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PURE MORPHISMS OF COMMUTATIVE RINGS ARE EFFECTIVE DESCENT MORPHISMS FOR MODULES – A NEW PROOF

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ABSTRACT. The purpose of this paper is to give a new proof of the Joyal-Tierney theorem (unpublished), which asserts that a morphism $f : R \to S$ of commutative rings is an effective descent morphism for modules if and only if f is pure as a morphism of R-modules.

Let R be a commutative ring with unit and $R-$mod the category of R-modules. Since, for any R-module M, the group $C(M) = \mathrm{Hom}_{Ab}(M, \mathbb{Q}/\mathbb{Z})$ (where Ab is the category of abelian groups and $\mathbb{Q}/\mathbb{Z}$ is the rational circle abelian group) becomes an R-module with the action of R on $C(M)$ by $(rf)(m) = f(rm)$, we can define a functor $C : (R-\mathrm{mod})^{op} \to R-\mathrm{mod}$, given by $C(M) = \mathrm{Hom}_{Ab}(M, \mathbb{Q}/\mathbb{Z})$. Since the abelian group $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator in the category of abelian groups (see, for example, [1]), the functor C is exact and reflects isomorphisms. We say that a morphism $f : M \to M'$ of R-modules is pure if for any R-module N,

$$1_N \otimes_R f : N \otimes_R M \to N \otimes_R M'$$

is monic. Let $f : M \to M'$ be a morphism of R-modules. The next lemma follows from the commutativity of the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_R(C(M), C(M')) & \xrightarrow{\mathrm{Hom}_R(C(M),C(f))} & \mathrm{Hom}_R(C(M, C(M))) \\
\downarrow \approx & & \downarrow \approx \\
C(C(M) \otimes_R M') & \xrightarrow[C(1_{C(M)} \otimes_R f)]{} & C(C(M) \otimes_R M),
\end{array}$$

where the vertical morphisms are the canonical isomorphisms.

1. LEMMA. *Let $f : M \to M'$ be a morphism of R-modules. The following conditions are equivalent:*

(a) f is a pure morphism of R-modules.

(b) $C(f)$ is a split epimorphism of R-modules.

Let $f : R \to S$ be a morphism of commutative rings. Recall that a descent datum on an object $M \in \mathrm{Ob}(S-\mathrm{mod})$ can be described as an S-module morphism $\theta : M \to S \otimes_R M$ such that θ makes

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$$\begin{array}{ccc} M & \xrightarrow{\theta} & S \otimes_R M \\ & \searrow\!\!\!\searrow & \downarrow{\scriptstyle \mu} \\ & & M \end{array}$$

and

$$\begin{array}{ccc} M & \xrightarrow{\quad\theta\quad} & S \otimes_R M \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle 1\otimes_R\theta} \\ S \otimes_R M & \xrightarrow[1\otimes_R i_M]{} & S \otimes_R S \otimes_R M \end{array}$$

commutative, where μ denotes the S-module structure on M, and $i_M : M \to S \otimes_R M$ is an R-morphism given by $i_M(m) = 1 \otimes_R m$.

Let $\mathrm{Des}(f)$ denote the category of pairs (M, θ), θ descent datum on $M \in \mathrm{Ob}(S-\mathrm{mod})$, in which morphisms $(M, \theta) \to (M', \theta')$ are just morphisms $g : M \to M'$ in $S-\mathrm{mod}$ which commute with descent data in the obvious sense (see, for example, [2]).

Any object $f^*(M) = (S \otimes_R M, i_M)$, $M \in \mathrm{Ob}(R-\mathrm{mod})$ can be equipped with descent data in a canonical way, and this gives rise to a commutative diagram

$$\begin{array}{ccc} R-\mathrm{mod} & \xrightarrow{f^*} & \mathrm{Des}(f) \\ & {\scriptstyle S\otimes_R -}\searrow & \downarrow{\scriptstyle U} \\ & & S-\mathrm{mod}, \end{array}$$

where U is the forgetful functor. f is said to be a descent morphism if f^* is full and faithful, and an effective descent morphism if f^* is an equivalence.

The functor f^* has a right adjoint f_* which is defined by requiring that

$$f_*(M, \theta) \xrightarrow{e} M \underset{i_M}{\overset{\theta}{\rightrightarrows}} S \otimes_R M$$

is an equalizer of S-modules for each $(M, \theta) \in \mathrm{Ob}(\mathrm{Des}(f))$. The counit of this adjunction is defined by $\delta_M = \mu(1 \otimes_R e)$. The unit $\epsilon_M : M \to f_*f^*(M)$ is obtained from the diagram

$$\begin{array}{ccccc} M & & & & \\ & \searrow^{i_M} & & & \\ {\scriptstyle\varepsilon_M}\,\vdots & & S \otimes_R M & \underset{i_{S\otimes_R M}}{\overset{1_S\otimes_R i_M}{\rightrightarrows}} & S \otimes_R S \otimes_R M. \\ & \nearrow & & & \\ f_*f^*(M) & & & & \end{array}$$

It does exist because i_M equalizes the two morphisms on the right hand side. From the description of ϵ and δ we obtain immediately the two following propositions.

2. Proposition. *$f : R \to S$ is a descent morphism if and only if f is pure as a morphism of R-modules.*

3. Proposition. *A descent morphism f is effective if and only if $S \otimes_R -$ preserves the equalizer*

$$f_*(M,\theta) \xrightarrow{\ e\ } M \underset{i_M}{\overset{\theta}{\rightrightarrows}} S \otimes_R M$$

for each $(M,\theta) \in Ob(Des(f))$.

Let $f : R \to S$ be pure as a morphism of R-modules. Then by Lemma 1 there is an R-module morphism $g : C(R) \to C(S)$ such that $c(f)g = 1_{C(R)}$.

If $(M,\theta) \in \mathrm{Ob}(\mathrm{Des}(f))$, then we have a commutative diagram

$$\begin{array}{ccccc}
f_*(M,\theta) & \xrightarrow{\ e\ } & M & \underset{i_M}{\overset{\theta}{\rightrightarrows}} & S \otimes_R M \\
 & & \downarrow{\scriptstyle i_M} & & \downarrow{\scriptstyle i_{S\otimes_R M}} \\
M & \xrightarrow{\ \theta\ } & S \otimes_R M & \underset{1_S \otimes_R i_M}{\overset{1_S \otimes_R \theta}{\rightrightarrows}} & S \otimes_R S \otimes_R M,
\end{array}$$

in which the rows are equalizer diagrams.

Applying the functor C to this diagram, we obtain the commutative diagram

$$\begin{array}{ccccc}
C(S \otimes_R S \otimes_R M) & \underset{C(1_S \otimes_R i_M)}{\overset{C(1_S \otimes_R \theta)}{\rightrightarrows}} & C(S \otimes_R M) & \xrightarrow{\ C(\theta)\ } & C(M) \\
{\scriptstyle C(i_{S\otimes_R M})}\downarrow & & \downarrow{\scriptstyle C(i_M)} & & \\
C(S \otimes_R M) & \underset{C(i_M)}{\overset{C(\theta)}{\rightrightarrows}} & C(M) & \xrightarrow[C(e)]{} & C(f_*(M,\theta)),
\end{array}$$

in which the rows are coequalizer diagrams. Now, since for any R-module P we have the isomorphism of functors $C(P \otimes_R -) \to \mathrm{Hom}_R(-, C(P))$, we obtain the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_R(S \otimes_R M, C(S)) & \underset{-\circ i_M}{\overset{-\circ\theta}{\rightrightarrows}} & \mathrm{Hom}_R(M, C(S)) & & \\
{\scriptstyle -\circ g}\uparrow\ \downarrow{\scriptstyle -\circ C(f)} & & {\scriptstyle -\circ g}\uparrow\ \downarrow{\scriptstyle -\circ C(f)} & & \\
\mathrm{Hom}_R(S \otimes_R M, C(R)) & \underset{-\circ i_M}{\overset{-\circ\theta}{\rightrightarrows}} & \mathrm{Hom}_R(M, C(R)) & \xrightarrow{\ -\circ e\ } & \mathrm{Hom}_R(f_*(M,\theta), C(R)).
\end{array}$$

Applying the above isomorphism of functors backwards, we deduce that there are R-morphisms h and h', such that the diagram

$$\begin{array}{ccccc}
C(S \otimes_R S \otimes_R M) & \underset{C(1_S \otimes_R i_M)}{\overset{C(1_S \otimes_R \theta)}{\rightrightarrows}} & C(S \otimes_R M) & \xrightarrow{\ C(\theta)\ } & C(M) \\
{\scriptstyle h'}\uparrow\ \downarrow{\scriptstyle C(i_{S\otimes_R M})} & & {\scriptstyle h}\uparrow\ \downarrow{\scriptstyle C(i_M)} & & \uparrow{\scriptstyle k} \\
C(S \otimes_R M) & \underset{C(i_M)}{\overset{C(\theta)}{\rightrightarrows}} & C(M) & \xrightarrow{\ C(e)\ } & C(f_*(M,\theta))
\end{array}$$

commutes. Since both left hand side squares commute, there is an R-morphism $k : C(f_*(M,\theta) \to C(M)$ such that $C(\theta)h = kC(e)$. It means that the bottom row becomes a split coequalizer diagram [3] in the category of R-modules,which is split by the morphisms

$$C(f_*(M,\theta)) \xrightarrow{k} C(M) \xrightarrow{h} C(S \otimes_R M).$$

Since split coequalizers are preserved by any functor, its image under the functor $\mathrm{Hom}(S,-)$ is a coequalizer diagram. So

$$\mathrm{Hom}_R(S, C(S \otimes_R M)) \underset{-\circ C(i_M)}{\overset{-\circ C(\theta)}{\rightrightarrows}} \mathrm{Hom}_R(S, C(M)) \xrightarrow{-\circ C(e)} \mathrm{Hom}_R(S, f_*(M,\theta))$$

is a coequalizer diagram, and hence so is

$$C(S \otimes_R S \otimes_R M) \underset{C(1_S \otimes_R i_M)}{\overset{C(1_S \otimes_R \theta)}{\rightrightarrows}} C(S \otimes_R M) \xrightarrow{C(1_S \otimes_R e)} C(S \otimes_R f_*(M,\theta)).$$

The functor C is exact and reflects isomorphisms. Therefore it also reflects coequalizers. It follows that

$$S \otimes_R f_*(M,\theta) \xrightarrow{1_S \otimes_R e} S \otimes_R M \underset{1_S \otimes_R i_M}{\overset{1_S \otimes_R \theta}{\rightrightarrows}} S \otimes_R S \otimes_R M$$

is an equalizer. But

$$M \xrightarrow{\theta} S \otimes_R M \underset{1_S \otimes_R i_M}{\overset{1_S \otimes_R \theta}{\rightrightarrows}} S \otimes_R S \otimes_R M$$

is also an equalizer diagram. Thus we have an isomorphism $S \otimes_R f_*(M,\theta) \to M$.

We obtain

4. THEOREM. *$f : R \to S$ is an effective descent morphism for modules if and only if f is pure as a morphism of R-modules.*

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