertainty of non-linear models. One way of overcoming the problem of parametric uncertainty is to use adaptive observers (Tyukina, et al., 2013; Farza, et al., 2014) in the particular case where the measurements are only available at discrete instants and have disturbances. Another approach (Mazenc & Dinh, 2014; Thabet, et al., 2014) consists of defining interval observers. Modeling observer systems by Takagi-Sugeno decomposition (Bezzaoucha, et al., 2013; Guerra, et al., 2015) is another possibility, as is the use of models using symmetries and semi-invariants (Menini & Tornambè, 2011), or the use of immersible techniques for systems transformed into a non-linear observer form (Back & Seo, 2008).

A large number of non-linear MISO systems with multiple inputs and a single output can be transformed into state equations using the form :

$$\dot{\underline{z}}(t) = \underline{s}(t) \quad (1a)$$

$$y(t) = \underline{d}^T \ \underline{z}(t) + \Phi(t) \quad (1b)$$

$$\underline{s}(t) = \begin{bmatrix} s_1 \left[\ \underline{z}(t), \underline{u}_1(t) \ \right] \\ \dots \\ s_n \left[\ \underline{z}(t), \underline{u}_1(t) \ \right] \end{bmatrix} \quad (1c)$$

$$\underline{d}^T = \left[\ d_1 \quad \dots \quad d_n \ \right] \quad (1d)$$

with the following definitions :
n : the order of the system of non-linear differential equations
m : number of independant inputs
$\underline{u}_1(t)^T$: the vector $[u_{11}(t), \dots u_{m1}(t)]$ of the m independent inputs
$y(t)$: the measurable output variable
$\underline{z}^T(t)$: the state vector $[\ z_1(t) \dots z_n(t)\ ]$
$\underline{d}^T$: the vector of the output parameters of the system
$\Phi(t)$: the non-linear function of vector $\underline{u}_1(t)$ of the inputs
$s_i \left[\ \underline{z}(t), \underline{u}_1(t)\ \right]$: one of the n non-linear functions of the state vector $\underline{s}(t)$.

Such systems are often found in nonlinear robotic systems in the form of trigonometric functions. Other systems contain non-linear polynomials (strange attractors, *Bernouilli* functions, non-linear springs), polynomial fractions, or various common simple functions . . .
The n non-linear functions of vector $\underline{s}(t)$ employ a vector of m independent inputs $\underline{u}_1(t)$, as well as the state vector $\underline{z}(t)$ as input variables. Such a procedure allows amongst other possibilities the description of bi-linear systems. We limited ourselves in this study to continuous functions in all points of type C^1.
One considers that the measurable output is a linear combination of $\underline{z}(t)$, superimposed on a non-linear function $\Phi \left[\ \underline{u}_1(t)\ \right]$.
For an engineer or physicist, many applications have such a form. Often, non-linear systems (1) are transformable in a regulation canonical form concieved by (Fliess, 1990), and are written :

$$\dot{\underline{x}}(t) = \underline{A} \ \underline{x}(t) + \underline{f}(t) \quad (2a)$$

$$y(t) = \underline{c}^T \ \underline{x}(t) + \Phi(t) \quad (2b)$$

$$\underline{A} = \delta_{ij} \quad j = i + 1, \quad i = 1 \dots n - 1 \quad (2c)$$

$$\underline{c}^T = \left[\ c_1 \quad \dots \quad c_n \ \right] \quad (2d)$$

$$\underline{f}(t)^T = \left[\ 0 \quad \dots \quad 0 \quad \Psi \left[\ \underline{x}(t), \ \underline{U}(t) \ \right] \ \right] \quad (2e)$$

with the following definitions :
$\underline{u}_i(t)$: the $(i-1)-th$ temporal derivative of the vector $\underline{u}_1(t)$, either $\underline{u}_i(t)^T = [\ u_{1i}(t), \dots u_{mi}(t)\ ] \quad i = 2 \dots n$
$\underline{U}(t) = \left[\ \underline{u}_1(t) \dots \underline{u}_n(t)\ \right]$: the $n \times m$ input matrix, with the group of n vectors $\underline{u}_i(t)$
$x_i(t)$: $(i-1)\ th$ temporal derivative of $x_1(t)$
$\underline{x}^T(t)$: state vector $[\ x_1(t), \dots x_n(t)\ ]$
$\underline{c}$: the output parameters vector of the transformed system
$\theta \leqslant n$: index of last coefficient $c_i \neq 0$
$\Psi \left[\ \underline{x}(t), \underline{U}(t)\ \right]$: a scalar non-linear C^1 function
$\underline{A}$: the $n \times n$ matrix of which the last line is zero.

Conversion of the transformed version (2) to the initial representation (1) is performed using :

$$\underline{z}(t) = \underline{g}(t) \quad (3a)$$

$$\underline{g}(t)^T = \left[\ g_1 \left[\ \underline{x}(t), \underline{U}(t) \ \right] \ \dots \ g_n \left[\ \underline{x}(t), \underline{U}(t) \ \right] \ \right] \quad (3b)$$

with $\underline{g}(t)$: the vector of n non-linear inverted transformation functions $g_i \left[\ \underline{x}(t),\ \underline{U}(t)\ \right]$ which link $\underline{x}(t)$ to $\underline{z}(t)$.

In (Schwaller, Ensminger, Dresp-Langley, & Ragot, 2016) a new observer was proposed which was adapted to this transformed form, and which provided non-biased robust estimates of $\underline{x}(t)$. This is not always the case for estimates of $\underline{z}(t)$. Functions $g_i\left[\underline{x}(t)\right]$ (1b) permit linking $\underline{x}(t)$ to $\underline{z}(t)$ (2c) and are called inverted transformations. Because of the non-linearity of $\underline{g}(t)$, small perturbations of estimates of $\underline{x}(t)$ may be considerably increased and strongly disturb estimates of $\underline{z}(t)$. The main aim of this study was to solve this type of situation, by introducing the inverted observer transform functions $g_i\,[\underline{x}(t)]$. Doing this, the resistance to observer noise is affected (Bodizs et al., 2011), and one obtains a tool capable of limiting its impact on estimates of $\underline{z}(t)$.

Definition 1 *Let us define, for the moment, a normalised pulse $\omega_o = 2\pi/T_o$, which introduces a new time scale τ for the representation of the transformed state of the system :*

$$\underline{\dot{x}}(\tau) = \underline{A}\ \underline{x}(\tau) + \underline{f}(\tau) \tag{4a}$$

$$y(\tau) = \underline{\widetilde{c}}^T\ \underline{x}(\tau) + \Phi(\tau) \tag{4b}$$

$$\underline{\widetilde{c}}^T = \left[\ \widetilde{c}_1 \quad \dots \quad \widetilde{c}_n\ \right] \tag{4c}$$

$$\underline{f}(\tau)^T = \left[\ 0 \quad \dots \quad 0 \quad \widetilde{\Psi}\,[\ \underline{x}(\tau), \underline{U}(\tau)]\ \right] \tag{4d}$$

and for the inverse transformation system :

$$\underline{z}(\tau) = \underline{g}(\tau) \tag{5a}$$

$$\underline{g}(\tau)^T = \left[\ g_1\,[\ \underline{x}(\tau), \underline{U}(\tau)\ ] \quad \dots \quad g_n\,[\ \underline{x}(\tau), \underline{U}(\tau)\ ]\ \right] \tag{5b}$$

with :

$$\tau = \omega_o\ t, \quad \dot{x}_n(t) = \dot{x}_n(\tau)\ {\omega_o}^n \tag{6a}$$

$$u_{ij}(t) = u_{ij}(\tau)\ {\omega_o}^{i-1}, \quad x_i(t) = x_i(\tau)\ {\omega_o}^{i-1} \tag{6b}$$

$$\widetilde{c}_i = c_i\ {\omega_o}^{i-1}, \quad z_i(t) = z_i(\tau), \quad i = 1 \dots n \tag{6c}$$

$\underline{f}(\tau)$ and $\underline{g}(\tau)$ are vectors with dimension n. In (4b), $\Phi(\tau) = \Phi\left[\ \underline{u}_1(\tau)\ \right] = \Phi\left[\ \underline{u}_1(t)\ \right]$. Equations (6) define time dilatation or retraction of the state representation and its new parameters, without changing the pattern of the signal $x_i(\tau)$. For the function Ψ, this is translated by the relation of changing the following scale representation :

$$\Psi\,[\ \underline{x}(t), \underline{U}(t)] = {\omega_o}^n\ \widetilde{\Psi}\,[\ \underline{x}(\tau), \underline{U}(\tau)] \tag{7}$$

The function $\widetilde{\Psi}\,[\ \underline{x}(\tau), \underline{U}(\tau)\ ]$ is obtained by replacing every state or command variable by the corresponding one in (6) and dividing everything by ${\omega_o}^n$.

Afterwards, the procedure can be separated into several steps: in section 2, the estimation of the state of the transformed system (4) is dealt with ; in section 3 a new observation method of the inverse transformation functions which permit estimation of state variables (1) is presented ; in section 4 this new approach is applied to observe a system of management of activated sludge in a purification station ; the study is concluded in section 5.

2. Structure of the Observer in Canonical Form

To begin with, let us isolate the componant $x_1(\tau)$ of (4b) which will subsequently serve to determine the observation error. To obtain $y_1(\tau)$, the estimation of variable $x_1(\tau)$, three cases may be distinguished. For $\theta = 1$:

$$y_1(\tau) = \frac{y(\tau) - \Phi(\tau)}{\widetilde{c}_1} \tag{8}$$

For $\theta = 2$, it becomes :

$$\dot{y}_1(\tau) = -\frac{\widetilde{c}_1}{\widetilde{c}_2}\ y_1(\tau) + \frac{y(\tau) - \Phi(\tau)}{\widetilde{c}_2} \tag{9}$$

In the most general case where $\theta > 2$, $y(\tau) - \Phi(\tau)$ is filtered by :

$$\underline{\dot{w}}(\tau) = \underline{K}\ \underline{w}(\tau) + \underline{k}\ \left[\ y(\tau) - \Phi(\tau)\ \right] \tag{10a}$$

$$\underline{K} = \begin{bmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \dots & 0 & 0 & 1 \\ -\frac{\widetilde{c}_1}{\widetilde{c}_\theta} & \dots & \dots & -\frac{\widetilde{c}_{\theta-1}}{\widetilde{c}_\theta} \end{bmatrix} \tag{10b}$$

$$\underline{w}(\tau)^T = \left[\ y_1(\tau) \quad \dots \quad y_{\theta-1}(\tau)\ \right], \quad \underline{w}(0) = \underline{0} \tag{10c}$$

$$\underline{k}^T = \left[\ 0 \quad \dots \quad 0 \quad 1/\widetilde{c}_\theta\ \right] \tag{10d}$$

To analyze the effect of the filter, we rewrite (4b) in scalar form, ignoring $\widetilde{c}_{\theta+1} \; \ldots \; \widetilde{c}_n$, which are all zero :

$$y(\tau) - \Phi(\tau) = \sum_{i=1}^{\theta} \widetilde{c}_i \; x_i(\tau) \tag{11}$$

If (11) is inserted in (10a), (9) or (8) as a function of θ, it becomes :

$$\sum_{i=1}^{\theta} \widetilde{c}_i \; y_i(\tau) = \sum_{i=1}^{\theta} \widetilde{c}_i \; x_i(\tau) \tag{12a}$$

$$\dot{y}_{\theta-1}(\tau) = y_\theta(\tau) \quad \theta \geqslant 2 \tag{12b}$$

The *Laplace* transformation of (12a) gives the transfer function :

$$y_1(s)/x_1(s) = 1 \tag{13}$$

To develop the rest, $y_1(\tau)$ is used to determine the observer error.

Definition 2 *To generate state estimates $\underline{v}(\tau)$ for the system (4), a PI observer structure is defined in (Schwaller, Ensminger, Dresp-Langley, & Ragot, 2016) with :*

$$\dot{\underline{\breve{x}}}(\tau) = \underline{A} \; \underline{\breve{x}}(\tau) + \underline{\widetilde{f}}(\tau) + \underline{\breve{h}} \; \Delta y_1(\tau) \tag{14a}$$

$$\dot{\underline{\widehat{x}}}(\tau) = \underline{A} \; \underline{\widehat{x}}(\tau) + \underline{\breve{A}} \; \underline{\breve{x}}(\tau) + \underline{\widehat{h}} \; \Delta y_1(\tau) \tag{14b}$$

$$\Delta y_1(\tau) = x_1(\tau) - \widehat{x}_1(\tau) \tag{14c}$$

$$\underline{\widetilde{f}}(\tau)^T = \begin{bmatrix} 0 & \ldots & 0 & \widetilde{f}(\tau) \end{bmatrix} \tag{14d}$$

$$\dot{I}_0(\tau) = h_0 \; \Delta y_1(\tau) \tag{14e}$$

$$\widetilde{f}(\tau) = I_0(\tau) + \widetilde{\Psi} \left[\; \underline{v}(\tau), \underline{U}(\tau) \; \right] \tag{14f}$$

$$\underline{\breve{x}}(\tau)^T = \begin{bmatrix} \breve{x}_2(\tau) & \ldots & \breve{x}_n(\tau) \end{bmatrix} \tag{14g}$$

$$\underline{\widehat{x}}(\tau)^T = \begin{bmatrix} \widehat{x}_1(\tau) & \ldots & \widehat{x}_{n-1}(\tau) \end{bmatrix} \tag{14h}$$

$$\underline{v}(\tau)^T = \begin{bmatrix} \widehat{x}_1(\tau) & \underline{\breve{x}}(\tau)^T \end{bmatrix} \tag{14i}$$

$$\underline{\widehat{x}}(0) = \underline{\breve{x}}(0) = \underline{0}, \quad I_0(0) = 0 \tag{14j}$$

$$\underline{\breve{h}}^T = \begin{bmatrix} \underline{0}^T & h_1 \end{bmatrix} \tag{14k}$$

$$\underline{\widehat{h}}^T = \begin{bmatrix} h_n & \ldots & h_2 \end{bmatrix} \tag{14l}$$

$$\underline{h}^T = \begin{bmatrix} \underline{\widehat{h}}^T, & h_1 \end{bmatrix} \tag{14m}$$

$$\underline{A} = \delta_{ij}, \quad j = i+1, \; i = 1 \ldots n-1 \tag{14n}$$

$$\underline{\breve{A}} = \begin{bmatrix} 0 & \ldots & \ldots & 0 \\ \vdots & \ddots & \ldots & \vdots \\ \vdots & & 0 & \vdots \\ 0 & \ldots & \ldots & 1 \end{bmatrix} \tag{14o}$$

with $\underline{\breve{x}}(\tau)$ (14g) and $\underline{\widehat{x}}(\tau)$ (14h) as two distinct state vectors of dimension $n-1$, coupled using the matrices $\underline{A}$ (14n) and $\underline{\breve{A}}$ (14o) of dimension $(n-1) \times (n-1)$. The vectors $\underline{\breve{h}}$ and $\underline{\widehat{h}}$ are also of dimension $n-1$. The matrix $\underline{A}$ is constructed using the Kronecker operator which puts the upper diagonal at 1. The parameters $h_i, \; i = 0 \ldots n$ are the gains of the observer.

Figure 1 illustrates the functional diagram of such an observer of third order.

The augmented vector $\underline{v}(\tau)$ (14i),(14h) and (14g) is used as estimation of $\underline{x}(\tau)$ and as variable of the function $\widetilde{\Psi} \left[\; \underline{v}(\tau), \underline{U}(\tau) \; \right]$ (14f). The state $\underline{\widehat{x}}(\tau)$ (14b) is an observer exploiting the observation error $\Delta y_1(\tau)$ (14c) *via* the gains h_i (14m) serving to correct the state distances between the system and its observer.

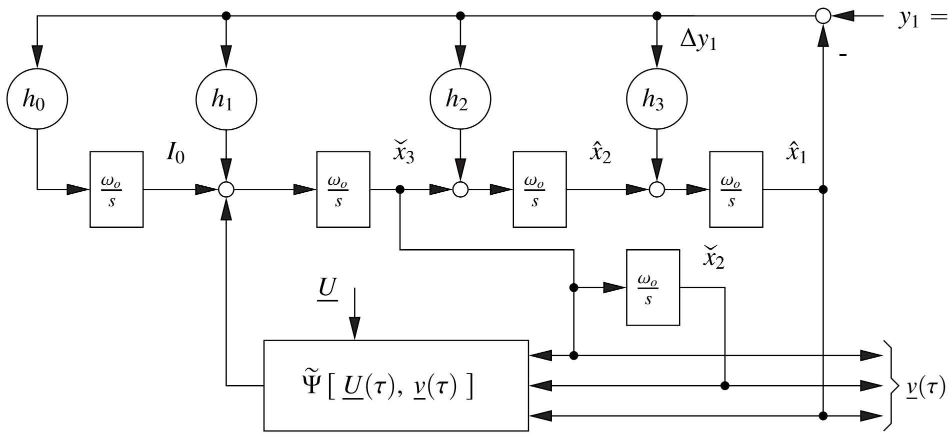


Figure 1. Third order observer

In figure 1, for example, we have :

$$\begin{aligned}
\underline{\widehat{x}}(\tau)^T &= [\ \widehat{x}_1(\tau) \quad \widehat{x}_2(\tau)\ ] \\
\underline{\check{x}}(\tau)^T &= [\ \check{x}_2(\tau) \quad \check{x}_3(\tau)\ ] \\
\underline{v}(\tau)^T &= [\ \widehat{x}_1(\tau) \quad \check{x}_2(\tau) \quad \check{x}_3(\tau)\ ] \\
\underline{\widehat{h}}^T &= [\ h_3 \quad h_2\ ] \\
\underline{\check{h}}^T &= [\ 0 \quad h_1\ ]
\end{aligned}$$

The choice of using two state variables $\underline{\widehat{x}}(\tau)$ and $\underline{\check{x}}(\tau)$ is motivated by the $n-1$ successive integrations of $\dot{\check{x}}_n(\tau)$ in which no $\underline{\widehat{h}}\ \Delta y_1(\tau)$ re-injection error is involved. This allows an increase in the robustness of the estimations to the measurement noise, which in general affects the variable $y_1(\tau)$. One thus overcomes a common weak point of high gain observations, i.e. their sensitivity to measurement noise. The second advantage comes from the non-linear function $\widetilde{\Psi}\,[\underline{v}(\tau), \underline{U}(\tau)]$ which is no longer subjected to the restrictive conditions used in (Schwaller, Ensminger, Dresp-Langley, & Ragot, 2013), and covers the ensemble of the systems described by (Fliess, 1990). The vector $\underline{\widetilde{f}}(\tau)$ (14d), of dimension $n-1$, compensates the effects of $\underline{f}(\tau)$, and of possible external exogenous disturbance of (2) using the integral component $I_0(\tau)$ (14e). One notes that at the second order, for a gain $h_0 = 0$ inhibiting the integrator I_0, the observer becomes similar to that proposed by (Gauthier, Hammouri, & Othman, 1992) for a SISO system.

In (Schwaller, Ensminger, Dresp-Langley, & Ragot, 2016), a full analysis was performed in order to determine the dynamics of the observation error $\Delta y_1(\tau)$ (14c) and its successive derivatives, to characterise stability conditions and also the exponential convergent nature of estimates $\underline{v}(\tau)$. A mthod to synthesize parameters $h_0 \ldots h_n$ was also proposed.

3. Observation of the Original System via the Inverted Transformation Functions

3.1 New observers definitions

In (5b), the inverted transform functions $\underline{g}(\tau)$ allow converting the system in the canonical form of regulation back to the original form (1). Using the estimates $\underline{v}(\tau)$ reconstructed by the observer (14), it is possible to define : (15)

$$\widehat{z}_i(t) = \widehat{g}_i\,[\ \underline{v}(\tau),\ \underline{U}(\tau)\ ] \quad i = 1 \ldots n \tag{15a}$$

$$\underline{\widehat{z}}(t)^T = [\ \widehat{z}_1(t) \quad \ldots \quad \widehat{z}_n(t)\ ] \tag{15b}$$

One thus obtains estimates $\underline{\widehat{z}}(t)$ of $\underline{z}(t)$ (1). If the stability conditions (Theorem 1 of (Schwaller, Ensminger, Dresp-Langley, & Ragot, 2016)) are respected, $\underline{\widehat{z}}(t) \to \underline{z}(t)$ when $\Delta y_1(t) \to 0$. Similarly, $\dot{\underline{\widehat{z}}}(t) \to \dot{\underline{z}}(t)$ when $\Delta y_1(t) \to 0$. One then has :

$$\lim_{\Delta y_1(t) \to 0} \dot{\underline{\widehat{z}}}(t) = \dot{\underline{z}}(t) \tag{16a}$$

$$\dot{\underline{\widehat{z}}}(t)^T = [\ \widehat{s}_1(t) \quad \ldots \quad \widehat{s}_n(t)\ ] \tag{16b}$$

$$\widehat{s}_i(t) = s_i\,[\ \underline{\widehat{z}}(t),\ \underline{u}_1(t)\ ] \quad i = 1 \ldots n \tag{16c}$$

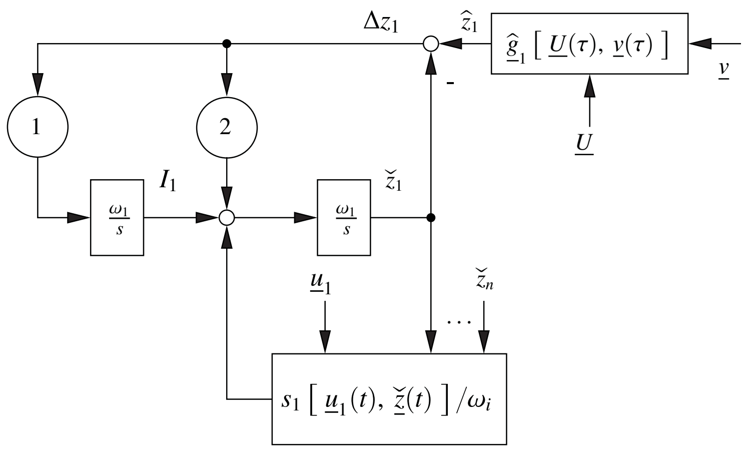


Figure 2. Observer of inverse function $\widehat{z}_1(\tau_1)$

The n estimates $\widehat{z}_i(t)$ can be used as reference inputs to observe n state variables $\breve{z}_i(t)$ which tend towards (15a). Their temporal derivatives tend towards $\dot{\underline{\widehat{z}}}(t)$, which themselves tend towards $\dot{\underline{z}}(t)$ (16). With the model (14), one defines n first order observers. Each is normalised by a pulse ω_i which leads to its dimensionless time definition (17e), possesses its own *Lipschitz* constant, and its specific stability conditions that we have to find. Synthesising the gains h_i (subsection 2.4 of (Schwaller, Ensminger, Dresp-Langley, & Ragot, 2016)) gives $h_0 = 1$ and $h_1 = 2$. The n observers are written :

$$\dot{\breve{z}}_i(\tau_i) = I_i(\tau_i) + 2\ \Delta z_i(\tau_i) + \breve{s}_i(\tau_i) \tag{17a}$$

$$\dot{I}_i(\tau_i) = \Delta z_i(\tau_i) \quad i = 1 \ldots n \tag{17b}$$

$$\Delta z_i(\tau_i) = \widehat{z}_i(\tau_i) - \breve{z}_i(\tau_i) \tag{17c}$$

$$\breve{s}_i(\tau_i) = s_i \left[\underline{\breve{z}}(t),\ \underline{u}_1(t) \right] \ / \ \omega_i \tag{17d}$$

$$\tau_i = \omega_i\ t \tag{17e}$$

with $\underline{\breve{z}}(t) = \left[\breve{z}_1(t) \quad \ldots \quad \breve{z}_n(t) \right]$ the vector of the estimations of $\underline{\widehat{z}}(t)$; $\breve{s}_i(\tau_i)$ is the normalised non-linear function of $\dot{\breve{z}}_i(\tau_i)$. Figure 2 illustrates (15) and (17).

The general calculation procedure is as follows :

- estimation of $\underline{v}(\tau)$ (14i) after treatment of (14);
- estimation of $\underline{\widehat{z}}(t)$ (15) ;
- estimation of the n state distances (17c) ;
- determination of the n non-linear functions $\breve{s}_i(\tau_i)$ (17d) to access the n terms $\dot{\breve{z}}_i(\tau_i)$ (17a) and $\dot{I}_i(\tau_i)$ (17b) ;
- integration of the n equations (17a) to obtain $\underline{\breve{z}}(t)$.

The temporal derivative of (17c) and inserting (17a) in the rest obtained enables one to obtain the expression of $\Delta\dot{z}_i(\tau)$:

$$\Delta\dot{z}_i(\tau_i) = \dot{\widehat{z}}_i(\tau_i) - \dot{\breve{z}}_i(\tau_i) \quad i = 1 \ldots n \tag{18a}$$

$$= \Delta\widetilde{\Psi}_i(\tau_i) - I_i(\tau_i) - 2\ \Delta z_i(\tau_i) \tag{18b}$$

$$\Delta\widetilde{\Psi}_i(\tau_i) = \widehat{s}_i(\tau_i) - \breve{s}_i(\tau_i) \tag{18c}$$

$$\widehat{s}_i(\tau_i) = s_i \left[\underline{\widehat{z}}(t),\ \underline{u}_1(t) \right] \ / \ \omega_i \tag{18d}$$

3.2 Dynamics of the Observer Errors

We now characterise the dynamics of the observer errors by searching the n differential equations of the distances $\Delta z_i(\tau_i)$. Due to the presence of integrators $I_i(\tau_i)$, an extra temporal derivative is necessary to obtain the differential equation of the distances $\Delta z_i(\tau_i)$. To do this, it is necessary to define the following augmented vectors :

$$\underline{\Upsilon}(\tau_i) = \left[\; \underline{u}_1(\tau_i) \quad \underline{u}_2(\tau_i) \; \right] \tag{19a}$$

$$\underline{\widehat{z}}_i(\tau_i)^T = \left[\; \widehat{z}_i(\tau_i) \quad \dot{\widehat{z}}_i(\tau_i) \; \right] \quad i = 1 \ldots n \tag{19b}$$

$$\underline{\breve{z}}_i(\tau_i)^T = \left[\; \breve{z}_i(\tau_i) \quad \dot{\breve{z}}_i(\tau_i) \; \right] \tag{19c}$$

$$\Delta\underline{z}_i(\tau_i)^T = \left[\; \Delta z_i(\tau_i) \quad \Delta\dot{z}_i(\tau_i) \; \right] \tag{19d}$$

$$\underline{\widehat{Z}}(\tau_i)^T = \left[\; \underline{\widehat{z}}_1(\tau_i) \quad \ldots \quad \underline{\widehat{z}}_n(\tau_i) \; \right] \tag{19e}$$

$$\underline{\breve{Z}}(\tau_i)^T = \left[\; \underline{\breve{z}}_1(\tau_i) \quad \ldots \quad \underline{\breve{z}}_n(\tau_i) \; \right] \tag{19f}$$

$$\Delta\underline{Z}(\tau_i)^T = \left[\; \Delta\underline{z}_1(\tau_i) \quad \ldots \quad \Delta\underline{z}_n(\tau_i) \; \right] \tag{19g}$$

The temporal derivative of (18b) is written :

$$\Delta\ddot{z}_i(\tau_i) = \Delta\dot{\tilde{\Psi}}_i(\tau_i) - \Delta z_i(\tau_i) - 2\,\Delta\dot{z}_i(\tau_i) \quad i = i \ldots n \tag{20a}$$

$$\Delta\dot{\tilde{\Psi}}_i(\tau_i) = \dot{s}_i \left[\; \underline{\widehat{Z}}(\tau_i), \underline{\Upsilon}(\tau_i) \; \right] - \dot{s}_i \left[\; \underline{\breve{Z}}(\tau_i), \underline{\Upsilon}(\tau_i) \; \right] \tag{20b}$$

and gives the scalar expression of the differential equations of the observation errors. Using notations (19) gives the matricial writing of (20a) in the form of state equations :

$$\Delta\dot{\underline{z}}_i(\tau_i) = \underline{A}_i \; \Delta\underline{z}_i(\tau_i) + \Delta\dot{\tilde{\underline{\Psi}}}_i(\tau_i) \quad i = 1 \ldots n \tag{21a}$$

$$\underline{A}_i = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \tag{21b}$$

$$\Delta\dot{\tilde{\underline{\Psi}}}_i(\tau_i) = \begin{bmatrix} 0 \\ \Delta\dot{\tilde{\Psi}}_i(\tau_i) \end{bmatrix} \tag{21c}$$

Assuming that the non-linear functions $\dot{s}_i$ are at least locally *Lipschitz* in $\underline{Z}(\tau_i)$, and uniformly bounded in $\underline{\Upsilon}(\tau_i)$ in an invariant set, they are associated with a *Lipschitz* constant L_i :

$$\left\| \; \Delta\dot{\tilde{\Psi}}_i(\tau_i) \; \right\| \leqslant L_i \; \| \; \Delta\underline{Z}(\tau_i) \; \| \quad i = 1 \ldots n \tag{22}$$

Applying the *Lipschitz* inequality to (20b) permits reduction to $\Delta\underline{Z}(\tau_i)$ the number of useful variables to characterise the perturbing difference $\Delta\dot{\tilde{\Psi}}_i(\tau_i)$. For many systems, if functions $\dot{s}_i$ are not globally of a *Lipschitz* type, they can be locally or be transformed adequately into the *Lipschitz* type.

3.3 Convergence of State Observations

Now let us try to analyse the globally asymptotic development of the observation errors and to characterise the limiting stability conditions of each observer (17).

Theorem 1 *Let us consider a MISO system decomposable as described in* (4)*, for which the observer structures* (14) *and* (17) *are used, and related to each other by the inverted transform function* (15a). *If the system function* $\dot{s}_i \left[\; \underline{\widehat{Z}}(\tau_i), \underline{\Upsilon}(\tau_i) \; \right]$ *is locally of the Lipschitz type in* $\underline{\widehat{Z}}(\tau_i)$ *and uniformly bounded in* $\underline{\Upsilon}(\tau_i)$ *in an invariant set, with a Lipschitz constant* L_i (22)*, then the observer* (17) *will be locally stable if the Lipschitz constant* L_i *satisfies the following conditions :*

$$L_i^{\,2} \leqslant \frac{2\,\sigma_i\,\phi_{i1} - 1}{4\,n\,\sigma_i^{\,2}\,\phi_{i1} \left(\dfrac{\lambda_i^{\,2}}{\phi_{i1}} + \dfrac{\phi_{i1}}{4} \right)} \tag{23a}$$

$$L_i^{\,2} \leqslant \frac{2\,\sigma_i\,(\,4\,\phi_{i2} - \phi_{i1}\,) - 1}{4\,n\,\sigma_i^{\,2}\,\phi_{i2} \left(\phi_{i2} + \dfrac{\phi_{i1}^{\,2}}{4\,\phi_{i2}} \right)} \tag{23b}$$

$$\lambda_i > 0, \quad \sigma_i > 0, \quad \phi_{i1} > 0, \quad \phi_{i2} > 0 \quad i = 0 \ldots n \tag{23c}$$

If the system function $\dot{s}_i\left[\underline{\widehat{Z}}(\tau_i),\underline{\Upsilon}(\tau_i)\right]$ *is globally of the Lipschitz type, and if the Lipschitz constant* L_i *satisfy* (23)*, then the observers* (17) *will be globally asymptotically stable.*

Proof. The proof of theorem 1 can be demonstrated by proving the stability of (21a) using an appropriate positive *Lyapunov* function, like the following quadratic function :

$$\breve{V}_n(\tau_i) = \sum_{i=1}^{n} v_i(\tau_i) \tag{24a}$$

$$v_i(\tau_i) = \Delta\underline{z}_i(\tau_i)^T\ \underline{P}_i\ \Delta\underline{z}_i(\tau_i) \tag{24b}$$

$$\underline{P}_i = \begin{bmatrix} \lambda_i & 0 \\ \phi_{i1} & \phi_{i2} \end{bmatrix} \tag{24c}$$

The $\underline{P}_i$ lower triangular matrix are defined as positive and satisfying the *Sylvester* criteria, with (24c). The proof of convergence is linked to the study of the sign of the derivative of the candidate for a *Lyapunov* function. This is obtained after temporal derivation of (24a), and after placing (21a) in the result obtained for terms $\Delta\dot{\underline{z}}_i(\tau_i)$:

$$\dot{\breve{V}}_n(\tau_i) = \sum_{i=1}^{n} \dot{v}_i(\tau_i) \tag{25a}$$

$$\dot{v}_i(\tau_i) = \Delta\underline{z}_i(\tau_i)^T\ \underline{Q}_i\ \Delta\underline{z}_i(\tau_i) + N_i(\tau_i) \quad i = 0\ldots n \tag{25b}$$

$$\begin{aligned} \underline{Q}_i &= \underline{A}_i^{\ T}\underline{P}_i + \underline{P}_i\ \underline{A}_i \\ &= \begin{bmatrix} -\phi_{i1} & 0 \\ 2(\lambda_i - \phi_{i1}) - \phi_{i2} & \phi_{i1} - 4\phi_{i2} \end{bmatrix} \end{aligned} \tag{25c}$$

$$N_i(\tau_i) = \Delta\underline{z}_i(\tau_i)^T\ \underline{S}_i\ \Delta\dot{\underline{\breve{\Psi}}}_i(\tau_i) \tag{25d}$$

$$\underline{S}_i = \underline{P}_i + \underline{P}_i^T \tag{25e}$$

An appropriate choice of $\phi_{i1},\ \phi_{i2}$ can provide negative diagonal coefficients for $\underline{Q}_i$. The criterion of semi-negativity of *Sylvester* is then respected, and the successive minors of $\underline{Q}_i$ will be of opposite sign, ensuring the semi-negativity of the first member on the right of (25b). Verifying the sign of the second member on the right of (25b) involves increasing $N_i(\tau_i)$ using the inequalities of *Schwartz* and *Lipschitz* (22) :

$$N_i(\tau_i) \leqslant \left\| \Delta\underline{z}_i(\tau_i)^T\ \underline{S}_i\ \Delta\dot{\underline{\breve{\Psi}}}(\tau_i) \right\| \tag{26a}$$

$$\leqslant \left\| \Delta\underline{z}_i(\tau_i)^T\ \underline{S}_i \right\| \left\| \Delta\dot{\underline{\breve{\Psi}}}(\tau_i) \right\| \tag{26b}$$

$$\leqslant \left\| \Delta\underline{z}_i(\tau_i)^T\ \underline{S}_i \right\| L \left\| \Delta\underline{z}_i(\tau_i) \right\| \tag{26c}$$

To determine the sign of $\dot{v}_i(\tau_i)$ function, one applies the following inequality :

$$\left\| \underline{a}(\tau_i)^T\underline{b}(\tau_i) \right\| \leqslant \frac{n\ \sigma_i}{2}\ \underline{a}(\tau_i)^T\underline{a}(\tau_i) + \frac{1}{2\ n\ \sigma_i}\ \underline{b}(\tau_i)^T\underline{b}(\tau_i) \tag{27a}$$

$$\underline{a}(\tau_i) = L\ \underline{S}_i^{\ T}\ \underline{z}_i(\tau_i) \tag{27b}$$

$$\underline{b}(\tau_i) = \Delta\underline{z}_i(\tau_i) \tag{27c}$$

to (26c) to obtain the desired increase of $N_i(\tau_i)$:

$$N_i(\tau_i) \leqslant \Delta\underline{z}_i(\tau_i)^T\ \underline{R}_i\ \Delta\underline{z}_i(\tau_i) \quad i = 1\ldots n \tag{28a}$$

$$\underline{R}_i = \frac{n\ \sigma_i L_i^2}{2}\ \underline{S}_i\ \underline{S}_i + \frac{I}{2\ n\ \sigma_i} \tag{28b}$$

In (28a) yields a positive lower triangular matrix $\underline{R}_i$ (28b), the diagonal elements of which are written :

$$r_{jj} = \begin{cases} 2\ n\ \sigma_i\ L_i^2\left(\lambda_i^2 + \phi_{i1}^2/4\right) + \dfrac{1}{2\sigma_i} & j = 1 \\ 2\ n\ \sigma_i\ L_i^2\left(\phi_{i2}^2 + \phi_{i1}^2/4\right) + \dfrac{1}{2\sigma_i} & j = 2 \end{cases} \tag{29}$$

The inequality (28a) permits deduction of (25b) :

$$\dot{v}_i(\tau_i) \leqslant \Delta\underline{z}_i(\tau_i)^T \ \underline{M}_i \ \Delta\underline{z}_i(\tau_i) \tag{30a}$$

$$\underline{M}_i = \underline{Q}_i + \underline{R}_i \tag{30b}$$

With negative functions $\dot{v}_i(\tau_i)$, adding together the diagonal terms of (25c) and (29), and imposing $\underline{Q}_i + \underline{R}_i \leqslant 0$, one obtains the conditions (23). The sum $\underline{Q}_i + \underline{R}_i$ yields an inferior triangular matrix that satisfies *Sylvester* criteria of semi-negativity if inequalities (23a) and (23b) are satisfied. Then, if $\Delta\dot{\tilde{\Psi}}_i(\tau_i)$ (20b) is *Lipschitz* (22), $\dot{v}_i(\tau_i)$ is semi-negative and (21a) is globally and asymptotically stable ; (21a) is locally stable if (22) is locally *Lipschitz* ■

Using the (theorem 2, section 2.3, (Schwaller, Ensminger, Dresp-Langley, & Ragot, 2016)), it is easy to demonstrate that the observers (17) will be exponentially convergent.

4. Application to a Sludge Activation Model

Let us now illustrate the proposed observation method by applying it to a non-linear example with multiple inputs.

4.1 Original Model

For this we choose a simplified treatment model for activated sludge ASM1 similar to that used by (Nagy-Kiss et al., 2010), and structurally of the same types as (1) :

$$\dot{z}_i(t) = s_i(t) \quad i = 1 \ldots 3 \tag{31a}$$

$$y_1(t) = z_1(t) \quad y_2(t) = z_2(t) \tag{31b}$$

with :

$$s_1(t) = k_1 \ \ell_1(t) + k_2 \ z_3(t) - k_3 \ \ell_2(t) \ z_3(t) \tag{32a}$$

$$s_2(t) = k_4 \ \ell_3(t) - k_1 \ \ell_4(t) - k_5 \ \ell_2(t) \ z_3(t) \tag{32b}$$

$$s_3(t) = k_9 \ \ell_5(t) - k_6 \ \ell_6(t) - k_7 \ z_3(t) + k_8 \ \ell_2(t) \ z_3(t) \tag{32c}$$

and non-linear functions :

$$\ell_1(t) = u_{11}(t) \ (\ u_{31}(t) - z_1(t) \ ) \tag{33a}$$

$$\ell_2(t) = \frac{z_1(t) \ z_2(t)}{(\ k_{10} + z_1(t) \ ) \ (\ k_{11} + z_2(t) \ )} \tag{33b}$$

$$\ell_3(t) = u_{21}(t) \ [\ k_{12} - z_2(t) \ ] \tag{33c}$$

$$\ell_4(t) = u_{11}(t) \ z_2(t) \tag{33d}$$

$$\ell_5(t) = u_{11}(t) \ u_{41}(t) \tag{33e}$$

$$\ell_6(t) = u_{11}(t) \ z_3(t) \tag{33f}$$

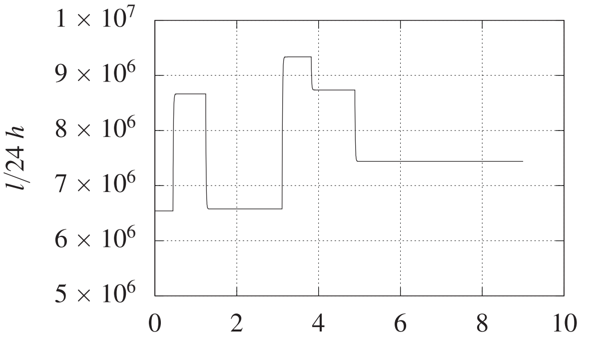
$$\underline{u}_1(t) = \left[\ u_{11}(t) \quad \ldots \quad u_{41}(t) \ \right] \tag{33g}$$

The constants used are given by :

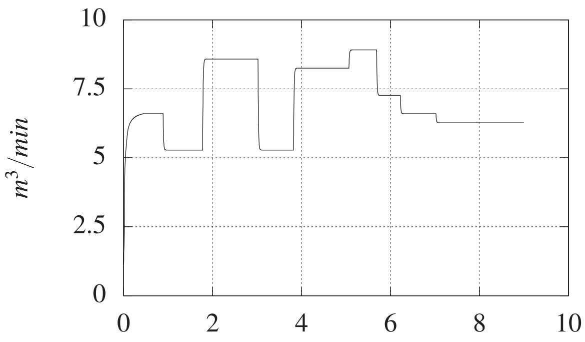
$$\begin{array}{lll} k_1 = 5 \cdot 10^{-11} & k_2 = 1,08 \cdot 10^{-5} & k_3 = 2.872 \cdot 10^{-4} \\ k_4 = 3.5 \ 10^{-4} & k_5 = 9 \cdot 10^{-5} & k_6 = 1.316 \cdot 10^{-12} \\ k_7 = 4.8 \cdot 10^{-6} & k_8 = 7.47 \cdot 10^{-5} & k_9 = 8 \cdot 10^{-11} \\ k_{10} = 20 & k_{11} = 0.2 & k_{12} = 10 \end{array} \tag{34}$$

$\underline{u}_1(t)$ (33g) represent the inputs of the system (figures 3(a),(b),(c),(d) page 10), respectively the input flow of waste water, the flow of injected air, the concentration soluble carbonated substrate recycled, the particle concentration of recycled heterotrophic biomass. All abscissas of the figures are expressed in hours.

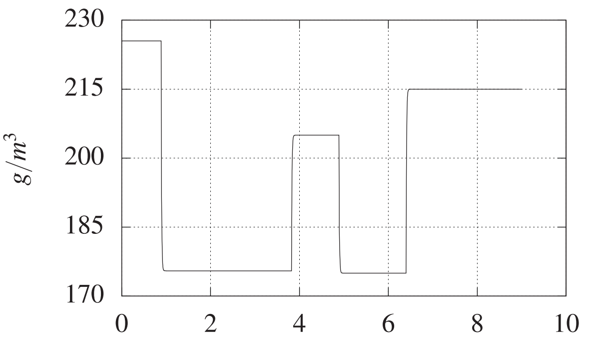
The variables $z_1(t)$, $z_2(t)$ $z_3(t)$ represent the state of the reactor (figures 3(e),(f),(g)), respectively the concentration of rapidly biodegradable substrate, the concentration of dissolved oxygen, the particle concentration of biomass, with (34) its parameters, all known, and $z_1(0) = 4.1$, $z_2(0) = 3.0$, $z_3(0) = 867$ the initial conditions. The sizes $y_1(t)$, $y_2(t)$ (31b) represent the measurable outputs.



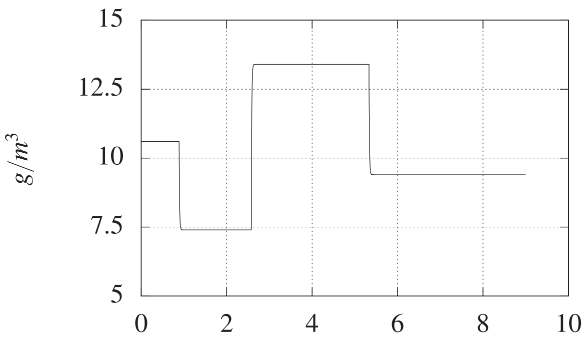
(a) $u_{11}(t)$, water flow



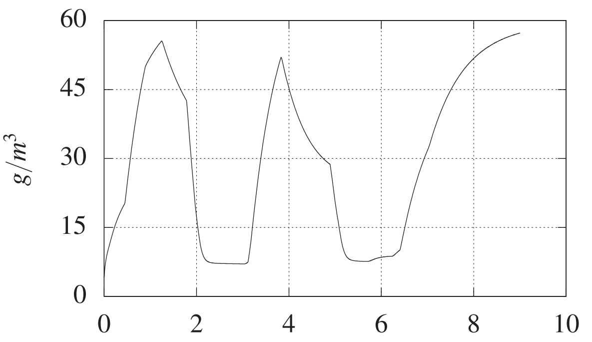
(b) $u_{21}(t)$, air flow



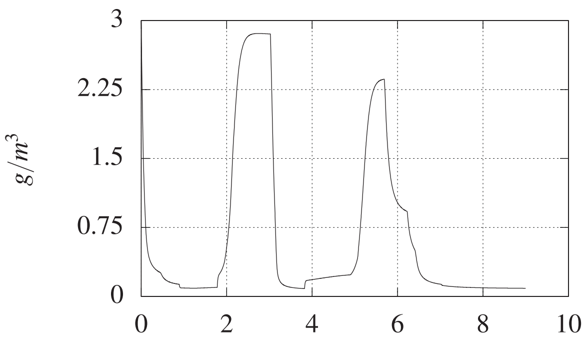
(c) $u_{31}(t)$, injection of carbonated substrate



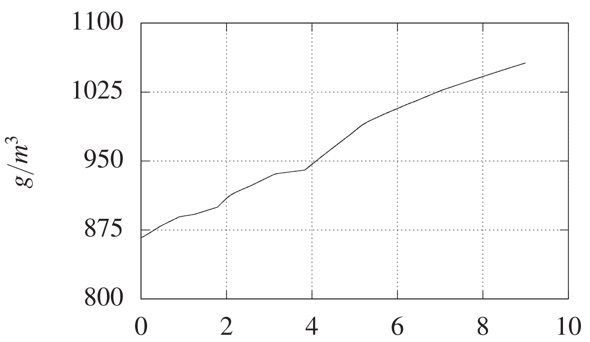
(d) $u_{41}(t)$, injection of biomass



(e) $z_1(t)$, concentration of biodegradable carbonated substrate



(f) $z_2(t)$, concentration of dissolved oxygen



(g) $z_3(t)$, concentration of biomass

Figure 3. Input variables and state variables of the bioreactor

4.2 Model Transformed in a Canonical Form of Regulation

We now transform the two first order differential equations (32b) and (32c) into a single second order differential equation. With this one we determine a function $\Psi[\,\underline{x}(t),\ \underline{U}(t)\,]$ and a new differential equation in a canonical form of regulation. As this application of the general procedure of transformation permits passage from systems (1) to (2) (Fliess, 1990) it will permit the use of an observer similar to that proposed in (14), associated with inverted transformation (15) and with observers (17).

In our example we try to observe the measured output $y_2(t)$ and its successive derivatives, to subsequently determine an estimation of the immeasurable variable $z_3(t)$. From $s_2(t)$ (32b) we deduce $z_3(t)$:

$$z_3(t) = \frac{k_4\ \ell_3(t) - k_1\ \ell_4(t) - \dot{z}_2(t)}{k_5\ \ell_2(t)} \tag{35}$$

The temporal derivative of (35) gives an expression of $\dot{z}_3(t)$ which can be equated to $s_3(t)$ (32c). We thus deduce :

$$\dot{x}_1(t) = x_2(t) \tag{36a}$$
$$\dot{x}_2(t) = \Psi[\,\underline{x}(t),\ \underline{U}(t)\,] \tag{36b}$$
$$\Psi[\,\underline{x}(t),\ \underline{U}(t)\,] = k_4\ \dot{\ell}_3(t) - k_1\ \dot{\ell}_4(t) - k_5\ \ell_7(t) \tag{36c}$$
$$\ell_2(t) = \frac{z_1(t)\ x_1(t)}{(\ k_{10} + z_1(t)\ )\ (\ k_{11} + x_1(t)\ )} = n(t)/d(t) \tag{36d}$$
$$\ell_3(t) = u_{21}(t)\ [\ k_{12} - x_1(t)\ ] \tag{36e}$$
$$\ell_4(t) = u_{11}(t)\ x_1(t) \tag{36f}$$
$$\ell_6(t) = u_{11}(t)\ z_3(t) \tag{36g}$$
$$\ell_7(t) = z_3(t)\ \dot{\ell}_2(t) + s_3(t)\ \ell_2(t) \tag{36h}$$
$$z_2(t) = x_1(t) \tag{36i}$$

System (36) is made up of a second order differential equation, in a canonical regulation form structurally of the same type as that described in (2). The derived functions $\dot{\ell}_3(t),\ \dot{\ell}_4(t)$ of (36c), and $\dot{\ell}_2(t),\ s_3(t)$ of (36h) are defined by :

$$\dot{\ell}_2(t) = \dot{n}(t)\ d(t) - \dot{d}(t)\ n(t)/d(t)^2 \tag{37a}$$
$$\dot{n}(t) = s_1(t)\ x_1(t) + z_1(t)\ x_2(t) \tag{37b}$$
$$\dot{d}(t) = s_1(t)\ (k_{11} + x_1(t)) + x_2(t)\ (k_{10} + z_1(t)) \tag{37c}$$
$$s_1(t) = k_1\ \ell_1(t) + k_2\ g_3(t) - k_3\ \ell_2(t)\ g_3(t) \tag{37d}$$
$$\dot{\ell}_3(t) = u_{22}(t)\ (\ k_{12} - x_1(t)\ ) - u_{21}(t)\ x_2(t) \tag{37e}$$
$$\dot{\ell}_4(t) = u_{12}(t)\ x_1(t) + u_{11}(t)\ x_2(t) \tag{37f}$$
$$s_3(t) = k_9\ \ell_5(t) - k_6\ \ell_6(t) - k_7\ g_3(t) + k_8\ \ell_2(t)\ g_3(t) \tag{37g}$$

Equation $\dot{z}_1(t) = s_1(t)$ defined in (31a) is conserved, and the integration of $s_1(t)$ provides $z_1(t)$, which is the measured output variable defined in (31b). The system of equation of functions of inverted transforms (3b) should permit in our example determination of $z_3(t)$. It is written :

$$z_1(t) = g_1(t) \tag{38a}$$
$$z_2(t) = g_2(t) \tag{38b}$$
$$z_3(t) = g_3(t) = \frac{k_4\ \ell_3(t) - k_1\ \ell_4(t) - x_2(t)}{k_5\ \ell_2(t)} \tag{38c}$$

and permits linking $\underline{x}(t)$ to $\underline{z}(t)$: $z_3(t)$ is a non-linear function of $\underline{U}(t)$ and $\underline{x}(t)$ through $\ell_2(t),\ \ell_3(t)$ and $\ell_4(t)$.

4.3 Time Scaling of $\widetilde{\Psi}\,[\underline{U}(t), \underline{x}(t)]$, Observation of $z_2(t)$ in Canonical Form of Regulation and Determination of the Inverted Transform System

Using (7), one can temporally normalise (36b), putting for $\widetilde{\Psi}(\tau)$ the definition of the following input-output variables :

$$\widetilde{\Psi}(\tau) = \widetilde{\Psi}\left[\ \underline{v}_a(t), \underline{U}(t)\ \right]/{\omega_o}^2 \tag{39a}$$

$$\underline{v}_a(t)^T = \left[\ \breve{z}_1(t) \quad \underline{v}(t)\ \right] \tag{39b}$$

$$\underline{v}(t)^T = \left[\ \widehat{x}_1(t) \quad \breve{x}_2(t)\ \right] \tag{39c}$$

$$\breve{x}_2(t) = \omega_o\ \breve{x}_2(\tau) \quad \widehat{x}_1(t) = \widehat{x}_1(\tau) \tag{39d}$$

$$\underline{U}(t) = \left[\ \underline{u}_1(t) \quad \underline{u}_2(t)\ \right] \tag{39e}$$

The scaling pulse chosen for (39) is $\omega_o = 3.927\ 10^{-2}\ rd/s$. In (39b) we define $\underline{v}_a(t)$ as the vector $\underline{v}(t)$ (14i) augmented by variable $\breve{z}_1(t)$, itself resulting from observation of the measured variable $y_1(t)$.
Note that $\underline{v}(t)$ contains two second order terms because of (36). Equation (39d) allows conversion of time scaled state variables to temporal variables. Taking (7) and definitions (39) into account, function $\widetilde{\Psi}(\tau)$ in scaled time used in (39a) is written :

$$\widetilde{\Psi}(\tau) = \left(k_4\ \dot{\widehat{\ell}}_3(t) - k_1\ \dot{\widehat{\ell}}_4(t) - k_5\ \widehat{\ell}_7(t)\right)/{\omega_o}^2 \tag{40a}$$

$$\widehat{\ell}_1(t) = u_{11}(t)\ (\ u_{31}(t) - \breve{z}_1(t)\ ) \tag{40b}$$

$$\widehat{\ell}_2(t) = \widehat{n}(t)/\widehat{d}(t) \tag{40c}$$

$$\widehat{\ell}_3(t) = u_{21}(t)\ \left[\ k_{12} - \widehat{x}_1(t)\ \right] \tag{40d}$$

$$\widehat{\ell}_4(t) = u_{11}(t)\ \widehat{x}_1(t) \tag{40e}$$

$$\widehat{\ell}_5(t) = u_{11}(t)\ u_{41}(t) \tag{40f}$$

$$\widehat{\ell}_6(t) = u_{11}(t)\ \widehat{z}_3(t) \tag{40g}$$

$$\widehat{\ell}_7(t) = \widehat{z}_3(t)\ \dot{\widehat{\ell}}_2(t) + \dot{\widehat{z}}_3(t)\ \widehat{\ell}_2(t) \tag{40h}$$

$$\widehat{n}(t) = \breve{z}_1(t)\ \widehat{x}_1(t) \tag{40i}$$

$$\widehat{d}(t) = \ (\ k_{10} + \breve{z}_1(t)\ )\ (\ k_{11} + \widehat{x}_1(t)\ ) \tag{40j}$$

The derived functions $\dot{\widehat{\ell}}_3(t)$, $\dot{\widehat{\ell}}_4(t)$ of (40a) and $\dot{\widehat{\ell}}_2(t)$, $\dot{\widehat{z}}_3(t)$ of (40h) are defined by :

$$\dot{\widehat{\ell}}_2(t) = \frac{\dot{\widehat{n}}(t)\ \widehat{d}(t) - \dot{\widehat{d}}(t)\ \widehat{n}(t)}{\widehat{d}(t)^2} \tag{41a}$$

$$\dot{\widehat{n}}(t) = \dot{\widehat{z}}_1(t)\ \widehat{x}_1(t) + \breve{z}_1(t)\ \breve{x}_2(t) \tag{41b}$$

$$\dot{\widehat{d}}(t) = \dot{\widehat{z}}_1(t)\ (\ k_{11} + \widehat{x}_1(t)\ ) + \breve{x}_2(t)\ (\ k_{10} + \breve{z}_1(t)\ ) \tag{41c}$$

$$\dot{\widehat{\ell}}_3(t) = u_{22}(t)\ (\ k_{12} - \widehat{x}_1(t)\ ) - u_{21}(t)\ \breve{x}_2(t) \tag{41d}$$

$$\dot{\widehat{\ell}}_4(t) = u_{12}(t)\ \widehat{x}_1(t) + u_{11}(t)\ \breve{x}_2(t) \tag{41e}$$

$$\dot{\widehat{z}}_1(t) = k_1\ \widehat{\ell}_1(t) + k_2\ \widehat{z}_3(t) - k_3\ \widehat{\ell}_2(t)\ \widehat{z}_3(t) \tag{41f}$$

$$\dot{\widehat{z}}_3(t) = k_9\ \widehat{\ell}_5(t) - k_6\ \widehat{\ell}_6(t) - k_7\ \widehat{z}_3(t) + k_8\ \widehat{\ell}_2(t)\ \widehat{z}_3(t) \tag{41g}$$

The observer in canonical form of regulation of the system(36) is written :

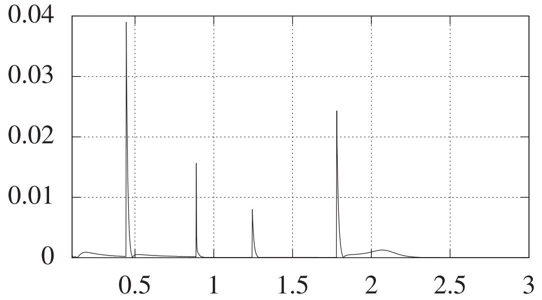
$$\dot{\widehat{x}}_1(\tau) = \breve{x}_2(\tau) + h_2\ \Delta y_1(\tau) \tag{42a}$$

$$\dot{\breve{x}}_2(\tau) = I_0(\tau) + h_1\ \Delta y_1(\tau) + \widetilde{\Psi}(\tau) \tag{42b}$$

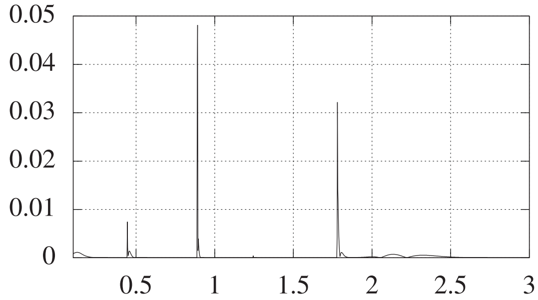
$$\dot{I}_0(\tau) = h_0\ \Delta y_1(\tau) \tag{42c}$$

$$\Delta y_1(\tau) = y_2(\tau) - \widehat{x}_1(\tau) \tag{42d}$$

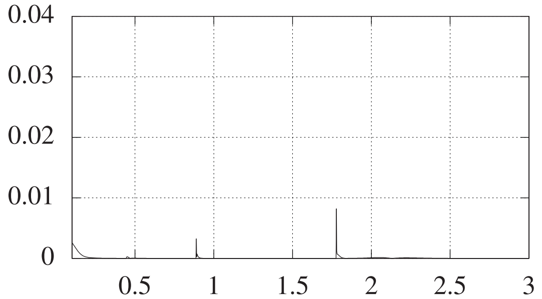
with $y_2(\tau)$ (31b) used to form the observation error $\Delta y_1(\tau)$ (14c).



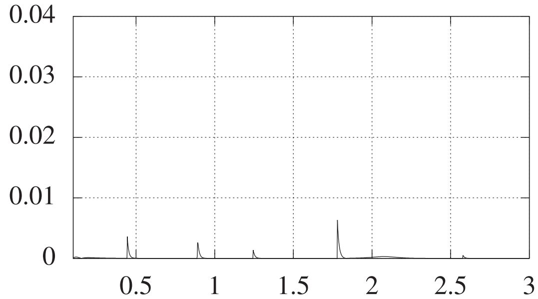
(h) Evolution of $\left\| \Delta\dot{\hat{\Psi}}_1(\tau_1) \right\| / \| \Delta\underline{Z}(\tau_1) \|$



(i) Evolution of $\left\| \Delta\dot{\hat{\Psi}}(\tau) \right\| / \left\| \Delta\underline{y}_a(\tau) \right\|$

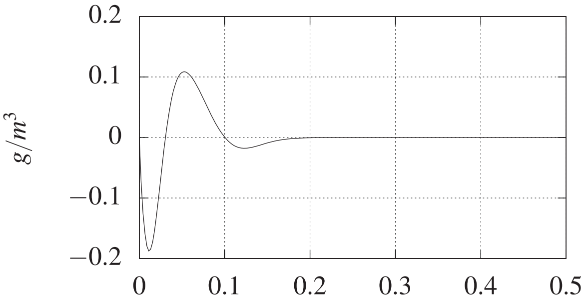


(j) Evolution of $\left\| \Delta\dot{\hat{\Psi}}_2(\tau_2) \right\| / \| \Delta\underline{Z}(\tau_2) \|$

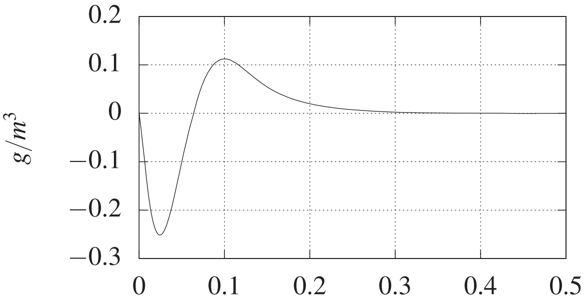


(k) Evolution of $\left\| \Delta\dot{\hat{\Psi}}_3(\tau_3) \right\| / \| \Delta\underline{Z}(\tau_3) \|$

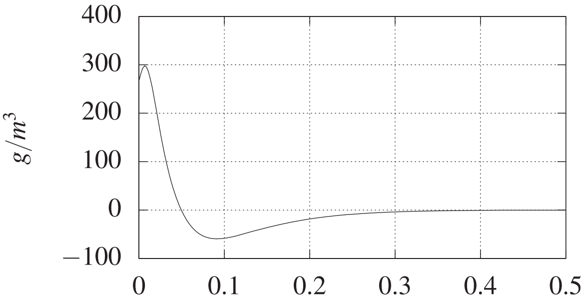
Figure 4. Search for *Lipschitz* constants



(l) Difference $z_1(t) - \breve{z}_1(t)$ in carbonated substrate

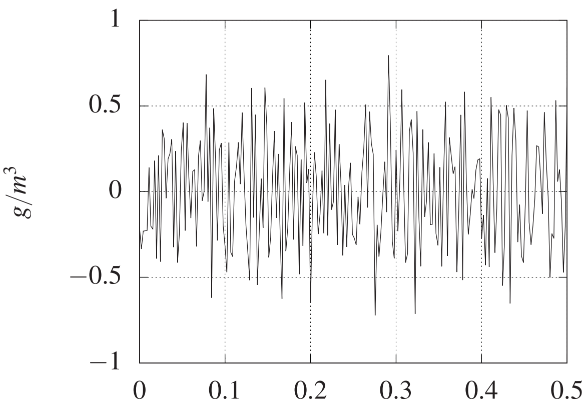


(m) Dissolved oxygen difference $z_2(t) - \breve{z}_2(t)$

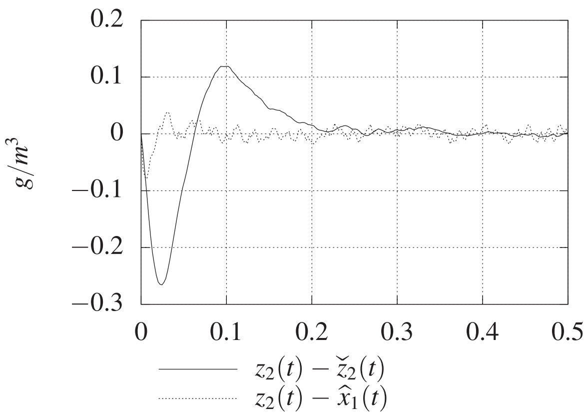


(n) Biomass difference $z_3(t) - \breve{z}_3(t)$

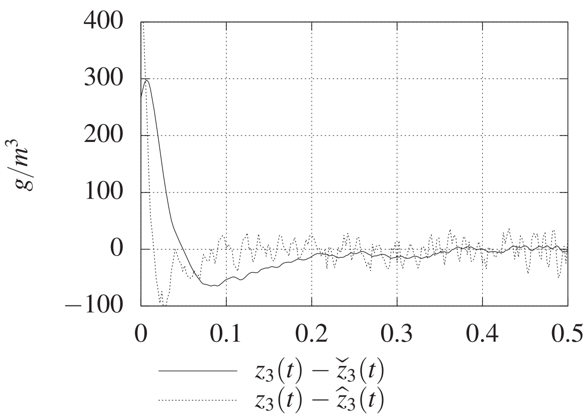
Figure 5. State distances without measurement noise



(o) Difference $z_1(t) - \breve{z}_1(t)$ in carbonated substrate



(p) Dissolved oxygen difference $z_2(t) - \breve{z}_2(t)$



(q) Biomass difference $z_3(t) - \breve{z}_3(t)$

Figure 6. State distances with measurement noise

The estimate $\widehat{z}_3(t)$ of $z_3(t)$ used in (40h), (41f) and (41g) and also the estimate $\widehat{z}_2(t)$ of $z_2(t)$ are defined by the following inverted transform functions :

$$\widehat{z}_1(t) = y_1(t) \tag{43a}$$

$$\widehat{z}_2(t) = \widehat{x}_1(t) \tag{43b}$$

$$\widehat{z}_3(t) = \frac{k_4\ \widehat{\ell}_3(t) - k_1\ \widehat{\ell}_4(t) - \breve{x}_2(t)}{k_5\ \widehat{\ell}_2(t)} \tag{43c}$$

$$\widehat{\underline{z}}(t) = \left[\ \widehat{z}_1(t) \quad \widehat{z}_2(t) \quad \widehat{z}_3(t)\ \right] \tag{43d}$$

4.4 Observation of the Inverted Transformation System

The inverted transformation system (43) serves to form the errors (14c) of three first order observers of the same type as those defined in (17), in order to estimate $\widehat{\underline{z}}(t)$.

The observed outputs and the scaling pulsation choices are given by :

$$\breve{\underline{z}}(t) = \left[\ \breve{z}_1(t) \quad \breve{z}_2(t) \quad \breve{z}_3(t)\ \right] \tag{44a}$$

$$\breve{s}_i(\tau_i) = \frac{s_i\left[\underline{u}_1(t), \breve{\underline{z}}(t)\right]}{\omega_i} \quad i = 1 \dots 3 \tag{44b}$$

$$\omega_1 = \omega_o \quad \omega_2 = \omega_3 = \omega_o/5 \tag{44c}$$

the observer of $\widehat{z}_1(t)$ is written :

$$\dot{\breve{z}}_1(\tau_1) = I_1(\tau_1) + 2\ \Delta z_1(\tau_1) + \breve{s}_1(\tau_1) \tag{45a}$$

$$\breve{s}_1(\tau_1) = \frac{k_1\ \widehat{\ell}_1(t) + k_2\ \breve{z}_3(t) - k_3\ \breve{\ell}_2(t)\ \breve{z}_3(t)}{\omega_1} \tag{45b}$$

$$\breve{\ell}_2(t) = \frac{\breve{z}_1(t)\ \breve{z}_2(t)}{(\ k_{10} + \breve{z}_1(t)\ )\ (\ k_{11} + \breve{z}_2(t)\ )} \tag{45c}$$

$$\dot{I}_1(\tau_1) = \Delta z_1(\tau_1) \tag{45d}$$

$$\Delta z_1(\tau_1) = \widehat{z}_1(t) - \breve{z}_1(\tau_1) \tag{45e}$$

That of $\widehat{z}_2(t)$:

$$\dot{\breve{z}}_2(\tau_2) = I_2(\tau_2) + 2\ \Delta z_2(\tau_2) + \breve{s}_2(\tau_2) \tag{46a}$$

$$\breve{s}_2(\tau_2) = \frac{k_4\ \breve{\ell}_3(t) - k_1\ \breve{\ell}_4(t) - k_5\ \breve{\ell}_2(t)\ \breve{z}_3(t)}{\omega_2} \tag{46b}$$

$$\breve{\ell}_3(t) = u_{21}(t)\ \left[\ k_{12} - \breve{z}_2(t)\ \right] \tag{46c}$$

$$\breve{\ell}_4(t) = u_{11}(t)\ \breve{z}_2(t) \tag{46d}$$

$$\dot{I}_2(\tau_2) = \Delta z_2(\tau_2) \tag{46e}$$

$$\Delta z_2(\tau_2) = \widehat{z}_2(t) - \breve{z}_2(\tau_2) \tag{46f}$$

that of $\widehat{z}_3(t)$:

$$\dot{\breve{z}}_3(\tau_3) = I_3(\tau_3) + 2\ \Delta z_3(\tau_3) + \breve{s}_3(\tau_3) \tag{47a}$$

$$\breve{s}_3(\tau_3) = \begin{bmatrix} k_9\ \widehat{\ell}_5(t) - k_6\ u_{11}(t)\ \breve{z}_3(t) \\ -k_7\ \breve{z}_3(t) + k_8\ \breve{\ell}_2(t)\ \breve{z}_3(t) \end{bmatrix} / \omega_3 \tag{47b}$$

$$\dot{I}_3(\tau_3) = \Delta z_3(\tau_3) \tag{47c}$$

$$\Delta z_3(\tau_3) = \widehat{z}_3(t) - \breve{z}_3(\tau_3) \tag{47d}$$

The observer (45) is there to counteract the effect of measurement noise superimposed on $z_1(t)$, which has sometimes a very great impact on the estimates $\widehat{z}_3(t)$ (43c), due to the term $\widehat{\ell}_2(t)$ in the denominator.
The scaling of (46b) and (47b), parts of (45b) is defined in (44c). This has for effect to strongly reduce the noise on estimates $\breve{z}_2(t)$ and $\breve{z}_3(t)$.

We now try to determine the *Lipschitz* constants that subsequently will allow defining the stability conditions of each observer. We thus start by looking for L in (36) using the same calculation method as that explained in ((Schwaller, Ensminger, Dresp-Langley, & Ragot, 2016), section 3.1).

With (31)-(34) and (38c) we get $x_2(t)$ by using $\underline{z}(t)$, $\ell_2(t)\ \ell_3(t)$ and $\ell_4(t)$. Then it is possible with (36c) and (38b) to calculate $\underline{x}(t)$ and then $\widetilde{\Psi}\left[\underline{x}(t), \underline{U}(t)\right]/{\omega_o}^2$.

Using the initial conditions $\underline{z}(0)$ with the same method of calculation, we can determine $\widetilde{\Psi}\left[\underline{x}(0), \underline{U}(t)\right]/{\omega_o}^2$. We then calculate the state distance $\Delta\underline{y}(\tau)$. Numerical derivation of $\widetilde{\Psi}\left[\underline{x}(t), \underline{U}(t)\right]/{\omega_o}^2 - \widetilde{\Psi}\left[\underline{x}(0), \underline{U}(t)\right]/{\omega_o}^2$ permits determination of the augmented vector $\underline{y}_a(\tau)$ of observation error and to obtain a *Lipschitz* constant adapted to the observer (42). Figure 4 (b) page 13 illustrates this procedure and allows choosing a constant $L = 0.15$. The abscissa is represented only for the first three hours of recording, the region where convergence of observers is expected. using the same stability conditions explained in ((Schwaller, Ensminger, Dresp-Langley, & Ragot, 2016), section 2.3 and 2.4), we fix parameter $\phi_3 = 2$, $\phi_2 = 2$, $\phi_1 = 4$. We choose $\lambda = 1/8$, $\sigma = 1$ and obtain the limiting conditions to respect to synthesise the gains h_i :

$$h_0 \geqslant 0.273 \quad h_1 \geqslant 0.512 \quad h_2 \geqslant 1.3175 \tag{48}$$

Using $\nu = 1$ and $n = 2$, we get :

$$\underline{h}_a = \left[\ 1 \quad 3 \quad 3\ \right] \tag{49}$$

which respects the conditions (48).

We now try to determine constants $L_1,\ L_2,\ L_3$ of inverted transformation observer functions. We use a similar procedure to estimate $\Delta\underline{Z}(\tau_i),\ \Delta\dot{\tilde{\Psi}}_i(\tau_i)$ and their respective modules. Figures 4(a),(c) and (d) allows the choice of *Lipschitz* constants $L_1 = 0.1,\ L_2 = L_3 = 0.02$ (22).

If we fix $\phi_{i1} = \phi_{i2} = 2,\ \sigma_i = 1,\ n = 3,\ \lambda_i = 1/8$ for $i = 1 \ldots 3$ the stability conditions (23) are respected, and the three observers of $\widehat{\underline{g}}(\tau)$ will be stable and properly damped.

4.5 Simulations and Results Obtained

The aim of the simulation is to observe the overall stabilisation of observers to an initial difference in biomass concentration.

The initial conditions of (39) are fixed at

$$I_0(0) = 0,\ \breve{x}_2(0) = -0.164,\ \widehat{x}_1(0) = z_2(0) = 3$$

Those of (45) at $\breve{z}_1(0) = z_1(0) = 4.1$, those of (46) at $\breve{z}_2(0) = 3$, those of (47) at $\breve{z}_3(0) = 600$.

In figures 5(a)- 5(c) the exponential convergent reduction of the state distances $z_i(t) - \breve{z}_i(t)$ are visualised for zero measured noise on the outputs $y_1(t)$ and $y_2(t)$. The same test is performed by adding two bandwidth limited white noise to outputs $y_1(t)$ and $y_2(t)$. These uncorrelated noises have an amplitude of 1% on each of the variables. Figure 4(h) permits verification that the dynamics of convergence of $\breve{z}_3(t)$ is conserved. The normative pulse ω_o chosen for (39) and (45) allow reduction of residual noise by about 10% compared with the measured variables and to contain that still present in $\widehat{z}_3(t)$. This setting permits however to have a rate of convergence of $\breve{z}_1(t)$ and $\widehat{x}_1(t)$ of the same order as the abrupt variations that are seen in $z_1 t)$ and $z_2(t)$.
In figures 6(a)-(c) page 14 the smoothing effect on the estimates of observer (46) and (47) is illustrated : division by 5 on the noise on $y_2(t)$ for $\breve{z}_2(t)$ and fluctuations of 2% on superimposed noise compared with the full scale for $\breve{z}_3(t)$.

5. Conclusions and Perspectives

Observation in canonical form of regulation that is proposed in (Schwaller, Ensminger, Dresp-Langley, & Ragot, 2016) did not take into account the effect of measurement noise on the inverted transformation, which allowed passing from the observation of transformed systems to the non-transformed state space. Certain non-linear functions, because of their nature, can greatly amplify the effect of extraneous perturbations on the final estimations. Observers of inverted transformation functions limit this type of effect. The time scale of each observer affects the stability conditions of each observer, *via* the value of the *Lipschitz* constant. This also greatly influences the existing noise on the estimated variables. By reducing the pulse ω_i of the observers (17), the *Lipschitz* constant L_i is reduced, and similarly the magnitude of remaining noise on estimates $\breve{z}_i(t)$ and one increases the convergence time. Setting the rate of convergence of each observer can be done independently.
Observer stability and synthesising observer gains employ demonstrations published in the previous study.
The proposed technique can be applied to other observers (Gauthier, Hammouri, & Othman, 1992) or to different high gain observers. Observation of inverted transformation functions opens the route to identification on line of parameters of n equation of the state of vector $\underline{s}(t)$ (1c). In fact, it is possible to consider using the n functional distances between $\dot{\widehat{z}}_i(t)$ and $s_i\left[\underline{u}_1(t), \underline{\breve{z}}(t)\right]$ to identify parameters of the n functions $s_i\left[\ \underline{u}_1(t), \underline{z}(t)\ \right]$ (1c). This could provide a means of dealing with parametric uncertainty in state equations of the system (1), as well as external perturbations, which are already compensated by the integral component of the observer (14e).
Finally, by slightly modifying the filter (8)-(10), it can be envisaged to extend the proposed method to multivariable MIMO systems with multiple outputs.

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The Cardinality of the Set of Zeros of Homogeneous Linear Recurring Sequences over Finite Fields

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Abstract

Consider homogeneous linear recurring sequences over a finite field $\mathbb{F}_q$, based on the irreducible characteristic polynomial of degree d and order m. We give upper and lower bounds, and in some cases the exact values of the cardinality of the set of zeros of the sequences within its least period. We also prove that the cyclotomy bound introduced here is the best upper bound as it is reached in infinitely many cases. In addition, the exact number of occurrences of zeros is determined using the correlation with irreducible cyclic codes when $(q^d-1)/m$ follows the quadratic residue conditions and also when it has the form $q^{2a}-q^a+1$ where $a \in \mathbb{N}$.

Keywords: linear recurring sequences, irreducible cyclic codes, weights of cyclic codes.

1. Introduction

Let $\mathbb{F}_q$ be the finite field with q elements where $q = p^m$ for prime p. Let k be a positive integer, and let $a_0, a_1, \ldots, a_{k-1}$ be given elements of $\mathbb{F}_q$. A sequence $s_0, s_1, \ldots$ of elements of $\mathbb{F}_q$ satisfying the relation

$$s_{n+k} = a_{k-1}s_{n+k-1} + a_{k-2}s_{n+k-2} + \ldots + a_0 s_n \qquad for \;\; n = 0, 1, \ldots \tag{1}$$

is called a (kth-order) homogeneous linear recurring sequence in $\mathbb{F}_q$. The terms $s_0, s_1, \ldots s_{k-1}$, which determine the complete sequence uniquely, are referred to as the initial values. A relation in the form of (1) is called a (kth-order) homogeneous linear recurrence relation. Let $s_0, s_1, \ldots$ be a kth order homogeneous linear recurring sequence in $\mathbb{F}_q$ satisfying the linear recurrence relation (1), where $a_j \in \mathbb{F}_q$ for $0 \leq j \leq k-1$. The polynomial

$$f(x) = x^k - a_{k-1}x^{k-1} - a_{k-2}x^{k-2} - \ldots - a_0 \in \mathbb{F}_q[x]$$

is called a *characteristic polynomial* of the linear recurring sequence. For $s_0, s_1, \ldots$ homogeneous linear recurring sequence in $\mathbb{F}_q$, $m(x) \in \mathbb{F}_q[x]$ is said to be the minimal polynomial of the sequence if it has the following property: a monic polynomial $f(x) \in \mathbb{F}_q[x]$ of positive degree is a characteristic polynomial of $s_0, s_1, \ldots$ if and only if $m(x)$ divides $f(x)$.

Definition *Let $f \in \mathbb{F}_q[x]$ be a non zero polynomial. If $f(0) \neq 0$, then the least positive integer e for which $f(x)$ divides $x^e - 1$ is called the order of f which is denoted by ord(f).*

Theorem 1. (Lidl & Niederreiter, 1994) *Let $s_0, s_1, \ldots$ be a homogeneous linear recurring sequence in $\mathbb{F}_q$ with minimal polynomial $m(x) \in \mathbb{F}_q[x]$. Then the least period of the sequence is equal to ord($m(x)$).*

Linear recurring sequences were discussed for many years with a substantial development in the study of examining zeros and determining effective bounds for the set of zeros over infinite fields (Everest, Poorten, Shparlinski & Ward, 2003). Here we will consider homogeneous linear recurring sequences over finite fields based on irreducible minimal polynomials of certain degree d and order m. Let $P(d, m)$ be the set of all irreducible polynomials over $\mathbb{F}_q$ of degree d and order m. For $f \in P(d, m)$ and $I \in (\mathbb{F}_q^d)^* = (\mathbb{F}_q^d) \setminus \{0\}$, let $S(I, f) := \{s_n(I, f) | 1 \leq n \leq m\}$ be the first m terms (terms within the least period) of the homogeneous linear recurring sequence S over $\mathbb{F}_q$. Let $\mathcal{A} := \{Z(S(I, f)) | I \in (\mathbb{F}_q^d)^*, f \in P(d, m)\}$ be the set of zeros. Let $t = (q^d-1)/m$. We will always assume that $t > 1$. If $t = 1$ then the polynomials in $P(d, m)$ are primitive and the number of zeros in the sequence is $q^{d-1} - 1$ (Lidl & Niederreiter, 1994). However, in the general case such an equitable distribution of zeros cannot be expected. Theorem 6.84 in Lidl and Niederreiter (1994) provides an estimate for the number of occurrences of zeros based on Gaussian sums and Mullen and Panario (2013) provides an improved bound. Table 1 gives some observations on the number of zeros of some linear recurring sequences over $\mathbb{F}_2$ computed via MAPLE (Kottegoda, 2010, Appendix I-VIII) with the degrees and orders of their corresponding irreducible minimal polynomials. In this paper, in addition to explaining why there are so few choices for the number of zeros, we will give an accurate bound for the cardinality of the set of zeros, also providing formulas for the exact number of zeros when t has the form $q^{2a} - q^a + 1$ where $a \in \mathbb{N}$.

Table 1. Zeros of some homogeneous linear recurring sequences over $\mathbb{F}_2$ based on degree d and order m irreducible minimal polynomials.

d	m	Number of zeros	Cardinality
8	51	27, 19	2
8	85	37, 45	2
9	73	33, 37, 45	3
10	93	45, 61	2
10	341	181, 165	2
11	89	49, 41, 33	3
12	65	39, 37, 35, 33, 31, 29, 27, 25	8
12	91	55, 51, 47, 43, 39	5
12	105	73, 57, 49	3
12	195	107, 99, 91	3
12	273	153, 141, 133, 129	4
12	315	155, 187	2
12	455	231, 199	2
12	585	305, 289, 281	3
12	819	435, 403	2
12	1365	693, 661	2
14	381	253, 189	2
14	5461	2773, 2709	2
15	1057	573, 553, 537, 525, 517, 513	6
15	4681	2361, 2345, 2265	3
16	3855	1807, 1935	2
16	771	411, 395, 387, 379, 363, 355	6
16	1285	669, 653, 645, 637, 621, 613, 581	7
16	4369	2225, 2185, 2177, 2169, 2097	5

Section 2 proves that the cardinality of the set of zeros is at most the number of q-cyclotomy classes in $\mathbb{Z}_t$, namely, the cyclotomy bound.

In section 3, results on irreducible cyclic codes are used to show $|\mathcal{A}| = 2$ if t has the form $q^{2a} - q^a + 1$ and also gives the exact values for $\mathcal{A}$ in this case. We also get a lower bound on $|\mathcal{A}|$ when $q = 2$ using results from Wolfmann (2005). Exact values for $|\mathcal{A}|$ when t follows the quadratic residue conditions are also discussed. Lastly, we show that the cyclotomy bound given in section 2 is the best bound as it is reached infinitely often, assuming the Generalized Riemann Hypothesis.

2. Cyclotomy Bound

2.1 Construction of the Cyclotomy Bound

First we will define the following equivalence relation on $\mathbb{Z}_t$.

Definition *For $a, b \in \mathbb{Z}_t$ define $a \sim b$ iff $q^u a \equiv b \pmod t$ for some $u \in \mathbb{Z}$.*

Definition *Let t be relatively prime to q. The cyclotomy class of q (or q-cylcotomy coset) modulo t containing i is defined by*

$$C_i = \{(iq^j \pmod t) \in \mathbb{Z}_t \mid j = 0, 1, \ldots\}$$

which is the equivalence class that contains i in the above mentioned equivalence relation.

Let C denote the set of all equivalence classes. The following theorem explains that when the characteristic polynomial is irreducible, a suitable trace form can be used to represent the terms of the linear recurring sequence S.

Theorem 2. (Lidl & Niederreiter, 1994) *Let $s_0, s_1, \ldots$ be a kth-order homogeneous linear recurring sequence in $K = \mathbb{F}_q$ whose characteristic polynomial $f(x)$ is irreducible over K. Let α be a root of $f(x)$ in the extension field $F = \mathbb{F}_{q^k}$. Then there exists a uniquely determined $\theta \in F$ such that*

$$s_n = Tr_{F/K}(\theta\alpha^n) \qquad for \ \ n = 0, 1, \ldots$$

Theorem 3 below gives the upper bound for the cardinality of the set of zeros.

Theorem 3. *Consider the homogeneous linear recurring sequences over $\mathbb{F}_q$ based on an irreducible minimal polynomial of degree d and order m. Set $t = (q^d - 1)/m$. Then for the set of numbers of zeros $\mathcal{A}$, we have $|\mathcal{A}| \leq |C|$.*

Proof.

Let $f \in P(d, m)$. By Theorem 2, there exists a root of f, $\beta \in \mathbb{F}_{q^d}$ and $\theta \in \mathbb{F}^*_{q^d}$ such that the nth term of the sequence S is given by,

$$s_n(I, f) = Tr_{\mathbb{F}_{q^d}/\mathbb{F}_q}(\theta\beta^n), \quad for \ \ all \ \ n, \quad 1 \leq n \leq m.$$

Fix a primitive element $\alpha \in \mathbb{F}_{q^d}$. Then order of $\beta = m$ and hence $\beta = \alpha^{rt}$ where $t = (q^d - 1)/m$ and $(r, m) = 1$. Define

$$s_n(\theta, t) := Tr_{\mathbb{F}_{q^d}/\mathbb{F}_q}(\theta\alpha^{tn}).$$

Hence

$$s_n(I, f) = Tr_{\mathbb{F}_{q^d}/\mathbb{F}_q}(\theta\beta^n) = Tr_{\mathbb{F}_{q^d}/\mathbb{F}_q}(\theta\alpha^{rtn}) = s_n(\theta, rt).$$

Therefore,

$$\mathcal{A} = \{Z(S(\theta, rt)) \mid \theta \in \mathbb{F}^*_{q^d}, t = (q^d - 1)/m, (r, m) = 1\} \tag{2}$$

Lemma 1. First Reduction : *For $(r, m) = 1$, $Z(S(\theta, t)) = Z(S(\theta, rt))$.*

Proof. Since $(r, m) = 1$, there exists a u such that $ur \equiv 1 \pmod m$ and then $urt \equiv 1 \pmod{q^n - 1}$. Hence

$$s_k(\theta, t) = Tr_{K/F}(\theta\alpha^{tk}) = Tr_{K/F}(\theta\alpha^{kurt}) = s_{ku}(\theta, rt)$$

and $s_k(\theta, rt)$ is simply $s_k(\theta, t)$ in a new order. Therefore

$$Z(S(\theta, t)) = Z(S(\theta, rt)).$$

□

Now $\mathcal{A}$ in (2) can be given as follows:

$$\mathcal{A} = \{Z(S(\theta, t)) \mid \theta \in \mathbb{F}^*_{q^d}, \ t = (q^d - 1)/m\} \tag{3}$$

Define

$$r_n(a, t) := Tr_{\mathbb{F}_{q^d}/\mathbb{F}_q}(\alpha^{a+tn}), \quad for \ some \ \ a \in \mathbb{N}.$$

Since $\theta \in \mathbb{F}^*_{q^d}$, let $\theta = \alpha^k$. Then

$$s_n(\theta, t) = Tr_{\mathbb{F}_{q^d}/\mathbb{F}_q}(\theta\alpha^{nt}) = Tr_{\mathbb{F}_{q^d}/\mathbb{F}_q}(\alpha^{k+tn}) = r_n(k, t).$$

Hence $\mathcal{A}$ in (3) can be written as the following

$$\mathcal{A} = \{Z(R(k, t)) \mid t = (q^d - 1)/m, \ 0 \leq k \leq q^d - 1\} \tag{4}$$

where R denotes the sequence $r_1, r_2, \ldots$

Lemma 2. Second Reduction : *If $k_1 \equiv k_2 (mod \ \ t)$ then $Z(R(k_1, t)) = Z(R(k_2, t))$.*

Proof. If $k_2 = k_1 + tu$ for some $u \in \mathbb{Z}$, then

$$r_n(k_2, t) = Tr_{\mathbb{F}_{q^d}/\mathbb{F}_q}(\alpha^{k_2+tn}) = Tr_{\mathbb{F}_{q^d}/\mathbb{F}_q}(\alpha^{k_1+(n+u)t}) = r_{n+u}(k_1, t)$$

Hence $r_n(k_2, t)$ is a shifted version of $r_n(k_1, t)$. Therefore,

$$Z(R(k_1, t)) = Z(R(k_2, t)).$$

□

Using Lemma 2, $\mathcal{A}$ in (4) can be given as follows:

$$\mathcal{A} = \{Z(R(k,t)) \mid t = (q^d - 1)/m,\ 0 \leq k < t\}$$

Therefore

$$\mid \mathcal{A} \mid \leq t.$$

Lemma 3. Third Reduction : *Z(R(k,t)) = Z(R(qk,t)).*

Proof.

$$r_n(k,t) = Tr_{\mathbb{F}_{q^d}/\mathbb{F}_q}(\alpha^{k+tn}) = Tr_{\mathbb{F}_{q^d}/\mathbb{F}_q}((\alpha^{k+tn})^q) = Tr_{\mathbb{F}_{q^d}/\mathbb{F}_q}(\alpha^{qk+qnt}) = r_{qn}(qk,t)$$

Hence

$$Z(R(k,t)) = Z(R(qk,t)).$$

Therefore

$$\mathcal{A} = \{Z(R(k,t)) \mid t = (q^d - 1)/m \ \ and \ \ C_k \in C\}.$$

Hence $\mid \mathcal{A} \mid \leq \mid C \mid$.

2.2 Properties of Cyclotomy Classes

Here we discuss some properties of the cyclotomy classes where we will be able to find the exact value for the cyclotomy bound and give the exact upper bound for the cardinality of the set of zeros $\mid \mathcal{A} \mid$, under specific conditions. Let $ord_a(b)$ be the smallest positive integer c such that $a^c \equiv 1 \pmod{a}$. By the equivalence relation defined in section 2, $C_1 = \{1, q, q^2, \ldots, q^{k-1}\} \pmod{t}$ where $k = ord_t(q)$. Hence $\mid C_1 \mid = ord_t(q)$.

Proposition 1. *If t is a composite and $l \mid t$, then there exists $C_l \in C$.*

Proof. Let $l \in C_a$ for some $a \in \mathbb{Z}_t$. Then by the definition of C_a, $l \geq a$ and $l \equiv q^r a \pmod{t}$ for some $r \in \mathbb{Z}$. Since $l \mid t \Rightarrow l \mid q^r a$ and $t \mid q^d - 1 \Rightarrow (t,q) = 1$, hence $(l,q) = 1$. Therefore $l \mid a$ and hence $l \leq a$. Hence $l = a$ and $C_l \in C$.

□

The following well known result and the corollaries give the exact values for the cyclotomy bound $\mid C \mid$ and hence the exact upper bound for the cardinality of the set of zeros $\mid \mathcal{A} \mid$.

Proposition 2. *Let $t \in \mathbb{N}$ and t and q are relatively prime. Then*

$$\mid C \mid = \sum_{d \mid t} \frac{\varphi(t/d)}{ord_{t/d}(q)}.$$

Corollary 1. *If t is a prime then* $\mid C \mid = \frac{t-1}{k} + 1$.

Corollary 2. *Let t be a prime power (say p^k) where p is an odd prime. If 2 is a primitive root of $\mathbb{Z}^*_{p^2}$, then* $\mid C \mid = k + 1$.

3. Coding Theory Approach

Weight distributions of irreducible cyclic codes were studied by Baumert and McEliece (1972), Baumert and Mykkeltveit (1973), Aubrey and Langevin (2005), Wolfmann (2005), Vega (2007), Aubrey and Langevin (2008) and Ding (2009). We will use these results to determine the exact occurrences of zeros in some cases, and determine the cardinality of the set of zeros of homogeneous linear recurring sequences based on irreducible minimal polynomials of fixed degree and order. First we set notations and review the basic facts as found on Lidl and Niederreiter (1994).

Let $f(x) \in \mathbb{F}_q[x]$ be an irreducible polynomial of degree d and order m. Let $S = \{s_n\}$ be a homogeneous linear recurring sequence over $\mathbb{F}_q$ based on f as its minimal polynomial. By Theorem 2, $s_n = Tr_{K/F}(\theta\alpha^n)$, where $F = \mathbb{F}_q$, $K = \mathbb{F}_{q^d}$, $\alpha \in K$ is a root of f and $\theta \in K^*$. Define the vector

$$c(\theta,\alpha) = [Tr_{K/F}(\theta\alpha), Tr_{K/F}(\theta\alpha^2), \ldots, Tr_{K/F}(\theta\alpha^m)],$$

where the entries represent the terms of the sequence S within its least period m. Set

$$C(\alpha) = \{c(\theta,\alpha) : \theta \in K\}.$$

$C(\alpha)$ is then a cyclic code whose words represent the terms of each sequence S within its least period, based on $f(x)$. Thus $C(\alpha)$ has length m and dimension d. The generator polynomial of $C(\alpha)$ is the reciprocal of $(x^m - 1)/f(x)$, so that $C(\alpha)$ is in fact an irreducible cyclic code.

Note that the weight $wt(c(\theta, \alpha))$, the number of non zero entries of the code word $c(\theta, \alpha)$ is $m - Z(S)$. The reductions of Theorem 3 show that all sequences based on irreducible minimal polynomials of degree d and order m have the same number of zeros. Hence

$$number\ of\ non-zero\ weights\ of\ C(\alpha) = \mid \mathcal{A} \mid .$$

We say a code is a *N-weight* code if it has N non-zero weights and hence for this case, $N = \mid \mathcal{A} \mid$.

3.1 Lower Bounds for the Cardinality of the Set of Number of Zeros of Hhomogeneous Linear Recurring Sequences

Theorem 4. (Wolfmann, 2005) *Let C be an $[n, k]$ linear code over $\mathbb{F}_q$. If C is a 1-weight code with weight w and if the weight of the dual code is at least 2, then there exists $\lambda \in \mathbb{N}$ such that*

$$n = \lambda \frac{q^k - 1}{q - 1}, \quad w = \lambda q^{k-1}.$$

Corollary 3. *Let C be an irreducible cyclic 1-weight code with length m and dimension d. Set $t = (q^d - 1)/m$. Then t divides $q - 1$.*

Proof. We first check that the dual code $C^\perp$ does not have minimal weight one. Suppose it has a minimal weight of one. As $C^\perp$ is also cyclic, the existence of a codeword of weight 1 in $C^\perp$ implies that all vectors of weight 1 are in $C^\perp$ and hence $C^\perp = \mathbb{F}_q^m$. But then $C = \{0\}$, which is not a 1-weight code.

We can thus apply Theorem 4 to get $m = \lambda(q^d - 1)/(q - 1)$ for some λ. Hence $q - 1 = \lambda(q^d - 1)/m = \lambda t$.

□

Corollary 4. *For $q = 2$, $\mid \mathcal{A} \mid \geq 2$ unless the minimal polynomial is primitive.*

Proof. Let $f(x)$ be an irreducible polynomial of degree d and order m. Set $t = (2^d - 1)/m$. If $\mid \mathcal{A} \mid = 1$ then $C(\alpha)$, where α is a root of $f(x)$, is a 1-weight irreducible cyclic code. By Corollary 3, t divides $q - 1 = 1$ so that $t = 1$ and f is primitive.

□

3.2 Kasami-Welch approach

Theorem 5. (Wolfmann, 2005) *Let C be an irreducible cyclic code of length m over $\mathbb{F}_q$. Let $\mathbb{F}_{q^d}$ be the splitting field of $x^m - 1$ over $\mathbb{F}_q$. Let t be the integer such that $mt = q^d - 1$. If $d = 2e$ and if there exists a divisor r of e such that $q^r \equiv -1 \pmod{t}$, then C is a 2-weight code with weights*

$$w_1 = (q-1)q^{e-1}\left(\frac{q^e + (t-1)\epsilon}{t}\right) \qquad w_2 = (q-1)q^{e-1}\left(\frac{q^e - \epsilon}{t}\right),$$

where ϵ is 1 or -1.

Theorem 6. *Let $q = 2$. Consider sequences based on an irreducible, non-primitive polynomial of degree d and order m. Set $t = (2^d - 1)/m$. Suppose t is prime and 2 is a primitive root modulo t. Then*

$$\mid \mathcal{A} \mid = 2 = \mid C \mid$$

where C is the set of 2-cylcotomic classes in $\mathbb{Z}_t$. In fact, d is even (say $d = 2e$) and $\mathcal{A}$ consists of

$$m - \frac{2^{e-1}(2^e + (t-1)\epsilon)}{t} \quad and \quad m - \frac{2^{e-1}(2^e - \epsilon)}{t},$$

where ϵ is 1 or -1, determined by $2^e \equiv \epsilon \pmod{t}$.

Proof. C_1 is the subgroup of $\mathbb{Z}_t^*$ generated by 2, hence $C_1 = \mathbb{Z}_t^*$. So there are exactly two cyclotomy classes, represented by 0 and 1. We have $ord_t(2) = t - 1$ is even and $2^d \equiv 1 \pmod{t}$ so that $t - 1$ divides d. Write $d = 2e$. For $r = \frac{t-1}{2}$ we have $r \mid e$ and $2^r \equiv -1 \pmod{t}$. So by Theorem 5, $\mid \mathcal{A} \mid = 2$ and its values are as given.

□

Example 1. Let $q = 2$. Consider sequences based on an irreducible polynomials of degree 10 and order 93 ($f(x) = x^{10} + x^5 + x^4 + x^2 + 1$ is one such polynomial). Then $t = (2^{10} - 1)/93 = 11$. As 2 is a primitive root modulo 11, Theorem 6 gives $\mid \mathcal{A} \mid = 2$. In fact, using $e = 5$ and $\epsilon = -1$, we have $\mathcal{A} = \{45, 61\}$. This explains the result on line 4 in Table 1.

Assuming the Generalized Riemann Hypothesis (GRH), Hooley (1967) proved the Artin Conjecture and in particular, that there are infinitely many primes t such that 2 is a primitive root modulo t. Together with Theorem 6, we thus get the following corollary that proves the cyclotomy bound determined in section 2 is the best bound for $|\mathcal{A}|$.

Corollary 5. *Assume the GRH. For $q = 2$, the cyclotomy bound is achieved infinitely often.*

Theorem 7. (Kasami-Welch case) *Consider sequences based on an irreducible polynomial over $\mathbb{F}_q$ of degree d and order m. Set $t = (q^d - 1)/m$. If t has the form $q^{2a} - q^a + 1$ for some integer a ($a \geq 2$ if $q = 2$) then $d = 2e$ is even and $|\mathcal{A}| = 2$. In fact: $\mathcal{A}$ consists of*

$$m - (q-1)q^{e-1}\left(\frac{q^e + (t-1)\epsilon}{t}\right) \qquad m - (q-1)q^{e-1}\left(\frac{q^e - \epsilon}{t}\right),$$

where $\epsilon = \pm 1$.

Proof.

Let $k = ord_t(q)$. We **Claim** that $k = 6a$. The basic equation is:

$$q^{3a} + 1 = (q^a + 1)(q^{2a} - q^a + 1) = (q^a + 1)t. \tag{5}$$

Then $q^{6a} \equiv 1 \pmod{t}$ and so $k \mid 6a$. Thus k has the form $x, 2x, 3x$ or $6x$ for some divisor x of a. Note that if $k = x$ or $3x$ then $q^{3a} \equiv 1 \pmod{t}$ while (5) gives $q^{3a} \equiv -1 \pmod{t}$. Hence $k = 2x$ or $6x$.

Suppose k has the form $2x$. Then $q^{2a} \equiv 1 \pmod{t}$ and since $t = q^{2a} - q^a + 1$, we have $q^{2a} \equiv q^a - 1 \pmod{t}$. So t divides $q^a - 2$. If $q = 2$, we assume that $a \geq 2$ and hence $q^a - 2 \neq 0$. Therefore,

$$t = q^{2a} - q^a + 1 \leq q^a - 2 \Rightarrow q^{2a} \leq 2q^a - 3 < 2q^a \Rightarrow q^a < 2,$$

which is impossible.

Thus k has the form $6x$. Write $a = xy$. We have $(q^{3x})^2 \equiv 1 \pmod{t}$ and by (2), $(q^{3x})^y \equiv -1 \pmod{t}$. Then y must be odd and $q^{3x} \equiv -1 \pmod{t}$. Then

$$t = q^{2a} - q^a + 1 \leq q^{3x} + 1 \Rightarrow q^a < q^a(q^a - 1) \leq q^{3x}.$$

Hence $a = xy < 3x$ and $y < 3$. Suppose $y = 2$. Then

$$q^{2x}(q^{2x} - 1) \leq q^{3x} \Rightarrow q^{2x} - 1 \leq q^x \Rightarrow q^x \leq 1 + q^{-x} < 2,$$

which is impossible. So $y = 1$, $a = x$ and $k = 6a$, proving the **Claim**.

Fix an irreducible polynomial $f(x) \in \mathbb{F}_q[x]$ of degree d and order m. Let α be a root of f. We wish to apply Theorem 5 to $C(\alpha)$. Now $t \mid q^d - 1$ so that the order of q modulo t, namely $6a$, divides d. So d is even; write $d = 2e$. Set $r = 3a$. Then r divides e and by (2), $q^r \equiv -1 \pmod{t}$. Thus $C(\alpha)$ is a 2-weight code and $|\mathcal{A}| = 2$. We have $wt[c(\theta, \alpha)] = m - Z(S(\theta, \alpha))$ so Theorem 5 proves the elements of $|\mathcal{A}|$ are as stated.

□

Remark 1 When F is a finite field of even characteristic, the terms of the homogeneous linear recurring sequence take the form of the well known Kasami-Welch function $Tr_{K/F}(x^{2^{2a}-2^a+1})$.

Example 2. Let $q = 2$. Consider sequences based on an irreducible polynomial of degree 12 and order 315 ($f = x^{12} + x^4 + x^2 + x + 1$ is one such polynomial). Then $t = (2^{12} - 1)/315 = 13$ has the form $2^{2a} - 2^a + 1$ for $a = 2$. The number of zeros in such a sequence is thus

$$315 - 2^5\left(\frac{2^6 + 12\epsilon}{13}\right) \qquad \text{or} \quad 315 - 2^5\left(\frac{2^6 - \epsilon}{13}\right),$$

where $\epsilon = \pm 1$. To get integers we must take $\epsilon = -1$. We get $|\mathcal{A}| = \{155, 187\}$. This explains the values on line 12 in Table 1.

Theorem 8. *Consider sequences based on an irreducible polynomial over $\mathbb{F}_q$ of degree d and order m. Set $t = (q^d - 1)/m$. Suppose*

1. *t is a prime where* $t \equiv 1 \pmod 4$*,*

2. $ord_t(q) = \frac{1}{2}(t-1)$.

Then $d = 2e$ *is even and* $|\mathcal{A}| = 2$ *and* $\mathcal{A}$ *consists of*

$$m - (q-1)q^{e-1}\left(\frac{q^e + (t-1)\epsilon}{t}\right) \qquad m - (q-1)q^{e-1}\left(\frac{q^e - \epsilon}{t}\right),$$

where $\epsilon = \pm 1$.

Proof.

Fix an irreducible polynomial $f(x) \in \mathbb{F}_q[x]$ of degree d and order m. Let α be a root of f. We will apply Theorem 5 to $C(\alpha)$. Now $t \mid q^d - 1$ and hence $ord_t(q) = \frac{1}{2}(t-1)$ divides d. Since $t \equiv 1 \pmod 4$, $ord_t(q)$ is even and hence d is even;

$$d = \frac{1}{2}(t-1)k = 2e$$

where $e = \frac{1}{4}(t-1)k$. Set $r = \frac{1}{4}(t-1)$. Then $r \mid e$. By the definition of t, $t \mid (q^{\frac{q-1}{2}} - 1)$ and since $(t, \frac{q-1}{4} + 1) = 1$, $t \mid (q^{\frac{q-1}{4}} + 1)$. Therefore $q^r \equiv -1 \pmod t$.

Then $C(\alpha)$ is a 2-weight code by Theorem 5 and $|\mathcal{A}| = 2$. We have $wt[c(\theta, \alpha)] = m - Z(S(\theta, \alpha))$ and Theorem 5 gives the elements of $\mathcal{A}$ as stated above.

□

Example 3. Let $q = 2$. Consider sequences based on an irreducible polynomial of degree 16 and order 3855. Then $t = (2^{16}-1)/3855 = 17$ and $ord_{17}(2) = \frac{1}{2}(17-1)$. A particular polynomial that can be considered is $f = x^{16}+x^{14}+x^{11}+x^3+1$. To get integers, take $\epsilon = 1$. Hence the values for $\mathcal{A}$ are:

$$3855 - 2^7\left(\frac{2^8 + 16}{17}\right) = 1807$$

$$3855 - 2^7\left(\frac{2^8 - 1}{17}\right) = 1935.$$

Hence $|\mathcal{A}| = 2$ which is explains another observation in Table 1.

So far we have only computed $\mathcal{A}$ using Theorem 5 which gives $|\mathcal{A}| = 2$. We will now discuss two other cases providing conditions for which $|\mathcal{A}| = 3$.

Theorem 9. *Let* $q = 2$*. Consider sequences based on an irreducible polynomial of degree* d *and order* m*. Set* $t = (2^d - 1)/m$*. Suppose*

1. *t is a prime not equal to 3,*

2. $t \equiv 3 \pmod 4$*,*

3. $ord_t(2) = \frac{1}{2}(t-1)$.

Then $|\mathcal{A}| = 3$.

Proof. We have $|C| = 3$ by Corollary 1 and hence $|\mathcal{A}| \leq 3$ by Theorem 3. $|\mathcal{A}| \neq 1$ by Corollary 4. Pick a particular polynomial f of degree d and order m. Let α be a root of f. The three conditions on t imply $C(\alpha)$ is not a 2-weight code by Proposition 2 in Aubrey and Langevin (2005). Hence $|\mathcal{A}| \neq 2$ and therefore $|\mathcal{A}| = 3$.

□

Example 4. Let $q = 2$ Consider sequences based on an irreducible polynomial of degree 9 and order 73 ($x^9 + x + 1$ is one such polynomial). Then $t = (2^9 - 1)/73 = 7$, which satisfies all three conditions of Theorem 9. Hence $|\mathcal{A}| = 3$. As given in the third observation of Table 1, a computer computation yields that in fact $\mathcal{A} = \{33, 37, 45\}$.

Theorem 10. *Consider sequences based on an irreducible polynomial of degree* d *and order* m *over* $\mathbb{F}_q$*. Set* $t = (q^d - 1)/m$*. Suppose*

1. *t is a prime not equal to 3,*

2. $t \equiv 3 \pmod 4$*,*

3. $ord_t(q) = \frac{1}{2}(t-1)$.

Then $|\mathcal{A}| = 2$ *or* 3.

Proof. By Corollary 1, $|C| = 3$. Hence $|\mathcal{A}| \leq 3$. Let f be a polynomial of degree d and order m and let α be a root of f. If the irreducible cyclic code $C(\alpha)$ of length m and dimension d is 1-weight, then by Corollary 3, $t \mid q-1$. Hence $ord_t(q) = \frac{t-1}{2} = 1 \Longrightarrow t = 3$ which contradicts the first condition above. Therefore $|\mathcal{A}| = 2$ or 3.

□

The following result can be given using Theorem 10 and Theorem 8 in Aubrey and Langevin (2008).

Corollary 3.4. *Suppose t satisfies the conditions given in Theorem 10. If* $t \equiv 7 \pmod 8$ *then* $|\mathcal{A}| = 3$.

4. Conclusion

The main purpose here was to give an accurate bound for the cardinality of the set of zeros of homogeneous linear recurring sequences over $\mathbb{F}_q$ based on irreducible minimal polynomials of given degree and order. This was achieved by the cyclotomy bound defined here and it was proved to be the best bound as it is reached in infinitely many cases. Besides determining a lower bound for sequences over $\mathbb{F}_2$, the exact number of zeros were given for Kasami Welch and the quadratic residue cases based on results on weights of irreducible cyclic codes. The work here was restricted to analyzing the conditions for the existence of $\mathcal{A} = 2$ and 3. This will be extended to an investigation of higher cardinality in the future.

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Asymptotic Properties of Longitudinal Weighted Averages for Strongly Mixing Data

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Abstract

We present general results of consistency and normality of a real-valued-longitudinal random variable. We suppose that this random variable is some formed weighted averages of α-mixing data. The results can be applied to within-subject covariance function.

Keywords: longitudinal data, α-mixing data, weighted averages, within-subject covariance function.

1. Introduction

Longitudinal data analysis involves irregularly-spaced and infrequent measurements. So, there is often relatively little information available about each subject. Repeated binary measurements models have been discusses in Heagerty (Heagerty, 1999). The repeated measurements take place on a few scartered observational times points for each subject.

Recent innovation in measurements recorded machine and data collected methods have facillited the collection of longitudinal data. Longitudinal data are observed at sparsely distributed time points and are often subject of experimental error (Diggle, et al., 2002, Yao, 2007).

The case of independent and identically distributed observations using kernel-based estimation has received considerable attention in recent years with contribution (Hart & Wehrly, 1986; Lin & Caroll, 2000; Yao, 2007; Hall, et al., 2008; Degras, 2008; Soro & Hili, 2012).

Yao (Yao, 2007) has proved the asymptotic normality of mean and covariance functions estimators. Also, Degras (Degras, 2008) has proved the asymptotic normality of estimator of the mean function under a mean-square continuous process.

However, the literature on influence of within-subject correlation on asymptotic results is not developped. For instance, see Hart & Wehrly (1986) for the study of Gasser-Müler estimator. Yao (Yao, 2007) has proved that the within-subject correlation can be ignored in deriving the asymptotic variance. His results are obtained for independent data with arguments that the data were formed by weighted averages of longitudinal or functional data. Soro & Hili (Soro & Hili, 2012) extended the results of Yao (Yao, 2007) for a continuous univariate stochastic process.

The main purpose of this article is to extend the results of Soro & Hili (Soro & Hili, 2012) to α-mixing longitudinal data. Our results can be applied to within-subject covariance function introduced by Soro & Hili (Soro & Hili, 2012) with mixing arguments.

We give general asymptotic properties for real-valued function that we assume to be formed from weighted averages of α-mixing data.

The paper is organised as follows. Section 2 contains the definition of the estimator and some assumptions. Sections 3 and 4 are the main results of the paper. They respectively establish the consistency and the asymptotic normality of the estimator.

2. Definition of the Estimator and Some Assumptions

We consider for $1 \leq i \leq n$, N triples $\{(T_{ij}, X_{ij}, Y_{ij}), 1 \leq j \leq N\}$ identically distributed as (T, X, Y) such that the sequence $(X_i, Y_{i.})$ is α-mixing. Y_{ij} is the jth observation of the random variable X_i, measured at the random time T_{ij}. The number of observations $N(n)$ depend on the sample size n. For simplicity, $N(n)$ will be noted N. We assume that X is defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ whereas Y is a real random variable. Let $\nu_i,\ 1 \leq i \leq 3$ and $k_i,\ 1 \leq i \leq 3$ be some given integers. Denote by ν, k the multi-indices $\nu = (\nu_1, \nu_2, \nu_3)$ and $k = (k_1, k_2, k_3)$. Let $|\nu| = \nu_1 + \nu_2 + \nu_3$, $|k| = k_1 + k_2 + k_3$; $\nu! = \nu_1!\nu_2!\nu_3!$ and $k! = k_1!k_2!k_3!$. As most kernel-based nonparametric estimators can be written as function of averages, then we consider averages (introduced in Soro & Hili (2012)) of the form:

$$\begin{aligned}\Gamma_{\lambda n} &= \Gamma_{\lambda n}(r, s, t)\\ &= \frac{1}{nN(N-1)(N-2)h_n^{|\nu|+3}} \sum_{i=1}^{n} \sum_{1 \leq j \neq k \neq l \leq N} \gamma_\lambda(T_{ij}, T_{ik}, T_{il}, Y_{ij}, Y_{ik}, Y_{il})\\ &\qquad \times K_3\left(\frac{r - T_{ij}}{h_n}, \frac{s - T_{ik}}{h_n}, \frac{t - T_{il}}{h_n}\right),\end{aligned}$$

for $1 \leq \lambda \leq l$.
For instance, the non-parametric regression model for repeated measurements, which is typically used for longitudinal data treatement, and dose-response curves:

$$Y_{ij} = X_i(T_{ij}) + \varepsilon_{ij},\ \ 1 \leq i \leq n, 1 \leq j \leq N.$$

Some applications of this model are given in Hart & Wehrly (1986) for biostatistics, Müler (1988) in human growth curve study, Ramsay & Ramsey (2002) for monthly index of nondurable goods production.
Let

$$\begin{aligned}\sigma_\lambda^2 &= \sigma_\lambda^2(r, s, t)\\ &= \|K_3\|^2 \int_{\mathbb{R}^3} \gamma_\lambda^2(r, s, t, y_1, y_2, y_3) g_3(r, s, t, y_1, y_2, y_3) dy_1 dy_2 dy_3,\ \text{ for }\ 1 \leq \lambda \leq l.\end{aligned}$$

(T_{ij}, Y_{ij}), $i = 1, ..., n,\ j = 1, ..., N$, are assumed to have the joint density $g(t, y)$. The observation times T_{ij} are assumed to be i.i.d. with a marginal density $f(t)$.

Let $f_3(r, s, t)$ be the joint density of (T_{ij}, T_{ik}, T_{il}), $g_3(r, s, t, y_1, y_2, y_3)$ be the joint density of $(T_{ij}, T_{ik}, T_{il}, Y_{ij}, Y_{ik}, Y_{il})$ and $g_6(r, s, t, r', s', t', y_1, y_2, y_3, y_1', y_2', y_3')$ be the joint density of the 12-uple $(T_{ij}, T_{ik}, T_{il}, T_{ij'}, T_{ik'}, T_{il'}, Y_{ij}, Y_{ik}, Y_{il}, Y_{ij'}, Y_{ik'}, Y_{il'})$ where $j \neq k \neq l$, and $(j, k, l) \neq (j', k', l')$.

To establish the properties of our random variable $\Gamma_{\lambda n}$, we need the following assumptions.

Assumptions K.

(K.1) $K_3(., ., .) : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is symmetric and has a compact support.

(K.2) $\|K_3\|_2^2 = \int_{\mathbb{R}^3} K_3^2(u, v, w) dudvdw < \infty$.

(K.3) K_3 is a kernel function of order $(|\nu|, |k|)$, that is,

$$\int_{\mathbb{R}^3} u^{\ell_1} v^{\ell_2} w^{\ell_3} K_3(u, v, w) dudvdw = \begin{cases} 0, & 0 \leq |\ell| < |k|, |\ell| \neq |\nu|. \\ (-1)^{|\nu|}\nu!, & |\ell| = |\nu|, \\ C, & |\ell| = |k|. \end{cases} \quad (1)$$

where C is a non null constant.

Assumptions B.

(B.1) $h_n \longrightarrow 0$, $nN(N-1)(N-2)h_n^{|\nu|+3} \longrightarrow \infty$, $nN(N-1)(N-2)h_n^{2|\nu|+3} \longrightarrow a^2$, where a is a positive constant, as $n \longrightarrow +\infty$.

(B.2) $nh_n^{|\nu|+3} \longrightarrow \infty$ and $N(N-1)(N-2)h_n^{|\nu|} \longrightarrow \infty$, as $n \longrightarrow \infty$.

Assumptions D.

The following conditions are assumed, where $N(r, s, t)$ is some neighborhood of $\{(r, s, t)\}$.

(D.1) $\frac{d^{|k|}}{du^{k_1} dv^{k_2} dw^{k_3}} f_3(u, v, w)$ exists and is continuous for $(u, v, w) \in N(r, s, t)$ and $f_3(u, v, w) > 0$ for all arguments $(u, v, w) \in N(r, s, t)$;

(D.2) $g_3(u, v, w, y_1, y_2, y_3)$ is continuous for $(u, v, w) \in N(r, s, t)$, uniformly for $(y_1, y_2, y_3) \in \mathbb{R}^3$;

(D.3) $\frac{d^{|k|}}{du^{k_1} dv^{k_2} dw^{k_3}} g_3(u, v, w, y_1, y_2, y_3)$ exists and is continuous for

$(u, v, w) \in N(r, s, t)$, uniformly for $(y_1, y_2, y_3) \in \mathbb{R}^3$;

(D.4) $g_6(u, v, w, u', v', w', y_1, y_2, y_3)$ is continuous for (u, v, w, u', v', w')

$\in N(r, s, t)^2$, uniformly for $(y_1, y_2, y_3) \in \mathbb{R}^3$.

The collection $\{\gamma_\lambda\}_{\lambda=1,...,l}$ of real functions $\gamma_\lambda : \mathbb{R}^6 \longrightarrow \mathbb{R}$, $\lambda = 1, ..., l$, satisfies:

(D.5) $\gamma_\lambda(r, s, t, y_1, y_2, y_3)$ is continuous for (r, s, t) uniformly for

$(y_1, y_2, y_3) \in \mathbb{R}^3$,

(D.6) $\frac{d^{|k|}}{dr^{k_1} ds^{k_2} dt^{k_3}} \gamma_\lambda(r, s, t, y_1, y_2, y_3)$ exists for all arguments $(r, s, t, y_1, y_2, y_3) \in \mathbb{R}^6$.

The process $\{X_i, Y_i\}$ is strongly mixing:

Let $\mathcal{F}_a^b$ be the sigma algebra generated by the random variables $\{X_i, Y_{i.}\}_{i=a}^b$. Set

$$\alpha(\ell) = \sup_t \sup_{A \in \mathcal{F}_{-\infty}^t, B \in \mathcal{F}_{t+\ell}^\infty} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|.$$

The mixing coefficient satistfies:

Assumption M.

(M.1) $\sum_{\ell=1}^{\infty} \ell^a [\alpha(\ell)]^{1-2/\delta} < \infty$ for some $a > 1 - 2/\delta$, for some $\delta > 2$.

3. Consistency of the Estimator

The folowing theorem gives the consistency of our estimator.

Theorem3.1. If assumptions (K), (B) and (D) are satisfied, we have

$$\Gamma_{\lambda n}(r, s, t) - m_\lambda(r, s, t) \quad \stackrel{\mathbb{P}}{\longrightarrow} \quad \mathcal{B}(r, s, t), \tag{2}$$

where

$$m_\lambda(r, s, t) = \frac{d^{|v|}}{dr^{v_1} ds^{v_2} dt^{v_3}} \int_{\mathbb{R}^3} \gamma_\lambda(r, s, t, y_1, y_2, y_3) g_3(r, s, t, y_1, y_2, y_3) dy_1 dy_2 dy_3,$$

$\lambda = 1, ..., l$ and

$$\mathcal{B}(r, s, t) = \frac{(-1)^{|k|}}{k!} \left\{ \int_{\mathbb{R}^3} u^{k_1} v^{k_2} w^{k_3} K_3(u, v, w) du dv dw \right.$$

$$\left. \times \frac{d^{|k|}}{dr^{k_1} ds^{k_2} dt^{k_3}} \int_{\mathbb{R}^3} \gamma_\lambda(r, s, t, y_1, y_2, y_3) g_3(r, s, t, y_1, y_2, y_3) dy_1 dy_2 dy_3 \right\}.$$

Proof.

We obtain the consistency of our estimator via the bias-variance decomposition which follows

$$\mathbb{E}[(\Gamma_\lambda(r, s, t) - m_\lambda(r, s, t))^2] \quad = \quad var(\Gamma_\lambda(r, s, t)) + \{\mathbb{E}[\Gamma_\lambda(r, s, t)] - m_\lambda(r, s, t)\}^2. \tag{3}$$

Let prove that the second term in (3) goes to 0 when n goes to $+\infty$. We have

$$\begin{aligned}
\mathbb{E}(\Gamma_{\lambda n}(r,s,t)) &= \frac{1}{nN(N-1)(N-2)h_n^{|\nu|+3}}\mathbb{E}\left\{\sum_{i=1}^{n}\sum_{1\le j\neq k\le l\neq N}\gamma_\lambda(T_{ij},T_{ik},T_{il},Y_{ij},Y_{ik},Y_{il})\right.\\
&\qquad \left.\times K_3\left(\frac{r-T_{ij}}{h_n}\frac{s-T_{ik}}{h_n},\frac{t-T_{il}}{h_n}\right)\right\}\\
&= \frac{1}{N(N-1)(N-2)h_n^{|\nu|+3}}\left\{\sum_{1\le j\neq k\le l\neq N}\mathbb{E}\left[\gamma_\lambda(T_{1j},T_{1k},T_{1l},Y_{1j},Y_{1k},Y_{1l})\right.\right.\\
&\qquad \left.\left.\times\ K_3\left(\frac{r-T_{1j}}{h_n},\frac{s-T_{1k}}{h_n},\frac{t-T_{1l}}{h_n}\right)\right]\right\}\\
&= \frac{1}{h_n^{|\nu|+3}}E\left\{\gamma_\lambda(T_{11},T_{12},T_{13},Y_{11},Y_{12},Y_{13})K_3\left(\frac{r-T_{11}}{h_n},\frac{s-T_{12}}{h_n},\frac{t-T_{13}}{h_n}\right)\right\}\\
&= m_\lambda(r,s,t)+\frac{(-1)^{|k|}}{k!}\left\{\int_{\mathbb{R}^3}u^{k_1}v^{k_2}w^{k_3}K_3(u,v,w)dudvdw\right.\\
&\quad \left.\times\frac{d^{|k|}}{dr^{k_1}ds^{k_2}dt^{k_3}}\int_{\mathbb{R}^3}\gamma_\lambda(r,s,t,y_1,y_2,y_3)g_3(r,s,t,y_1,y_2,y_3)dy_1dy_2dy_3\times h_n^{|k|-|\nu|}\right\}\\
&\qquad +o(h_n^{|k|-|\nu|}).
\end{aligned}\tag{4}$$

So

$$\mathbb{E}\Gamma_{\lambda n}(r,s,t)-m_\lambda(r,s,t)\quad\longrightarrow\quad\mathcal{B}(r,s,t).\tag{5}$$

Now, we prove that $var(\Gamma_{\lambda n}(r,s,t))\longrightarrow 0$.

$$var(\Gamma_{\lambda n}(r,s,t)) =$$

$$\frac{1}{nN(N-1)(N-2)h_n^{2|\nu|+6}}var\left[\gamma_\lambda(T_{11},T_{12},T_{13},Y_{11},Y_{12},Y_{13})K_3\left(\frac{r-T_{11}}{h_n},\frac{s-T_{12}}{h_n},\frac{t-T_{13}}{h_n}\right)\right]+$$

$$\frac{1}{(nN(N-1)(N-2)h_n^{|\nu|+3})^2}\sum_{\substack{i=1\\ i\neq i'}}^{n}\sum_{i'=1}^{n}\sum_{1\le j\neq k\neq l\le N}\sum_{1\le j'\neq k'\neq l'\le N}$$

$$cov\left\{\gamma_\lambda(T_{ij},T_{ik},T_{il},Y_{ij},Y_{ik},Y_{il})K_3\left(\frac{r-T_{ij}}{h_n},\frac{s-T_{ik}}{h_n},\frac{t-T_{il}}{h_n}\right),\right.$$

$$\left.\gamma_\lambda(T_{i'j'},T_{i'k'},T_{i'l'},Y_{i'j'},Y_{i'k'},Y_{i'l'})K_3\left(\frac{r-T_{i'j'}}{h_n},\frac{s-T_{i'k'}}{h_n},\frac{t-T_{i'l'}}{h_n}\right)\right\}=$$

$$I_1+I_2$$

$$
\begin{aligned}
I_1 &= \frac{1}{nN(N-1)(N-2)h_n^{2|\nu|+6}} var\left[\gamma_\lambda(T_{11},T_{12},T_{13},Y_{11},Y_{12},Y_{13})K_3\left(\frac{r-T_{11}}{h_n},\frac{s-T_{12}}{h_n},\frac{t-T_{13}}{h_n}\right)\right] \\
&= \frac{1}{nN(N-1)(N-2)h_n^{|\nu|+3}} \\
&\times \left\{\mathbb{E}\left[\frac{1}{h_n^{|\nu|+3}}\gamma_\lambda^2(T_{11},T_{12},T_{13},Y_{11},Y_{12},Y_{13})K_3^2\left(\frac{r-T_{11}}{h_n},\frac{s-T_{12}}{h_n},\frac{t-T_{13}}{h_n}\right)\right]\right. \\
&- \left.\mathbb{E}^2\left[\frac{1}{h_n^{|\nu|+3}}\gamma_\lambda(T_{11},T_{12},T_{13},Y_{11},Y_{12},Y_{13})K_3\left(\frac{r-T_{11}}{h_n},\frac{s-T_{12}}{h_n},\frac{t-T_{13}}{h_n}\right)\right]\right\} \\
&= \frac{1}{nN(N-1)(N-2)h_n^{|\nu|+3}} \\
&\times \left\{\frac{1}{h_n^{|\nu|+3}}\int_{\mathbb{R}^6} g_3(t_1,t_2,t_3,y_1,y_2,y_3)\gamma_\lambda^2(t_1,t_2,t_3,y_1,y_2,y_3)\right. \\
&\quad K_3^2\left(\frac{r-t_1}{h_n},\frac{s-t_2}{h_n},\frac{t-t_3}{h_n}\right)dt_1dt_2dt_3dy_1dy_2dy_3 \\
&- \left[\frac{1}{h_n^{|\nu|+3}}\int_{\mathbb{R}^6} g_3(t_1,t_2,t_3,y_1,y_2,y_3)\gamma_\lambda(t_1,t_2,t_3,y_1,y_2,y_3)\right. \\
&\quad \left.\left. K_3\left(\frac{r-t_1}{h_n},\frac{s-t_2}{h_n},\frac{t-t_3}{h_n}\right)dt_1dt_2dt_3dy_1dy_2dy_3\right]^2\right\} \\
&= \frac{1}{nN(N-1)(N-2)h_n^{|\nu|+3}}\left\{\frac{1}{h_n^{|\nu|}}\int_{\mathbb{R}^6} g_3(r-h_nu,s-h_nv,t-h_nw,y_1,y_2,y_3)\times\right. \\
&\quad \gamma_\lambda^2(r-h_nu,s-h_nv,t-h_nw,y_2,y_3)K_3^2\,(u,v,w)\,dudvdwdy_1dy_2dy_3 \\
&- h_n^3\left[\frac{1}{h_n^{|\nu|}}\int_{\mathbb{R}^6} g_3(r-h_nu,s-h_nv,t-h_nw,y_1,y_2,y_3)\right. \\
&\quad \left.\left.\gamma_\lambda(r-h_nu,s-h_nv,t-h_nw,y_2,y_3)K_3\,(u,v,w)\,dudvdwdy_1dy_2dy_3\right]^2\right\} \\
&= \frac{1}{nN(N-1)(N-2)h_n^{|\nu|+3}}\left\{\sigma_\lambda^2(r,s,t)+o(1)\right\} \\
&\longrightarrow 0, n\longrightarrow+\infty. \qquad (6)
\end{aligned}
$$

Let consider I_2. We use the fact that triples $\{Y_{ij},Y_{ik},Y_{il}\}$ and $\{Y_{ij'},Y_{ik'},Y_{il'}\}$ are independent and equidistributed.

$$
\begin{aligned}
I_2 = \frac{[N(N-1)(N-2)]^2}{[nN(N-1)(N-2)h_n^{|\nu|+3}]^2}\sum_{\substack{i=1\\ i\neq i'}}^{n}\sum_{i'=1}^{n} \\
cov\left\{\gamma_\lambda(T_{i1}^{(1)},T_{i1}^{(2)},T_{i1}^{(3)},Y_{i1}^{(1)},Y_{i1}^{(2)},Y_{i1}^{(3)})K_3\left(\frac{r-T_{i1}^{(1)}}{h_n},\frac{s-T_{i1}^{(2)}}{h_n},\frac{t-T_{i1}^{(3)}}{h_n}\right),\right. \\
\left.\gamma_\lambda(T_{i'2}^{(1)},T_{i'2}^{(2)},T_{i'2}^{(3)},Y_{i'2}^{(1)},Y_{i'2}^{(2)},Y_{i'2}^{(3)})K_3\left(\frac{r-T_{i'2}^{(1)}}{h_n},\frac{s-T_{i'2}^{(2)}}{h_n},\frac{t-T_{i'2}^{(3)}}{h_n}\right)\right\} \\
= \frac{1}{n^2h_n^{2|\nu|+6}}\sum_{\substack{i=1\\ i\neq i'}}^{n}\sum_{i'=1}^{n} cov(R_{\lambda,i},R_{\lambda,i'}).
\end{aligned}
$$

Let $S=\{(i,i'):0\leq|i-i'|<d_n, i,i'=1,...,n, i\neq i'\}$.

$$
\begin{aligned}
I_2 &= \frac{1}{n^2h_n^{2|\nu|+6}}\sum_{\substack{i=1\\ i\neq i'}}^{n}\sum_{i'=1}^{n} cov(R_{\lambda,i},R_{\lambda,i'}) \\
&= \frac{1}{n^2h_n^{2|\nu|+6}}\left\{\sum_{i,i'=1}^{n}\sum_{(i,i')\in S} cov(R_{\lambda,i},R_{\lambda,i'}) + \sum_{i,i'=1}^{n}\sum_{(i,i')\notin S} cov(R_{\lambda,i},R_{\lambda,i'})\right\} \\
&= I_{21}+I_{22}.
\end{aligned}
$$

By Holder inequality,

$$|cov(R_{\lambda,i}, R_{\lambda,i'})| \leq (\mathbb{E}[R^2_{\lambda,i}]\mathbb{E}[R^2_{\lambda,i'}])^{1/2} + [\mathbb{E}|R_{\lambda,i'}|]^2,$$

so

$$\begin{aligned}|I_{21}| &\leq \frac{1}{n^2 h_n^{|\nu|+3}} \sum_{i,i'=1}^{n} \sum_{(i,i')\in S} \left\{ \frac{1}{h_n^{|\nu|+3}} (\mathbb{E}[R^2_{\lambda,i}]\mathbb{E}[R^2_{\lambda,i'}])^{1/2} + \frac{1}{h_n^{|\nu|+3}} [\mathbb{E}|R_{\lambda,i'}|]^2 \right\} \\ &= \frac{1}{n^2 h_n^{|\nu|+3}} \sum_{i,i'=1}^{n} \sum_{(i,i')\in S} \left\{ \sigma^2_\lambda(r,s,t) + o(1) \right\}.\end{aligned}$$

Since $Card(S) \leq nd_n$, we obtain

$$\begin{aligned}|I_{21}| &\leq \frac{nd_n}{n^2 h_n^{|\nu|+3}} \left\{ \sigma^2_\lambda(r,s,t) + o(1) \right\} \\ &\leq \frac{d_n}{nh_n^{|\nu|+3}} \left\{ \sigma^2_\lambda(r,s,t) + o(1) \right\}.\end{aligned}$$

Choosing $d_n = (\ln\ln n)^2 \ln n$, $h_n = \frac{\ln\ln n}{\ln n}$, it comes,

$$d_n \longrightarrow \infty, \;\; h_n \longrightarrow 0, \;\; nh_n^{|\nu|+3} \longrightarrow \infty \;\text{ and }\; \frac{d_n}{nh_n^{|\nu|+3}} \longrightarrow 0.$$

Hense

$$I_{21} \longrightarrow 0, \;\text{ as }\; n \longrightarrow +\infty. \tag{7}$$

Now consider I_{22}. By Davydov's lemma (see Hall & Heyde, Corrollary A.2), and (K.1) we have

$$\begin{aligned}|cov(R_{\lambda,i}, R_{\kappa,i'})| &\leq 8[\mathbb{E}|R_{\lambda,i}|^\delta]^{2/\delta}[\alpha(|i-i'|)]^{1-2/\delta} \\ &\leq 8C\left[h_n^{|\nu|+3}\right]^{2/\delta}[\alpha(|i-i'|)]^{1-2/\delta}.\end{aligned}$$

It follows that

$$\begin{aligned}|I_{22}| &\leq \frac{8C\left[h_n^{|\nu|+3}\right]^{2/\delta}}{n^2 h_n^{2|\nu|+6}} \sum_{i,i'=1}^{n} \sum_{(i,i')\notin S} [\alpha(|i-i'|)]^{1-2/\delta} \\ &\leq \frac{8C}{n^2 h_n^{(2|\nu|+3)(3-1/\delta)}} \sum_{i,i'=1}^{n} \sum_{(i,i')\notin S} [\alpha(|i-i'|)]^{1-2/\delta}.\end{aligned}$$

Reducing the double sum above to a single sum, it follows that

$$\begin{aligned}|I_{22}| &\leq \frac{8C}{n^2 h_n^{(2|\nu|+3)(3-1/\delta)}} \sum_{\ell=d_n+1}^{n} \ell^a[\alpha(\ell)]^{1-2/\delta} \\ &\leq \frac{8Cn}{n^2 h_n^{(2|\nu|+3)(3-1/\delta)}} \sum_{\ell=d_n+1}^{n} \ell^a[\alpha(\ell)]^{1-2/\delta} \\ &\leq \frac{8C}{nh_n^{(2|\nu|+3)(3-1/\delta)}} \sum_{\ell=d_n+1}^{\infty} \ell^a[\alpha(\ell)]^{1-2/\delta}.\end{aligned}$$

Since $\delta \geq 2$, then $(3-1/\delta) > 0$ and from assumption (M), one has

$$I_{22} \longrightarrow 0, \;\text{ as }\; n \longrightarrow +\infty. \tag{8}$$

Combining (6), (7) and (8), we conclude that $var(\Gamma_\lambda(r,s,t))$ goes to 0 as n goes to $+\infty$. So Theorem 3.1 is proved. □

4. Asymptotic Normality of the Estimator

The asymptotic normality of our estimator is given by the following theorem.

Theorem 4.1. If assumptions (K), (B), (D) and (M) are satisfied, we have

$$\sqrt{nN(N-1)(N-2)h_n^{|\gamma|+3}}(\Gamma_{\lambda n} - \mathbb{E}\Gamma_{\lambda n}) \quad \longrightarrow \quad \mathcal{N}(0, \sigma^2_\lambda(r,s,t)). \tag{9}$$

Proof.

First, recall that

$$\sqrt{nN(N-1)(N-2)h_n^{|\gamma|+3}}(\Gamma_{\lambda n} - \mathbb{E}\Gamma_{\lambda n}) = \frac{\sqrt{nN(N-1)(N-2)h_n^{|\gamma|+3}}}{nN(N-1)(N-2)h_n^{|\gamma|+3}} \sum_{i=1}^n$$

$$\sum_{1\neq j\neq k\neq l\leq N}\left[\gamma_\lambda(T_{ij},T_{ik},T_{il},Y_{ij},Y_{ik},Y_{il})K_3\left(\frac{r-T_{ij}}{h_n}\frac{s-T_{ik}}{h_n},\frac{t-T_{il}}{h_n}\right)\right.$$

$$\left. - \mathbb{E}\gamma_\lambda(T_{ij},T_{ik},T_{il},Y_{ij},Y_{ik},Y_{il})K_3\left(\frac{r-T_{ij}}{h_n}\frac{s-T_{ik}}{h_n},\frac{t-T_{il}}{h_n}\right)\right]$$

$$= \frac{1}{\sqrt{nN(N-1)(N-2)h_n^3}}\sum_{i=1}^n$$

$$\sum_{1\leq j\neq k\neq l\leq N}\left[\gamma_\lambda(T_{ij},T_{ik},T_{il},Y_{ij},Y_{ik},Y_{il})K_3\left(\frac{r-T_{ij}}{h_n}\frac{s-T_{ik}}{h_n},\frac{t-T_{il}}{h_n}\right)\right.$$

$$\left. - \mathbb{E}\gamma_\lambda(T_{ij},T_{ik},T_{il},Y_{ij},Y_{ik},Y_{il})K_3\left(\frac{r-T_{ij}}{h_n}\frac{s-T_{ik}}{h_n},\frac{t-T_{il}}{h_n}\right)\right]$$

$$= \sum_{i=1}^n\sum_{1\leq j\neq k\neq l\leq N}\frac{1}{\sqrt{nN(N-1)(N-2)h_n^{|\gamma|+3}}}$$

$$\left[\gamma_\lambda(T_{ij},T_{ik},T_{il},Y_{ij},Y_{ik},Y_{il})K_3\left(\frac{r-T_{ij}}{h_n}\frac{s-T_{ik}}{h_n},\frac{t-T_{il}}{h_n}\right)\right.$$

$$\left. - \mathbb{E}\gamma_\lambda(T_{ij},T_{ik},T_{il},Y_{ij},Y_{ik},Y_{il})K_3\left(\frac{r-T_{ij}}{h_n}\frac{s-T_{ik}}{h_n},\frac{t-T_{il}}{h_n}\right)\right].$$

Denote

$$Z_{ijkl} = \frac{1}{\sqrt{nN(N-1)(N-2)h_n^{|\gamma|+3}}}\gamma_\lambda(T_{ij},T_{ik},T_{il},Y_{ij},Y_{ik},Y_{il})K_3\left(\frac{r-T_{ij}}{h_n}\frac{s-T_{ik}}{h_n},\frac{t-T_{il}}{h_n}\right).$$

Then

$$\sqrt{nN(N-1)(N-2)h_n^{|\gamma|+3}}(\Gamma_{\lambda n} - \mathbb{E}\Gamma_{\lambda n}) = \sum_{i=1}^n\sum_{1\leq j\neq k\neq l\leq N}(Z_{ijkl} - \mathbb{E}Z_{ijkl}).$$

Denote $Z_{n,i} = \sum_{1\leq j\neq k\neq l\leq N}(Z_{ijkl} - \mathbb{E}Z_{ijkl})$. Hence

$$\sqrt{nN(N-1)(N-2)h_n^{|\gamma|+3}}(\Gamma_{\lambda n} - \mathbb{E}\Gamma_{\lambda n}) = \sum_{i=1}^n Z_{n,i}.$$

We now introduce Bernstein's big-block and small-block decomposition. We partition the set $\{1,2,...,n\}$ into $2k_n+1$ subsets with large blocks of size u_n and small blocks of size v_n and we set $k_n = \left\lfloor \frac{n}{u_n+v_n}\right\rfloor$, where $u_n = \left\lfloor nN(N-1)(N-2)h_n^{|\gamma|+3}\right\rfloor$ and $v_n = o(nN(N-1)(N-2)h_n^{|\gamma|+3})$. The symbol $\lfloor . \rfloor$ is integer part. Using (B.2), one has

$$\frac{v_n}{u_n}\longrightarrow 0, \frac{u_n}{n}\longrightarrow 0 \quad , \quad \frac{nN(N-1)(N-2)}{u_nh_n^3}\longrightarrow 0, \frac{n}{u_n}\alpha(v_n)\longrightarrow 0, \text{ as } n\longrightarrow +\infty. \tag{10}$$

Let U_m, V_m and W_m be defined as follows:

$$U_m = \sum_{i=m(u_n+v_n)+1}^{m(u_n+v_n)+u_n} Z_{n,i}, \quad 0 \le m \le k_n - 1 \tag{11}$$

$$V_m = \sum_{i=m(u_n+v_n)+u_n+1}^{(m+1)(u_n+v_n)} Z_{n,i}, \quad 0 \le m \le k_n - 1 \tag{12}$$

$$W_m = \sum_{i=k_n(u_n+v_n)+1}^{n} Z_{n,i}. \tag{13}$$

Then, we obtain the decomposition

$$T_n = \sum_{i=1}^{n} Z_{n,i} = \sum_{m=0}^{k_n-1} U_m + \sum_{m=0}^{k_n-1} V_m + W_m \tag{14}$$

$$= S_{n,1} + S_{n,2} + S_{n,3}. \tag{15}$$

Now, let start the proof of theorem 4.1.

The main idea is to show that as $n \longrightarrow \infty$,

$$A_1 = \mathbb{E}[S_{n,2}^2] \longrightarrow 0 \tag{16}$$

$$A_2 = \mathbb{E}[S_{n,3}^2] \longrightarrow 0 \tag{17}$$

$$A_3 = \left| \mathbb{E}[\exp(iuS_{n,1}) - \prod_{m=0}^{k_n-1} \mathbb{E}[\exp(iuU_m)] \right| \longrightarrow 0 \tag{18}$$

$$A_4 = \mathbb{E}[U_m^2] \longrightarrow \sigma_\lambda^2(r,s,t) \tag{19}$$

$$A_5 = \sum_{m=0}^{k_n-1} \mathbb{E}\left[U_m^2 I\{|U_m| > \varepsilon\sigma_\lambda(r,s,t)\}\right] \longrightarrow 0, \forall \varepsilon > 0. \tag{20}$$

Remark: Relations (16) and (17) imply that $S_{n,2}$ and $S_{n,3}$ are asymptotically negligeable; (18) shows that the summands $\{U_m\}$ in $S_{n,1}$ are asymptotically independent; (19) and (20) are Lindeberg-Feller conditions for asymptotic normality of $S_{n,1}$ under dependence. Expressions (16)-(18) entail the asymptotic normality

$$T_n \xrightarrow{L} \mathcal{N}(0, \sigma_\lambda^2(r,s,t)) \tag{21}$$

(i) **Proof of (16)**

$$\mathbb{E}[S_{n,2}^2] = var\left(\sum_{m=0}^{k_n-1} V_m\right)$$
$$= \sum_{m=0}^{k_n-1} var(V_m) + \sum_{\substack{m=0 \\ m\neq m'}}^{k_n-1} \sum_{m'=0}^{k_n-1} cov(V_m, V_{m'})$$
$$= A_{11} + A_{12}. \tag{22}$$

To control A_{11}, we get

$$var(V_m) = var\left(\sum_{i=m(u_n+v_n)+u_n+1}^{(m+1)(u_n+v_n)} Z_{n,i}\right)$$
$$= \sum_{i=m(u_n+v_n)+u_n+1}^{(m+1)(u_n+v_n)} var(Z_{n,i}) + \sum_{\substack{i=m(u_n+v_n)+u_n+1 \\ i\neq i'}}^{(m+1)(u_n+v_n)} \sum_{i'=m(u_n+v_n)+u_n+1}^{(m+1)(u_n+v_n)} cov(Z_{n,i}, Z_{n,i'}) \tag{23}$$

and using the second-order stationarity and the fact that $\{Z_{ijkl}\}$ and $\{Z_{ij'k'l'}\}$ are independent,

$$\begin{aligned} var(V_m) &= \sum_{i=1}^{v_n} var(Z_{n,i}) + \sum_{\substack{i=1\\ i\neq i'}}^{v_n}\sum_{i'=1}^{v_n} cov(Z_{n,i}, Z_{n,i'}) \\ &= v_n var(Z_{n,1}) + \sum_{\substack{i=1\\ i\neq i'}}^{v_n}\sum_{i'=1}^{v_n} cov(Z_{n,i}, Z_{n,i'}) \\ &= \frac{v_n}{n}\sigma_\lambda^2(r,s,t)(1+o(1)). \end{aligned} \tag{24}$$

Because

$$\begin{aligned} var(Z_{n,1}) &= var\left(\sum_{1\leq j\neq k\neq l\leq N}(Z_{1jkl} - \mathbb{E}Z_{1jkl})\right) \\ &= var\left(\sum_{1\leq j\neq k\neq l\leq N}(Z_{jkl} - \mathbb{E}Z_{jkl})\right) \\ &= \sum_{1\leq j\neq k\neq l\leq N} var\left(Z_{jkl} - \mathbb{E}Z_{jkl}\right) \\ &= N(N-1)(N-2)var(Z_{111} - \mathbb{E}Z_{111}) \end{aligned}$$

$$\begin{aligned} &= N(N-1)(N-2)\left\{\mathbb{E}(Z_{111} - (\mathbb{E}Z_{111})^2\right\} \\ &= N(N-1)(N-2)\left\{\frac{\sigma_\lambda^2(r,s,t)}{nN(N-1)(N-2)}(1+o(1))\right\} \\ &= \frac{\sigma_\lambda^2(r,s,t)}{n}(1+o(1)). \end{aligned}$$

And also,

$$\begin{aligned} |cov(Z_{n,i}, Z_{n,i'})| &\leq \frac{N(N-1)(N-2)}{n}\left|\frac{1}{h_n^{|\nu|+3}}cov(R_{\lambda,i}, R_{\lambda,i'})\right| \\ &\leq \frac{N(N-1)(N-2)}{n}\left\{\sigma_\lambda^2(r,s,t) + o(1)\right\} \\ \sum_{\substack{i=1\\ i\neq i'}}^{v_n}\sum_{i'=1}^{v_n}|cov(Z_{n,i}, Z_{n,i'})| &\leq \frac{v_n}{n}\left\{v_n N(N-1)(N-2)[\sigma_\lambda^2(r,s,t) + o(1)]\right\} \\ &= \frac{v_n}{n}\{o(n)\} \\ &= v_n o(1). \end{aligned}$$

Then, we get

$$\begin{aligned} |A_{11}| &\leq \sum_{m=0}^{k_n-1}\left\{v_n\frac{\sigma_\lambda^2(r,s,t)}{n}(1+o(1)) + v_n o(1)\right\} \\ &= k_n\left\{v_n\frac{\sigma_\lambda^2(r,s,t)}{n}(1+o(1)) + v_n o(1)\right\} \\ &= k_n v_n\left\{\frac{\sigma_\lambda^2(r,s,t)}{n}(1+o(1)) + o(1)\right\} \\ &= k_n v_n\left\{\frac{\sigma_\lambda^2(r,s,t)}{n}(1+o(1))\right\} \\ &= k_n\frac{v_n}{n}\sigma_\lambda^2(r,s,t)(1+o(1)) \end{aligned}$$

$$
\begin{aligned}
&= \left\lfloor \frac{n}{u_n+v_n} \right\rfloor \frac{v_n}{n}\sigma_\lambda^2(r,s,t)(1+o(1))\\
&\sim \frac{n}{u_n}\frac{v_n}{n}\sigma_\lambda^2(r,s,t)(1+o(1))\\
&= \frac{v_n}{u_n}\sigma_\lambda^2(r,s,t)(1+o(1))\\
&\longrightarrow 0, \quad \text{by } (10).
\end{aligned}
$$

Now

$$
\begin{aligned}
A_{12} &= \sum_{\substack{m=0\\ m\neq m'}}^{k_n-1}\sum_{m'=0}^{k_n-1} cov(V_m, V_{m'})\\
&= \sum_{\substack{m=0\\ m\neq m'}}^{k_n-1}\sum_{m'=0}^{k_n-1}\sum_{\substack{i=m(u_n+v_n)+u_n+1\\ i\neq i'}}^{(m+1)(u_n+v_n)}\sum_{i=m'(u_n+v_n)+u_n+1}^{(m'+1)(u_n+v_n)} cov(Z_{n,i}, Z_{n,i'})\\
&= \sum_{\substack{m=0\\ m\neq m'}}^{k_n-1}\sum_{m'=0}^{k_n-1}\sum_{\substack{i=1\\ i\neq i'}}^{v_n}\sum_{i=1}^{v_n} cov(Z_{n,m(u_n+v_n)+u_n+i}, Z_{n,m'(u_n+v_n)+u_n+i'})\\
&= \sum_{\substack{m=0\\ m\neq m'}}^{k_n-1}\sum_{m'=0}^{k_n-1}\sum_{\substack{i=1\\ i\neq i'}}^{v_n}\sum_{i=1}^{v_n} cov(Z_{n,\lambda_m+i}, Z_{n,\lambda_{m'}+i'})
\end{aligned}
$$

since $|\lambda_m - \lambda_{m'} + i - i'| \geq u_n$ then we reduce the sums and we write

$$
\begin{aligned}
|A_{12}| &\leq \sum_{\substack{i=1\\ |i-i'|\geq u_n}}^{n}\sum_{i=1}^{n} |cov(Z_{n,i}, Z_{n,i'})|\\
&\leq \frac{N(N-1)(N-2)}{nh_n^{|\nu|+3}} 8C[h_n^{|\nu|+3}]^{2/\delta}\sum_{\ell=1}^{\infty}\ell^a[\alpha(\ell)]^{1-2/\delta}.\\
&= \frac{8CN(N-1)(N-2)}{nh_n^{(|\nu|+3)(1-2/\delta)}}\sum_{\ell=1}^{\infty}\ell^a[\alpha(\ell)]^{1-2/\delta}\\
&= o(1).\\
\text{Therefore } A_{12} &\longrightarrow 0, \text{ as } n\longrightarrow +\infty. \qquad (25)
\end{aligned}
$$

Combining (23) and (24), it follows that $\mathbb{E}[S_{n,2}^2] \longrightarrow 0$ and

$$S_{n,2} \longrightarrow 0 \text{ in probability.}$$

This achieves the proof of (16).

(ii) **Proof of (17)** Using the same arguments as in the proof of (16), one has

$$
\begin{aligned}
\mathbb{E}[S_{n,3}^2] &= var(\sum_{m=0}^{k_n-1} U_m)\\
&\leq \frac{u_n+v_n}{n}\left\{\sigma_\lambda^2(r,s,t)+o(1)\right\}.\\
&\sim \frac{u_n}{n}\left\{\sigma_\lambda^2(r,s,t)+o(1)\right\}.\\
&\longrightarrow 0. \qquad (26)
\end{aligned}
$$

(iii) **Proof of (18)** The proof is based on the Lemma of Volkonskii & Rozanov (1959).

Here note that U_m is $\{\mathcal{F}_{i_1,\ldots,i_{u_n}}\}$-mesurable with $i_1 = m(u_n+v_n)+1$ and $i_{u_n} = m(u_n+v_n)+u_n$ and taking $V_m = \exp(iuU_m)$

as in the Lemma of Volkonskii & Rozanov, we have

$$\begin{aligned}\left|\mathbb{E}[\exp(iuS_{n,1}) - \mathbb{E}[\exp(iuU_m)]\right| &\leq 16k_n\alpha(v_n+1)\\ &\sim 16\frac{n}{u_n}\alpha(v_n+1)\\ &\longrightarrow 0 \text{ by (10).}\end{aligned} \tag{27}$$

(iv) **Proof of (19)** Replacing u_n by v_n we have

$$\begin{aligned} var(U_m) &= var\left(\sum_{i=m(u_n+v_n)+1}^{m(u_n+v_n)+u_n} Z_{n,i}\right)\\ &= \sum_{i=m(u_n+v_n)+1}^{m(u_n+v_n)+u_n} var(Z_{n,i}) + \sum_{\substack{i=m(u_n+v_n)+1\\ i\neq i'}}^{m(u_n+v_n)+u_n}\ \sum_{i'=m(u_n+v_n)+1}^{m(u_n+v_n)+u_n} cov(Z_{n,i},Z_{n,i'})\\ &= u_n\sigma_\lambda^2(r,s,t)(1+o(1)).\end{aligned} \tag{28}$$

So that

$$\begin{aligned}\sum_{m=0}^{k_n-1}\mathbb{E}[U_m^2] &= k_n\frac{u_n}{n}\sigma_\lambda^2(r,s,t)(1+o(1))\\ &\sim \frac{u_n}{u_n}\sigma_\lambda^2(r,s,t)(1+o(1))\\ &\longrightarrow \sigma_\lambda^2(r,s,t).\end{aligned}$$

(v) **Proof of (20)** We need a truncation argument. Let τ_n be a fixed truncation point. We can replace $\gamma_\lambda(T_{ij},T_{ik},T_{il},Y_{ij},Y_{ik},Y_{il})$ with the truncated process
$\gamma_\lambda(T_{ij},T_{ik},T_{il},Y_{ij},Y_{ik},Y_{il})I(|\gamma_\lambda(T_{ij},T_{ik},T_{il},Y_{ij},Y_{ik},Y_{il})|\leq\tau_n)$ in (Y_{ij},Y_{ik},Y_{il}). Denote

$$\begin{aligned} Z_{ijkl}^{\tau_n} &= \frac{1}{\sqrt{nN(N-1)(N-2)h_n^{|\nu|+3}}}\gamma_\lambda(T_{ij},T_{ik},T_{il},Y_{ij},Y_{ik},Y_{il})I(|\gamma_\lambda(T_{ij},T_{ik},T_{il},Y_{ij},Y_{ik},Y_{il})|\leq\tau_n)\\ &\quad K_3\left(\frac{r-T_{ij}}{h_n}\frac{s-T_{ik}}{h_n},\frac{t-T_{il}}{h_n}\right),\\ Z_{n,i}^{\tau_n} &= \sum_{1\leq j\neq k\neq l\leq N}(Z_{ijkl}^{\tau_n}-\mathbb{E}Z_{ijkl}^{\tau_n}).\end{aligned}$$

Define $T_n^{\tau_n} = \sum_{i=1}^n Z_{n,i}^{\tau_n}$ and

$$T_n^{*\tau_n} = \sum_{i=1}^n (Z_{n,i}-Z_{n,i}^{\tau_n}) = \sum_{i=1}^n Z_{n,i}I(|\gamma_\lambda(T_{ij},T_{ik},T_{il},Y_{ij},Y_{ik},Y_{il})|>\tau_n). \tag{29}$$

Since $|\gamma_\lambda(T_{ij},T_{ik},T_{il},Y_{ij},Y_{ik},Y_{il})|\leq\tau_n$ and from (K.1), it follows that

$$|Z_{n,i}^{\tau_n}|\leq 2C\frac{N(N-1)(N-2)\tau_n}{\sqrt{nN(N-1)(N-2)h_n^{|\nu|+3}}}$$

and

$$\max_{0\leq m\leq k_n-1}|U_m^{\tau_n}|\leq 2C\frac{N(N-1)(N-2)u_n\tau_n}{\sqrt{nN(N-1)(N-2)h_n^{|\nu|+3}}}.$$

Therefore if we take τ_n and u_n such that

$$u_n\tau_n = \frac{n^{1/2}h_n^{|\nu|+3}}{(N(N-1)(N-2))^{1/2}},$$

then,

$$\max_{0\le m\le k_n-1}|U_m^{\tau_n}| \le 2C\frac{N(N-1)(N-2)u_n\tau_n}{\sqrt{nN(N-1)(N-2)h_n^{|\nu|+3}}} \longrightarrow 0.$$

Hence, for n sufficiently large, the set $\{|U_m^{\tau_n}| > \epsilon\sigma_\lambda^2(r,s,t)\}$ becomes empty for all $\varepsilon > 0$. Thus, $\mathbb{P}(|U_m^{\tau_n}| > \epsilon\sigma_\lambda^2(r,s,t)) = 0$ for large n, for all $\varepsilon > 0$ so

$$\sum_{m=0}^{k_n-1}\mathbb{E}\left[U_m^2 I\{|U_m| > \varepsilon\sigma_r(r,s,t)\}\right] = 0, \text{ for all } \varepsilon > 0.$$

Hence

$$T_n^{\tau_n} \quad \overset{L}{\longrightarrow} \quad \mathcal{N}(0,\sigma^2_{\lambda,\tau_n}(r,s,t)). \tag{30}$$

In order to complete the proof, namely to establish. (21) for the general case, it suffices to show that as first $n \longrightarrow +\infty$ and $\tau_n \longrightarrow +\infty$ (see Masry, 2005 or Fan & Masry, 1992) we have

$$var(T_n^{*\tau_n}) \quad \longrightarrow \quad 0. \tag{31}$$

Indeed,

$$\begin{aligned}
&\left|\mathbb{E}\exp(iuT_n) - \exp(-u^2\sigma_\lambda^2(r,s,t)/2)\right| \\
= \;& \left|\mathbb{E}\exp(iu(T_n^{\tau_n} + T_n^{*\tau_n})) - \exp(-u^2\sigma^2_{\lambda,\tau_n}(r,s,t)/2)\right. \\
& \left. + \exp(-u^2\sigma^2_{\lambda,\tau_n}(r,s,t)/2) - \exp(-u^2\sigma_\lambda^2(r,s,t)/2)\right| \\
\le \;& \left|\mathbb{E}\exp(iuT_n^{\tau_n}) - \exp(-u^2\sigma^2_{\lambda,\tau_n}(r,s,t)/2)\right| + \mathbb{E}\left|\exp(iu(T_n^{*\tau_n}) - 1\right| \\
& + \left|\exp(-u^2\sigma^2_{\lambda,\tau_n}(r,s,t)/2) - \exp(-u^2\sigma_\lambda^2(r,s,t)/2)\right|.
\end{aligned}$$

Letting $n \longrightarrow +\infty$, the first term goes to zero by (30), for every $\tau_n > 0$; the second term converges to zero by (31), because first $n \longrightarrow +\infty$ and then $\tau_n \longrightarrow +\infty$; the third term goes to zero as $\tau_n \longrightarrow +\infty$ by the dominated convergence theorem.

Therefore, it remains to prove (31). Note that by (29), $T_n^{*\tau_n}$ has the same structure as $T_n^{\tau_n}$ except that $Z_{n,i}^{\tau_n}$ is replace by $(Z_{n,i} - Z_{n,i}^{\tau_n})$. Applying the Lemma 2.3 in Fan & Masry (1992) or the same arguments as in Masry (2005) we conclued that, for all fixed $\tau_n > 0$, one has (31).

Then, it suffices to choose τ_n sufficiently large, such that the non-troncated part becomes asymptotically negligeable. □

Theorem 4.2. Under assumptions of theorems 3.1 and 4.1, we have

$$\sqrt{nN(N-1)(N-2)h_n^{|\nu|+3}}(\Gamma_{\lambda n} - m_{\lambda n}) \quad \longrightarrow \quad \mathcal{N}(\mathcal{B}(r,s,t),\sigma_\lambda^2(r,s,t)). \tag{32}$$

Proof. Theorem 4.2 follows from theorem 3.1 and theorem 4.1.□

Under the assumptions of theorem 3.1 and theorem 4.1, we rewrite theorem 2.1 in Soro & Hili (2012) with mixing arguments.

Let $H : \mathbb{R}^l \longrightarrow \mathbb{R}$ be a function with continuous second order derivatives. We denote the gradient vector $(\frac{\partial H}{\partial x_1}(v), ..., \frac{\partial H}{\partial x_l}(v))^T$ by $DH(v)$.

Let

$$m_\lambda = m_\lambda(r,s,t) = \frac{d^{|\nu|}}{dr^{\nu_1}ds^{\nu_2}dt^{\nu_3}}\int_{\mathbb{R}^3}\gamma_\lambda(r,s,t,y_1,y_2,y_3)g_3(r,s,t,y_1,y_2,y_3)dy_1dy_2dy_3,$$

$1 \le \lambda \le l$,

$$\begin{aligned}
\mathbf{B}(r,s,t) \quad = \quad & \frac{(-1)^{|k|}a}{k!}\sum_{\lambda=1}^{l}\left\{\int_{\mathbb{R}^3}u^{k_1}v^{k_2}w^{k_3}K_3(u,v,w)dudvdw\right. \\
& \left.\times\frac{d^{|k|}}{dr^{k_1}ds^{k_2}dt^{k_3}}\int_{\mathbb{R}^3}\gamma_\lambda(r,s,t,y_1,y_2,y_3)g_3(r,s,t,y_1,y_2,y_3)dy_1dy_2dy_3\right\} \\
& \times\left\{\frac{\partial H}{\partial m_\lambda}(m_1,...,m_l)^T\right\}
\end{aligned}$$

and

$$\begin{aligned}\delta_{\lambda k} &= \delta_{\lambda k}(r,s,t)\\ &= \|K_3\|^2 \int_{\mathbb{R}^3} \gamma_\lambda(r,s,t,y_1,y_2,y_3)\gamma_k(r,s,t,y_1,y_2,y_3)g_3(r,s,t,y_1,y_2,y_3)dy_1dy_2dy_3,\\ \Xi &= (\delta_{k\lambda})_{1\le\lambda,k\le l} \text{ the variance-covariance matrix,}\end{aligned}$$

Theorem 4.3. Assume assumptions of theorems 3.1 and 4.1 hold. Then

$$\begin{aligned}&\sqrt{nN(N-1)(N-2)h_n^{|\gamma|+3}}[H(\Gamma_{1n},...,\Gamma_{ln}) - H(m_1,...,m_l)]\\ &\quad\xrightarrow{L} \quad \mathcal{N}(\mathbf{B}(r,s,t),[DH(m_1,...,m_l)]^T\Xi[DH(m_1,...,m_l)]).\end{aligned} \tag{33}$$

Proof.

A l-dimensional Taylor expansion of H around $(m_1,...,m_l)^T$ of order 1 combined with (2) gives

$$\sqrt{nN(N-1)(N-2)h_n^{|\gamma|+3}}[H(\mathbb{E}\Gamma_{1n},...,\mathbb{E}\Gamma_{ln}) - H(m_1,...,m_l)] \quad \xrightarrow{\mathbb{P}} \quad \mathbf{B}(r,s,t). \tag{34}$$

Applying the Cramér-Wold device to (9) it comes

$$\begin{aligned}&\sqrt{nN(N-1)(N-2)h_n^{|\gamma|+3}}(H(\Gamma_{1n},...,\Gamma_{ln}) - H(\mathbb{E}\Gamma_{1n},...,\mathbb{E}\Gamma_{ln}))\\ &\quad\longrightarrow \quad \mathcal{N}(0,[DH(m_1,...,m_l)]^T\Xi[DH(m_1,...,m_l)]).\end{aligned} \tag{35}$$

Finally, (34) and (35) lead to (33). □

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Nilpotency of the Ordinary Lie-algebra of an *n*-Lie Algebra

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Abstract

In this paper, we generalize to n-Lie algebras a corollary of the well-known Engel's theorem which offers some justification for the terminology "nilpotent" and we construct a nilpotent ordinary Lie algebra from a nilpotent n-Lie algebra.

Keywords: Lie algebra, n-Lie algebra, nilpotency

1. Introduction

(Filipov, 1985) Introduced a generalization of a Lie algebra, which he called an n-Lie algebra. The Lie product is taken between n elements of the algebra instead of two. This new bracket is n-linear, anti-symmetric and satisfies a generalization of the Jacobi identity.

(Bossoto,Okassa, & Omporo, 2013) Associate to an n-Lie algebra, a Lie algebra called the ordinary Lie algebra.

In this paper, we generalize to n-Lie algebras a corollary of the well-known Engel's theorem and we construct a nilpotent ordinary Lie algebra from a nilpotent n-Lie algebra.

1.1 n-Lie Algebra Structure

In the following, K will denote a commutative field with characteristic zero.

An n-Lie algebra $\mathcal{G}$ over K is a vector space together with a multilinear fully skewsymmetric map

$$\{,...,\}:\mathcal{G}^n=\mathcal{G}\times\mathcal{G}\times...\times\mathcal{G}\longrightarrow\mathcal{G},(x_1,x_2,...,x_n)\longmapsto\{x_1,x_2,...,x_n\},$$

such that

$$\{x_1,x_2,...,x_{n-1},\{y_1,y_2,...,y_n\}\}=\sum_{i=1}^{n}\{y_1,y_2,...,y_{i-1},\{x_1,x_2,...,x_{n-1},y_i\},y_{i+1},...,y_n\}$$

for all $x_1,x_2,...,x_{n-1},y_1,y_2,...,y_n$ elements of $\mathcal{G}$.

The above equation is called the generalized Jacobi Identity.

A subspace $\mathcal{G}_0$ of $\mathcal{G}$ is called an n-Lie subalgebra if for any $y_1,y_2,...,y_n\in\mathcal{G}_0,\{y_1,y_2,...,y_n\}\in\mathcal{G}_0$.

Let $\mathcal{G}_1,\mathcal{G}_2,...,\mathcal{G}_n$ be subalgebras of n-Lie algebra $\mathcal{G}$ and let $\{\mathcal{G}_1,\mathcal{G}_2,...,\mathcal{G}_n\}$ denote the subspace of $\mathcal{G}$ generated by all vectors $\{x_1,x_2,...,x_n\}$, where $x_i\in\mathcal{G}_i$ for $i=1,2,...,n$. The subalgebra $\{\mathcal{G},\mathcal{G},...,\mathcal{G}\}$ is called the derived algebra of $\mathcal{G}$, and is denoted by $\mathcal{G}^1$. If $\mathcal{G}^1=0$, then $\mathcal{G}$ is called an abelian n-Lie algebra.

Using the derivation $ad(x_1,x_2,...,x_{n-1}):\mathcal{G}\longrightarrow\mathcal{G},y\longmapsto\{x_1,x_2,...,x_{n-1},y\}$, we can rephrase this definition as follows:

A vector subspace $\mathcal{G}_0$ of $\mathcal{G}$ is an n-Lie subalgebra of $\mathcal{G}$ if $ad(x_1,x_2,...,x_{n-1})(\mathcal{G}_0)\subset\mathcal{G}_0$ for any $x_1,x_2,...,x_{n-1}\in\mathcal{G}_0$.That is, $ad(\mathcal{G}_0,\mathcal{G}_0,...,\mathcal{G}_0)(\mathcal{G}_0)\subset\mathcal{G}_0$.

A subspace $\mathcal{I}$ of $\mathcal{G}$ is called an ideal if $\{x,y_1,y_2,...,y_{n-1}\}\in\mathcal{I}$ for any $x\in\mathcal{I}$, and for any $y_1,y_2,...,y_{n-1}\in\mathcal{G}$. That is equivalent to say that $ad(\mathcal{G},...,\mathcal{G})(\mathcal{I})\subset\mathcal{I}$.

1.2 The Ordinary Lie Algebra of an n-Lie Algebra

Let $\mathcal{G}$ be an n-Lie algebra over a field K. (Bossoto et al., 2013) associate to $\mathcal{G}$ a Lie algebra called the ordinary Lie algebra. This construction goes as presented below:

Consider the map

$$\mathcal{G}^{n-1}\longrightarrow Der_K(\mathcal{G}),(x_1,x_2,...,x_{n-1})\longmapsto ad(x_1,x_2,...,x_{n-1}),$$

where $Der_K(\mathcal{G})$ denote the set of $K-$derivations of $\mathcal{G}$.

Denote by $\Lambda_K^{n-1}(\mathcal{G})$, the $(n-1)$-exterior power of the K-vector space $\mathcal{G}$, there exists a unique K-linear map

$$ad_{\mathcal{G}} : \Lambda_K^{n-1}(\mathcal{G}) \longrightarrow Der_K(\mathcal{G})$$

such that

$$ad_{\mathcal{G}}(x_1\Lambda x_2\Lambda...\Lambda x_{n-1}) = ad(x_1, x_2, ..., x_{n-1})$$

for all $x_1, x_2, ..., x_{n-1} \in \mathcal{G}$.

When $f : W \longrightarrow W$ is an endomorphism of a K-vector space W and when $\Lambda_K(W)$ is the K-exterior algebra of W, then there exists a unique derivation of degree *zero*

$$D_f : \Lambda_K(W) \longrightarrow \Lambda_K(W)$$

such that, for $p \in \mathbb{N}$,

$$D_f(w_1\Lambda w_2\Lambda...\Lambda w_p) = \sum_{i=1}^{p} w_1\Lambda w_2\Lambda...\Lambda w_{i-1}\Lambda f(w_i)\Lambda w_{i+1}\Lambda...\Lambda w_p$$

for all $w_1, w_2, ..., w_p$ elements of W.

Proposition 1 For all s_1 and s_2 elements of $\Lambda_K^{n-1}(\mathcal{G})$, then we have simultaneously

$$[ad_{\mathcal{G}}(s_1), ad_{\mathcal{G}}(s_2)] = ad_{\mathcal{G}}\left(D_{ad_{\mathcal{G}}(s_1)}(s_2)\right)$$

and

$$[ad_{\mathcal{G}}(s_1), ad_{\mathcal{G}}(s_2)] = ad_{\mathcal{G}}\left(-D_{ad_{\mathcal{G}}(s_2)}(s_1)\right)$$

where [,] denotes the usual bracket of endomorphisms.

We denote by $\mathcal{V}_K(\mathcal{G})$ the K-subspace of $\Lambda_K^{n-1}(\mathcal{G})$ generated by the elements of the form $D_{ad_{\mathcal{G}}(s_1)}(s_2) + D_{ad_{\mathcal{G}}(s_2)}(s_1)$ where s_1 and s_2 describe $\Lambda_K^{n-1}(\mathcal{G})$.

Let

$$\Lambda_K^{n-1}(\mathcal{G}) \longrightarrow \Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G}), s \longmapsto \overline{s},$$

be the canonical surjection. Given the foregoing, we conclude that

$ad_{\mathcal{G}}[\mathcal{V}_K(\mathcal{G})] = 0$.

We denote by

$$\widetilde{ad_{\mathcal{G}}} : \Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}(\mathcal{G}) \longrightarrow Der_K(\mathcal{G})$$

the unique linear map such that

$$\widetilde{ad_{\mathcal{G}}}(\overline{s}) = ad_{\mathcal{G}}(s)$$

for all $s \in \Lambda_K^{n-1}(\mathcal{G})$.

Theorem 2 When $(\mathcal{G}, \{, ..., \})$ is a n-Lie algebra, then the map

$$[,] : \left[\Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G})\right]^2 \longrightarrow \Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G}), (\overline{s_1}, \overline{s_2}) \longmapsto \overline{D_{ad_{\mathcal{G}}(s_1)}(s_2)},$$

depends only on $\overline{s_1}$ and $\overline{s_2}$, and defines an ordinary Lie algebra structure on $\Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G})$.

Proposition 3 If a subspace $\mathcal{G}_0$ of an n-Lie algebra $\mathcal{G}$ is stable for the representation

$$\widetilde{ad_{\mathcal{G}}} : \Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G}) \longrightarrow Der_K(\mathcal{G}), \overline{s} \longmapsto ad_{\mathcal{G}}(s),$$

then $\mathcal{G}_0$ is an ideal of the n-Lie algebra $\mathcal{G}$.

2. Nilpotency of the Ordinary Lie Algebra

An n-Lie algebra $\mathcal{G}$ is nilpotent if $\mathcal{G}$ satisfies $\mathcal{G}^r = 0$ for some $r \geq 0$, where $\mathcal{G}^0 = \mathcal{G}$ and $\mathcal{G}^r$ is defined by induction, $\mathcal{G}^{r+1} = [\mathcal{G}^r, \mathcal{G}, \mathcal{G}, \cdots, \mathcal{G}]$ for $r \geq 0$.

Proposition 4 Let $\mathcal{G}$ be an n-Lie algebra over a field K. If $\mathcal{G} \neq 0$ is nilpotent then $\mathcal{Z}(\mathcal{G}) \neq 0$.

Proof. Let us suppose $\mathcal{Z}(\mathcal{G}) = 0$.

Nilpotency of $\mathcal{G}$ implies that there exists an integer $k \geq 0$ such that $\mathcal{G}^{k-1} \neq 0$ and $\mathcal{G}^k = 0$.

$$\begin{aligned} 0 = \mathcal{G}^k &= \left\{\mathcal{G}^{k-1}, \mathcal{G}, \mathcal{G}, ..., \mathcal{G}\right\} \\ &= \left\{\mathcal{G}, \mathcal{G}, ..., \mathcal{G}, \mathcal{G}^{k-1}\right\} \\ &= ad(\mathcal{G}, \mathcal{G}, ..., \mathcal{G})(\mathcal{G}^{k-1}) \\ &= 0 \end{aligned}$$

Then $\mathcal{G}^{k-1} \subset \mathcal{Z}(\mathcal{G})$.

Therefore $0 \neq \mathcal{G}^{k-1} \subset \mathcal{Z}(\mathcal{G}) = 0$ which is impossible.

Thus $\mathcal{Z}(\mathcal{G}) \neq 0$.

Below we give the statement of the Engel's theorem and its corollary for Lie algebras:

Theorem 5 (Engel) Let $\rho : \mathcal{G} \rightarrow End(V)$ be a linear representation of $\mathcal{G}$ on the vector space V such that $\rho(x)$ is nilpotent for each $x \in \mathcal{G}$. If $V \neq (0)$, then there

exists $v \in V, v \neq 0$ such that $\rho(x)v = 0$ for all $x \in \mathcal{G}$.

Corollary 6 $\mathcal{G}$ is nilpotent if and only if adx is nilpotent for each $x \in \mathcal{G}$.

Now we're going to give a generalization to n-Lie algebras of the above corollary:

Theorem 7 Let $\mathcal{G}$ be an n-Lie algebra over a field K. $\mathcal{G}$ is nilpotent if and only if $ad(x_1, x_2, ..., x_{n-1})$ is nilpotent for any $x_1, x_2, ..., x_{n-1} \in \mathcal{G}$.

To prove the Theorem, one needs some Lemmas:

Lemma 8 Let $\mathcal{G}$ be an n-Lie algebra, $\mathcal{Z}(\mathcal{G})$ the center of $\mathcal{G}$ and $\pi : \mathcal{G} \rightarrow \mathcal{G}/\mathcal{Z}(\mathcal{G})$ the canonical surjection. For any $x_1, x_2, ..., x_{n-1} \in \mathcal{G}$, if $ad(x_1, x_2, ..., x_{n-1}) : \mathcal{G} \rightarrow \mathcal{G}$ is nilpotent, then the unique linear map

$$\overline{ad_{\mathcal{G}}(x_1, x_2, ..., x_{n-1})} : \mathcal{G}/\mathcal{Z}(\mathcal{G}) \rightarrow \mathcal{G}/\mathcal{Z}(\mathcal{G}), \bar{y} \mapsto \overline{\{x_1, x_2, ..., x_{n-1}, y\}}$$

such that $\pi \circ \overline{ad_{\mathcal{G}}(x_1, x_2, ..., x_{n-1})} = ad_{\mathcal{G}}(x_1, x_2, ..., x_{n-1}) \circ \pi$ is nilpotent.

Proof. It's clear that $ad(x_1, x_2, ..., x_{n-1})[\mathcal{Z}(\mathcal{G})] = 0$. We denote by

$$\overline{ad_{\mathcal{G}}(x_1, x_2, ..., x_{n-1})} : \mathcal{G}/\mathcal{Z}(\mathcal{G}) \rightarrow \mathcal{G}/\mathcal{Z}(\mathcal{G}), \bar{y} \mapsto \overline{\{x_1, x_2, ..., x_{n-1}, y\}}$$

the unique linear map such that $\pi \circ \overline{ad_{\mathcal{G}}(x_1, x_2, ..., x_{n-1})} = ad_{\mathcal{G}}(x_1, x_2, ..., x_{n-1}) \circ \pi$.

$ad(x_1, x_2, ..., x_{n-1})$ nilpotent, then there exists $k \geq 0$ such that $(ad_{\mathcal{G}}(x_1, x_2, ..., x_{n-1}))^k = 0$. We have: $(\overline{ad_{\mathcal{G}}h})^k \circ \pi = \pi \circ (ad_{\mathcal{G}}h)^k = 0$. Since π is surjective $\Rightarrow (\overline{ad_{\mathcal{G}}(x_1, x_2, ..., x_{n-1})})^k = 0$ ie $\overline{ad_{\mathcal{G}}(x_1, x_2, ..., x_{n-1})}$ is nilpotent.

Lemma 9 If for any $x_1, x_2, ..., x_{n-1} \in \mathcal{G}$, $ad(x_1, x_2, ..., x_{n-1}) : \mathcal{G} \rightarrow \mathcal{G}$ is nilpotent, then $\mathcal{Z}(\mathcal{G}) \neq (0)$.

Proof. Using the well-known Engel's theorem, there exists $u \in \mathcal{G}, u \neq 0$, such that $ad(x_1, x_2, ..., x_{n-1})(u) = 0$, for any $x_1, x_2, ..., x_{n-1} \in \mathcal{G}$. That implies $u \in \mathcal{Z}(\mathcal{G})$. And as $u \neq 0$, thus $\mathcal{Z}(\mathcal{G}) \neq (0)$.We are done.

The set $\{ad(x_1, x_2, ..., x_{n-1})/ad(x_1, x_2, ..., x_{n-1})$ is nilpotent for any $x_1, x_2, ..., x_{n-1} \in \mathcal{G}\}$ is a Lie subalgebra of $End_{\mathbb{k}}(\mathcal{G})$.

Proof. " $\Rightarrow$ ":

$\mathcal{G}$ nilpotent implies that there exists $k \geq 0$ such that $\mathcal{G}^{k-1} \neq 0$ and $\mathcal{G}^k = 0$.

$$
\begin{aligned}
0 &= \mathcal{G}^k = \left\{\mathcal{G}, \mathcal{G}, ..., \mathcal{G}, \mathcal{G}^{k-1}\right\} \\
&= ad(\mathcal{G}, \mathcal{G}, ..., \mathcal{G})(\mathcal{G}^{k-1}) \\
&= ad(\mathcal{G}, \mathcal{G}, ..., \mathcal{G})\left\{\mathcal{G}, \mathcal{G}, ..., \mathcal{G}, \mathcal{G}^{k-2}\right\} \\
&= ad(\mathcal{G}, \mathcal{G}, ..., \mathcal{G})\left[ad(\mathcal{G}, \mathcal{G}, ..., \mathcal{G})(\mathcal{G}^{k-2})\right] \\
&= \underbrace{[ad(\mathcal{G}, \mathcal{G}, ..., \mathcal{G}) \circ ad(\mathcal{G}, \mathcal{G}, ..., \mathcal{G}) \circ ad(\mathcal{G}, \mathcal{G}, ..., \mathcal{G}) \circ ... \circ ad(\mathcal{G}, \mathcal{G}, ..., \mathcal{G})]}_{k-times}(\mathcal{G}) \\
&= [ad(\mathcal{G}, \mathcal{G}, ..., \mathcal{G})]^k (\mathcal{G})
\end{aligned}
$$

i.e $[ad(x_1, x_2, ..., x_{n-1})]^k = 0$ for any $x_1, x_2, ..., x_{n-1} \in \mathcal{G}$.

Thus $ad(x_1, x_2, ..., x_{n-1})$ is nilpotent.

"$\Longleftarrow$ " we prove by induction on the dimension of $\mathcal{G}$.

• $\dim \mathcal{G} = 1$, $ad(x_1, x_2, ..., x_{n-1}) : \mathcal{G} \to \mathcal{G}$ is nilpotent $\Rightarrow ad(x_1, x_2, ..., x_{n-1})(y) = 0$ for any $x_1, x_2, ..., x_{n-1}, y \in \mathcal{G}$, that is $\mathcal{G}$ is commutative. Thus $ad(\mathcal{G}^{n-1})\ (\mathcal{G}) = 0$ ie $\mathcal{G}^1 = 0$.Therefore $\mathcal{G}$ is nilpotent.

• Suppose the assumption true for $\dim \mathcal{G} = n$.Let's verify the assumption for $\dim \mathcal{G} = n + 1$.

$ad(x_1, x_2, ..., x_{n-1})$ nilpotent for any $x_1, x_2, ..., x_{n-1} \in \mathcal{G}$, then from *Lemma* 8,$\overline{ad_{\mathcal{G}}(x_1, x_2, ..., x_{n-1})} : \mathcal{G}/\mathcal{Z}(\mathcal{G}) \to \mathcal{G}/\mathcal{Z}(\mathcal{G})$ is nilpotent for any $x_1, x_2, ..., x_{n-1} \in \mathcal{G}.\Rightarrow \mathcal{G}/\mathcal{Z}(\mathcal{G})$ is nilpotent and $\mathcal{Z}(\mathcal{G}) \neq 0$ from *Lemma* 9. $\mathcal{G}/\mathcal{Z}(\mathcal{G})$ nilpotent, there exists $k \geq 0$ such that $[\mathcal{G}/\mathcal{Z}(\mathcal{G})]^k = 0$. As $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{Z}(\mathcal{G})$, then $[\mathcal{G}/\mathcal{Z}(\mathcal{G})]^k = \pi(\mathcal{G}^k) = 0$ since π is surjective. Thus $\mathcal{G}^k \subset \mathcal{Z}(\mathcal{G})$. $\mathcal{G}^{k+1} = ad(\mathcal{G}^{n-1})(\mathcal{G}^k) \subset ad(\mathcal{G}^{n-1})(\mathcal{Z}(\mathcal{G})) = 0$.Therefore $\mathcal{G}$ is nilpotent.That ends the proof.

Below we give the statement of the main theorem we obtained:

Theorem 10 If $\mathcal{G}$ is a nilpotent n-Lie algebra over a field $\mathbb{k}$ and if $\widetilde{ad_{\mathcal{G}}} : \Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G}) \longrightarrow Der_K(\mathcal{G}), \overline{s} \longmapsto ad_{\mathcal{G}}(s)$,is the canonical representation of $\Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G})$ in $\mathcal{G}$, then $\left[\Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G})\right]/Ker\widetilde{ad_{\mathcal{G}}}$ is a nilpotent Lie algebra.

Proof. Let $\mathcal{G}$ be an n-Lie algebra. Then the mapping

$$\mathcal{G}^{n-1} \longrightarrow Der_K(\mathcal{G}), (x_1, x_2, ..., x_{n-1}) \longmapsto ad(x_1, x_2, ..., x_{n-1}),$$

induces a representation $\widetilde{ad_{\mathcal{G}}} : \Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G}) \longrightarrow Der_K(\mathcal{G}), \overline{s} \longmapsto ad_{\mathcal{G}}(s)$ of $\Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G})$ in $\mathcal{G}$. When $\mathcal{G}$ is a nilpotent n-Lie algebra then $\widetilde{ad_{\mathcal{G}}}(\Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G}))$ is a Lie subalgebra of $Der_K(\mathcal{G})$ whose all elements are nilpotent.Thus $\widetilde{ad_{\mathcal{G}}}(\Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G}))$ is a nilpotent Lie algebra. Therefore $\left[\Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G})\right]/Ker\widetilde{ad_{\mathcal{G}}}$ is a nilpotent Lie algebra.

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Bounds for Covering Symmetric Convex Bodies by a Lattice Congruent to a Given Lattice

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Abstract

In this paper, we focus on lattice covering of centrally symmetric convex body on $\mathbb{R}^2$. While there is no constraint on the lattice in many other results about lattice covering, in this study, we only consider lattices congruent to a given lattice to retain more information on the lattice. To obtain some upper bounds on the infimum of the density of such covering, we will say a body is a coverable body with respect to a lattice if such lattice covering is possible, and try to suggest a function of a given lattice such that any centrally symmetric convex body whose area is not less than the function is a coverable body. As an application of this result, we will suggest a theorem which enables one to apply this to a coverable body to suggesting an efficient lattice covering for general sets, which may be non-convex and may have holes.

Keywords: lattice covering, centrally symmetric convex body, density of covering, minkowski sum

1. Introduction

The covering problem of centrally symmetric convex bodies, especially related to the density of covering, is a famous problem in discrete geometry. In this paper, we will deal with lattice coverings, which is fundamental when we deal with centrally symmetric convex bodies. For a body A and a lattice Λ, $C = \{A + \lambda | \lambda \in \Lambda\}$ is called a lattice arrangement. If the members of C cover the whole plane, C is called a lattice covering. The density of a lattice covering can be expressed as $\frac{S(A)}{\det \Lambda}$ (Pach & Agarwal, 2011), where $S(A)$ is the area of A and $\det \Lambda$ is the area of the smallest lattice parallelogram of Λ. There are many studies about the upper bounds on the infimum of the density of lattice covering when A is a given body and Λ is any lattice. Because we may choose an appropriate Λ for minimizing the density of covering, the upper bounds are near 1(Fary, 1950). Especially when A is a centrally symmetric convex body, it is well known that it is $\frac{2\pi}{\sqrt{27}} \simeq 1.2092$ (Pach & Agarwal, 2011). In this study, we will consider the same problem when A is a given centrally symmetric convex body and Λ is any lattice congruent with a given lattice Λ_0. Since the condition of Λ is stronger, this upper bound is a lot bigger than 1.2092. This cannot be a constant, since it can be arbitrarily big depending on the given lattice. Thus, we aim to suggest a function of Λ_0 and $S(A)$ which is always less than or equal to

$$\inf_{A+\Lambda=\mathbb{R}^2, \Lambda \equiv \Lambda_0} \frac{S(A)}{\det \Lambda}$$

This is equivalent to suggesting a function f of lattice Λ such that for every centrally symmetric convex body A whose area is not less than $f(\Lambda)$, there exists $\Lambda' \equiv \Lambda$ such that $A + \Lambda' = \mathbb{R}^2$. We will call A a coverable body with respect to Λ if A is a centrally symmetric convex body and there exists a lattice $\Lambda' \equiv \Lambda$ such that $A + \Lambda' = \mathbb{R}^2$.

To suggest the function f, we will first prove some properties of centrally symmetric convex bodies. Then, we will define several new functions related to Λ and prove some properties of them. Using these, we will prove the main result of this paper, which gives the function f.

The condition that the lattice is congruent to a given lattice can be used to suggesting an efficient lattice covering of general sets which need not be convex and may have holes. This will be discussed in the application chapter of this paper.

2. Results

2.1 Geometric Properties of Centrally Symmetric Convex Bodies

In this section, some properties of centrally symmetric convex bodies, which are important lemmas for the main results, are suggested.

The next lemma states a method to determine whether a given set A satisfies $A + \Lambda = \mathbb{R}^2$.

Lemma 1. *Given a lattice $\Lambda \subset \mathbb{R}^2$, the followings hold:*

(i) *Given a closed connected set A, if* $A + \Lambda = \mathbb{R}^2$, *there exists* $\Lambda' \equiv \Lambda$ *and a lattice triangle XYZ of* Λ' *such that* $X, Y, Z \in A$.

(ii) *Given a centrally symmetric convex set A, if there exists a lattice triangle* $XYZ \subset A$, $A + \Lambda = \mathbb{R}^2$.

Proof. (i) For any set S, denote its boundary by ∂S. For any two distinct points P, Q, $\overleftrightarrow{PQ}$ denotes the line containing both of them, and $\overline{PQ}$ may denote either the segment connecting P, Q or the length of such segment.

Since A is closed, there exist $\lambda_1, \lambda_2 \in \Lambda$ such that $(A + \lambda_1) \cap (A + \lambda_2) \neq \emptyset$. Let L be $\overleftrightarrow{\lambda_1\lambda_2} \cap \Lambda$. Let A_1 be a connected component of $A + L$ which includes $A + \{\lambda_1, \lambda_2\}$. Since $A + \{\lambda_1, \lambda_2\} \subset A_1$ and $\partial A_1 \subset \partial(A + L) \subset \partial A + L$, it can be shown that there exist $u, v \in L$ such that $(\partial A + u) \cap (\partial A + v) \cap \partial A_1 \neq \emptyset$. Let p be an element of the intersection. Then since $p \in \partial A_1 \subset \partial(A + L)$, any neighborhood of p contains a point p' such that $p' \notin A + L$, while $p' \in \mathbb{R}^2 = A + \Lambda$. Then $p \in A + (\Lambda \setminus L)$, there exists $w \in \Lambda \setminus L$ such that $p \in A + w$. Then $p \in A + u, A + v, A + w$, thus $-u + p, -v + p, -w + p \in A$. Also, since $w \notin L = \overleftrightarrow{uv} \cap \Lambda$, u, v, w form a triangle. Thus, $-u + p, -v + p, -w + p$ form a lattice triangle of $-\Lambda + p \equiv \Lambda$.

(ii) Since A is centrally symmetric, it can be shown that there exists a hexagon $H = XY'ZX'YZ'$ such that $H \subset A$, $XYZ \equiv X'Y'Z'$ and $\overline{XY} \parallel \overline{X'Y'}$, which shall be degenerated. Then $\mathbb{R}^2 = H + \Lambda \subset A + \Lambda$ can be shown as the following figure.

□

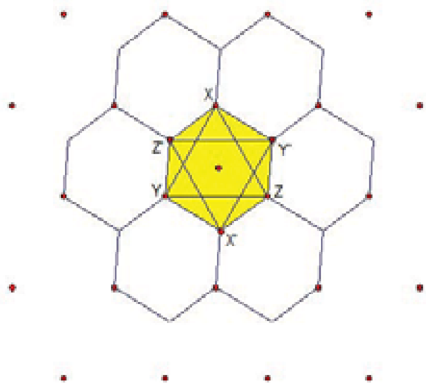


Figure 1. $H + \Lambda = \mathbb{R}^2$

The following is a corollary of Lemma 1 (ii).

Corollary 2. *If A is a centrally symmetric convex body and there exists a triangle in A which is congruent to a lattice triangle of a lattice* Λ, *A is a coverable body with respect to* Λ.

From now, we will denote Ω as a centrally symmetric convex body.

Lemma 3. *There exist polar coordinates such that the origin O is the center of* Ω *and the four rays* $\theta = \frac{\pi k}{2}, k = 0, 1, 2, 3$ *divide* Ω *into four parts of the same area.*

Proof. First consider polar coordinates whose origin is O. For $\phi \in \mathbb{R}$, let $f(\phi)$ be $S(\Omega \cap \{(r, \theta) | \theta \in (\phi, \phi + \frac{\pi}{2})\}) - S(\Omega \cap \{(r, \theta) | \theta \in (\phi - \frac{\pi}{2}, \phi)\})$. Since Ω is centrally symmetric, $f(0) = -f(\frac{\pi}{2})$. Thus there exists $t \in [0, \frac{\pi}{2}]$ such that $f(t) = 0$. Therefore, by rotating the polar coordinates through t, we obtain the polar coordinates satisfying this lemma. □

In this section, we will always use the polar coordinates suggested in Lemma 3.

Lemma 4. *If* $S(\Omega) = \frac{\pi}{2}$, *there exists an inscribed rhombus PQRS such that* $\overline{PQ} = 1$.

Proof. Since Ω is centrally symmetric,

$$\frac{\pi}{2} = S(\Omega) = \frac{1}{2}\int_0^{2\pi} r(\theta)^2 d\theta = \int_0^{\frac{\pi}{2}} r(\theta)^2 + r(\theta + \frac{\pi}{2})^2 d\theta,$$

thus there exists ϕ such that $r(\phi)^2 + r(\phi + \frac{\pi}{2})^2 = 1$. Let W, X, Y, Z the intersections of the boundary of Ω and the rays $\theta = \phi + \frac{\pi k}{2}, k = 0, 1, 2, 3$. Then $\overline{WX} = \overline{XY} = \overline{YZ} = \overline{ZW} = 1$, since

$$\sqrt{r(\phi)^2 + r(\phi + \frac{\pi}{2})^2} = \sqrt{r(\phi + \frac{\pi}{2})^2 + r(\phi + \pi)^2} = 1$$

□

The following lemma is a key theorem in showing the existence of a certain inscribed parallelogram.

Lemma 5. *For any function $f : [0, \frac{\pi}{8}] \to (0, \pi)$ such that its derivative f' exists and is continuous on $[0, \frac{\pi}{8}]$, and $f(0) = f(\frac{\pi}{8}) = \frac{\pi}{2}$, the following holds:*

$$\int_0^{\frac{\pi}{8}} \sqrt{64\sin^2 f(x) + f'(x)^2} dx \geq \pi$$

Proof. Since

$$\int_0^{\frac{\pi}{8}} \sqrt{64\sin^2 f(x) + f'(x)^2} dx = \int_0^{\frac{\pi}{16}} \sqrt{64\sin^2 f(x) + f'(x)^2} dx + \int_0^{\frac{\pi}{16}} \sqrt{64\sin^2 f\left(\frac{\pi}{8} - x\right) + f'\left(\frac{\pi}{8} - x\right)^2} dx,$$

it is sufficient to prove $\int_0^{\frac{\pi}{16}} \sqrt{64\sin^2 y + y'^2} dx \geq \frac{\pi}{2}$ for all function $y : [0, \frac{\pi}{16}] \to (0, \pi)$ such that its derivative y' exists and is continuous on $[0, \frac{\pi}{16}]$ and $y(0) = \frac{\pi}{2}$.

Let $y_0(x)$ be $\frac{\pi}{2} - \int_0^x |y'(t)| dt$. If $\int_0^{\frac{\pi}{16}} |y'(t)| dt \geq \frac{\pi}{2}$, $\int_0^{\frac{\pi}{16}} \sqrt{64\sin^2 y + y'^2} dx \geq \frac{\pi}{2}$ also holds, thus we will suppose $\int_0^{\frac{\pi}{16}} |y'(t)| dt < \frac{\pi}{2}$. Then for all $x \in [0, \frac{\pi}{16}]$,

$$\left|\frac{\pi}{2} - y_0(x)\right| = \int_0^x |y'(t)| dt \geq \left|\int_0^x y'(t) dt\right| = \left|\frac{\pi}{2} - y(x)\right|,$$

thus $\sin y(x) \geq \sin y_0(x)$. For all x, since $|y'(x)| = |y_0{}'(x)|$, $\sqrt{64 \sin^2 y + y'^2} \geq \sqrt{64 \sin^2 y_0 + y_0{}'^2}$. Therefore, it is sufficient to prove $\int_0^{\frac{\pi}{16}} \sqrt{64\sin^2 y_0 + y_0{}'^2} dx \geq \frac{\pi}{2}$. Suppose that this y_0 doesn't satisfy this inequality.

For any t, let y_t be $y_t(x) = y_0(x) - tx$. Since

$$\lim_{t \to 0} \int_0^{\frac{\pi}{16}} \sqrt{64 \sin^2 y_t + y_t{}'^2} dx = \int_0^{\frac{\pi}{16}} \sqrt{64 \sin^2 y_0 + y_0{}'^2} dx < \frac{\pi}{2}$$

there exists $a > 0$ such that

$$\int_0^{\frac{\pi}{16}} \sqrt{64 \sin^2 y_a + y_a{}'^2} dx < \frac{\pi}{2}$$

Since y_0 is a decreasing function, y_a is a strictly decreasing function.

Let z be $\frac{\pi}{2} - y_a$ and let h be $z(\frac{\pi}{16})$. Since z is a strictly increasing function and $z(0) = \frac{\pi}{2} - y_a(0) = \frac{\pi}{2} - y_0(0) = 0$,

$$\int_0^{\frac{\pi}{16}} \sqrt{64\sin^2 y_a + \left(\frac{dy_a}{dx}\right)^2} dx = \int_0^{\frac{\pi}{16}} \sqrt{64\cos^2 z + \left(\frac{dz}{dx}\right)^2} dx = \int_0^h \sqrt{64\left(\frac{dx}{dz}\right)^2 \cos^2 z + 1} dz$$

Define a function v of z as $\sqrt{64\left(\frac{dx}{dz}\right)^2 \cos^2 z + 1}$. Since $\frac{dz}{dx} = -\left(\frac{dy_0}{dx} - a\right)$ is bounded and continuous, v is bounded, continuous and $\inf_{z \in [0,h]} v \geq 1$.

Since $\int_0^h \sec z \sqrt{v^2 - 1} dz = 8 \int_0^h \left(\frac{dx}{dz}\right) dz = \frac{\pi}{2}$, it is sufficient to prove the following statement for bounded continuous function v whose infimum is at least 1.

$$\int_0^h \sec z \sqrt{v^2 - 1} dz = \frac{\pi}{2} \Rightarrow \int_0^h v dz \geq \frac{\pi}{2}$$

Then, it is sufficient to prove the following:

$$\int_0^h v dz < \frac{\pi}{2} \Rightarrow \int_0^h \sec z \sqrt{v^2 - 1} dz < \frac{\pi}{2}$$

Since $\int_0^h v dz < \frac{\pi}{2}$ and $v \geq 1$, $h < \frac{\pi}{2}$. Thus there exists $\tau \in \left(\max\left\{h, \int_0^h v dz\right\}, \frac{\pi}{2}\right)$. For all $n \in \mathbb{N}$, let

$$D_n := \left\{(a_1, ..., a_n) | \frac{h}{n}\sum_{k=1}^{n} a_k \leq \tau, a_1, ..., a_n \geq 1\right\}$$

Since D_n is a compact set, there exists a pair $(b_1, ..., b_n) \in D_n$ such that for all $(a_1, ..., a_n) \in D_n$,

$$\sum_{k=1}^{n} \sec\frac{hk}{n}\sqrt{b_k^2 - 1} \geq \sum_{k=1}^{n} \sec\frac{hk}{n}\sqrt{a_k^2 - 1}$$

It can be easily shown that $\frac{h}{n}\sum_{k=1}^{n} b_k = \tau$. If there exist $i, j \in \{1, ..., n\}$ such that

$$\frac{d\sec\frac{hi}{n}\sqrt{b_i^2 - 1}}{db_i} > \frac{d\sec\frac{hj}{n}\sqrt{b_j^2 - 1}}{db_j}$$

then for sufficiently small $\epsilon > 0$, it can be shown that

$$\sum_{k=1}^{n} \sec\frac{hk}{n}\sqrt{b_k^2 - 1} < \sum_{k<n, k\neq i,j} \sec\frac{hk}{n}\sqrt{b_k^2 - 1} + \sec\frac{hi}{n}\sqrt{(b_i + \epsilon)^2 - 1} + \sec\frac{hj}{n}\sqrt{(b_j - \epsilon)^2 - 1}$$

Therefore, the values of $\frac{d\sec\frac{hk}{n}\sqrt{b_k^2-1}}{db_k}$, $1 \leq k \leq n$ should be a constant c_n(possibly infinite). If $c_n = \infty$, then $b_1 = ... = b_n = 1$, $h = \tau$. Thus c_n is a finite constant.

Solving the equation we obtain $b_k = \frac{c_n}{\sqrt{c_n^2 - \sec^2\frac{hk}{n}}}$, where c_n is the solution of $\frac{h}{n}\sum_{k=1}^{n}\frac{c_n}{\sqrt{c_n^2-\sec^2\frac{hk}{n}}} = \tau$.

As n goes to infinity, c_n converges to the solution c of $\int_0^h \frac{c}{\sqrt{c^2-\sec^2 z}}dz = \tau$. Here since $\tau < \frac{\pi}{2}$, $c > \sec h$. Thus

$$\lim_{n\to\infty}\frac{h}{n}\sum_{k=1}^{n}\sec\frac{hk}{n}\sqrt{b_k^2 - 1} = \int_0^h \sec z\sqrt{\left(\frac{c}{\sqrt{c^2 - \sec^2 z}}\right)^2 - 1}dz < \frac{\pi}{2}$$

can be shown.

Meanwhile, since $\int_0^h v dz < \tau$, $\frac{h}{n}\sum_{k=1}^{n} v(\frac{hk}{n}) \leq \tau$ holds for sufficiently big n. Therefore

$$\int_0^h \sec z\sqrt{v^2 - 1}dz = \lim_{n\to\infty}\frac{h}{n}\sum_{k=1}^{n}\sec\frac{hk}{n}\sqrt{v\left(\frac{hk}{n}\right)^2 - 1} \leq \lim_{n\to\infty}\frac{h}{n}\sum_{k=1}^{n}\sec\frac{hk}{n}\sqrt{b_k^2 - 1} < \frac{\pi}{2}$$

□

Theorem 6. *If $S(\Omega) = \frac{\pi}{2}$, there exists an inscribed parallelogram $PQRS$ such that $S(PQRS) \geq 1$ and $\overline{PR}, \overline{QS}$ divide Ω into four parts of the same area.*

Proof. Let $f(x)$ be $\frac{1}{2}\int_0^x r^2(\theta)d\theta$, let g be its inverse, let $\psi(x)$ be the parallelogram whose vertices are the intersections of the lines whose directions are $g(x), g(x + \frac{\pi}{8})$ and the boundary of Ω, and let $s(x)$ be $S(\psi(x))$.

Suppose that $s(x) < 1$ holds for all x. Define functions p, q as $p(x) = g(x) + g(x + \frac{\pi}{8})$, $q(x) = g(x + \frac{\pi}{8}) - g(x)$. Then since

$$s(x) = 4\cdot\frac{1}{2}\sin\left(g(x + \frac{\pi}{8}) - g(x)\right)\cdot\frac{1}{\sqrt{\frac{1}{2}g'(x + \frac{\pi}{8})}}\cdot\frac{1}{\sqrt{\frac{1}{2}g'(x)}} = \frac{8\sin q(x)}{\sqrt{p'(x)^2 - q'(x)^2}},$$

$\sqrt{64\sin^2 q(x) + q'(x)^2} < p'(x)$ always holds. Therefore,

$$\int_0^{\frac{\pi}{2}}\sqrt{64\sin^2 q(x) + q'(x)^2}dx < \int_0^{\frac{\pi}{2}} p'(x)dx = p(\frac{\pi}{2}) - p(0) = 4\pi$$

Since $q(\frac{n\pi}{8}) = \frac{\pi}{2}$ holds by Lemma 3 for all $n \in \mathbb{Z}$, by Lemma 5, the following inequality holds:

$$\int_0^{\frac{\pi}{2}}\sqrt{64\sin^2 q(x) + q'(x)^2}dx = \sum_{n=0}^{3}\int_{\frac{n\pi}{8}}^{\frac{(n+1)\pi}{8}}\sqrt{64\sin^2 q(x) + q'(x)^2}dx \geq 4\pi$$

This is a contradiction, thus there exists x such that $s(x) \geq 1$. It can be easily shown that $\psi(x)$ satisfies the theorem. □

As it is proved above that there exists an inscribed parallelogram ψ such that $S(\psi) \geq 1$ and the two diagonals of ψ divide a given centrally symmetric convex body Ω into four parts of equal areas, we will try to constrict ψ to satisfy $S(\psi) = 1$. However, not all Ω satisfy such property, thus we will call Ω admissible if there exists an inscribed parallelogram ψ such that $S(\psi) = 1$ and the two diagonals of ψ divide Ω into four parts of equal areas. From now, we will focus on the properties of the bodies which are not admissible.

Let $S(XY)$ denote the area of the arc XY for any chord XY of Ω and let X^* denote the reflection of X with respect to O for any point X. By Theorem 4 and Theorem 6, there exists an inscribed rhombus $P_1Q_1P_1{}^*Q_1{}^*$ such that $\overline{P_1Q_1} = 1$ and an inscribed parallelogram $P_2Q_2P_2{}^*Q_2{}^*$ such that $S(P_2Q_2P_2{}^*Q_2{}^*) \geq 1$, $S(P_2Q_2) = S(Q_2P_2{}^*)$.

In the following lemmas, for all t, the intersection of the boundary of Ω and the ray $\theta = t$ is denoted by $X(t)$.

Lemma 7. *If $S(\Omega) = \frac{\pi}{2}$ and Ω is not admissible, for all chord PQ such that $S(PQ) \geq \frac{\pi}{8} - \frac{1}{4}$, $S(P^*Q) < \frac{\pi}{8} - \frac{1}{4}$.*

Proof. Suppose that $S(P^*Q) \geq \frac{\pi}{8} - \frac{1}{4}$. Let T be a point on $\widehat{PQ} \cup \widehat{QP^*}$ such that $S(PT) = S(P^*T)$. Since $S(PT) \geq S(PQ)$ or $S(P^*T) \geq S(P^*Q)$, $S(PT) = S(P^*T) \geq \frac{\pi}{8} - \frac{1}{4}$, thus $S(PTP^*T^*) \leq 1$. For all t, let $Y(t)$ be the point on the boundary of Ω such that $S(X(t)Y(t)) = S(X^*(t)Y(t))$, then let $\psi(t)$ be $X(t)Y(t)X^*(t)Y^*(t)$. Let α, β be the angles such that $\psi(\alpha) = PTP^*T^*, \psi(\beta) = P_2Q_2P_2{}^*Q_2{}^*$. Since $S(\psi(\alpha)) \leq 1 \leq S(\psi(\beta))$, there exists γ such that $S(\psi(\gamma)) = 1$, thus Ω is admissible. □

Lemma 8. *If $S(\Omega) = \frac{\pi}{2}$ and Ω is not admissible, there exists an inscribed parallelogram PQP^*Q^* such that $\overline{PQ} = 1$ and $S(PQP^*Q^*) \geq 1$.*

Proof. Without loss of generality, suppose $\angle P_1Q_1P_1{}^* \geq \frac{\pi}{2}$. Since $\overline{P_1Q_1} = \overline{P_1{}^*Q_1} = 1$, $S(P_1Q_1P_1{}^*Q_1{}^*) \leq 1$, thus without loss of generality we may assume $S(P_1Q_1) \geq \frac{\pi}{8} - \frac{1}{4}$. By Lemma 7, $S(P_1{}^*Q_1) < \frac{\pi}{8} - \frac{1}{4}$. Let α, β be the angles such that $X(\alpha) = P_1$, $X(\beta) = Q_1$. For θ between α and β, since $\angle P_1Q_1P_1{}^* \geq \frac{\pi}{2}$, there exists $Y(\theta) \in \widehat{P_1{}^*Q_1}$ such that $\overline{X(\theta)Y(\theta)} = 1$. Let $\psi(\theta)$ be $X(\theta)Y(\theta)X^*(\theta)Y^*(\theta)$. Since $S(X(\alpha)Y(\alpha)) \geq \frac{\pi}{8} - \frac{1}{4} \geq S(X(\beta)Y(\beta))$, there exists ϕ such that $S(X(\phi)Y(\phi)) = \frac{\pi}{8} - \frac{1}{4}$. By Lemma 7, $S(X(\phi)Y^*(\phi)) \leq \frac{\pi}{8} - \frac{1}{4}$, thus $S(\psi(\phi)) \geq 1$. Therefore, $\psi(\phi)$ satisfies this lemma. □

Lemma 9. *If $S(\Omega) = \frac{\pi}{2}$, there exists an inscribed parallelogram PQP^*Q^* such that $\overline{P^*Q} \geq 1$, $S(PQP^*Q^*) \geq 1$, $\frac{S(PQ)}{S(PQP^*Q^*)} = \frac{\pi}{8} - \frac{1}{4}$.*

Proof. If Ω is admissible, there exists ABA^*B^* such that $S(AB) = S(BA^*) = \frac{\pi}{8} - \frac{1}{4}$. Since $\overline{AB} \cdot \overline{A^*B} \geq S(ABA^*B^*) = 1$, without loss of generality assume that $\overline{A^*B} \geq 1$. Then ABA^*B^* satisfies this lemma. Therefore, we will suppose that Ω is not admissible.

Without loss of generality, suppose $S(P_1Q_1) \geq S(P_1{}^*Q_1)$. Since $S(P_1Q_1P_1{}^*Q_1{}^*) \leq \overline{P_1Q_1} \cdot \overline{P_1{}^*Q_1} = 1$, $S(P_1Q_1) \geq \frac{\pi}{8} - \frac{1}{4}$. By Lemma 7, $S(P_1{}^*Q_1) < \frac{\pi}{8} - \frac{1}{4}$. Let α be the angle such that $X(\alpha) = P_1$. For all t, let $Y(t)$ be the point on the boundary of Ω such that $\overline{X^*(t)Y(t)} \parallel \overline{P_1{}^*Q_1}$ and let $\psi(t)$ be the parallelogram $X(t)Y(t)X^*(t)Y^*(t)$. Since $S(X^*(\alpha)Y(\alpha)) < \frac{\pi}{8} - \frac{1}{4}$, there exists β such that $S(X^*(\beta)Y(\beta)) = \frac{\pi}{8} - \frac{1}{4}$, $\widehat{X^*(\alpha)Y(\alpha)} \subset \widehat{X^*(\beta)Y(\beta)}$. Then by Lemma 7, $S(X(\beta)Y(\beta)) < \frac{\pi}{8} - \frac{1}{4}$, $S(\psi(\beta)) \geq 1$. Since $\frac{S(X(\alpha)Y(\alpha))}{S(\psi(\alpha))} \geq \frac{\pi}{8} - \frac{1}{4} \geq \frac{S(X(\beta)Y(\beta))}{S(\psi(\beta))}$, there exists γ between α, β such that $\frac{S(X(\gamma)Y(\gamma))}{S(\psi(\gamma))} = \frac{\pi}{8} - \frac{1}{4}$. Since $S(X^*(\beta)Y(\beta)) = \frac{\pi}{8} - \frac{1}{4}$ and $\widehat{X^*(\gamma)Y(\gamma)} \subset \widehat{X^*(\beta)Y(\beta)}$, $S(X^*(\gamma)Y(\gamma)) \leq \frac{\pi}{8} - \frac{1}{4}$. Then $2S(X(\gamma)Y(\gamma)) + S(\psi(\gamma)) \geq \frac{\pi}{4} + \frac{1}{2}$, $S(\psi(\gamma)) \geq 1$. Since $\widehat{X^*(\alpha)Y(\alpha)} \subset \widehat{X^*(\gamma)Y(\gamma)}$ and $\overline{X^*(\alpha)Y(\alpha)} \parallel \overline{X^*(\gamma)Y(\gamma)}$, $\overline{X^*(\gamma)Y(\gamma)} \geq \overline{X^*(\alpha)Y(\alpha)} = 1$. Thus $\psi(\gamma)$ satisfies all conditions of this lemma. □

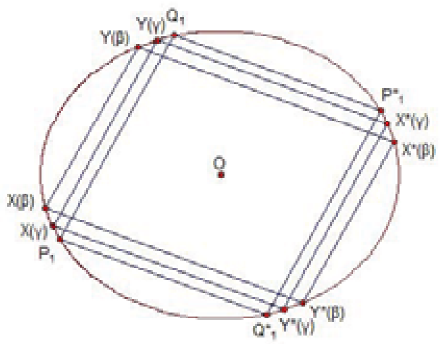
2.2 Upper Bounds on the Area of Non-coverable Set

In this section, we will suggest a function f such that for any given lattice Λ, any centrally symmetric convex body Ω is a coverable body with respect to Λ if $S(\Omega) \geq f(\Lambda)$. Also, for more efficient covering, we will suggest a certain lattice Λ^* such that $\det \Lambda^* = 1$ and any centrally symmetric convex body Ω is a coverable body with respect to Λ^* if $S(\Omega) \geq \frac{\pi}{2}$.

The followings are definitions related to the lattice, which are required to construct the function f.

Definition 10. *An elementary segment is a segment connecting two lattice points X, Y such that no lattice point exists on $\overline{XY} \setminus \{X, Y\}$. An elementary triangle is a triangle whose vertices are lattice points X, Y, Z such that no lattice point exists on $XYZ \setminus \{X, Y, Z\}$.*

For any lattice Λ, define elementary segments $d_1, d_2, \ldots$ as follows:



For all $i \in \mathbb{N}$, d_i is a shortest segment among all the elementary segments which are not parallel with $d_1, ..., d_{i-1}$.

Definition 11. *For any lattice Λ, $D(\Lambda)$ is the set of the lengths of $d_2, d_3, d_4, d_5 ....$*

For any set S of positive real numbers, if $S = \{s_1, s_2, ...\}$ and $s_1 < s_2 < ...$, $\mu(S) := \sup \frac{s_{i+1}}{s_i}$.

The length of d_1 is excluded from $D(\Lambda)$ to make $\mu(D(\Lambda))$ be bounded. The next theorem shows an upper bound of $\mu(D(\Lambda))$.

Theorem 12. *For all lattice Λ, $\mu(D(\Lambda)) \leq \sqrt{3}$.*

Proof. Let X, Y be the points such that $\overline{OX} = d_1, \overline{OY} = d_2, \overline{OX} \parallel d_1, \overline{OY} \parallel d_2$, $0 < \angle XOY \leq \frac{\pi}{2}$. For all k, denote Y_k as $Y + kX$. Since $\overline{OY} \leq \overline{OY_{-1}} \leq \overline{OY_1} \leq \overline{OY_{-2}} \leq \overline{OY_2} \leq ...$ and all these segments are in D, it is sufficient to show $\frac{\overline{OY_k}}{\overline{OY_{-k}}}, \frac{\overline{OY_{-k}}}{\overline{OY_{k-1}}} \leq \sqrt{3}$ for every $k \in \mathbb{N}$. Let Y' be the midpoint of $\overline{YY_{-1}}$. Let W, W' be the points such that $WW'Y'Y$ is a rectangle and $W \in \overline{OX}$. Since $\overline{W'Y'}^2 = \overline{WY}^2 = \overline{OY}^2 \sin^2 \angle XOY \geq \frac{3}{4}\overline{OX}^2$, the followings can be shown:

$$\frac{\overline{OY_k}}{\overline{OY_{-k}}} \leq \frac{\overline{W'Y_k}}{\overline{W'Y_{-k}}} = \frac{\sqrt{(k+\frac{1}{2})^2\overline{OX}^2 + \overline{W'Y'}^2}}{\sqrt{(k-\frac{1}{2})^2\overline{OX}^2 + \overline{W'Y'}^2}} \leq \frac{\sqrt{(k+\frac{1}{2})^2 + \frac{3}{4}}}{\sqrt{(k-\frac{1}{2})^2 + \frac{3}{4}}} \leq \sqrt{3}$$

$$\frac{\overline{OY_{-k}}}{\overline{OY_{k-1}}} \leq \frac{\overline{WY_{-k}}}{\overline{WY_{k-1}}} = \frac{\sqrt{k^2\overline{OX}^2 + \overline{WY}^2}}{\sqrt{(k-1)^2\overline{OX}^2 + \overline{WY}^2}} \leq \frac{\sqrt{k^2 + \frac{3}{4}}}{\sqrt{(k-1)^2 + \frac{3}{4}}} \leq \sqrt{3}$$

□

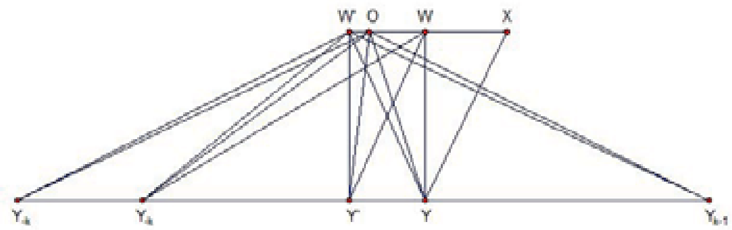


Figure 2. Proof of Theorem 12

Lemma 13. *Let O be a point and let l be a line such that $O \notin l$. Let H be the foot of the perpendicular from O to l. Let $A, B, C, D \in l$ be the points in order A, B, H, C, D, such that $\overline{AB} = \overline{CD}$, $\overline{AH} \leq \overline{DH}$. If $\angle BOD \geq \frac{\pi}{2}$, $\frac{\overline{OD}}{\overline{OA}} \leq \frac{\overline{OC}}{\overline{OB}}$.*

Proof. Let a, b, c, d, h be $\overline{AH}, \overline{BH}, \overline{CH}, \overline{DH}, \overline{OH}$, respectively. Since $\angle BOD \geq \frac{\pi}{2}$, $h^2 \leq bd$. Also, $a - b = d - c$ and $b \leq c \leq a \leq d$ hold by the given conditions. Thus $(h^2 + b^2)(h^2 + d^2) \leq (h^2 + a^2)(h^2 + c^2)$ can be shown, and this is equivalent to $\frac{\overline{OD}}{\overline{OA}} \leq \frac{\overline{OC}}{\overline{OB}}$. □

The following theorem shows how to find $\mu(D(\Lambda))$ in finite steps.

Theorem 14. *Let center O be a lattice point and let $\overline{OX}, \overline{OY}$ be the shortest two elementary segments such that $\overline{OX} \le \overline{OY}$ and $\angle XOY \le \frac{\pi}{2}$. Let $D'(\Lambda) = D(\Lambda) \cap \{\overline{OP} | \overline{OP} < 12d_2\}$. Then $\mu(D(\Lambda)) = \mu(D'(\Lambda))$.*

Proof. For $k \in \mathbb{N}$, let Y_{2k-1} be $Y - kX$ and Y_{2k} be $Y + kX$. Let Z be $Y + Y_1$. Let n be the integer such that $\overline{OY_n} \le \overline{OZ} < \overline{OY_{n+1}}$. Suppose there exists $k \ge \max\{4, n\}$ such that $\angle Y_{k-2}OY_{k+1} < \frac{\pi}{2}$. Since $\overline{OY_{k+1}} > \overline{OZ} \ge 2\overline{OH}$, $\angle OY_{k+1}H < \frac{\pi}{6}$. Then $\angle OY_{k-2}H > \frac{\pi}{3}$, thus $\overline{Y_{k+1}H} > 3\overline{Y_{k-2}H}$. This contradicts $\overline{Y_{k+1}H} \le \frac{k+2}{2}\overline{YY_1}$ and $\overline{Y_{k-2}H} \ge \frac{k-2}{2}\overline{YY_1}$. Thus $\angle Y_{k-2}OY_{k+1} \ge \frac{\pi}{2}$, and by Lemma 13, $\frac{\overline{OY_{k+1}}}{\overline{OY_k}} \le \frac{\overline{OY_{k-1}}}{\overline{OY_{k-2}}}$ holds for all $k \ge \max\{4, n\}$.

Meanwhile, it can be shown that d_1, d_2, d_3, d_4, d_5 are $\overline{OX}, \overline{OY}, \overline{OY_1}, \overline{OY_2}, \overline{OY_3}$, respectively. Thus we only need to consider the following cases.

(i) $n \ge 4$: Since $\overline{OY_1}, ..., \overline{OY_n}$ are the smallest elements of $D(\Lambda)$ and $\frac{\overline{OY_n}}{\overline{OY_{n-1}}} \ge \frac{\overline{OY_{n+2}}}{\overline{OY_{n+1}}} \ge ...$ and $\frac{\overline{OY_{n-1}}}{\overline{OY_{n-2}}} \ge \frac{\overline{OY_{n+1}}}{\overline{OY_n}} \ge ...$ hold, $\mu(D(\Lambda)) = \mu(\{\overline{OY_1}, ..., \overline{OY_n}\})$. Since $\overline{OY_n} \le \overline{OZ} \le 2d_2 + d_1 < 12d_2$, $\mu(D'(\Lambda)) = \mu(D(\Lambda))$.

(ii) $n = 3$: $\frac{\overline{OY_5}}{\overline{OY_4}} \le \frac{\overline{OY_3}}{\overline{OY_2}} \le \mu(D(\Lambda))$. Also, it can be easily shown that $d_5 = \overline{OY_3}$, $d_6 = \overline{OZ}$, $d_7 = \overline{OY_4}$. Thus $\frac{\overline{OY_5}}{\overline{OY_3}} = \frac{\overline{OY_5}}{\overline{OY_4}} \frac{\overline{OY_4}}{\overline{OZ}} \frac{\overline{OZ}}{\overline{OY_3}} \le \mu(D(\Lambda))^3$. Since $\overline{HY_5}^2 = \overline{OY_5}^2 - \overline{OH}^2 > \overline{OZ}^2 - \overline{OH}^2 \ge 3\overline{OH}^2$ and $\overline{HY_5} \ge \frac{3}{2}\overline{HY_3}$,

$$\mu(D(\Lambda)) \ge \sqrt[3]{\frac{\overline{OY_5}}{\overline{OY_3}}} = \sqrt[6]{\frac{\overline{HY_5}^2 + \overline{OH}^2}{\overline{HY_3}^2 + \overline{OH}^2}} \ge \sqrt[6]{\frac{\overline{HY_5}^2 + \frac{1}{3}\overline{HY_5}^2}{\overline{HY_3}^2 + \frac{1}{3}\overline{HY_5}^2}} \ge \sqrt[6]{\frac{12}{7}} > \frac{12}{11}$$

Let S be $\{\overline{OY_{2k}} | k \ge 11\}$. Then since $S \subset D(\Lambda)$, $S \cap D'(\Lambda) \ne \emptyset$ and $\mu(S) \le \frac{12}{11} < \mu(D(\Lambda))$, $\mu(D'(\Lambda)) = \mu(D(\Lambda))$.

□

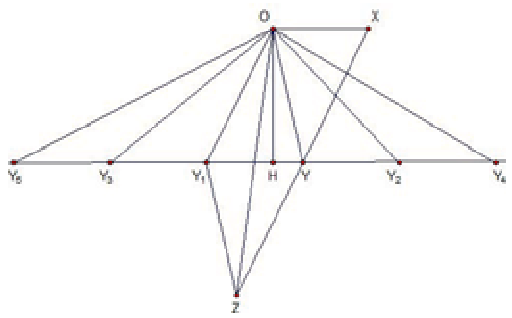


Figure 3. Arrangement of $O, X, Y_1, Y_2, Y_3...$

Example 15. *Let Λ_3 be $\{m[1, 0] + n[\frac{1}{2}, \frac{\sqrt{3}}{2}] | m, n \in \mathbb{Z}\}$ and let Λ_4 be $\mathbb{Z}^2$. Then $\mu(D(\Lambda_3)) = \sqrt{3}$, $\mu(D(\Lambda_4)) = \frac{\sqrt{5}}{\sqrt{2}}$ can be shown using Theorem 14.*

The next lemma shows two inequalities related to the chords of Ω. For any two sets $X, Y \subset \mathbb{R}^2$ we will denote $d(X, Y)$ as the distance between X, Y.

Lemma 16. *Suppose $S(\Omega) = \frac{\pi}{2}$. Let $PQRS$ be an inscribed parallelogram such that $S(PQRS) \ge 1$, $\frac{S(PQ)}{S(PQRS)} = \frac{\pi}{8} - \frac{1}{4}$. Given $\alpha \in [1, \frac{\pi}{4} + \frac{1}{2}]$ and $\beta \in [\frac{1}{2}, 1]$, let $\overline{U_1V_1}$ be a chord between $\overleftrightarrow{PQ}$ and $\overleftrightarrow{MN}$ such that $\overline{U_1V_1} \parallel \overline{PQ}$, $\overline{U_1V_1} = \alpha\overline{PQ}$ and let $\overline{XY}$ be a chord such that $\overline{XY} \parallel \overline{PQ}$, $\overline{XY} = \beta\overline{PQ}$, which is nearer to $\overleftrightarrow{RS}$ than $\overleftrightarrow{PQ}$. Then the followings hold:*

$$d(\overleftrightarrow{U_1V_1}, \overleftrightarrow{PQ}) \le \frac{\alpha - 1}{\pi - 2} S(PQRS) \cdot \frac{1}{\overline{PQ}}, \ d(\overleftrightarrow{XY}, \overleftrightarrow{RS}) \ge (1 - \beta)\left(\frac{\pi}{2} - \left(\frac{\pi}{4} + \frac{1}{2}\right) S(PQRS)\right) \cdot \frac{1}{\overline{PQ}}$$

Proof. Let l_0 be the line such that $l_0 \parallel \overline{PQ}$ and $O \in l_0$. Let $\overline{UV}$ be a chord between $\overleftrightarrow{PQ}$ and l_0 such that $\overline{UV} \parallel \overline{PQ}$, $d(\overleftrightarrow{UV}, \overleftrightarrow{PQ}) = \frac{\alpha-1}{\pi-2} d(\overleftrightarrow{PQ}, \overleftrightarrow{RS})$. Let L, M, N be $\overleftrightarrow{RX} \cap \overleftrightarrow{SY}$, $l_0 \cap \overleftrightarrow{PU}$, $l_0 \cap \overleftrightarrow{QV}$, respectively.

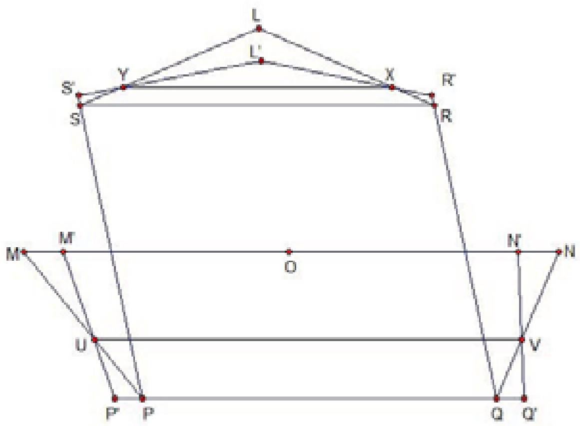


Figure 4. Proof of Lemma 16

Let u, v, x, y be the tangent lines of Ω at U, V, X, Y, respectively. Let $M', N', P', Q', R', S', L'$ be $u \cap \overleftrightarrow{MN}, v \cap \overleftrightarrow{MN}, u \cap \overleftrightarrow{PQ}, v \cap \overleftrightarrow{PQ}, x \cap \overleftrightarrow{QR}, y \cap \overleftrightarrow{PS}, x \cap y$, respectively. Since $d(\overleftrightarrow{PQ}, \overleftrightarrow{UV}) \leq d(\overleftrightarrow{UV}, \overleftrightarrow{MN})$,

$$\frac{1}{2}(\overline{MN}+\overline{PQ})d(\overleftrightarrow{PQ}, \overleftrightarrow{MN}) = S(MNQP) \geq S(M'N'Q'P') \geq \frac{S(\Omega)}{2} - S(PQ) = \left(\frac{\pi}{8}+\frac{1}{4}\right)S(PQRS) = \left(\frac{\pi}{4}+\frac{1}{2}\right)\overline{PQ}\cdot d(\overleftrightarrow{PQ}, \overleftrightarrow{MN})$$

Thus $\overline{MN} \geq \frac{\pi}{2}\overline{PQ}$. Then since

$$\overline{UV} = \frac{2(\alpha-1)}{\pi-2}\overline{MN} + \left(1 - \frac{2(\alpha-1)}{\pi-2}\right)\overline{PQ} \geq \alpha\overline{PQ} = \overline{U_1V_1},$$

$d(\overleftrightarrow{U_1V_1}, \overleftrightarrow{PQ}) \leq d(\overleftrightarrow{UV}, \overleftrightarrow{PQ}) = \frac{\alpha-1}{\pi-2}d(\overleftrightarrow{PQ}, \overleftrightarrow{RS}) = \frac{\alpha-1}{\pi-2}S(PQRS)\cdot\frac{1}{\overline{PQ}}$.

Meanwhile, since $2\overline{XY} \geq \overline{RS}$,

$$\overline{RS}\cdot d(L, \overleftrightarrow{RS}) = 2S(SLR) \geq 2S(SS'L'R'R) \geq 2S(RS) = \frac{\pi}{2} - S(PQRS)\left(\frac{\pi}{4}+\frac{1}{2}\right),$$

$$d(\overleftrightarrow{RS}, \overleftrightarrow{XY}) = (1-\beta)d(L, \overleftrightarrow{RS}) \geq (1-\beta)\left(\frac{\pi}{2} - S(PQRS)\left(\frac{\pi}{4}+\frac{1}{2}\right)\right)\cdot\frac{1}{\overline{RS}}$$

□

Definition 17. *Given an elementary segment $\overline{XY}$ of a lattice Λ, let l be a line such that $l \parallel \overline{XY}$ and $d(l, \overline{XY}) = \frac{1}{\overline{XY}}\det\Lambda$. Let T be the union of $l \cap \Lambda$ and its reflection with respect to the orthogonal bisector of $\overline{XY}$. Let k be the maximum distance between two adjacent points in T. Then the lattice rate of $\overline{XY}$ is $\frac{k}{\overline{XY}}$.*

Remark 18. *Let Z be a point on $l \cap \Lambda$ such that $\angle ZXY, \angle ZYX \leq \frac{\pi}{2}$ and let H be the point on $\overline{XY}$ such that $\overline{ZH} \perp \overline{XY}$. Let H' be the reflection of H with respect to the midpoint of $\overline{XY}$. Then since the projection of T onto $\overleftrightarrow{XY}$ is $\{H + i(Y-X) | i \in \mathbb{Z}\} \cup \{H' + i(Y-X) | i \in \mathbb{Z}\}$, the lattice rate of $\overline{XY}$ is*

$$\frac{\max\{\overline{HH'}, \overline{XY} - \overline{HH'}\}}{\overline{XY}}$$

Theorem 19. *For any lattice Λ, Ω is a coverable body if $S(\Omega)$ is not less than $f(\Lambda) = \frac{\pi}{2}\max\left\{\left(\frac{\det\Lambda}{d_1}\right)^2, {d_1}^2, \left(\frac{d_2}{\tau\mu}\right)^2, \frac{\det\Lambda}{\tau^2\mu}\right\}$, where $\mu := \mu(D(\Lambda))$, $\tau := \frac{\pi}{\pi-2+2\mu}$.*

Proof. Consider a scaling which transforms the area of Ω to $\frac{\pi}{2}$. It is sufficient to prove that Ω is a coverable body with respect to Λ if $S(\Omega) = \frac{\pi}{2}$ and $\max\left\{\left(\frac{\det\Lambda}{d_1}\right)^2, {d_1}^2, \left(\frac{d_2}{\tau\mu}\right)^2, \frac{\det\Lambda}{\tau^2\mu}\right\} \leq 1$.

Suppose that Ω is not admissible. Then by Lemma 8, there exists a parallelogram $P_0Q_0R_0S_0 \subset \Omega$ such that $\overline{P_0Q_0} = 1$ and $S(P_0Q_0R_0S_0) \geq 1$. Since $d_1 \leq 1$ and $d_1 \geq \det\Lambda$, it can be shown that there exists a parallelogram $WXYZ \subset P_0Q_0R_0S_0$

such that $S(WXYZ) = \det\Lambda$ and $\overline{WX} = d_1$. Since the lattice rate of d_1 is at most 1 and $d(\overleftrightarrow{WX}, \overleftrightarrow{YZ}) = \frac{1}{d_1}\det\Lambda$, there exists a point $T \in \overline{YZ}$ such that WXT is congruent to a lattice triangle. Then by Corollary 2, Ω is a coverable body, thus we will now suppose Ω is admissible.

Since Ω is admissible, there exists an inscribed parallelogram $PQRS$ such that $S(PQRS) = 1$ and $S(PQ) = S(QR)$. Without loss of generality, suppose $\overline{PQ} \geq \overline{QR}$. Since $\overline{PQ} \cdot \overline{QR} \geq S(PQRS) = 1$, $\overline{PQ} \geq 1$. Since $\frac{d_2}{\tau\mu} \leq 1 \leq \overline{PQ}$, there exists $u \in D(\Lambda)$ such that $\alpha := \frac{u}{\overline{PQ}} \in [\tau, \tau\mu]$. We will consider two cases : when $\alpha \geq 1$ and when $\alpha < 1$.

(i) When $\alpha \geq 1$: Since $1 < \mu < \sqrt{3}$, $\alpha \leq \tau\mu < \frac{1}{2} + \frac{\pi}{4}$. Thus by Lemma 16, there exists a chord X_1Y_1 such that $\overline{X_1Y_1} \parallel \overline{PQ}$, $\overline{X_1Y_1} = u$, $d(\overleftrightarrow{X_1Y_1}, \overleftrightarrow{PQ}) \leq \frac{\alpha-1}{\pi-2} \cdot \frac{1}{\overline{PQ}}$. Then $S(X_1Y_1{X_1}^*{Y_1}^*) = \overline{X_1Y_1} d(\overleftrightarrow{X_1Y_1}, \overleftrightarrow{{X_1}^*{Y_1}^*}) = \alpha\overline{PQ}(d(\overleftrightarrow{PQ}, \overleftrightarrow{RS}) - 2d(\overleftrightarrow{X_1Y_1}, \overleftrightarrow{PQ})) \geq \alpha\left(1 - 2\cdot\frac{\alpha-1}{\pi-2}\right) = \frac{\alpha(\pi-2\alpha)}{\pi-2} \geq \frac{\tau\mu(\pi-2\tau\mu)}{\pi-2} = \tau^2\mu \geq \det\Lambda$.

(ii) When $\alpha < 1$: Since $1 < \mu < \sqrt{3}$, $\frac{1}{2} < \tau \leq \alpha$. Thus by Lemma 16, there exists a chord X_2Y_2 such that $\overline{X_2Y_2} \parallel \overline{PQ}$, $\overline{X_2Y_2} = u$, $d(\overleftrightarrow{X_2Y_2}, \overleftrightarrow{RS}) \geq (1-\alpha)\left(\frac{\pi}{4} - \frac{1}{2}\right) \cdot \frac{1}{\overline{PQ}}$. Then $S(X_2Y_2{X_2}^*{Y_2}^*) = \overline{X_2Y_2} d(\overleftrightarrow{X_2Y_2}, \overleftrightarrow{{X_2}^*{Y_2}^*}) = \alpha\overline{PQ}(d(\overleftrightarrow{PQ}, \overleftrightarrow{RS}) + 2d(\overleftrightarrow{X_2Y_2}, \overleftrightarrow{RS})) = \alpha\left(1 + (1-\alpha)\left(\frac{\pi}{2} - 1\right)\right) \geq \tau\left(1 + (1-\tau)\left(\frac{\pi}{2} - 1\right)\right) = \tau^2\mu \geq \det\Lambda$.

Therefore, there exists a parallelogram $XYX'Y' \subset \Omega$ such that $\overline{XY} = u$, $S(XYX'Y') = \det\Lambda$. Since $\overline{XY} \in D(\Lambda)$, $d(\overleftrightarrow{XY}, \overleftrightarrow{X'Y'}) = \frac{1}{\overline{XY}}\det\Lambda$ and the lattice rate of $\overline{XY}$ is at most 1, there exists a point $W \in \overline{X'Y'}$ such that WXY is congruent to a lattice triangle. Therefore, by Corollary 2, Ω is a coverable body. □

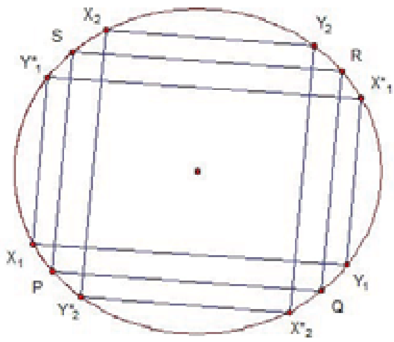


Figure 5. Proof of Theorem 19

The following example shows how we apply this theorem and the theorem's accuracy.

Example 20. *If* $S(\Omega) \geq \frac{(\pi-2+2\sqrt{3})^2}{4\pi} \simeq 1.69$*, by Theorem 19 and Example 15,* Ω *is a coverable body with respect to* Λ_3*. Similarly, if* $S(\Omega) \geq \frac{(\pi-2+\sqrt{10})^2}{\sqrt{10}\pi} \simeq 1.86$*, by Theorem 19 and Example 15,* Ω *is a coverable body with respect to* Λ_4*.*

Let Ω_3 *be* $\left\{(x,y)|x^2+y^2 < \frac{3}{4}, y^2 < \frac{3}{16}\right\}$ *and let* Ω_4 *be* $\left\{(x,y)|x^2+y^2 < \frac{1}{2}\right\}$*. Then it can be shown that no lattice triangle can be inscribed in each of these, thus* Ω_3*,* Ω_4 *are not coverable bodies. Since* $S(\Omega_3) = \frac{\pi}{4} + \frac{3\sqrt{3}}{8} > 1.43$ *and* $S(\Omega_4) = \frac{\pi}{2} > 1.57$*,* $S(\Omega)$ *should be at least* 1.43*,* 1.57 *to certify that* Ω *is always a coverable body with respect to* Λ_3*,* Λ_4*, respectively, while the constants we obtained from Theorem 19 were* 1.69 *and* 1.86*.*

To find out an efficient covering, we may apply Theorem 19 to an appropriate lattice. However, there exists a certain lattice which enables us get a more efficient covering. The followings are the processes of suggesting such lattice, denoted by Λ^*, and showing that Ω whose area is $\frac{\pi}{2}$ is always a coverable body with respect to Λ^*.

Definition 21. Λ^* *is a lattice such that* $\det\Lambda^* = 1$*,* $d_2 = \sqrt{2}\, d_1$ *and* $\|\boldsymbol{d}_1 + \boldsymbol{d}_2\| = \sqrt[4]{2}\, \|\boldsymbol{d}_1 - \boldsymbol{d}_2\|$*, where* $\boldsymbol{d}_1, \boldsymbol{d}_2$ *are the vectors satisfying* $\boldsymbol{d}_1 \parallel d_1$*,* $\|\boldsymbol{d}_1\| = d_1$*,* $\boldsymbol{d}_2 \parallel d_2$*,* $\|\boldsymbol{d}_2\| = d_2$*,* $\boldsymbol{d}_1 \cdot \boldsymbol{d}_2 > 0$*.*

Theorem 22. *A centrally symmetric convex body* Ω *is a coverable body with respect to* Λ^* *if* $S(\Omega) = \frac{\pi}{2}$*.*

Proof. Let Φ be $A \cup B$, where A, B are the following sets :

$$A := \{p\mathbf{d}_1 + q\mathbf{d}_2 | 0 \leq p \leq 6, q = \pm 1, p, q \in \mathbb{Z}\} \cup \{4\mathbf{d}_2 + (4p \pm 1)\mathbf{d}_1 | 1 \leq p \leq 3, p \in \mathbb{Z}\},$$

$$B := \{4\mathbf{d}_2 + (4p \pm 1)\mathbf{d}_1 | p \geq 3, p \in \mathbb{Z}\}$$

For all $t \geq 6$, since $\|4\mathbf{d}_2 + (2t+1)\mathbf{d}_1\| - \|4\mathbf{d}_2 + (2t-1)\mathbf{d}_1\| \leq 2\|\mathbf{d}_1\| < \frac{1}{5}\|4\mathbf{d}_2 + 11\mathbf{d}_1\|$, $\|4\mathbf{d}_2 + (2t+1)\mathbf{d}_1\| < \frac{6}{5}\|4\mathbf{d}_2 + (2t-1)\mathbf{d}_1\|$. Thus $\mu(B) < \frac{6}{5}$. Also, $\mu(A) < \frac{6}{5}$ can be shown by checking all elements. Therefore, $\mu(\Phi) < \frac{6}{5}$.

For any $p \geq 3$, let X, Y, Z be the lattice points such that $\overleftrightarrow{XY} = 4\mathbf{d}_2 + (4p \pm 1)\mathbf{d}_1$, $\overleftrightarrow{XZ} = \mathbf{d}_2 + p\mathbf{d}_1$ and let H be the point on $\overline{XY}$ such that $\overline{ZH} \perp \overline{XY}$. Let H' be the reflection of H with respect to the midpoint of $\overline{XY}$. Since $S(XYZ) = \frac{1}{2}$, $d(Z, \overleftrightarrow{XY}) = \frac{1}{\overline{XY}}$. Since

$$\left|\frac{1}{4} - \frac{\overline{XH}}{\overline{XY}}\right| = \left|\frac{1}{4} - \frac{(4\mathbf{d}_2 + (4p \pm 1)\mathbf{d}_1) \cdot (\mathbf{d}_2 + p\mathbf{d}_1)}{\|4\mathbf{d}_2 + (4p \pm 1)\mathbf{d}_1\|^2}\right| = \frac{(4\mathbf{d}_2 + (4p \pm 1)\mathbf{d}_1) \cdot \mathbf{d}_1}{4\,\|4\mathbf{d}_2 + (4p \pm 1)\mathbf{d}_1\|^2} \leq \frac{\|\mathbf{d}_1\|}{4\,\|4\mathbf{d}_2 + (4p \pm 1)\mathbf{d}_1\|} \leq \frac{1}{16},$$

$\max\{\overline{HH'}, \overline{XY} - \overline{HH'}\} \leq \frac{3}{4}\overline{XY}$. Also, since $\overleftrightarrow{XZ} \cdot \overleftrightarrow{ZY} \geq (\mathbf{d}_2 + p\mathbf{d}_1) \cdot (3\mathbf{d}_2 + (3p \pm 1)\mathbf{d}_1) \geq 0$, $\angle XYZ \geq \frac{\pi}{2}$, $\angle ZXY, \angle ZYX \leq \frac{\pi}{2}$. Thus the lattice rate of $\overline{XY}$ is at most $\frac{3}{4}$. Also, it can be shown that the lattice rate of any element of A is at most $\frac{3}{4}$ by checking all elements. Therefore, every element of Φ has lattice rate not bigger than $\frac{3}{4}$.

By Lemma 9, there exists an inscribed parallelogram $PQRS$ such that $S(PQRS) \geq 1$, $\frac{S(PS)}{S(PQRS)} = \frac{\pi}{8} - \frac{1}{4}$ and $\overline{PQ} \geq 1$. Let s be $S(PQRS)$. Since $\frac{5}{6}d_2 < 1 \leq \overline{PQ}$ and $d_2 \in \Phi$ and $\mu(\Phi) < \frac{6}{5}$, there exists $d_i \in \Phi$ such that $\overline{PQ} \leq d_i < \frac{6}{5}\overline{PQ}$.

Let XY be a chord between $\overleftrightarrow{PQ}$ and O such that $\overline{XY} \parallel \overline{PQ}$ and $\overline{XY} = d_i$. Let $X'Y'$ be a chord such that $\overline{X'Y'} \parallel \overline{PQ}$ and $\overline{X'Y'} = \frac{3}{4}d_i$. Let t be $\frac{d_i}{\overline{PQ}}$. Then by Lemma 16,

$$d(\overleftrightarrow{XY}, \overleftrightarrow{X'Y'}) = d(\overleftrightarrow{X'Y'}, \overleftrightarrow{RS}) + d(\overleftrightarrow{PQ}, \overleftrightarrow{RS}) - d(\overleftrightarrow{XY}, \overleftrightarrow{PQ}) \geq \left(1 - \frac{3}{4}t\right)\left(\frac{\pi}{2} - \left(\frac{\pi}{4} + \frac{1}{2}\right)s\right)\frac{1}{\overline{PQ}} + \frac{s}{\overline{PQ}} - \frac{t-1}{\pi - 2} \cdot \frac{s}{\overline{PQ}}$$

Thus, $d(\overleftrightarrow{XY}, \overleftrightarrow{X'Y'})d_i \geq t\left((1 - \frac{3}{4}t)\left(\frac{\pi}{2} - \left(\frac{\pi}{4} + \frac{1}{2}\right)s\right) + s - \frac{t-1}{\pi-2}s\right)$ and this is always bigger than 1, since $t \in [1, \frac{6}{5})$ and $s \geq 1$.

Since Ω is convex, there exists $\overline{X_1Y_1} \subset \Omega$ such that $d(\overleftrightarrow{X_1Y_1}, \overleftrightarrow{XY})d_i = 1$, $\overline{X_1Y_1} \parallel \overline{XY}$ and $\overline{X_1Y_1} = \overline{X'Y'} = \frac{3}{4}d_i$. Since the lattice rate of d_i is at most $\frac{3}{4}$ and $\overline{X_1Y_1} = \frac{3}{4}\overline{XY}$, it can be shown that there exists a point $Z \in \overline{X_1Y_1}$ such that XYZ is congruent to a lattice triangle. Since $XYZ \subset \Omega$, by Corollary 2, Ω is a coverable body. □

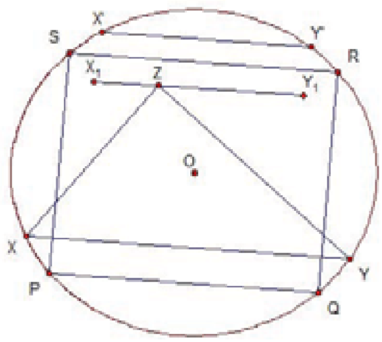


Figure 6. Proof of Theorem 22

2.3 Application

An interesting property of the coverable body is that we can suggest a reasonable upper bound on the infimum of the density of lattice covering with the minkowski sum of a coverable body and an uniformly coverable set with respect to the same lattice. Here, the uniformly coverable set is a new definition, which indicates any bounded closed set $A \subset \mathbb{R}^2$ such that for all $\Lambda' \equiv \Lambda$, $A + \Lambda' = \mathbb{R}^2$.

Theorem 23. *Let A be a coverable body and let B be an uniformly coverable set with respect to the same given lattice Λ. Then there exists a lattice covering of $A + B$ whose density is $\frac{S(A+B)}{3\det\Lambda}$.*

Proof. Since A is a coverable body with respect to Λ, there exists $\Lambda' \equiv \Lambda$ such that $A + \Lambda' = \mathbb{R}^2$. By Lemma 1, there exists $\Lambda_1 \equiv \Lambda$ such that A includes a lattice triangle of Λ_1. Since A is convex, there also exists an elementary triangle $LMN \subset A$. Let T be the lattice $\{pL + qM + rN | p + q + r = 0, p \equiv q \equiv r \mod 3\}$. Then since $\Lambda_1 = \{pL + qM + rN | p + q + r = 1\}$, $\Lambda_1 = T + \{L, M, N\}$, thus $\Lambda_1 \subset T + A$. Therefore, $\mathbb{R}^2 = B + \Lambda_1 \subset B + T + A = (A + B) + T$, $\{A + B + t | t \in T\}$ is a covering whose density is $\frac{S(A+B)}{\det T} = \frac{S(A+B)}{3\det\Lambda_1}$. □

This theorem is beneficial to general sets, since the uniformly coverable set needs not be connected and may have holes. The following is an example of this.

Example 24. *Let A be* $\Gamma \setminus \Gamma'$*, where* $\Gamma := \left\{P|\overline{OP} \le \frac{2}{\sqrt{3}}\right\}$, $\Gamma' := \left\{P|\overline{OP} < \frac{\sqrt{3}}{2}\right\}$. *We will show that A is an uniformly coverable body with respect to* Λ_3*. Let X be any point on the plane. For* $i, j \in \{0, 1\}$*, let* $\Lambda(i, j)$ *be the lattice* $\{(2m + i)[1, 0] + (2n + j)[\frac{1}{2}, \frac{\sqrt{3}}{2}] \,|\, m, n \in \mathbb{Z}\}$*. Since a right triangle congruent to a lattice triangle of* $\Lambda(i, j)$ *can be inscribed in* Γ*, by Corollary 2, there exists* $\lambda \in \Lambda(i, j)$ *such that* $X \in \Gamma + \lambda$*. Meanwhile, since the diameter of* Γ' *is* $\sqrt{3}$*, it can be shown that there are at most three elements of* $\{\lambda | X \in \Gamma' + \lambda, \lambda \in \Lambda_3\}$*. Therefore, there exists a lattice point* λ *such that* $X \in (\Gamma \setminus \Gamma') + \lambda = A + \lambda$*. Thus* $A + \Lambda_3 = \mathbb{R}^2$*. Since A is the region between two concentric circles,* $A + \Lambda' = \mathbb{R}^2$ *holds for all* $\Lambda' \equiv \Lambda_3$*, thus A is an uniformly coverable set with respect to* Λ_3*.*

Let B be any centrally symmetric convex body whose area is bigger than $\frac{(\pi-2+2\sqrt{3})^2}{4\pi}$*. B is a coverable body with respect to* Λ_3*, as it was shown in Example 20. Thus by Theorem 23, there exists a lattice covering of* $A + B$ *whose density is* $\frac{S(A+B)}{3\det\Lambda_3} = \frac{2}{3\sqrt{3}} S(A + B)$*.*

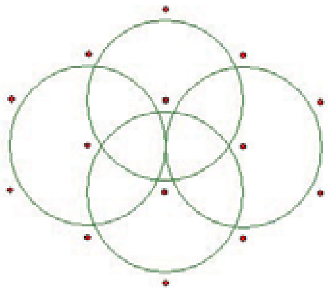


Figure 7. Covering by Γ'

3. Conclusion

In this paper, we suggested a function f such that any centrally symmetric convex body Ω is a coverable body with respect to a lattice Λ if $S(\Omega) \ge f(\Lambda)$. Also, we discovered a lattice Λ^* such that any centrally symmetric convex body Ω is a coverable body with respect to Λ^* if $S(\Omega) \ge \frac{\pi}{2}$. To apply the coverable body to more general problems, we also suggested a method to prove the existence of an efficient lattice covering using a coverable body.

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On the Automorphisms of the Four-dimensional Real Division Algebras

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Abstract

In this paper, we study partially the automorphisms groups of four-dimensional division algebra. We have proved that there is an equivalence between $Der(A) = su(2)$ and $Aut(A) = SO(3)$. For an unitary four-dimensional real division algebra, there is an equivalence between $\dim(Der(A)) = 1$ and $Aut(A) = SO(2)$.

Keywords: division algebra, derivations, automorphisms, mutation, isotope.

1. Introduction

The finited-dimensional real division algebra A, an actuel problem, takes its origin with the quaternion's discovery $\mathbb{H}$, by Hamilton in 1843. One of the fundementals results of a n-dimensional real division algebra affirms that $n \in \{1, 2, 4, 8\}$(Bott & Milnor, 1958; Kervaire, 1958). For $n \in \{1, 2\}$, the real division algebra A is known (Althoen & Kugler, 1983; Hübner & Peterson, 2004; Dieterich, 2005). However the problem persists for the others cases. One of the method of determining the algebra A is to know its derivations and/or its automorphisms. Benkart and Osborn have classified Lie algebra of derivations $Der(A)$ (Benkart & Osborn, 1981). It's well known that if A is finite dimensional ,then the automorphism group $Aut(A)$ is a group of Lie, whose associated Lie algebra and Lie algebra $Der(A)$ coincide. In dimension 1, the group $Aut(A)$ is trivial. In dimension 2, Dieterich has classified $Aut(A)$, (Dieterich, 2005). However the problem persists for the others cases. This paper is a contribution to the advancement of the determination of the group $Aut(A)$. In the first part, we give some preliminaries results on the automorphism of an algebra A. In the second part, we characterize the 4-dimensional real division algebra A whose $Aut(A) = SO(3)$. Finally, we characterize also an unitary 4-dimensional real division algebra whose $Aut(A) = SO(2)$.

2. Preliminary

An algebra is said to be mutation α of A denoted A^α, the vector space A which has as product: $x \bullet_\alpha y = \alpha xy + (1-\alpha)yx$, x, $y \in A$. If $\lambda, \mu \in \mathbb{R}$ we have $(A^\lambda)^\mu = A^\alpha$ with $\alpha = 2\lambda\mu - \lambda - \mu + 1$. The product of $\mathbb{H}^\lambda$ in the basic $e = 1$, $e_1 = \frac{i}{2\lambda-1}$, $e_2 = \frac{j}{2\lambda-1}$, $e_3 = \frac{k}{2\lambda-1}$, is given by: $ee_n = e_ne = e_n$; $e_n^2 = \dfrac{1}{(2\lambda-1)^2}e$; $e_1e_2 = -e_2e_1 = e_3$; $e_1e_3 = -e_3e_1 = -e_2$; $e_2e_3 = -e_3e_2 = e_1$. Where $\{1, i, j, k\}$ in the canonical basis of the quaternions algebra $\mathbb{H}$. We denote $Aut(A) = \{f : A \longrightarrow A$, linear bijection: $f(xy) = f(x)f(y)$, $\forall x\ y \in A\}$ the automorphism group of A. We denote $Der(A) = \{\partial : A \longrightarrow A$, linear mapping: $\partial(xy) = \partial(x)y + x\partial(y)$, $\forall x\ y \in A\}$ the Lie algebra of derivations of A. The algebra A is called division if for all $x \in A - \{0\}$ the linears mapping L_x and R_x are bijective. Let $x, y \in A$, $[x, y] = xy - yx$ is the commutator of x and y. We recall that $I(A) = \{x \in A : \ x^2 = x\}$. Let ϕ, ψ the linears bijections, we call isotopy of A denoted $A_{\phi,\psi}$, the algebra whose product is: $x \odot y = \phi(x)\psi(y)$, $x, y \in A$.

Example *The mutation* $\lambda \in \mathbb{R}$ *of* $\mathbb{C}$, $\mathbb{C}^\lambda$ *is isomorphic to* $\mathbb{C}$. *The mutation* $\frac{1}{2}$ *of* $\mathbb{H}$, $\mathbb{H}^{\frac{1}{2}}$ *is commutative and it's not of division, called the symtrization, one notes it* $\mathbb{H}^+$

Lemma 1 *Let A be a real algebra, then the following assertions are equivalent:*

1. $f \in Aut(A)$ *and* $[f, \varphi] = [f, \psi] = 0$;

2. $f \in Aut(A_{\phi,\psi})$ *and* $[f, \varphi] = [f, \psi] = 0$.

Proof. Let $f \in Aut(A_{\phi,\psi})$, for all x and $y \in A$ we have:

$$\begin{aligned} f(x \odot y) &= f(x) \odot f(y) \\ \Leftrightarrow f(\phi(x) \cdot \psi(y)) &= \varphi(f(x)) \cdot \psi(f(y)) \\ \Leftrightarrow f(\phi(x) \cdot \psi(y)) &= f(\phi(x)) \cdot f(\psi(y)). \quad Then \quad f \in Aut(A). \end{aligned}$$

Lemma 2 *Let A be an algebra and $\lambda \in \mathbb{R}$, so $Aut(A) \subset Aut(A^{(\lambda)})$. Furthermore if $\lambda \neq \frac{1}{2}$ then $Aut(A) = Aut(A^{(\lambda)})$.*

Proof. It's easy to show that $Aut(A) \subset Aut(A^{(\lambda)})$. If $\lambda \neq \frac{1}{2}$, we have $Aut(A^{(\lambda)}) \subset Aut((A^{(\lambda)})^{\frac{\lambda}{2\lambda-1}}) = Aut(A)$

3. Characterization of Four-dimensional Real Division Algebra with $SO(3)$ as Its Automorphic Group

In (Benkart & Osborn, $(1981)_2$), we have the following result:

Theorem 1 *A is an four-dimensional real division algebra with su(2) as its derivation algebra if and only if A has a basis $\{e, e_1, e_2, e_3\}$ with multiplication given by* (**1.1**) *for some real numbers α, β, γ such that $\alpha\beta\gamma > 0$.*

$$\begin{aligned} &e^2 = e, \quad ee_i = \alpha e_i, \quad e_i e = \beta e_i \quad e_i^2 = -\gamma e \;\; for \;\; all \;\; i \in \{1,2,3\} \\ &e_1e_2 = -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = e_1, \quad e_3e_1 = -e_1e_3 = e_2 \qquad . \end{aligned} \qquad (\mathbf{1.1})$$

Remark 1 *Let $x = \lambda_0 e + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$, $y = \lambda'_0 e + \lambda'_1 e_1 + \lambda'_2 e_2 + \lambda'_3 e_3 \in A$, we have:*

$$\begin{aligned} xy &= \Big(\lambda_0\lambda'_0 - \gamma\lambda_1\lambda'_1 - \gamma\lambda_2\lambda'_2 - \gamma\lambda_3\lambda'_3\Big)e + \Big(\alpha\lambda_0\lambda'_1 + \beta\lambda_1\lambda'_0 + \lambda_2\lambda'_3 - \lambda_3\lambda'_2\Big)e_1 \\ &+ \Big(\alpha\lambda_0\lambda'_2 + \beta\lambda_2\lambda'_0 + \lambda_3\lambda'_1 - \lambda_1\lambda'_3\Big)e_2 + \Big(\alpha\lambda_0\lambda'_3 + \beta\lambda_3\lambda'_0 + \lambda_1\lambda'_2 - \lambda_2\lambda'_1.\Big)e_3 \end{aligned}$$

We defined $\psi_\alpha : A \longrightarrow A$; $\psi_\alpha(\lambda e + u) = \lambda e + \frac{1}{\alpha}u$ with $(\alpha, \lambda) \in \mathbb{R}^* \times \mathbb{R}$ and $u \in lin\{e_1, e_2, e_3\}$.

Theorem 2 *Let A be an 4-dimensional real division algebra with su(2) as its derivation algebra, then the isotope $A_{\psi_\alpha,\psi_\beta}$ of A is isomorphic to $\mathbb{H}^\mu$ with $\mu = \frac{1}{2\sqrt{\alpha\beta\gamma}} + \frac{1}{2}$.*

Proof. Let A be an algebra of theorem 1. The multiplication of $A_{\psi_\beta,\psi_\alpha}$ in the basis $\{e, e_1, e_2, e_3\}$ is given by (**1.2**)

$$\begin{aligned} &e \odot e = e, \quad e \odot e_i = e_i \odot e = e_i, \quad e_i \odot e_i = -\frac{\gamma}{\alpha\beta}e \;\; for \;\; all \;\; i \in \{1,2,3\} \\ &e_1 \odot e_2 = -e_2 \odot e_1 = \frac{1}{\alpha\beta}e_3, \quad e_2 \odot e_3 = -e_3 \odot e_2 = \frac{1}{\alpha\beta}e_1, \quad e_3 \odot e_1 = -e_1 \odot e_3 = \frac{1}{\alpha\beta}e_2 \qquad . \end{aligned} \qquad (\mathbf{1.2})$$

Setting $e' = e$, $e'_1 = \alpha\beta e_1$, $e'_2 = \alpha\beta e_2$ and $e'_3 = \alpha\beta e_3$, we obtain, an algebra isomorphic to $\mathbb{H}^\mu$ with $\mu = \frac{1}{2\sqrt{\alpha\beta\gamma}} + \frac{1}{2}$.

Corollary 1 *Every four-dimensional real division algebra with su(2) as its derivation algebra is isotope to the algebra $\mathbb{H}^\lambda$.*

Lemma 3 *Let A be an 4-dimensional real division algebra with su(2) as its derivation algebra. Then A has a basis $\{e, e_1, e_2, e_3\}$ with multiplication given by* (**1.1**). *Then we have*

$$\begin{aligned} I(A) &= \Big\{e\Big\} \cup \Big\{\frac{1}{\alpha+\beta}e + \sum_{i=1}^{3}\lambda_i e_i; \sum_{i=1}^{3}\lambda_i^2 = \frac{1-(\alpha+\beta)}{\gamma(\alpha+\beta)^2}\Big\}, \; if \;\; \alpha+\beta \neq 0 \;\; and \;\; \frac{1-(\alpha+\beta)}{\gamma} > 0, \\ I(A) &= \Big\{e\Big\}, \;\; otherwise. \end{aligned}$$

Proof. Let $x = \lambda_0 e + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in A$, we have:

$$x^2 = x \iff \begin{cases} \lambda_0^2 - \gamma(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) = \lambda_0 \\ \lambda_i\Big((\alpha+\beta)\lambda_0 - 1\Big) = 0, \quad i \in \{1,2,3\} \end{cases}$$

We obtain $I(A)$ by resolving the system and discussing on $\alpha + \beta$ and $\frac{1-(\alpha+\beta)}{\gamma}$.

Corollary 2 *Let A be an real algebra of theorem 1. Let u and $v \in A$ linearly independent. Then the following assertions are equivalent:*

1. $x \in I(A)$, $u^2 = v^2 = -\gamma x$, $xu = \alpha u$, $ux = \beta u$, $xv = \alpha v$, *and* $vx = \beta v$
2. $x = e$ *and* $u, v \in \{\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3;$ *with* $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1\}$.

Proof. (1) $\Longrightarrow$ (2) the proof will be reduce in the case $\alpha + \beta \neq 0$ *and* $\frac{1-(\alpha+\beta)}{\gamma} > 0$.
Suppose that $x = \frac{1}{\alpha+\beta}e + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in I(A)$ with $\sum_{i=1}^{3} \lambda_i^2 = \frac{1-(\alpha+\beta)}{\gamma(\alpha+\beta)^2}$.
Let $u = \Sigma_{i=0}^{3}\lambda'_i e_i$, and $v = \Sigma_{i=0}^{3}\lambda''_i e_i \in A$ satisfied the equations of (a). We have:

$u^2 = v^2 = -\gamma x \Longrightarrow \lambda_i = -\frac{\alpha+\beta}{\gamma}\lambda'_0\lambda'_i = -\frac{\alpha+\beta}{\gamma}\lambda''_0\lambda''_i \quad i \in \{1, 2, 3\}.$ (**E.1**)

And $xu = \alpha u$, $xv = \alpha v \Longrightarrow \lambda'^2_0 = \lambda''^2_0 = \frac{\alpha\gamma\left(1-(\alpha+\beta)\right)}{\beta(\alpha+\beta)^2}$. Consequently $\lambda'_0 = \varepsilon\lambda''_0$ with $\varepsilon^2 = 1$. We have $u = \varepsilon v$ according to (**E.1**), which is adsurd since u and v are linearly independent, then $x = e$. It's easily shown that the equations $u^2 = v^2 = -\gamma e$, $eu = \alpha u$, $ue = \beta u$, $ev = \alpha v$, and $ve = \beta v$ gives $\lambda'_0 = \lambda''_0 = 0$ and $\Sigma_{i=0}^{3}\lambda'^2_i = \Sigma_{i=0}^{3}\lambda''^2_i = 1$.

(2) $\Longrightarrow$ (1) the proof is evident.

Proposition 1 *Let A be a 4-dimensional real division algebra with su(2) as its derivation algebra and* $f \in Aut(A)$*, then* $f(e) = e$ *and* $f(lin\{e_1, e_2, e_3\}) \subseteq lin\{e_1, e_2, e_3\}$*. Moreover* $[f, \psi_\alpha] = 0$.

Proof. We notice that $f(e) \in I(A)$ and $f(e_i)$ for all $i \in \{1, 2, 3\}$, satisfy to (a) of corollary 1. Then $f(e) = e$ and $f(e_i) \in lin\{e_1, e_2, e_3\}$. It's easy to show that $[f, \psi_\alpha] = 0$.

Theorem 3 *Let A be a 4-dimensional real division algebra with su(2) as its derivation algebra, then the following propositions are equivalent:*

1. $Aut(A) \cong SO(3)$;
2. $Der(A) \cong su(2)$;
3. $A_{\psi_\alpha,\psi_\beta}$ *is isomorphic to* $\mathbb{H}^\mu$ *with* $\mu = \frac{1}{2\sqrt{\alpha\beta\gamma}} + \frac{1}{2}$.

Proof. (1) $\Longrightarrow$ (2) $Der(A) = Lie(Aut(A)) = Lie(SO(3)) \cong so(3) \cong su(2)$.

(2) $\Longrightarrow$ (3) See the Theorem 2.

(3) $\Longrightarrow$ (1) All automorphisms of A commute with ψ_α and ψ_β according to Proposition 1 and also all automorphisms of $A_{\psi_\alpha,\psi_\beta}$ commute with ψ_α and ψ_β according to theorem 2, then $Aut(A) = Aut(A_{\psi_\alpha,\psi_\beta})$.
The Lemmas 1 and 2 give $Aut(A) = Aut(A_{\psi_\beta,\psi_\alpha}) = Aut(\mathbb{H}^\mu) = Aut(\mathbb{H}) \cong SO(3)$.

4. Characterization Unitary 4-dimensional Real Division Algebra with $SO(2)$ as Its Automorphisms Groups

In (Diabang & all, $(2016)_1$), we have the following result:

Theorem 4 *Let A be an unital 4-dimensional real division algebra having a non-trivial derivation* ∂*, then there exists a basis* $\mathcal{B}_1 = \{e, e_1, e_2, e_3\}$ *of A for which the multiplication is given by the table* (**1.3**)*:*

$\odot$	e	e_1	e_2	e_3
e	e	e_1	e_2	e_3
e_1	e_1	$-e$	$\alpha_1 e_2 + \alpha_2 e_3$	$-\alpha_2 e_2 + \alpha_1 e_3$
e_2	e_2	$\alpha_3 e_2 + \alpha_4 e_3$	$\alpha_5 e + \alpha_6 e_1$	$\alpha_7 e + \alpha_8 e_1$
e_3	e_3	$-\alpha_4 e_2 + \alpha_3 e_3$	$-\alpha_7 e - \alpha_8 e_1$	$\alpha_5 e + \alpha_6 e_1$

(**1.3**)

for some real numbers α_i, $i \in \{1, \ldots, 7\}$.

Corollary 3 *Let A be an four-dimensional real unital division algebra A having a non-trivial derivation, then the following propositions are equivalent:*

1. $\alpha_1 = \alpha_3 = \alpha_6 = \alpha_7 = 0$, $\alpha_5 < 0$, $\alpha_2 = -\alpha_4 \neq 0$ *and* $\alpha_8 = -\alpha_2\alpha_5 \neq 0$;
2. *A is quadratic and flexible;*
3. $Der(A) = su(2)$;

4. *$Aut(A) = SO(3)$;*

5. *A is isotope to $\mathbb{H}^{\mu}$.*

Proof. (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) results of Theorem 2 in (Diabang & all, $(2016)_1$).

(3) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (5) results of Theorem 3.

Lemma 4 *Let A be an unital four-dimensional real division algebra having a non-trivial derivation ∂ such that A isn't quadratic or isn't flexible. If $f \in Aut(A)$, then $f(e) = e$ and $f(e_1) = \varepsilon e_1$ with $\varepsilon^2 = 1$.*

Proof. f being bijective then for all $y \in A$ there is $x \in A$ such that $f(x) = y$. We have $f(e)y = f(e)f(x) = f(ex) = f(x) = y$ and $yf(e) = f(x)f(e) = f(xe) = f(x) = y$, then $f(e)$ is an unitary element of A, therefore $f(e) = e$. The subalgebra of A generated by $f(e_1)$, denoted $< f(e_1) >$, is isomorphic to $B_0 = \ker\partial$. As $\dim(Der(A)) = 1$ then for all $x \in< f(e_1) >$, $\partial(x) = 0$ consequently $f(e_1) \in B_0$. The equation $f(e_1)^2 = -e$ gives $f(e_1) = \varepsilon e_1$.

Remark 2 *Let A be an unital 4-dimensional real division algebra having a non-trivial derivation. Let $x = \lambda_0 e + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in A$, we have:*

$$x^2 = -e \quad \Longleftrightarrow \quad \begin{cases} \lambda_0^2 - \lambda_1^2 + \alpha_5(\lambda_2^2 + \lambda_3^2) = -1 & (\mathbf{E.2}) \\ 2\lambda_0\lambda_1 + \alpha_6(\lambda_2^2 + \lambda_3^2) = 0, & (\mathbf{E.3}) \\ 2\lambda_0\lambda_2 + (\alpha_1 + \alpha_3)\lambda_1\lambda_2 - (\alpha_2 + \alpha_4)\lambda_1\lambda_3 = 0, & (\mathbf{E.4}) \\ 2\lambda_0\lambda_3 + (\alpha_2 + \alpha_4)\lambda_1\lambda_2 + (\alpha_1 + \alpha_3)\lambda_1\lambda_3 = 0. & (\mathbf{E.5}) \end{cases}$$

$$\lambda_2\mathbf{E.4} + \lambda_3\mathbf{E.5} \Longrightarrow \Big(2\lambda_0 + (\alpha_1 + \alpha_3)\lambda_1\Big)(\lambda_2^2 + \lambda_3^2) = 0 \qquad (\mathbf{E.6})$$

$$\lambda_3\mathbf{E.4} + \lambda_2\mathbf{E.5} \Longrightarrow (\alpha_2 + \alpha_4)\lambda_1(\lambda_2^2 + \lambda_3^2) = 0 \qquad (\mathbf{E.7})$$

There are four possible cases:
Cas 1. *If $\alpha_6(\alpha_2 + \alpha_4) \neq 0$, then $x^2 = -e \Longleftrightarrow x = \varepsilon e_1$.*
Cas 2. *If $\alpha_6 = 0$ and $\alpha_2 + \alpha_4 \neq 0$, then*

$$x^2 = -e \quad \Longleftrightarrow \quad \begin{cases} x \in \{\varepsilon e_1\} \cup \{\lambda_2 e_2 + \lambda_3 e_3;\ \lambda_2^2 + \lambda_3^2 = -\frac{1}{\alpha_5}\}, & If \quad \alpha_5 < 0 \\ x = \varepsilon e_1 \qquad otherwise, \end{cases}$$

Cas 3. *If $\alpha_6 = \alpha_2 + \alpha_4 = 0$, then*

$$x^2 = -e \quad \Longleftrightarrow \quad \begin{cases} x \in \{\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3;\ \lambda_1^2 = 1 + \alpha_5\lambda_2^2 + \alpha_5\lambda_3^2\}, & If \quad \alpha_1 + \alpha_3 = 0 \\ x \in \{\varepsilon e_1\} \cup \{\lambda_2 e_2 + \lambda_3 e_3;\ \lambda_2^2 + \lambda_3^2 = -\frac{1}{\alpha_5}\}, & If \quad \alpha_1 + \alpha_3 \neq 0 \quad and \quad \alpha_5 < 0 \\ x = \varepsilon e_1, \quad If \quad \alpha_1 + \alpha_3 \neq 0 \quad and \quad \alpha_5 \geq 0 \end{cases}$$

Cas 4. *If $\alpha_6 \neq 0$ and $\alpha_2 + \alpha_4 = 0$, then*

$$x^2 = -e \quad \Longleftrightarrow \quad \begin{cases} x \in \{\varepsilon e_1\} \cup \{k_o e + \varepsilon\sqrt{k_1} e_1 + \lambda_2 e_2 + \lambda_3 e_3;\ \lambda_2^2 + \lambda_3^2 = \frac{\alpha_1+\alpha_3}{\alpha_6}k_1\}, & If \quad \frac{\alpha_1+\alpha_3}{\alpha_6} > 0, k_1 > 0 \\ x = \varepsilon e_1, \quad otherwise \end{cases}$$

with $k_1 = \dfrac{4\alpha_6}{4\alpha_6 - 4\alpha_5(\alpha_1 + \alpha_3) - \alpha_6(\alpha_1 + \alpha_3)^2}$, $k_0 = -\dfrac{\varepsilon(\alpha_1 + \alpha_3)\sqrt{k_1}}{2}$ *and* $\varepsilon \in \{-1, 1\}$.

Proposition 2 *Let A be an unital four-dimensional real division algebra having a non-trivial derivation ∂ such that A isn't quadratic or isn't flexible. If $f \in Aut(A)$, then*

$$M(f, \mathcal{B}_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta) & -\sin(\theta) \\ 0 & 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

with $\theta \in \mathbb{R}$ so $Aut(A) \cong SO(2)$.

Proof. The lemma 4, gives $f(e) = e$ and $f(e_1) = \varepsilon e_1$. By the definition of the automorphism f and the equations $(E.2), \ldots (E.7)$, we obtains the result.

Definition 1 *(Unit-duplication process) Let B be an real algebra having an unit element e and let* $\rho, \sigma, \phi, \psi : B \longrightarrow B$ *be linear mappings such that* $\phi(e) = \psi(e) = e$*. We define on the space* $B \times B$ *the produit:*

$$(x, y) \odot (x', y') = (xx' + \rho(\sigma(y')y); y\phi(x') + y'\psi(x)) \qquad \textbf{(2.1)}$$

The algebra resulting has an unit element $(e, 0)$ *and contains* $B \times \{0\}$ *as sub-algebra. It is said to be obtained from B and* ϱ*, by unit-duplication process and is denoted by* $UDP_B(\rho, \sigma, \phi, \psi)$*. This generalizes the classical Cayley-Dickson process as-well as the process given.*

Theorem 5 *Let A be an unital 4-dimensional real division algebra having a non-trivial derivation such that A isn't quadratic or isn't flexible, then the following propositions are equivalent:*

1. $Aut(A) \cong SO(2)$*;*
2. $dim(Der(A)) = 1$*;*
3. *A is obtained from the unital real algebra* $\mathbb{C}$ *by unit-duplication process.*

Proof. (1) $\Longrightarrow$ (2) $Der(A) = Lie(Aut(A)) = Lie(SO(2)) = so(2)$, so $dim(Der(A)) = 1$.

(2) $\Longrightarrow$ (3) See Corollary 1 in (Diabang & all, $(2016)_1$).

(3) $\Longrightarrow$ (1) A admits a nonzero derivation, then A satisfies the hypotheses of the Theoreml 4. The proposition 2 completes the proof.

Remark 3 *Let A be a finite-dimensional real division algebra, whose Lie algebra of derivations is trivial, then the group* $Aut(A)$ *is finite.*

Problem 1 *Let A be an four-dimensional real division algebra, whose group* $Aut(A)$ *is finite. Is there an upper limit to the order of the group* $Aut(A)$*?.*

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Arithmetic Triangle

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Abstract

The product of the first n terms of an arithmetic progression may be developed in a polynomial of n terms. Each one of them presents a coefficient C_{nk} that is independent from the initial term and the common difference of the progression.

The most interesting point is that one may construct an ”Arithmetic Triangle”, displaying these coefficients, in a similar way one does with Pascal's Triangle. Moreover, some remarkable properties, mainly concerning factorials, characterize the Triangle. Other related 'triangles' – eventually treated as matrices – also display curious facts, in their linear *modus operandi*, such as successive ”descendances”.

Keywords: arithmetic progression, factorials of order k, arithmetic triangle, descendant matrices, progressive matrix, matrix column product, harmonic numbers.

1. Introduction

What is the product of the first n terms of an arithmetic progression? Contrary to geometrical progressions, where simple equations allow to calculate the sum or the product of its first n terms, the issue concerning the product for an arithmetic progression is not so easy. However, its study brings some fascinating results and, to my knowledge, new concepts in Linear Algebra. I propose a little journey on this subject.

First of all, I discovered that it relates to the starting point of a paper of mine, on “Integer Binomial Plan”, published in this same journal, in August 2010 (Ferreira), presenting a generalization on factorials and binomial coefficients to all integers. Since then I never returned to the subject till recently, when a student asked me if I knew a formula for the product of the first n terms of an arithmetic progression:

$$a_1\ ,\ a_2 = a_1 + d\ ,\ a_3 = a_1 + 2d\ ,\ \ldots\ ,\ a_n = a_1 + (n-1)\,d,\ \ldots\ . \tag{1}$$

I didn't; but then we began thinking on the problem. Let $P_n = a_1 \cdot a_2 \cdot \ \ldots\ \cdot a_n$. Following a simple procedure, remarking that $P_n = P_{n-1} \cdot [a_1 + (n-1)d]$ and beginning with $P_1 = a_1$, we soon found out that P_n may be expressed by a sum of n terms concerning powers of a_1 and d multiplied by n coefficients, which are independent from those variables. We followed the procedure up to $n = 5$:

$$\begin{aligned}
P_1 &= 1 \cdot a_1\\
P_2 &= 1 \cdot a_1^2 + 1 \cdot a_1 d\\
P_3 &= 1 \cdot a_1^3 + 3 \cdot a_1^2 d + 2 \cdot a_1 d^2\\
P_4 &= 1 \cdot a_1^4 + 6 \cdot a_1^3 d + 11 \cdot a_1^2 d^2 + 6 \cdot a_1 d^3\\
P_5 &= 1 \cdot a_1^5 + 10 \cdot a_1^4 d + 35 \cdot a_1^3 d^2 + 50 \cdot a_1^2 d^3 + 24 \cdot a_1 d^4,
\end{aligned}$$

so we could establish the coefficients triangle

$$\begin{array}{llllll}
1 & & & & & \\
1 & 1 & & & & \\
1 & 3 & 2 & & & \\
1 & 6 & 11 & 6 & & \\
1 & 10 & 35 & 50 & 24 & \\
1 & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}$$

This reminds Newton's Binomial and Pascal's Triangle. The question which naturally arises is: may one built this "Arithmetic Triangle" in a similar way one does with binomial coefficients? To eventually achieve this goal, one must find and understand the pattern underneath, which is not self-evident. But the answer to the question is affirmative. In fact, later that day, I found out that we may indeed construct the triangle progressively, in a similar way we do with Pascal's triangle. To my surprise, however, this issue brought back complete and incomplete factorials.

The search for an answer and some collateral results are the subject of this article.

I must point out that – also surprisingly – the solution found corresponds closely to the *unsigned Stirling numbers of the first kind*, which I didn't know. Several papers have been recently produced concerning Stirling numbers and cyclic groups, for instance (Broder, 1984) and (Deveci & Akuzum, 2014); or Pascal matrices (Deveci & Karaduman, 2012) and (Hiller, 2016). But they don't seem especially relevant to this article, which essentially deals with basic stuff.

2.1. Factorial of Order k

Consider a real number x and an integer k, along with the special case $0! = 1$; the $\boldsymbol{x}$ **factorial of order** $\boldsymbol{k}$ is defined by (read *"x k factorial"*)

$$x_{(k)}! = \frac{x!}{(x-k)!}\,, \tag{2}$$

or, for $k > 0$,

$$x_{(k)}! = \prod_{i=1}^{k}(x-i+1) = x\cdot(x-1)\cdot\ \ldots\ \cdot(x-k+1)\,; \tag{2.a}$$

in this case, $x_{(k)}!$ is an 'incomplete factorial', the product of the k major values of $x!$.

We assume that the definition above may be extended to negative order numbers; but then we conclude that

$$x_{(-k)}! = \frac{1}{(x+k)_{(k)}!}. \tag{3}$$

Remark that

$$\begin{cases} x_{(0)}! = 1 \\ x_{(x)}! = x! \\ x_{(1)}! = x\,, & \text{for } x \neq 0 \\ x_{(-1)}! = \frac{1}{x+1}\,, & \text{for } x \neq -1. \end{cases}$$

This proofs to be quite consistent. Generally speaking, as I explained in (Ferreira), factorials of order k must be seen as limits of a quotient. Besides, as presupposed in the generalization of $x_{(k)}!$ from positive to negative numbers x, if $k > 0$ this factorial represents the product of the k values presented in equation (2.a) even for $x < 0$, where $x! = \infty$. For instance,

$$(-8)_{(3)}! = \frac{(-8)!}{(-11)!} = \frac{(-8)\cdot(-9)\cdot(-10)\cdot(-11)!}{(-11)!} = (-8)\cdot(-9)\cdot(-10) = -720\,;$$

if $k < 0$, then, according to (3), $n_{(k)}!$ represents the inverse of a product of k factors:

$$(-8)_{(-3)}! = \frac{1}{(-5)_{(3)}!} = \frac{1}{(-5)\cdot(-6)\cdot(-7)} = -\frac{1}{210}.$$

In these terms, of course

$$\binom{n}{k} = \frac{n_{(k)}!}{k!} \tag{4}$$

is a *generalized binomial coefficient* and so, the development in MacLaurin series of the function $y = (1+x)^n$ may be written as

$$(1+x)^n = \sum_{k=0}^{+\infty}\binom{n}{k}.\,x^k = \sum_{k=0}^{+\infty}\frac{n_{(k)}!}{k!}x^k.$$

2.2 Rising and Falling Factorials

In a rapid web research, these days, I found out that the concept of $x_{(k)}!$ was already known, with other name and symbol(s), since the end of the 19th Century; it has been introduced by Leo A. Pochhammer with the notation $(x)_k$. I was quite surprised but, anyway, Pochhammer doesn't consider negative integers for k.

I also discovered the useful concept of **rising factorial** – in contrast with **falling factorial**, this is, $x_{(k)}!$ –, which I'll note by $x^{(k)}!$; for $x \neq 0$ and positive integer k:

$$x^{(k)}! = \prod_{i=1}^{k} (x + i - 1) = x \cdot (x + 1) \cdot \ \ldots \ \cdot (x + k - 1)\,. \tag{5}$$

It's easy to see that

$$x^{(k)}! = (x + k - 1)_{(k)}! \quad \text{or} \quad x_{(k)}! = (x - k + 1)^{(k)}!\,; \tag{5.a}$$

For instance, $7^{(4)}! = 10_{(4)}! = 7 \cdot 8 \cdot 9 \cdot 10$. Based on the assumption that the equality (5.a) must stand for any values of x and k and keeping in mind the equation (2), we'll generically define the **rising factorial** $x^{(k)}!$ by

$$x^{(k)}! = \frac{(x + k - 1)!}{(x - 1)!}. \tag{6}$$

Therefore, as particular cases, we obtain

$$\begin{cases} x^{(0)}! = 1 \\ 1^{(k)}! = k!. \end{cases}$$

Besides, we'll consider (5.a) also valid for negative order number $-k$; but then, according to (3):

$$x^{(-k)}! = \frac{1}{(x - 1)_{(k)}!}\,. \tag{7}$$

Finally: to simplify, we may read $x_{(k)}!$ and $x^{(k)}!$ respectively as "***x* down *k* factorial**" and "***x* up *k* factorial**".

2.3 New Generalization on Factorials

Let's resume the arithmetic progression

$$a_1 = a\ ,\ a_2 = a_1 + d\ ,\ a_3 = a_1 + 2d\ ,\ \ldots\ ,\ a_n = a_1 + (n - 1)\,d,\ \ldots\ ,$$

where a_1 is the *initial term*. The constant d is usually known as the *common difference* of successive members; I will call it the ***increment*** of the progression.

Let P_n be the product of the first n terms of this arithmetic progression. It's quite evident that:

- for $\boxed{a_1 = d = 1}$: $P_n = n!$;
  We'll say that this is an **unitary arithmetic progression (u.a.p.) on basis 1** (its initial term).
- for $\boxed{a_1 = a \ \text{ and } \ d = 1}$: $P_n = a^{(n)}! = (a + n - 1)_{(n)}!$.

Now, we may put the n-th term of the progression above in the form

$$a_n = d\left[\frac{a}{d} + (n - 1)\right]$$

and the expression between brackets corresponds to an u.a.p. on basis a/d. In this case,

$$P_n = \prod_{i=1}^{n} a_i = d^n \prod_{i=1}^{n} \left[\frac{a}{d} + (i - 1)\right]$$

and we'll identify P_n with the ***a* rising factorial of increment *d* and order *n***, noted by $a^{(d\,;\,n)}!$ (read "***a* up *d* *n* factorial**"):

$$a^{(d\,;\,n)}! = d^n \left(\frac{a}{d}\right)^{(n)}!\,. \tag{8}$$

This relates to the ***a* falling factorial of increment *d* and order *n***, noted by $a_{(d\,;\,n)}!$ (read "***a* down *d* *n* factorial**") as follows:

$$a_{(d\,;\,n)}! = d^n \left(\frac{a}{d}\right)_{(n)}!\,, \tag{9}$$

in such a way that

$$a_{(d\,;\,n)}! = [a - (n-1)\,d]^{(d\,;\,n)}! \quad \text{and} \quad a^{(d\,;\,n)}! = [a + (n-1)\,d]_{(d\,;\,n)}!\ ,$$

As a particular case, from (8):

$$a = d \quad \Rightarrow \quad a^{(a\,;\,n)}! = a^n\, 1^{(n)}! = a^n n!\ .$$

Of course, making $d = 1$, we recover the simple rising and falling factorials $a^{(1\,;\,n)}! = a^{(n)}!$ and $a_{(1\,;\,n)}! = a_{(n)}!$.

3. Arithmetic Coefficients

The key point to discover the rule for building the Arithmetic Triangle is to notice that, because of the linear nature of an arithmetic progression, each P_n proceeds from the precedent P_{n-1} in a similar linear way.

Our goal is to discover the n coefficients C_{nk} (or $C_{n;k}$) that allow us to write, making $a_1 = a$, the product P_n as

$$P_n = \sum_{k=1}^{n} C_{nk}\, a^{n-k+1} d^{k-1} = C_{n1}\, a^n d^0 + C_{n2}\, a^{n-1} d^1 + \cdots + C_{nk}\, a^{n-k+1} d^{k-1} + \cdots + C_{nn}\, a^1 d^{n-1}. \tag{10}$$

We'll call these C_{nk} the **arithmetic coefficients** for the product P_n. Since the coefficients C_{nk} are independent from a and d, we'll make $d = 1$ in the following deduction [the numbers in bold correspond to $n - 1$ in each case].

$$P_1 = a \qquad \Rightarrow \quad C_{11} = 1.$$

$$P_2 = C_{11}\, a \cdot \left(a + \underbrace{n-1}_{1} \right) = C_{11}\, a^2 + \mathbf{1}\, C_{11}\, a \qquad \Rightarrow \quad \begin{cases} C_{21} = C_{11} = 1 \\ C_{22} = (n-1)\, C_{11} \end{cases}$$

$$P_3 = \Big(C_{21}\, a^2 + C_{22}\, a \Big) \cdot \left(a + \underbrace{n-1}_{2} \right)$$

$$= C_{21}\, a^3 + (\mathbf{2}\, C_{21} + C_{22})\, a^2 + \mathbf{2}\, C_{22} a \qquad \Rightarrow \quad \begin{cases} C_{31} = C_{21} = 1 \\ C_{32} = (n-1)\, C_{21} + C_{22} \\ C_{33} = (n-1)\, C_{22} \end{cases}$$

$$P_4 = \Big(C_{31}\, a^3 + C_{32}\, a^2 + C_{33}\, a \Big) \cdot \left(a + \underbrace{n-1}_{3} \right)$$

$$= C_{31}\, a^4 + (\mathbf{3}\, C_{31} + C_{32})\, a^3 + (\mathbf{3}\, C_{32} + C_{33})\, a^2 + \mathbf{3}\, C_{33} a \quad \Rightarrow \quad \begin{cases} C_{41} = C_{31} = 1 \\ C_{42} = (n-1)\, C_{31} + C_{32} \\ C_{43} = (n-1)\, C_{32} + C_{33} \\ C_{44} = (n-1)\, C_{33}. \end{cases}$$

This is enough to recognize the pattern. We see that $C_{n1} = 1$, for every n, and we obtain the general rule:

$$\begin{cases} C_{n1} = 1 \\ C_{nk} = (n-1)\, C_{n-1;\,k-1} + C_{n-1;\,k} \\ C_{nn} = (n-1)\, C_{n-1;\,n-1}, \ \text{for } n > 1. \end{cases} \tag{11}$$

A more formal demonstration follows the same path, as we'll see in the sequence. As a matter of fact, the single line

$$C_{nk} = (n-1)\, C_{n-1;\,k-1} + C_{n-1;\,k}$$

is sufficient if we understand that it must be

$$C_{nk} = 0 \quad \text{for} \quad k < 1 \quad \text{or} \quad k > n\ . \tag{12}$$

This is because $k < 1$ corresponds to the 'left wing' – this is, the 'negative-increment' terms – and $k > n$ to upper terms in the progression; in both cases, these terms are out of the product P_n. The last condition also corresponds to the empty spaces in the Arithmetic Triangle.

Furthermore, we'll generalize $C_{n1} = 1$ to $n = 0$, resulting $C_{11} = \underbrace{0 \cdot C_{00}}_{0} + \underbrace{C_{01}}_{1} = 1$.

Theorem 1. *If P_n is the product of the first n terms of the arithmetic progression given by $a_k = a_1 + (n-1)\,k$, then*

$$\boxed{P_n = \sum_{k=1}^{n} C_{nk}\, a^{n-k+1} d^{k-1} \quad with \quad C_{nk} = (n-1)\, C_{n-1;\,k-1} + C_{n-1;\,k}\,.} \tag{13}$$

Proof. It's a proof by induction:

1) The equation above is obviously valid for $P_1 = 1 \cdot a$.

2) Let $P_{n-1} = \sum_{k=1}^{n-1} C_{n-1;\,k}\, a^{n-k} d^{k-1}$.

3) Then, $P_n = P_{n-1} \cdot [a - (n-1)\,d]$

For each k in the resulting sum, we'll have two terms:

$$C_{n-1;\,k}\, a^{n-k+1} d^{k-1} + C_{n-1;\,k}\, a^{n-k} d^{k}$$

So, if we make $k = i-1$ and $k = i$, we'll get

$$\begin{cases} k = i-1: & C_{n-1;\,i-1}\, a^{n-1+2} d^{1-2} + (n-1)\, C_{n-1;\,i-1}\, a^{n-i+1} d^{i-1} \\ k = i: & C_{n-1;\,i}\, a^{n-i+1} d^{i-1} + (n-1)\, C_{n-1;\,i}\, a^{n-i} d^{i} \end{cases}$$

Remark that the second term for $(i-1)$ combines with the first one for i :

$$\left[(n-1)\, C_{n-1;\,i-1} + C_{n-1;\,i}\right] a^{n-i+1} d^{i-1}$$

This is valid for every i, therefore for every k ; but the indexes of C_{nk} relates to the product $a^{n-k+1} d^{k-1}$ and, so, the expression between brackets corresponds to C_{nk}, which means that $C_{nk} = (n-1)\, C_{n-1;\,k-1} + C_{n-1;\,k}$. □

Based on (13) or (11), one can progressively construct the Arithmetic Triangle:

n=1	1									
n=2	1	1								
n=3	1	3	2							
n=4	1	6	11	6						
n=5	1	10	35	50	24					
n=6	1	15	85	225	274	120				
n=7	1	21	175	735	1624	1764	720			
n=8	1	28	322	1960	6769	13132	13068	5040		
n=9	1	36	546	4536	22449	67284	118124	109584	40320	
n=10	1	45	870	9450	63273	269325	723680	1172700	1026576	362880
...	...	...	...	...	...	...	...	...	...	...

(14)

Each set of Arithmetic Coefficients – this is, for each value of n – displays the following remarkable properties:

- $\boxed{\sum_{k=1}^{n} C_{nk} = n!}$
- $\boxed{C_{nn} = (n-1)!}$
- $\boxed{\text{For } n > 1 : \sum_{k=1}^{n-1} C_{nk} = n! - (n-1)! = (n-1) \cdot (n-1)!}$
- $\boxed{\text{For } n > 1 : \sum_{k=1}^{n} (-1)^{(k-1)}\ C_{nk} = 0}$.

The first equality is self-evident because, as we have seen, making $a_1 = d = 1$ (u.a.p. on basis 1), P_n is the product of the n integers from 1 to n, this is, n factorial. The second comes from a recurrence relation: from (13) and (12), remembering that $C_{11} = 1 = 0!$, we get, for $n > 1$, since $C_{n-1;\,n} = 0$: $\quad C_{nn} = (n-1)\,C_{n-1;\,n-1}$; so,

$$\begin{aligned}
&C_{22} = 1 \cdot C_{11} = 1!\\
&C_{33} = 2 \cdot C_{22} = 2 \cdot 1! = 2!\\
&C_{44} = 3 \cdot C_{33} = 3 \cdot 2! = 3!\\
&\dots\\
&C_{nn} = (n-1) \cdot C_{n-1;\,n-1} = (n-1)!\ .
\end{aligned}$$

The third equality results from combining the previous two. Finally, the fourth equality – which we may call an **alternate sum** – comes from the fact that, if we consider the 'left wing' of the u.a.p. on basis 1 (by simply making $d = -1$), it results $a_2 = 0$ and, so, for $n > 1$, each product P_n includes the zero.

For instance, making $n = 6$:

- $\sum_{k=1}^{6} C_{6k} = 1 + 15 + 85 + 225 + 274 + 120 = 720 = 6!$
- $C_{66} = 5! = 120$
- $\sum_{k=1}^{5} C_{6k} = 1 + 15 + 85 + 225 + 274 = 600 = 5 \cdot 5!$
- $\sum_{k=1}^{6} (-1)^{(k-1)}\ C_{6k} = 1 - 15 + 85 - 225 + 274 - 120 = 0.$

An important feature here is that the use of the Arithmetic Coefficients *transform* ***products**, n!, into* **sums**, both with n terms. Concerning these coefficients, here's another interesting fact:

- $\boxed{\prod_{k=1}^{n} C_{kk} = \prod_{k=1}^{n-1} (n-k)! = \prod_{k=1}^{n-1} k!.}$

  This is the product of the diagonal terms of the Arithmetic Triangle up from 1 to n. Following Neil Sloane and Simon Plouffe, it may be useful to define, for a positive integer n, the n **superfactorial** – I'll note $n!^*$ instead of $\mathrm{sf}(n)$ – as

  $$n!^* = n!\,(n-1)!^* = \prod_{k=1}^{n} k! = n!\ (n-1)!\ (n-2)!\ \dots\ 2!\ 1!\ , \tag{15}$$

  which is equivalent to

  $$n!^* = \prod_{k=1}^{n} k^{n-k+1} = n \cdot (n-1)^2 \cdot (n-2)^3\ \dots\ 2^{n-1} \cdot 1^n. \tag{15.a}$$

  This last equality also results from a recurrence relation; informally:

  $$\begin{aligned}
  &2!^* = 2!\ 1!\\
  &3!^* = \underbrace{3!}_{3\cdot 2!}\ 2!\ 1! = 3 \cdot (2!)^2\ \ 1! = 3 \cdot (\underbrace{2!}_{2\cdot 1!})^2\ \ 1! = 3 \cdot 2^2 \cdot 1^3\\
  &4!^* = \underbrace{4!}_{4\cdot 3!}\ 3!^* = 4\,(3 \cdot 2 \cdot 1)\left(3 \cdot 2^2 \cdot 1^3\right) = 4 \cdot 3^2 \cdot 2^3 \cdot 1^4\\
  &\dots\\
  &n!^* = \underbrace{n!}_{n\cdot(n-1)!}\ (n-1)!^* = n\ (n-1)!\ (n-1)!^* = n \cdot (n-1)^2 \cdot (n-2)^3\ \dots\ 2^{n-1} \cdot 1^n.
  \end{aligned}$$

  We may write, then: $\boxed{\prod_{k=1}^{n} C_{kk} = (n-1)!^*}$

  For instance, for $n = 6$: $\prod_{k=1}^{6} C_{kk} = 5!^* = 5 \cdot 4^2 \cdot 3^3 \cdot 2^4 \cdot 1^5 = 34560$.

  Consider now some other cases for the arithmetic progression:

  1. $\boxed{a = d}$: $a^{n-k+1} d^{k-1} = a^n$ and $\boxed{P_n = a^n \sum_{k=1}^{n} C_{nk} = a^n n!}$.
  2. As a particular case, $\boxed{a = d = -1}$: $\boxed{P_n = (-1)^n \sum_{k=1}^{n} C_{nk} = (-1)^n\, n! = (-1)_{(n)}! = (-n)^{(n)}!}$ according to Theorem 2 in (Ferreira) and equation (5.a) above.

3. As another particular case, $\boxed{a = d = 2}$:
   2 4 6 8 10 ... 2k ... ;

   this is the set of **even numbers** and it results $\boxed{2^{(2;n)}! = (2n)!! = P_n = 2^n \sum_{k=1}^{n} C_{nk} = 2^n n!}$ [where $(2n)!!$ is the *double factorial* of $m = 2n$].
   For instance, $8!! = P_4 = 2^4\,(1 + 6 + 11 + 6) = 2^4 4! = 16 \cdot 24 = 384$.
4. $\boxed{a = 1 \text{ and } d = 2}$:
   1 3 5 7 9 ... 2k − 1 ... ;

   this is now the set of **odd numbers** and it results $\boxed{1^{(2;n)}! = (2n-1)!! = P_n = \sum_{k=1}^{n} 2^{k-1} C_{nk}}$ [where $(2n-1)!!$ is the *double factorial* of $m = 2n - 1$].
   For instance, $9!! = P_5 = 1 \cdot 1 + 2 \cdot 10 + 4 \cdot 35 + 8 \cdot 50 + 16 \cdot 24 = 1 + 20 + 140 + 400 + 384 = 945$.
5. $\boxed{\text{Making } d = 1}$: $\boxed{P_n = \sum_{i=1}^{n} C_{ni}\, a^{n-i+1} = a_{n\,(n)}! = a^{(n)}!}$, for a given a or $a_n = a + n - 1$.
   For instance, making $a_5 = 10$, then $a = 6$, this corresponding to the progression
   6 7 8 9 $\boxed{10}$ 11 ... ,
   then $P_5 = \sum_{i=1}^{5} C_{5i} \cdot 6^{6-i} = 7776 + 12960 + 7560 + 1800 + 144 = 30240 = 10_{(5)}! = 6^{(5)}!$

4. Direct Determination of Arithmetic Coefficients

4.1 Sum-factorials

My goal here was to establish an explicit equation (or equations) allowing to directly obtain whatever coefficient C_{nk}, independently from the Arithmetic Triangle (as one does with binomial numbers). Unfortunately, I haven't been able to induce a simple formula; in the end, for $k > 4$, we stumble on a recursive method of the kind. Anyway, some interesting results appear. So, first of all, it is convenient to define the ***x* sum-factorial** (also known as *"x-th triangular number"*) as

$$x\,\dagger = \sum_{i=1}^{x} i = 1 + 2 + 3 + \cdots + (x-1) + x = \frac{(x+1)_{(2)}!}{2}\,, \tag{16}$$

or $x\,\dagger = x + (x-1)\,\dagger$, together with the special case $0\dagger = 0$; and, for a non-negative integer k, the ***x* sum-factorial of order *k*** as

$$\begin{aligned} x_{(k)}\dagger &= x\,\dagger - (x-k)\,\dagger \\ &= \textstyle\sum_{i=1}^{k} (x-i+1) = x + (x-1) + \cdots + (x-k+1) \\ &= \tfrac{k}{2}\,(2x-k+1)\,, \end{aligned} \tag{17}$$

this is, the sum of the k major values of $x\,\dagger$. As for product-factorials $x_{(k)}!$, we'll consider that the definition of $x_{(k)}\,\dagger$ also stands for negative integers $-k$; the result is

$$\begin{aligned} x_{(-k)}\dagger &= -\,[(x+k)\,\dagger - x\,\dagger] \\ &= -\,(x+k)_{(k)}\,\dagger \\ &= -\tfrac{k}{2}\,(2x+k+1)\,. \end{aligned} \tag{18}$$

It's easy to see that, as particular cases,

$$\begin{cases} x_{(0)}\dagger = 0 \\ x_{(x)}\dagger = x\,\dagger \\ x_{(1)}\dagger = x \\ x_{(-1)}\dagger = -\,(x+1)\,. \end{cases}$$

The infinite sequence of $u_j = j\,\dagger$ is

1 3 6 10 15 21 28 36 45 55 66 78 ...

Finally, as a curiosity, if we consider $x_{(k)}\,\dagger$ as a **falling sum-factorial**, then the ***x* rising sum-factorial** is

$$\begin{aligned} x^{(k)}\dagger &= \textstyle\sum_{i=1}^{k} (x+i-1) = x + (x+1) + \cdots + (x+k-1) \\ &= \tfrac{k}{2}\,(2x+k-1)\,. \end{aligned} \tag{19}$$

4.2 Obtaining Arithmetic Coefficients

Coming back to the C_{nk} issue, rather than providing exhaustive proofs, we'll take an inductive path. This, I think, is enough for the moment and more instructive. We'll proceed step by step, each column at a time, increasing the value of k, always keeping in mind that $C_{nk} = 0$ for $k > n$ and the recursive relation

$$C_{nk} = (n-1)\, C_{n-1;\,k-1} + C_{n-1;\,k} \ .$$

1. First of all, we know that $C_{n1} = 1$. Then, it's easy to conclude that, for $k = 2$:

$$\begin{cases} C_{22} = 1 + 0 = 1 \\ C_{32} = 2\ C_{21} + C_{22} = 2 + 1 \\ C_{42} = 3\ C_{31} + C_{32} = 3 + 2 + 1 \\ C_{52} = 4\ C_{41} + C_{42} = 4 + 3 + 2 + 1 \\ \dots , \end{cases}$$

this is,

$$C_{n2} = (n-1)\,\dagger = \tfrac{n-1}{2}\,(1 + n - 1) \quad \Rightarrow \quad C_{n2} = \tfrac{1}{2}\, n_{(2)}! = \begin{pmatrix} n \\ 2 \end{pmatrix}. \tag{20}$$

2. We'll proceed in the same way for $k = 3$:

$$\begin{cases} C_{23} = 0 \\ C_{33} = 2\ C_{22} + C_{23} = \tfrac{1}{2}\ 2 \cdot 2_{(2)}! \\ C_{43} = 3\ C_{32} + C_{33} = \tfrac{1}{2}\ [3 \cdot 3_{(2)!} + 2 \cdot 2_{(2)}!] \\ C_{53} = 4\ C_{42} + C_{43} = \tfrac{1}{2}\ [4 \cdot 4_{(2)}! + 3 \cdot 3_{(2)}! + 2 \cdot 2_{(2)}!] \\ \dots , \end{cases}$$

this is,

$$C_{n3} = \tfrac{1}{2}\ \textstyle\sum_{j=2}^{n-1} j \cdot j_{(2)}! = \sum_{j=2}^{n-1} F_j \ , \tag{21}$$

where

$$F_j = \frac{1}{2}\ j \cdot j_{(2)}! = j \cdot \begin{pmatrix} j \\ 2 \end{pmatrix}. \tag{22}$$

These are the values for the F_j , j from 1 to 12:

$$\begin{array}{llllll} F_1 = 0 & F_2 = 2 & F_3 = 9 & F_4 = 24 & F_5 = 50 & F_6 = 90 \\ F_7 = 142 & F_8 = 224 & F_9 = 324 & F_{10} = 450 & F_{11} = 605 & F_{12} = 792. \end{array} \tag{22.a}$$

So, for instance, $C_{73} = \sum_{j=2}^{6} F_j = 2 + 9 + 24 + 50 + 90 = 175.$

3. Let's examine now the case of $k = 4$:

$$\begin{cases} C_{34} = 0 \\ C_{44} = 3\ C_{33} + C_{34} = 3F_2 \\ C_{54} = 4\ C_{43} + C_{44} = 4F_3 + (4+3)\, F_2 = 4_{(1)}\dagger \cdot F_3 + 4_{(2)}\dagger \cdot F_2 \\ C_{64} = 5\ C_{53} + C_{54} = 5_{(1)}\dagger \cdot F_4 + 5_{(2)}\dagger \cdot F_3 + 5_{(3)}\dagger \cdot F_2 \\ \dots , \end{cases}$$

this is,

$$C_{n4} = \textstyle\sum_{i=1}^{n-3} (n-1)_{(i)}\, \dagger \cdot F_{n-i-1} \ . \tag{23}$$

We'll write this as

$$C_{n4} = \sum_{i=1}^{n-3} A_{n4}^{i} \cdot F_{n-i-1} \ ; \tag{23.a}$$

making

$$A^i_{n4} = (n-1)_{(i)} \dagger = \frac{i}{2}(2n-i-1)\,. \tag{24}$$

where i is not an exponent but an upper index. Remark that

$$A^0_{n4} = 0, \quad A^1_{n4} = n-1 \quad \text{and, for } n > 4, \quad A^i_{n4} = A^{i-1}_{n4} + (n-i)\,.$$

The values for A^i_{n4} (where $1 \le i \le n-3$) are:

(25)

i	1	2	3	4	5	6	7	8	9	...
n=4	3									
n=5	4	7								
n=6	5	9	12							
n=7	6	11	15	18						
n=8	7	13	18	22	25					
n=9	8	15	21	26	30	33				
n=10	9	17	24	30	35	39	42			
n=11	10	19	27	34	40	45	49	52		
n=12	11	21	30	38	45	51	56	60	63	
...	...	...	...	...	...	...	...	...	...	...

For instance, making $n = 8$, we get

$$\begin{aligned} C_{84} &= \textstyle\sum_{i=1}^{5} A^i_{84} \cdot F_{n-i-1} \\ &= A^1_{84} \cdot F_6 + A^2_{84} \cdot F_5 + A^3_{84} \cdot F_4 + A^4_{84} \cdot F_3 + A^5_{84} \cdot F_2 \\ &= 7 \cdot 90 + 13 \cdot 50 + 18 \cdot 24 + 22 \cdot 9 + 25 \cdot 2 = 1960. \end{aligned}$$

OBS.: we'll necessarily note $C_{n;k}$ or $A^i_{n;k}$ in the case of n or k being two digits numbers, like $A^5_{12;\,4} = 45$.

4. Following a similar path, we'll come now to $k = 5$; and then generalize the result to $k \ge 4$.

$$\begin{cases} C_{45} = 0 \\ C_{55} = 4\,C_{44} + C_{45} = 4A^1_{44} \cdot F_2 \\ C_{65} = 5\,C_{54} + C_{55} = 5A^1_{54}F_3 + \left(5A^2_{54} + 4A^1_{44}\right)F_2 \\ C_{75} = 6\,C_{64} + C_{65} = 6A^1_{64}F_4 + \left(6A^2_{64} + 5A^1_{54}\right)F_3 + \left(6A^3_{64} + 5A^2_{54} + 4A^1_{44}\right)F_2 \\ \dots\,. \end{cases}$$

Take for instance, $n = 7$; if we put

$$\begin{cases} A^1_{75} = 6A^1_{64} \\ A^2_{75} = 6A^2_{64} + 5A^1_{54} \\ A^3_{75} = 6A^3_{64} + 5A^2_{54} + 4A^1_{44}, \end{cases}$$

this corresponding to *diagonal product/sums* of i terms in the table (25), we'll write

$$C_{75} = A^1_{75} \cdot F_4 + A^2_{75} \cdot F_3 + A^3_{75} \cdot F_2,$$

which is an analogous equation to (23.a).

We deal here with a recursive process; so, for $k \ge 5$, it must be

$$A^i_{nk} = \sum_{j=1}^{i} (n-j)\,A^{i-j+1}_{n-j\,;\,k-1}\,. \tag{26}$$

Remark that the number of terms for C_{nk} is always equal to $\boxed{n-k+1}$; on the other hand, the smallest term of the F_b is F_2 and, so, the biggest one is F_{n-k+2}. We are therefore able to write a general recursive equation, for $k \geq 4$:

$$\begin{aligned} C_{nk} &= \sum_{i=1}^{n-k+1} A_{nk}^{i} \cdot F_{(n-k+3)-i} \\ &= A_{nk}^{1} \cdot F_{n-k+2} + A_{nk}^{2} \cdot F_{n-k+1} + \cdots + A_{nk}^{n-k+1} \cdot F_2. \end{aligned} \tag{27}$$

Because of its recursive nature, this is mainly of theoretical interest, from an algebraic point of view. Its application to real cases becomes quite fastidious, as k increases. Consider, for instance, the above-mentioned case C_{75}; the first step is easy, deriving directly from table (25):

$$\begin{cases} A_{75}^{1} = 6 \cdot 5 = 30 \\ A_{75}^{2} = 6 \cdot 9 + 5 \cdot 4 = 74 \\ A_{75}^{3} = 6 \cdot 12 + 5 \cdot 7 + 4 \cdot 3 = 119 \end{cases} \quad \Rightarrow \quad C_{75} = 30 \cdot 48 + 74 \cdot 9 + 119 \cdot 2 = 1624.$$

Now, for $k = 6$, to compute the two A_{76}^{i}, we are obliged to previously calculate A_{65}^{1}, A_{65}^{2} and A_{55}^{1} because

$$\begin{cases} A_{76}^{1} = 6A_{65}^{1} \\ A_{76}^{2} = 6A_{65}^{2} + 5A_{55}^{1}. \end{cases}$$

We get

$$\begin{cases} A_{65}^{1} = 5A_{54}^{1} = 5 \cdot 4 = 20 \\ A_{65}^{2} = 5A_{54}^{2} = 5 \cdot 7 + 4 \cdot 3 = 47 \\ A_{55}^{1} = 4A_{44}^{1} = 4 \cdot 3 = 12 \end{cases} \quad \Rightarrow \quad \begin{cases} A_{76}^{1} = 6 \cdot 20 = 120 \\ A_{76}^{2} = 6 \cdot 47 + 5 \cdot 12 = 342; \end{cases}$$

so, finally:

$$C_{76} = A_{76}^{1} \cdot F_3 + A_{76}^{2} \cdot F_2 = 120 \cdot 9 + 342 \cdot 2 = 1764.$$

For the last term, C_{77}, there is a single A:

$$C_{77} = A_{77}^{1} \cdot \underbrace{F_2}_{2} \;;$$

but

$$A_{77}^{1} = 6 \underbrace{A_{66}^{1}}_{5A_{55}^{1}} = 6_{(2)}! \underbrace{A_{55}^{1}}_{4A_{44}^{1}} = 6_{(3)}! \cdot 4 = 6_{(4)}! \;,$$

which means that $C_{77} = 6_{(4)}! \cdot 2 = 6!$, as we already knew.

4.3 Matrix Calculation

Of course, the calculus of a certain C_{nk} becomes less fastidious if one previously (and automatically) calculates the several A_{nk}^{i} by means of appropriate definitions and rules.

Let $\mathbf{A}_k$ be an infinite square matrix, where we'll put A_{nk}^{i} for its entry in the *supposed* n-th row and the *supposed* i-th column; *supposed* because we'll make

$$A_{nk}^{i} = A_{n_0+\Delta n\,;\,k}^{i_0+\Delta i} \;,$$

where Δn and Δi are integers we'll call **row** and **column increments** respectively; n_0 and i_0 are the **effective row** and **column** numbers. For our purposes, we'll make $\Delta n = 3$ and $\Delta i = 0$; therefore, A_{4k}^{1} is the entry in the effective first row and first column.

We'll define $\mathbf{A}_{k+1}$ as the **(first) diagonal descendant matrix** of $\mathbf{A}_k$ – which is its **diagonal ascendant** – if

$$A_{n;\,k+1}^{i} = \sum_{j=1}^{i} (n-j)\, A_{n-j\,;\,k}^{i-j+1} \,. \tag{28}$$

This is an equivalent equation to (26). The idea is to present $\mathbf{A}_5$ as the descendant of $\mathbf{A}_4$ [the matrix corresponding to the table (25)]. Naturally, the **second diagonal descendant** of $\mathbf{A}_k$ is $\mathbf{A}_{k+2}$, the descendant of $\mathbf{A}_{k+1}$; generally speaking, $\mathbf{A}_{k+r}$ is the ***r*-th diagonal descendant** of $\mathbf{A}_k$.

So, if we define the matrix $\mathbf{A}_4$ following the equation (24), just like the table (25), in which we'll make all empty entries equal to zero, we get

$$\mathbf{A}_4 = \begin{bmatrix} \boxed{\boxed{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \boxed{4} & \boxed{\boxed{7}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 5 & \boxed{9} & \boxed{\boxed{12}} & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 6 & 11 & 15 & 18 & 0 & 0 & 0 & 0 & 0 & \dots \\ 7 & 13 & 18 & 22 & 25 & 0 & 0 & 0 & 0 & \dots \\ 8 & 15 & 21 & 26 & 30 & 33 & 0 & 0 & 0 & \dots \\ 9 & 17 & 24 & 30 & 35 & 39 & 42 & 0 & 0 & \dots \\ 10 & 19 & 27 & 34 & 40 & 45 & 49 & 52 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}. \tag{29}$$

The first diagonal descendant of $\mathbf{A}_4$ is, then, the matrix

$$\mathbf{A}_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 20 & 47 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 30 & \boxed{74} & \boxed{\boxed{119}} & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 42 & 107 & 179 & 618 & 0 & 0 & 0 & 0 & 0 & \dots \\ 56 & 146 & 251 & 1361 & 29858 & 0 & 0 & 0 & 0 & \dots \\ 72 & 191 & 235 & 2688 & 6136 & 9616 & 0 & 0 & 0 & \dots \\ 90 & 242 & 431 & 4857 & 11702 & 19214 & 26036 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}. \tag{30}$$

The *modus operandi*, for A_{75}^2 and A_{75}^3, as we saw before, is signalized respectively by the box and the double box on both matrices. Using the same simple routine, we obtain the second diagonal descendant:

$$\mathbf{A}_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 60 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 120 & 342 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 210 & 638 & 1175 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 336 & 1066 & 2070 & 1361 & 22697 & 0 & 0 & 0 & 0 & \dots \\ 504 & 1650 & 3325 & 75841 & 265632 & 0 & 0 & 0 & 0 & \dots \\ 720 & 2414 & 5000 & 208899 & 815505 & 4240576 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}; \tag{31}$$

and so on. We see that the null rows increase from the top, one by one, as we 'descend' from the original matrix; related to that, at the same time the number of non-null entries diminishes in each row.

As an example, with a spreadsheet, we may use the matrix $\mathbf{A}_6$ to (automatically) compute C_{76} and C_{96}:

$$\begin{cases} C_{76} = \sum_{i=1}^{2} A_{76}^i \cdot F_{4-i} = 120 \cdot 9 + 342 \cdot 2 = 1764 \\ C_{96} = \sum_{i=1}^{4} A_{96}^i \cdot F_{6-i} = 336 \cdot 50 + 1066 \cdot 24 + 2070 \cdot 9 + 3135 \cdot 2 = 67284. \end{cases}$$

5. Stirling and Harmonic Numbers

Some days after obtaining most of the precedent results (this is, except all the previous section), I learned about the so-called Stirling numbers. Then I found out that the arithmetic coefficients are equivalent to the *unsigned Stirling numbers of the first kind* $\left[\begin{array}{c} n \\ k \end{array}\right]$ as follows:

$$\left[\begin{array}{c} n \\ k \end{array}\right] = C_{n;\,n-k+1} \quad \text{or} \quad C_{nk} = \left[\begin{array}{c} n \\ n-k+1 \end{array}\right] \tag{32}$$

and that one may generate a triangle formed by these numbers, using a similar recurrence relation to the one established by theorem 1, which is essentially symmetrical to the Arithmetic Triangle!

I discovered these peculiar numbers – concerning permutations according to their number of cycles – by chance, related to the *n-th harmonic number* H_n, this is, the sum of the reciprocals of the first n natural numbers:

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}.$$

And I looked for harmonic numbers precisely because they corresponds to the sequence of the reciprocals of natural numbers and I had just proved the following theorem, which I found interesting.

Theorem 2. *If H_n is the n-th harmonic number, then, for $n > 1$,*

$$H_n = \frac{C_{n+1;\,n}}{C_{n+1;\,n+1}} = \frac{C_{n+1;\,n}}{n!}. \tag{33}$$

Proof. It is, once again, a proof by induction:

1) The equation is valid for $n = 2$: $H_2 = 1 + \frac{1}{2} = \frac{3}{2} = \frac{C_{32}}{C_{33}}$.

2) Suppose $H_{n-1} = \frac{C_{n;\,n-1}}{C_{n;\,n}} = \frac{C_{n;\,n-1}}{(n-1)!}$.

3) Then,

$$H_n = H_{n-1} + \frac{1}{n} = \frac{n\, C_{n;\,n-1} + C_{nn}}{n!} = \frac{C_{n+1;\,n}}{C_{n+1;\,n+1}},$$

according to theorem 1. □

For instance, $H_5 = \frac{C_{65}}{C_{66}} = \frac{274}{120} = \frac{137}{60} = 2,2833\ldots$.

It has been proved that the harmonic series roughly approximate the natural logarithm function (Harmonic). In fact, it is quite simple to prove, using the MacLaurin-Cauchy integral test, that the series is divergent. But this means that

$$\lim_{n\to\infty} H_n = \lim_{n\to\infty} \frac{C_{n;\,n-1}}{C_{nn}} + \underbrace{\lim_{n\to\infty} \frac{1}{n}}_{0} = \infty \quad \Rightarrow \quad \lim_{n\to\infty} \frac{C_{n;\,n-1}}{C_{nn}} = \infty,$$

which is coherently obvious since the last quotient represents H_{n-1}. But one may also understand that this means that $C_{n;\,n-1}$ grows faster then C_{nn}; and this is because of the double recurrence relation:

$$\left\{\begin{array}{l} C_{n;\,n-1} = \underbrace{(n-1)\, C_{n-1;\,n-2}}_{\neq 0} + \underbrace{C_{n-1;\,n-1}}_{\neq 0}; \quad \text{and} \\ C_{nn} = \underbrace{(n-1)\, C_{n-1;\,n-1}}_{\neq 0} + \underbrace{C_{n-1;\,n}}_{0} = (n-1)\, C_{n-1;\,n-1} \end{array}\right. \quad \Rightarrow \quad \frac{C_{n;\,n-1}}{C_{n;\,n}} = \frac{C_{n-1;\,n-2}}{C_{n-1;\,n-1}} + \frac{1}{n-1}.$$

The last equation correspond to $H_{n-1} = H_{n-2} + \frac{1}{n-1}$, this is, (33) applied to H_{n-1}. But one clearly sees that there is a surplus of $\frac{1}{n-1}$ for $\frac{C_{n;\,n-1}}{C_{n;\,n}}$ in relation to $\frac{C_{n-1;\,n-2}}{C_{n-1;\,n-1}}$. This generically makes $C_{n;\,n-1}$ grow faster then C_{nn}, with $n \to \infty$. On the other hand, the growth is attenuated as n increases because the same happens with $1/n$.

6. Further Matrix Representation

If we form a lower triangular matrix of order n with the C_{ij}, $1 \le i, j \le n$,

$$\mathbf{C}_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 3 & 2 & \dots & 0 \\ 1 & \dots & \dots & \dots & \dots \\ 1 & C_{n2} & C_{n3} & \dots & C_{nn} \end{bmatrix} \tag{34}$$

then $\prod_{i=1}^{n} C_{ii} = (n-1)!^*$ is its determinant.

We may use a matrix representation for the first equation in (13), either in the 'individual' form (for each n)

$$\mathbf{P}_n = \mathbf{C}_{nn} \cdot \mathbf{A}_n \quad \text{this is,} \quad [P_n] = \begin{bmatrix} C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix} \cdot \begin{bmatrix} a^n \\ a^{n-1}d \\ \dots \\ ad^{n-1} \end{bmatrix},$$

or using the square matrix $\mathbf{C}_n$ above. For this purpose, however, we must introduce the concepts of *progressive matrix* and *matrix column product*.

1. A **progressive matrix** $\check{\mathbf{M}}_n$ on variables x and y is an upper triangular matrix of order n defined by

$$\begin{cases} \check{M}_{ij} = x^{j-i} y^{i-1} & \text{for } j \ge i\ ; \\ \check{M}_{ij} = 0 & \text{for } j < i. \end{cases} \tag{35}$$

   Explicitly:

$$\check{\mathbf{M}}_n = \begin{bmatrix} 1 & x & x^2 & x^3 & \dots & x^{n-1} \\ 0 & y & xy & x^2y & \dots & x^{n-2}y \\ 0 & 0 & y^2 & xy^2 & \dots & x^{n-3}y^2 \\ 0 & 0 & 0 & y^3 & \dots & x^{n-4}y^3 \\ 0 & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & y^{n-1} \end{bmatrix} \tag{16.a}$$

   We see that the determinant of this matrix is given by $\left|\check{\mathbf{M}}\right| = \prod_{i=0}^{n-1} y^{\,i} = y^{\frac{n_{(2)}!}{2}}$.

   Two special cases deserve attention:

   (a) For $x = 0$ and $y = 1$, $\check{\mathbf{M}}_n$ turns into the *identity matrix* of order n.

   (b) Making $x = y = 1$, we obtain $\check{\mathbf{L}}_n$:

$$\begin{cases} \check{L}_{ij} = 1 & \text{for } j \ge i\ ; \\ \check{L}_{ij} = 0 & \text{for } j < i. \end{cases}$$

2. Given two square matrices of the same order n, $\mathbf{B}$ and $\mathbf{A}$, the **column product** $\mathbf{B} \diamond \mathbf{A}$ is the $n \times 1$ matrix given by

$$(\mathbf{B} \diamond \mathbf{A})_i = \sum_{j=1}^{n} B_{ij} A_{ji}\,. \tag{36}$$

   This means that each row i of $\mathbf{B}$ doesn't multiply but the column of the same order i of $\mathbf{A}$, thus producing a column matrix.

Now, coming back to the use of the Arithmetic Triangle in the form of $\mathbf{C}_n$, we'll make $x = a$ and $y = d$; furthermore, we'll multiply the matrix $\check{\mathbf{M}}_n$ by the initial term a, in such a way that we'll write for the $n \times 1$ matrix $\mathbf{P}_n$, which elements are P_i $(1 \le i \le n)$:

$$\mathbf{P}_n = \mathbf{C}_n \diamond a \cdot \check{\mathbf{M}}_n, \tag{37}$$

keeping in mind that

$$a \cdot \check{\mathbf{M}}_{ij} = \begin{bmatrix} a^{j-i+1} \\ a^{j-i}d^1 \\ a^{j-i-1}d^2 \\ \dots \\ ad^{j-1} \\ 0 \\ \dots \\ 0 \end{bmatrix}.$$

On the other hand:

$$\mathbf{C}_n \diamond \check{\mathbf{L}}_n = [i\,!]_n \;. \tag{38}$$

As an example, let $a = 2$ and $d = 3$; well have the progression

$$2 \quad 5 \quad 8 \quad 11 \quad 14 \quad 17 \quad 20 \quad 23 \quad 26 \quad 29 \quad 32 \quad 35 \quad \dots$$

and, for limit $n = 6$,

$$\mathbf{P}_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 2 & 0 & 0 & 0 \\ 1 & 6 & 11 & 6 & 0 & 0 \\ 1 & 10 & 35 & 50 & 24 & 0 \\ 1 & 15 & 85 & 225 & 274 & 120 \end{bmatrix} \diamond \begin{bmatrix} 2 & 4 & 8 & 16 & 32 & 64 \\ 0 & 6 & 12 & 24 & 48 & 96 \\ 0 & 0 & 18 & 36 & 72 & 144 \\ 0 & 0 & 0 & 54 & 108 & 216 \\ 0 & 0 & 0 & 0 & 162 & 324 \\ 0 & 0 & 0 & 0 & 0 & 486 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \\ 80 \\ 880 \\ 12320 \\ 209440 \end{bmatrix}.$$

Remark that these concepts of *progressive matrix* and *matrix column product* are quite useful, for example, to represent Newtons binomial, $S_n = (x + y)^n$, as

$$\mathbf{S}_n = \mathbf{B}_{n+1} \diamond \check{\mathbf{M}}_{n+1},$$

where $\mathbf{B}_{n+1}$ is the $(n + 1) \times (n + 1)$ matrix of the binomial coefficients $\begin{pmatrix} n \\ k \end{pmatrix} = B_{n+1;\,k+1}$, called *Pascal matrix of order* $n + 1$; or yet to express the equality, for $n > 0$,

$$Q_n = x^n - y^n = (x - y) \sum_{i=1}^{n} x^{n-i} y^{i-1}$$

as

$$\mathbf{Q}_n = [x - y]_n \diamond \check{\mathbf{M}}_n,$$

$[x - y]_n$ being the $n \times n$ matrix which elements are all equal to $x - y$.

7. Conclusion

The study of the product of the first n terms of an arithmetic progression leads to an infinite numerical triangle – called Arithmetic Triangle – formed by certain coefficients, C_{nk} and which may be progressively built. One discovers that the C_{nk} correspond to the so-called Unsigned Stirling Numbers of the First Kind, which also generate a numerical triangle, essentially symmetrical to the one displaying the C_{nk}.

We established several proprieties and propositions concerning the Arithmetic Triangle and factorials. On the way, we generalized the concepts of factorial of order k, rising and falling factorials, and introduced the concepts of sum-factorial, descendant matrix, progressive matrix and matrix column product.

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On The Twistor Method for Treating Differential Equations

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Abstract

In this research we utilized complex structure in R^3 to construct geometrical solutions for Laplace equation, wave equation and monopole equation. The complex space used is the so called mini – twistor space and the solutions in all the above cases is given by a contour integral of a twistor function over a bundle space of one – dimensional complex projective space.

Keywords: Laplace equation, wave equation, monopole equation, the complex space, mini–twistor space, a twistor function.

1. Introduction

Twistors were introduced by Sir Roger Penrose and his associates since 1960, as a new way of describing the geometry of space-time where the ordinary space – time concepts can be translated into twistor terms. The primary geometrical object is not a point in Minkowiski space but a null straight line (a twistor) or, more generally, a twisting congruence of null lines. It turns out that twitor algebra has the same type of universality in relation to the Lorentz group. Thus, twistor theory is a applicable to quantum field theory and free fields of zero- rest- mass. It also formulates other fields such as Yang Mills fields. Recently the twistor programme has been utilized in the integrability of differential equations. It was initiated by Atiyah and Ward (Ward, R. S., 1977; Ward, R. & Tabor, M., 1985) and further extended by Nick Woodhouse, Lionel Mason, George, Sparling and others (Murray, M. K., 2002; Hitchin, N., 1982).

In this paper, we discuss the twistor space and some applications for differential equations representing the non Abielian monopole equation. The structure of this paper is as follows. In section (1) we introduced the basic concepts used in this paper, such as complex projective space $\mathrm{CP_n}$ and holomorphic line bundle. Section (2) dealt with a complex structure on R^3. In this section we defined the twistor space to be the space of oriented lines in R^3, it is infact the non- trivial tangent bundle $\mathrm{T\,S}^2$. Differential equations in R^3 in terms of twistor functions have been treated in section (3). In this section we motivated Penrose transform by introducing the solution of the wave equation by a closed contour integral of a twistor function. Similarly integrating of an appropriate twistor function along a closed contour integral delivers a solution of a harmonic equation. The closed contour on both cases is in the one – dimensional complex projective space. The last section provided a twistor solution to the monopole equation. This equation is infact shown to be the itegrability conditions for linear Lax equations that were interpreted geometrically as null 2- planes that correspond to the points of the twistor space T via the incidence relation given by equation (30) that yields two affine coordinates $(\lambda\,,\eta)$ where $\lambda = \pi_0/\pi_1$ and $\eta = \omega/\pi\hat{}2$ correspond to the homogenous coordinates $(\omega\,,\pi_0\,,\pi_1\,)$ on the twistor space T. Thus we constructed holomorphic vector bundle over the twistor space T.

2. Preliminaries

2.1 Complex Projective Space

Consider the set of all complex lines through the origin. It forms a complex differentiable manifold which is the n-dimensional complex projective space denoted by $\mathrm{CP_n}$. The complex line through z is denoted by [z](Barth, W., et al., 2015), and it is in fact

$$[\mathrm{z}] = [\mathrm{z}^0, \dots, \mathrm{z}^n] = [\lambda \mathrm{z}^0, \dots, \lambda \mathrm{z}^n]\ ,\ \ \lambda \in \mathrm{C} - \{0\} \quad (1)$$

The numbers $(\mathrm{z}^0, \dots, \mathrm{z}^n)$ are called the homogeneous co-ordinates of the line. It can be shown that $\mathrm{CP_n}$ is a complex differentiable manifold.

2.2 Holomorphic Line Bundle (Jacob, A. & Yau, S.-T., 2014)

A holomorphic line bundle is defined by a triple (M, L, π) such that $\pi : L \rightarrow M$ satisfies the following properties: (i) $\pi^{-1}(m) = L_m$ is called a fiber over the base manifold. It is one-dimensional complex vector space

(ii) M is covered with open sets U_α, such that there exist a bi-holomorphic maps $\psi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times C$ and $\psi_{\alpha|L_m}$ is a linear isomorphism

If L is holomorphic bundle then we define the holomorphic section of L as a holomorphic map:

$$\psi : M \rightarrow L \text{ with } \psi(m) \in L_m \tag{2}$$

for $m \in U_\alpha \cap U_\beta$ and $\psi_\alpha(m)$, $\psi_\beta(m)$ in L_m .

There are holomorphic maps

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow C - \{0\} \tag{3}$$

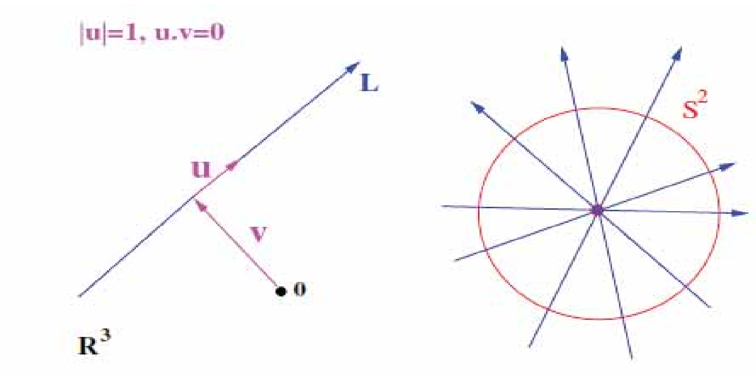
Called the transition functions such that

$$\psi_\beta = g_{\alpha\beta}\psi_\alpha \quad \text{on} \quad U_\alpha \cap U_\beta. \tag{4}$$

3. Complex Structure on R^3

The three-dimensional Euclidean space R^3 may be represented as a two-dimensional complex manifold which in fact interpreted as a simple twistor space. To see this, consider the space of all oriented lines L in R^3 of the form $L = v + su$ where $s \in R$, u is a unit vector in the direction of L and v is orthogonal to u. Then let

$$T = \{(u, v) \in s^2 \times R^3, u \cdot v = 0\} \tag{5}$$



It is a four –dimensional space which may be regarded as TS^2(Glover, R. & Sawon, J., 2014).

Reversing the orientation of lines induces a map $\tau : T \rightarrow T$ given by $\tau(u, v) = (-u, v)$. The points $p = (x, y, z)$ in R^3 correspond to two spheres in T given by τ -invariant maps

$$u \rightarrow (u, v(u) = p - (p \cdot u)u) \in T \tag{6}$$

which are sections of the projection $T \rightarrow S^2$.

4. Differential Equations and Twistor Functions

On an open set $U \subset T$ not containing the point $(0, 0, 1)$ define a local holomorphic coordinates by

$$\lambda = \frac{u_1 + iu_2}{1 - u_3} \in CP_1 = S^2, \; \eta = \frac{v_1 + iv_2}{1 - u_3} + \frac{u_1 + iu_2}{(1 - u_3)^2} v_3 \tag{7}$$

the corresponding complex coordinates $(\tilde{\lambda}, \tilde{\eta})$ in $\tilde{U}$ containing $(0, 0, 1)$ may also be defined On the overlap region

$$\tilde{\lambda} = 1/\lambda,$$

$$\tilde{\eta} = -\eta/\lambda^2 \tag{8}$$

Then

$$\tau(\lambda,\ \eta) = (-\frac{1}{\lambda}, -\frac{\bar{\eta}}{\bar{\lambda}^2}). \tag{9}$$

From equation (6) we get the τ -invariant holomorphic map

$$\lambda \rightarrow (\lambda,\ \eta = (x + iy) + 2\lambda z - \lambda^2(x - iy)). \tag{10}$$

This is map $CP^1 \rightarrow T\,CP^1$(Dunajski, M., 2009). For real valued function f on R^3, and an oriented line L in R^3 We define $\phi(L)$ as

$$\phi(L) = \int_L f \tag{11}$$

Equivalently

$$\phi(\alpha_1, \alpha_2, \beta_1, \beta_2) = \int_{-\infty}^{\infty} f(\alpha_1 s + \beta_1, \alpha_2 s + \beta_2, s)ds \tag{12}$$

so we have

$$\frac{\partial^2 \emptyset}{\partial\alpha_1 \partial\beta_2} - \frac{\partial^2 \emptyset}{\partial\alpha_2 \partial\beta_1} = 0 \tag{13}$$

We see that smooth solutions to above equation arise from some function on R^3. In twistor theory a twistor function yields solution to a differential equation on space-time. After the change of coordinates

$$\alpha_1 = x + y,$$

$$\alpha_2 = t + z,$$

$$\beta_1 = t - z,$$

$$\beta_2 = x - y \tag{14}$$

Produce the wave equation.

4.1 Penrose Transforms

The following formula for solutions to the wave equation in Minkowski space was provided by penrose

$$\phi(x, y, z, t) = \oint_{\Gamma\subset CP^1} f((z + t) + (x + iy)\lambda, (x - iy) - (z - t)\lambda, \lambda)\, d\lambda \tag{15}$$

Here $\Gamma \subset CP^1$ is a closed contour and the function f is holomorphic on CP^1 except some number of poles. Differentiating the RHS verifies that

$$\frac{\partial^2 \emptyset}{\partial t^2} - \frac{\partial^2 \emptyset}{\partial x^2} - \frac{\partial^2 \emptyset}{\partial y^2} - \frac{\partial^2 \emptyset}{\partial z^2} = 0. \tag{16}$$

One could modify a contour and add a holomorphic function inside the contour to f without changing the solution $\emptyset$. The proper description uses sheaf cohomology which considers equivalence classes of functions and contours(Baston, R. J. & M. G., 2015).

4.2 Harmonic Functions(Karp, L., 2016)

To find a harmonic function at $P = (x, y, z)$, restrict a twistor function f (λ, η) defined on $U \cap \widetilde{U}$ to a line $\breve{P} = CP^1 = S^2$ and Integrate along a closed contour integral we have

$$\emptyset(x, y, z) = \oint_{\Gamma\subset\breve{P}} f(\lambda, (x + iy) + 2\lambda z - \lambda^2(x - iy))d\lambda \tag{17}$$

Then Differentiate under the integral to verify

$$\frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2} + \frac{\partial^2 \emptyset}{\partial z^2} = 0 \tag{18}$$

4. 3 Abelian Monopole Equation(Atiyah, M. F. & Hitchin, N., 2014)

We can now consider the A belian monopole equation a function $\emptyset$ and a magnetic potential $A = (A_1, A_2, A_3)$ of the form

$$\nabla\emptyset = \nabla \wedge A \tag{19}$$

This a first order linear equation that is related to our above construction of the twistor contour integral

Geometrically, the one-form $A = A_j dx^j$ is a connection on a U(1)principal bundle over R^3, and $\emptyset$ is a section of the adjoin bundle. Taking the curl of both sides of this equation implies that $\emptyset$ is harmonic, and conversely given a harmonic function $\emptyset$ locally one can always find a one-form A such that the Abelian monopole equation holds.

4.4 Non-abelian Monopoles and Hitchin Correspondence(Shibata, A., et al., 2015)

We can generalize equation (19) using a non Abelian lie group such as SU(n). The generalized equations in R^3 results if we consider the anti- Hermitian $n \times n$ matrices $(A_j, \emptyset)$. The generalized non-abelian monopole equation is given by

$$\frac{\partial\emptyset}{\partial x^j} + [A_j, \emptyset] = \frac{1}{2}\varepsilon_{jkl}F_{kl} \tag{20}$$

where F_{kl} is the non-abelian magnetic field

$$F_{kl} = \frac{\partial A_l}{\partial x^k} - \frac{\partial A_k}{\partial x^l} + [A_k, A_l], \quad k, l = 1,2,3 \tag{21}$$

The pair $(A, \emptyset)$ transform as

$$A \to gAg^{-1} - dgg^{-1},$$

$$\emptyset \to g\emptyset g^{-1}$$

for

$$g = g(x, y, t) \in SU(n) \tag{22}$$

5. Twistor Solution to the Monopole Equation

A brief description of the twistor solution to the monopole equation goes as follows(Shibata, A., et al., 2015):

For the potentials $(A_j(X), \emptyset(X))$ we solve the matrix ODE along each line $x(s) = v + su$

$$\frac{dV}{ds} + (u^j A_j + i\emptyset)V = 0 \tag{23}$$

The space of solutions at $p \in R^3$ is a complex vector space C^n, thus giving rise to a complex vector bundle over T with patching matrix $(\lambda, \bar{\lambda}, \eta, \bar{\eta}) \in GL(n, C)$.

The monopole equation (20) on R^3 holds if and only if this vector bundle is holomorphic, i.e. the Cauchy–Riemann equations

$$\frac{\partial F}{\partial \bar{\lambda},} = 0,$$

$$\frac{\partial F}{\partial \bar{\eta}} = 0 \tag{24}$$

hold.

We now introduce a metric and a volume forms on $R^{2,1}$

$$h = dx^2 - 4dudv,$$

$$vol = du \wedge dx \wedge dv \tag{25}$$

where the coordinates $(x, u, v)$are real

With $D_\mu = \partial_\mu + A_\mu$we define $D_\emptyset = d\emptyset + [A, \emptyset]$. The monopole equations become

$$D_x\emptyset = \frac{1}{2} F_{uv}$$

$$D_u\emptyset = F_{ux},$$

$$D_v\emptyset = F_{xv} \quad (26)$$

Where $F_{\mu v} = [D_\mu , D_v]$

We notice that the above equations are the integrability conditions for an overdetermined system of linear Lax equations

$$L_0\Psi = 0, \; L_1\Psi = 0$$

where

$$L_0 = D_u - \lambda(D_x + \emptyset), \quad L_1 = D_x - \emptyset - \lambda D_v \quad (27)$$

And $\Psi = \Psi(x , u , v , \lambda)$ takes values in $GL(n, C)$

For $G = U(n)$, equation (27) provide a gauge $A_v = 0$, and $A_x = -\phi$, with matrix $J : R^{2,1} \to U(n)$ such that

$$A_u = J^{-1}\partial_u J,$$

$$A_x = -\emptyset = \frac{1}{2} J^{-1}\partial_x J \quad (28)$$

The above gauge and (26) yield the integrable chiral model

$$\partial_v(J^{-1}\partial_u J) - \partial_x(J^{-1}\partial_x J) = 0 \quad (29)$$

The Lax representation (27) can be interpreted geometrically: given a pair of real numbers (η, λ) the plane

$$\eta = v + x\lambda + u\lambda^2 \quad (30)$$

is null with respect to the Minkowski metric on $R^{2,1}$, infact all null planes are of this form with $\lambda = \infty$.

We see that M is the two-dimensional complex twistor space $T = TCP^1$ in which points of T are the 2-planes in M via the incidence relation

$$x^{AB}\pi_A\pi_B = \omega \quad (31)$$

Here (ω , π_0 , π_1) are homogeneous coordinates on T as $(\omega, \pi_A) \sim (c^2 w, c\pi_A)$, where $c \in C^*$. In the affine coordinates $\lambda := \frac{\pi 0}{\pi 1}$, $\eta := \omega/(\pi)^2$ equation (31) gives (30).

The homogeneous coordinates are denoted by $\pi_A = (\pi_0 , \pi_1)$, and the two-set covering of CP^1 lifts to a covering of the twistor space T

$$U = \{(w, \pi_A), \pi_1 \neq 0\} ,$$

$$\widetilde{U} = \{(w, \pi_A), \pi_0 \neq 0\} \quad (32)$$

The functions $\lambda = \pi_0/\pi_1$, $\tilde{\lambda} = 1/\lambda$ are the inhomogeneous coordinates in U and $\widetilde{U}$, respectively. It then follows that $\lambda = -\pi^1/\pi^0$

Conversely for a holomorphic vector bundle we can construct a monopole. The construction is as in the following theorm.

5.1 Theorem

There exists a one-to-one correspondence between the gauge equivalence classes of complex solutions to (26) in the complexified Minkowski space M with the gauge group $GL(n, C)$ and holomorphic rank n vector bundles E over the twistor space T which are trivial on the holomorphic sections of $TCP^1 \to CP^1$(*Dunajski, M.*, 2009).

Proof

We first outline how a holomorphic rank n vector bundle with connection $(A, \emptyset)$ can be constructed. Have $(A, \emptyset)$ is a solution to (26). Integrating the pair of linear PDEs $L_0 V = L_1 V = 0$, where L_0, L_1 are given by (27), we get an n-dimensional vector space to each null plane Z in a complexified Minkowski space. The null plane Z corresponds to a point in T which is a fiber of a holomorphic vector bundle μ: E → T. The fibres of $E \mid L_p$ at Z_0, Z_1 can be identified and therefore E is trivial on each section.

Conversely if E is a vector bundle over T which is trivial on each sectional $L_p \cong CP_1$ we can utilize Birkhoff–Grothendieck theorem to get

$$E \mid L_p = O \oplus O \dots \dots \dots \dots \dots \oplus O \tag{33}$$

where the space of sections restricted to L_p is C^n. Let us now construct a pair (A, ∅) on this bundle that satisfies (26). First cover the twistor space with two open sets U and $\widehat{U}$ so that we have local trivializations

$$\aleph : \mu^{-1}(U) \to U \times C^n,$$

$$\widetilde{\aleph} : \mu^{-1}(\widetilde{U}) \to \widetilde{U} \times C^n \tag{34}$$

The holomorphic patching is simply $F = \widetilde{\aleph} \circ \aleph : C^n \to C^n$ on $U \cap \widehat{U}$. F can be split:

$$F = \widetilde{H} H^{-1}, \tag{35}$$

Then $\delta_A F = 0$ implies that

$$H^{-1}\delta_A H = \widetilde{H}^{-1}\delta_A \widetilde{H} = \pi^B \Phi_{AB} \tag{36}$$

Since both sides of the above equation are homogenous of degree 1 in π^A and holomorphic around $\lambda = 0$ and $\lambda = \infty$, respectively we see that the decomposing of Φ_{AB} as

$$\Phi_{AB} = \Phi_{(AB)} + \varepsilon_{AB}\emptyset \tag{37}$$

gives a one-form $\Phi_{AB}dx^{AB}$ and a scalar field $\phi = (1/2)\varepsilon^{AB}\Phi_{AB}$ on the complexified Minkowski space, i.e.

$$\Phi_{AB} = \begin{pmatrix} A_u & A_x + \emptyset \\ A_x - \emptyset & A_v \end{pmatrix} \tag{38}$$

The Lax pair (27) becomes

$$\mathrm{L_A} = \delta_\mathrm{A} + \mathrm{H}^{-1}\delta_\mathrm{A} \tag{39}$$

Where $\delta_A = \pi^B \partial_{AB}$, so that

$$\mathrm{L_A}(\mathrm{H}^{-1}) = -\mathrm{H}^{-1}(\delta_\mathrm{A}\mathrm{H})\mathrm{H}^{-1} + \mathrm{H}^{-1}(\delta_\mathrm{A}\mathrm{H})\mathrm{H}^{-1} = 0 \tag{40}$$

and $\Psi = H^{-1}$ is a solution to the Lax equations regular around $\lambda = 0$. Let us show explicitly that (26) holds. Differentiating (36) with respect to δ_A yields

$$\delta^A(\mathrm{H}^{-1}\delta_\mathrm{A}\mathrm{H}) = -(\mathrm{H}^{-1}\delta^A H)(\mathrm{H}^{-1}\delta_\mathrm{A}\mathrm{H}) \tag{41}$$

which holds for all π^A if

$$D_A(C\Phi_B^A) = 0 \tag{42}$$

Where $D_{AC} = \partial_{AC} + \Phi_{AC}$. Equation (42) is the Yang Mills spinor form equation.

The vector bundle E need to be compatible with (10) and therefore $\det F = 1$, this a mounts to Euclidean reality conditions for non abelian monopole. We shoud also have

$$F^*(Z) = F(\tau(Z)) \tag{43}$$

Where $Z \in T$ and * denotes the Hermitian conjugation.

To determent the Lorentzian reality conditions, the bundle must be invariant under the involution (41). Below we shall demonstrate how the gauge choices leading to the integrable chiral model (29) can be made at the twistor level.

Let

$$h = H(x^\mu, \pi^A = O^A) ,$$

$$\tilde{h} = \widetilde{H}(x^\mu, \pi^A = l^A) \tag{44}$$

So that

$$\Phi_{A0} = h^{-1}\,\partial_{A0}\,h \ ,$$
$$\Phi_{A1} = \tilde{h}^{-1}\,\partial_{A1}\,\tilde{h} \tag{45}$$

H is defined up to a multiple by an inverse of a non-singular matrix $g = g(x^{\mu})$ independent of π^{A}

$$H = Hg^{-1} \ ,$$
$$\widetilde{H} = \widetilde{H}g^{-1} \tag{46}$$

We choose g such that $\widetilde{h} = 1$ so

$$\Phi_{A1} = l^{A}\,\Phi_{AB} = 0 \tag{47}$$

And

$$\Phi_{AB} = -l_{B}O^{C}h^{-1}\,\partial_{AC}\,h \tag{48}$$

i.e.

$$A_{x} + \emptyset = A_{v} = 0 \tag{49}$$

giving the Ward gauge with $J(x^{\mu}) = h$. With respect to this gauge, the system (42) becomes

$$\partial_{1}^{A}\Phi_{A0} = 0 \tag{50}$$

which is (29). The solution is given by

$$J\,(x^{\mu}) = \Psi^{-1}\,(x^{\mu}, \lambda = 0) \tag{51}$$

Where $\Psi = H^{-1}$ is a solution of the Lax pair.

Setting $\mathrm{F} = \exp(\mathrm{f})$ for some f, the nonlinear splitting (35) reduces to the additive splitting of f. This can be done using Cauchy integral formula, taking Γ as a real contour in a rational curve $w = x^{AB}\pi_{A}\,\pi_{B}$. The Higgs field satisfies the wave equation and given by

$$\phi = \int_{\Gamma}\frac{\partial f}{\partial w}\,\rho\,.\,d\rho \tag{52}$$

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Over Absolute Valued Algebras with Central Element not Necassary Idempotent

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Abstract

We study the absolute valued algebras containing a central element non necessary idempotent. We determine the absolute valued algebras containing a central element if we add some requirements. Also we gives a classification of finite-dimensional absolute valued algebras containing a generalized left unit and central element.

Keywords: Absolute valued algebra, central element, left unit and generalized left unit.

Mathematics Subject Classification: 17A35, 17A36

1. Introduction

The absolute valued algebras are introduced by Ostrowski in 1918. It's the normed algebra A such that $\|xy\| = \|x\|\|y\|$ for all x, y in A. An algebra is called division if and only if R_x and L_x are bijective for all x in A. The category of finite-dimensional absolute valued algebra is a full subcategory of the category of division algebra. If A is a finite dimensional absolute valued algebra, then A has dimension 1, 2, 4 or 8 (Bott, et al., 1958; Kervaire, 1958), A is isotopic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ and the norm of A comes from an inner product(Albert, 1947). We have in (Beslimane & Moutassim, 2011; Diankha, et al., 2013) a classification of absolute valued algebras with left unit and containing a central element. The norm of absolute valued algebra containing a central idempotent c, comes from to an inner product and the isometric map $x \mapsto x^\star := 2(x|c)c - x$ is an involution (El-Mallah, 1990). For $\|u\| = 1$, we recall the following notations $\mathbb{H}_u := \mathbb{H}_{T_{u,\overline{u}}}$, and $\mathbb{O}_u := \mathbb{O}_{T_{u,\overline{u}}}$. Let $a, b \in \mathbb{H}$ such that $\|a\| = \|b\| = 1$, we recall that $\mathbb{H}(a, b) := (\mathbb{H}, \star_1)$, with $x \star_1 y = axyb$ and ${}^\star\mathbb{H}(a, b) := (\mathbb{H}, \star_2)$, with $x \star_2 y = \overline{x}ayb$ (Ramirez, 1999). Let A be an algebra, we note that $Z(A) = \{a \in A : ax = xa \ for \ all \ x \in A\}$. In this work we give a characterization of finite dimensional absolute valued algebra containing a central element. We determine the finite-dimensional absolute algebra containing a genaralized left unit and central element. We classify the absolute valued algebra containing a central element if we add some conditions.

2. Preliminary

Let f, g, f', g' be linear isometries of euclidean space $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ fixing 1, and let $\Phi : \mathbb{A} \to \mathbb{A}$ be a linear mapping. Then it is easy to see that $\Phi : \mathbb{A}_{f,g} \to \mathbb{A}_{f',g'}$ is an algebra isomorphism fixing 1 if and only if $\Phi : \mathbb{A} \to \mathbb{A}$ is an algebra automorphism and $(f', g') = (\Phi \circ f \circ \Phi^{-1}, \Phi \circ g \circ \Phi^{-1})$ (Calderon, et al., 2011).

Let $\mathbb{A}$ be one of the unital absolute valued algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ of dimension m. Consider the caley dickson product $\odot$ in $\mathbb{A} \times \mathbb{A}$, we define on the space $\mathbb{A} \times \mathbb{A}$ the product

$$(x, y) \star (x', y') = (f_1(x), f(x)) \odot (g_1(x'), g(y')).$$

With f_1, g_1, f, g be linear isometries of $\mathbb{A}$ and $f_1(1) = g_1(1) = 1$. We obtain a $2m$-dimensional absolute valued real algebra $\mathbb{A} \times \mathbb{A}_{(f_1,f),(g_1,g)}$. The process is called the duplication process (Calderon, & et al., 2011). Note that the algebra is left unit if $g_1 = g = I_{\mathbb{A}}$ and this case we note the algebra by $\mathbb{A} \times \mathbb{A}_{(f_1,f)}$ (Rochdi, 2003).

Theorem 1 *The finite-dimensional absolute valued real algebras with left unit are precisely those of the form $\mathbb{A}_\varphi$, where $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ and φ is an isometric of the euclidien espace $\mathbb{A}$ fixed 1, and $\mathbb{A}_\varphi$ denotes the absolute valued real algebra obtained by endowing the normed space of $\mathbb{A}$ with the product $x \odot y := \varphi(x)y$. Moreover, given linear isometries $\varphi, \phi : \mathbb{A} \to \mathbb{A}$ fixing 1, the algebras $\mathbb{A}_\varphi$ and $\mathbb{A}_\phi$ are isomorphic if and only if there exists an algebra automorphism ψ of $\mathbb{A}$ satisfying $\phi = \psi \circ \varphi \circ \psi^{-1}$* ((Rochdi, 2003)).

3. Finite Dimensional Absolute Valued Algebra Containing a Central Element

An element c in A is called central if $L_c = R_c$. In this paragraph, the central element is non necessary idempotent. As A isalternative, Artin's theorem (Schafer, 1996) shows that for any $x, y \in A$, the set $\{x, y, \overline{x}, \overline{y}\}$ is contained in an associative subalgebra of A.

Theorem 2 *Let A be an finite dimensional absolute valued algebra with nonzero central element c. Then A is precisely $\mathbb{R}$, $\mathbb{C}$, $\overset{\star}{\mathbb{C}}$ or of the form $\mathbb{A}_{\varphi,\psi}$, with $\mathbb{A} = \{\mathbb{H}, \mathbb{O}\}$, φ a linear isometry of the euclidien space $\mathbb{A}$ fixing 1 and $\psi = L_{\overline{\varphi(c)}} \circ R_{\varphi(c)} \circ \varphi$. Moreover for $dim(A) \geq 4$, if $\psi = I_A$, then A is isomorphic to $\mathbb{H}(c, 1)$ or $\mathbb{O}_c$.*

Proof. If $dim(A) \leq 2$, the result is clear. Assume now $dim(A) \geq 4$. Then the algebra A is of the form $\mathbb{A}_{\varphi,\psi}$, where ψ, φ are the linear isometries of the euclidien space $\mathbb{A} \in \{\mathbb{H}, \mathbb{O}\}$ such that $\psi(1) = \varphi(1) = 1$ (Calderon, at al., 2011). Using now $x \odot c = c \odot x$, for all x in $\mathbb{A} \Leftrightarrow \varphi(x)\psi(c) = \varphi(c)\psi(x)$, for all x in $\mathbb{A}$.
For $x = 1$, we have $\psi(c) = \varphi(c)$.

$$\begin{aligned}
\varphi(x)\psi(c) = \varphi(c)\psi(x),\ for\ all\ x\ in\ \mathbb{A} \quad &\Rightarrow \quad \varphi(x)\varphi(c) = \varphi(c)\psi(x),\ for\ all\ x\ in\ \mathbb{A} \\
&\Rightarrow \quad \psi(x) = \overline{\varphi(c)}\varphi(x)\varphi(c),\ for\ all\ x\ in\ \mathbb{A} \\
&\Rightarrow \quad \psi(x) = (L_{\overline{\varphi(c)}} \circ R_{\varphi(c)} \circ \varphi)(x),\ for\ all\ x\ in\ \mathbb{A} \\
&\Rightarrow \quad \psi = L_{\overline{\varphi(c)}} \circ R_{\varphi(c)} \circ \varphi.
\end{aligned}$$

Moreover if $\psi = I_A$, then A is left unit and $\varphi = L_c \circ R_{\overline{c}}$ (Diankha, et al., 2013). For the algebra $\mathbb{H}_c$, we have the following isomorphism of algebra $\Phi : \mathbb{H}(c, 1) \to \mathbb{H}_c \ \ x \mapsto xc$.∎

Theorem 3 *Let A be an finite-dimensional absolute valued algebra containing a central idempotent c. Then $c \in \{1\} \cup \{-\frac{1}{2} + u\text{: } u \in Im(\mathbb{A})$ and $\|u\| = \frac{\sqrt{3}}{2}\}$.*

Proof. Using Theorem 3.3., A is of the form $\mathbb{A}_{\varphi,\psi}$, where φ is a linear isometric of $\mathbb{A}$ fixing 1 and $\psi = L_{\overline{\varphi(c)}} \circ R_{\varphi(c)} \circ \varphi$. We remark also $\varphi(c) = \psi(c)$, hence $c \odot c = \varphi(c)\psi(c) = \varphi(c)^2$. Assume now $c = \alpha + u \in S(\mathbb{A})$ (with $\mathbb{A} = \mathbb{R} \oplus Im(\mathbb{A})$: Frobenius decomposition). We note that if $u \in 1^{\perp} = Im(\mathbb{A})$, $< 1, \varphi(u) >=< \varphi(1), \varphi(u) >=< 1, u >= 0$. Hence we have $\varphi^n(1^{\perp}) \subseteq 1^{\perp}$, with $n \in \mathbb{N}$ and $\varphi(1^{\perp})^n \in \mathbb{R}$ if and only if $n \in 2\mathbb{N}$.

Hence $c \odot c = c$ and $\|c\| = 1$ are equivalent to $\left\{\begin{array}{rcl} \alpha^2 + \varphi(u)^2 = & \alpha & (1) \\ 2\alpha\varphi(u) = & u & (2) \\ \|\varphi(c)\| = & 1 & (3) \end{array}\right.$

The assertions (2) and (3) imply that $\frac{1}{4\alpha^2}u^2 = \alpha^2 - 1$ (4). Otherwise the assertions (1) and (2) implies that $\alpha^2 + (\frac{1}{2\alpha}u)^2 = \alpha$ (5). The equality between (4) and (5) gives $\alpha = 1$ or $-\frac{1}{2}$.
If $\alpha = 1$, this is equivalent to $c = 1$.
Assume now $\alpha = -\frac{1}{2}$, hence $c = -\frac{1}{2} + u$. Then

$$\begin{aligned}
\|c\|^2 &= < -\frac{1}{2} + u, -\frac{1}{2} + u > \\
&= < -\frac{1}{2}, -\frac{1}{2} > + < u, u > \\
&= \frac{1}{4} + \|u\|^2 \\
&= 1
\end{aligned}$$

This implies that $\|u\| = \frac{\sqrt{3}}{2}$.∎

Lemma 1 *Let A be an absolute valued algebras containing a nonzero central element c. The following assertions are equivalent:*

1. *$x^2c = x^2$, for all $x \in A$*
2. *A is finite dimensional and is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.*

Proof. 2) $\Rightarrow$ 1) is clear.
Now assume 1), Using the equality $(x+c)^2c = (x+c)^2$ for all x in A, we have $(xc-x)c = 0$ for all x in A. Then $L_c = R_c = I_A$ and A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ (Urbanik & Wright, 1960).

The group G_2 acts transitively on the sphere $S(Im(\mathbb{O})) := S^6$, that is the mapping $G_2 \rightarrow S^6\ \Phi \mapsto \Phi(i)$ is surjective (Postnikov, 1985).

Definition 1 *An element $e \in A$ is called strongly left unit, if it's left unit and square root of right unit: $L_e = R_e^2 = I_A$* (Diouf, 2017).

Theorem 4 *Let A be an absolute valued algebra with strongly left unit and containing a central element c. Then $c \in S(\mathbb{R}) \cup 1^{\perp}$ and A is finite dimensional and isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{H}(i,1)$, $\mathbb{O}$ or $\mathbb{O}_i$.*

Proof. It's clear that A is of finite dimensional. If $dim(A) \leq 2$, the result is clear that A is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. Assume now $dim(A) \geq 4$ and A contains a central element c (Diankha, et al., 2013) proves that A is of the form $\mathbb{A}_{\varphi}$, where $\mathbb{A} \in \{\mathbb{H}, \mathbb{O}\}$, $c \in S(\mathbb{A})$ and $\varphi = L_c \circ R_{\overline{c}}$.
Otherwise we have $R_e^2 = I_A \Leftrightarrow (x \odot 1) \odot 1 = x$. Hence

$$\begin{aligned} x &= (x \odot 1) \odot 1 \\ &= c(cx\overline{c})\overline{c} \\ &= c^2 x \overline{c}^2 \ Artin's\ theorem \end{aligned}$$

The equality $c^2 x = xc^2$, implies that $c^2 \in S(\mathbb{R}) = \{-1, 1\}$.
If $c^2 = 1$, then $c = \pm 1$ and A is isomorphic to $\mathbb{H}$ or $\mathbb{O}$.
If $c^2 = -1$, then $c \in S(Im(\mathbb{A}))$.
There exists $u \in S(Im(\mathbb{A}))$ such that $uc\overline{u} = i$ and let the automorphism $\Phi := T_{u,\overline{u}}$ of $\mathbb{A} = \{\mathbb{H}, \mathbb{O}\}$, with $\Phi^{-1} = T_{\overline{u},u}$. We have

$$\begin{aligned} \Phi \circ T_{c,\overline{c}} \circ \Phi^{-1} &= T_{u,\overline{u}} \circ T_{c,\overline{c}} \circ T_{\overline{u},u} \\ &= T_{uc\overline{u}, \overline{uc\overline{u}}} \\ &= T_{i,\overline{i}} \end{aligned}$$

Then $\mathbb{A}_{T_{c,\overline{c}}}$ and $\mathbb{A}_{T_{i,\overline{i}}}$ are isomorphic (Theorem 2.1) and (Diouf, 2017), we have $\mathbb{H}_{T_{i,\overline{i}}}$ is isomorphic to $\mathbb{H}(i,1)$.
I's clear that if $dim(A) \geq 2$, theirs algebras can be obtained by using the duplication process.

Corollary 1 *Let A be an absolute valued algebra containing two elements e and c. The following assertions are equivalent:*

1. *e is left unit and c central orthogonal to e,*
2. *A is isomorphic to $\mathbb{C}$, $\mathbb{H}_i$ or $\mathbb{O}_i$.*∎

Definition 2 *An element e is called generalyzed left unit if it satisfies to $[L_e, L_x] = 0$, for all x in A* (Chandid & Rochdi, 2008).

We give a generalisation of the papier (Diankha, et al., 2013).

Theorem 5 *Let A be an finite dimensional absolute valued algebra contains generalized left unit e and central element c. Then A is precisely $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}(a,b)$ or $\mathbb{O}_c$.*

Proof. If $dim(A) \in \{1, 2, 8\}$, then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{O}_c$ ((Diankha & all, 2013), (Chandid & Rochdi, 2008)). The algebras $\mathbb{H}(a,b)$ and $^{\star}\mathbb{H}(a,b)$ are the unique four-dimensional absolute valued algebras containing a generalized left unit (Diouf, 2014). Without loss of generality, assume that $\|c\| = 1$.
For the algebra $\mathbb{H}(a,b)$,

$$\begin{aligned} c\ is\ central &\Leftrightarrow x \star_1 c = c \star_1 x,\ for\ all\ x\ in\ \mathbb{H} \\ &\Leftrightarrow axcb = acxb,\ for\ all\ x\ in\ \mathbb{H} \\ &\Leftrightarrow xc = cx,\ for\ all\ x\ in\ \mathbb{H} \\ &\Leftrightarrow c \in Z(\mathbb{H}) \cap S(\mathbb{R}) = \{-1, 1\}. \end{aligned}$$

Then the algebra $\mathbb{H}(a,b)$ contains a central element.
For the algebra $^{\star}\mathbb{H}(a,b)$,

$$\begin{aligned} c\ is\ central &\Leftrightarrow x \star_2 c = c \star_2 x,\ for\ all\ x\ in\ \mathbb{H} \\ &\Leftrightarrow \overline{x}acb = \overline{c}axb,\ for\ all\ x\ in\ \mathbb{H} \\ &\Leftrightarrow \overline{x}ac = \overline{c}ax\ (*),\ for\ all\ x\ in\ \mathbb{H} \end{aligned}$$

For $x = 1$, we have $ac = \overline{c}a$ and $(*)$ imply $\overline{x}ac = acx$ $(**)$, $for\ all\ x\ in\ \mathbb{H}$.
New put $x = ac$, we have $(ac)^2 = \|ac\|^2 = \|a\|^2\|c\|^2 = 1$. Hence $ac = \pm 1$ and $(**)$ imply $\overline{x} = x$, for all x in $\mathbb{H}$, which is absurd. Then the algebra $^\star\mathbb{H}(a, b)$ does not contain a central element.**.**

Proposition 1 *Let A be an absolute valued algebra containing a generalized left unit e and a central idempotent element c such that* $e \in c^\perp$. *Then* $A(e, c)$ *is finite dimensional and isomorphic to* $\mathbb{C}$.

Proof. The norm $\|.\|$ of A comes from an inner product and $x \mapsto x^\star := 2 < c, x > c - x$ is an involution (El-Mallah, 1990). Without loss of generality, assume that $\|e\| = 1$. We have $ce = ec = ec^2 = c(ec)$. This implies $ec = ce = e$. The element e is orthogonal to c, then $e^2 = -\|e\|^2 c = -c$ (El-Mallah, 1990).

Problem 1 Let A be an absolute valued algebra containing a generalized left unit e and a central element c. Is A a finite dimensional? This problem is solved partialy if e is idempotent (Calderon, et al., Preprint).**.**

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A Pilot Included Column Mean Vanishing Matrix

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Abstract

In this paper, we study a special matrix used in OFDM technology including the pilot vector. This is based on the property of 'column mean vanishing' and orthogonal columns. We study the spectral decomposition. Using this, we suggest a new method of generating such matrices. Numerical examples are included.

Keywords: pilot, orthogonal columns, column mean vanishing property, SVD

1. Introduction

Recently, there have arisen a large necessity of developing a new technology in wireless communications. An OFDM or its generalization is a big trend (O. Edfors, et al., 1998; Frederiksen, F. B. & Prasad, R., 2002; Myungsup, K. & Kwak, D. Y.). In this paper, we review the algorithm developed in (Myungsup, K. & Kwak, D. Y.) and study some properties of the OFDM matrix. Based on this we propose a simple method to generate the matrix. In the resulting matrix, we see the pilot column has only two nonzero entries which correspond to zero rows, so that the pilot does not interfere with other data.

Definition 1.1. *We say a matrix A has a column mean vanishing (CMV) property if the sum of each column is zero.*

2. Generation of CMV Matrix Having Orthonormal Columns

Let $L = n \gg N$. Recall the scheme introduced in (Myungsup, K. & Kwak, D. Y.):

Algorithm Orth-pilot

1. Given a $N \times (N-2)$ initial matrix K_p with orthonormal columns.
2. Multiply by $L \times N$ matrix $\mathbf{P}$ obtaining $A = \mathbf{P}K_p$.
3. Perform IFFT to obtain $\mathbf{F}^{-1}(\mathbf{P}K_p)$.
4. Subtract the first row from all the rows, the result is $\mathbf{\Phi} \circ \mathbf{F}^{-1}(\mathbf{P}K_p)$.
5. Perform FFT to get $\mathbf{F} \circ \mathbf{\Phi} \circ \mathbf{F}^{-1}(\mathbf{P}K_p)$.
6. Multiply $\mathbf{P}^T$ to obtain $\hat{K} := \mathbf{P}^T \circ \mathbf{F} \circ \mathbf{\Phi} \circ \mathbf{F}^{-1}(\mathbf{P}K_p)$.
7. Let $G = UV^H$ where $U\Sigma V^H$ is the SVD of $\hat{K}$.

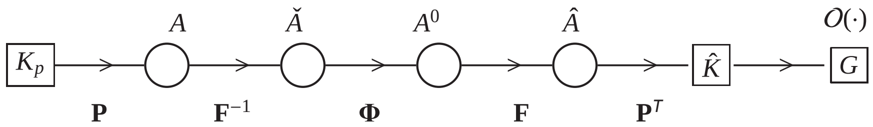

Figure 1. Signal flow diagram for matrix generation. $O(\cdot)$ is orthonormalization operator.

Now we will explain more details of the algorithm:

Step (2). Permute and Pad Zeros

Assume $N = 2m + 1$ is odd. Set $M = N - 2,\ L = n \gg N$. Starting from an $N \times M$ initial matrix, we construct an $L \times M$ matrix as follows: Move the last $m + 1$ rows of K_p to the first $m + 1$ rows of K_p. Next fill it with pad with $L - M$ zero rows (called zero padding). This process can be expressed as $\mathbf{P}K_p$ where

$$\mathbf{P} = \left[\begin{array}{c|c} \mathbf{0}_{(m+1)\times m} & \mathbf{I}_{m+1} \\ \hline 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \hline \mathbf{I}_m & \mathbf{0}_{m\times(m+1)} \end{array}\right]. \tag{1}$$

Steps (3) and (4) : IFFT Followed by Subtraction of the First Row

Let us use the notation $K = (k_{ij})$ and $K_1 = (k^1_{ij}) := \mathbf{P}K_p$. Let $\check{K}_1 = \mathbf{F}^{-1}(\mathbf{P}K_p)$ be the inverse FFT of $\mathbf{P}K_p$. By definition of IFFT the first row of $\check{K}_1$ is

$$\check{\mathbf{k}}_1 = \left[\check{k}_{11}, \check{k}_{12}, \cdots, \check{k}_{1M}\right] = \frac{1}{n}\left[\sum_{i=0}^{n-1} k^1_{i1}, \sum_{i=0}^{n-1} k^1_{i2}, \cdots, \sum_{i=0}^{n-1} k^1_{iM}\right]. \tag{2}$$

Hence the matrix after step (4) is

$$\check{K}_1{}' = \begin{bmatrix} \check{k}_{11} & \check{k}_{12} & \cdots & \check{k}_{1M} \\ \check{k}_{21} & \check{k}_{22} & \cdots & \check{k}_{2M} \\ \vdots & \vdots & \cdots & \vdots \\ \check{k}_{n,1} & \check{k}_{n,2} & \cdots & \check{k}_{n,M} \end{bmatrix} - \begin{bmatrix} \check{k}_{11} & \check{k}_{12} & \cdots & \check{k}_{1M} \\ \check{k}_{11} & \check{k}_{12} & \cdots & \check{k}_{1M} \\ \vdots & \vdots & \cdots & \vdots \\ \check{k}_{11} & \check{k}_{12} & \cdots & \check{k}_{1M} \end{bmatrix} \equiv \check{K}_1 - \check{K}_1^*. \tag{3}$$

Here $\check{K}_1^*$ is the matrix all of whose rows are the vector $\check{\mathbf{k}}_1$.

Lemma 2.1. *The sum of all entries of each column of the matrix $\hat{K}$ is zero.*

Proof. Clear from (2) and (3). □

Step (7) - Nearest Orthogonal Matrix

The scheme to find the nearest orthogonal matrix (Higham, N. J. 1986; R. -C. Li., 1995; Ji-Guang, S., 1995; Banerjee, 2014) is given by

$$G = UV^H = \hat{K}\left(\hat{K}^H\hat{K}\right)^{-1/2}, \tag{4}$$

where $U\Sigma V^H$ is the SVD of $\hat{K}$.

Theorem 2.1. *The matrix G obtained in step (7) satisfies CMV property:*

Proof. Let $\vec{\mathbf{1}} = [1, \cdots, 1]$. Then by Lemma 2.1, we have

$$\vec{\mathbf{1}} \cdot \hat{K} = [0, 0, \cdots, 0].$$

Hence by (4) we see

$$\vec{\mathbf{1}} \cdot G = \vec{\mathbf{1}} \cdot \hat{K}\left(\hat{K}^H\hat{K}\right)^{-1/2} = [0, 0, \cdots, 0].$$

□

3. Simplification of the Algorithm

In this paper we simplify the algorithm above. We apply the algorithm to an initial matrix having CMV property. First consider the case $N = 2m + 1$ is odd. We will explain with $m = 2$, general case follows easily from this. Consider the following $N \times (N-1)$ initial matrix.

$$K^* = \frac{1}{\sqrt{2}} \left[\begin{array}{cc|c|ccc} 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & -1 \end{array} \right] \tag{5}$$

Using this matrix we will generate a pilot included matrix having the desired properties. We remove a specified column (3rd, say) consisting of two 1's and move the next column to the first to get $N \times (N-2)$ matrix

$$K_{t1} = \frac{1}{\sqrt{2}} \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & -1 \end{array} \right] \Rightarrow K_{t2} = \frac{1}{\sqrt{2}} \left[\begin{array}{c|cccc} 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & -1 \end{array} \right] \tag{6}$$

Then subtract the row vector $\begin{bmatrix} 0 & 2 & 0 & 2 & 0 \end{bmatrix}$ from the center row, to get

$$K_{p,e} = \frac{1}{\sqrt{2}} \left[\begin{array}{c|cccc} 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & -1 \end{array} \right] \tag{7}$$

In general it looks like this:

$$\mathbf{K}_{p,e} = \frac{1}{\sqrt{2}} \left[\begin{array}{c|ccccccc} & & & & & & 1 & 1 \\ 1 & & & & & \iddots & & \\ \hline & & & 1 & 1 & & & \\ & 1 & 1 & & & & & \\ & -2 & 0 & & & & -2 & \\ & 1 & -1 & & & & & \\ & & & 1 & -1 & & & \\ -1 & & & & & \ddots & & \\ \hline & & & & & & 1 & -1 \end{array} \right] \tag{8}$$

Next, we see the case $N = 2m$ (even). We start from $(N+1) \times N$ matrix. For example, when we want 8×6 matrix, we start from a 9×8 matrix

$$\frac{1}{\sqrt{2}} \left[\begin{array}{cccc|c|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{array} \right] \tag{9}$$

Remove the zero row in the center from (9) and remove first column. Then we remove a specified column (5-th, say) and move the next one to first to get $N \times (N-2)$ matrix

$$K_{t1} = \frac{1}{\sqrt{2}} \left[\begin{array}{ccc|c|ccc} 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \Rightarrow K_{t2} = \frac{1}{\sqrt{2}} \left[\begin{array}{c|ccccc} 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \tag{10}$$

Then subtract the row vector $[0, 0, 1, 0, 1, 0]$ from 4-th and 5-th rows. The resulting matrix is the initial for pilot included matrix.

$$K_{p,e} = \frac{1}{\sqrt{2}} \left[\begin{array}{c|ccccc} 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \tag{11}$$

Lemma 3.1. *If the initial matrix K_p ($K_{p,e}$ or $K_{p,o}$) satisfies CMV property, then steps (1)-(7) is simplified as*

Algorithm Orth-pilot-CMV

1. *Given a $N \times (N-2)$ initial CMV matrix K_p with orthonormal columns.*
2. *Let $G = UV^H$ where $U\Sigma V^H$ is the SVD of $\hat{K} = K_p$.*

4. Property of Odd Columns

We assume N is odd. The case of even is similar. Let $\mathbf{k}_i$ and $\mathbf{g}_i$ and denote the i-th column of the matrix K_p and G respectively.

Lemma 4.1. *The odd columns of K_p are orthogonal to all other columns of K_p. As a consequence, for all odd j, the vector $\mathbf{e}_j = [0, \cdots, 1, \cdots, 0]^T$ is an eigenvector of $K_p^H K_p$ corresponding to the eigenvalue 1.*

Proof. The orthogonality of odd columns of K_p comes from that of K^* of (5) since during the transformation of K^* to K_p in (5), (7), the odd columns did not change essentially(only the orders are permuted). Let $K_p = [\mathbf{k}_1, \cdots, \mathbf{k}_{N-2}]$. Then $K_p \mathbf{e}_j = \mathbf{k}_j$ and hence the j -th column of $\hat{K}_p^H \hat{K}_p$ satisfies

$$\hat{K}_p^H \hat{K}_p \mathbf{e}_j = K_p^H K_p \mathbf{e}_j = \begin{bmatrix} \mathbf{k}_1^T \\ \vdots \\ \mathbf{k}_{N-1}^T \end{bmatrix} \mathbf{k}_j = \begin{bmatrix} \mathbf{k}_1^T \cdot \mathbf{k}_j \\ \vdots \\ \mathbf{k}_{N-1}^T \cdot \mathbf{k}_j \end{bmatrix} = \mathbf{e}_j. \tag{12}$$

This means that when j is odd, the j-th columns of K_p are orthogonal to all other columns of K_p. Clearly (12) implies the second assertion of the lemma. □

Example 4.1. *For $N = 5$ we see*

$$K_p^H K_p = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ \boxed{0} & -2 & \boxed{0} \\ 0 & 1 & -1 \\ -1 & 0 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{13}$$

Note that the zeros in the box keep the odd columns of K_p orthogonal to other columns. In view of (12), $K_p^H K_p$ has two eigenvectors $\mathbf{e}_j$, $j = 1, 3$ corresponding to the eigenvalue 1.

Theorem 4.2. *The odd columns of* $G = K_p(K_p^H K_p)^{-1/2}$ *are the same as those of* K_p.

Proof. From the spectral decomposition of $K_p = U\Sigma V^H$ we have that of $K_p^H K_p$:

$$K_p^H K_p = V\Sigma^H \Sigma V^H := V\Lambda V^H (V^H = V^{-1}), \tag{14}$$

where by (12) Λ and V have the following form:

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \cdots & & \cdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \ \Lambda^{-1/2} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2} & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \cdots & & \cdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \ V = [\mathbf{e}_1, \mathbf{v}_2, \cdots, \mathbf{e}_{M-1}, \mathbf{v}_M]. \tag{15}$$

Note that for j odd $V\mathbf{e}_j = \mathbf{e}_j$ and for each odd j,

$$\begin{aligned} K_p(K_p^H K_p)^{-1/2}\mathbf{e}_j &= K_p V \Lambda^{-1/2} V^{-1}\mathbf{e}_j \\ &= K_p V \Lambda^{-1/2}\mathbf{e}_j \\ &= K_p V \mathbf{e}_j \\ &= K_p \mathbf{e}_j. \end{aligned}$$

This is the same as j-th column of K_p (normalization does not change even columns). □

5. Numerical Example

Example 5.1. *Let* $N = 7$. *We have, from initial matrix (7)*

$$G_p = \begin{bmatrix} 0.0000 & -0.1954 & 0.0000 & 0.5117 & 0.7071 \\ 0.7071 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.5117 & 0.7071 & -0.1954 & 0.0000 \\ 0.0000 & -0.6325 & 0.0000 & -0.6325 & 0.0000 \\ 0.0000 & 0.5117 & -0.7071 & -0.1954 & 0.0000 \\ -0.7071 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & -0.1954 & 0.0000 & 0.5117 & -0.7071 \end{bmatrix}$$

Example 5.2 (Even N)**.** *When* $N = 8, M = 6$ *we start from the initial matrix*

$$K_p = \frac{1}{\sqrt{2}} \left[\begin{array}{c|ccccc} 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

to get

$$G = \begin{bmatrix} 0.0000 & 0.0000 & -0.1494 & 0.0000 & 0.5577 & 0.7071 \\ 0.7071 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.5577 & 0.7071 & -0.1494 & 0.0000 \\ 0.0000 & 0.7071 & -0.4082 & 0.0000 & -0.4082 & 0.0000 \\ 0.0000 & -0.7071 & -0.4082 & 0.0000 & -0.4082 & 0.0000 \\ 0.0000 & 0.0000 & 0.5577 & -0.7071 & -0.1494 & 0.0000 \\ -0.7071 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & -0.1494 & 0.0000 & 0.5577 & -0.7071 \end{bmatrix}$$

In these matrices, we observe that rows corresponding to the nonzero entries of pilot vector (blue) are zeros (red) except the first entry.

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Numerical Solutions of a Quadratic Integral Equations by Using Variational Iteration and Homotopy Perturbation Methods

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Abstract

In this paper, the approximate solutions for quadratic integral equations (QIEs) are given by the variational iteration method (VIM) and homotopy perturbation method (HPM). These methods produce the solutions in terms of convergent series without needing to restrictive assumptions, to illustrate the ability and credibility of the methods, we deal with some examples that show simplicity and effectiveness.

Keywords: Quadratic Integral Equations, Variational Iteration Method, Homotopy Perturbation Method.

Subject Code: 45G10.

1. introduction

Quadratic integral equations (QIEs) are often applied in the radiative transfer, neutron transport, kinetic theory of gases and in the traffic theories.

The QIEs are studied in many papers and monographs (Bana's, et al., 2005; Bana's, et al., 1998; Bana's & Martinon, 2004; El-sayed & Hashem, 2009a; El-Sayed & Hashem, 2009b).

Recently, the different analytical and numerical methods are applied to reach the approximate solutions of QIEs. As there is no exact solutions for the most QIEs, many different kinds of researches are focusing on the effective of QIEs properties like the existence, uniqueness, positive solutions and monotonic solutions of this class of equations (Argyros, 1985; Bana's et, al., 1998; Bana's & Martinon, 2004; El-Sayed & Rzepka, 2006). There are few papers which have dealt with the numerical solutions of QIEs such as Elsayed (El-Sayed et al., 2010) used the classical method of successive approximations Picard and Adomian decomposition method for solving QIEs, Avazzadeh (Avazzadeh, 2012) used the radial basis functions to obtain the approximate solutions of QIEs of Urysohn's type. (He, 1999a; He, 1999b; He, 2000; He, 2003) was the first one who proposed the VIM and HPM to find the solutions of linear and nonlinear problems.

Widely, the VIM is used in the literature in different scientific applications in (Abdou & Soliman, 2005; Abulwafa et al., 2006; He & Wu, 2007). This method presents significant enhancements over existing numerical and analytic technique like the perturbation, Adomian, Galerkin, finite differences methods, etc. These methods have dealt with ordinary, partial differential equations, the integro-differential equations (IDEs) and integral equations, in a direct way without needing to any specific restriction which may give the closed form of exact solution if there is an exact solution. The VIM has no specific restrictions for nonlinear terms which involve in the equation.

The homotopy perturbation method deforms a difficult problem under study into a simple one which is easy to solve. Most perturbation methods assume there is a small parameter, but there is no small parameter at all in the most nonlinear problems. Many new methods are proposed to eliminate the small parameter (He, 1999b; Liao, 1995). Also, the HPM is employed for solving several kinds of integral equations. Such as, Fredholm, nonlinear Volterra-Fredholm integral equations and Volterra integro-differential equations.

The aim of the present paper is extending the application of HPM and VIM to give some approximate solutions for the following QIE where $\mathcal{A}(t)$ is given and $\mathcal{F}(s, x(s))$ is any nonlinear functions. We want to point out that this work is applied for first time on these kind of equations.

$$x(t) = \mathcal{A}(t) + \mathcal{G}(t, x(t)) \int_0^t \mathcal{F}(s, x(s))ds. \tag{1}$$

It is clear that the results reveal the effectively and simplicity for the presented two methods.

2. Variational Iteration Method

Consider the following differential equation where $\mathcal{L}$ and $\mathcal{N}$ are linear and nonlinear operators respectively, and $g(x)$ is the inhomogeneous source term

$$\mathcal{L}[u(x)] + \mathcal{N}[u(x)] = g(x). \tag{2}$$

The VIM presents a correction functional for eq.(2) in the following form:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\zeta)[\mathcal{L}u_n(\zeta) + \mathcal{N}\widetilde{u_n}(\zeta) - g(\zeta)]d\zeta, \tag{3}$$

where λ is a general Lagrange multiplier, noting that in this method λ may be a constant or a function, which can be identified perfectly by the variational theory and the subscript n denotes the nth-order approximation, $\widetilde{u_n}$ is considered as a restricted value that means it behaves as a constant, i.e. $\delta\widetilde{u_n} = 0$.
It was found in (Abdou & Soliman, 2005; Abulwafa et al., 2006; He & Wu, 2007). the general formula for $\lambda(x)$ for the nth order differential equation

$$u^{(n)} + f\left(u(\zeta), u'(\zeta), u''(\zeta), ..., u^{(n)}(\zeta)\right) = 0, \tag{4}$$

has the form

$$\lambda(x) = (-1)^n \frac{1}{(n-1)!}(\zeta - x)^{(n-1)}. \tag{5}$$

The solution given by

$$u(x) = \lim_{n\to\infty} u_n(x).$$

3. Homotopy Perturbation Method

Consider the differential equation (2) with following the boundary conditions where B is a boundary operator, Γ is the boundary of the domain Ω and $x \in \Omega$

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad x \in \Gamma. \tag{6}$$

The He's homotopy perturbation technique (He, 1999a), (He, 2000) defines the homotopy $v(x, p) : \Omega \times [0, 1] \to \Re$ which satisfies

$$\mathcal{H}(v, p) = (1 - p)[\mathcal{L}(v) - \mathcal{L}(u_0)] + p[\mathcal{L}(v) + \mathcal{N}(v) - g(x)] = 0, \tag{7}$$

or

$$\mathcal{H}(v, p) = [\mathcal{L}(v) - \mathcal{L}(u_0)] + p\mathcal{L}(u_0) + p[\mathcal{N}(v) - g(x)] = 0, \tag{8}$$

where $x \in \Omega$ and $p \in [0, 1]$ is an impeding parameter, u_0 is an initial approximation which satisfies the boundary conditions, from eq's.(7) and (8), we have

$$\mathcal{H}(v, 0) = \mathcal{L}(v) - \mathcal{L}(u_0) = 0, \tag{9}$$
$$\mathcal{H}(v, 1) = \mathcal{L}(v) + \mathcal{N}(v) - g(x) = 0. \tag{10}$$

The p process of changing from zero to unity is just that of $v(x, p)$ from u_0 to $u(x)$. In topology, this is called deformation, $\mathcal{L}(v) - \mathcal{L}(u_0)$ and $\mathcal{L}(v) + \mathcal{N}(v) - g(x)$ are homotopic. The solutions of eq.(7) and eq.(8) can be defined as a power series in p

$$v = v_0 + pv_1 + p^2v_2 + ..., \tag{11}$$

when $p \to 1$, corresponding to (7) becomes the approximate solution is

$$u = v_0 + v_1 + v_2 + ..., \tag{12}$$

the convergence of the series (12) has been proved in (He, 1999a; He, 2000).

4. Numerical Examples

In this part, we study some examples and apply the VIM and HPM methods for comparison reasons.

Example 1. solve the QIE (El-Sayed et al., 2010)

$$x(t) = \left(t^2 - \frac{t^{10}}{35}\right) + \frac{t}{5}x(t)\int_0^t s^2x^2(s)ds, \tag{13}$$

with exact solution $x(t) = t^2$.
as beginning we have to convert volterra QIE to an equivalent volterra IDE. We can do this by differentiating two sides of the QIEs, we should used the Leibnitz rule for differentiating the QIEs at the right side.

$$x'(t) = 2t - 10\left(\frac{t^9}{35}\right) + \frac{t^3}{5}x^3(t) + \left(\frac{1}{5}x(t) + \frac{t}{5}x'(t)\right)\int_0^t s^2x^2(s)ds, \quad x(0) = 0, \tag{14}$$

we can get the initial condition $x(0) = 0$ by substituting $x = 0$ in eq.(13), the correction functional for Equation (14) is

$$\begin{aligned} x_{n+1}(t) &= x_n(t) + \int_0^t \lambda(\zeta)\left(x_n'(\zeta) - 2\zeta + 10\frac{\zeta^9}{35} - \frac{\zeta^3}{5}x_n^3(\zeta) - \left(\frac{1}{5}x_n(\zeta) + \frac{\zeta}{5}x_n'(\zeta)\right)\right. \\ &\left.\int_0^\zeta r^2x^2(r)dr\right)d\zeta. \end{aligned} \tag{15}$$

We substitute the value of $\lambda(\zeta) = -1$ in eq.(15) which is identified by the variational theory, also, we can use the initial value $x(0) = 0$ to obtain the zeroth approximation $x_0(t)$ and by using Equation (15) we get the successive approximations,

$$x_0(t) = 0, \tag{16}$$

$$x_1(t) = t^2 - \frac{1}{35}t^{10}, \tag{17}$$

$$x_2(t) = t^2 - \frac{1}{4930625}t^{34} + \frac{61}{2113125}t^{26} - \frac{29}{18375}t^{18}, \tag{18}$$

and so on, and the solution given by

$$x(t) = \lim_{n\to\infty} x_n(t).$$

Table 1. Comparison of the numerical results with the exact solution $x(t)$

t	Approximate Solution	Exact Solution	Absolute error
0.10	0.01000000	0.01000000	3.008×10^{-40}
0.20	0.04000000	0.04000000	5.168×10^{-30}
0.30	0.09000000	0.09000000	5.017×10^{-24}
0.40	0.16000000	0.16000000	8.879×10^{-20}
0.50	0.25000000	0.25000000	1.751×10^{-16}
0.60	0.36000000	0.36000000	8.617×10^{-14}
0.70	0.49000000	0.49000000	1.626×10^{-11}
0.80	0.64000000	0.64000000	1.521×10^{-9}
0.90	0.80999992	0.81000000	8.311×10^{-8}
1.00	0.99999704	1.00000000	0.0000029606

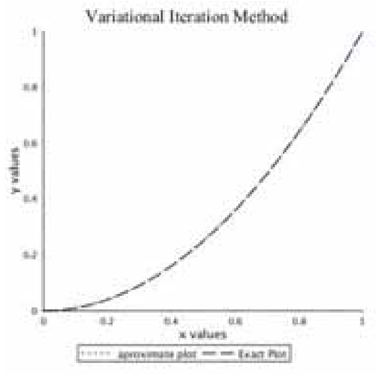


Figure 1. Comparison of the approximate solution by VIM with the exact solution

Table 1 shows the approximate solution for $n = 4$, also it is obvious that we can improve the accuracy of solutions by computing more terms of the approximate solutions. We can construct the following homotopy according to HPM,

$$\mathcal{H}(\mathtt{u}, p) = (1-p)(\mathtt{u}(t) - \mathtt{g}(t)) + p\left(\mathtt{u}(t) - \left(t^2 - \frac{t^{10}}{35}\right) - \frac{t}{5}\mathtt{u}(t)\int_0^t s^2\mathtt{u}^2(s)ds\right) = 0, \tag{19}$$

such that $\mathtt{g}(t) = \left(t^2 - \frac{t^{10}}{35}\right)$ then

$$\mathcal{H}(\mathtt{u}, p) = \mathtt{u}(t) - \left(t^2 - \frac{t^{10}}{35}\right) - p\frac{t}{5}\mathtt{u}(t)\int_0^t s^2\mathtt{u}^2(s)ds = 0, \tag{20}$$

substituting (11) into (20) and equating the terms with the same identical powers of p we have

$$p^0 : \mathtt{u}_0(t) = \left(t^2 - \frac{t^{10}}{35}\right), \tag{21}$$

$$p^1 : \mathtt{u}_1(t) = \frac{t}{5}\mathtt{u}_0(t)\int_0^t s^2 H_0(s)ds, \tag{22}$$

$$p^2 : \mathtt{u}_2(t) = \frac{t}{5}\mathtt{u}_0(t)\int_0^t s^2 H_1(s)ds + \frac{t}{5}\mathtt{u}_1(t)\int_0^t s^2 H_0(s)ds, \tag{23}$$

$$p^3 : \mathtt{u}_3(t) = \frac{t}{5}\mathtt{u}_0(t)\int_0^t s^2 H_2(s)ds + \frac{t}{5}\mathtt{u}_1(t)\int_0^t s^2 H_1(s)ds + \frac{t}{5}\mathtt{u}_2(t)\int_0^t s^2 H_0(s)ds, \tag{24}$$

and so on, where H_i are He's polynomials of the nonlinear term x^2, and the solution will be,

$$\mathtt{u}(t) = \sum_{i=0}^{n}\mathtt{u}_i(t).$$

Tbale 2. Comparison of the numerical results with the exact solution $x(t)$

t	Approximate Solution	Exact Solution	Absolute error
0.10	0.01000000	0.01000000	4.992×10^{-49}
0.20	0.04000000	0.04000000	2.195×10^{-36}
0.30	0.09000000	0.09000000	5.462×10^{-29}
0.40	0.16000000	0.16000000	9.655×10^{-24}
0.50	0.25000000	0.25000000	1.134×10^{-19}
0.60	0.36000000	0.36000000	2.398×10^{-16}
0.70	0.49000000	0.49000000	1.549×10^{-13}
0.80	0.64000000	0.64000000	4.181×10^{-11}
0.90	0.80999999	0.81000000	5.745×10^{-9}
1.00	0.99999954	1.00000000	4.554×10^{-7}

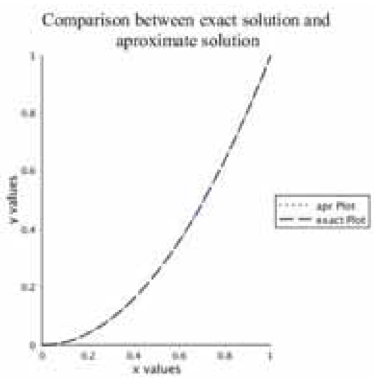


Figure 2. Comparison of the approximate solution by HPM with the exact solution

Table 2 shows the approximate solution for $n = 4$, also it is obvious that we can improve the accuracy of solutions by computing more terms of the approximate solutions.

Example 2. Solve the QIE (El-Sayed et al., 2010)

$$x(t) = \left(t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110}\right) + \frac{t^3}{10}x^2(t)\int_0^t (s+1)x^3(s)ds, \tag{25}$$

with exact solution $x(t) = t^3$.
as beginning we have to convert volterra QIE to an equivalent volterra IDE. We can do this by differentiating two sides of the QIEs, we should used the Leibnitz rule for differentiating the QIEs at the right side.

$$\begin{aligned} x'(t) &= \left(3t^2 - 19\frac{t^{18}}{100} - 20\frac{t^{19}}{110}\right) + 3\frac{t^2}{10}x^2(t)\int_0^t (s+1)x^3(s)ds + \frac{t^3}{10}(2x(t)x'(t))\int_0^t (s+1)x^3(s)ds \\ &\quad + \frac{t^3}{10}(t+1)x^5(t)ds, \quad x(0) = 0, \end{aligned} \tag{26}$$

we can get the initial condition $x(0) = 0$ by substituting $x = 0$ in eq.(25), the correction functional for Equation (26) is

$$\begin{aligned} x_{n+1}(t) &= x_n(t) + \int_0^t \lambda(\zeta)\left(x_n'(\zeta) - 3\zeta^2 + 19\frac{\zeta^{18}}{100} + 20\frac{\zeta^{19}}{110} - 3\frac{\zeta^2}{10}x_n^2(\zeta)\int_0^\zeta (r+1)x_n^3(r)dr\right. \\ &\quad \left. - \frac{\zeta^3}{10}(2x_n(\zeta)x_n'(\zeta))\int_0^\zeta (r+1){x_n}^3(r)dr - \frac{\zeta^3}{10}(\zeta+1)x_n^5(\zeta)\right)d\zeta, \end{aligned} \tag{27}$$

We substitute the value of $\lambda(\zeta) = -1$ in eq.(27) which is identified by the variational theory, also, we can use the initial value $x(0) = 0$ to obtain the zeroth approximation $x_0(t)$ and the solution given by

$$x(t) = \lim_{n\to\infty} x_n(t).$$

Table 3. Comparison of the numerical results with the exact solution $x(t)$

t	Approximate Solution	Exact Solution	Absolute error
0.10	0.00100000	0.00100000	1.112×10^{-56}
0.20	0.00800000	0.00800000	3.189×10^{-41}
0.30	0.02700000	0.02700000	3.813×10^{-32}
0.40	0.06400000	0.06400000	1.105×10^{-25}
0.50	0.12500000	0.12500000	1.175×10^{-20}
0.60	0.21600000	0.21600000	1.540×10^{-16}
0.70	0.34300000	0.34300000	4.749×10^{-13}
0.80	0.51200000	0.51200000	5.066×10^{-10}
0.90	0.72899976	0.72900000	2.393×10^{-7}
1.00	0.99994142	1.00000000	0.0000585810

Figure 3. Comparison of the approximate solution by VIM with the exact solution

Table 3 shows the approximate solution for $n = 3$, also it is obvious that we can improve the accuracy of solutions by computing more terms of approximate solutions. We can construct the following homotopy according to HPM,

$$\begin{aligned}\mathcal{H}(\mathrm{u}, p) &= (1-p)(\mathrm{u}(t) - \mathrm{g}(t)) + p\left(\mathrm{u}(t) - \left(t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110}\right)\right. \\ &\left. -\frac{t^3}{10}x^2(t)\int_0^t (s+1)x^3(s)ds\right) = 0, \end{aligned} \tag{28}$$

such that $\mathrm{g}(t) = \left(t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110}\right)$, then

$$\mathcal{H}(\mathrm{u}, p) = \mathrm{u}(t) - \left(t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110}\right) - p\frac{t^3}{10}x^2(t)\int_0^t (s+1)x^3(s)ds = 0, \tag{29}$$

substituting (11) into (29) and equating the terms with identical powers of p we have

$$p^0 : \mathtt{u}_0(t) = t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110}, \tag{30}$$

$$p^1 : \mathtt{u}_1(t) = \frac{t^3}{10} A_0(t) \int_0^t (s+1) B_0(s) ds, \tag{31}$$

$$p^2 : \mathtt{u}_2(t) = \frac{t^3}{10} A_0(t) \int_0^t (s+1) B_1(s) ds + \frac{t^3}{10} A_1(t) \int_0^t (s+1) B_0(s) ds, \tag{32}$$

$$p^3 : \mathtt{u}_3(t) = \frac{t^3}{10} A_0(t) \int_0^t (s+1) B_2(s) ds + \frac{t^3}{10} A_1(t) \int_0^t (s+1) B_1(s) ds + \frac{t^3}{10} A_2(t) \int_0^t (s+1) B_0(s) ds, \tag{33}$$

and so on, where A_i and B_i are He's polynomials of the nonlinear terms x^2 and x^3 respectively, and the solution will be,

$$\mathtt{u}(t) = \sum_{i=0}^{n} \mathtt{u}_i(t),$$

Table 4. Comparison of the numerical results with the exact solution $x(t)$

t	Approximate Solution	Exact Solution	Absolute error
0.10	0.00100000	0.00100000	1×10^{-73}
0.20	0.00800000	0.00800000	1.611×10^{-53}
0.30	0.02700000	0.02700000	1.363×10^{-41}
0.40	0.06400000	0.06400000	4.227×10^{-33}
0.50	0.12500000	0.12500000	1.704×10^{-26}
0.60	0.21600000	0.21600000	4.389×10^{-21}
0.70	0.34300000	0.34300000	1.688×10^{-16}
0.80	0.51200000	0.51200000	1.607×10^{-12}
0.90	0.72899999	0.72900000	5.210×10^{-9}
1.00	0.99999316	1.00000000	0.0000068386

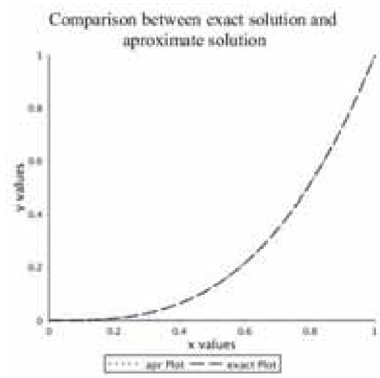


Figure 4. Comparison of the approximate solution by HPM with the exact solution

Table 4 shows the approximate solution for $n = 3$, also it is obvious that we can improve the accuracy of solutions by computing more terms of approximate solutions.

Example 3. Solve the QIE (Bana's et al., 2005)

$$x(t) = t^3 + \left(\frac{1}{4}x(t) + \frac{1}{4}\right)\left(\int_0^t t + cos\left(\frac{x(s)}{1 + x^2(s)}\right)\right)ds. \tag{34}$$

According to the VIM, differentiating both sides of eq.(34) ones with respect to t gives the IDE

$$\begin{aligned} x'(t) &= 3t^2 + \frac{1}{2}tx(t) + \frac{1}{4}t^2x'(t) + \frac{1}{4}x(t)cos\left(\frac{x(t)}{1 + x^2(t)}\right) \\ &+\frac{1}{4}x'(t)\int_0^t cos\left(\frac{x(s)}{1 + x^2(s)}\right)ds + \frac{1}{2}t + \frac{1}{4}cos\left(\frac{x(t)}{1 + x^2(t)}\right), \end{aligned} \tag{35}$$

The correction functional for eq.(35) is

$$\begin{aligned} x_{n+1}(t) &= x_n(t) - \int_0^t \left(x_n'(\zeta) - 3\zeta^2 - \frac{1}{2}\zeta x_n(\zeta) - \frac{1}{4}\zeta^2 x_n'(\zeta) - \frac{1}{4}x_n(\zeta)cos\left(\frac{x_n(\zeta)}{1 + x_n^2(\zeta)}\right)\right. \\ &\left. -\frac{1}{4}x_n'(\zeta)\int_0^\zeta cos\left(\frac{x_n(r)}{1 + x_n^2(r)}\right)dr - \frac{1}{2}\zeta - \frac{1}{4}cos\left(\frac{x_n(\zeta)}{1 + x_n^2(\zeta)}\right)\right)d\zeta, \end{aligned} \tag{36}$$

we can use the initial value $x(0) = 0$ to obtain the zeroth approximation $x_0(t)$ and by using the eq.(36) we get the successive approximations

$$x_0(t) = 0, \tag{37}$$

$$x_1(t) = \frac{1}{4}t + t^3 + \frac{1}{4}t^2, \tag{38}$$

$$\begin{aligned} x_2(t) &= \frac{1}{4}t + \frac{431}{384}t^3 + \frac{5}{16}t^2 - \frac{1}{56}t^{10} - \frac{5}{336}t^9 - \frac{71}{3360}t^8 - \frac{1499}{53760}t^7 \\ &-\frac{269}{15360}t^6 + \frac{3599}{15360}t^5 + \frac{473}{1536}t^4, \end{aligned} \tag{39}$$

and so on, and the solution given by

$$x(t) = \lim_{n\to\infty} x_n(t).$$

We can construct the following homotopy according to HPM,

$$\mathcal{H}(\mathrm{u}, p) = (1 - p)(\mathrm{u}(t) - t^3) + p\left(\mathrm{u}(t) - t^3 - \left(\frac{1}{4}\mathrm{u}(t) + \frac{1}{4}\right)\int_0^t t + \cos\left(\frac{\mathrm{u}(s)}{1 + \mathrm{u}^2(s)}\right)ds\right) = 0, \tag{40}$$

$$\mathcal{H}(\mathrm{u}, p) = \mathrm{u}(t) - t^3 - p\left(\frac{1}{4}\mathrm{u}(t) + \frac{1}{4}\right)\int_0^t t + \cos\left(\frac{\mathrm{u}(s)}{1 + \mathrm{u}^2(s)}\right)ds = 0, \tag{41}$$

substituting (11) into (41) and equating the term with identical powers of p we have

$$p^0 : u_0(t) = t^3, \tag{42}$$

$$p^1 : u_1(t) = \frac{1}{4}u_0(t)\int_0^t (t + H_0(s))ds + \frac{1}{4}\int_0^t (t + H_0(s))ds, \tag{43}$$

$$p^2 : u_2(t) = \frac{1}{4}u_0(t)\int_0^t H_1(s)ds + \frac{1}{4}\int_0^t H_1(s)ds + \frac{1}{4}u_1(t)\int_0^t (t + H_0(s))ds, \tag{44}$$

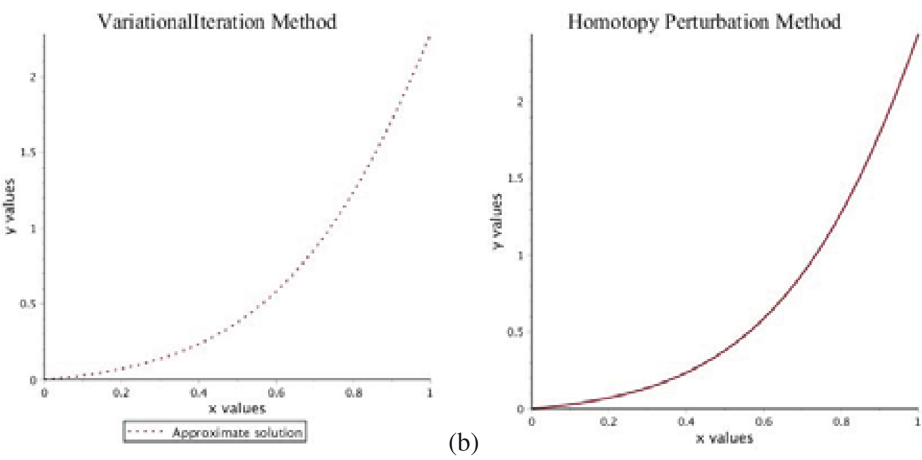
$$p^3 : u_3(t) = \frac{1}{4}u_0(t)\int_0^t H_2(s)ds + \frac{1}{4}\int_0^t H_2(s)ds + \frac{1}{4}u_1(t)\int_0^t H_1(s)ds + \frac{1}{4}u_2(t)\int_0^t (t + H_0(s))ds, \tag{45}$$

and so on, where H_i are He's polynomials of the nonlinear term $cos(\frac{x(s)}{1+x^2(s)})$ and the solution will be

$$u(t) = \sum_{i=0}^{n} u_i(t),$$

Table 5. Approximate solution $x(t)$ by VIM and HPM for $n = 1$

t	VIM solution	HPM solution
0.10	0.02930184	0.02930216
0.20	0.07228471	0.07229271
0.30	0.13761286	0.13768016
0.40	0.23575023	0.23610759
0.50	0.37941576	0.38085479
0.60	0.58374728	0.58851618
0.70	0.86560555	0.87921321
0.80	1.24089607	1.27522943
0.90	1.71800041	1.79569813
1.00	2.28475926	2.44303370



(a) (b)

Figure 5. Approximate solutions by using (VIM) and (HPM)

Example 4. Solve QIE (Bana's & Martinon, 2004)

$$x(t) = e^{-t} + x(t)\int_0^t \frac{t^2 \ln(1 + s|x(s)|)}{2e^{(t+s)}} ds, \quad 0 < t \le 2. \tag{46}$$

According to VIM, differentiating both sides of Equation (46) ones with respect to t gives the IDE

$$\begin{aligned} x'(t) &= -e^{-t} + x'(t)\int_0^t \frac{t^2 \ln(1 + s|x(s)|)}{2e^{(t+s)}} + x(t)\Bigg(\frac{t^2}{2e^{2t}} \ln(1 + t|x(t)|) \\ &+ \int_0^t \frac{-2t^2 e^{(t+s)} + 4te^{(t+s)}}{4e^{(t+s)^2}} \ln(1 + s|x(s)|)\Bigg), \quad x(0) = 1. \end{aligned} \tag{47}$$

The correction functional for eq.(47) is

$$\begin{aligned} x_{n+1} &= x_n(t) - \int_0^t \Bigg(x_n'(\zeta) + e^{-\zeta} - x_n'(\zeta)\int_0^\zeta \frac{\zeta^2}{2e^{(\zeta+r)}} \ln(1 + r|x_n(r)|)dr \\ &- x_n(\zeta)\left[\frac{\zeta^2}{2e^{2\zeta}} \ln(1 + \zeta|x_n(\zeta)|) + \int_0^\zeta \frac{-2\zeta^2 e^{(\zeta+r)} + 4\zeta e^{(\zeta+r)})}{4e^{(\zeta+r)2}} \ln(1 + r|x_n(r)|)dr\right]\Bigg) d\zeta, \end{aligned}$$

the zeroth approximation $x_0(t)$ can be selected by using the initial value $x(0) = 1$. We can construct the following homotopy according to HPM,

$$\mathcal{H}(\mathrm{u}, p) = \mathrm{u}(t) - e^{-t} - p\mathrm{u}(t)\int_0^t \frac{t^2 \ln(1 + s|\mathrm{u}(s)|)}{2e^{(t+s)}} = 0, \tag{48}$$

substituting (11) into (48) and equating the terms with identical powers of p we have

$$p^0 : \mathrm{u}_0(t) = e^{-t}, \tag{49}$$

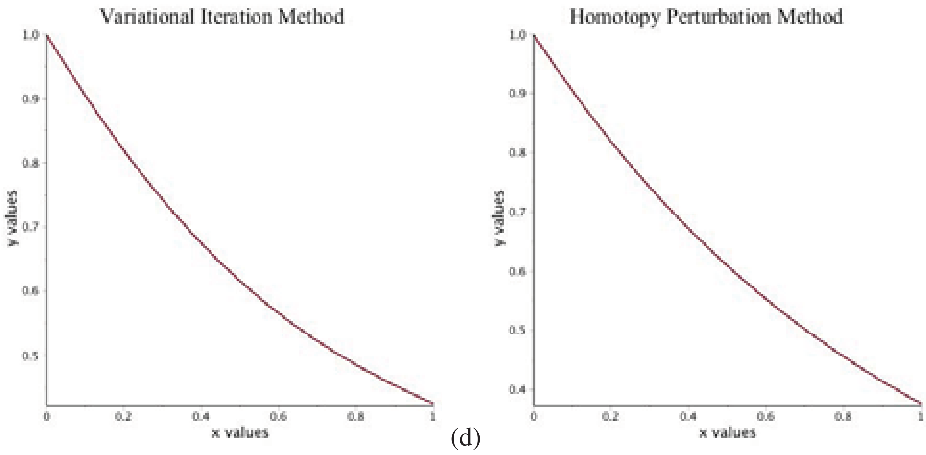
$$p^1 : \mathrm{u}_1(t) = \mathrm{u}_0(t)\int_0^t \frac{t^2}{2e^{(t+s)}} H_0(s)ds, \tag{50}$$

$$p^2 : \mathrm{u}_2(t) = \mathrm{u}_0(t)\int_0^t \frac{t^2}{2e^{(t+s)}} H_1(s)ds + \mathrm{u}_1(t)\int_0^t \frac{t^2}{2e^{(t+s)}} H_0(s)ds, \tag{51}$$

and so on, where H_i are He's polynomials of the nonlinear term $\ln(1 + s|x(s)|)$.

Table 6. Approximate solution $x(t)$ by VIM and HPM for $n = 1$

t	VIM solution	HPM solution
0.10	0.90486075	0.90485481
0.20	0.81907560	0.81892557
0.30	0.74240511	0.74151336
0.40	0.67478739	0.67187791
0.50	0.61603248	0.60924288
0.60	0.56560069	0.55284340
0.70	0.52254959	0.50196629
0.80	0.48565135	0.45597253
0.90	0.45360274	0.41430435
1.00	0.42522512	0.37648238



(c) (d)

Figure 6. Approximate solution by using (VIM) and (HPM)

5. Conclusion

We have been successfully applied the VIM and HPM to find the approximate solutions for nonlinear QIEs. We have found out that the two methods are applicable and efficient technique, also the HPM is better than VIM in finding the accurate solutions. We have been observing that the accuracy can be improved by computing more n-terms off approximate solutions or by taking more terms in the Taylor expansion of the nonlinear terms. To find the calculations we have used the Maple package (2015).

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Boundary Value Problems at Infinite Genus

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Abstract

Boundary value problems are formulated on infinite-genus surfaces. These are solved for a variety of boundary conditions. The symbol calculus for differential operators is developed further for solution of parabolic differential equations at infinite genus.

MSC: 30E25, 30F35, 34B09, 35K51

Keywords: parabolic boundary problem, infinite genus surfaces, classical symbols

1. Introduction

The theory of partial differential equations with given boundary conditions has developed from series solutions and integral transforms to an operator calculus (Hormander, 1985). The matrix algebra formed from the symbols representing a set of operators for a pseudodifferential equation defined at the interior and the boundary can be used to evaluate the inverse for elliptic boundary value problems (Boutet de Monvel, 1971). A framework for this derivation may be given with this symbolic calculus for parabolic boundary value problems. The Volterra pseudodifferential differential equations were solved for a given set of boundary conditions (Piriou, 1970). It had been proven that an isomorphism existed between a normed space of solutions to elliptic boundary value problems with these boundary conditions in the complex plane on the real line to a normed space of solutions to a corresponding parabolic differential equation (Agronovich and Vishik, 1964). The exponential long-time asymptotics on a noncompact manifold in the elliptic problem (Schuss, 1973) similarly may be transformed to asymptotics in a parabolic problem.

The existence and uniqueness of solutions to parabolic differential equations with exponential asymptotics in the $t \to \infty$ limit have been established (Krainer, 2002). External states in string amplitudes are known to be described by semi-infinite cylinders and the solutions to equations on these ends in the Euclidean formalism may be related to the parabolic boundary value problems through a generalized inverse Laplace transform. Therefore, the asymptotics of solutions in string theory would be represented. The $t \to \infty$ limit, however, does not necessarily arise for infinite-genus surfaces with accumulating handles. Exponentially decaying solutions at $t = -\infty$ are not required for the semi-infinite cylinders representing the propagation of external states. Nevertheless, a change of coordinates equivalent to $r = e^{-t}$ again maps the boundary conditions at $t = \infty$ to the origin of a punctured disk at $r = 0$.

The methods for solving differential equations of field theories at the ideal boundaries of surfaces will be given. Ideal boundaries may consist of discrete sets of points or a continuum. An example of a series solution to an elliptic differential equation on a surface with a boundary with an infinite number of ends is provided in §2. The uniqueness of the function which represents the harmonic measure with a given set of boundary values follows. The path to ideal boundary will be parameterized by a coordinate t tending to infinity, and the mapping from infinity to the origin can have an image that is a discrete set or the real line. When it is not the real line, by the uniformization of surfaces of genus $g \geq 2$, there is a formalism based on the automorphic functions defined on the entire upper half plane instead of a fundamental region for a Fuchsian group, and boundary conditions may be specified on the real line. Infinite-genus surfaces in the class O_G are parabolic and have a countable number of ends. The mapping to a surface with one end to a finite region would yield a surface with handles accumulating to only one point. Therefore, the boundary is not identified with a continuum. The boundary value problem shall be solved through the above method of functions invariant under the uniformizing group.

Analytic function theory on infinite-genus surfaces is necessary for the set of conditions (Widom,1971) required for a convergent representation of the Green function as a product over elements of the uniformizing group (Pommerenke, 1976) The convergence of the series for theta functions similarly can be proven for spectral curves of the parabolic heat equation. Nontrivial solutions $\psi \in L^\infty_{loc}(\mathbb{R}^2)$ to the heat equation $\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x_2^2}\right)\psi + q(x_1, x_2)\psi = 0$ and $\psi(x_1 + \omega_1, x_2 + \omega_2) = \xi_1\psi(x_1, x_2)$ and $\psi(x_1, x_2 + 2\pi) = \xi_2\psi(x_1, x_2)$ for $q \in L^2(\mathbb{R}^2/\Gamma)$, where $\Gamma = (0, 2\pi)\mathbb{Z} \oplus (\omega_1, \omega_2)\mathbb{Z}$, may be mapped to functions that satisfy the Kadomtsev-Petviashvili equation on $\mathbb{R}^2/\Gamma$ (Feldman, Knorrer Trubowitz, 2003). The pseudodifferential calculus for this parabolic boundary value problem in §3 then yields a solution for the Kadomtsev-Petviahshvili equation in terms of this theta function through the mapping to the hyperbolic space the covering transformation of the surface.

2. The Harmonic Measure of the Ideal Boundary

The Dirichlet problem may be formulated for an infinite-genus surface. If $\triangle u = 0$ with specified values at the boundary, $w(z, \partial\Sigma, \Sigma)$, which equals the Perron solution at z, gives the probability of random motion beginning at z and exiting through $\partial\Sigma$ (Kakutani, 1944). It equals the harmonic measure of the ideal boundary. Random motion does not produce a flux to the embedding space unless the the capacity is non-zero, and the surface does not belong to O_G. This result would complement the introduction of exceptional group gauge symmetries through the intersection matrix (Davis, 2014).

The capacity of the ideal boundary is $c_\beta = e^{-k_\beta}$, where

$$k_\beta = lim_{n\to\infty} \int_{\partial E_n} s_n * ds_n, \tag{1}$$

with $s_n = ln|z| + \varphi_n(z)$ relative to some origin $z = 0$ and $\varphi_n(z)$ being harmonic on E_n, an n^{th} order approximation of the end E (Sario Nakai, 1970). When there is a null boundary, there is no second source for the Green function and $\varphi_n(z)$ will not cancel $ln|z|$ in the limit $z \to \partial E_n$ and $n \to \infty$. If the ideal boundary has non-zero harmonic measure, the equivalent of the second source is sufficient for a cancellation with $ln|z|$ and the remainder is finite. The integral $\frac{1}{2\pi}\int_\beta s_\beta * ds_\beta$ would be finite and $c_\beta \neq 0$.

The harmonic measure of an end with respect to the ideal boundary is defined to be a solution to $\triangle w = 0$ with $w|_\alpha = 0$ and $w|_\beta = 1$. Uniqueness of the harmonic measure follows if it is an $\widetilde{HD}$-minimal function, since another function may be selected to be either less than or greater than u in an entire neighbourhood of the ideal boundary. Any harmonic function with finite Dirichlet norm in $\widetilde{HD}(\Sigma)$ may be expressed as $\int_D P(z, p) f(p) d\mu(p)$, where $P(z, p)$ is the harmonic kernel, $f(p)$ is a boundary function and $\mu(p)$ is the harmonic measure (Sario Nakai, 1970).

An example of a surface in $O_{HD} - O_G$ is Toki's surface. If D_0 is the slit disk $D - \cup_{m,n,\nu} S^{\nu}_{m,n}$, where $S^{\nu}_{mn} = \{z = re^{i\theta} | -2^{-2\mu} \leq log\ r \leq -2^{-2\mu+1}, \theta = \nu \cdot 2\pi \cdot 2^{-2\mu}, \nu = 1, ..., 2^{2\mu}, \mu = 2^{m-1}(2n-1)\}$, the Riemann surface is constructed by joining the copies of the disks $\Sigma(i + mj)$ with $\Sigma'(i + m + mj)$ for even j and $\Sigma(i + mj)$ with $\Sigma'(i - m + mj)$ for odd j cross along every slit S^{ν}_{mn}, $n = 1, 2, ..., \nu = 1, ..., 2^{2\mu}$ (Toki, 1962). The Dirichlet problem may be solved on this surface by the integral

$$\varphi(z) = -\int_{\partial\Sigma} d\sigma \varphi_0(\sigma) \frac{\partial G}{\partial n} \tag{2}$$

where $\varphi|_{\partial\Sigma} = \varphi_0$ and $G(z, z')$ is the Green function (Poincare, 1890). The solution on Tôki's surface for the harmonic measure would satisfy the boundary condition $\phi_0(\sigma) = 1$ at each of the slits in S^{ν}_{mn}.

Theorem 1. The harmonic measure of Tôki's surface may be given in series form.

Proof.

Given the Green function on the upper half plane $G(z, z') = -\frac{1}{4\pi} ln \frac{(x-x')^2+(y-y')^2}{(x-x')^2+(y+y')^2}$, the normal derivative may be found on the unit disk. Let us define a group Γ_T generated by translations $\theta_\nu \to \theta_{\nu+1}$, $\nu = 1, ..., 2^{2\nu} - 1$, $\theta_{2^{2\mu}} \to \theta_1$ and $m \to m + 1$, $n \to n + 1$. The Green function then can be evaluated by the method of images

$$G(z, z') = -\frac{1}{4\pi} \sum_{\gamma\in\Gamma_T} ln \frac{(x - \gamma x')^2 + (y - \gamma y')^2}{(x - \gamma x')^2 + (y + \gamma y')^2}, \tag{3}$$

where the slit, $-\frac{1}{4} \leq log\ r \leq \frac{1}{8}$, $\theta = \frac{\pi}{4}$ is aligned with the y axis. At each slit, the normal vector is perpendicular to the direction given by $\gamma(S^1{}_{11})$ or S^{ν}_{mn}. It is sufficient to establish the angle of the perpendicular relative to the adjusted y-axis, which is $\frac{\pi}{2} + \theta^{\nu}{}_{mn} - \frac{\pi}{4} = \frac{\pi}{4} + \nu(2\pi)2^{-2^m(2n-1)}$. The effect of $\gamma_\nu^{s_1}\gamma_m^{s_2}\gamma_n^{s_3}$ on this angle is

$$\begin{aligned} \gamma_\nu^{s_1}\gamma_m^{s_2}\gamma_n^{s_3}(\theta^\nu_{mn}) &= \theta^{(\nu+s_1)_{\sigma(2^{2^{(m+s_2)(2(n+s_3)-1)}})}} \\ &= \frac{\pi}{4} + (\nu + s_1)_{\sigma(2^{2^{(m+s_2)(2(n+s_3)-1)}})}(2\pi)2^{-2^{(m+s_2)(2(n+s_3)-1)}} \end{aligned} \tag{4}$$

where $\sigma(N)$ represents cyclic permutation with respect to N. The gradient vector perpendular to $\gamma_\nu^{s_1}\gamma_m^{s_2}\gamma_n^{s_3}(S^{\nu}{}_{mn})$ is

$$cos\left(\frac{\pi}{4}+(\nu+s_1)_{\sigma(2^{2^{(m+s_2)(2(n+s_3)-1)}})}(2\pi)2^{-2^{(m+s_2)(2(n+s_3)-1)}}\right)\frac{\partial}{\partial x} \qquad (5)$$
$$+\, sin\left(\frac{\pi}{4}+(\nu+s_1)_{\sigma(2^{2^{(m+s_2)(2(n+s_3)-1)}})}(2\pi)2^{-2^{(m+s_2)(2(n+s_3)-1)}}\right)\frac{\partial}{\partial y}$$

Consequently,

$$\frac{\partial G}{\partial n}=-\frac{1}{4\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\nu=1}^{2^m(2n-1)} \qquad (6)$$
$$\left\{cos\left(\frac{\pi}{4}+(\nu+s_1)_{\sigma(2^{2^{(m+s_2)(2(n+s_3)-1)}})}(2\pi)2^{-2^{(m+s_2)(2(n+s_3)-1)}}\right)\frac{\partial}{\partial x}\right.$$
$$\left.+\, sin\left(\frac{\pi}{4}+(\nu+s_1)_{\sigma(2^{2^{(m+s_2)(2(n+s_3)-1)}})}(2\pi)2^{-2^{(m+s_2)(2(n+s_3)-1)}}\right)\frac{\partial}{\partial y}\right\}$$
$$ln\frac{(x-\gamma_\nu^{s_1}\gamma_m^{s_2}\gamma_n^{s_3}x')^2+(y-\gamma_\nu^{s_1}\gamma_m^{s_2}\gamma_n^{s_3}y')^2}{(x-\gamma_\nu^{s_1}\gamma_m^{s_2}\gamma_n^{s_3}x')^2+(y+\gamma_\nu^{s_1}\gamma_m^{s_2}\gamma_n^{s_3}y')^2}.$$

where the action of $\gamma_\nu^{s_1}\gamma_m^{s_2}\gamma_n^{s_3}$ on points in the disk may be defined by the rotation and radial translation required from the mapping of the midpoint of slit ${S^1}_{11}$ to the midpoint of ${S^{1+s_1}}_{1+s_2,\,1+s_3}$. Then the formula (2) with the Dirichlet boundary condition yields

$$\phi(z)=\frac{1}{4\pi}\int_{(x',y')\in {S^1}_{11}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\nu=1}^{2^m(2n-1)} \qquad (7)$$
$$\left\{cos\left(\frac{\pi}{4}+(\nu+s_1)_{\sigma(2^{2^{(m+s_2)(2(n+s_3)-1)}})}(2\pi)2^{-2^{(m+s_2)(2(n+s_3)-1)}}\right)\frac{\partial}{\partial x}\right.$$
$$\left.+\, sin\left(\frac{\pi}{4}+(\nu+s_1)_{\sigma(2^{2^{(m+s_2)(2(n+s_3)-1)}})}(2\pi)2^{-2^{(m+s_2)(2(n+s_3)-1)}}\right)\frac{\partial}{\partial y}\right\}$$
$$ln\frac{(x-\gamma_\nu^{s_1}\gamma_m^{s_2}\gamma_n^{s_3}x')^2+(y-\gamma_\nu^{s_1}\gamma_m^{s_2}\gamma_n^{s_3}y')^2}{(x-\gamma_\nu^{s_1}\gamma_m^{s_2}\gamma_n^{s_3}x')^2+(y+\gamma_\nu^{s_1}\gamma_m^{s_2}\gamma_n^{s_3}y')^2}.$$

The uniqueness of this solution follows from evaluating the difference of two harmonic measures with same boundary conditions to be a harmonic function vanishing everywhere on the boundary. By the maximum modulus principle, this harmonic function vanishes and the two series are equal.

3. Solutions Spaces for Parabolic Surfaces of Infinite Genus

The identification of theta function on the Riemann surface of infinite genus and periodic solutions to the Kadomtsev-Petviashvilli equation on $\mathbb{R}^2$ reflects an embedding into a calculus of symbols for this parabolic differential equation that would unify the two analytic function spaces.

The solutions to differential equations on surfaces of infinite genus belong to Sobolev spaces W_s^p. Given that the Hilbert spaces $\mathcal{H}$ and $\tilde{\mathcal{H}}$ admit a scaling group action, Λ is a conical manifold and the norm is $\langle\xi,\lambda\rangle_\ell=(1+|\xi|^{2\ell}+||^2)^{\frac{1}{2\ell}}$, which satisfies a linear convexity relation, the space of $\mathcal{L}(\mathcal{H},\tilde{\mathcal{H}})$ anisotropic symbols of order μ, Λ (Krainer, 2002) is
$S^{\mu;\ell}(\mathbb{R}^n\times\Lambda;\mathcal{H},\tilde{\mathcal{H}})=\Big\{a\in C^\infty(\mathbb{R}^n\times\Lambda,\mathcal{L}(\mathcal{H},\tilde{\mathcal{H}});\forall k\in\mathbb{N}_0:$
$sup_{\substack{(x,\lambda)\in\mathbb{R}^n\times\Lambda\\ |\beta|_\ell\le k}}\ \|\ \tilde{\kappa}^{-1}_{\langle\xi,\lambda\rangle_\ell}\partial^\beta_{(\xi,\lambda)}a(\xi,\lambda)\kappa_{\langle\xi,\lambda\rangle_\ell}\ \|\ \langle\xi,\lambda\rangle_\ell^{-\mu+|\beta|_\ell}<\infty\Big\}$ where $\kappa_\rho\in\mathcal{L}(\mathcal{H})$ and $\tilde{\kappa}_\rho\in\mathcal{L}(\tilde{\mathcal{H}})$ with $\rho\in\mathbb{R}^+$, while the space of classical symbols is $S^{\mu;\ell}_{cl}(\mathbb{H}^2\times\Lambda;\mathcal{H},\tilde{\mathcal{H}})=\Big\{a\in S^{\mu;\ell}(\mathbb{H}^2\times\Lambda,\mathcal{H},\tilde{\mathcal{H}});a\sim\sum_{k=0}^\infty\chi a_{(\mu-k)}\Big\}$ and $\chi\in C^\infty(\mathbb{H}^2\times\Lambda)=\begin{cases}0 & (x,\lambda)=(0,0)\\ 1 & x\to\infty\end{cases}$ and $a_{(\mu-k)}\in C^\infty(\mathbb{H}^2\times\Lambda\backslash\{(0,0)\},\mathcal{L}(\mathcal{H},\tilde{\mathcal{H}})$ consist of anisotropic functions of degree $\mu-k$, with $f(\rho\xi,\rho^\ell\lambda)=\rho^\mu\tilde{\kappa}_\rho f(\xi,\lambda)\kappa_\rho^{-1}$ (Krainer, 2002).

The norm may be defined on the product of the fundamental domain of the uniformizing Fuchsian group of the surface with Λ, $\mathcal{F}_\Gamma\times\Lambda=\mathbb{H}^2/\Gamma\times\Lambda$ to be

$$\langle\xi,\lambda\rangle_{\Sigma,\ell}=(1+|\xi|^{2\ell}_{\mathbb{H}}+|\lambda|^2)^{\frac{1}{2\ell}}. \qquad (8)$$

The metric in $\mathbb{H}^2$ is $\frac{dx^2+dy^2}{|y|^2}$ and has isometries given by the fractional linear transformations $z\to\frac{az+b}{cz+d}$ and fractional antilinear transformations $z\to\frac{c\bar{z}+d}{a\bar{z}+b}$ with $ad-bc=1$. The composition of two pure translations $z\to z+b_1$ and

$z \to z + b_2$ yields the pure translation $z \to z + b_1 + b_2$ with $b_1,\ b_2 \in \mathbb{R}$. The product of the two groups is isomorphic to $PSL(2;\mathbb{R}) \times PSL(2;\mathbb{R})$. It may be noted that fractional linear transformations with complex coefficients are isometries of $\mathbb{H}^3$. Since hyperbolic Riemann surfaces would represent boundary components of the space $\mathbb{H}^3/G$, where G is a discrete subgroup of $PSL(2;\mathbb{C})$, the restriction to the boundary yields a complex isometry on the covering space $\mathbb{H}^2$ if it preserves the upper half plane. The set of translations $z \to z + b$ where $Im\ b > 0$ have an image in $\mathbb{H}^2$ the composition defines a group. The composition of two translations along geodesics, by contrast, yields a mapping along another geodesic which is not related directly by translation either in $\mathbb{H}^2$ or the unit disk with the hyperbolic metric. The first type of translations will be considered in proving the convexity of the anisotropic norm.

Lemma 1. The norm on $\mathbb{H}^2/\Gamma$ satisfies the inequality

$$\langle \xi_1 + \xi_2, \lambda_1 + \lambda_2 \rangle_{\Sigma,\ell}^{|s|} \le c^{|s|} \langle \xi_1, \lambda_1 \rangle_\ell^{|s|} \langle \xi_2, \lambda_2 \rangle_\ell^{|s|}.$$

with $c = max\left(3, 1 + \frac{3c_1}{max(|\xi_1|_{\mathbb{H}}^2, |\xi_2|_{\mathbb{H}}^2, 2|\xi_1|_{\mathbb{H}}|\xi_2|_{\mathbb{H}})}\right)$ and $\left[|\xi_1|_{\mathbb{H}}^{2\ell_1}|\xi_2|_{\mathbb{H}}^{2\ell_2}(\sqrt{2}|\xi_1|_{\mathbb{H}}^{\frac{1}{2}}|\xi_2|_{\mathbb{H}})^{2\ell_3}\right]^{\frac{1}{\ell}} \le c_1$.

Proof.

The absolute value of $\gamma(\xi_1 + \xi_2)$ in the hyperbolic plane is

$$|\gamma(\xi_1 + \xi_2)|_{\mathbb{H}} = \frac{|\xi_1 + \xi_2|_E}{|Im(\xi_1 + \xi_2)|_E^2} \le \frac{|\xi_1|_E}{|Im(\xi_1)|^2} + \frac{|\xi_2|_E}{|Im(\xi_2)|^2} = |\gamma(\xi_1)|_{\mathbb{H}} + |\gamma(\xi_2)|_{\mathbb{H}}. \tag{9}$$

for any $\gamma \in \Gamma$. It follows that the norm $\langle \xi, \lambda \rangle_{\Sigma,\ell}$ is invariant under the action of Γ and can be defined on the fundamental domain. The inequalities

$$\begin{aligned}
\langle \xi_1 + \xi_2, \lambda_1 + \lambda_2 \rangle_\ell^{|s|} &= (1 + |\xi_1 + \xi_2|^{2\ell} + |\lambda_1 + \lambda_2|^2)^{\frac{|s|}{2\ell}} \\
&= \left[1 + \left(\frac{|\xi_1 + \xi_2|_E}{|Im(\xi_1 + \xi_2)|^2}\right)^{2\ell} + |\lambda_1 + \lambda_2|^2\right]^{\frac{|s|}{2\ell}} \\
&\le \left[1 + \left(\frac{|\xi_1|_E}{|Im(\xi_1)|_E^2} + \frac{|\xi_2|_E}{|Im(\xi_2)|_E^2}\right)^{2\ell}\right. \\
&\qquad \left. + 1 + |\lambda_1|^2 + |\lambda_2|^2 + |\lambda_1\lambda_2|^2\right]^{\frac{|s|}{2\ell}} \\
&\le \left[2 + \left(\frac{|\xi_1|_E^2}{|Im(\xi_1)|^4} + \frac{|\xi_2|_E^2}{|Im(\xi_2)|^4} + 2\frac{|\xi_1\xi_2|_E}{|Im(\xi_1)Im(\xi_2)|_E^2}\right)^\ell\right. \\
&\qquad \left. + |\lambda_1|^2 + |\lambda_2|^2 + |\lambda_1\lambda_2|^2\right]^{\frac{|s|}{\ell}} \\
&\le \left[2 + \left(|\xi_1|_{\mathbb{H}}^2 + |\xi_2|_{\mathbb{H}}^2 + 2|\xi_1||_{\mathbb{H}}|\xi_2|_{\mathbb{H}}\right)^\ell\right. \\
&\qquad \left. + |\lambda_1|^2 + |\lambda_2|^2 + |\lambda_1\lambda_2|\right]^{\frac{|s|}{2\ell}}
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
|\xi_1 + \xi_2|_{\mathbb{H}}^{2\ell} \le &\left[|\xi_1|_{\mathbb{H}}^2 + |\xi_2|_{\mathbb{H}}^2 + 2|\xi_1|_{\mathbb{H}}|\xi_2|_{\mathbb{H}}\right]^\ell \\
&\left(1 + \frac{3c_1}{|\xi_1|_{\mathbb{H}}^2 + |\xi_2|_{\mathbb{H}}^2 + 2|\xi_1|_{\mathbb{H}}|\xi_2|_{\mathbb{H}}}\right)^\ell,
\end{aligned} \tag{11}$$

where

$$|\xi_1|_{\mathbb{H}}^{2\ell_1}|\xi_2|^{2\ell_2}(|\xi_1|_{\mathbb{H}}|\xi_2|_{\mathbb{H}})^{2\ell_3} \le c_1^\ell, \tag{12}$$

yield

$$\langle \xi_1 + \xi_2, \lambda_1 + \lambda_2 \rangle_\ell^{|s|} \leq c^{|s|} \Big[3 + |\lambda_1|^2 + |\lambda_2|^2 + |\lambda_1 \lambda_2|^2 + |\xi_1|_{\mathbb{H}}^{2\ell} + |\xi_2|_{\mathbb{H}}^{2\ell} + (|\xi_1|_{\mathbb{H}} \xi_2|_{\mathbb{H}})^{2\ell} + |\lambda_1|^2 |\xi_2|_{\mathbb{H}}^{2\ell} + |\lambda_2|^2 |\xi_1|_{\mathbb{H}}^{2\ell} \Big]^{\frac{|s|}{2\ell}} \leq c^{|s|} \langle \xi_1, \lambda_1 \rangle_\ell^{|s|} \langle \xi_2, \lambda_2 \rangle_\ell^{|s|} \tag{13}$$

with $c = max\left(3, 1 + \frac{3c_1}{max(|\xi_1|_{\mathbb{H}}^2, |\xi_2|_{\mathbb{H}}^2, 2|\xi_1|_{\mathbb{H}} |\xi_2|_{\mathbb{H}})}\right)$.

□

The space of Volterra symbols is defined by the space of classical symbols with the parameter space Λ equal to $\mathbb{H}$. The Volterra symbols are given by
$a(\xi, \zeta) = \sum_{k=1}^{\infty} \chi\left(\frac{\xi}{c_k}, \frac{\zeta}{c_k^\ell}\right) a_k(\xi, \zeta) = \sum_{k=1}^{\infty} (H(\varphi(c_k t) a_k)(\xi, \zeta)$, $c_k \to \infty$ as $k \to \infty$, and

$$(H(\varphi)b)(\xi, \zeta) = \int_{\mathbb{H}^2} e^{-it\tau} \varphi(t) b(\xi, \eta - \tau) dt d\tau \underset{V}{\sim} \sum_{j=0}^{\infty} \left(\frac{-1)^j}{j!} D_t^j \varphi(0)\right) \partial_\zeta^j b(\xi, \zeta) \tag{14}$$

and the translation operator is $(T_{i\tau} a)(\xi, \zeta) = a(\xi, \zeta + i\tau)$ (Krainer, 2002). The operator product for the symbols would modified to

$$a\#b(x, \xi, \zeta) = \int \int e^{-iy\eta} a(x, \xi + \eta, \zeta) b(x + y, \xi, \zeta) \frac{dy}{|Im(y)|^2} d\eta \underset{V}{\sim} \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} (\partial^\alpha a)(D_x b). \tag{15}$$

The operator algebra is defined such that each symbol a has an inverse p with $a\#p - 1$ and $p\#a - 1$ is a symbol of order $-\infty$ belonging to $S^{-\infty;\ell}(\mathbb{H}^2 \times \Lambda, H, \tilde{H}) = \Big\{a \in C^\infty(\mathbb{H}^2 \times \Lambda, \mathcal{L}(H, \tilde{H})); \forall k \in \mathbb{N}_0 : \ sup_{\substack{(\xi,\lambda) \in \mathbb{H}^2 \times \Lambda \\ |\beta|_\ell \leq k}} \parallel \tilde{\kappa}^{-1}_{\langle \xi, \lambda \rangle_\ell} \partial^\beta_{(\xi,\lambda)} a(\xi, \lambda) \kappa_{\langle \xi, \lambda \rangle_\ell} \parallel \langle \xi, \lambda \rangle_\ell^\infty < \infty\Big\}$. Given that $\parallel \tilde{\kappa}^{-1}_{\langle \xi, \lambda \rangle_\ell^{-1}} \partial^\beta_{(\xi,\lambda)} a(\xi, \lambda) \kappa_{\langle \xi, \lambda \rangle_\ell} \parallel > 0$, $\langle \xi, \lambda \rangle_\ell \leq 1$ and $|\xi|_{\mathbb{H}}^{2\ell} = |\lambda| = 0$. Since $|\xi|_{\mathbb{H}} = \frac{|\xi|_E}{|Im(\xi)|^2}$, either $Im\ \xi = \infty$. Consequently, it is only necessary to establish that

$$\parallel \tilde{\kappa}^{-1}_{\langle \sigma \pm i\infty, 0 \rangle} \partial^\beta_{(\sigma + i\infty, 0)} a(\sigma + i\infty, 0) \kappa_{\langle \sigma + i\infty, 0 \rangle} \parallel = \parallel \tilde{\kappa}_1^{-1} \partial_\sigma^\beta a(\sigma + i\infty, 0) \kappa_1 \parallel = \parallel \partial_\sigma^\beta a(\sigma + i\infty, 0) \parallel < \infty. \tag{16}$$

since $1 + |\beta|^2 \leq k$ includes $\beta = 0$. Then $a(\xi, 0)$ must be bounded at infinity in all directions in the upper half plane. Furthermore, if $\xi \in \mathbb{F}_\Gamma$ has finite coordinates, $\langle \xi, \lambda \rangle_\ell > 1$ and

$$\parallel \tilde{\kappa}^{-1}_{\langle \xi, \lambda \rangle_\ell} \partial^\beta_{(\xi,\lambda)} a(\xi, \lambda) \kappa_{\langle \xi, \lambda \rangle} \parallel = 0 \tag{17}$$

and $a(\xi, \lambda)$ must vanish in this space.

The operator symbol with respect to a boundary parameterized by x is

$$op_x(a) : S_{cl}^{\mu;\ell}(\mathbb{H}^2 \times \mathbb{H}^2 \times \Lambda; \mathbb{C}^{N_-}, \mathbb{C}^-) \to S^{\mu;\ell}(\mathbb{R} \times \mathbb{R} \times \Lambda; H^{s,\delta}(\mathbb{R}_+, \mathbb{C}^{N_-}), H^{s-\mu,\delta}(\mathbb{C}^{N_-})) \quad s > -\frac{1}{2},\ \delta \in \mathbb{R}. \tag{18}$$

Near $r = 0$, the smoothing Mellin operator is

$$op_M^{\gamma-1}(h)u(r) = \frac{1}{2\pi i} \int_{\Gamma_{\frac{3}{2}-\gamma}} \int_{\mathbb{R}_+} \left(\frac{r}{r'}\right)^{-s} h(r, z) u(r') \frac{dr'}{r'} dz \tag{19}$$

The calculus of classical symbols includes singular Green, trace, potential and boundary symbols (Schrohe, 2001). The operator for the Green function can be expanded as $g = \sum_{j=0}^d g_j \partial_x^j \in S_{cl}^{\mu;\ell}(\mathbb{R} \times \mathbb{R} \times \Lambda, S'(\mathbb{R}_+), S(\mathbb{R}^+) \otimes \mathcal{L}(\mathbb{C}^{N_-}, \mathbb{C}^{N_+}))$, with $\mathcal{S}(\mathbb{R}_+) \simeq proj - lim_{s,\delta \in \mathbb{R}} H^{s,\delta}(\mathbb{R}_+)$ and $S'(\mathbb{R}_+) \simeq ind - lim_{s,\delta \in \mathbb{R}} H_0^{s,\delta}(\bar{\mathbb{R}}_+)$ and g_j is a symbol of order $\mu - j$ and type 0, the trace symbol of order μ and type d equals $t = \sum_{j=0}^d t_j \partial_x^j \in S_{cl}^{\mu;\ell}(\mathbb{R} \times \mathbb{R} \times \Lambda, \mathcal{S}'(\mathbb{R}_+), \mathbb{C}) \otimes \mathcal{L}(\mathbb{C}^{N_-}, \mathbb{C}^{M_+}))$, where t_j is a symbol of order $\mu - j$ and type 0, the potential symbol of order μ, $k \in S_{cl}^{\mu;\ell}(\mathbb{R} \times \mathbb{R}; \mathbb{C}, S(\mathbb{R}_+) \otimes \mathcal{L}(\mathbb{C}^{M_-}, \mathbb{C}^{N_+})$, with N_-, N_+ and M_-, M_+ being the complex dimension of the domain and range of symbols satisfies the transmission condition (Boutet de Monvel, 1971), requiring an upper bound for derivatives with homogeneous components, and the more general class, respectively, and the boundary symbol of order μ and type d is given by

$$a_0 = \begin{pmatrix} op_x(a) + g & k \\ t & s \end{pmatrix} \tag{20}$$

with the classical symbols being defined to be automorphic with respect to Γ, $a(\Gamma\xi, \lambda) = a(\xi, \lambda)$ (Krainer, 2002). It can be proven that $g = \sum_{j=0}^{d-1} k_j\gamma_j + g_0$, where k_j is a potential symbol of order $\mu - j - \frac{1}{2}$, $\gamma_\nu(f) = \partial_x^\nu f$ is a trace symbol of order $\nu + \frac{1}{2}$,

$$a_0\#b_0 = \begin{pmatrix} op_x(a\#_x b) + \tilde{g} & \tilde{k} \\ \tilde{t} & \tilde{s} \end{pmatrix} \tag{21}$$

a boundary symbol of order $\mu_1+\mu_2$ and type $d = max(\mu_2+d_1, d_2)$ with $a_0 = \begin{pmatrix} op_x(a) + g_1 & k_1 \\ t_1 & s_1 \end{pmatrix}$ and $b_0 = \begin{pmatrix} op_x(b) + g_2 & k_2 \\ t_2 & s_2 \end{pmatrix}$ are boundary symbols of order and type (μ_1, d_1) and (μ_2, d_2) respectively.

Boundary value problems on a manifold X of dimension n would be formulated on a boundary Y of dimension $n - 1$. The space of classical symbols have been defined with X and Y chosen to be $\mathbb{H}^2$ and $\mathbb{R}$ respectively. The ideal boundary of a Riemann surface is given by $(\mathcal{F} \cap \mathbb{R})/\Gamma$. A differential operator $A = \sum_{j=0}^{M} A_j(t)\partial_t^j = \sum_{j=0}^{M} A_j(-ln\ r)(-r\partial_r)^j$ and the solutions to $Au = f$ has the form $\sum_j \sum_{k=0}^{m_j} \tilde{c}_{j,k} log^k(r) r^{-p_j}$ as $r \to 0$. The data at the boundary determine the Mellin asymptotic type $\{(p_j, m_j, L_j),\ j \in \mathbb{Z}\}$, where $m_j \in \mathbb{N}_0$, L_j are finite-dimensional operators of $\mathcal{B}^{-\infty,d}(X)$ and $p_j \in \mathbb{C}$ such that $a(z) = \sum_{k_j=0}^{m_j} \nu_{k_j}(z - p_j)^{-(k_j+1)} + a_0(z)$.

Given a matrix

$$\begin{pmatrix} A & K \\ T & Q \end{pmatrix} \tag{22}$$

and the inverse

$$\begin{pmatrix} \hat{P} & \hat{K} \\ \hat{T} & \hat{Q} \end{pmatrix}. \tag{23}$$

provides a solution to the differential equation. Letting $\omega A\tilde{\omega} = op_M^{\gamma-1}(h) + A_{M+G}$ near $r = 0$, ω, $\tilde{\omega} \in C_0^\infty(\bar{R}_+)$, A_{M+G} is a Green operator of type d, $\tilde{h} = (H_{\gamma-1}h')(r, z)$ is the interior symbol, $P' = \omega_1 op_M^{\gamma-1}(\tilde{h})\omega_2 + (1 - \omega_1)op_r(\tilde{a}(1 - \omega_3)$, with $\chi_{[0,\tilde{T}_1]} < \omega_3 < \omega_1 < \omega_2 < \chi_{0,\tilde{T}_1]}$, $\tilde{P} = P' + \omega op_M^{\gamma-1}(g)\omega$, where $g = \sigma_M^0(A)^{-1} - \sigma_M^0(P') \in M_{V,Q}^{-\infty,t'}(X; \mathbb{H}_{\frac{3}{2}-\gamma})$, $P = \tilde{P}(1 + D_1)$ or $P = (1 + D_2)\tilde{P}$ satisfies $PA = 1$ (Krainer, 2002). Then $u = P^{-1}f$.

This technique easily transposes between $\mathbb{R}^2$ and $\mathbb{H}^2$ since the classical symbols had been defined initially for $X = \mathbb{R}^n$. The following lemma provides a mapping between solutions of elliptic and parabolic boundary value problems restricted by conditions on the function in the upper half plane.

Lemma 2. The solution to a parabolic boundary value problem with conditions on the real line is given by a generalized Fourier transform of a function satisfying an elliptic differential equation.

Proof.

Let $u(x, t)$ be a solution to an elliptic equation

$$\frac{\partial^2 u}{\partial t^2} + k\frac{\partial^2 u}{\partial x^2} = 0. \tag{24}$$

Let $U(x, p)$ be defined such that $\frac{\partial U(x,p)}{\partial p}$ is the inverse Laplace transform of $\frac{\partial u(x,t)}{\partial t}$ such that

$$U(x, p) = \frac{1}{2\pi i}\int_0^p d\tilde{p}\int_{\rho-i\infty}^{\rho+i\infty} e^{\tilde{p}t}\frac{\partial u(x, t)}{\partial t}dt. \tag{25}$$

where t is generalized to be a complex variable and the line $Re\ t = \rho$ is located to the right of any singularities of $\frac{\partial u(x,t)}{\partial t}$. Suppose that

$$\frac{\partial U(p, x)}{\partial p} = \kappa\frac{\partial^2 U(p, x)}{\partial x^2}. \tag{26}$$

Then

$$\int_{\rho-i\infty}^{\rho+i\infty} e^{pt}\frac{\partial u(x, t)}{\partial t}dt = \kappa\int_{\rho-i\infty}^{\rho+i\infty} e^{pt}\frac{\partial^2 u(x, t)}{\partial t^2}dt \tag{27}$$

and

$$\frac{1}{p}\int_{\rho-i\infty}^{\rho+i\infty}\frac{\partial}{\partial t}(e^{pt})\frac{\partial u(x,t)}{\partial t}dt = \kappa\int_{\rho-i\infty}^{\rho+i\infty}e^{pt}\frac{\partial^2 u(x,t)}{\partial t^2}dt. \tag{28}$$

After integration by parts, setting the boundary terms equal to zero,

$$-\frac{1}{p}\int_{\rho-i\infty}^{\rho+i\infty}e^{pt}\frac{\partial^2 u(x,t)}{\partial t^2}dt = \kappa\int_{\rho-i\infty}^{\rho+i\infty}e^{pt}\frac{\partial^2 u(x,t)}{\partial t^2}dt. \tag{29}$$

The equation derived from the integrand is

$$\frac{1}{p}\frac{\partial^2 u(x,t)}{\partial t^2} + \kappa\frac{\partial^2 u(x,t)}{\partial x^2} = 0 \tag{30}$$

which is the elliptic equation with $k = \kappa p$. The space of functions with $\frac{\partial u(x,t)}{\partial t}$ defined in the plane $Re\ t > \gamma$ such that $\left\|\frac{\partial u(x,t)}{\partial t}\right\|_\alpha^2 = sup_{\sigma>\gamma}\int\left|\frac{\partial u(x,t)}{\partial t}|_{t=\sigma+i\tau}\right|^2|\sigma+i\tau|^{2\alpha}d\tau < \infty$ is mapped to functions of the form $\frac{\partial U(x,p)}{\partial p}$ defined on the real line and equal to zero for $Re\ p < 0$, with $e^{-\gamma p}\frac{\partial U(x,p)}{\partial p} \in H_\alpha^1(\mathbb{R})$ for $Re\ \gamma > \rho$ and $\left\|\frac{\partial U(x,p)}{\partial p}\right\|_\alpha^2 = \left(\int_{\mathbb{R}}\left(1+|\xi|^{2\alpha}\right)\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\xi p}\frac{\partial U(x,p)}{\partial p}\right|^2 d\xi\right)^2 < \infty$ (Agronovich & Vishik, 1964).

□

The region of support in the p plane can be rotated by $\frac{\pi}{2}$ to the upper half-plane. It follows that the method may be used to invert the differential operator on $\mathbb{R}^2/\Gamma_1$ and $\mathbb{H}^2/\Gamma_2$. The mapping between the solutions with the specified asymptotic data follows from the relation between the integral transformations.

Consequently, it is necessary to consider the eigenvalue spectrum of the heat equation on $\mathbb{R}^2/\Gamma$ and the Kadomtsev-Petviashvilli equation. The mapping between the calculus of symbols for the this equation and the heat curve is given in the following theorem.

Theorem 2. There exists a transformation Φ from the class symbols of the heat equation operator on a curve of infinite genus and the Kadomtsev-Petviashvili equation on $\mathbb{R}^2$ with periodic boundary conditions.

Proof. The Kadomtsev-Petviashvili equation with periodic boundary conditions and the differential equation defining the heat curve may be formulated on $\mathbb{R}^2/\Gamma_1$ and $\mathbb{H}^2/\Gamma_2$ respectively for discrete group Γ_1 and Γ_2. The group Γ_1 is the infinite tensor product of $\Gamma_{per.}$ generated by two lattice vectors $(0, 2\pi)$ and (ω_1, ω_2) for each admissible value of (ξ_1, ξ_2). It is not possible to formulate the solution on $\mathbb{R}^2/\Gamma_{per.}$ because iterations of the boundary condition for $\xi_1 \neq 1$ and $\xi_2 \neq 1$ yield a different values of $\psi(x_1, x_2)$ at each lattice point. Instead, the quotient will be a surface consisting of an infinite sequence of genus-one components with the same monodromy factor for each solution. The Fuchsian group Γ_2 defined by the set of periodicity factors $(\xi_1, \xi_2) \in \mathbb{C}^* \times \mathbb{C}^*$ corresponding to nontrivial $\psi(x_1, x_2) \in L^2(\mathbb{R}^2/\Gamma_1)$ in the heat equation, requires dual group $\Gamma^\# = \left(\frac{2\pi}{\omega_1}, 0\right)\mathbb{Z} \oplus \left(-\frac{\omega_2}{\omega_1}, 1\right)\mathbb{Z}$, the operator $H_k = e^{-i\langle k,x\rangle}\left(\frac{\partial}{\partial x_1} - \frac{\partial^2}{\partial x_2^2}\right)e^{i\langle k,x\rangle} = \frac{\partial}{\partial x_1} - 2ik_1\frac{\partial}{\partial x_2} - \frac{\partial^2}{\partial x_2^2} + ik_1 + k_2^2$ and the union of the parabolas $\mathcal{H}(0) = \cup_{b\in\Gamma^\#}\mathcal{P}_b$, $\mathcal{P}_b = \{(k_1, k_2) \in \mathbb{C}^2|\mathcal{P}_b(k_1, k_2) = i(k_1 + b_1) + (k_2 + b_2)^2 = 0\}$ (Feldman, Knorrer Trubowitz, 2003). Therefore, condition on the periodicity factors is translated to a condition on eigenvalues of a related differential operator.

Following the integral representation of operator symbols and products on $\mathbb{R}^2$ and $\mathbb{H}^2$, the mapping Φ will be defined by

$$\Phi : \begin{pmatrix} A & K \\ T & Q \end{pmatrix}_{\mathbb{R}^2/\Gamma_1} \to \begin{pmatrix} A & K \\ T & Q \end{pmatrix}_{\mathbb{H}^2/\Gamma_2} \tag{31}$$

such that the transformation of the boundary symbol

$$\Phi_B : \begin{pmatrix} op_x(a) + g & k \\ t & s \end{pmatrix}_{\mathbb{R}^2/\Gamma_1} \to \begin{pmatrix} op_x(a) + g & k \\ t & s \end{pmatrix}_{\mathbb{H}^2/\Gamma_2} \tag{32}$$

is a continuous limit of Φ at the boundary $\mathbb{R}^2/\Gamma_1 \cap \mathbb{R}$. For a parabolic symbol, the boundary symbol is defined on a set of null harmonic measure. The inverse of the mapping

$$\Phi^{-1} : \begin{pmatrix} A & K \\ T & Q \end{pmatrix}^{-1}_{\mathbb{H}^2/\Gamma_2} \equiv \begin{pmatrix} \hat{P} & \hat{K} \\ \hat{T} & \hat{Q} \end{pmatrix}_{\mathbb{H}^2/\Gamma_2} \to \begin{pmatrix} A & K \\ T & Q \end{pmatrix}^{-1}_{\mathbb{R}^2/\Gamma_1} \equiv \begin{pmatrix} \hat{P} & \hat{K} \\ \hat{T} & \hat{Q} \end{pmatrix}_{\mathbb{R}^2/\Gamma_1} \tag{33}$$

with $\hat{P}$ yielding the inverse P of the differential operator A. It follows that solutions to the heat equation on the infinite-genus curve $\mathbb{H}^2/\Gamma_2$, including the theta function, would be mapped to a solution of the Kadomtsev-Petviashvili equation on $\mathbb{R}^2$ with periodic boundary conditions. The independence of the eigenvalue spectrum and the heat curve with respect to the time parameter allows an interpretation of the Green function of the differential operator in terms of the eigenvalues through $G(x_1, x_2) = \sum \frac{\psi_n^*(x_1)\psi_n(x_2)}{\lambda_m - \lambda}$. Consequently, the differential operator also can be represented by the eigenvalue spectrum and the infinite-genus curve. The coordinates on the Riemann surface may be derived from local complex coordinates with an expansion of the holomorphic one-forms in terms of the differentials of these coordinates together with vectors representing multiplicative coefficients (Feldman, Knorrer Trubowitz, 2003). It follows that the map Φ from the theta function on the Riemann surface to the solution to the Kadomtsev-Petviashvili equation only requires that the surface satisfies standard geometric hypotheses and bounds on the coefficients vectors in the theta function.

□

The introduction of a map from one differential system to another defined on a different domain may yield a method for solving a similar class of equations.

4. Conclusion

There exist infinite-genus surfaces with ideal boundaries that are represented as images under an infinite group of a single component. Then the Dirichlet problem may be solved through the method of images. The Green function can be expressed as an infinite sum of functions on the upper half plane with an infinite number of sources. The solution to the boundary value problem is an integral of the product of the field on a single component and the normal derivative of the Green function.

The formulation of the class of symbols directly on the hyperbolic plane facilitates the study of differential equations on Riemann surfaces of genus $g \geq 2$. The class symbols has been given on a manifold of the form $X \times [t_0, \infty)$, where X is $\mathbb{R}^n$. The dimension has been set equal to 2 and the integrals representing the symbols have been generalized to $\mathbb{H}^2$. Norm conditions have been demonstrated to be valid in hyperbolic space, which is necessary for the definition of equivalence and symbols with order $-\infty$. Consequently, it is possible to transform symbols and operators from $\mathbb{R}^2$ to $\mathbb{H}^2$.

Together with the quotient by the discrete uniformizing group, it has been found that there exists a transformation between a theta function on a Riemann surface of infinite genus and the solution to the Kadomtsev-Petviashvili equation satisfying periodic conditions. It is the operator between the two domains. The form of the solution requires the mapping to represent the definition of the holomorphic one-forms and the theta functions in terms of coordinates on the complex plane. This technique may have some degree of generality because an infinite symmetry with respect to one component of the surface is not necessary and mappings to domains with tractable boundary value problems would provide a method for deriving the solution.

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