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COHERENCE FOR FACTORIZATION ALGEBRAS

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ABSTRACT. For the 2-monad $((-)^{\mathbf{2}}, I, C)$ on **CAT**, with unit I described by identities and multiplication C described by composition, we show that a functor $F : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$ satisfying $FI_{\mathcal{K}} = 1_{\mathcal{K}}$ admits a unique, normal, pseudo-algebra structure for $(-)^{\mathbf{2}}$ if and only if there is a mere natural isomorphism $F \cdot F^{\mathbf{2}} \xrightarrow{\simeq} F \cdot C_{\mathcal{K}}$. We show that when this is the case the set of all natural transformations $F \cdot F^{\mathbf{2}} \to F \cdot C_{\mathcal{K}}$ forms a commutative monoid isomorphic to the centre of $\mathcal{K}$.

1. Preliminaries

1.1. When we speak of 'the 2-monad $(-)^{\mathbf{2}}$ on **CAT**' we understand the canonical monad that arises by exponentiation of the cocommutative comonoid structure

$$\mathbf{1} \xleftarrow{\;!\;} \mathbf{2} \xrightarrow{\;\Delta\;} \mathbf{2} \times \mathbf{2}$$

on the ordinal **2** in **CAT**. We write $I_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}^{\mathbf{2}}$ for the $\mathcal{K}$-component of the unit; it sends an object to its identity arrow. We write $C_{\mathcal{K}} : (\mathcal{K}^{\mathbf{2}})^{\mathbf{2}} \to \mathcal{K}^{\mathbf{2}}$ for the $\mathcal{K}$-component of the multiplication; it sends a commutative square to its composite arrow. This monad was very carefully described in [K&T], wherein it was shown that the normal pseudo-algebras for $(-)^{\mathbf{2}}$ are equivalent to factorization systems. In [R&W] Coppey's result [COP] that strict algebras for $(-)^{\mathbf{2}}$ are strict factorization systems was rediscovered (in the context of distributive laws). Mindful of the inflection terminology of [K&S] we call a normal pseudo-algebra for $(-)^{\mathbf{2}}$ a *factorization algebra* and call a strict algebra for $(-)^{\mathbf{2}}$ a *strict factorization algebra*. It is convenient in this context to call a mere functor $F : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$ a *factorization pre-algebra*. In the event that $FI_{\mathcal{K}} = 1_{\mathcal{K}}$, we say that F is a *normal* factorization pre-algebra (although this terminology does not entirely conform with that of [K&S]). (We are also aware that the lax factorization algebras of [R&T] are *certain* of the *oplax* algebras for $(-)^{\mathbf{2}}$ in the terminology of [K&S] but this presents no difficulty for the terminology employed here.)

1.2. REMARK. It was pointed out in [K&T] that the normality equation $FI_{\mathcal{K}} = 1_{\mathcal{K}}$ imposes no real loss of generality for a $(-)^{\mathbf{2}}$-pseudo-algebra. Given an isomorphism $\iota : 1_{\mathcal{K}} \xrightarrow{\simeq} FI_{\mathcal{K}}$, [K&T] explains how to define a new functor $F' : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$ with $F' \xrightarrow{\simeq} F$

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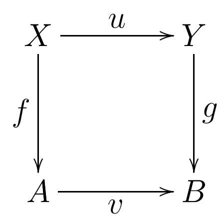
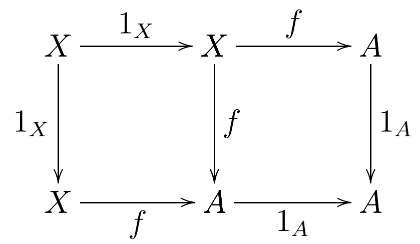
and $F'I_{\mathcal{K}} = 1_{\mathcal{K}}$. This certainly conforms with practice but it may be worth pointing out that intuitionistically the definition of F' requires that the identity arrows of $\mathcal{K}$ be a complemented subset of the set of all arrows of $\mathcal{K}$. While we will hypothesize normality throughout, we will include methodological remarks about the general case when appropriate.

1.3. Most of the notation here will be similar to that of [K&T] and [R&W]. In particular, for $F:\mathcal{K}^{\mathbf{2}}\to\mathcal{K}$ a factorization pre-algebra and an arrow



in $\mathcal{K}^{\mathbf{2}}$ with domain f and codomain g, we write $F(u,v):F(f)\to F(g)$. Since it is $(f;u,v;g)$, not (u,v), which is an arrow of $\mathcal{K}^{\mathbf{2}}$, this notation requires care. Also, for any arrow $f:X\to A$ in $\mathcal{K}$, we have the following factorization of $I_{\mathcal{K}}(f)$ in $\mathcal{K}^{\mathbf{2}}$:



and if $F : \mathcal{K}^{\mathbf{2}}\to\mathcal{K}$ is a normal factorization pre-algebra, we follow [K&T] in writing $X\xrightarrow{e_f} F(f)\xrightarrow{m_f} A$ for the result of applying F to these factors of $I_{\mathcal{K}}(f)$. Thus in this context, $F(1_X, f) = e_f$ and $F(f, 1_A) = m_f$ (but it is not true that we always have $F(1_S, f) = e_f$ and $F(f, 1_T) = m_f$). When the first square of this section is regarded as an object of $(\mathcal{K}^{\mathbf{2}})^{\mathbf{2}}$ we often write $S = (f;u,v;g)$ and it follows that $FF^{\mathbf{2}}(S) = F(F(u,v))$. If the composite $uf = gv$ is $c:X\to B$ then $FC_{\mathcal{K}}(S) = F(c)$. For $F:\mathcal{K}^{\mathbf{2}}\to\mathcal{K}$ a normal factorization pre-algebra, F is a strict factorization algebra if $FF^{\mathbf{2}} = FC_{\mathcal{K}}$, that is if, for all S in $(\mathcal{K}^{\mathbf{2}})^{\mathbf{2}}$, $F(F(u,v)) = F(c)$.

1.4. For a normal factorization pre-algebra $F:\mathcal{K}^{\mathbf{2}}\to\mathcal{K}$, a factorization algebra *structure* on F is an isomorphism $\gamma:FF^{\mathbf{2}}\xrightarrow{\simeq} FC_{\mathcal{K}}$ which satisfies:

$$\gamma I_{\mathcal{K}^{\mathbf{2}}} = 1_F \tag{1}$$

$$\gamma(I_{\mathcal{K}})^{\mathbf{2}} = 1_F \tag{2}$$

$$\gamma C_{\mathcal{K}^{\mathbf{2}}}\cdot\gamma(F^{\mathbf{2}})^{\mathbf{2}} = \gamma(C_{\mathcal{K}})^{\mathbf{2}}\cdot F\gamma^{\mathbf{2}} \tag{3}$$

these equations being the specialization of the equations in §2 of [STR] to the case at hand.

1.5. Remark. In the absence of normality, a $(-)^{\mathbf{2}}$-pseudo-algebra structure further requires an isomorphism $\iota : 1_{\mathcal{K}} \xrightarrow{\simeq} FI_{\mathcal{K}}$ and equations (1) and (2) above must then be replaced by:

$$\gamma I_{\mathcal{K}^{\mathbf{2}}} \cdot \iota F = 1_F \qquad (1')$$

$$\gamma (I_{\mathcal{K}})^{\mathbf{2}} \cdot F\iota^{\mathbf{2}} = 1_F \qquad (2')$$

2. The main results

For a normal factorization pre-algebra $F:\mathcal{K}^{\mathbf{2}} \to \mathcal{K}$, we will show that existence of a mere isomorphism $\alpha : FF^{\mathbf{2}} \xrightarrow{\simeq} FC_{\mathcal{K}}$, subject to no equations (other than naturality), is equivalent to the existence of a unique factorization algebra structure $\gamma : FF^{\mathbf{2}} \xrightarrow{\simeq} FC_{\mathcal{K}}$, and that such an α may itself fail to be an algebra structure. Of course this result shows that it is in fact a *property* of a factorization pre-algebra to be a factorization algebra, as is implicit from the conjunction of Theorems A and B in [K&T]. In general there is no comparison arrow joining $FF^{\mathbf{2}}$ and $FC_{\mathcal{K}}$ in either direction. But there is both a comparison span and a comparison cospan joining them. The span and cospan form a commutative square that one might call a *comparison diamond* and which is the subject of our first Lemma.

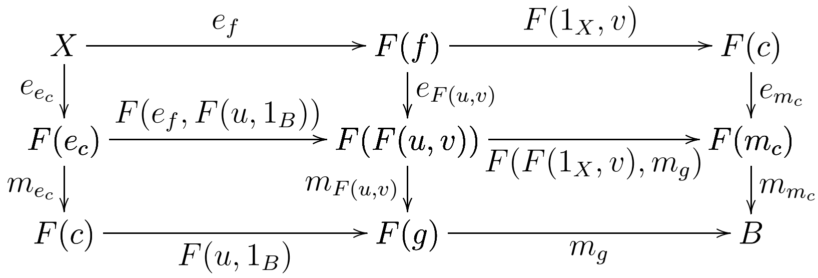
2.1. Lemma. *For $F:\mathcal{K}^{\mathbf{2}} \to \mathcal{K}$ a normal factorization pre-algebra and $S =$*

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[v]{} & B \end{array}$$

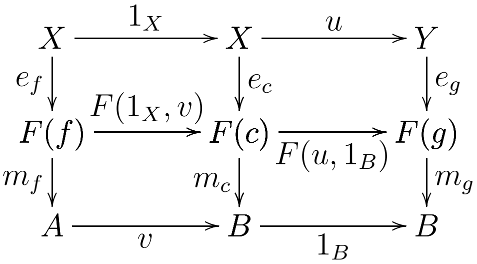
a typical object of $(\mathcal{K}^{\mathbf{2}})^{\mathbf{2}}$ with $C_{\mathcal{K}}(S) = X \xrightarrow{c} B$, the following diagram commutes:

$$\begin{array}{ccccc} & & F(e_c) & & \\ & {\scriptstyle F(e_f, F(u,1_B))} \swarrow & & \searrow {\scriptstyle m_{e_c}} & \\ F(F(u,v)) & & & & F(c) \\ & {\scriptstyle F(F(1_X,v),m_g)} \searrow & & \swarrow {\scriptstyle e_{m_c}} & \\ & & F(m_c) & & \end{array}$$

PROOF. Observe first that the diagram of of the statement is meaningfully labelled and that the left-most arrows of it are defined by



where, in turn, $F(1_X, v)$ and $F(u, 1_B)$ are defined by



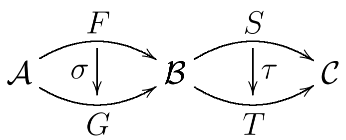
From the definition of e_- and m_-, we have $F(1_X, v) \cdot e_f = e_c$ and $m_g \cdot F(u, 1_B) = m_c$. Thus by functoriality of F, $F(F(1_X, v), m_g) \cdot F(e_f, F(u, 1_B)) = F(e_c, m_c)$. On the other hand, the right-most composite is

$$e_{m_c} \cdot m_{e_c} = F(1_{F(c)}, m_c) \cdot F(e_c, 1_{F(c)}) = F(e_c, m_c)$$

■

Our second Lemma is almost a triviality and, like much of what we have to say here, is generalizable in many ways. However, along with obvious variations, this Lemma is quite useful for a number of coherence questions.

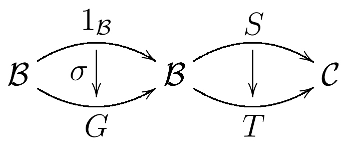
2.2. LEMMA. *For natural transformations*



if $T\sigma$ *is invertible then* τF *is determined by* τG*, in the sense that* $\tau F = (T\sigma)^{-1} \cdot \tau G \cdot S\sigma$*. Similarly, if* $S\sigma$ *is invertible then* τG *is determined by* τF*.* ■

To give an example of the application of Lemma 2.2, suppose that $\mathcal{X}$ is a reflective subcategory of $\mathcal{B}$ with defining adjunction $\eta, \epsilon : A \dashv I : \mathcal{X} \to \mathcal{B}$. Then, if T inverts η, τ is determined by its components of the form τIX. In fact, in situations like this we can say a little more:

2.3. Corollary. *For*



with $T\sigma$ invertible and G well-pointed by σ (meaning that $G\sigma = \sigma G$), precomposition with G provides a bijection

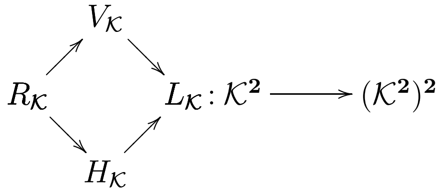
$$\mathbf{CAT}(\mathcal{B},\mathcal{C})(S,T) \xrightarrow{(-)G} \mathbf{CAT}(\mathcal{B},\mathcal{C})(SG,TG)$$

Proof. For $\omega : SG \to TG$, consider the following squares:

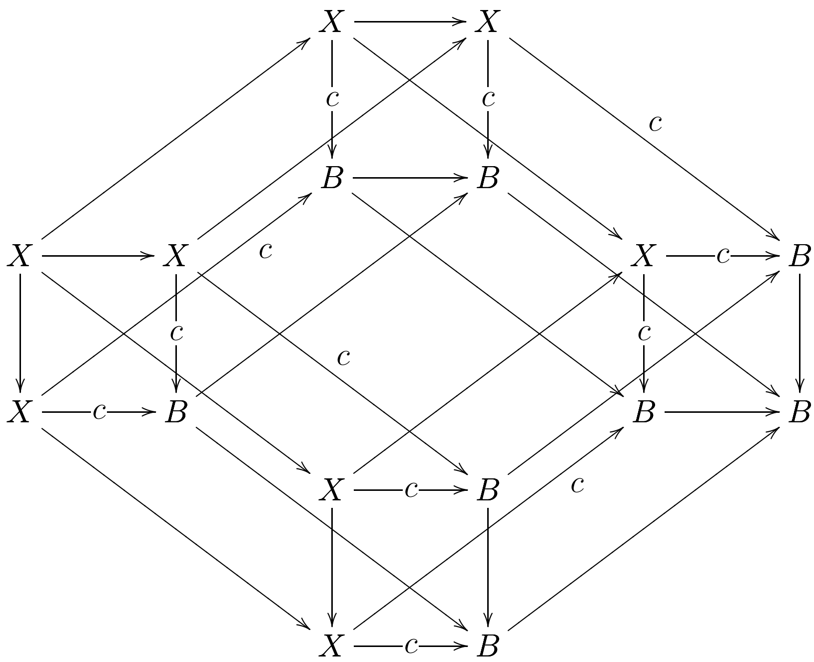
$$\begin{array}{ccc} S & \xrightarrow{S\sigma} & SG \\ {\scriptstyle g(\omega)}\downarrow & & \downarrow{\scriptstyle \omega} \\ T & \xleftarrow[(T\sigma)^{-1}]{} & TG \end{array} \qquad \begin{array}{ccc} SG & \xrightarrow{S\sigma G} & SGG \\ {\scriptstyle g(\omega)G}\downarrow & & \downarrow{\scriptstyle \omega G} \\ TG & \xrightarrow[T\sigma G]{} & TGG \end{array} \qquad \begin{array}{ccc} SG & \xrightarrow{SG\sigma} & SGG \\ {\scriptstyle \omega}\downarrow & & \downarrow{\scriptstyle \omega G} \\ TG & \xrightarrow[TG\sigma]{} & TGG \end{array}$$

The first square defines a function $g : \mathbf{CAT}(\mathcal{B},\mathcal{C})(SG,TG) \to \mathbf{CAT}(\mathcal{B},\mathcal{C})(S,T)$ which by Lemma 2.2 admits $(-)G$ as a section. The second square commutes by instantiating an evidently commutative square at G. The third commutes by naturality of ω. Since $\sigma G = G\sigma$, the top arrows of the second and third squares are equal and similarly so are the bottom arrows of the second and third squares. Since $T\sigma$ is invertible, it follows that $g(\omega)G = \omega$ which completes the proof that $(-)G$ is a bijection. ∎

2.4. We return now to $(-)^{\mathbf{2}}$. In addition to $C_{\mathcal{K}} : (\mathcal{K}^{\mathbf{2}})^{\mathbf{2}} \to \mathcal{K}^{\mathbf{2}} \leftarrow \mathcal{K} : I_{\mathcal{K}}$, it is convenient to name four embeddings $\mathcal{K}^{\mathbf{2}} \to (\mathcal{K}^{\mathbf{2}})^{\mathbf{2}}$ and indicate natural transformations between them as below:



These are defined on an object $c\colon X \rightarrow B$ of $\mathcal{K}^{\mathbf{2}}$ by the following commutative square in $(\mathcal{K}^{\mathbf{2}})^{\mathbf{2}}$

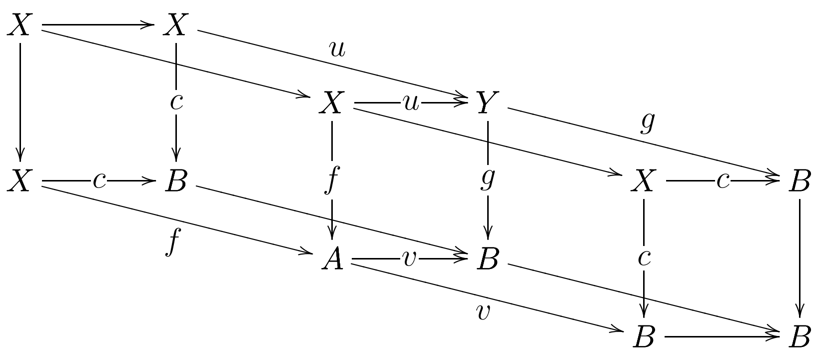


it being understood that unlabelled arrows are identities. However, $V_{\mathcal{K}} = I_{\mathcal{K}^{\mathbf{2}}}$, $H_{\mathcal{K}} = (I_{\mathcal{K}})^{\mathbf{2}}$ and $R_{\mathcal{K}} \dashv C_{\mathcal{K}} \dashv L_{\mathcal{K}}$.

2.5. The unit for $R_{\mathcal{K}} \dashv C_{\mathcal{K}}$ and the counit for $C_{\mathcal{K}} \dashv L_{\mathcal{K}}$ are identities. For $S =$

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[v]{} & B \end{array}$$

a typical object of $(\mathcal{K}^{\mathbf{2}})^{\mathbf{2}}$ with $C_{\mathcal{K}}(S) = X \xrightarrow{c} B$, the S-component of the counit for $R_{\mathcal{K}} \dashv C_{\mathcal{K}}$ and the S-component of the unit for $C_{\mathcal{K}} \dashv L_{\mathcal{K}}$ are shown below as a composable pair in $(\mathcal{K}^{\mathbf{2}})^{\mathbf{2}}$



Now for F a normal factorization pre-algebra, let $\alpha : FF^{\mathbf{2}} \to FC_{\mathcal{K}} : (\mathcal{K}^{\mathbf{2}})^{\mathbf{2}} \to \mathcal{K}$ be *any* natural transformation. Application of α to the composable pair above gives

$$\begin{array}{ccccc} F(e_c) & \xrightarrow{F(e_f, F(u, 1_B))} & F(F(u,v)) & \xrightarrow{F(F(1_X, v), m_g)} & F(m_c) \\ {\scriptstyle \alpha R_{\mathcal{K}} c = \alpha R_{\mathcal{K}} C_{\mathcal{K}} S}\downarrow & & \downarrow{\scriptstyle \alpha S} & & \downarrow{\scriptstyle \alpha L_{\mathcal{K}} C_{\mathcal{K}} S = \alpha L_{\mathcal{K}} c} \\ F(c) & \xrightarrow[1_{F(c)}]{} & F(c) & \xrightarrow[1_{F(c)}]{} & F(c) \end{array}$$

as is seen by consulting the definitions of e_f and the like in 1.3. For an arbitrary natural transformation $\beta : FC_{\mathcal{K}} \to FF^{\mathbf{2}}$ we get a diagram similar to that above but with the vertical arrows reversed. From these diagrams several observations follow almost immediately.

2.6. Lemma. *For F a normal factorization pre-algebra, any natural transformation $\alpha : FF^{\mathbf{2}} \to FC_{\mathcal{K}}$ is determined by $\alpha L_{\mathcal{K}} : FF^{\mathbf{2}} L_{\mathcal{K}} \to F : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$. Any natural transformation $\beta : FC_{\mathcal{K}} \to FF^{\mathbf{2}}$ is determined by $\beta R_{\mathcal{K}} : FF^{\mathbf{2}} R_{\mathcal{K}} \to F : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$. If α is an isomorphism then $F(e_f, F(u, 1_B))$ and $F(F(1_X, v), m_g)$ are isomorphisms.*

Proof. The right hand square immediately above the statement of the Lemma shows that the hypotheses of Lemma 2.2 are satisfied — with τ the α under consideration and σ the unit for $C_{\mathcal{K}} \dashv L_{\mathcal{K}}$. The second statement follows from a similar consideration while the third also follows from the diagram. ∎

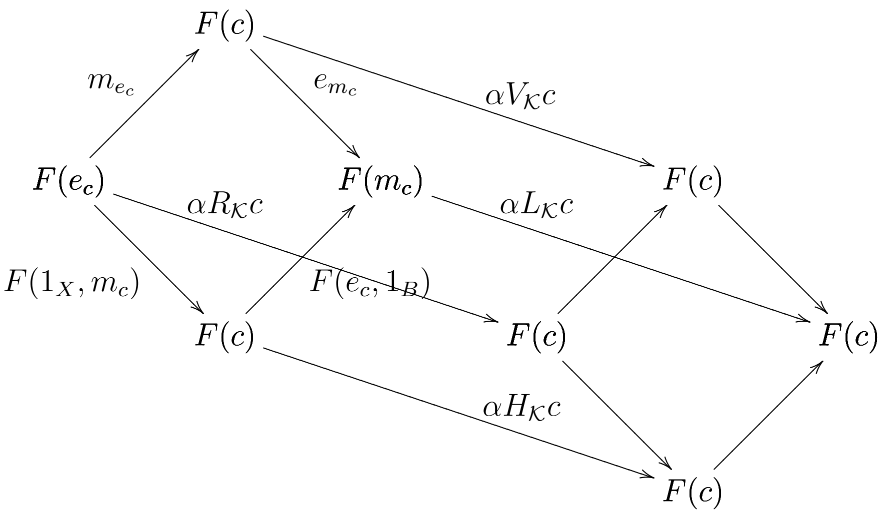
In fact, we can improve Lemma 2.6 by applying Corollary 2.3, with the role of $\sigma : 1_{\mathcal{B}} \to G$ taken by $1_{(\mathcal{K}^{\mathbf{2}})^{\mathbf{2}}} \to L_{\mathcal{K}} C_{\mathcal{K}}$, and noting that $C_{\mathcal{K}}$ is cofully faithful (since $L_{\mathcal{K}}$ is fully faithful).

2.7. Corollary. *For F a normal factorization pre-algebra, precomposition with $L_{\mathcal{K}}$ provides a bijection*

$$\mathbf{CAT}((\mathcal{K}^{\mathbf{2}})^{\mathbf{2}}, \mathcal{K})(FF^{\mathbf{2}}, FC_{\mathcal{K}}) \xrightarrow{(-)L_{\mathcal{K}}} \mathbf{CAT}(\mathcal{K}^{\mathbf{2}}, \mathcal{K})(FF^{\mathbf{2}} L_{\mathcal{K}}, F)$$

∎

We now apply our arbitrary $\alpha : FF^{\mathbf{2}} \to FC_{\mathcal{K}}$ to the second diagram in 2.4 resulting in the commutativity of



from which further observations follow.

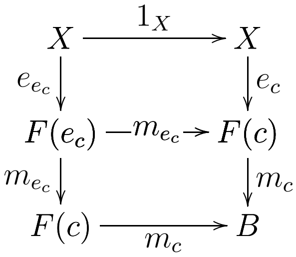
2.8. Lemma. *For F a normal factorization pre-algebra and any natural transformation $\alpha\colon FF^{\mathbf{2}}\to FC_{\mathcal{K}}$, $\alpha R_{\mathcal{K}}$ is determined by $\alpha V_{\mathcal{K}}$ which in turn is determined by $\alpha L_{\mathcal{K}}$ and also $\alpha R_{\mathcal{K}}$ is determined by $\alpha H_{\mathcal{K}}$ which in turn is determined by $\alpha L_{\mathcal{K}}$. If $\alpha : FF^{\mathbf{2}}\to FC_{\mathcal{K}}$ is an isomorphism then, for every arrow c in $\mathcal{K}$, m_{e_c}, e_{m_c}, $F(1_X, m_c)$ and $F(e_c, 1_B)$ are isomorphisms and*

$$m_{e_c} = F(1_X, m_c) \quad \text{and} \quad e_{m_c} = F(e_c, 1_B)$$

Proof. All aspects of the first sentence follow from the fact that all four arrows in the right-most diamond of the diagram above are identities and thereby enable four applications of Lemma 2.2. From the diagram it is clear that if α is an isomorphism then all m_{e_c}, e_{m_c}, $F(1_X, m_c)$ and $F(e_c, 1_B)$ are isomorphisms. Now from [J&T], merely knowing that the m_{e_c} are epimorphisms and the e_{m_c} are monomorphisms is enough to ensure that, for each object $S = (f; u, v; g)$ of $(\mathcal{K}^{\mathbf{2}})^{\mathbf{2}}$, $F(u, v)$ is the unique solution s of the equations

$$\begin{aligned} s \cdot e_f &= e_g \cdot u \\ m_g \cdot s &= v \cdot m_f \end{aligned}$$

Thus commutativity of



shows that $m_{e_c} = F(1_X, m_c)$ and a similar diagram provides $e_{m_c} = F(e_c, 1_B)$. ∎

In fact, from the diagram preceding Lemma 2.8 and the Lemma itself we have:

2.9. COROLLARY. *For F a normal factorization pre-algebra and any natural isomorphism $\alpha : FF^{\mathbf{2}} \rightarrow FC_{\mathcal{K}}$, the following are equivalent:*

i) $\alpha V_{\mathcal{K}} = 1_F$;

ii) $\alpha L_{\mathcal{K}} = (e_{m_-})^{-1}$;

iii) $\alpha H_{\mathcal{K}} = 1_F$;

iv) $\alpha R_{\mathcal{K}} = m_{e_-}$.

■

Recall now that $V_{\mathcal{K}} = I_{\mathcal{K}^{\mathbf{2}}}$ and $H_{\mathcal{K}} = (I_{\mathcal{K}})^{\mathbf{2}}$.

2.10. THEOREM. *For a normal factorization pre-algebra $F : \mathcal{K}^{\mathbf{2}} \rightarrow \mathcal{K}$, if $\gamma : FF^{\mathbf{2}} \xrightarrow{\simeq} FC_{\mathcal{K}}$ is an isomorphism satisfying any of the equivalent conditions of Corollary 2.9 then, for $S = (f; u, v; g)$ in $(\mathcal{K}^{\mathbf{2}})^{\mathbf{2}}$ with $C_{\mathcal{K}}(S) = X \xrightarrow{c} B$, γS is given equally by*

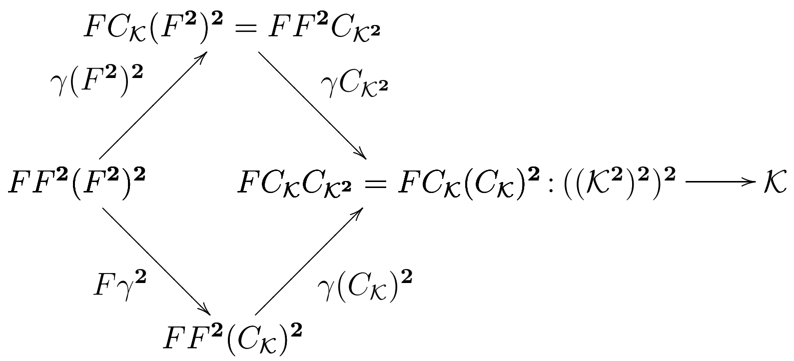
$$\gamma(S) = m_{e_c} \cdot (F(e_f, F(u, 1_B)))^{-1}$$

and by

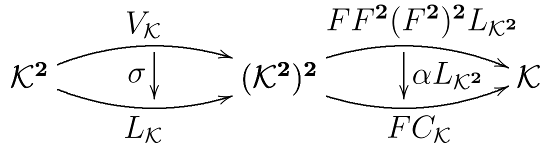
$$\gamma(S) = (e_{m_c})^{-1} \cdot F(F(1_X, v), m_g)$$

Moreover, γ satisfies all of (1),(2) and (3) of 1.4, thereby making (F, γ) a factorization algebra.

PROOF. Assume that $\gamma : FF^{\mathbf{2}} \xrightarrow{\simeq} FC_{\mathcal{K}}$ satisfies condition i) of Corollary 2.9. This is (1) of 1.4. The equations for $\gamma(S)$ follow immediately from the diagrams preceding Lemma 2.6 and Lemma 2.8. For example, we have $\gamma(S) = \gamma L_{\mathcal{K}} c \cdot F(F(1_X, v), m_g)$ by the righthand rectangle preceding Lemma 2.6, which is equal to $(e_{m_c})^{-1} \cdot F(F(1_X, v), m_g)$ using the back parallelogram in the diagram preceding Lemma 2.8 with $\gamma V_{\mathcal{K}} c = 1_{F(c)}$. Also iii) of Corollary 2.9 is (2) of 1.4. For (3) of 1.4 observe that the relevant configuration is



To see that the two composite natural transformations are equal we begin by applying Corollary 2.7 twice. First, the adjunction $C_{\mathcal{K}^{\mathbf{2}}} \dashv L_{\mathcal{K}^{\mathbf{2}}} : (\mathcal{K}^{\mathbf{2}})^{\mathbf{2}} \rightarrow ((\mathcal{K}^{\mathbf{2}})^{\mathbf{2}})^{\mathbf{2}}$ has its unit taken by $C_{\mathcal{K}^2}$ to an identity because, for *any* $\mathcal{K}$, the adjunction $C_{\mathcal{K}} \dashv L_{\mathcal{K}}$ has its unit taken by $C_{\mathcal{K}}$ to an identity. Thus $FC_{\mathcal{K}}C_{\mathcal{K}^2}$ takes the unit of $C_{\mathcal{K}^2} \dashv L_{\mathcal{K}^2}$ to an identity and Corollary 2.7 applies to show that *any* natural transformation $\alpha : FF^{\mathbf{2}}(F^{\mathbf{2}})^{\mathbf{2}} \to FC_{\mathcal{K}}C_{\mathcal{K}^2}$ is uniquely determined by $\alpha L_{\mathcal{K}^2}C_{\mathcal{K}^2}$ and hence by $\alpha L_{\mathcal{K}^2} : FF^{\mathbf{2}}(F^{\mathbf{2}})^{\mathbf{2}}L_{\mathcal{K}^2} \to FC_{\mathcal{K}} : (\mathcal{K}^{\mathbf{2}})^{\mathbf{2}} \to \mathcal{K}$ (since $C_{\mathcal{K}^2}$ is cofully faithful). Next, we repeat the argument for $C_{\mathcal{K}} \dashv L_{\mathcal{K}}$ to show that $\alpha L_{\mathcal{K}^2}$ is uniquely determined by $\alpha L_{\mathcal{K}^2}L_{\mathcal{K}} : FF^{\mathbf{2}}(F^{\mathbf{2}})^{\mathbf{2}}L_{\mathcal{K}^2}L_{\mathcal{K}} \to F : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$. Consider now



where σ is the natural transformation $V_{\mathcal{K}} \to L_{\mathcal{K}}$ defined in 2.4. We claim $FF^{\mathbf{2}}(F^{\mathbf{2}})^{\mathbf{2}}L_{\mathcal{K}^2}\sigma$ is invertible. To see this, use $(F^{\mathbf{2}})^{\mathbf{2}}L_{\mathcal{K}^2} = L_{\mathcal{K}}F^{\mathbf{2}}$ and compute

$$FF^{\mathbf{2}}L_{\mathcal{K}}F^{\mathbf{2}}\sigma c = e_{m_{m_c}} \cdot e_{m_c}$$

which is an isomorphism, since all e_{m_f} are so by Lemma 2.8. Using the second clause of Lemma 2.2 we conclude that $\alpha L_{\mathcal{K}^2}L_{\mathcal{K}}$, and hence our arbitrary α, is determined by $\alpha L_{\mathcal{K}^2}V_{\mathcal{K}}$. Since each $\gamma V_{\mathcal{K}}c$ is $1_{F(c)}$ it follows that each $(\gamma C_{\mathcal{K}^2} \cdot \gamma(F^{\mathbf{2}})^{\mathbf{2}})L_{\mathcal{K}^2}V_{\mathcal{K}}c$ and each $(\gamma(C_{\mathcal{K}})^{\mathbf{2}} \cdot F\gamma^{\mathbf{2}})L_{\mathcal{K}^2}V_{\mathcal{K}}c$ is also $1_{F(c)}$ showing that $\gamma C_{\mathcal{K}^2} \cdot \gamma(F^{\mathbf{2}})^{\mathbf{2}} = \gamma(C_{\mathcal{K}})^{\mathbf{2}} \cdot F\gamma^{\mathbf{2}}$. ∎

The theorem shows that if a normal factorization pre-algebra $F : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$ admits a factorization algebra structure, γ then there is no question about what it is. Said otherwise, being a factorization algebra is a *property* for a normal factorization pre-algebra. We summarize.

2.11. Theorem. *For $F : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$ a normal factorization pre-algebra, the following are equivalent:*

i) F admits a necessarily unique factorization algebra structure;

ii) there is an isomorphism $FF^{\mathbf{2}} \xrightarrow{\simeq} FC_{\mathcal{K}}$;

iii) all m_{e_f} and all e_{m_f} are isomorphisms.

Proof. Clearly, i) implies ii) is trivial. That ii) implies iii) is contained in Lemma 2.8. Assume iii). It is shown in [K&T] that for $\mathcal{E}_F = \{h | m_h \text{ is invertible}\}$ and $\mathcal{M}_F = \{h | e_h \text{ is invertible}\}$, $(\mathcal{E}_F, \mathcal{M}_F)$ is a factorization system for $\mathcal{K}$. Consider the first diagram in the proof of Lemma 2.1. Inspection of the top left square shows that $F(e_f, F(u, 1_B))$ is in $\mathcal{E}_F$ — since arrows of the form e_h are in $\mathcal{E}_F$ and $\mathcal{E}_F$ is closed with respect to composition and the cancellation rule. Since $m_g \cdot F(u, 1_B) = m_c$ and arrows of the form m_h are in $\mathcal{M}_F$ and $\mathcal{M}_F$ is closed with respect to cancellation, $F(u, 1_B)$ is in $\mathcal{M}_F$. Inspection of the bottom left square now shows that $F(e_f, F(u, 1_B))$ is in $\mathcal{M}_F$. It follows then that $F(e_f, F(u, 1_B))$, being in $\mathcal{E}_F \cap \mathcal{M}_F$, is an isomorphism. (Of course it then follows from Lemma 2.1 that $F(F(1_X, v), m_g)$ is an isomorphism.) Now, for $S = (f; u, v; g)$

with $C_{\mathcal{K}}S = c$, *define* $\gamma S = m_{e_c} \cdot (F(e_f, F(u, 1_B)))^{-1}$. This provides an isomorphism $\gamma : FF^{\mathbf{2}} \xrightarrow{\simeq} FC_{\mathcal{K}}$. For $c : X \to B$, it follows that

$$\gamma I_{\mathcal{K}^{\mathbf{2}}} c = \gamma V_{\mathcal{K}} c = m_{e_c} \cdot F(e_c, F(1_X, 1_B))^{-1} = m_{e_c} \cdot F(e_c, 1_{F(c)})^{-1} = m_{e_c} \cdot m_{e_c}^{-1} = 1_{F(c)}$$

so that by Theorem 2.10 (F, γ) is a factorization algebra. ∎

2.12. Remark. Without normality there is a little to change. For example, from $(1')$ in 1.5 it follows that in Corollary 2.9 we should replace i) by

i′) $\alpha V_{\mathcal{K}} = (\iota F)^{-1}$

and continue with similar adjustments — after first redefining $e_{\_}$ and $m_{\_}$ to absorb ι. We leave the details of this and subsequent modifications that need to be made in the absence of normality to the interested reader.

2.13. It should not be supposed however that in the situation hypothesised by Theorem 2.11 there is at most one isomorphism $FF^{\mathbf{2}} \xrightarrow{\simeq} FC_{\mathcal{K}}$. From the diagram preceding Lemma 2.8 we see using Lemma 2.2 that if F is a factorization algebra then *any* $\alpha : FF^{\mathbf{2}} \to FC_{\mathcal{K}} : (\mathcal{K}^{\mathbf{2}})^{\mathbf{2}} \to \mathcal{K}$ is determined equally by $\alpha V_{\mathcal{K}} = \alpha I_{\mathcal{K}^{\mathbf{2}}} : F \xrightarrow{\simeq} F : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$, $\alpha L_{\mathcal{K}}$, $\alpha H_{\mathcal{K}}$ and $\alpha R_{\mathcal{K}}$. In fact it is convenient to note:

2.14. Lemma. *For a factorization algebra* $F : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$, *precomposition with* $V_{\mathcal{K}} : \mathcal{K}^{\mathbf{2}} \to (\mathcal{K}^{\mathbf{2}})^{\mathbf{2}}$ *provides a bijection*

$$\mathbf{CAT}((\mathcal{K}^{\mathbf{2}})^{\mathbf{2}}, \mathcal{K})(FF^{\mathbf{2}}, FC_{\mathcal{K}}) \xrightarrow{(-)V_{\mathcal{K}}} \mathbf{CAT}(\mathcal{K}^{\mathbf{2}}, \mathcal{K})(F, F)$$

Moreover, both $(-)V_{\mathcal{K}}$ *and its inverse preserve invertibility.*

Proof. Let $\beta : F \to F$ and define $v(\beta) = \beta C_{\mathcal{K}} \cdot (FF^{\mathbf{2}}\sigma C_{\mathcal{K}})^{-1} \cdot FF^{\mathbf{2}}\eta$, where η is the unit for $C_{\mathcal{K}} \dashv L_{\mathcal{K}}$ and σ is again the natural transformation $V_{\mathcal{K}} \to L_{\mathcal{K}}$ of 2.4. Note that for $S = (f; u, v; g)$ in $(\mathcal{K}^{\mathbf{2}})^{\mathbf{2}}$ with $C_{\mathcal{K}}S = c$ we have $FF^{\mathbf{2}}\sigma C_{\mathcal{K}} S = e_{m_c}$. For $\alpha : FF^{\mathbf{2}} \to FC_{\mathcal{K}}$ $v(\alpha V_{\mathcal{K}}) = \alpha$ has been shown in earlier diagrams. To see that $v(\beta)V_{\mathcal{K}} = \beta$ it suffices to show that $(FF^{\mathbf{2}}\sigma C_{\mathcal{K}})^{-1} V_{\mathcal{K}} \cdot FF^{\mathbf{2}}\eta V_{\mathcal{K}} = 1_F$. This follows immediately from $\eta V_{\mathcal{K}} = \sigma$ which can be seen by inspection of η as displayed in 2.5 and σ as displayed in 2.4. It is clear that $(-)V_{\mathcal{K}}$ preserves invertibility. By Lemma 2.6 we know that $FF^{\mathbf{2}}\eta$ is invertible for a factorization algebra and it then follows from the explicit description of v that $v(\beta)$ is invertible when β is so. ∎

Now the fully faithful $I_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}^{\mathbf{2}}$ has both left and right adjoints, given respectively by ‘codomain’$= \partial_1$ and ‘domain’$= \partial_0$ but neither the unit for $\partial_1 \dashv I_{\mathcal{K}}$ nor the counit for $I_{\mathcal{K}} \dashv \partial_0$ are in general inverted by F so that Lemma 2.2 is not applicable. In fact it is easy to see — and we will use — that F applied to the c-component of the unit for $\partial_1 \dashv I_{\mathcal{K}}$ is m_c and F applied to the c-component of the counit for $I_{\mathcal{K}} \dashv \partial_0$ is e_c. However:

2.15. LEMMA. *For a factorization algebra $F : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$ precomposition with $I_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}^{\mathbf{2}}$ provides a bijection*

$$\mathbf{CAT}(\mathcal{K}^{\mathbf{2}}, \mathcal{K})(F, F) \xrightarrow{(-)I_{\mathcal{K}}} \mathbf{CAT}(\mathcal{K}, \mathcal{K})(1_{\mathcal{K}}, 1_{\mathcal{K}})$$

which is an isomorphism of monoids. The inverse function is given by $F(-)^{\mathbf{2}}$.

PROOF. That $(-)I_{\mathcal{K}}$ provides a homomorphism of monoids is immediate from the definitions. Consider a $\beta : F \to F : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$. For any object $c : X \to B$ of $\mathcal{K}^{\mathbf{2}}$ we have the composable pair

$$I_{\mathcal{K}}X = I_{\mathcal{K}}\partial_0 c \xrightarrow{(1_X, c)} c \xrightarrow{(c, 1_B)} I_{\mathcal{K}}\partial_1 c = I_{\mathcal{K}}B$$

in $\mathcal{K}^{\mathbf{2}}$ consisting of the counit for $I_{\mathcal{K}} \dashv \partial_0$ and the unit for $\partial_1 \dashv I_{\mathcal{K}}$. Application of $\beta : F \to F$ gives

$$\begin{array}{ccc} X & \xrightarrow{\beta I_{\mathcal{K}}X} & X \\ {\scriptstyle e_c}\downarrow & & \downarrow{\scriptstyle e_c} \\ F(c) & \xrightarrow{\beta c} & F(c) \\ {\scriptstyle m_c}\downarrow & & \downarrow{\scriptstyle m_c} \\ B & \xrightarrow[\beta I_{\mathcal{K}}B]{} & B \end{array}$$

and, again using [J&T], the mere fact that each m_{e_f} is an epimorphism and each e_{m_f} is a monomorphism ensures that $\beta c = F(\beta I_{\mathcal{K}}X, \beta I_{\mathcal{K}}B)$. Clearly we have $\beta = F(\beta I_{\mathcal{K}})^{\mathbf{2}}$. For $\alpha : 1_{\mathcal{K}} \to 1_{\mathcal{K}}$ and any arrow $c : X \to B$ in $\mathcal{K}$ we have

$$\begin{array}{ccc} X & \xrightarrow{\alpha X} & X \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c} \\ B & \xrightarrow[\alpha B]{} & B \end{array}$$

which can be seen as the c-component of $\alpha^{\mathbf{2}}$. It follows that the c-component of $F\alpha^{\mathbf{2}}$ is $F(\alpha X, \alpha B) : F(c) \to F(c)$. Hence the X-component of $F\alpha^{\mathbf{2}}I_{\mathcal{K}}$ is $F(\alpha X, \alpha X) : F(1_X) \to F(1_X)$ which is $\alpha X : X \to X$, showing that $F\alpha^{\mathbf{2}}I_{\mathcal{K}} = \alpha$ and completing the proof that $(-)I_{\mathcal{K}}$ is a bijection. ■

2.16. Recall that for any 2-category $\mathbf{K}$ and object $\mathcal{K}$ therein, the set $\mathbf{K}(\mathcal{K}, \mathcal{K})(1_{\mathcal{K}}, 1_{\mathcal{K}})$ is a commutative monoid under (either) composition of transformations (2-cells). It is the familiar centre of $\mathcal{K}$ when $\mathcal{K}$ is a monoid. We speak simply of the *centre* of $\mathcal{K}$ in the full generality of the last sentence and write $\mathbf{Z}\mathcal{K}$ for the centre of $\mathcal{K}$. An early, unpublished, but readily available reference is [WD]. The following theorem is an obvious summary of our observations but note carefully the statement — our extension of an isomorphism $1_{\mathcal{K}} \xrightarrow{\simeq} 1_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}$ to a particular isomorphism $FF^{\mathbf{2}} \xrightarrow{\simeq} FC_{\mathcal{K}} : (\mathcal{K}^{\mathbf{2}})^{\mathbf{2}} \to \mathcal{K}$ requires the *existence* of *some* isomorphism $FF^{\mathbf{2}} \xrightarrow{\simeq} FC_{\mathcal{K}}$ (or the equivalent statements of Theorem 2.11).

2.17. THEOREM. *For $F : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$ a factorization algebra, precomposition with $I_{\mathcal{K}^{\mathbf{2}}} I_{\mathcal{K}} : \mathcal{K} \to (\mathcal{K}^{\mathbf{2}})^{\mathbf{2}}$ provides a bijection*

$$\mathbf{CAT}((\mathcal{K}^{\mathbf{2}})^{\mathbf{2}}, \mathcal{K})(FF^{\mathbf{2}}, FC_{\mathcal{K}}) \xrightarrow{(-)I_{\mathcal{K}^{\mathbf{2}}} I_{\mathcal{K}}} \mathbf{Z}\mathcal{K}$$

and both $(-)I_{\mathcal{K}^{\mathbf{2}}} I_{\mathcal{K}}$ and its inverse preserve invertibility. ∎

2.18. Of course one might wonder if the existence of a factorization algebra on a category $\mathcal{K}$ forces the set of invertible elements of its centre to be trivial. This is not the case. For example, the category of abelian groups admits several factorization algebras and has as its centre the monoid of integers under multiplication. In fact the set of invertible elements in the centre of a category with a factorization algebra can be arbitrarily large. For if α is an invertible element in the centre of $\mathcal{K}$ then $\alpha^{\mathbf{2}}$ is an invertible element in the centre of $\mathcal{K}^{\mathbf{2}}$, which carries the free strict factorization algebra provided by $C_{\mathcal{K}}$. To finish the argument it suffices to take $\mathcal{K}$ to be a commutative group and observe that such $\mathcal{K}$ may be arbitrarily large.

We close with what is evidently the core coherence requirement for factorization algebras.

2.19. THEOREM. *For a factorization algebra $F : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$, if $\gamma : FF^{\mathbf{2}} \xrightarrow{\simeq} FC_{\mathcal{K}}$ is an isomorphism satisfying*

$$\gamma I_{\mathcal{K}^{\mathbf{2}}} I_{\mathcal{K}} \quad = \quad 1_{1_{\mathcal{K}}}$$

then (F, γ) is the unique factorization algebra structure on F. ∎

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