

## FINITE SETS AND SYMMETRIC SIMPLICIAL SETS

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ABSTRACT. The category of finite cardinals (or, equivalently, of finite sets) is the symmetric analogue of the category of finite ordinals, and the ground category of a relevant category of presheaves, the *augmented symmetric simplicial sets*. We prove here that this ground category has characterisations similar to the classical ones for the category of finite ordinals, by the existence of a universal symmetric monoid, or by generators and relations. The latter provides a definition of symmetric simplicial sets by faces, degeneracies and transpositions, under suitable relations.

### Introduction

The category  $\Delta^{\sim}$  of finite ordinals (and monotone mappings) is the basis of the presheaf category  $\mathbf{Smp}^{\sim}$  of augmented simplicial sets. It has well known characterisations, as:

- (a) *the free strict monoidal category with an assigned internal monoid;*
- (b) *the subcategory of  $\mathbf{Set}$  generated by finite ordinals, their faces and degeneracies;*
- (c) *the category generated by faces and degeneracies, under the cosimplicial relations.*

The last characterisation is currently used in the usual description of an augmented simplicial set as a sequence of sets with faces and degeneracies, subject to the (dual) simplicial relations. The restriction of this characterisation (c) to the category  $\Delta$  of *positive* finite ordinals plays the same role for *ordinary* (non augmented) simplicial sets (while (a) cannot be so restricted).

Here, in Theorems 4.1 and 4.2, we give similar characterisations for the "symmetric analogue", the category  $!\Delta^{\sim}$  of *finite cardinals*, with the same objects  $n \geq 0$  and all mappings, equivalent to the (large) category of finite sets.  $!\Delta^{\sim}$  is thus:

- (a') *the free strict monoidal category with an assigned symmetric monoid;*
- (b') *the subcategory of  $\mathbf{Set}$  generated by faces, degeneracies and main transpositions;*
- (c') *the category generated by faces, degeneracies and main transpositions, under the symmetric cosimplicial relations (Section 3).*

Again, the last characterisation gives a presentation of the non-augmented symmetric simplicial site  $!\Delta$ , and provides a definition of symmetric simplicial sets by faces, degeneracies and transpositions, under the dual relations.

To motivate the interest of such characterisations, let us recall that *symmetric simplicial sets*, i.e. the presheaves  $X: !\Delta^{op} \rightarrow \mathbf{Set}$  on finite positive cardinals, have been studied

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in [4, 5], where a combinatorial homotopy theory has been introduced for their category  $\mathbf{!Smp} = \mathbf{Set}^{\Delta^{op}}$ , extending a previous theory for simplicial complexes [3].

As a crucial advantage of the extension, we have a *fundamental  $n$ -groupoid* functor  $\Pi_n: \mathbf{!Smp} \rightarrow n\text{-}\mathbf{Gpd}$  ( $n \leq \omega$ ) left adjoint to a *symmetric nerve*  $M_n$ , which yields a strong, simple version of the Seifert-van Kampen theorem:  $\Pi_n$  preserves all colimits. Analogously, a notion of (non-reversible) *directed homotopy* has been developed in the ordinary simplicial topos  $\mathbf{Smp}$ , with applications to image analysis as in [3]; we have now a *fundamental  $n$ -category functor*  $\uparrow\Pi_n$ , left adjoint to a *nerve*  $N_n$ .

A classical reference on simplicial sets is May's book [11]. The characterisations of the category  $\Delta^\sim$  of finite ordinals can be found in Mac Lane's text [10]. For monoidal categories, see [10] and Kelly's book [8]. The case  $n = 1$  of the adjunction  $\Pi_n \dashv M_n$  was already noted by Lawvere [9], and was at the origin of this research.

NOTATION. As usual, finite ordinals and finite cardinals coincide, and are constructed as  $0 = \emptyset$ ,  $n = (n - 1) \cup \{n - 1\} = \{0, 1, \dots, n - 1\}$ . The term "graph" stands for *oriented graph*.

## 1. Reviewing the simplicial site

The category  $\Delta^\sim$  of finite ordinals has a rich structure (cf. [10], VII.5, where  $\Delta^\sim, \Delta$  are written as  $\Delta, \Delta^+$ , respectively); it is reviewed here as a leading frame for our symmetric analogue.

To begin with,  $\Delta^\sim$  is a strict monoidal category, with respect to the ordinal sum  $m + n$  (non-symmetric). The object 1 is an internal monoid

$$\begin{aligned} \partial : 0 \rightarrow 1 \leftarrow 2 : e \\ e(\partial + 1) = \text{id} = e(1 + \partial), \quad e(e + 1) = e(1 + e) \end{aligned} \tag{1}$$

with unit (or face)  $\partial$  and multiplication (or degeneracy)  $e$ . Then, the terminal mapping  $e^{(k)}: k \rightarrow 1$  appears to be an iterated multiplication, with

$$e^{(0)} = \partial, \quad e^{(1)} = \text{id}, \quad e^{(2)} = e, \quad e^{(3)} = e(e + 1) = e(1 + e), \dots$$

and each monotone mapping  $f: m \rightarrow n$  can be uniquely decomposed as a sum  $f = e^{(m_0)} + \dots + e^{(m_{n-1})}$  of iterated multiplications, where  $m_i = \#(f^{-1}\{i\})$ , and  $m = m_0 + \dots + m_{n-1}$ .

The usual (co)faces and (co)degeneracies can be constructed with the structural maps  $\partial, e$  and the monoidal structure (for  $0 \leq i \leq n$ )

$$\partial_i^n = i + \partial + (n - i): n \rightarrow n + 1, \quad e_i^n = i + e + (n - i): n + 2 \rightarrow n + 1 \tag{2}$$

(the injective monotone map which omits  $i$  and the surjective monotone map which repeats  $i$ , respectively); the *cosimplicial relations* follow easily from the previous formulae:

$$\begin{aligned} \partial_i \partial_j &= \partial_{j+1} \partial_i \quad (i \leq j), & e_j e_i &= e_i e_{j+1} \quad (i \leq j), \\ e_j \partial_i &= \partial_i e_{j-1}, \text{ or } 1, \text{ or } \partial_{i-1} e_j & (i < j \text{ or } i = j, j + 1 \text{ or } i > j + 1). \end{aligned} \tag{3}$$

A monotone mapping  $f: m \rightarrow n$  has a canonical factorisation

$$f = \partial_{j_1} \cdot \partial_{j_2} \cdot \dots \cdot e_{i_2} \cdot e_{i_1} \quad (m - 1 > i_1 > i_2 > \dots \geq 0; \quad n > j_1 > j_2 > \dots \geq 0) \quad (4)$$

by faces and degeneracies; every composite of faces and degeneracies can be put in canonical form, using the cosimplicial relations as rewriting rules (from the left).

Taking advantage of all this, the category  $\Delta^\sim$  of finite ordinals is characterised as:

- (a) the free strict monoidal category with an assigned internal monoid, 1;
- (b) the subcategory of **Set** generated by finite ordinals, their faces and degeneracies;
- (c) the category generated by the graph (2), subject to the cosimplicial relations (3).

## 2. Symmetric monoids

The category  $!\Delta^\sim$  of finite cardinals (equivalent to the category of finite sets) has a strict monoidal structure  $m + n$ , the (categorical) sum of cardinals, with a canonical symmetry (provided by the sum)

$$s: m + n \rightarrow n + m, \quad (5)$$

$$s(i) = n + i \quad (0 \leq i < m), \quad s(m + j) = j \quad (0 \leq j < n)$$

which is not strict (note that the identity  $m + n = n + m$  is not natural). Now,  $(1; \partial, e)$  is a commutative monoid within this enriched structure, satisfying the obvious axioms

$$e(\partial + 1) = \text{id} = e(1 + \partial), \quad e(e + 1) = e(1 + e), \quad es = e. \quad (6)$$

However, we want to be able to deal with "commutative" (or *symmetric*) monoids, within a mere monoidal category without symmetry (which is necessary for symmetric monads, cf. Section 6); this can be done by transferring the symmetry to the monoid itself. The object 1 is now viewed as an internal symmetric monoid, with a *unit* (or face)  $\partial$ , *multiplication* (or degeneracy)  $e$  and *symmetry*  $s$ ,

$$\partial : 0 \rightarrow 1 \leftarrow 2 : e \quad s : 2 \rightarrow 2 \quad (s(t) = 1 - t), \quad (7)$$

satisfying the axioms below (containing a Yang-Baxter condition on  $s$ , see [7] and references therein)

$$\begin{aligned} e(\partial + 1) &= \text{id} = e(1 + \partial), & e(e + 1) &= e(1 + e), \\ ss &= 1, & (s + 1)(1 + s)(s + 1) &= (1 + s)(s + 1)(1 + s), \\ s(\partial + 1) &= 1 + \partial, & es = e, & \quad s(1 + e) = (e + 1)(1 + s)(s + 1). \end{aligned} \quad (8)$$

(In the previous case the four new identities hold automatically, by the coherence theorem of symmetric monoidal categories and by naturality of  $s$ .)

### 3. The symmetric site

After higher faces and degeneracies, we can also construct in  $!\Delta^\sim$  the main transpositions  $s_i$  (the permutation which exchanges  $i, i + 1$ , for  $0 \leq i \leq n$ )

$$s_i = s_i^n = i + s + (n - i): n + 2 \rightarrow n + 2 \tag{9}$$

subject to the *Moore relations*:

$$s_i \cdot s_i = 1, \quad s_i \cdot s_j \cdot s_i = s_j \cdot s_i \cdot s_j \ (i = j - 1), \quad s_i \cdot s_j = s_j \cdot s_i \ (i < j - 1). \tag{10}$$

This is precisely the usual *Moore presentation* of the symmetric group  $S_{n+2}$ , the group of automorphisms of the set  $n + 2$ : generators  $s_i = (i, i + 1)$ , subject to the relations (10); see Coxeter-Moser [2], 6.2; or Johnson [6], Section 5, Thm. 3. ( $S_{n+2}$  also admits systems of two generators, e.g. the cyclic permutation  $(0, 1, \dots, n + 1)$  and  $s_0 = (0, 1)$ ; but then, the relations are complicated, cf. [2].)

Now, *faces, degeneracies and main transpositions form a system of generators for  $\Delta^\sim$* : an arbitrary mapping  $f: m \rightarrow n$  can be factorised as

$$f = h \cdot \rho, \quad h = f_0 + \dots + f_{n-1}, \tag{11}$$

$$(f_j = e^{(m_j)}: m_j \rightarrow 1, \quad m_j = \#(f^{-1}\{j\}))$$

where  $\rho: m \rightarrow m$  is a permutation and  $h$  is monotone; the latter is uniquely determined by  $f$ , as above, while  $\rho$  is not unique, generally:  $h\rho = h\sigma$  iff  $h\rho\sigma^{-1} = h$ , iff  $\rho\sigma^{-1}$  can be decomposed in a sum of permutations  $\sigma_0 + \dots + \sigma_{n-1}$ , coherently with the set-decomposition  $m = m_0 + \dots + m_{n-1}$ . (However,  $\rho$  is uniquely determined if we ask that  $\rho^{-1}$  be strictly monotone on each interval of the preceding decomposition of  $m$ ; then, depending on conventions,  $\rho$  and  $\rho^{-1}$  are respectively called an  $(m_0, \dots, m_{n-1})$ -*shuffle* and an  $(m_0, \dots, m_{n-1})$ -*deal*, or vice versa.)

Our generators satisfy the *symmetric cosimplicial relations*, consisting:

- of the usual *cosimplicial relations* for faces and degeneracies (3),
- of the *Moore relations* for transpositions (10),
- of the following *mixed relations*

$$\begin{aligned} s_i \partial_j &= \partial_j s_i, & s_i e_j &= e_j s_i & (i < j - 1), \\ s_i \partial_i &= \partial_{i+1}, & s_i e_i &= e_{i+1} s_i s_{i+1}, \\ s_i \partial_j &= \partial_j s_{i-1}, & s_i e_j &= e_j s_{i+1} & (i > j), \end{aligned} \tag{12}$$

$$e_i s_i = e_i, \tag{13}$$

which again follow easily from the structural properties (8).

It follows easily that  $s_i \partial_{i+1} = \partial_i$  and  $s_i e_{i+1} = e_i s_{i+1} s_i$ , so that the previous relations (12) can be viewed as rewriting rules for  $s_i \partial_j: n + 1 \rightarrow n + 2$  and  $s_i e_j: n + 3 \rightarrow n + 2$  ( $i \leq n; j \leq n + 1$ ), which allow one to transfer permutations to the right of monotone maps.

In general, given a strict monoidal category  $(\mathbf{A}, +, 0)$  and an internal symmetric monoid  $(a; \partial, e, s)$ , the "multiples"  $a + \dots + a$  are linked by a system of maps (for  $0 \leq i \leq n$ )

$$\begin{aligned} \partial_i^n &= ia + \partial + (n - i)a: na \rightarrow (n + 1)a, \\ e_i^n &= ia + e + (n - i)a: (n + 2)a \rightarrow (n + 1)a, \\ s_i^n &= ia + s + (n - i)a: (n + 2)a \rightarrow (n + 2)a, \end{aligned} \tag{14}$$

which satisfies the symmetric cosimplicial relations. As in Section 1, we write  $e^{(n)} : na \rightarrow a$  the  $n$ -ary multiplication, inductively defined as

$$e^{(0)} = \partial, \quad e^{(1)} = \text{id}, \quad e^{(n+1)} = e(e^{(n)} + 1) = e(1 + e^{(n)}); \tag{15}$$

one can easily deduce from (13), by induction, that  $e^{(n)} \cdot s_i^{n-2} = e^{(n)}$ .

## 4. Main results

4.1. THEOREM. (The internal symmetric monoid)  $!\Delta^\sim$  can be characterised as:

(a) the free strict monoidal category with an assigned symmetric monoid, 1.

PROOF. Let a strict monoidal category  $(\mathbf{A}, +, 0)$  be given, together with an internal symmetric monoid  $(a; \partial, e, s)$ ; we have to show that there is a unique strictly monoidal functor  $F: !\Delta^\sim \rightarrow \mathbf{A}$  sending 1 to  $a$  and preserving the structure. We already know that the "multiples"  $a + \dots + a$  form a symmetric cosimplicial object (14). From Section 1, there is a unique strictly monoidal functor  $F: \Delta^\sim \rightarrow \mathbf{A}$  sending 1 to  $a$  and preserving unit and multiplication; it operates in the obvious way on generators

$$F(n) = na, \quad F(\partial_i^n) = \partial_i^n, \quad F(e_i^n) = e_i^n. \tag{16}$$

Consider now the group  $S_{n+2}$  of automorphisms of  $n + 2$  in  $!\Delta^\sim$ ; on the main transpositions  $s_i^n = i + s + (n - i): n + 2 \rightarrow n + 2$  we must have

$$F(s_i^n) = ia + s + (n - i)a = s_i^n: (n + 2)a \rightarrow (n + 2)a; \tag{17}$$

on the other hand, since this setting is consistent with the Moore relations (10), we have defined a sequence of group-homomorphisms  $F: S_{n+2} \rightarrow \text{Aut}((n + 2)a)$ , and extended the functor  $F$  to all bijections of  $!\Delta^\sim$ .

Take now an arbitrary mapping  $f = h\rho: m \rightarrow n$ , factorised as above (11):  $h$  is monotone and  $\rho$  is a permutation; we must set  $Ff = Fh \cdot F\rho$ ; to show that the definition is correct, it is sufficient to verify that  $Fh = Fh \cdot Fs_i$ , for each main transposition  $s_i$  "acting within a summand" of  $m_0 + \dots + m_{n-1}$ ; taking for instance  $0 \leq i < m_0 - 1$ , we have (also by the identity  $e^{(n)} \cdot s_i^{n-2} = e^{(n)}$ , at the end of Section 3)

$$Fh \cdot Fs_i = (Ff_0 + \dots + Ff_{n-1}) \cdot Fs_i = (Ff_0 \cdot Fs_i) + \dots + Ff_{n-1} = Fh. \tag{18}$$

Last, we must prove that the extended mapping  $F$  preserves composition; let us begin showing that, for each monotone mapping  $h: m \rightarrow n$  and each permutation  $\sigma: n \rightarrow n$

$$\begin{array}{ccc}
 m & \xrightarrow{\tau} & m \\
 h \downarrow & & \downarrow k \\
 n & \xrightarrow{\sigma} & n
 \end{array} \tag{19}$$

one can find a permutation  $\tau: m \rightarrow m$  and a monotone  $k: m \rightarrow n$  such that the square above commutes, as well as its  $F$ -image in  $\mathbf{A}$ . One can assume that  $\sigma = s_i$ ; let  $h = h_u \cdot \dots \cdot h_1$  be the canonical factorisation of a monotone map (4); applying the rewriting rules deriving from the mixed relations (12), we obtain a factorisation  $\sigma h = k\tau$ , with a monotone map  $k = k_v \cdot \dots \cdot k_1$  (canonical factorisation (4)) and a permutation  $\tau = \tau_w \cdot \dots \cdot \tau_1$  (product of main transpositions). The same relations hold in  $\mathbf{A}$  (for its  $\partial_i^n, e_i^n, s_i^n$ ) and  $F$  preserves the composition within monotone maps and within bijections, whence

$$F\sigma \cdot Fh = F\sigma \cdot Fh_u \cdot \dots \cdot Fh_1 = Fk_v \cdot \dots \cdot Fk_1 \cdot F\tau_w \cdot \dots \cdot F\tau_1 = Fk \cdot F\tau. \tag{20}$$

Now, the functorial property for  $F$  follows easily: if the mappings  $f = h\rho: m \rightarrow n$  and  $f' = h'\sigma: n \rightarrow p$  are factorised as above, in (11), we rewrite  $\sigma h = k \cdot \tau$  as in diagram (19), and

$$Ff' \cdot Ff = Fh' \cdot F\sigma \cdot Fh \cdot F\rho = Fh' \cdot Fk \cdot F\tau \cdot F\rho = F(h'k) \cdot F(\tau\rho) = F(f'f). \tag{21}$$

(For the last equality, note that  $(h'k) \cdot (\tau\rho) = h'\sigma \cdot h\rho = f'f$  is an admissible factorisation of  $f'f$ , i.e. a permutation followed by a monotone map.) ■

4.2. THEOREM. (Presentation). *The category  $!\Delta^\sim$  can also be characterised as:*

(*b'*) *The subcategory of  $\mathbf{Set}$  generated by faces ( $\partial_i: n \rightarrow n + 1$ ), degeneracies ( $e_i: n + 2 \rightarrow n + 1$ ) and main transpositions ( $s_i: n + 2 \rightarrow n + 2$ ), where  $0 \leq i \leq n$ .*

(*c'*) *The category generated by faces, degeneracies and main transpositions, under the symmetric cosimplicial relations (Section 3).*

PROOF. (*b'*) is already known and (*c'*) follows easily from the previous characterisation (Theorem 4.1). Let  $!\Delta^\sim$  be defined by the presentation above. It is strictly monoidal: define the sum-functor in the obvious way ( $\partial_i + q = \partial_i, p + \partial_i = \partial_{p+i}$ , etc.) and check the consistency with relations. Then take a strict monoidal category  $(\mathbf{A}, +, 0)$  with an internal symmetric monoid  $(a; \partial, e, s)$ ; a strict monoidal functor  $F: !\Delta^\sim \rightarrow \mathbf{A}$  sending 1 to  $a$  and preserving the structure is uniquely determined on generators, as in (16), (17):  $F(n) = na, F(\partial_i^n) = ia + \partial + (n - i)a$ , etc. Conversely, defining  $F$  in this way is obviously consistent with relations, since all of them can be deduced from the axioms (8) and the monoidal structure. ■

### 5. Symmetric simplicial sets

Of course, the (non-augmented) symmetric simplicial site  $!\Delta$  can be presented as the category generated by faces, degeneracies and main transpositions between positive cardinals, under the restricted relations.

Therefore, a symmetric simplicial set  $X:!\Delta^{op} \rightarrow \mathbf{Set}$  can be assigned by the corresponding data  $(X_n, \partial_i^n, e_i^n, s_i^n)$ , where we write  $X_n = X[n] = X(n + 1)$ , as usual; faces, degeneracies and main transpositions (for  $0 \leq i \leq n$ )

$$\partial_i^n: X_n \rightarrow X_{n-1}, \quad e_i^n: X_n \rightarrow X_{n+1}, \quad s_i^n: X_{n+1} \rightarrow X_{n+1} \tag{22}$$

are to satisfy the symmetric simplicial relations (dual to the ones considered in Section 3). Equivalently, one can assign a simplicial set  $(X_n, \partial_i^n, e_i^n)$  and a right action of each symmetric group  $S_{n+1}$  on the component  $X_n$  ( $x\rho = \rho^*(x)$ ), coherently with faces and degeneracies (i.e., the latter have to satisfy the dual *mixed relations* with the main transpositions, cf. (12), (13)).

The usual embedding of  $\Delta$  in  $\mathbf{Top}$  extends easily to  $!\Delta$ , forming a symmetric cosimplicial object with the same components, the standard topological simplices  $!\Delta_n = \Delta_n$ , and extended actions  $\lambda^*$  (for all mappings  $\lambda: m \rightarrow n$ )

$$(!\Delta_n, \lambda^*): !\Delta \rightarrow \mathbf{Top}, \quad \lambda^*((t_i)_{i=0,\dots,n}) = (\sum_{\lambda i=j} t_i)_{j=0,\dots,m}. \tag{23}$$

This model of  $!\Delta$  in  $\mathbf{Top}$  gives rise to the functor  $!S_*$  of *symmetric singular simplices*

$$!S_*: \mathbf{Top} \rightarrow \mathbf{!Smp}, \quad !S_n(X) = \mathbf{Top}(!\Delta_n, X) \tag{24}$$

where the transposition  $s_i^n: !S_{n+1}(X) \rightarrow !S_{n+1}(X)$  amounts to a reflection of simplices, with respect to the symmetry hyperplane of the  $i$ -th,  $(i + 1)$ -th vertices of  $\Delta_n$ .

Its left adjoint is the (symmetric) geometric realisation functor  $\mathbf{!Smp} \rightarrow \mathbf{Top}$ : the realisation of the symmetric simplicial set  $X$  is the coend  $\int^{[n]} X_n \cdot !\Delta_n$  (of the inner functor  $!\Delta^{op} \times !\Delta \rightarrow \mathbf{Top}$ ). By Yoneda, the realisation of  $!\Delta[n]$  is  $!\Delta_n = \Delta_n$ .

### 6. Symmetric comonads

A comonad  $(K, \partial, e)$  in the category  $\mathbf{A}$  is a comonoid in the category  $\mathbf{End}(\mathbf{A})$  of endomorphisms of  $\mathbf{A}$ , with the strict monoidal structure of composition

$$\begin{aligned} \partial: K \rightarrow 1, & & e: K \rightarrow K^2, \\ \partial K \cdot e = \text{id}K = K\partial \cdot e & & eK \cdot e = Ke \cdot e; \end{aligned} \tag{25}$$

it generates an augmented simplicial object in  $\mathbf{End}(\mathbf{A})$ , and - by evaluation - a functor  $K_*: \mathbf{A} \rightarrow \mathbf{Smp}\tilde{(\mathbf{A})}$  with values in the category of augmented simplicial objects on  $\mathbf{A}$  (cf. [1])

$$\begin{aligned} K_*(X) = ((K_{n+1}(X)), (\partial_i^n), (e_i^n)) & \quad (n \geq -1; 0 \leq i \leq n), \\ \partial_i^n = K^{n-i}\partial K^i: K^{n+1} \rightarrow K^n, & \quad e_i^n = K^{n-i}eK^i: K^{n+1} \rightarrow K^{n+2}. \end{aligned} \tag{26}$$

For a category  $\mathbf{C}$ , the category of augmented simplicial objects  $Smp\tilde{\mathbf{C}}$  has a well known comonad, given by the shift (or decalage)  $\mathbf{K}$

$$\begin{aligned} \mathbf{K}: Smp\tilde{\mathbf{C}} &\rightarrow Smp\tilde{\mathbf{C}}, & \mathbf{K}X &= ((X_{n+1}), (\partial_{i+1}^{n+1}), (e_{i+1}^{n+1}))_{n \geq -1}, \\ \partial: \mathbf{K}X &\rightarrow X, & \partial &= \partial_0^{n+1}: X_{n+1} \rightarrow X_n, \\ e: \mathbf{K}X &\rightarrow \mathbf{K}^2 X, & e &= e_0^{n+1}: X_{n+1} \rightarrow X_{n+2} \end{aligned} \tag{27}$$

with counit  $\partial$  and comultiplication  $e$  consisting of the discarded faces and degeneracies.

In fact,  $Smp\tilde{\mathbf{C}}$  is the cofree category-with-comonad on  $\mathbf{C}$ , with respect to the forgetful functor  $|-|$  from categories with a comonad to categories. The counit-component  $|Smp\tilde{\mathbf{C}}| \rightarrow \mathbf{C}$  sends the augmented simplicial object  $X$  to  $X_{-1}$ , while the unit-component  $\mathbf{A} \rightarrow Smp\tilde{(|\mathbf{A}|)}$  is the functor  $K_*$  considered above.

Similarly, a *symmetric comonad*  $(K, \partial, e, s)$  will be a symmetric comonoid in  $\text{End}(\mathbf{A})$ : with respect to the previous structure, in (25), we have to add a *symmetry*  $s: K^2 \rightarrow K^2$ , satisfying

$$\begin{aligned} ss &= 1, & sK \cdot Ks \cdot sK &= Ks \cdot sK \cdot Ks, \\ \partial K \cdot s &= K\partial, & se &= e, & eK \cdot s &= Ks \cdot sK \cdot Ke. \end{aligned} \tag{28}$$

By the characterisation theorems of Section 4, it generates an augmented symmetric simplicial object in  $\text{End}(\mathbf{A})$  and a functor  $K_*: \mathbf{A} \rightarrow !Smp\tilde{\mathbf{A}}$ . Again,  $!Smp\tilde{\mathbf{C}}$  is the cofree category with symmetric comonad over  $\mathbf{C}$ .

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