Reprints in Theory and Applications of Categories, No. 2, 2003, pp. 1–59.

TRIPLES, ALGEBRAS AND COHOMOLOGY

JONATHAN MOCK BECK

Contents

Editors' Preface

It is with great pleasure that the editors of *Theory and Applications of Categories* make this dissertation generally available. Although the date on the thesis is 1967, there was a nearly complete draft circulated in 1964. This thesis was a revelation to those of us who were interested in homological algebra at the time.

Although the world's very first triple (now more often called "monad") in the sense of this thesis was non-additive and used to construct flabby resolutions of sheaves ([Godement (1958)]), the then-prevailing belief was that the theory of triples had a use in homological algebra only via additive triples on abelian categories, typically something like $\Lambda \otimes_{\Lambda\otimes\Lambda} -$, on the category of Λ-Λ bimodules. In fact, [Eilenberg & Moore (1965b)] went so far as to base their relative homological algebra on triples that were additive and preserved kernels. Thus there was considerable astonishment when Jon Beck, in the present work, was able not only to define cohomology by a triple on the category of objects of interest (rather than the abelian category of coefficient modules) but even prove in wide generality that the first cohomology group classifies singular extensions by a module. Not the least of Beck's accomplishments in this work are his telling, and general, axiomatic descriptions of module, singular extension, and derivation into a module. The simplicity and persuasiveness of these descriptions remains one of the more astonishing features of this thesis.

Transmitted by R. Paré, J. Stasheff, and R.J. Wood. Reprint published on 2003-06-17.
2000 Mathematics Subject Classification: 18C15,18C20.
Key words and phrases: Cotriple cohomology, Beck modules.

We hope that this publication will provide an informative and useful resource for workers in our field. We have left the thesis unchanged except for (surprisingly few) typographical corrections. In addition, the editors have made a few notes in places where we have updated references, corrected the original manuscript, or, in one place, clarified things somewhat. But what you have before you is basically the thesis presented in 1967.

Acknowledgments. One of the gratifying aspects of this reprint effort was that within 24 hours of asking for volunteers for the retyping project, there were more volunteers than we could use. Eventually, we chose 11, more or less at random, and asked each one to type 10 pages of the 109 pages (plus bibliography). There were logistical problems getting the pages to the volunteers, but not one of them took more than a few days to do his or her bit once they had it. The typists we thank are Robert Dawson, Robert L. Knighten, Francisco Marmolejo, Shane O'Conchuir, Valeria de Paiva, Dorette Pronk, Robert Rosebrugh, Robert Seely, Andrew Tonks, Charles Wells, and Noson Yanofsky. In addition we would like to thank Jack Duskin, Jeff Egger, and Maria Manual Clementino who were equally prepared to volunteer their labor. Joan Wick-Pelletier gave us a copy of the original thesis, without which we could not have proceeded. Finally, we thank Donovan Van Osdol for a masterly proof-reading job that went beyond the finding of variations from the original, but also found errors in the original.

While working on this we were reminded how we used to type mathematics in those days. Putting in symbols by hand, sometimes fabricating them by overtyping two characters and so on. Mathematics was called "penalty copy" and linotypists would charge double or more to do it. So we also owe a big debt of gratitude to Donald Knuth, to Leslie Lamport, the LaTeX 2_ε team, Kris Rose, and Ross Moore (the creator and current developer, resp., of XY-pic).

The editors of *Theory and Applications of Categories*

Triples, Algebras and Cohomology

Jonathan Mock Beck

1967

Submitted in Partial Fulfillment of the Requirements
For the Degree of Doctor of Philosophy,
In the Faculty of Pure Science,
Columbia University

0. Introduction

This thesis is intended to complete the exposition in [Eilenberg & Moore (1965a)] with regard to certain points. In §1 we recall the definitions of *triple*, *algebra* over a triple, and give our main (original) definition, that of *tripleable adjoint pair of functors*. In §2 we show how to obtain a cohomology theory from an adjoint pair of functors. In §3, when the adjoint pair is tripleable, we prove that the cohomology group H^1 classifies principal homogeneous objects. When coefficients are in a *module*, principal objects are interpreted as algebra extensions. §4 is devoted to examples. Many categories occurring in algebra are shown to be tripleable. The corresponding cohomology and extension theories, ranging from groups and algebras to the classical $\mathrm{Ext}(A, C)$, are discussed. Many new theories arise.

A method for proving coincidence of triple cohomology with certain standard theories has been given by [Barr & Beck (1966)]. That paper contains a summary of the present work.

I should like to express my most profound gratitude to Professor S. Eilenberg, with the help of whose energetic criticism and encouragement these results were obtained.

1. Triples and Algebras.

DEFINITION 1. $\mathbf{T} = (T, \eta, \mu)$ is a *triple* in a category $\underline{A}$ if T is a functor $\underline{A} \to \underline{A}$, η and μ are natural transformations $\underline{A} \to T$ and $TT \to T$ respectively, and the diagrams

$$\begin{tikzcd} T \arrow[r, "T\eta"] \arrow[dr, "T"'] & TT \arrow[d, "\mu"] \\ & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, "T"'] & TT \arrow[d, "\mu"] \\ & T \end{tikzcd}$$

$$\begin{tikzcd} TTT \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & TT \arrow[d, "\mu"] \\ TT \arrow[r, "\mu"'] & T \end{tikzcd}$$

commute. Thus η is a right and left *unit* for the *multiplication* μ, and μ is *associative*. Dually, $\mathbf{G} = (G, \epsilon, \delta)$ is a *cotriple* in $\underline{B}$ if $G\colon \underline{B} \to \underline{B}$ is a functor, $\epsilon\colon G \to \underline{B}$ and $\delta\colon G \to GG$

are natural transformations, and

$$\begin{tikzcd} G \arrow[r, "\delta"] \arrow[dr, "G"'] & GG \arrow[d, "G\epsilon"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\delta"] \arrow[dr, "G"'] & GG \arrow[d, "\epsilon G"] \\ & G \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, "\delta"] \arrow[d, "\delta"'] & GG \arrow[d, "\delta G"] \\ GG \arrow[r, "G\delta"'] & GGG \end{tikzcd}$$

commute.

Triples and cotriples usually arise from adjoint functors. We recall that an *adjoint pair of functors* consists of functors $F: \underline{A} \longrightarrow \underline{B}$, $U: \underline{B} \longrightarrow \underline{A}$ together with a natural isomorphism

$$(A, BU) \xrightarrow[\approx]{\alpha} (AF, B)$$

where $A \in |\underline{A}|$, $B \in |\underline{B}|$. The functor U is the *right adjoint* (or *adjoint*), and F is the *left adjoint* (or *coadjoint*). The relation between F and U is often symbolized by $\alpha: F \dashv U$. Taking $A = BU$, $(BU)\alpha: BUF \longrightarrow B$ defines a natural transformation $\epsilon: UF \longrightarrow \underline{B}$. Taking $B = AF$, $(AF)\alpha^{-1}: A \longrightarrow AFU$ defines a natural transformation $\eta: \underline{A} \longrightarrow FU$. η and ϵ are called the *unit* and *counit* of the adjointness, and satisfy the relations [Kan (1958)]

$$\begin{tikzcd} F \arrow[r, "\eta F"] \arrow[dr, "F"'] & FUF \arrow[d, "F\epsilon"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "U\eta"] \arrow[dr, "U"'] & UFU \arrow[d, "\epsilon U"] \\ & U \end{tikzcd}$$

The adjoint pair α induces a triple $\mathbf{T} = (T, \eta, \mu)$ in the category $\underline{A}$, defined by

$$\mathbf{T} \quad \begin{cases} T = FU : \underline{A} \longrightarrow \underline{A} \\ \eta : \underline{A} \longrightarrow T \\ \mu = F\epsilon U : TT \longrightarrow T \end{cases}$$

and a cotriple in the category $\underline{B}$, defined by

$$\mathbf{G} \quad \begin{cases} G = UF : \underline{B} \longrightarrow \underline{B} \\ \epsilon : G \longrightarrow \underline{B} \\ \delta = U\eta F : G \longrightarrow GG \end{cases}$$

Indeed, $T\eta{\cdot}\mu = FU\eta{\cdot}F\epsilon U = F(U\eta{\cdot}\epsilon U) = FU = T$ and $\eta T{\cdot}\mu = \eta FU{\cdot}F\epsilon U = (\eta F{\cdot}F\epsilon)U = FU = T$ dispose of the unitary axiom. For associativity, $T\mu \cdot \mu = FUF\epsilon U \cdot F\epsilon U = F(UF\epsilon \cdot \epsilon)U$ and $\mu T \cdot \mu = F\epsilon UFU \cdot F\epsilon U = F(\epsilon UF \cdot \epsilon)U$. These coincide by naturality of ϵ. The proof for $\mathbf{G}$ is dual.

The triple $\mathbf{T}$, with its unit and multiplication, is something like a *monoid*. The next definition formalizes the intuitive idea of such a monoid's operating on an object of the category $\underline{A}$ [Eilenberg & Moore (1965a)].

DEFINITION 2. (X, ξ) is a $\mathbf{T}$-*algebra* if $\xi\colon XT \to X$ is a map in $\underline{A}$ and the diagrams

$$\begin{tikzcd} X \arrow[r, "X\eta"] \arrow[dr, "X"'] & XT \arrow[d, "\xi"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} XTT \arrow[r, "X\mu"] \arrow[d, "\xi T"'] & XT \arrow[d, "\xi"] \\ XT \arrow[r, "\xi"'] & X \end{tikzcd}$$

commute. ξ is called the $\mathbf{T}$-*structure* of the algebra and the above diagrams state that ξ is *unitary* and *associative*. $f\colon (X, \xi) \to (Y, \theta)$ is a *map* of $\mathbf{T}$-algebras if $f\colon X \to Y$ in $\underline{A}$ and is compatible with $\mathbf{T}$-structures:

$$\begin{tikzcd} XT \arrow[r, "fT"] \arrow[d, "\xi"'] & YT \arrow[d, "\theta"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

$\mathbf{T}$-algebras form a category which we denote by $\underline{A}^{\mathbf{T}}$.

We have a canonical *adjoint pair of functors*

$$\underline{A} \xrightarrow{F^{\mathbf{T}}} \underline{A}^{\mathbf{T}} \xrightarrow{U^{\mathbf{T}}} \underline{A}$$

$U^{\mathbf{T}}$ is the *underlying* $\mathbf{A}$-*object* functor, which maps $(X, \xi) \Rightarrow X$ and $f \Rightarrow f$. $F^{\mathbf{T}}$ is the *free* $\mathbf{T}$-*algebra* functor, which maps $A \Rightarrow (AT, A\mu)$ and $a\colon A \longrightarrow A' \Rightarrow aT\colon (AT, A\mu) \longrightarrow (A'T, A'\mu)$. It follows from the axioms for η and μ that $(AT, A\mu)$ actually is a $\mathbf{T}$-algebra. The formula for the adjointness isomorphism

$$(A, (Y, \theta)U^{\mathbf{T}}) \xrightarrow[\simeq]{\alpha^{\mathbf{T}}} (AF^{\mathbf{T}}, (Y, \theta))$$

is $y\alpha^{\mathbf{T}} = yT \cdot \theta$, $f(\alpha^{\mathbf{T}})^{-1} = A\eta \cdot f$.

To justify the definition we only need to verify the adjointness, all the other assertions being obvious. First, let $y\colon A \longrightarrow Y$ be a map in $\underline{A}$, and let us check that $y\alpha^{\mathbf{T}}$ is a

$\mathbf{T}$-algebra map. We have the diagram

$$\begin{tikzcd}
ATT \arrow[r, "yTT"] \arrow[d, "A\mu"'] & YTT \arrow[r, "\theta T"] \arrow[d, "Y\mu"] & YT \arrow[d, "\theta"] \\
AT \arrow[r, "yT"'] & YT \arrow[r, "\theta"'] & Y
\end{tikzcd}$$

The top and bottom compositions are $(y\alpha)T$ and $y\alpha$ (if α stands for $\alpha^{\mathbf{T}}$, as it will for the rest of this proof). The second square commutes by the associative law for θ. Now we have $y\alpha\alpha^{-1} = A\eta \cdot yT \cdot \theta = y \cdot Y\eta \cdot \theta = y$, by the unitary property of θ, and if $f\colon AF^{\mathbf{T}} = (AT, A\mu) \longrightarrow (Y, \theta)$ in $\underline{A}^{\mathbf{T}}$, then $f\alpha^{-1}\alpha = (A\eta \cdot f)T \cdot \theta = A\eta T \cdot fT \cdot \theta = A\eta T \cdot A\mu \cdot f = f$.

$$\begin{tikzcd}
A \arrow[r, "y"] \arrow[d, "A\eta"'] & Y \arrow[d, "Y\eta"'] \arrow[dr, "Y"] & \\
AT \arrow[r, "yT"'] & YT \arrow[r, "\theta"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
AT \arrow[r, "A\eta T"] \arrow[dr, "AT"'] & ATT \arrow[r, "fT"] \arrow[d, "A\mu"] & YT \arrow[d, "\theta"] \\
 & AT \arrow[r, "f"'] & Y
\end{tikzcd}$$

We will skip the proof that $\alpha^{\mathbf{T}}$ is natural, which is easy.

Now let $\alpha\colon F \dashv U$ be any adjoint pair, where $F\colon \underline{A} \longrightarrow \underline{B}$ and $U\colon \underline{B} \longrightarrow \underline{A}$. α generates a triple $\mathbf{T} = (T, \eta, \mu)$ in $\underline{A}$, where $T = FU, \ldots$, hence a category of $\mathbf{T}$-algebras $\underline{A}^{\mathbf{T}}$ with the above free and underlying object functors. The relation between the old adjoint pair $\underline{A} \longrightarrow \underline{B} \longrightarrow \underline{A}$ and the new one, $\underline{A} \longrightarrow \underline{A}^{\mathbf{T}} \longrightarrow \underline{A}$, is expressed in terms of a canonical functor $\Phi\colon \underline{B} \longrightarrow \underline{A}^{\mathbf{T}}$ which we will be interested in throughout. Φ exists because any object in $\underline{B}$ naturally induces a $\mathbf{T}$-algebra structure on the object in $\underline{A}$ underlying it, and fits into the following commutative diagram of functors:

$$\begin{tikzcd}
\underline{A}^{\mathbf{T}} \arrow[dr, shift left, "U^{\mathbf{T}}"] & & \underline{B} \arrow[ll, "\Phi"'] \arrow[dl, shift left, "U"] \\
 & \underline{A} \arrow[ul, shift left, "F^{\mathbf{T}}"] \arrow[ur, shift left, "F"] &
\end{tikzcd}$$

$$(\Phi U^{\mathbf{T}} = U, F\Phi = F^{\mathbf{T}})$$

Φ is defined by the formulas $Y\Phi = (YU, Y\epsilon U)$, $y\Phi = yU\colon (YU, Y\epsilon U) \longrightarrow (Y'U, Y'\epsilon U)$ if $y\colon Y \longrightarrow Y'$. Intuitively, the counit $Y\epsilon\colon YUF \longrightarrow Y$ is the natural map of the free object generated by Y onto Y (it need not be an epimorphism in this general context) and the $\mathbf{T}$-structure on $Y\Phi$ is the $\underline{A}$-map underlying this. It is clear, of course, from the adjointness identities that $(YU, Y\epsilon U)$ is really a $\mathbf{T}$-algebra.

The construction of $\underline{A}^{\mathbf{T}}$ does not in general give back the original adjoint pair, that is, $\Phi\colon \underline{B} \longrightarrow \underline{A}^{\mathbf{T}}$ is not always an equivalence. We want to isolate as particularly tractable the situation in which it is:

DEFINITION 3. The adjoint pair $\alpha\colon F \dashv U$ is *tripleable* if $\Phi\colon \underline{B} \longrightarrow \underline{A}^{\mathbf{T}}$ is an equivalence of categories.

If U is held fixed and $F, F' \dashv U$ are two left adjoints, then the canonical isomorphism $F' \xrightarrow{\simeq} F$ [Kan (1958)] can be used to show that $F \dashv U$ is tripleable $\Leftrightarrow$ $F' \dashv U$ is. Indeed, $F' \xrightarrow{\simeq} F$ induces an isomorphism of triples $\mathbf{T}' \xrightarrow{\simeq} \mathbf{T}$, hence an isomorphism of the algebra categories $\underline{A}^{\mathbf{T}} \xrightarrow{\simeq} \underline{A}^{\mathbf{T}'}$ which one finds commutes with the canonical functors $\Phi\colon \underline{B} \longrightarrow \underline{A}^{\mathbf{T}}$, $\Phi'\colon \underline{B} \longrightarrow \underline{A}^{\mathbf{T}'}$. Thus Φ is an equivalence $\Leftrightarrow$ Φ' is. The details of the reasoning can be left to the reader, but as a consequence of it we can state:

DEFINITION 3′. A *functor* $U\colon \underline{B} \longrightarrow \underline{A}$ is *tripleable* if U has a left adjoint F and the adjoint pair $F \dashv U$ is tripleable.

In practice this language is convenient because often the underlying object functor is the main item of interest. Its coadjoint free functor has to be present, but needn't be brought explicitly into the discussion. Of course, such expressions as "$\underline{B}$ is tripleable over $\underline{A}$" can be used if a definite underlying object functor $\underline{B} \longrightarrow \underline{A}$ is understood. As a rule the category $\underline{B}$ will be fixed and various underlying object functors $\underline{B} \longrightarrow \underline{A}_i$ will be considered. If one of them, $U\colon \underline{B} \longrightarrow \underline{A}$, is tripleable, this means that $\underline{B}$ is exactly recoverable as the category of objects in $\underline{A}$ which have the structure of algebras over the triple in $\underline{A}$ induced by the left adjoint of U. To say that a functor $\underline{B} \longrightarrow \underline{A}$ is tripleable is therefore to say it is "forgetful" in a rather precise sense: there is a uniquely determined triple in $\underline{A}$ whose algebras are $\underline{B}$ and U is exactly the functor which forgets these algebra structures (up to categorical equivalences). Incidentally, the term "tripleable" cannot be replaced by "forgetful" because there remain many functors that are intuitively "forgetful", that is, drop structure, but which are not tripleable. Speaking in general and vague terms, tripleableness implies algebraicity of some kind. In Example 1, §4, we will return to the question of just what sort of structure tripleableness entails. In [J. Beck (to appear)] we shall give a (rather complicated) necessary and sufficient condition for a functor to be tripleable. It is a refinement of the following (rather weak) theorem which we will apply to some examples later.

THEOREM 1. *Let $\alpha\colon F \dashv U$ be an adjoint pair.*

$$\begin{tikzcd}
\underline{A}^{\mathbf{T}} \arrow[rr, dashed, shift left, "\check{\Phi}"] \arrow[dr, shift left, "U^{\mathbf{T}}"] & & \underline{B} \arrow[ll, shift left, "\Phi"] \arrow[dl, shift left, "U"] \\
& \underline{A} \arrow[ul, shift left, "F^{\mathbf{T}}"] \arrow[ur, shift left, "F"] &
\end{tikzcd}$$

$$(\Phi U^{\mathbf{T}} = U, F\Phi = F^{\mathbf{T}})$$

(1) If $\underline{B}$ has coequalizers, then there exists a left adjoint $\check{\Phi} \dashv \Phi$.

Assuming the existence of $\check{\Phi}$*:*

(2) If U *preserves coequalizers, then the unit of* $\check{\Phi} \dashv \Phi$ *is an isomorphism* $\underline{A}^{\mathbf{T}} \xrightarrow{\simeq} \check{\Phi}\Phi$*.*

(3) If U *reflects coequalizers, then the counit is an isomorphism* $\Phi\check{\Phi} \xrightarrow{\simeq} \underline{B}$*.*

Finally, in the presence of (2), (3) can be replaced by:

(3′) If U *reflects isomorphisms, then the counit is an isomorphism* $\Phi\check{\Phi} \xrightarrow{\simeq} \underline{B}$*.*

Proof. $\check{\Phi}$ exists if and only if the following coequalizer diagram (which is used as the definition of $(X,\xi)\check{\Phi}$) exists in $\underline{B}$:

$$XFUF \underset{XF\epsilon}{\overset{\xi F}{\rightrightarrows}} XF \xrightarrow{\pi} (X,\xi)\check{\Phi}$$

(In effect $(X,\xi)\check{\Phi}$ is $(X,\xi) \otimes_{\mathbf{T}} F$, and Φ can be thought of as $\mathrm{Hom}(F,\ )$ with right $\mathbf{T}$-operators.) Assuming the coequalizer exists, the adjointness $\check{\Phi} \dashv \Phi$ is demonstrated by verifying the following sequence of 1-1 correspondences:

$$\begin{array}{rl} & \text{maps } (X,\xi) \xrightarrow{f} Y\Phi \text{ in } \underline{A}^{\mathbf{T}} \\ \longrightarrow & \text{maps } X \xrightarrow{f} YU \text{ such that } \xi f = fFU \cdot Y\epsilon U \\ \xrightarrow{\alpha} & \text{maps } XF \xrightarrow{g} Y \text{ such that } \xi F \cdot g = XF\epsilon \cdot g \\ \longrightarrow & \text{maps } (X,\xi)\check{\Phi} \xrightarrow{g_1} Y \text{ in } \underline{B}. \end{array}$$

The adjointness isomorphism α can be retrieved for ϵ by the well-known formula [Kan (1958)] $f\alpha = fF \cdot Y\epsilon$. Thus, given the condition on f, we can write

$$\begin{aligned} \xi F \cdot g &= \xi F \cdot fF \cdot Y\epsilon = (\xi f)F \cdot Y\epsilon = (fFU \cdot Y\epsilon U)F \cdot Y\epsilon \\ &= fFUF \cdot Y\epsilon UF \cdot Y\epsilon = XF\epsilon \cdot fF \cdot Y\epsilon = XF\epsilon \cdot g, \end{aligned}$$

using mainly naturality of ϵ. Since also $f = g\alpha^{-1} = X\eta \cdot gU$, one is able similarly to reverse the correspondence.

Note that (1) has been proved. To complete the remark at the beginning of the proof, if $\check{\Phi} \dashv \Phi$ exists, the unit $\underline{A}^{\mathbf{T}} \longrightarrow \check{\Phi}\Phi$ composed with $-U^{\mathbf{T}}$ gives a map $U^{\mathbf{T}} \longrightarrow \check{\Phi}U$ (since $\Phi U^{\mathbf{T}} = U$). The adjoint of this last transformation is a map $\pi\colon U^{\mathbf{T}}F \longrightarrow \check{\Phi}$. One can now prove directly (but we will omit this) that

$$XFUF \underset{XF\epsilon}{\overset{\xi F}{\rightrightarrows}} XF \xrightarrow{\pi} (X,\xi)\check{\Phi}$$

is a coequalizer diagram in $\underline{B}$. From now on, we assume $\check{\Phi} \dashv \Phi$ exists, and make use of this diagram to prove (2), (3) and (3′).

Let $\varphi\colon (X,\xi) \longrightarrow (X,\xi)\check{\Phi}\Phi$ be the unit. $\varphi\colon X \longrightarrow (X,\xi)\check{\Phi}U$ in $\underline{A}$ and is compatible with $\mathbf{T}$-structures. Explicitly, φ is obtained by working the above 1-1 correspondences backwards starting from the identity map of $(X, \xi)\check{\Phi}$. This yields $\varphi = X\eta \cdot \pi U$. Consider the following diagram.

$$\begin{tikzcd}
 & X \arrow[d, "X\eta"'] \arrow[dr, "X"] & \\
XFUFU \arrow[r, shift left, "\xi FU"] \arrow[r, shift right, "XF\epsilon U"'] & XFU \arrow[r, "\xi"] \arrow[dr, "\pi U"'] & X \arrow[d, "\varphi"] \\
 & & (X, \xi)\check{\Phi}U
\end{tikzcd}$$

ξ is a coequalizer of ξFU and $XF\epsilon U$. For if $z \colon XFU \longrightarrow Z$ in $\underline{A}$ and $\xi FU \cdot z = XF\epsilon U \cdot z$, then $X\eta \cdot z \colon X \longrightarrow Z$ is uniquely determined by the fact that it satisfies the equation $\xi \cdot X\eta \cdot z = z$. Uniqueness is evident from the retraction property of ξ, $X\eta \cdot \xi = X$, and from the equation itself: $\xi \cdot X\eta \cdot z = XFU\eta \cdot \xi FU \cdot z = XFU\eta \cdot XF\epsilon U \cdot z = XF(U\eta \cdot \epsilon U) \cdot z = z$.

$$\begin{tikzcd}
 & XFU \arrow[r, "\xi"] \arrow[dl, "XFU"'] \arrow[d, "XFU\eta"] & X \arrow[d, "X\eta"] \\
XFU \arrow[dr, "z"'] & XFUFU \arrow[l, "XF\epsilon U"'] \arrow[r, "\xi FU"] & XFU \arrow[dl, "z"] \\
 & Z &
\end{tikzcd}$$

πU is also a coequalizer of ξFU and $XF\epsilon U$, if U preserves coequalizers. Moreover $\varphi \colon X \longrightarrow (X, \xi)\check{\Phi}U$ is compatible with ξ and πU, because $\xi\varphi = \xi \cdot X\eta \cdot \pi U = XFU\eta \cdot \xi FU \cdot \pi U = XFU\eta \cdot XF\epsilon U \cdot \pi U = \pi U$. Thus φ is an isomorphism in $\underline{A}$. Trivially, if a map of $\mathbf{T}$-algebras is an isomorphism in the underlying category, then it is an isomorphism of $\mathbf{T}$-algebras (its inverse also respects $\mathbf{T}$-structures). In other words, the functor $U^{\mathbf{T}} \colon \underline{A}^{\mathbf{T}} \longrightarrow \underline{A}$ reflects isomorphisms. This proves (2).

The counit $\psi \colon Y\Phi\check{\Phi} \longrightarrow Y$ is obtained from the identity map of $Y\Phi$ via the above 1-1 correspondences, and is uniquely determined by its appearance in the following diagram, the top line of which is a coequalizer.

$$\begin{tikzcd}
YUFUF \arrow[r, shift left, "Y\epsilon UF"] \arrow[r, shift right, "YUF\epsilon"'] & YUF \arrow[r, "\pi"] \arrow[d, "Y\epsilon"'] & Y\Phi\check{\Phi} \arrow[dl, "\psi"] \\
 & Y &
\end{tikzcd}$$

We proved above that the $\mathbf{T}$-structure of an algebra is a coequalizer in $\underline{A}$. (Recall that $Y\epsilon U$ is the $\mathbf{T}$-structure of $Y\Phi$.) If U reflects coequalizers, then $Y\epsilon$ is a coequalizer of $Y\epsilon UF$ and $YUF\epsilon$, and ψ is an isomorphism.

If U preserves coequalizers, both πU and $Y\epsilon U$ are coequalizers, hence ψU is an isomorphism. If U reflects isomorphisms, so is ψ. This proves the remark pertaining to $(3')$, and completes the proof of the theorem.

Remarks on the functor Φ. It would be interesting to know an example of a functor $\Phi\colon \underline{B} \longrightarrow \underline{A}^{\mathbf{T}}$ with $\underline{B}$ having arbitrary limits (limit = projective limit, in our terminology) wherein no left adjoint $\check{\Phi}$ exists. Counterexamples apparently exist when $\underline{B}$ is not complete.

Whether an adjoint or not, Φ preserves all an adjoint should preserve, for example, limits and algebraic objects. Indeed, in the commutative diagram

$$\begin{tikzcd} \underline{A}^{\mathbf{T}} \arrow[rr, "\Phi"] \arrow[dr, "U^{\mathbf{T}}"'] & & \underline{B} \arrow[dl, "U"] \\ & \underline{A} & \end{tikzcd}$$

U, being an adjoint, preserves the property involved, and one finds that $U^{\mathbf{T}}$ *reflects* the property. We saw above that $U^{\mathbf{T}}$ reflects isomorphisms. Similarly, one can show that $U^{\mathbf{T}}$ reflects all other limits. In §2 we will need the fact, and will then prove in detail, that Φ preserves several kinds of algebraic objects.

Φ can also be given an interpretation in terms of "structure" and "semantics". Let $\mathrm{Ad}(\underline{A})$ be the category of adjoint pairs over $\underline{A}$, that is, pairs $F \dashv U$ where U has $\underline{A}$ as range, with functors which commute with the right adjoints (like Φ itself) as maps. Let $\mathrm{Trip}(\underline{A})$ denote the category of triples in $\underline{A}$, a map $\mathbf{S} \longrightarrow \mathbf{T}$ being a natural transformation of the functors commuting with the units and multiplications, and let $\mathrm{Trip}(\underline{A})^*$ denote the dual category. Then functors

$$\mathrm{Ad}(\underline{A}) \underset{\sigma}{\overset{\check{\sigma}}{\rightleftarrows}} \mathrm{Trip}(\underline{A})^*$$

exist, because adjoint pairs give rise to triples and triples give rise to categories $\underline{A}^{\mathbf{T}}$ with adjoint free and underlying object functors. In fact, $\check{\sigma} \dashv \sigma$, $\sigma\check{\sigma} \xrightarrow{\simeq} \mathrm{Trip}(\underline{A})^*$, and the unit of this adjointness is precisely $\Phi\colon \mathrm{Ad}(\underline{A}) \longrightarrow \check{\sigma}\sigma$. All of this is proved in [Eilenberg & Moore (1965a)], and is reminiscent of the structure-semantics situation studied by [Lawvere (1963)] in the case of algebraic categories and by [Linton (1966)] in the case of equational categories. We should remark that by means of a more elaborate construction the domain of $\check{\sigma}$ can be extended to the category whose objects are functors $\underline{B} \longrightarrow \underline{A}$ ($\underline{B}$ variable) which are "small" in an appropriate sense but do not need to have left adjoints.

It is useful to think of the triple $\mathbf{T}$ induced by an adjoint pair of functors as a structural invariant of the adjoint pair. Various properties of the adjoint pair only depend on $\mathbf{T}$ and the functor Φ plays a role in setting up the relevant isomorphisms. We shall see this illustrated in the next section in the case of cohomology.

2. Cohomology

Adjoint functors, it is now well known, lead to cohomology ([Eilenberg & Moore (1965a), Godement (1958), Mac Lane (1963)]—or to homotopy [Huber (1961)]). If

$$\underline{A} \xrightarrow{F} \underline{B} \xrightarrow{U} \underline{A} \qquad (F \dashv U)$$

is an adjoint pair, objects of the form $AF \in |\underline{B}|$ are regarded as *"free" relative to the underlying object functor* U. The counit

$$XUF \xrightarrow{X\epsilon} X$$

is intuitively the first step of a functorial free resolution of any object $X \in |\underline{B}|$. By iterating UF one extends $X\epsilon$ to a *free simplicial resolution* of X, and defines derived functors as usual in homological algebra. Here we only consider the simplest case, that of defining *cohomology groups*

$$H^n(X, Y), \qquad n \geq 0,$$

of an object $X \in |\underline{B}|$ *with coefficients in an abelian group object* $Y \in |\underline{B}|$, *relative to the given underlying object functor* $U\colon \underline{B} \longrightarrow \underline{A}$ (having a left adjoint). Tripleableness of $F \dashv U$ will not play any appreciable role until we discuss special properties of the cohomology in §3. We now recall the details of the construction of the cohomology groups. Some of the terms used are clarified in the proof of Theorem 2, which summarizes the main properties the cohomology possesses.

Let (G, ϵ, δ) be the cotriple in $\underline{B}$ induced by $F \dashv U$; thus $G = UF$ and $\epsilon\colon G \longrightarrow \underline{B}$. The following simplicial object in $\underline{B}$ is called the *standard (free simplicial) resolution* of the object X:

$$X \xleftarrow{\epsilon_0} XG \underset{\epsilon_1}{\overset{\epsilon_0}{\leftleftarrows}} XG^2 \; \overleftarrow{\overleftarrow{\overleftarrow{\quad}}} \cdots \longleftarrow XG^n \xleftarrow{\epsilon_i} XG^{n+1} \longleftarrow \cdots$$

We abbreviate this by XG^* if necessary. Here XG^{n+1} is the term of degree n, and the face operator $\epsilon_i\colon XG^{n+1} \longrightarrow XG^n$ is $G^i\epsilon G^{n-i}$, $0 \leq i \leq n$. X itself is in dimension -1 and the last ϵ_0 augmenting the simplicial object into X is just ϵ. The simplicial identities

$$\epsilon_i\epsilon_j = \epsilon_{j+1}\epsilon_i, \qquad i \leq j,$$

can easily be verified. ($\delta\colon G \longrightarrow G^2$ induces degeneracy operators but these will play no role in our theory.)

An *n-cochain of* X *with coefficients in* Y is a map $XG^{n+1} \longrightarrow Y$. If Y is an abelian group object in the category $\underline{B}$, the n-cochains (XG^{n+1}, Y) form an abelian group, and the face operators ϵ_i induce abelian group maps $(X\epsilon_i, Y)\colon (XG^n, Y) \longrightarrow (XG^{n+1}, Y)$. Thus

$$0 \longrightarrow (XG, Y) \xrightarrow{d^1} (XG^2, Y) \xrightarrow{d^2} \cdots \longrightarrow (XG^{n+1}, Y) \xrightarrow{d^{n+1}} (XG^{n+2}, Y) \longrightarrow \cdots$$

is a cochain complex of abelian groups, where $d^{n+1} = \sum(-1)^i(X\epsilon_i, Y)$, $0 \le i \le n+1$, $dd = 0$ because of the simplicial identities, and the augmentation term has been dropped. To distinguish it from another complex that will be introduced later, this complex is called the *homogeneous complex*. We define

$$H^n(X,Y), \qquad n \ge 0,$$

as the n-th cohomology group of this complex. Obviously maps $X' \longrightarrow X$ in $\underline{B}$ and $Y \longrightarrow Y'$ in the category of abelian group objects in $\underline{B}$ induce maps

$$H^n(X,Y) \longrightarrow H^n(X',Y), \quad H^n(X,Y) \longrightarrow H^n(X,Y')$$

and the cohomology is functorial in the usual way.

I do not know how to characterize the cohomology theory $H(X,Y) = (H^n(X,Y))$, $n \ge 0$, axiomatically. However, acyclicity of free objects and the exact cohomology sequence are properties that can easily be established:

THEOREM 2.

$$H^n(AF,Y) = \begin{cases} (AF,Y), & n = 0, \\ 0, & n > 0\cdot \end{cases}$$

If $0 \longrightarrow Y' \longrightarrow Y \longrightarrow Y'' \longrightarrow 0$ *is a U-exact sequence of abelian group objects in $\underline{B}$, then there is an exact cohomology sequence*

$$\begin{tikzcd}
 & \cdots & & \\
 & H^n(X,Y') \arrow[r] & H^n(X,Y) \arrow[r] & H^n(X,Y'') \arrow[ull, "d"'] \\
0 \arrow[r] & H^0(X,Y') \arrow[r] & \cdots \arrow[r] & H^{n-1}(X,Y'') \arrow[ull, "d"']
\end{tikzcd}$$

PROOF. For the first part, we notice that the standard resolution of a free object AF has a simplicial contracting homotopy

$$AF \xrightarrow{s_{-1}} AFG \xrightarrow{s_0} AFG^2 \longrightarrow \cdots \longrightarrow AFG^{n+1} \xrightarrow{s_n} AFG^{n+2} \longrightarrow \cdots$$

given by $s_n = A\eta FG^{n+1}$ (see [Huber (1961), p. 248], for example; this homotopy satisfies the relations $s_n\epsilon_0 =$ identity, $s_n\epsilon_i = \epsilon_{i-1}s_{n-1}$, $1 \le i \le n+1$). If we let $t^n = (s_n, Y)$, then

$$0 \longleftarrow (AF,Y) \xleftarrow{t^{-1}} (AFG,Y) \xleftarrow{t^0} (AFG^2,Y) \longleftarrow \cdots \xleftarrow{t^n} (AFG^{n+2},Y) \longleftarrow \cdots$$

is a contracting homotopy of the augmented cochain complex, that is, we have $dt + td =$ identity. Hence $H^0(AF,Y)$ is the term of degree -1 and all higher cohomology vanishes.

For the second part we must explain the concept of U-exactness; this requires recalling some facts about abelian group objects in categories (see [Eckmann & Hilton (1962)] for a fuller treatment). Y is an *abelian group object* in $\underline{B}$ if the hom set (B,Y) has an abelian group structure for every object B in the category $\underline{B}$, and naturality holds in that every induced map $(B,Y) \longrightarrow (B',Y)$ is an abelian group map, where $B' \longrightarrow B$ in $\underline{B}$. $Y \longrightarrow Y'$ is a *map* of abelian group objects in $\underline{B}$ if it is a map in $\underline{B}$ and every induced $(B,Y) \longrightarrow (B,Y')$ is an abelian group map. The abelian group objects form a category $\mathrm{Ab}\underline{B}$, with an obvious forgetful functor $\mathrm{Ab}\underline{B} \longrightarrow \underline{B}$. Of course, one can define in the same way other types of algebraic objects in categories, such as objects with base points, nonabelian groups, rings, in fact models for any algebraic theory [Lawvere (1963)], and we will need some of these categories later. However, what general theory we will use is adequately illustrated by the case of abelian groups. The following lemma can be paraphrased by saying that adjoints preserve abelian group objects:

LEMMA 1. *Let $\underline{A} \xleftarrow{U} \underline{B}$ be a functor which has a left adjoint, and let $Y \in |\mathrm{Ab}\underline{B}|$. Then there exists a unique abelian group structure on $YU \in |A|$ such that*

$$(BU, YU) \xleftarrow{U} (B,Y)$$

is an abelian group map for all $B \in |\underline{B}|$. In fact, the abelian group structure on YU is such that if $\alpha\colon F \dashv U$ is any adjointness then

$$(A, YU) \xrightarrow{\alpha} (AF, Y)$$

is an abelian group map.

PROOF. Picking any α, the last displayed map, which is an isomorphism, defines an abelian group structure on YU. Since

$$\begin{tikzcd} (BU,YU) & & (B,Y) \arrow[ll, "U"'] \arrow[dl, "{(B\epsilon,Y)}"] \\ & (BUF,Y) \arrow[ul, "\alpha^{-1}"] & \end{tikzcd}$$

commutes, U is an abelian group map. If another left adjoint[1] $\alpha'\colon F' \dashv U$ is given, there is an isomorphism $F' \xrightarrow{\simeq} F$ such that

$$\begin{tikzcd} & (AF,Y) \arrow[d, "\simeq"] \\ (A,YU) \arrow[ur, "\alpha"] \arrow[r, "\alpha'"'] & (AF',Y) \end{tikzcd}$$

[1]Editors' note: The original starts with $(\alpha'\colon)$, but there is no discernible reason for the parentheses and we have chosen to omit it.

commutes [Kan (1958)]. Since the vertical map is an abelian group isomorphism, the addition defined in (A, YU) is independent of the selection of the left adjoint. Finally, taking any $\alpha\colon F \dashv U$, since

$$\begin{tikzcd}
(A,YU) & & (AF,Y) \arrow[ll, "\alpha^{-1}"'] \arrow[dl, "U"] \\
& (AFU,YU) \arrow[ul, "(A\eta{,}YU)"] &
\end{tikzcd}$$

commutes and $A\eta$ induces an abelian group map, any group law on YU such that U is an abelian group map must inevitably be definable in terms of α in the above manner. This proves uniqueness.

This lemma allows us to define U-exactness. The idea is that like the cohomology itself, the notion of exactness in $\underline{B}$ should be "relativized" by means of the underlying object functor U. That is, $0 \longrightarrow Y' \longrightarrow Y \longrightarrow Y'' \longrightarrow 0$ is a *U-exact sequence in* Ab$\underline{B}$ if $0 \longrightarrow Y'U \longrightarrow YU \longrightarrow Y''U \longrightarrow 0$ is an exact sequence of abelian group objects in $\underline{A}$, and this we in turn define to mean that $0 \longrightarrow (A, Y'U) \longrightarrow (A, YU) \longrightarrow (A, Y''U) \longrightarrow 0$ is an exact sequence of ordinary abelian groups, for every $A \in |\underline{A}|$.

To see how this exactness concept works in practice, the reader is referred to the Examples. However, the proof of Theorem 2 can now be completed. Applying adjointness to the last sequence above, $0 \longrightarrow (AF, Y') \longrightarrow (AF, Y) \longrightarrow (AF, Y'') \longrightarrow 0$ is an exact sequence of abelian groups for any free object AF. Since the simplicial resolution $X\mathbf{G}^*$ consists of free objects, the sequence of cochain complexes $0 \longrightarrow (X\mathbf{G}^*, Y') \longrightarrow (X\mathbf{G}^*, Y) \longrightarrow (X\mathbf{G}^*, Y'') \longrightarrow 0$ is exact. The long exact sequence in cohomology is now standard. Theorem 2 is proved.

Remark. It is in the sense of U-exactness that $X\mathbf{G}^*$ is a resolution of X. In the underlying category $\underline{A}$, the augmented simplicial object $X\mathbf{G}^*U$ is contractible, with

$$XU \xrightarrow{h_{-1}} XGU \xrightarrow{h_0} XG^2U \xrightarrow{h_1} \cdots \longrightarrow XG^{n+1}U \xrightarrow{h_n} XG^{n+2}U \longrightarrow \cdots$$

as contracting homotopy where $h_n = XG^{n+1}U\eta$. (In this case $h_n\epsilon_{n+1} =$ identity, $h_n\epsilon_i = \epsilon_i h_{n-1}$, $0 \le i \le n$, are the identities satisfied.)

The above properties of the cohomology are purely formal, depending only on adjointness. To interpret the cohomology groups, at least in the lowest dimensions, one must invoke, as far as I know, the assumption that the adjoint pair is tripleable. Indeed, it is interesting to note, as mentioned in §1, that the cohomology itself is only a function of the triple $\mathbf{T}$ induced on $\underline{A}$ by the adjoint pair, in the following sense. Let $F \dashv U$ and recall the standard diagram

$$
\begin{tikzcd}
\underline{A}^{\mathbf{T}} \arrow[dr, "U^{\mathbf{T}}"'] & & \underline{B} \arrow[ll, "\Phi"'] \arrow[dl, "U"] \\
& \underline{A} &
\end{tikzcd}
$$

THEOREM 3. Φ *induces a cohomology isomorphism*

$$H(X\Phi, Y\Phi) \xleftarrow[\simeq]{H(\Phi)} H(X,Y)$$

Here $X \in |\underline{B}|$, $Y \in |\mathrm{Ab}\underline{B}|$, $H(X,Y)$ is the graded group $(H^n(X,Y))$, $n \geq 0$, and the cohomologies are taken with respect to $U^{\mathbf{T}}$ and U.

To give meaning to this theorem we have to describe how $Y\Phi$ is treated as an abelian group object in $\underline{A}^{\mathbf{T}}$, and then how the cohomology map $H(\Phi)$ is induced by Φ. We actually establish more than is asserted in the theorem, namely, we show that Φ induces an isomorphism of cochain complexes $\left(X\left(G^{\mathbf{T}}\right)^*, Y\right) \xleftarrow{\simeq} (XG^*, Y)$. The abelian group structure on $Y\Phi$ results from the following two lemmas. The first, Lemma 2, strengthens Lemma 1 in the tripleable case. It asserts that abelian group objects are not only preserved but also reflected by the underlying object functor $\underline{A}^{\mathbf{T}} \longrightarrow \underline{A}$:

LEMMA 2. *Let (Y, θ) be an abelian group object in $\underline{A}^{\mathbf{T}}$. There is a unique abelian group structure on $Y \in |\underline{A}|$ such that the forgetful*

$$(X, Y) \xleftarrow{U^{\mathbf{T}}} ((X,\xi), (Y,\theta)) \tag{1}$$

is an abelian group map for every (X,ξ) in $\left|\underline{A}^{\mathbf{T}}\right|$. This abelian group structure satisfies

$$(y_0 + y_1)T \cdot \theta = y_0 T \cdot \theta + y_1 T \cdot \theta \tag{2}$$

for all y_0, $y_1 \colon A \longrightarrow Y$ in $\underline{A}$. Conversely, given an abelian group law on Y in $\underline{A}$ satisfying (2), there exists a unique abelian group law on (Y,θ) in $\underline{A}^{\mathbf{T}}$ such that (1) is an abelian group map.

PROOF. By lemma 1, $\alpha^{\mathbf{T}} \colon (A, Y) \xrightarrow{\simeq} \left(AF^{\mathbf{T}}, (Y,\theta)\right)$ must be an abelian group map. Since $y\alpha^{\mathbf{T}} = yT \cdot \theta$, (2) follows. For the converse, let y_0, $y_1 \colon (X,\xi) \longrightarrow (Y,\theta)$ be maps in $\underline{A}^{\mathbf{T}}$ and let $y_0 + y_1 \colon X \longrightarrow Y$ be their sum in $\underline{A}$. We have to show that this is a $\mathbf{T}$-algebra map. But $(y_0 + y_1)T \cdot \theta = y_0 T \cdot \theta + y_1 T \cdot \theta = \xi y_0 + \xi y_1 = \xi(y_0 + y_1)$ by naturality of the group law and condition (2). The other group operations lift similarly. Uniqueness of course is a result of the fact that $U^{\mathbf{T}}$ is faithful.

The next lemma states that the functor Φ shares with adjoints the property of preserving group objects. Its proof is based on the U-preserves—$U^{\mathbf{T}}$ reflects principle enunciated in §1.

LEMMA 3. *If* $Y \in |\mathrm{Ab}\underline{B}|$, *then there is a unique abelian group law on* $Y\Phi \in \left|\underline{A}^{\mathbf{T}}\right|$ *such that*

$$(X\Phi, Y\Phi) \xleftarrow{\Phi} (X, Y)$$

is an abelian group map for all $X \in |\underline{B}|$.

PROOF. Let us write $\alpha\colon F \dashv U$ for the adjointness isomorphism. An abelian group law exists on $Y\Phi$ because $Y\Phi U^{\mathbf{T}} = YU$ is an abelian group in $\underline{A}$ and the $\mathbf{T}$-structure of $Y\Phi$, $Y\epsilon U\colon YUT \longrightarrow YU$, satisfies the linearity condition of Lemma 2: $(y_0 + y_1)T \cdot Y\epsilon U = ((y_0 + y_1)\alpha)\, U = (y_0\alpha + y_1\alpha)U = (y_0\alpha)U + (y_1\alpha)U = y_0T \cdot Y\epsilon U + y_1T \cdot Y\epsilon U$. Lemma 1 has been used to achieve linearity of α and U, and of course $(y\alpha)U = (yF \cdot Y\epsilon)U = yT \cdot T\epsilon U$ for any map $y\colon A \longrightarrow YU$. Now,

$$\begin{tikzcd}
(X\Phi, Y\Phi) \arrow[dr, "U^{\mathbf{T}}"'] & & (X, Y) \arrow[ll, "\Phi"'] \arrow[dl, "U"] \\
& (XU, YU) &
\end{tikzcd}$$

commutes, $U^{\mathbf{T}}$ and U are abelian group maps, and $U^{\mathbf{T}}$ is faithful, so it follows that the Φ indicated is an abelian group map. For uniqueness, let (A, ξ) be any $\mathbf{T}$-algebra, and consider the following diagram:

$$((A, \xi), Y\Phi) \xrightarrow{(A\epsilon^{\mathbf{T}}, Y\Phi)} \left(AF^{\mathbf{T}}, Y\Phi\right) \xleftarrow{\Phi} (AF, Y)$$

(labeling everything pertaining to $\underline{A} \longrightarrow \underline{A}^{\mathbf{T}} \longrightarrow \underline{A}$ with superscript $\mathbf{T}$ and recalling that Φ preserves "free" functors, $F\Phi = F^{\mathbf{T}}$). We are supposing that $Y\Phi$ has an abelian group structure in $\underline{A}^{\mathbf{T}}$, hence $\left(A\epsilon^{\mathbf{T}}, Y\Phi\right)$ has to be an abelian group map. It is injective, because $\epsilon^{\mathbf{T}}$ is an epimorphism (even a coequalizer; or use faithfulness of $U^{\mathbf{T}}$). Thus if the addition in $\left(AF^{\mathbf{T}}, Y\Phi\right)$ is uniquely determined by the condition that Φ should be an abelian group map, then uniqueness will be proved. But the Φ indicated is an isomorphism because of the following commutative diagram:

$$\begin{tikzcd}
(AF^{\mathbf{T}}, Y\Phi) = (AF\Phi, Y\Phi) & & (AF, Y) \arrow[ll, "\Phi"'] \\
(A, Y\Phi U^{\mathbf{T}}) = (A, YU) \arrow[u, "\alpha^{\mathbf{T}}"] \arrow[urr, "\alpha"'] & &
\end{tikzcd} \tag{3}$$

Indeed, if $y\colon A \longrightarrow YU$, then $(y\alpha)\Phi = (y\alpha)U = (yF \cdot Y\epsilon)U = yT \cdot Y\epsilon U = y\alpha^{\mathbf{T}}$.

Now we can finish the proof of Theorem 3. Since Φ preserves both free and underlying object functors, we have $\Phi G^{\mathbf{T}} = G\Phi$, where $G^{\mathbf{T}}$ is the standard cotriple in $\underline{A}^{\mathbf{T}}$, $(X, \xi)G^{\mathbf{T}} =$

$(XT, X\mu)$, with counit given by the $\mathbf{T}$-algebra structures. Moreover, both counits are compatible with the above equality:

$$\begin{tikzcd} \Phi G^{\mathbf{T}} = G\Phi \arrow[d, shift right, "\Phi\epsilon^{\mathbf{T}}"'] \arrow[d, shift left, "\epsilon\Phi"] \\ \Phi \end{tikzcd}$$

Thus we have commutative diagrams

$$\begin{tikzcd} (X\Phi(G^{\mathbf{T}})^{n+2}, Y\Phi) & (XG^{n+2}, Y) \arrow[l, "\Phi"'] \\ (X\Phi(G^{\mathbf{T}})^{n+1}, Y\Phi) \arrow[u, "(X\Phi\epsilon_i^{\mathbf{T}}, Y\Phi)"] & (XG^{n+1}, Y) \arrow[l, "\Phi"] \arrow[u, "(X\epsilon_i, Y)"'] \end{tikzcd}$$

for all $n \geq 0$ and all $0 \leq i \leq n+1$. By Lemma 3 and diagram (3) above, the horizontal Φ's are abelian group isomorphisms. Thus Φ defines an isomorphism of cochain complexes, which on the cohomology level we denote by $H(\Phi)$. This completes the proof of Theorem 3.

As complements to the material covered in this section, we present that stump of the cohomology which can be defined when the coefficients are in a non-abelian group object, as well as a non-homogeneous complex used in making calculations of cohomology. These topics will be needed in §3.

NON-ABELIAN COHOMOLOGY. We will define $H^0(X, Y)$ and $H^1(X, Y)$ when Y is a group object in $\underline{B}$, more or less as is usually done ([Serre (1965), p. I-56 and ff.], for example). We are in the situation $U\colon \underline{B} \longrightarrow \underline{A}$ with left adjoint F, and Y a *group object* in $|\underline{B}|$ means that every hom set (X, Y) where $X \in |\underline{B}|$ has a group structure which is (contravariantly) natural in X. The group operations in the hom set will be written

$$X \xrightarrow{y_0, y_1, y} Y \Longrightarrow \begin{cases} y_0 \circ y_1 : X \longrightarrow Y \text{ (product)} \\ y^{-1} : X \longrightarrow Y \text{ (inverse)} \\ 1 = 1_X : X \longrightarrow Y \text{ (neutral element)} \end{cases}$$

We will construe the 1-cocycles and the 1-coboundaries as objects and maps in a category $\underline{Z}^1(X, Y)$. A (*non-abelian*) 1-*cocycle* is a map $a\colon XG^2 \longrightarrow Y$ such that $\epsilon_2 a \circ \epsilon_0 a = \epsilon_1 a$ as maps $XG^3 \longrightarrow Y$ (regarding $\epsilon_i\colon XG^3 \longrightarrow XG^2$, $i = 0, 1, 2$). $b\colon a \longrightarrow a'$ is a 1-*coboundary* or a *map of* 1-*cocycles* if b is a map $XG \longrightarrow Y$ such that $a \circ \epsilon_0 b = \epsilon_1 b \circ a'$. The identity map $a \longrightarrow a$ is given by the neutral map $1_{XG}\colon XG \longrightarrow Y$, and

$$a \xrightarrow{b} a' \xrightarrow{b'} a'' \qquad \Longrightarrow \qquad a \xrightarrow{b \circ b'} a''$$

that is, composition in $\underline{Z}^1(X, Y)$ is induced by multiplication in the group of maps $XG \longrightarrow Y$. $\underline{Z}^1(X, Y)$ is therefore a category. In fact, it is a groupoid, a category in which every map is an isomorphism. The objects of $\underline{Z}^1(X,Y)$, i. e., the 1-cocycles, therefore fall into equivalence classes, and one can speak of the automorphisms of any one 1-cocycle in this category. We define

$H^1(X,Y) =$ the set of isomorphism classes of 1-cocycles,
$H^0(X,Y) =$ the automorphism group of the trivial 1-cocycle $1_{XG^2} \in |\underline{Z}^1(X,Y)|$.

$H^1(X,Y)$ is a set with distinguished element $[1_{XG^2}]$, the isomorphism class of the neutral 1-cocycle, and $H^0(X,Y)$ is a group. Clearly if Y is an abelian group object, both H^1 and H^0 are abelian groups and coincide with the cohomology groups defined earlier. The non-abelian cohomology shares the properties of the abelian theory to the degree that it is defined. $H^1(AF,Y)=1$, and there is a six-term "exact" sequence. Theorem 3 also continues to hold.

Nonhomogeneous complex. This cochain complex $C(X,Y) = (C^n(X,Y))$, $n \geq 0$, is derived by adjointness from the standard (homogeneous) complex $(X\mathbf{G}^*, Y)$. The cochains in the nonhomogeneous complex will be maps in the underlying category $\underline{A}$. The word "nonhomogeneous" refers to the varied forms of the terms occurring in the coboundary formula. It will be evident that the nonhomogeneous complex exists whenever one has adjoint functors $\underline{A} \longrightarrow \underline{B} \longrightarrow \underline{A}$. However in stating the formulas we shall confine ourselves to the tripleable case ($\underline{B} = \underline{A}^{\mathbf{T}}$) which is the only one we will need. We will then prove that the adjointness isomorphism $\alpha^{\mathbf{T}}$ gives an isomorphism of complexes $C(X,Y) \xrightarrow{\simeq} (X(\mathbf{G}^{\mathbf{T}})^*, Y)$.

Let $X = (X,\xi)$ be a $\mathbf{T}$-algebra and (Y,θ) an abelian group object in the category of $\mathbf{T}$-algebras. (From now on we often suppress the algebra structures from the notation for brevity). We define

$$C^n(X,Y) = (XT^n, Y), \qquad n \geq 0,$$

and the coboundary $d\colon C^n(X,Y) \longrightarrow C^{n+1}(X,Y)$ is given by $d = \sum(-1)^i d^i$, $0 \leq i \leq n+1$, where

$$ad^i = \begin{cases} \xi T^n \cdot a, & i = 0 \\ XT^{i-1}\mu T^{n-i} \cdot a, & 1 \leq i \leq n \\ aT \cdot \theta, & i = n+1 \end{cases}$$

for any n-cochain $a\colon XT^n \longrightarrow Y$ (a map in $\underline{\mathrm{A}}$). This defines $C(X,Y)$. We now have that

$$C^n(X,Y) \xrightarrow{\alpha} ((X,\xi)G^{n+1}, (Y,\theta)), \qquad n \geq 0,$$

is an isomorphism of cochain complexes of abelian groups.

(Superscript $\mathbf{T}$ will be omitted for the time being, $\alpha = \alpha^{\mathbf{T}}, \cdots$. The above form of words is not accurate since $C(X,Y)$ is not yet known to be a cochain complex. But if the α's are additive isomorphisms and commute with the coboundary operators, as we shall show, then $C(X,Y)$ is a complex and as such isomorphic to the homogeneous complex).

Indeed,

$$\begin{aligned}
C^n(X,Y) &= (XT^n, Y)\\
&= (XF(UF)^{n-1}U, (Y,\theta)U)\\
&\xrightarrow{\alpha} (X(FU)^nF, (Y,\theta))\\
&= ((XT^{n+1}, XT^n\mu), (Y,\theta))\\
&= ((X,\xi)G^{n+1}, (Y,\theta))
\end{aligned}$$

for $n \geq 0$. Moreover each diagram

$$\begin{tikzcd}
C^{n+1}(X,Y) \arrow[r, "\simeq", "\alpha"'] & ((X,\xi)G^{n+2}, (Y,\theta)) \\
C^n(X,Y) \arrow[r, "\simeq", "\alpha"'] \arrow[u, "d^i"] & ((X,\xi)G^{n+1}, (Y,\theta)) \arrow[u, "(\epsilon_i, (Y,\theta))"']
\end{tikzcd}$$

commutes, $0 \leq i \leq n+1$. Recalling that $\epsilon_i = G^i\epsilon G^{n-i}$, we note

$$(X,\xi)\epsilon_i = \begin{cases} \xi T^{n+1}, & i = 0 \\ XT^{i-1}\mu T^{n-i+1}, & 1 \leq i \leq n+1 \end{cases}$$

If $a\colon XT^n \longrightarrow Y$ is a nonhomogeneous n-cochain, then[2]

$$\begin{aligned}
a\alpha \cdot ((X,\xi)\epsilon_0, (Y,\theta)) &= \xi T^{n+1} \cdot aT \cdot \theta\\
&= (\xi T^n \cdot a)T \cdot \theta\\
&= (ad^0)\alpha
\end{aligned}$$

which checks the square for $i = 0$; we omit the verification for $1 \leq i \leq n+1$.

As a sample, here are the coboundaries of $a \in C^1(X,Y)$, $b \in C^0(X,Y)$:

$$\begin{aligned}
&(a)d = \xi T \cdot a - X\mu \cdot a + aT \cdot \theta,\\
&(b)d = \xi b - bT \cdot \theta
\end{aligned} \tag{4}$$

Finally, we need the *nonhomogeneous category of (non-abelian) 1-cocycles*, $\underline{Z}^1(X,Y)$ (same notation as in the homogeneous case). Confining ourselves to the tripleable case, a 1-cocycle $a\colon (X,\xi)G^2 \longrightarrow (Y,\theta)$ can be replaced using adjointness by a nonhomogeneous 1-cocycle $a\colon XT \longrightarrow Y$. Similarly, a 1-coboundary $b\colon a \longrightarrow a'$ in the category of 1-cocycles, formerly a map $b\colon (X,\xi)G \longrightarrow (Y,\theta)$, can now be thought of as a map $b\colon X \longrightarrow Y$ in $\underline{A}$. The homogeneous cocycle and coboundary conditions now have to be translated into nonhomogeneous terms using the formulas from the foregoing discussion of $C(X,Y)$. We find that $a\colon XT \longrightarrow Y$ is a 1-cocycle, and $b\colon a \longrightarrow a'$ is a map of 1-cocycles (where $b\colon X \longrightarrow Y$ in $\underline{A}$) if the following formulas hold:

$$\begin{aligned}
(aT \cdot \theta) \circ (\xi T \cdot a) &= X\mu \cdot a,\\
a \circ \xi b &= (bT \cdot \theta) \circ a'
\end{aligned} \tag{5}$$

[2]Editors' note: In the original thesis, the third line in the display was $(ad_0)\alpha$ but the subscripted 0 seemed inconsistent with previous notation

Abelianized, these formulas read: $(a)d = 0$, $(b)d = 0$ (cf. (4) above). Of course, in the last two formulas we have used multiplication in the group of maps $XT^2 \longrightarrow Y$, $XT \longrightarrow Y$ in $\underline{A}$. We assumed to start with that $(Y, \theta) \in |\mathrm{Gp}\underline{A}^{\mathbf{T}}|$ (the category of group objects in $\underline{A}^{\mathbf{T}}$); since adjoints preserve groups as well as abelian groups, the image of (Y, θ) under $\underline{A}^{\mathbf{T}} \longrightarrow \underline{A}$, namely Y, is a group object in $\underline{A}$.

3. Interpretation of Cohomology in Dimensions 0 and 1

For a general adjoint pair $\underline{A} \longrightarrow \underline{B} \longrightarrow \underline{A}$ there is a map of the interpretation into the cohomology (at least in the dimensions we will consider, 0 and 1). The main result in this section is that this map is an isomorphism if the adjoint pair is tripleable. Here "interpretation" means the hom set in dimension 0 and a concept of principal object for the coefficient group in dimension 1. The latter can be made to specialize to extensions of algebras of the type usually classified by Ext^1 or H^2. In this sense the triple cohomology generalizes Ext (it always classifies extensions in dimension 1).

From now on we assume $F\colon \underline{A} \longrightarrow \underline{B}$, $U\colon \underline{B} \longrightarrow \underline{A}$, $\alpha\colon F \dashv U$ a fixed adjointness. $G = UF$ is the functor part of the standard cotriple in $\underline{B}$ used to define cohomology with respect to the underlying object functor U.

In dimension 0 a natural map

$$(X, Y) \longrightarrow H^0(X, Y)$$

where (,) denotes the hom set in $\underline{B}$, is defined as follows. If $y\colon X \longrightarrow Y$, consider the diagram

$$XG^2 \underset{\epsilon_1}{\overset{\epsilon_0}{\rightrightarrows}} XG \xrightarrow{\epsilon} X \xrightarrow{y} Y$$

Then ϵy is in $\underline{Z}^0(X, Y)$ since $\epsilon_0\epsilon = \epsilon_1\epsilon$. The desired map sends $y \Longrightarrow [\epsilon y]$, the corresponding 0-dimensional cohomology class.

THEOREM 4.

$$(X, Y) \longrightarrow H^0(X, Y)$$

is an isomorphism if the adjoint pair $F \dashv U$ is tripleable.

PROOF. In this case we write $X = (X, \xi), Y = (Y, \theta)$ in $\underline{A}^{\mathbf{T}}$ (replacing $\underline{B}$ by $\underline{A}^{\mathbf{T}}$ to which it is equivalent via $\Phi\colon \underline{B} \xrightarrow{\simeq} \underline{A}^{\mathbf{T}}$). The above diagram becomes, in this case,

$$(XT^2, XT\mu) \underset{X\mu}{\overset{\xi T}{\rightrightarrows}} (XT, X\mu) \xrightarrow{\xi} (X, \xi) \xrightarrow{y} (Y, \theta)$$

The first three terms constitute a coequalizer diagram, so the result follows. (Given any $z\colon (XT, X\mu) \longrightarrow (Z, \zeta)$ such that $\xi T \cdot z = X\mu \cdot z$ then $X\eta \cdot z\colon (X, \xi) \longrightarrow (Z, \zeta)$ is uniquely determined by its satisfying the equation $z = \xi(X\eta \cdot z)$. The same calculation, showing that a **T**-structure is a coequalizer in the underlying category, appeared in the proof of Theorem 1.)

Of course if Y is a pointed object, group or abelian group object, $(X, Y) \longrightarrow H^0(X, Y)$ preserves the structures that arise.

In dimension 1 the cohomology classifies objects which resemble principle bundles trivialized by passage to the underlying category.

DEFINITION 4. $E \xrightarrow{p} X$ is a *Y-principal object over X, in $\underline{B}$, (with given trivialization) relative to the underlying object functor U*, if

(1) The group object Y operates on E, that is, there is a natural transformation

$$(\ , Y) \times (\ , E) \xrightarrow{\circ} (\ , E)$$

satisfying $(y_0 \circ y_1) \circ e = y_0 \circ (y_1 \circ e)$, $\quad (1 \circ e) = e$. Here y_0, y_1 are any maps $B \longrightarrow Y$, e is any map $B \longrightarrow E$ in $\underline{B}$, and 1 is the neutral element in the group of maps (B, Y).

(2) The operation of Y is compatible with the projection p. That is, if maps $y\colon B \longrightarrow Y$, $e\colon B \longrightarrow E$ are given, then $(y \circ e)p = ep$.

(3) Y operates in the following simply-transitive fashion: given two maps $e_0, e_1\colon B \longrightarrow E$ such that $e_0 p = e_1 p$, then there exists one and only one map $y\colon B \longrightarrow Y$ such that $y \circ e_0 = e_1$.

(4) There is given as part of the structure a *section* $s\colon XU \longrightarrow EU$ in the underlying category $\underline{A}$, splitting the projection, $s \cdot pU = XU$. By adjointness s can also be taken as a map $s\colon XUF = XG \longrightarrow E$. The condition that it should split the projection is then $sp = X\epsilon : XG \longrightarrow X$. In the following we shall use whichever version of s is convenient.

$E \xrightarrow{f} E'$ is defined to be a *map* of Y-principal objects over X if f preserves the projections

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

and commutes with the operations of Y, $(y \circ e)f = y \circ ef$.

Y-principal objects over X trivialized with respect to U form a category which we shall denote by

$$\underline{\mathrm{PO}}(X, Y) \qquad \text{(relative to U)}$$

A remark about the sections: one should think of $s\colon XU \longrightarrow EU$ in $\underline{A}$ as the analogue of a local trivialization of a principal bundle. Intuitively, passage to the underlying category restricts the principal object to a covering of X which it is required to become trivial with respect to. For principal bundles triviality means a global product structure.

Here similarly, *if* $p\colon E \longrightarrow X$ *has a section* $s\colon XU \longrightarrow EU$*, then* EU *is isomorphic to the product* $XU \times YU$ *in the underlying category* $\underline{A}$*, with the following maps as inverse isomorphisms:*

$$EU \xrightarrow{(pU,\sigma)} XU \times YU \xrightarrow{\tau} EU$$

σ *and* τ *being defined as follows.* It is easy to show à la Lemma 1 that $U\colon \underline{B} \longrightarrow \underline{A}$ takes Y-principal objects $E \longrightarrow X$ into YU-principal objects $EU \longrightarrow XU$. Using the given section s, the two maps

$$EU \underset{EU}{\overset{pU\cdot s}{\rightrightarrows}} EU$$

are equal when followed by pU. σ arises as the unique map $EU \longrightarrow YU$ such that $\sigma \circ pU \cdot s = EU$. τ is defined by the composition $(x, y)\tau = y \circ xs$, for any maps $x\colon A \longrightarrow XU$, $y\colon A \longrightarrow YU$. Taking the identities we can also write $\tau = \pi_{YU} \circ \pi_{XU} s$, where the π's are the projections of the product. To prove that (pU, σ) and τ are inverses of each other, one may as well assume, which will simplify the writing, that $E \longrightarrow X$ splits in $\underline{B}$ itself, and ignore the underlying category. Then $(p, \sigma)\tau = \sigma \circ ps = E$. Also $\tau(p, \sigma) = X \times Y$, since $\tau(p, \sigma)\pi_X = \pi_X$ is easy to see by compatibility of the group operation with the projection, and $\tau(p, \sigma)\pi_Y = \pi_Y$ since both of these maps operate in the same way on $\tau ps\colon X \times Y \longrightarrow E$ (using the simply-transitive character of the operation on Y):

$$\begin{aligned} \tau(p,\sigma)\pi_Y \circ \tau ps &= \tau\sigma \circ \tau ps \\ &= \tau(\sigma \circ ps) \\ &= \tau, \end{aligned}$$

$$\begin{aligned} \pi_Y \circ \tau ps &= \pi_Y \circ (\pi_Y \circ \pi_X s)ps \\ &= \pi_Y \circ \pi_X s \\ &= \tau. \end{aligned}$$

Of course, no assumption about the existence of products in the underlying category is involved here. The lemma could be rephrased to assert that pU and σ have the universal mapping property of projections.

A further remark: the reason for including the section in the structure of a principal object and not merely assuming its existence, as is customary, is that this affords us a well-defined map PO's $\longrightarrow$ 1-cocycles. Note that the postulated sections do not have to be preserved by maps of principal objects. Two principal objects identical except for their sections will be isomorphic in the category $\underline{\mathrm{PO}}(X, Y)\cdot$

Cohomology Classification of Principal Objects. We define a functor

$$\underline{\mathrm{PO}}(X, Y) \xrightarrow{\Theta} \underline{Z}^1(X, Y) \qquad \text{(relative to } U\text{)}$$

where $\underline{Z}^1(X, Y)$ is the category of non-abelian 1-cocycles (homogeneous, for the moment) described in our discussion of cohomology.

The functor $\underline{\Theta}$ is defined in the following way. If $p\colon E \longrightarrow X$ is a given principal object, consider the diagram

$$\begin{tikzcd} XG^2 \arrow[r, shift left, "\epsilon_0"] \arrow[r, shift right, "\epsilon_1"'] & XG \arrow[r, "s"] \arrow[dr, "\epsilon"'] & E \arrow[d, "p"] \\ & & X \end{tikzcd}$$

s being the assumed section. Then $(\epsilon_0 s)p = \epsilon_0(sp) = \epsilon_0\epsilon = \epsilon_1\epsilon = (\epsilon_1 s)p$, so we know that there is a unique map $a\colon XG^2 \longrightarrow Y$ such that $a \circ \epsilon_0 s = \epsilon_1 s$. A calculation which is given below shows that a is a non-abelian 1-cocycle. We define $\underline{\Theta}$ on objects by $(E)\underline{\Theta} = a$. If $f\colon E \longrightarrow E'$ is a map in $\underline{\mathrm{PO}}(X, Y)$, form the diagram

$$\begin{tikzcd} XG \arrow[r, shift left, "sf"] \arrow[r, shift right, "s'"'] \arrow[dr, "\epsilon"'] & E' \arrow[d, "p'"] \\ & X \end{tikzcd}$$

Since $s'p' = \epsilon = sp = sfp'$, there is a unique $b\colon XG \longrightarrow Y$ such that $b \circ s' = sf$. By calculation b is a map of 1-cocycles $a \longrightarrow a'$. We define $(f)\underline{\Theta} = b$. One easily sees that $\underline{\Theta}$ is a functor, the main verification needed being that $(ff')\underline{\Theta} = b \circ b'$, the product in the group of maps $XG \longrightarrow Y$ and the composition in the category $\underline{Z}^1(X, Y)$.

Clearly $\underline{\Theta}$ induces a map

$$\mathrm{PO}(X, Y) \xrightarrow{\Theta} H^1(X, Y) \qquad \text{(relative to } U\text{)}$$

where $\mathrm{PO}(X, Y)$ is the set of connected components of the category $\underline{\mathrm{PO}}(X, Y)$. (Two objects can be connected if there is a string of morphisms pointing in either direction leading from one to the other. These morphisms become composable isomorphisms when mapped into the groupoid $\underline{Z}^1(X, Y)$.)

Now suppose that the cartesian product $X \times Y$ exists in $\underline{B}$. Because of the projection $\pi_X\colon X \times Y \longrightarrow X$ and the left operation of Y on the second factor, $X \times Y$ is a Y-principal object over X. Its section of course is the map $(X, 1)\colon X \longrightarrow X \times Y$ where $1\colon X \longrightarrow Y$ is the neutral map. This principal object is *trivial*, or *split*. Any Y-principal object which is split in $\underline{B}$ is isomorphic to $X \times Y$. We refer to the component of $\underline{\mathrm{PO}}(X, Y)$ that $X \times Y$ lies in as the *trivial element* of $\mathrm{PO}(X, Y)$. (Not all principal objects in the trivial component will be split, since $\underline{\mathrm{PO}}(X, Y)$ is not necessarily a groupoid.) Clearly

$$\underline{\mathrm{PO}}(X, Y) \xrightarrow{\underline{\Theta}} \underline{Z}^1(X, Y) \qquad \text{(relative to } U\text{)}$$

preserves the trivial object, which in $\underline{Z}^1$ is the neutral 1-cocycle $XG^2 \longrightarrow Y$. Thus the induced map

$$\mathrm{PO}(X, Y) \xrightarrow{\Theta} H^1(X, Y)$$

also preserves the trivial element, and another map is induced, denoted by

$$\mathrm{Aut}(X \times Y) \xrightarrow{\mathrm{Aut}(\underline{\Theta})} H^0(X, Y).$$

The automorphism group is that of $X \times Y$ in the category of Y-principal objects.

Before proving that these maps $\underline{\Theta}$ are equivalences in the tripleable case, we give the calculations necessary in order to know that the functor $\underline{\Theta}$ takes its values in the category of 1-cocycles.

To prove that $a = (E)\underline{\Theta}$ is a 1-cocycle, we let the maps $\epsilon_2 a \circ \epsilon_0 a$ and $\epsilon_1 a$ operate on the map $\epsilon_0\epsilon_0 s\colon XG^3 \longrightarrow E$, and note that they give the same result.

$$\begin{aligned}
\epsilon_2 a \circ \epsilon_0 a \circ \epsilon_0\epsilon_0 s &= \epsilon_2 a \circ \epsilon_0(a \circ \epsilon_0 s) \\
&= \epsilon_2 a \circ \epsilon_0\epsilon_1 s \\
&= \epsilon_2 a \circ \epsilon_2\epsilon_0 s \\
&= \epsilon_2(a \circ \epsilon_0 s) \\
&= \epsilon_2\epsilon_1 s,
\end{aligned}$$

$$\begin{aligned}
\epsilon_1 a \circ \epsilon_0\epsilon_0 s &= \epsilon_1 a \circ \epsilon_1\epsilon_0 s \\
&= \epsilon_1(a \circ \epsilon_0 s) \\
&= \epsilon_1\epsilon_1 s.
\end{aligned}$$

$b = (f)\underline{\Theta}$ a 1-coboundary $a \longrightarrow a'$ is proved similarly. We let $a \circ \epsilon_0 b$ and $\epsilon_1 b \circ a'$ operate on $\epsilon_0 s'\colon XG^2 \longrightarrow E'$.

$$\begin{aligned}
a \circ \epsilon_0 b \circ \epsilon_0 s' &= a \circ \epsilon_0(b \circ s') \\
&= a \circ \epsilon_0 s f \\
&= (a \circ \epsilon_0 s) f \\
&= \epsilon_1 s f,
\end{aligned}$$

$$\begin{aligned}
\epsilon_1 b \circ a' \circ \epsilon_0 s' &= \epsilon_1 b \circ \epsilon_1 s' \\
&= \epsilon_1(b \circ s') \\
&= \epsilon_1 s f.
\end{aligned}$$

The tripleable case. When the adjoint pair is $\underline{A} \longrightarrow \underline{A}^{\mathbf{T}} \longrightarrow \underline{A}$, then $\underline{\Theta}\colon \underline{\mathrm{PO}} \longrightarrow \underline{Z}^1$ is an equivalence, implying the desired cohomology classification. The principal objects considered are those trivialized by $\underline{A}^{\mathbf{T}} \longrightarrow \underline{A}$ and the cohomology is also taken with respect to this underlying object functor. In general, if $\underline{A} \longrightarrow \underline{B} \longrightarrow \underline{A}$ is an arbitrary adjoint pair, one should still observe that $\underline{\Theta}\colon \underline{\mathrm{PO}} \longrightarrow \underline{Z}^1$ is compatible with the process of "tripleization," in the sense of commutativity of the following diagram

$$\begin{array}{ccc} \underline{\mathrm{PO}}(X,Y) & \xrightarrow{\ \underline{\Theta}\ } & \underline{Z}^1(X,Y) & \qquad \text{(relative to } U\text{)} \\ \Big\downarrow{\scriptstyle \underline{\mathrm{PO}}(\Phi)} & & \simeq \Big\downarrow{\scriptstyle \underline{Z}^1(\Phi)} & \\ \underline{\mathrm{PO}}(X\Phi,Y\Phi) & \xrightarrow[\underline{\Theta}^{\mathbf{T}}]{\ \simeq\ } & \underline{Z}^1(X\Phi,Y\Phi) & \qquad \text{(relative to } U^{\mathbf{T}}\text{)} \end{array}$$

Here $\underline{\Theta}^{\mathbf{T}}$ is the functor which will be proved to be an equivalence, and Φ is the "unit of structure" (§1). (Φ is easily shown to map principal objects, after the manner of Lemma 3.) This diagram shows that an arbitrary adjoint pair $\underline{A} \longrightarrow \underline{B} \longrightarrow \underline{A}$ can be canonically "closed up" via Φ, to a tripleable adjoint pair $\underline{A} \longrightarrow \underline{A}^{\mathbf{T}} \longrightarrow \underline{A}$, wherein the interpretation of H^1 as isomorphism classes of principal objects always succeeds (as well as the interpretation of H^0 as the hom functor). Of course if $\underline{A} \longrightarrow \underline{B} \longrightarrow \underline{A}$ is tripleable to start with (Φ an equivalence) then H^1 already classifies principal objects in $\underline{B}$.

Let us assume we are in the situation $\underline{A} \longrightarrow \underline{A}^{\mathbf{T}} \longrightarrow \underline{A}$, $X = (X,\xi)$ is a $\mathbf{T}$-algebra, $Y = (Y,\theta)$ is a group object in the category of $\mathbf{T}$-algebras, and let us write $\underline{\Theta} = \underline{\Theta}^{\mathbf{T}}$.

THEOREM 5. *If the cartesian product $X \times Y$ exists in $\underline{A}$, then*

$$\underline{\mathrm{PO}}(X,Y) \xrightarrow{\ \underline{\Theta}\ } \underline{Z}^1(X,Y) \qquad \textit{(relative to } U^{\mathbf{T}}\textit{)}$$

is an equivalence of categories, inducing isomorphisms

$$\left.\begin{array}{r} \mathrm{PO}(X,Y) \xrightarrow[\simeq]{\ \Theta\ } H^1(X,Y) \\ \mathrm{Aut}(X\times Y) \xrightarrow[\simeq]{\mathrm{Aut}(\Theta)} H^0(X,Y) \end{array}\right. \qquad \textit{(relative to } U^{\mathbf{T}}\textit{)}$$

For algebra extensions the special case $X = 1$ (terminal object) is important (see Theorem 6 below). In that case the product assumption can be dropped since $1 \times Y \simeq Y$. The statement about $\mathrm{Aut}(X \times Y)$ above is related to Theorem 4. In general, $\mathrm{Aut}(X\times Y) \simeq (X,Y)$ as groups since any principal object endomorphism of $X \times Y$ satisfies $(x,y)f = (y\circ(x,1))f = y\circ(x,1)f$ and is therefore determined by $(x,1)f$ which is a map $X \longrightarrow Y$. Any endomorphism is thus an automorphism, although $\underline{\mathrm{PO}}(X,Y)$ as a whole is not necessarily a groupoid.

PROOF. We consider (Y,θ)-principal $\mathbf{T}$-algebras $p\colon (E,\psi) \longrightarrow (X,\xi)$, with sections $s\colon X \longrightarrow E$ in $\underline{A}$. What tripleableness does for us is allow us to express the structures of principal algebras wholly in terms of the underlying category. We have the following lemma, which is proved like Lemma 2.

LEMMA 4. *$(E,\psi) \xrightarrow{\ p\ } (X,\xi)$ is a (Y,θ)-principal $\mathbf{T}$-algebra $\Longleftrightarrow$ $p\colon E \longrightarrow X$ is a Y-principal object in $\underline{A}$ and the $\mathbf{T}$-structure $\psi\colon ET \longrightarrow E$ satisfies*

$$(y\circ e)T\cdot\psi = (yT\cdot\theta)\circ(eT\cdot\psi)$$

in the set of maps $AT \longrightarrow E$*, where* $y\colon A \longrightarrow Y$*,* $e\colon A \longrightarrow E$ *are any maps in* $\underline{A}$*.*

$f\colon (E,\psi) \longrightarrow (E',\psi')$ *is a map of* (Y,θ)*-principal algebras over* (X,ξ) $\iff$ f *is a* $\mathbf{T}$*-algebra map, preserves the projections into* X*, and satisfies*

$$(y \circ e)f = y \circ ef$$

for any $y\colon A \longrightarrow Y$*,* $e\colon A \longrightarrow E$ *in* $\underline{A}$*.*

For the next step in the proof let us write, momentarily, $\overline{a}\colon XG^2 \longrightarrow Y$ in $\underline{A}^{\mathbf{T}}$ for the homogeneous 1-cocycle $\overline{a} = (E,\psi)\underline{\Theta}$, and $\overline{s}\colon XG \longrightarrow E$ in $\underline{A}^{\mathbf{T}}$ for the section. Then $\overline{a}$ is determined by the relation $\overline{a} \circ \epsilon_0\overline{s} = \epsilon_1\overline{s}$. Let $a\colon XT \longrightarrow Y$ correspond to $\overline{a}$ under adjointness. Then a is a nonhomogeneous 1-cocycle. The given section $s\colon X \longrightarrow E$ corresponds to $\overline{s}$ under adjointness also. What relation between a and s corresponds to the relation between $\overline{a}$ and $\overline{s}$? Now, in proving Lemma 4, one uses adjointness to push the Y-operation on E down to the underlying category. One employs the following type of commutative diagram where we have fixed $XG^2 = (XT^2, XT\mu)$ in the first variable —

$$\begin{tikzcd}
(XG^2,Y)\times(XG^2,E) \arrow[r, "\circ"] & (XG^2,E) & \text{in } \underline{A}^{\mathbf{T}} \\
(XT,Y)\times(XT,E) \arrow[u, "\alpha\times\alpha"] \arrow[r, "\circ"'] & (XT,E) \arrow[u, "\alpha"'] & \text{in } \underline{A}
\end{tikzcd}$$

The following elements match up under the vertical isomorphisms:

$$\begin{tikzcd}
(\overline{a},\epsilon_0\overline{s}) \arrow[r] & \epsilon_1\overline{s} \\
(a,\xi s) \arrow[u, "\alpha\times\alpha"] \arrow[r] & sT\cdot\psi \arrow[u, "\alpha"']
\end{tikzcd}$$

ξs and $sT \cdot \psi$ correspond to the compositions of $\overline{s}$ with ϵ_0 and ϵ_1 under adjointness (cf. description of the nonhomogeneous coboundary operator in §2). Thus, nonhomogeneously, if $a\colon XT \longrightarrow Y$ in $\underline{A}$ is $(E,\psi)\underline{\Theta}$, then a is determined by the relation

(5) $$a \circ \xi s = sT \cdot \psi \qquad \text{in the set of maps } XT \longrightarrow E.$$

Of course, (5) could be used to define a directly, since the two maps

$$\begin{tikzcd}
 & X \arrow[dr, "s"] & \\
XT \arrow[ur, "\xi"] \arrow[dr, "sT"'] & & E \\
 & ET \arrow[ur, "\psi"'] &
\end{tikzcd}$$

are equal after composition with $p: E \longrightarrow X$. The argument we sketched shows that this definition of a is consistent with the earlier, homogeneous, definition, in the not necessarily tripleable case. Since a is a 1-cocycle, we have the coboundary zero relation $(aT \cdot \theta) \circ (\xi T \cdot a) = X\mu \cdot a$ in the group of maps $XT^2 \longrightarrow Y$.

The effect of $\underline{\Theta}: \underline{\mathrm{PO}}(X,Y) \longrightarrow \underline{Z}^1(X,Y)$ on maps is as follows (with $\underline{Z}^1$ treated nonhomogeneously). If $f: (E,\psi) \longrightarrow (E',\psi')$, then the corresponding coboundary $b = (f)\underline{\Theta}: a \longrightarrow a'$ is given by the formula

$$b \circ s' = sf \qquad \text{as maps } X \longrightarrow E \text{ in } \underline{A}, \tag{6}$$

where s and s' are the sections of E and E'.

Now we shall define the inverse functor

$$\underline{\mathrm{PO}}(X,Y) \xleftarrow[\underline{\Theta}^{-1}]{} \underline{Z}^1(X,Y) \qquad \text{(relative to } U^{\mathbf{T}}).$$

If $a: XT \longrightarrow Y$ is any 1-cocycle, let $a\underline{\Theta}^{-1}$ be the principal $\mathbf{T}$-algebra given by $X \times Y$ as an object in $\underline{A}$, with left Y-operation (in $\underline{A}$) on the second factor, $\pi_X: X \times Y \longrightarrow X$ as projection, $(X,1): X \longrightarrow X \times Y$ as section, and the composition

$$(X \times Y)T \xrightarrow{\pi} XT \times YT \xrightarrow{\xi \times \theta a} X \times Y$$

as $\mathbf{T}$-structure. Here π is induced by the projections, $\pi = (\pi_X T, \pi_Y T)$, and $\xi \times \theta a$ is an abbreviation which we shall use consistently for the map $(t_0,t_1)(\xi \times \theta a) = (t_0\xi, t_1\theta \circ t_0 a)$ for any maps $t_0: A \longrightarrow XT$, $t_1: A \longrightarrow YT$ in $\underline{A}$. Diagrammatically, $\xi \times \theta a$ is the composition

$$XT \times YT \xrightarrow{(\xi,a)\times\theta} X \times Y \times Y \xrightarrow{X\times(\pi_2,\pi_1)} X \times Y \times Y \xrightarrow{X\times\text{mult.}} X \times Y$$

One can verify easily that the projection $\pi_X: (X \times Y, \pi(\xi \times \theta a)) \longrightarrow (X,\xi)$ is indeed a $\mathbf{T}$-algebra map, although the section $(X,1)$ is only a map in $\underline{A}$ (unless $a = 1$, or a coboundary, of course, as follows from the theorem; if $a = 1$, then $\pi(\xi \times \theta a)$ is the $\mathbf{T}$-structure on the product algebra $(X,\xi) \times (Y,\theta)$). The rest of the verification that $a\underline{\Theta}^{-1}$ is a (Y,θ)-principal $\mathbf{T}$-algebra will be left till the end.

If $b: a \longrightarrow a'$ is a map of 1-cocycles, we set $b\underline{\Theta}^{-1} = X \times Y \circ b$

$$a\underline{\Theta}^{-1} = (X \times Y, \pi(\xi \times \theta a)) \xrightarrow{X\times Y\circ b} (X \times Y, \pi(\xi \times \theta a')) = a'\underline{\Theta}^{-1}$$

where $X \times Y \circ b$ is given by the formula $(t_0,t_1)(X \times Y \circ b) = (t_0, t_1 \circ t_0 b)$ for $t_0: A \longrightarrow X$, $t_1: A \longrightarrow Y$ in $\underline{A}$. We will sketch later a proof that this is a map of principal algebras.

We can now show that

$$\underline{\mathrm{PO}}(X,Y) \xrightarrow{\simeq} \underline{\Theta}\underline{\Theta}^{-1}\ , \underline{\Theta}^{-1}\underline{\Theta} = \underline{Z}^1(X,Y)\ .$$

Let $(E,\psi) \xrightarrow{p} (X,\xi)$ be a (Y,θ)-principal algebra with section $s: X \longrightarrow E$. Then we know that $E \xrightarrow{\simeq} X \times Y$ in $\underline{A}$,

$$E \xrightarrow{(p,\sigma)} X \times Y \xrightarrow{\tau} E$$

where $\sigma \circ ps = E$, $(x,y)\tau = y \circ xs$. To prove that $\underline{\mathrm{PO}} \simeq \underline{\Theta}\Theta^{-1}$, all we have to do is show that the $\underline{A}$-isomorphism

$$(E,\psi) \xrightarrow{(p,\sigma)} (X \times Y, \pi(\xi \times \theta a))$$

is a $\mathbf{T}$-algebra map. Consider the diagram

$$\begin{tikzcd}
ET \arrow[r, "(p,\sigma)T"] \arrow[d, "ET"'] & (X \times Y)T \arrow[d, "\pi"] \\
ET \arrow[r, "(pT,\sigma T)"] \arrow[d, "\psi"'] & XT \times YT \arrow[d, "\xi\times\theta a"] \\
E \arrow[r, "(p,\sigma)"'] & X \times Y
\end{tikzcd}$$

Since $\psi p = pT \cdot \xi$ (p is a $\mathbf{T}$-algebra map) and

$$(pT, \sigma T)(\xi \times \theta a) = (pT \cdot \xi, (\sigma T \cdot \theta) \circ (pT \cdot a))$$

we have to prove that $\psi\sigma = (\sigma T \cdot \theta) \circ (pT \cdot a)$ as maps $ET \longrightarrow Y$. We let them both operate on $\psi ps \colon ET \longrightarrow E$:

$$\begin{aligned}
(\sigma T \cdot \theta) \circ (pT \cdot a) \circ \psi ps &= (\sigma T \cdot \theta) \circ (pT \cdot a) \circ (pT \cdot \xi s) \\
&= (\sigma T \cdot \theta) \circ pT \cdot (a \circ \xi s) \\
&= (\sigma T \cdot \theta) \circ (pT \cdot sT \cdot \psi) \\
&= (\sigma T \cdot \theta) \circ ((ps)T \cdot \psi) \\
&= (\sigma \circ ps)T \cdot \psi \\
&= \psi \ ,
\end{aligned}$$

using Lemma 4, and

$$\begin{aligned}
\psi\sigma \circ \psi ps &= \psi(\sigma \circ ps) \\
&= \psi
\end{aligned}$$

Verification of $f\underline{\Theta}\underline{\Theta}^{-1} = f \colon E \longrightarrow E'$ will be left out; it also uses Lemma 4.

To prove $\underline{\Theta}^{-1}\underline{\Theta} = \underline{Z}^1(X,Y)$, let a be a 1-cocycle. Then $a\underline{\Theta}^{-1}\underline{\Theta}$ is determined by the equation

$$(X,1)T \cdot \pi(\xi \times \theta a) = a\underline{\Theta}^{-1}\underline{\Theta} \circ \xi(X,1) \ ,$$

$(X,1)$ being the section in $a\underline{\Theta}^{-1}$. Computing, we have

$$\begin{aligned}
(X,1)T \cdot \pi(\xi \times \theta a) &= (XT, 1T)(\xi \times \theta a) \\
&= (\xi, (1T \cdot \theta) \circ a) \\
&= (\xi, a)
\end{aligned}$$

because $1T \cdot \theta$ is the neutral map $X \longrightarrow Y$, the operation $(\ )T \cdot \theta$ preserving the group law (adaptation of Lemma 2 to the non-abelian case), whereas

$$\xi(X, 1) = (\xi, 1_{XT}),$$

by naturality of neutral maps. Manifestly, $a\underline{\Theta}^{-1}\underline{\Theta} = a$.

It remains, finally, to prove that the values of $\underline{\Theta}^{-1}$ are principal $\mathbf{T}$-algebras and maps thereof. We begin by showing that the proposed $\mathbf{T}$-structure $\pi(\xi \times \theta a)$ on $a\underline{\Theta}^{-1}$ is associative. This is done by computing both sides of the diagram

$$\begin{tikzcd}
(X \times Y)TT \arrow[r, "\mu"] \arrow[d, "\pi T"'] & (X \times Y)T \arrow[d, "\pi"] \\
(XT \times YT)T \arrow[d, "(\xi \times \theta a)T"'] & XT \times YT \arrow[d, "\xi \times \theta a"] \\
(X \times Y)T \arrow[r, "\pi(\xi \times \theta a)"'] & X \times Y
\end{tikzcd}$$

We present the result for the left side in the table which follows. Each entry on the right is the accumulated composition of the maps down to that stage.

$$\begin{array}{ll}
(X \times Y)TT & \\
\Big\downarrow{\scriptstyle \pi T} & \\
(XT \times YT)T & (\pi_X T, \pi_Y T)T \\
\Big\downarrow{\scriptstyle (\xi \times \theta a)T} & \\
(X \times Y)T & (\pi_X T \cdot \xi, (\pi_Y T \cdot \theta) \circ (\pi_X T \cdot a))T \\
\Big\downarrow{\scriptstyle \pi} & \\
XT \times YT & (\pi_X TT \cdot \xi T, ((\pi_Y T \cdot \theta) \circ (\pi_X T \cdot a))T) \\
\Big\downarrow{\scriptstyle \xi \times \theta a} & \\
X \times Y & (\pi_X TT \cdot \xi T \cdot \xi, ((\pi_Y T \cdot \theta) \circ (\pi_X T \cdot a))T \cdot \theta \circ (\pi_X TT \cdot \xi T \cdot a))
\end{array}$$

which is $(\pi_X TT \cdot \xi T \cdot \xi, (\pi_Y TT \cdot \theta T \cdot \theta) \circ (\pi_X TT \cdot aT \cdot \theta) \circ (\pi_X TT \cdot \xi T \cdot a))$ by the multiplicative property of $(\ )T \cdot \theta$. The right side of the diagram is, similarly[3],

$$
\begin{array}{ll}
(X \times Y)TT & \\
\big\downarrow{\scriptstyle \mu} & \\
(X \times Y)T & \\
\big\downarrow{\scriptstyle \pi} & \\
XT \times YT & (\pi_X TT \cdot X\mu, \pi_Y TT \cdot Y\mu) \\
\big\downarrow{\scriptstyle \xi \times \theta a} & \\
X \times Y & (\pi_X TT \cdot X\mu \cdot \xi, (\pi_Y TT \cdot Y\mu \cdot \theta) \circ (\pi_X TT \cdot X\mu \cdot a))
\end{array}
$$

Since ξ and θ are associative, and a satisfies the 1-cocycle identity, both sides of the diagram give the same map. Thus the structure $\pi(\xi \times \theta a)$ is associative.

To prove the structure is unitary, consider the diagram

$$
\begin{tikzcd}
X \times Y \arrow[r, "(X\times Y)\eta"] \arrow[dr, "X\eta\times Y\eta"'] & (X \times Y)T \arrow[d, "\pi"] \\
& XT \times YT \arrow[d, "\xi\times\theta a"] \\
& X \times Y
\end{tikzcd}
$$

The composition $\eta\pi(\xi \times \theta a) = (X\eta \cdot \xi, (Y\eta \cdot \theta) \circ (X\eta \cdot a))$. Since ξ and θ are unitary, it suffices to prove that $X\eta \cdot a \colon X \longrightarrow Y$ is the trivial map 1. (In a sense, the cocycle a is already "normalized".) This follows from the following peculiar computation. Let $X\eta\eta \colon X \longrightarrow XTT$ denote either of $X\eta \cdot XT\eta$, $X\eta \cdot X\eta T$. Then

$$
\begin{aligned}
X\eta\eta \cdot X\mu &= X\eta\ , \\
X\eta\eta \cdot aT \cdot \theta &= X\eta \cdot XT\eta \cdot aT \cdot \theta \\
&= X\eta \cdot a \cdot Y\eta \cdot \theta \\
&= X\eta \cdot a\ , \\
X\eta\eta \cdot \xi T &= X\eta \cdot X\eta T \cdot \xi T \\
&= X\eta.
\end{aligned}
$$

[3]Editors' note: The editors believe that the "a" at the right end of the last line is correct, although the original shows a ξ.

Hence

$$\begin{aligned} X\eta \cdot a &= X\eta\eta \cdot X\mu \cdot a \\ &= X\eta\eta((aT \cdot \theta) \circ (\xi T \cdot a)) \\ &= (X\eta\eta \cdot aT \cdot \theta) \circ (X\eta\eta \cdot \xi T \cdot a) \\ &= (X\eta \cdot a) \circ (X\eta \cdot a) \end{aligned}$$

In a group, this implies $X\eta \cdot a = 1$.

It must also be proved that if $b\colon a \longrightarrow a'$ is a map of 1-cocycles, then $b\underline{\Theta}^{-1} = X \times Y \circ b\colon a\underline{\Theta}^{-1} \longrightarrow a'\underline{\Theta}^{-1}$ in $\underline{\mathrm{PO}}(X, Y)$. One must dissect the diagram

$$\begin{tikzcd} (X \times Y)T \arrow[r, "(X\times Y \circ b)T"] \arrow[d, "\pi"'] & (X \times Y)T \arrow[d, "\pi"] \\ XT \times YT \arrow[d, "\xi\times\theta a"'] & XT \times YT \arrow[d, "\xi\times\theta a'"] \\ X \times Y \arrow[r, "X\times Y \circ b"'] & X \times Y \end{tikzcd}$$

as in the proof of associativity, and at the end invoke the coboundary relation $a \circ \xi b = (bT \cdot \theta) \circ a'$. Also one has to use the fact that $X \times Y \circ b$ can be written as $X \times (\pi_Y \circ \pi_X b)$. ($b$ is a map $X \longrightarrow Y$ in $\underline{A}$ to start with.)

From all the foregoing we know that $\underline{\Theta}^{-1}$ is correctly defined with regard to $\mathbf{T}$-structures, both on objects and on maps. However, we do not quite know yet that the values of $\underline{\Theta}^{-1}$ are actually (Y, θ)-principal algebras. We have to verify that the $\mathbf{T}$-structure and the Y-operation on $X \times Y$ in $a\underline{\Theta}^{-1}$ are connected by the multiplicativity relation stated in Lemma 4. We take $e\colon A \longrightarrow X \times Y$ in $\underline{A}$, that is, $e = (e_0, e_1)$ where $e_0\colon A \longrightarrow X$ and $e_1\colon A \longrightarrow Y$, and recalling that Y operates on the left of the second factor of $X \times Y$, we have

$$\begin{aligned} (y \circ (e_0, e_1))T \cdot \pi(\xi \times \theta a) &= (e_0T, (y \circ e_1)T)(\xi \times \theta a) \\ &= (e_0T \cdot \xi, ((y \circ e_1)T \cdot \theta) \circ (e_0T \cdot a)) \\ &= (e_0T \cdot \xi, (yT \cdot \theta) \circ (e_1T \cdot \theta) \circ (e_0T \cdot a)), \\ (yT \cdot \theta) \circ (e_0, e_1)T \cdot \pi(\xi \times \theta a) &= (yT \cdot \theta) \circ (e_0T \cdot \xi, (e_1T \cdot \theta) \circ (e_0T \cdot a)) \\ &= (e_0T \cdot \xi, (yT \cdot \theta) \circ (e_1T \cdot \theta) \circ (e_0T \cdot a)). \end{aligned}$$

The multiplicativity condition to be satisfied by $b\underline{\Theta}^{-1} = X \times Y \circ b$ (Lemma 4) can be similarly verified. This completes the proof of Theorem 5.

If there were an exact sequence in the first variable of the cohomology, all of this might be avoided. (See [Barr & Rinehart (1966)], for such a possibility.)

Modules and Extensions. We shall now consider a special case of the foregoing theory, in which principal objects are interpreted as algebra extensions of a given algebra by one of its modules. This specialization is the version of triple cohomology outlined in [Barr & Beck (1966)]. In this part we limit ourselves to discussing tripleable adjoint pairs, for brevity.

Recall that the "*comma category*" $(\underline{A}^{\mathbf{T}}, X)$, where X is a $\mathbf{T}$-algebra, has maps $Z \longrightarrow X$ in $\underline{A}^{\mathbf{T}}$ as its objects, and commutative triangles

$$\begin{tikzcd} Z \arrow[rr] \arrow[dr] & & Z' \arrow[dl] \\ & X & \end{tikzcd} \qquad \text{in } \underline{A}^{\mathbf{T}}$$

as its maps. This is also called the *category of objects over* X. (See [Lawvere (1966)] for the general definition of this useful notation. Nothing depends on our having $\mathbf{T}$-algebras in this definition. X could be an object in any category. The same is true of the following definition.)

DEFINITION 5. An X*-module* is an abelian group object in the category $(\underline{A}^{\mathbf{T}}, X)$.

If $\underline{A}^{\mathbf{T}}$ has pullbacks (fibered products) the addition in an X-module will be represented by a binary operation

$$\begin{tikzcd} Y \times_X Y \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & X & \end{tikzcd}$$

(The fibered product is the ordinary cartesian product in $(\underline{A}^{\mathbf{T}}, X)$. It exists if fibered products exist in $\underline{A}$.) The identity map X is always terminal in $(\underline{A}^{\mathbf{T}}, X)$, since an object $p: Z \longrightarrow X$ admits the unique map

$$\begin{tikzcd} Z \arrow[rr, "p"] \arrow[dr, "p"'] & & X \arrow[dl, "X"] \\ & X & \end{tikzcd}$$

Since a terminal object is a 0-fold product, the nullary operation consisting of the zero element in an X-module $p: Y \longrightarrow X$ will be represented by a map of the terminal object:

$$\begin{tikzcd} X \arrow[rr, "s"] \arrow[dr, "X"'] & & Y \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

Thus an X-module always has a *zero section*, which splits the projection, $sp = X$, in the category $\underline{A}^{\mathbf{T}}$. In effect, our procedure is to identify X-modules with split extensions.

Now let $\underline{A} \xrightarrow{F} \underline{A}^{\mathbf{T}} \xrightarrow{U} \underline{A}$ be a tripleable adjoint pair. Then we get adjoint functors on comma categories

$$(\underline{A}, X) \xrightarrow{(F,X)} (\underline{A}^{\mathbf{T}}, X) \xrightarrow{(U,X)} (\underline{A}, X)$$

where $X = (X, \xi)$ is a given $\mathbf{T}$-algebra. (U, X) is the obvious forgetful functor. (F, X) is the functor which on objects:

$$\begin{tikzcd} Z \arrow[d, "p"'] & & (ZT, Z\mu) \arrow[d, "pT"] \\ X & \Longrightarrow & (XT, X\mu) \arrow[d, "\xi"] \\ & & (X, \xi) \end{tikzcd}$$

This is the usual free algebra functor lifted up to the comma categories. The composition $(F, X)(U, X)$ induces a *triple* $(\mathbf{T}, X)$ in $(\underline{A}, X)$ given by

$$\begin{tikzcd} Z \arrow[rr, "(\eta,X)"] \arrow[dr, "p"'] & & ZT \arrow[dl, "pT\cdot\xi"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} ZTT \arrow[rr, "(\mu,X)"] \arrow[dr, "pTT\cdot\xi T\cdot\xi"'] & & ZT \arrow[dl, "pT\cdot\xi"] \\ & X & \end{tikzcd}$$

on a typical object $Z \xrightarrow{p} X$. The *adjoint pair* $(F, X) \dashv (U, X)$ *is tripleable*, that is, the canonical functor

$$\begin{tikzcd} (\underline{A}, X)^{(\mathbf{T},X)} \arrow[dr, "U^{(\mathbf{T},X)}"'] & & (\underline{A}^{\mathbf{T}}, X) \arrow[ll, "\Phi"'] \arrow[dl, "(U,X)"] \\ & (\underline{A}, X) & \end{tikzcd}$$

is an isomorphism. For it is a triviality to verify that a $(\mathbf{T}, X)$-structure on $p: Z \longrightarrow X \in |(\underline{A}, X)|$

$$\begin{tikzcd} ZT \arrow[rr, "\theta"] \arrow[dr, "pT\cdot\xi"'] & & Z \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

is precisely equivalent to a $\mathbf{T}$-algebra map $(Z, \theta) \xrightarrow{p} (X, \xi)$, i.e., an object of $(\underline{A}^{\mathbf{T}}, X)$.

We have a cohomology theory

$$H^n(Z, Y)_X\,, \qquad n \geq 0\,,$$

defined for Z a given $\mathbf{T}$-algebra over X (suppressing the algebra structures and the projection $Z \longrightarrow X$ from the notation) and Y a given X-module. The cohomology is relative to the underlying object functor $(U, X)\colon (\underline{A}^{\mathbf{T}}, X) \longrightarrow (\underline{A}, X)$. The subscript X is put in as a reminder that all algebras, ... are being considered with given projections into X. The complexes used to define these groups resemble the usual ones (homogeneous or nonhomogeneous), but all cochains have to be maps over X.

By tripleableness, $H^0(Z, Y)_X$ is the abelian group of maps

$$\begin{tikzcd} Z \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & X & \end{tikzcd} \qquad \text{in } (\underline{A}^{\mathbf{T}}, X)\,\cdot$$

$H^1(Z, Y)_X$ classifies Y-principal $\mathbf{T}$-algebras $E \longrightarrow Z$ over X. We will return to this in a moment, giving a separate interpretation to the case $Z = X$. First we point out that there is a sort of co-Shapiro lemma by which one can in effect always assume that $Z = X$, that is, there is a cohomology isomorphism

$$H(Z, Yp^{-1})_Z \xrightarrow{\simeq} H(Z, Y)_X\,\cdot$$

This results from the adjoint pair

$$(\underline{A}^{\mathbf{T}}, Z) \underset{p^{-1}}{\overset{(\underline{A}^{\mathbf{T}}, p)}{\rightleftarrows}} (\underline{A}^{\mathbf{T}}, X)$$

where if Y is an algebra over X, Yp^{-1} is the pullback (existing, as remarked above, if $\underline{A}$ has pullbacks). The coadjoint is just composition with p, the fixed structural map $Z \longrightarrow X$. Since p^{-1} is an adjoint, it preserves abelian group objects:

$$Z\text{-module}\left\{\begin{tikzcd} Yp^{-1} \arrow[r, dashed] \arrow[d] & Y \arrow[d] \\ Z \arrow[r, "p"'] & X \end{tikzcd}\right\}X\text{-module}$$

This explains the appearance of Yp^{-1} as coefficients above. Once everything is defined, the proof of the co-Shapiro lemma is trivial (the two complexes involved are isomorphic under the above adjointness). In this sense it is sufficient to consider $H^1(X, Y)_X$, which evidently provides a cohomology classification for the objects described in the following definition:

DEFINITION 6. An *extension of X by the X-module $Y \longrightarrow X$* (in the category $(\underline{A}^{\mathbf{T}}, X)$) is a $(Y \longrightarrow X)$-principal $\mathbf{T}$-algebra over the terminal object $X \longrightarrow X$, which is trivialized relative to the underlying object functor $(U, X)\colon (\underline{A}^{\mathbf{T}}, X) \longrightarrow (\underline{A}, X)$.

Explicitly, such an extension consists of the following data:

(1) A $\mathbf{T}$-algebra map $E \xrightarrow{p} X$.

The definition insists on our giving a $\mathbf{T}$-algebra in $(\underline{A}^{\mathbf{T}}, X)$ together with a projection map p into the terminal object:

$$\begin{tikzcd} E \arrow[rr, "p"] \arrow[dr] & & X \arrow[dl, "X"] \\ & X & \end{tikzcd} \qquad \text{in } \underline{A}^{\mathbf{T}} \cdot$$

Therefore the unlabelled structural map of E as an object in $(\underline{A}^{\mathbf{T}}, X)$ also has to be p.

(2) An operation of the X-module $Y \longrightarrow X$ on the object $p: E \longrightarrow X$ in the category $(\underline{A}^{\mathbf{T}}, X)$. This means that there is a pairing

$$(Z, Y)_X \times (Z, E)_X \xrightarrow{\circ} (Z, E)_X$$

which is (contravariantly) natural in the variable $Z \longrightarrow X$, $(\ ,\ )_X$ being the hom functor in $(\underline{A}^{\mathbf{T}}, X)$. This operation is compatible with the projection p $((y \circ e)p = ep)$, and is simply transitive: given any two maps

$$\begin{tikzcd} Z \arrow[rr, shift left, "e_0"] \arrow[rr, shift right, "e_1"'] \arrow[dr] & & E \arrow[dl] \\ & X & \end{tikzcd}$$

there exists a unique

$$\begin{tikzcd} Z \arrow[rr, "y"] \arrow[dr] & & Y \arrow[dl] \\ & X & \end{tikzcd}$$

such that $y \circ e_0 = e_1$.

(3) There is a section $X \xrightarrow{s} E$ in $\underline{A}$. Thus *the extension* $E \xrightarrow{p} X$ *is split in the underlying category.*

One further easily checks that *maps* of extensions are $\mathbf{T}$-algebra maps over X

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

which commute with the $(Y \longrightarrow X)$-operations (and do not need to respect the sections, which, as with principal algebras, are considered as given parts of the structure).

Extensions $E \longrightarrow X$ as defined clearly form a category, which we denote by $\underline{\mathrm{Ex}}(X,Y)$. Let $\mathrm{Ex}(X,Y)$ be the set of isomorphism classes in this category (which is in fact a groupoid). Let $\mathrm{Aut}(Y)$ be the automorphisms of the trivial extension, which is just the X-module $Y \longrightarrow X$ itself. Then Theorem 5 in this context becomes:

THEOREM 6.

$$\underline{\mathrm{Ex}}(X,Y) \xrightarrow{\Theta} \underline{Z}^1(X,Y)_X$$

is an equivalence of categories, inducing isomorphisms

$$\mathrm{Ex}(X,Y) \xrightarrow{\simeq} H^1(X,Y)_X$$

$$\mathrm{Aut}(Y) \xrightarrow{\simeq} H^0(X,Y)_X$$

Both the extensions and the cohomology are taken relative to the underlying object functor $(U,X)\colon (\underline{A}^{\mathbf{T}},X) \longrightarrow (\underline{A},X)$.

It is of some interest to make $\underline{\Theta}$ explicit in this context. An extension $p\colon E \longrightarrow X$ gives rise to a nonhomogeneous 1-cocycle

$$\begin{tikzcd} XT \arrow[rr, "a"] \arrow[dr, "\xi"'] & & Y \arrow[dl] \\ & X & \end{tikzcd}$$

which is determined by the formula $a \circ \xi s = sT \cdot \psi$, where $\psi\colon ET \longrightarrow E$ is the $\mathbf{T}$-structure of the extension. Note that the two maps

$$\begin{tikzcd} & X \arrow[dr, "s"] & \\ XT \arrow[ur, "\xi"] \arrow[dr, "sT"'] & & E \\ & ET \arrow[ur, "\psi"'] & \end{tikzcd} \qquad \text{in } (\underline{A}, X)$$

agree when followed by p, so one can be carried into the other by a map into Y ($Y \longrightarrow X$ is also an X-module in the underlying category). Conversely, identifying E with Y (as an object in $\underline{A}$) for simplicity, the $\mathbf{T}$-structure on E must be of the form

$$\psi = \theta + pT \cdot a\colon ET \longrightarrow E\,.$$

The sum of these maps exists because they are both compatible with the ever-present projections into X.

Incidentally, nothing requires the module $Y \longrightarrow X$ to be an abelian group object. We have assumed it above only because abelianness is usually present in examples, and there is no convenient terminology for the other notion.

A final comment concerning the application of these ideas: in practice one starts not with $\underline{A}$ and $\mathbf{T}$, but with category $\underline{B}$ for which one seeks tripleable underlying object functors $U: \underline{B} \longrightarrow \underline{A}$. Given one such, the cohomology $H^1(X, Y)_X$ relative to U classifies extensions which are split in the underlying category in which U takes values. For instance, associative K-algebras are tripleable over K-modules, the extensions come out K-linearly split, and the cohomology which classifies them turns out to be Hochschild's. If sets are taken as the underlying objects of K-algebras, the extensions only have to be split in the category of sets, i.e., they are not really split at all, and the cohomology which emerges to classify them is a theory, it turns out, developed by Shukla. But these are subjects that we will discuss in much more detail in the Examples.

4. Examples

We begin with an archetype of a large number of algebraic examples.

EXAMPLE 1. *Groups tripleable over sets.* We let $\mathscr{G}$ be the category of groups, $\underline{A}$ the category of sets, and $U: \mathscr{G} \longrightarrow \underline{A}$ the usual underlying set functor. We have adjointness $F \dashv U$, where F is the free group functor. If X is a set, we write the elements of XF as words spelled by means of formal group operations in generators (x), where $x \in X$. The empty word is denoted by $(\ )$. By adjointness we get a triple $\mathbf{T} = (T, \eta, \mu)$ in $\underline{A}$ where $XT = XFU$ is the underlying set of the free group generated by X. One further verifies (since these maps are determined by the adjointness) that $X\eta: X \longrightarrow XT$ is the map $x \Rightarrow (x)$, while $X\mu: XTT \longrightarrow XT$ is the map described as follows. The elements of XTT are words spelled with generators (w), where $w \in XT$. Then $X\mu$ maps $(w) \Rightarrow w$ and is extended to other elements by multiplicativity. For example, if $w_1 = (x_1)(x_2)$, $w_2 = (x_2)^{-1}$, then $W = (w_1)(w_2) \in XTT$ (it can also be written $((x_1)(x_2))((x_2)^{-1})$), and $W \cdot X\mu = w_1 w_2 = (x_1) \in XT$. Intuitively, this is the only map μ could be; in fact, this map is correct, because by construction of $\mathbf{T}$, $X\mu$ underlies the counit $XF\epsilon: XFUF \longrightarrow XF$, which is defined by multiplication in the free group.

We will now show that the underlying set functor is tripleable, by demonstrating that the category of $\mathbf{T}$-algebras $\underline{A}^{\mathbf{T}}$ is equivalent to the category of groups, $\mathscr{G}$. We do this by exhibiting a 1–1 correspondence between $\mathbf{T}$-algebra structures on a set X and group laws on X. Indeed, one could think of XT merely as a list of all the group operations which could conceivably be performed on elements of X. The $\mathbf{T}$-structure $\xi: XT \longrightarrow T$ is then a function which can be thought of as telling us what the values of these operations are. For example, given any two elements $x_0, x_1 \in X$, and given a function $\xi: XT \longrightarrow X$, the formal, juxtaposition, product $(x_0)(x_1)$ exists in XT, and the value of ξ on this element, $[(x_0)(x_1)]\xi$, naturally suggests itself as the definition of a binary operation $x_0 \cdot x_1$. In fact, just using any function $\xi: XT \longrightarrow X$, we can define a candidate for a group law on X:

$$\begin{aligned} x_0 \cdot x_1 &= [(x_0)(x_1)]\xi && \text{(multiplication)} \\ x^{-1} &= [(x)^{-1}]\xi && \text{(inversion)} \\ 1 &= [(\ )]\xi && \text{(neutral element)} \end{aligned}$$

The assumption that ξ is unitary and associative implies that these operations satisfy the group axioms. For example, here is the proof that the multiplication $x_0 \cdot x_1$ is associative. Let W_1, $W_2 \in XTT$ be the following words:

$$W_1 = ((x_0)(x_1))((x_2)),\ W_2 = ((x_0))((x_1)(x_2)).$$

Pursuing them around the associativity diagram

$$\begin{tikzcd} XTT \arrow[r, "X\mu"] \arrow[d, "\xi T"'] & XT \arrow[d, "\xi"] \\ XT \arrow[r, "\xi"'] & X \end{tikzcd}$$

we get

$$W_1 \cdot \xi T \cdot \xi = [(x_0 \cdot x_1)(x_2)]\xi = (x_0 \cdot x_1) \cdot x_2$$
$$W_2 \cdot \xi T \cdot \xi = [(x_0)(x_1 \cdot x_2)]\xi = x_0 \cdot (x_1 \cdot x_2)$$

whereas $W_1 \cdot X\mu \cdot \xi = W_2 \cdot X\mu \cdot \xi = [(x_0)(x_1)(x_2)]$, q.e.d. Note that $[(x)]\xi = x$ because of the unitary axiom

$$\begin{tikzcd} X \arrow[r, "X\eta"] \arrow[dr, "X"'] & XT \arrow[d, "\xi"] \\ & X \end{tikzcd}$$

The other group axioms can be verified in a similar manner.

Conversely, it is quite clear that a given group law on X defines a map $\xi\colon XT \longrightarrow X$ by just performing the indicated group operations using the group law, and this map will be unitary and associative. More precisely, the canonical functor

$$\begin{tikzcd} \underline{A}^{\mathbf{T}} \arrow[dr, "U^{\mathbf{T}}"'] & & \mathscr{G} \arrow[ll, "\Phi"'] \arrow[dl, "U"] \\ & \underline{A} & \end{tikzcd}$$

maps a group π into its underlying set πU with the $\mathbf{T}$ -structure $\pi\epsilon U\colon \pi UT \longrightarrow \pi U$ which employs the given operations in π to evaluate in πU all the proposed group operations formally present in πUT. The foregoing procedure of constructing a group out of a $\mathbf{T}$-algebra defines a functor $\Phi^{-1}\colon \underline{A}^{\mathbf{T}} \longrightarrow \mathscr{G}$ with the properties $\Phi\Phi^{-1} = \mathscr{G}$, $\Phi^{-1}\Phi = \underline{A}^{\mathbf{T}}$. (One gets an actual isomorphism of categories.) The functor Φ, leaving underlying sets unchanged, simply interchanges two equivalent formulations of the notion of a group structure on a set.)

The example of groups is typical. It is known that all *algebraic categories* in the sense of [Lawvere (1963)] are tripleable over sets, with respect to their usual underlying set functors. [Linton (1966)] has shown that over sets this is almost the whole story: admitting infinitary operations one gets *equational categories* of algebras, and over the base category of sets tripleableness is equivalent to equationality.

Over other base categories, tripleableness does not seem to have any such standard interpretations. It is the proposal of this paper that tripleableness be regarded as a new type of mathematical structure, parallel to but not necessarily definable in terms of other known types of structure, such as algebraic, equational, topological, ordered, ...

Inasmuch as it has been insinuated all along that tripleableness is a restriction—

EXAMPLE 2. *A non-tripleable adjoint pair.* Let $\underline{\mathrm{Top}}$ be the category of spaces, $\underline{A}$ the category of sets, and $U\colon \underline{\mathrm{Top}} \longrightarrow \underline{A}$ the usual forgetful or underlying set functor. Left adjoint to U is the discrete space functor, $XF =$ the topological space of underlying set X with the discrete topology. The composition FU is just $\underline{A}$. Indeed, the triple $\mathbf{T}$ in $\underline{A}$ induced by the adjoint pair $F \dashv U$ is just the *identity triple* consisting of the identity functor with its identity natural transformation as unit and multiplication. The corresponding category of algebras is nothing but $\underline{A}$. (An algebra structure $\xi\colon X \longrightarrow X$ has to be the identity by the unitary axiom.) In the canonical diagram

$$\begin{tikzcd} \underline{A} \arrow[dr, "\underline{A}"'] & & \underline{\mathrm{Top}} \arrow[ll, "\Phi"'] \arrow[dl, "U"] \\ & \underline{A} & \end{tikzcd}$$

we have $\Phi = U$, not an equivalence.

As we have mentioned, tripleableness implies some sort of algebraicity, and tends to exclude topological structures. However, replacing $\underline{\mathrm{Top}}$ by the category of compact[4] spaces, with the usual underlying set functor (the left adjoint is the Stone-Čech compactification of the discrete space), the resulting adjoint pair *is* tripleable [Linton (1966)].

We shall now give some examples involving groups and monoids, and corresponding cohomology theories which come from our general theory.

EXAMPLE 3. *The category of π-sets.* Let π be a monoid (group, in the original Eilenberg-Mac Lane presentation [Eilenberg & Mac Lane (1947)]). Using π we get a *triple* in the category of sets, $\underline{A}$, by

$$A \xrightarrow{(a,1)} A \times \pi,\ \ A \times \pi \times \pi \xrightarrow{(a, x_1 x_2)} A \times \pi.$$

This triple is also denoted by π. The category of π-*algebras*, $\underline{A}^\pi$, consists of sets equipped with unitary, associative π-structures $A \times \pi \longrightarrow A$, that is, *right π-sets.* A *group object* in $\underline{A}^\pi$ consists of a group object in $\underline{A}$, i.e., an ordinary group, G, with a π-structure $G \times \pi \longrightarrow G$ which is compatible with the multiplication in $G\colon (g_1 g_2)x = (g_1 x)(g_2 x)$. Such an object is called a *right π-group.* An *abelian group object* in $\underline{A}^\pi$ is then just a *right-π-module*

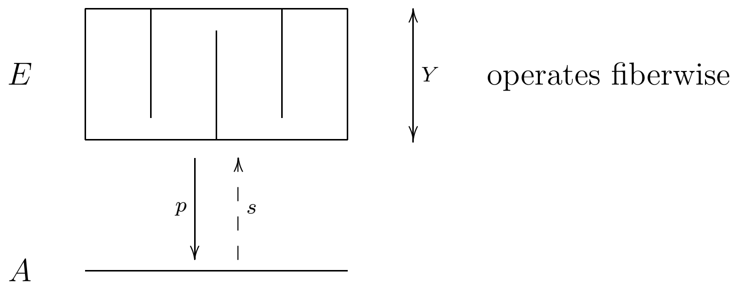
[4]Editors' note: At the time the thesis was written, compact meant compact Hausdorff, under the influence of Bourbaki. Both earlier and later, this was not the standard usage.

[Cartan & Eilenberg (1956), Mac Lane (1963), . . .]. The cohomology groups $H^n(A, Y)$ are defined for a right π-set A and a right π-module Y. They can be calculated from the *nonhomogeneous complex* (§2) which in this case becomes: $C^n(A, Y) =$ all functions $f\colon A \times \pi^n \longrightarrow Y$,

$$\begin{aligned}(a, x_1, \dots, x_{n+1})(fd) = {} & (ax_1, x_2, \dots, x_{n+1})f \\ & + \sum_{i=1}^{n} (-1)^i (a, x_1, \dots, x_i x_{i+1}, \dots, x_{n+1})f \\ & + (-1)^{n+1}(a, x_1, \dots, x_n)f x_{n+1}\end{aligned}$$

being the coboundary of an n-cochain $f \in C^n(A, Y)$. This is identical with the Eilenberg-Mac Lane complex [Eilenberg & Mac Lane (1947)], except that usually one takes $A = 1$, the trivial right π-set (in which case terms of the form ax drop out). The groups $H^n(1, Y)$, relative to the triple π are usually written $H^n(\pi, Y)$ and called the (*Eilenberg-Mac Lane*) *cohomology groups* of π.

As we know, $H^0(A, Y)$ is isomorphic to the group of right π-maps $A \longrightarrow Y$ (i.e., equivariant maps). When $A = 1$, this reduces to the *invariant elements* of Y (those $y \in Y$ such that $yx = y$ for all $x \in \pi$). The interpretation of $H^1(A, Y)$ is well known when $A = 1$. Then H^1 classifies principal homogeneous π-sets for the π-module Y (see [Serre (1965), I-56 ff.]). When $A \neq 1$, a representative of an element of $H^1(A, Y)$ can be pictured as



E, p, A, Y are all right π-maps, and the operation of Y is consistent with π-structures. The fibers are all non-canonically isomorphic with Y (via the section s which is not preserved by maps), and A is the quotient set of the Y-operation.

H^1 can also be interpreted in terms of derivations. A 1-cocycle in the above complex is a map $f\colon A \times \pi \longrightarrow Y$ such that $(a, x_1x_2)f = (ax_1, x_2)f + (a, x_1)fx_2$. If $A = 1$, f can be regarded as a function $\pi \longrightarrow Y$ and is a derivation in the usual sense. The 1-coboundaries are precisely the inner derivations.

EXAMPLE 4. *Cohomology in the category of groups.* We will consider the cohomology groups $H^n(\pi, Y)$ where π is a group and Y is an abelian group object in the category of groups, $\mathscr{G}$. As is well known [Brinkman & Puppe (1965), Eckmann & Hilton (1962), for example], this just means that Y is an abelian group. In this setting π does not operate on Y (or operates trivially). For the cohomology of π with coefficients in a π-module Y, we refer to Example 5; we want to study this example first, in order to filter the complications. Of course, "cohomology" in $\mathscr{G}$ is not defined except with reference to an underlying object functor. For this we choose the usual underlying set functor $U: \mathscr{G} \longrightarrow \underline{A}$, which we know is tripleable, by Example 1.

$$\underline{A} \underset{U}{\overset{F}{\rightleftarrows}} \mathscr{G} \qquad (\underline{A} = \text{sets})$$

The standard cotriple **G** has $G = UF$ as its functor, and the natural epimorphism of the free group $\pi\epsilon: \pi G \longrightarrow \pi$ as its counit ($\pi \in \mathscr{G}$). $H^n(\pi, Y)$ is the n^{th} cohomology group of either of the following two complexes. The *homogeneous complex* (§2) is

$$0 \longrightarrow (\pi G, Y) \xrightarrow{d} (\pi G^2, Y) \xrightarrow{d} \cdots \xrightarrow{d} (\pi G^{n+1}, Y) \xrightarrow{d} \cdots$$

where $d = \sum_{i=0}^{n}(-1)^i(\pi G^i \epsilon G^{n-i}, Y)$ and the hom is in the category of groups. The *nonhomogeneous complex* is

$$0 \longrightarrow (\pi, Y) \longrightarrow (\pi T, Y) \longrightarrow \cdots \longrightarrow (\pi T^n, Y) \longrightarrow \cdots$$

where we have written π, Y for the underlying sets, $T: \underline{A} \longrightarrow \underline{A}$ is the underlying triple (Example 1), and the hom is in the category of sets. The coboundary formula can be found in §2.

Since U is tripleable, $H^0(\pi, Y)$ is the abelian group of homomorphisms $\pi \longrightarrow Y$.

$H^1(\pi, Y)$ classifies Y-principal groups over π, $p: E \longrightarrow \pi$, as follows. The group structures of π and Y will be written as $\xi: \pi T \longrightarrow \pi$ and $\theta: YT \longrightarrow Y$. The group structure of the trivial principal group $\pi \times Y$ will be componentwise (§3)[5]:

$$(\pi \times Y)T \xrightarrow{\text{proj.}} \pi T \times YT \xrightarrow{\xi \times \theta} \pi \times Y.$$

For example, if we take the word $(x_1, y_1)(x_2, y_2) \in (\pi \times Y)T$, its image in $\pi \times Y$ will give the binary operation of multiplication:

$$(x_1, y_1)(x_2, y_2) \Rightarrow ((x_1)(x_2), (y_1)(y_2)) \Rightarrow (x_1x_2, y_1y_2).$$

If $E \longrightarrow \pi$ is any Y-principal group over π, we will have $E \simeq \pi \times Y$ as a set but its group structure will be of the form $\xi \times (\theta + a)$, in the additive form of the notation in §3, where $a: \pi T \longrightarrow Y$ is a 1-cocycle:

$$(\pi \times Y)T \longrightarrow \pi T \times YT \xrightarrow{\xi \times (\theta + a)} \pi \times Y$$

$$(w_0, w_1) \Rightarrow (w_0\xi, w_1\theta + w_0 a)$$

[5]Editors' note: The second arrow label was simply θ in the original. We have changed it because it seems correct to do so and also because the spacing in the original suggests that something was to have been added by hand

Thus $\xi \times (\theta + a)$ instructs us to multiply in E (identifying E with $\pi \times Y$ as a set) according to the rule

$$(x_1, y_1)(x_2, y_2) = (x_1 x_2, y_1 + y_2 + [(x_1)(x_2)]a)$$

where $(x_1)(x_2)$ is the formal product in πT. Evidently, Y-principal groups over π are just group extensions of π by Y in the ordinary sense (with π operating trivially on Y). We have an isomorphism

$$H^1(\pi, Y) \xrightarrow{\simeq} \mathrm{EM}^2(\pi, Y)$$

because of this classification of extensions (EM is the Eilenberg-Mac Lane theory). Using the method of [Barr & Beck (1966)], one proves

$$H^n(\pi, Y) \xrightarrow{\simeq} \mathrm{EM}^{n+1}(\pi, Y)$$

for $n \geq 0$ ($n = 0$ was covered above).

Some further remarks about principal groups: note that the cocycle $a\colon \pi T \longrightarrow Y$ "twists" the whole group structure on $\pi \times Y$, not just the multiplication. For example, the new neutral element is $(1, 0 + [(\ )]a)$. On the other hand, the cocycle a will always satisfy $[(x)]a = 0$. (This is the relation $X\eta \cdot a = 1$, which appeared in the proof of Theorem 5.) The identity operation never is twisted.

$\mathscr{G}$ is also tripleable over the category $(1, \underline{A})$, the comma category, better known as the category of pointed sets. The free functor $(X, x_0)F$ is the free group on the set X modulo the single relation $(x_0) = 1$. The whole example can be carried through similarly in this setting. One finds that the cocycle satisfies $[(x_0)]a = 0$. Thus the underlying pointed structure of the principal group over π, say, is untwisted. (The pointed structure is the neutral element.) In any principal object, in this theory, the underlying category structure remains that of a product. Only the added, $\mathbf{T}$-, structure is twisted.

EXAMPLE 5. *Cohomology of a group with coefficients in a module.* A *π-module* means an abelian group object in the comma category $(\mathscr{G}, \pi)$. We first show that this general categorical definition reduces to the ordinary notion of right π-module (or left, with a different choice of product representation below). We shall prove:

$$\pi - \underline{\mathrm{Mod}} = \mathrm{Ab}(\mathscr{G}, \pi) \xrightarrow{\ker} \underline{\mathrm{Right}}\ \pi - \underline{\mathrm{modules}}$$

is an equivalence of categories.

This functor maps a π-module $Y \longrightarrow \pi$ into $M = \ker(Y \longrightarrow \pi)$ with right π-operators defined by conjugation. We shall give the details. Since $Y \longrightarrow \pi$ is an abelian group object it must have a zero section (§3)

$$\begin{tikzcd} Y \arrow[d, shift right, "p"'] \\ \pi \arrow[u, shift right, "s"'] \end{tikzcd} \qquad (sp = \pi)$$

which is a map in the category of groups. Thus $Y \xrightarrow{\simeq} \pi \times M$ as a set. Explicitly, $y\sigma = (yp, y(yps)^{-1})$ and $(x, m)\tau = m(xs)$ define inverse isomorphisms $Y \longrightarrow \pi \times M \longrightarrow Y$.

Viewing σ as an identification for simplicity, we now express the group multiplication in Y in terms of the product representation. Knowing that s, p, and the injection $M \longrightarrow Y$ are group maps, we have

$$\begin{aligned}(x_1, 1)(x_2, 1) &= (x_1x_2, 1)\\ (1, m_1)(1, m_2) &= (1, m_1m_2)\\ (x_1, m_1)(x_2, m_2) &= (x_1x_2, \ldots)\end{aligned}$$

and from the formula for τ,

$$(1, m)(x, 1) = (x, m) \ .$$

Conjugation induces right π-operators in M, since the kernel is an invariant subgroup:

$$(x, 1)(1, m)(x, 1)^{-1} = (1, mx) \ .$$

Thus M is a right π-group (Example 3). The full multiplication table for Y now emerges:

$$\begin{aligned}(x_1, m_1)(x_2, m_2) &= (1, m_1)(x_1, 1)(1, m_2)(x_2, 1)\\ &= (1, m_1)(1, m_2x_1)(x_1, 1)(x_2, 1)\\ &= (1, m_1(m_2x_1))(x_1x_2, 1)\\ &= (x_1x_2, m_1(m_2x_1)) \cdot \end{aligned} \tag{1}$$

Thus Y is isomorphic to the crossed product of π by M.

Up to this point we have only used the zero element of the module. Now we introduce the addition which, being a binary operation, is represented by a map

$$\begin{tikzcd} Y \times_\pi Y \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & \pi & \end{tikzcd}$$

Replacing Y by $\pi \times M$, we have that $(\pi \times M) \times_\pi (\pi \times M)$ is universal for pairs of morphisms $(x_1, m_1), (x_2, m_2)$ such that $x_1 = x_2$. Thus $Y \times_\pi Y$ is isomorphic to $\pi \times M \times M$ as a set, and the above diagram becomes

$$\begin{tikzcd} \pi \times M \times M \arrow[rr] \arrow[dr] & & \pi \times M \arrow[dl] \\ & \pi & \end{tikzcd}$$

The induced map of kernels $M \times M \longrightarrow M$ is a group law on M *as a right π-group*. Thus

$$\begin{aligned}M \in \mathrm{Gp}(\underline{\mathrm{Right}}\ \pi - \underline{\mathrm{groups}}) &= \mathrm{Gp}\ \mathrm{Gp}\ \underline{A}^\pi\\ &= \mathrm{Ab}\ \underline{A}^\pi\\ &= \underline{\mathrm{Right}}\ \pi - \underline{\mathrm{modules}} \ ;\end{aligned}$$

moreover, the map of kernels $M \times M \longrightarrow M$ must coincide with the group law in M (as a subgroup of Y) and be abelian. (These facts follow from [Brinkman & Puppe (1965)] and [Eckmann & Hilton (1962)]. Notice that in this case there are no properly non-abelian π-"modules".) This proves that the kernel functor takes its values in the category of right π -modules, as required.

The multiplication in Y can now be written as

$$(x_1, m_1)(x_2, m_2) = (x_1x_2, m_1 + m_2x_1) \ , \tag{2}$$

the usual formula for multiplication in the split extension. Of course, for the inverse of the kernel functor, take any right π-module and construct a π-module in our sense by formula (2). Thus we have established the desired equivalence.

For the rest of this example we use the notation $Y \longrightarrow \pi$ for a π-module, M for the kernel, and we identify Y with $\pi \times M$ with multiplication (2) when convenient.

The cohomology theory which arises is written $H^n(Z, Y)_\pi$ (§3), and is defined for $Z \longrightarrow \pi$ any group over π (object in the comma category $(\mathscr{G}, \pi)$ and $Y \longrightarrow \pi$ a π-module. It is the n-th cohomology group of either of the complexes

$$0 \longrightarrow (ZG, Y)_\pi \longrightarrow (ZG^2, Y)_\pi \longrightarrow \cdots \longrightarrow (ZG^{n+1}, Y)_\pi \longrightarrow \cdots$$

$$0 \longrightarrow (Z, Y)_\pi \longrightarrow (ZT, Y)_\pi \longrightarrow \cdots \longrightarrow (ZT^n, Y)_\pi \longrightarrow \cdots$$

The cochains are maps in $(\mathscr{G}, \pi)$ or $(\underline{A}, \pi)$ respectively ($\underline{A} = \text{Sets}$), T is really an abbreviation for (T, π), the functor part of the triple induced on the comma category (§3), and both of the displayed general terms are in dimension n.

$H^0(Z, Y)_\pi$ is the abelian group of maps $Z \longrightarrow Y$ in $(\mathscr{G}, \pi)$. By formula (2) such a map is the same thing as a function $f\colon Z \longrightarrow M$ satisfying

$$(z_1z_2)f = z_1f + (z_2f)z_1 \ .$$

Thus f is a *derivation* of $Z \longrightarrow M$, where M is treated as a right Z-module via the fixed map $Z \longrightarrow \pi$. We have

$$H^0(Z, Y)_\pi \simeq \operatorname{Der}(Z, M) \ .$$

This isomorphism depends on the fact that the underlying set-over-π functor $(\mathscr{G}, \pi) \longrightarrow (\underline{A}, \pi)$ is tripleable; this is a consequence of tripleableness of $\mathscr{G} \longrightarrow \underline{A}$, as remarked in §3.

The cohomology group $H^0(\pi, Y)_\pi$ classifies *extensions* $E \longrightarrow \pi$ *of* π *by the* π-module $Y \longrightarrow \pi$. Using the given set section $s\colon \pi \longrightarrow E$ we get an isomorphism $E \simeq \pi \times M$ in which s corresponds to the map $x \rightrightarrows (x, 0)$. The group structure of E will be determined by a 1-cocycle $a\colon \pi T \longrightarrow Y$ in $(\underline{A}, \pi)$, and it is sufficient to know the M-component of a. We regard the cocycle as a map $a\colon \pi T \longrightarrow M$. The addition in the π-module being carried out in the M-component, we find that the multiplication in $E \simeq \pi \times M$ is:

$$(x_1, m_1)(x_2, m_2) = (x_1x_2, m_1 + m_2x_1 + [(x_1)(x_2)]a)$$

(by the remarks after Theorem 6, $\psi = \theta + pT \cdot a$). These are the usual extensions of the group π by the π-module M. In general, maps of extensions will not commute with the product representations, which depend on the sections. A map of two extensions will be of the form $(x, m) \Longrightarrow (x, m + xb)$. Then $b\colon \pi \longrightarrow M$ is the 0-cochain whose coboundary puts the two extensions into the same 1-cohomology class. One conjectures as a result (taking dimension 0 considered above into account):

$$H^n(\pi, Y)_\pi \xrightarrow{\simeq} \begin{cases} \mathrm{Der}(\pi, M), & n = 0 \\ \mathrm{EM}^{n+1}(\pi, M), & n > 0 \end{cases}$$

(EM being the Eilenberg-Mac Lane theory). This is proved by an acyclic models method in [Barr & Beck (1966)].

We shall now give a series of examples involving linear algebras.

EXAMPLE 6. *Associative K-algebras with identity.* K being a commutative ring, we let $\mathscr{A}$ denote the category described. $\mathscr{A}$ has few group objects. Indeed, if $Y \in \mathrm{Ab}\mathscr{A}$, then the zero operation in Y must be represented by a map of the terminal object $0 \longrightarrow Y$. Since this map must preserve identity elements, $1 = 0$ in Y; therefore $Y = 0$. To get abelian group objects, we must consider categories of modules. Let Λ be a K-algebra. Then

$$\Lambda\text{-}\underline{\mathrm{Mod}} = \mathrm{Ab}(\mathscr{A}, \Lambda) \xrightarrow{\ker} \Lambda\text{-}\Lambda\text{-}\underline{\mathrm{Bimodules}}$$

is an equivalence of categories.

Let $Y \longrightarrow \Lambda$ be a Λ-module. There is a zero section $\Lambda \longrightarrow Y$ in $\mathscr{A}$, so we can write $Y \simeq \Lambda \oplus M$ as a K-module, and its multiplication will be of the form

$$\begin{aligned}(\lambda_1, m_1)(\lambda_2, m_2) &= [(\lambda_1, 0) + (0, m_1)][(\lambda_2, 0) + (0, m_2)] \\ &= (\lambda_1, 0)(\lambda_2, 0) + (\lambda_1, 0)(0, m_2) + (0, m_1)(\lambda_2, 0) + (0, m_1)(0, m_2) \\ &= (\lambda_1\lambda_2, \lambda_1 m_2 + m_1\lambda_2 + m_1 m_2)\ .\end{aligned}$$

The first component must be $\lambda_1\lambda_2$ since the projection is an algebra map, $\lambda_1 m_2$ is *defined* as $(\lambda_1, 0)(0, m_2)$ and is an element of the kernel, $m_1\lambda_2$ similarly, and $m_1 m_2$ appears because the kernel is multiplicatively closed. The addition map $Y \times_\Lambda Y \longrightarrow Y$ in $(\mathscr{A}, \Lambda)$ will be given by addition in the kernel, *i.e.*, it is equivalent to $\Lambda \oplus M \oplus M \longrightarrow \Lambda \oplus M$. This addition must be an algebra map. We leave to the reader the easy task of showing that this imposes the condition $m_1 m_2 = 0$. Thus, we obtain that Y is equivalent to the split extension of Λ by the two-sided Λ-module M, with multiplication

$$(\lambda_1, m_1)(\lambda_2, m_2) = (\lambda_1\lambda_2, \lambda_1 m_2 + m_1\lambda_2)$$

Some cohomology theories in $\mathscr{A}$ come from the following underlying object functors:

$$\begin{tikzcd}
 & \mathscr{A} \arrow[r, "U_5"] \arrow[dl, "U_4"'] \arrow[ddl, "U_3"] \arrow[dd, "U_0"] \arrow[ddr, "U_1"'] \arrow[dr, "U_2"] & \underline{A}_5 \\
\underline{A}_4 \arrow[d] & & \underline{A}_2 \arrow[d] \\
\underline{A}_3 \arrow[r] & \underline{A}_0 & \underline{A}_1 \arrow[l]
\end{tikzcd}$$

all of which are tripleable. $\underline{A}_2$ is K-modules, and the coadjoint $F_2 \dashv U_2$ is the tensor algebra functor $AF_2 = K + A + A \otimes_K A + \cdots$ with juxtaposition as multiplication. A $\mathbf{T}_2$ structure $\xi \colon AT_2 \longrightarrow A$ is equivalent to an associative-algebra-with-1 structure on the K-module A. $\underline{A}_1$ is the category of abelian groups. $F_1 \dashv U_1$ is given by $AF_1 = (A \otimes_{\mathbf{Z}} K)F_2$, coadjoint of a composition being the composition of the coadjoints. $\underline{A}_0$ is the category of sets. AF_0 is the polynomial K-algebra with the elements of the set A as noncommuting variables. $\underline{A}_4$ is the category of rings. U_4 forgets K-structure, F_4 puts it back in; $AF_4 = A \otimes_{\mathbf{Z}} K$. $\underline{A}_3$ is the category of monoids. U_3 remembers only the multiplication and the multiplicative identity. AF_3 is the K-monoid algebra (free K-module with basis A and the obvious multiplication.) Note that a $\mathbf{T}_3$-structure is a *monoid* map $AT_3 \longrightarrow A$, so that a $\mathbf{T}_3$-algebra keeps its original multiplicative structure and receives the new operations of addition and K-scalar multiplication; similar comments apply for all the other cases.

These triples give cohomology theories, of which we shall only consider the groups $H_i^n(\Lambda, Y)_\Lambda$ for $i = 0, \ldots, 5$. These are the cohomology groups of, for instance, the homogeneous complexes

$$0 \longrightarrow (\Lambda G_i, Y)_\Lambda) \longrightarrow (\Lambda (G_i)^2, Y)_\Lambda) \longrightarrow \cdots$$

where the cotriples $G_i = U_i F_i$ are being used to build up free resolutions of varying depths of freeness, as it were.

The theories arising when $i = 2$ or 0 are known. Relative to U_2 the extensions classified are K-linearly split, and their K-algebra structures are only twisted with regard to multiplication. Thus an extension of Λ by $Y \longrightarrow \Lambda$ relative to U_2 is isomorphic to $\Lambda \oplus M$ as a K-module and has a multiplication given by (3) plus a bilinear function of two variables with values in M:

$$(\lambda_1, m_1)(\lambda_2, m_2) = (\lambda_1\lambda_2, \lambda_1 m_2 + m_1 \lambda_2 + (\lambda_1, \lambda_2)a) \ .$$

Thus the extensions and, one supposes, the whole cohomology theory agree with that defined by [Hochschild (1945)]. In fact, an isomorphism

$$H_2^n(\Lambda, Y)_\Lambda \xrightarrow{\ \simeq\ } \begin{cases} \operatorname{Der}(\Lambda, M), & n = 0 \\ \operatorname{Hoch}^{n+1}(\Lambda, M), & n > 0 \end{cases}$$

is obtained in [Barr & Beck (1966)], where $M = \ker(Y \longrightarrow \Lambda)$. In the U_0-cohomology theory, the extensions are isomorphic to $\Lambda \times M$ only as sets. All three structures - addition,

K-scalar multiplication, and multiplication - are twisted. These extensions have been classified by a cohomology theory devised by Shukla [Shukla (1961)]. Barr has shown [Barr (1967)] that the H_0^n groups are isomorphic to the Shukla groups, with the same degree 0 value and shift in dimension as above. Note that whatever underlying category we descend to, if $\mathscr{A}$ is tripleable over it, H^0 will always be the hom functor in $(\mathscr{A}, \Lambda)$, which is derivations, by formula (3).

The theories H_1^n,H_3^n,H_4^n have not been studied. In dimension 1 they classify extensions which are additively, multiplicatively, and both additively and multiplicatively split, respectively.

We have refrained from speaking until now about the cohomology theory given by the underlying object functor U_5, which is a little bizarre. We take $\underline{A}_5$ to be the category of K-Lie algebras. As a K- module ΛU_5 is the same as Λ, and has the Lie algebra operation $[\lambda_1, \lambda_2] = \lambda_1\lambda_2 - \lambda_2\lambda_1$. In [Lawvere (1963)] it is proved that such "algebraic functors" always have coadjoints. In the triple context one proves that they are in fact tripleable. Thus we get a cohomology theory H_5^n in $\mathscr{A}$ whose 1-dimensional part classifies algebra extensions which are split with respect to their Lie algebra structures. This theory is nonzero, because when applied to a commutative K-algebra, and a commutative module (see below), it gives the ordinary K-split commutative theory.

We conclude Example 6 with a few remarks about exactness in the category of Λ-modules and the choice of the underlying object functor $U\colon \mathscr{A} \longrightarrow \underline{A}$, or better, $(U, \Lambda)\colon (\mathscr{A}, \Lambda) \longrightarrow (\underline{A}, \Lambda)$. Recall that a sequence of Λ-modules

(3)
$$\begin{tikzcd} 0 \arrow[r] & Y' \arrow[r] \arrow[dr] & Y \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[dl] & 0 \\ & & \Lambda & & \end{tikzcd}$$

is (U, Λ)-exact in Λ-Mod $= \mathrm{Ab}(\mathscr{A}, \Lambda)$ if

$$0 \longrightarrow (A, Y')_\Lambda \longrightarrow (A, Y)_\Lambda \longrightarrow (A, Y'')_\Lambda \longrightarrow 0$$

is an exact sequence of abelian groups for every object $A \longrightarrow \Lambda$ in the underlying category $(\underline{A}, \Lambda)$. ($(\ ,\ )_\Lambda$ denotes the hom functor in $(\underline{A}, \Lambda)$, and strictly speaking we should introduce YU's into the above sequence.) It is in this situation that a long exact sequence arises in the U-cohomology (see §2, Theorem 2). For convenience identify Y with $\Lambda \oplus M$ where M is the kernel of $Y \longrightarrow \Lambda$ regarded as a Λ-Λ-bimodule, and the same for Y', Y'' (see the computation of Λ-Mod at the beginning of this example). Then (3) gives rise to the sequence of Λ-Λ-bimodules

(4)
$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

Let us now take $U = U_i\colon \mathscr{A} \longrightarrow \underline{A}_i$ as above, and interpret some of the resulting (U_i, Λ)-exactness. We shall find that (3) is (U_i, Λ)-exact $\iff$ (4) is an ordinary exact sequence of Λ-Λ-bimodules and has an additional splitting property relative to the underlying category $\underline{A}_i$.

First consider $U_2\colon \mathscr{A} \longrightarrow \underline{A}_2$, the category of K-modules. Letting $A \longrightarrow \Lambda$ be an object of $(\underline{A}_2, \Lambda)$ with zero projection, and choosing A variously, one sees that (4) must be exact in the usual sense; taking $A = Y'' \longrightarrow \Lambda$ one finds a map $Y \longleftarrow Y''$ that splits $Y \longrightarrow Y''$ in $(\underline{A}_2, \Lambda)$. Thus (3) is (U_2, Λ)-exact $\iff$ (4) is a K-split exact sequence of Λ-Λ-bimodules. This is the type of coefficient sequence usually considered in Hochschild cohomology, (cf. [Mac Lane (1963), p. 287]).

Similarly, (3) is (U_1, Λ)-exact ($\underline{A}_1$ being the category of abelian groups) $\iff$ (4) is a $\mathbf{Z}$-split exact sequence of Λ-Λ-bimodules, and (3) is (U_0, Λ)-exact ($\underline{A}_0$ = sets) $\iff$ (4) is exact in the ordinary sense; here no additional splitting condition enters, except set-theoretically, in showing that $M \longrightarrow M''$ is onto. (U_4, Λ)-exactness ($\underline{A}_4$ = rings) is equivalent to $M \cong M' \oplus M''$ as bimodules. We leave the formulation of the curious (U_3, Λ)- and (U_5, Λ)-exactness to the reader.

The same study of exactness can be carried out in all the other examples, but we omit it.

EXAMPLE 7. *Lie Algebras.* We include this divertissement as an elementary demonstration of the fact that our theory does not discriminate against non-associative systems. Let $\mathscr{L}$ be the category of K-Lie algebras, $\underline{A}$ the category of K-modules, K a commutative ring, and let $U\colon \mathscr{L} \longrightarrow \underline{A}$ be the usual underlying. The free Lie algebra functor $F \dashv U$ is described in [Cartan & Eilenberg (1956), p. 285]. This generates a triple $\mathbf{T}$ in $\underline{A}$ as usual, with $T = FU$. If $X \in \underline{A}$, we write the map $X \longrightarrow XT$ as $x \Rightarrow (x)$, so that the symbol (x) obeys $(x_0) + (x_1) = (x_0 + x_1)$, $k(x) = (kx)$. Other elements of XT are written in terms of Lie algebra operations applied to the generators (x). For example, $[(x_0), (x_1)]$ is in XT, while $([(x_0), (x_1)])$ and

$$\begin{aligned} W_0 &= [([(x_0), (x_1)]), ((x_2))], \\ W_1 &= [([(x_0), (x_2)]), ((x_1))] + [((x_0)), ([(x_1), (x_2)])] \end{aligned}$$

are in XTT. Under the triple multiplication $XTT \longrightarrow XT$,

$$\begin{aligned} W_0 &\Rightarrow [[(x_0), (x_1)], (x_2)], \\ W_1 &\Rightarrow [[(x_0), (x_2)], (x_1)] + [(x_0), [(x_1), (x_2)]], \end{aligned}$$

and these are equal as elements of the free Lie algebra XT.

Now let (X, ξ) be a $\mathbf{T}$-algebra. X is a K-module, and we introduce a possible Lie bracket in X by defining

$$[x_0, x_1] = [(x_0), (x_1)]\xi$$

(cf. Example 1 on groups). Clearly, under $\xi T\colon XTT \longrightarrow XT$,

$$\begin{aligned} W_0 &\Rightarrow [([x_0, x_1]), (x_2)], \\ W_1 &\Rightarrow [([x_0, x_2]), (x_1)] + [(x_0), ([x_1, x_2])]. \end{aligned}$$

Thus we have

$$[[x_0, x_1], x_2] = W_0 \cdot \xi T \cdot \xi$$

$$
\begin{aligned}
&= W_0 \cdot X\mu \cdot \xi \\
&= W_1 \cdot X\mu \cdot \xi \\
&= W_1 \cdot \xi T \cdot \xi \\
&= [[x_0, x_2], x_1] + [x_0, [x_1, x_2]].
\end{aligned}
$$

Since ξ is associative, [,] satisfies Jacobi's identity (!).

Of course the canonical $\Phi\colon \mathscr{L} \xrightarrow{\simeq} \underline{A}^{\mathsf{T}}$. Modules are (antisymmetric) as usual. H^1 classifies K-split extensions [Mac Lane (1963)]. $\mathscr{L}$ is also tripleable over sets with extensions as in [Dixmier (1957)].

Other types of linear algebras are also tripleable and have triple cohomology, for example, the algebras in [Eilenberg (1948)], Jordan algebras [McCrimmon (1966)], Lie triple systems [Harris (1961)] (obviously there is no restriction to binary systems), ...

I am indebted to Michael Barr for showing me the following pathological case.

EXAMPLE 8. *Commutative Algebras.* Let $\mathscr{C}$ be the category of commutative K-algebras (associative, with identity), $\underline{A}$ the category of K-modules, $U\colon \mathscr{C} \longrightarrow \underline{A}$ the usual underlying. The free commutative algebra functor $F \dashv U$ is given by the symmetric algebra construction (symmetrized tensor algebra)

$$XF = K \oplus X \oplus \frac{X \otimes X}{S(2)} \oplus \frac{X \otimes X \otimes X}{S(3)} \cdots .$$

We certainly have $\mathscr{C} \xrightarrow{\simeq} \underline{A}^{\mathsf{T}}$, where $T = FU$.

If Λ is a commutative algebra, then

$$\Lambda\text{-}\underline{\mathrm{Mod}} = \mathrm{Ab}(\mathscr{C}, \Lambda) \xrightarrow{\ker} \underline{\mathrm{Right}}\ \Lambda\text{-}\underline{\mathrm{Modules}}$$

([Cartan & Eilenberg (1956), Mac Lane (1963)]) *is an equivalence of categories.* As in Example 6, a Λ-module $Y \longrightarrow \Lambda$ must be of the form $\Lambda \oplus M \longrightarrow \Lambda$ as a K-module, where M is the kernel of $Y \longrightarrow \Lambda$, and have multiplication $(\lambda_1, m_1)(\lambda_2, m_2) = (\lambda_1\lambda_2, \lambda_1 m_2 + m_1\lambda_2)$. However since Y is commutative, the bimodule M must be *symmetric.* We view M indifferently as a right module, left module, or symmetric bimodule, over Λ.

$H^1(\Lambda, Y)_\Lambda$ classifies K-split commutative algebra extensions of Λ by Y. Such an extension must have the form $\Lambda \oplus M \longrightarrow \Lambda$ as a K-module, with multiplication

$$(\lambda_1, m_1)(\lambda_2, m_2) = (\lambda_1\lambda_2, \lambda_1 m_2 + m_1\lambda_2 + (\lambda_1 \otimes \lambda_2)f)$$

where $\Lambda \otimes \Lambda \xrightarrow{f} M$ is a *factor set* satisfying, in general, whatever identities are needed in order to make $\Lambda \oplus M$ into a commutative algebra (such as the symmetry $(\lambda_1 \otimes \lambda_2)f) = (\lambda_2 \otimes \lambda_1)f)$.

Now take $\Lambda = K[x]/(x^2 = 0)$, $M = \Lambda$ as a Λ-module, $Y = \Lambda \oplus M \longrightarrow \Lambda$ via projection as the module, and to heighten the drama let K be a field. Let f be the factor set

$$(1 \otimes 1)f = (1 \otimes x)f = (x \otimes 1)f = 0, \quad (x \otimes x)f = 1.$$

The extension $E = \Lambda \oplus M \longrightarrow \Lambda$ constructed by means of f represents a nonzero element in $H^1(\Lambda, Y)_\Lambda$. (Otherwise there would be an isomorphism

$$\begin{tikzcd} E \arrow[rr, "\simeq"] \arrow[dr] & & Y \arrow[dl] \\ & \Lambda & \end{tikzcd} \quad \text{in } (\mathscr{C}, \Lambda)$$

Since $Y \longrightarrow \Lambda$ is split by its zero section $\lambda \Rightarrow (\lambda, 0)$, which is an algebra map, $E \longrightarrow \Lambda$ would also be split by an algebra map. Thanks to the choice of f this is impossible. Always, in triple cohomology, an extension represents the zero cohomology class $\Longleftrightarrow$ it is *inessential*, i.e., split in the category of algebras.) Thus we know that $H^1(\Lambda, Y)_\Lambda \neq 0$.

However, $Y \longrightarrow \Lambda$, or equivalently its kernel M, is *injective* in the category of Λ-modules. Indeed, as a Λ-module, $M \simeq \mathrm{Hom}_K(M, K)$ (use the obvious 1-1 correspondence the K-base $1, x$ of M and the dual base), which proves injectivity, as K is a field, over which everything is injective.[6] We conclude that *triple cohomology need not vanish on injective coefficients.*

This example shows that algebra cohomology cannot both classify extensions and be a derived functor on the module category in the sense of [Cartan & Eilenberg (1956)] or [Mac Lane (1963)].

Barr also knows an example of a commutative H^2 which fails to vanish on injective coefficients.[7] There seems to be no reason why the same thing cannot happen in any dimension.

If $\mathscr{C}$ is tripled over sets, and the ground ring K is not a field, such examples are even easier to come by. Consider $\mathbf{Z}/4\mathbf{Z} \longrightarrow \mathbf{Z}/2\mathbf{Z}$, here $K = \mathbf{Z}$, and the kernel is actually a vector space over $\Lambda = \mathbf{Z}/2\mathbf{Z}$. Even the relative homological algebra in the module category does not seem to offer much hope (see [Eilenberg & Moore (1965b)], [Heller (1958)], or [Mac Lane (1963), Chapter IX]).

There is a close relationship between the theories of triples and of sites, or Grothendieck topologies, which it is beyond the scope of this paper to explore. Using this insight, one observes that it is possible to write the triple cohomology $H^n(X, Y)$ as a derived functor $R^n H^0(X, Y)$ in the category of functors (presheaves) $(\mathrm{Im} G)^* \longrightarrow \mathrm{Ab}$, where $G\colon \underline{A}^{\mathbf{T}} \longrightarrow \underline{A}^{\mathbf{T}}$ is the free $\mathbf{T}$-algebra cotriple. This result is analogous to Theorem (3.1) of [Artin (1962)]. I am indebted to S. U. Chase for showing me this.

EXAMPLE 9. *Additive Categories.* In additive categories the notion of module simplifies. Indeed, if $\underline{B}$ is additive, $X \in \underline{B}$, we have

$$X\text{-}\underline{\mathrm{Mod}} = \mathrm{Ab}(\underline{B}, X) \xrightarrow{\ker} \underline{B}$$

---

[6]Editors' note: the proof of injectivity is a little terse. The point is that when R is a K-algebra, then for any R-projective P and K-injective Q, the R-module $\mathrm{Hom}(P, Q)$ is R-injective.

[7]Editors' note: Subsequently, this example was published: M. Barr, A note on commutative algebra cohomology. Bull. Amer. Math. Soc. **74** (1968), 310–313.

is an equivalence of categories. ker is the functor $Y \longrightarrow X \Rightarrow M = \ker(Y \longrightarrow X)$. Because of the zero section of the module, we must have $Y \simeq X \oplus M$, q.e.d. (We are assuming that additive categories have a $\oplus$ and kernels, i.e., finite projective limits.)

Thus in an additive category every object "is" a module over every other object, in a unique manner. A typical cohomology theory arising in the additive context is the classical $\mathrm{Ext}^n_\Lambda(A, C)$ of (right) Λ-modules. The two variables in Ext give the illusion of being on the same footing, in contrast with group cohomology, say, where one variable is a group and the other is a module. But in view of the above proposition, C is equally an A-module, so there is no real contrast between Ext and the group case.

Phrased differently, there is no need to pass to the comma category $(\underline{B}, X)$ in order to obtain enough abelian group objects. Thus the only cohomology theory we are concerned with is of the type $H^n(X, Y)$, where $X, Y \in \underline{B}$, which is the same as $\mathrm{Ab}\underline{B}$. This arises as follows. Let

$$\underline{A} \xrightarrow{F} \underline{B} \xrightarrow{U} \underline{A} \qquad (F \dashv U)$$

be an adjoint pair and let $G\colon \underline{B} \longrightarrow \underline{B}$ be UF, the functor part of the standard cotriple in $\underline{B}$ arising from adjointness. ($\underline{A}$ is not assumed additive, hence G need not be additive, nor need $0G = 0 \in \underline{B}$.) Form the standard resolution

$$0 \longleftarrow X \xleftarrow{\partial_0} XG \xleftarrow{\partial_1} \cdots \longleftarrow XG^n \xleftarrow{\partial_n} XG^{n+1} \longleftarrow \cdots$$

where X is in dimension -1 and $\partial_n = \Sigma(-1)^i X\epsilon_i, 0 \leq i \leq n, \epsilon_i = G^i \epsilon G^{n-i}$, using additivity of $\underline{B}$ to add up the face operators in advance. Applying the functor $(\ , Y)\colon \underline{B}^* \longrightarrow \mathrm{Ab}$, one gets a cochain complex

$$0 \longrightarrow (XG, Y) \xrightarrow{d^1} (XG^2, Y) \longrightarrow \cdots \longrightarrow (XG^n, Y) \xrightarrow{d^n} (XG^{n+1}, Y) \longrightarrow \cdots$$

where $d^n = (\partial_n, Y)$. $H^n(X, Y)$, relative to $F \dashv U$ of course, is the n-th cohomology group of this complex.

If $F \dashv U$ is tripleable, $H^0(X, Y)$ is the hom functor and $H^1(X, Y)$ classifies U-exact sequences $0 \longrightarrow Y \longrightarrow E \longrightarrow X \longrightarrow 0$. U-exactness is defined in Theorem 2 above, and these facts are contained in the interpretation of the cohomology given in §3, provided one can identify short U-exact sequences with U-split principal homogeneous objects. But if $0 \longrightarrow Y \longrightarrow E \longrightarrow X \longrightarrow 0$ is U-exact, then Y operates on E by additivity, moreover simply-transitively and compatibly with $E \longrightarrow X$, and $EU \simeq XU \times YU$. Conversely, given such an $E \longrightarrow X$ operated on by Y, a U-exact sequence is defined by

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[rr] \arrow[dr, "{(Y,0)}"'] & & E \arrow[r] & X \arrow[r] & 0 \\ & & Y \oplus E \arrow[ur, "\circ"'] & & & \end{tikzcd}$$

Now, doing the obvious, let $E^n(X, Y)$ be the set of Yoneda equivalence classes of U-exact sequences (n-dimensional extensions relative to U) of the form $0 \longrightarrow Y \longrightarrow Y_{n-1}$

$\longrightarrow \cdots Y_0 \longrightarrow X \longrightarrow 0$ [Mitchell (1965)]. The standard resolution XG^{n+1}, ∂_n $(n \geq 0)$ is F-free (as well as U-exact). Thus, given an n-dimensional extension, we can construct a map of complexes over X in the usual manner:

$$\begin{tikzcd} 0 & X \arrow[l] \arrow[d, equal] & XG \arrow[l, "\partial_0"'] \arrow[d] & XG^2 \arrow[l, "\partial_1"'] \arrow[d] & \cdots \arrow[l] & XG^n \arrow[l] \arrow[d] & XG^{n+1} \arrow[l, "\partial_n"'] \arrow[d, "a"] & XG^{n+2} \arrow[l, "\partial_{n+1}"'] \arrow[d] & \cdots \arrow[l] \\ 0 & X \arrow[l] & Y_0 \arrow[l] & Y_1 \arrow[l] & \cdots \arrow[l] & Y_{n-1} \arrow[l] & Y \arrow[l] & 0 \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

The component a is an n-cocycle. This defines the map needed in the following additive extension of Theorems 5, 6 of §3:

THEOREM 7. *The natural map*

$$E^n(X,Y) \longrightarrow H^n(X,Y)$$

is an isomorphism if $F \dashv U$ is tripleable.

Several proofs of this result are possible, we will omit all of them. One proof involves breaking up long U-exact sequences into composites of short ones (this requires kernels in $\underline{B}$), and then using the fact that short ones are classified by H^1, or more precisely, determined by 1-cocycles $XT \longrightarrow Y$ in $\underline{A}$. In an additive category it is possible to characterize $H^n(X,Y)$ by a fairly obvious set of axioms, as a functor of X. Another proof of Theorem 7 then proceeds by verifying these axioms for $E^n(X,Y)$.

Note the following gap. In the general, nonadditive case, Barr's examples referred to above show that $H^n(X,Y)$, $X \in \underline{B}$, $Y \in \mathrm{Ab}\underline{B}$, does *not* classify mixed additive-nonadditive "extensions" of the form

$$\begin{cases} 0 \longrightarrow Y \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_0 \longrightarrow 0, & U\text{-exact} \\ E \xrightarrow{Y_0} X, & U\text{-split principal homogeneous object.} \end{cases}$$

(Such have been considered, from the point of view of cohomology classification, in [Barr (1965), Barr & Rinehart (1966), Gerstenhaber (1964)].) Otherwise $H^n(X,Y)$ would vanish when Y is injective. In general, does $H^n(X,Y)$ classify any concept of n-dimensional extensions of X by Y? Should these "extensions" perhaps be required to have a simplicial structure?

As examples of the situation envisaged in Theorem 7, we cite the following:

(a) The category of right Λ-modules tripleable over the category of sets $\underline{A}$. We have $F\colon \underline{A} \longrightarrow \mathscr{M}_\Lambda$, $U\colon \mathscr{M}_\Lambda \longrightarrow \underline{A}$, where $AF = A\cdot\Lambda$ (A-fold coproduct of Λ's) and XU is the usual underlying set of X; $F \dashv U$. U-exactness is the ordinary abelian-category exactness, the standard complex XG^* is a Λ-free resolution of X, hence (as Theorem 7 also shows)

$$H^n(X,Y) \xrightarrow{\simeq} \mathrm{Ext}^n_\Lambda(X,Y), \qquad n \geq 0\ .$$

(b) If $\Lambda \xrightarrow{\varphi} \Gamma$ is a ring map, we get a tripleable adjoint pair $- \otimes_\Lambda \Gamma\colon \mathscr{M}_\Lambda \longrightarrow \mathscr{M}_\Gamma$, $\mathscr{M}_\varphi\colon \mathscr{M}_\Gamma \longrightarrow \mathscr{M}_\Lambda$ ([Cartan & Eilenberg (1956)], p. 29). As cohomology we obtain Hochschild's relative Ext [Hochschild (1956)]:

$$H^n(X,Y) \xrightarrow{\simeq} \mathrm{Ext}^n_\varphi(X,Y), \qquad n \geq 0 \; .$$

We have been emphasizing cohomology. But one can take coefficients in functors other than hom functors, for example, the tensor product with a fixed Λ-module. Thus Tor^Λ, Tor^φ can be introduced into our theory, as well as a general homology theory of algebras (which we pass over in silence).

(c) If $\underline{A}$ is a graded abelian group and $\mathscr{C}$ is the category of chain complexes, adjoint functors (tripleable)

$$\underline{A} \xrightarrow{F} \mathscr{C} \xrightarrow{U} \underline{A}$$

are defined by: $A = (A_n) \Longrightarrow AF = (AF_n)$ with $AF_n = A_n \oplus A_{n+1}$ and boundary operator $AF_n \longrightarrow AF_{n-1}$ by shifting $(a_n, a_{n+1}) \Longrightarrow (0, a_n)$; U forgets the boundary operator. Then the cohomology theory $H^n(X,Y)$ classifies sequences of chain complexes $0 \longrightarrow Y \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_0 \longrightarrow X \longrightarrow 0$ which are split exact in $\underline{A}$, *i.e.*, ignoring the boundary operator.

If negative degrees are permitted (the reader can make the categories precise), this example coalesces with the obvious graded variant of (b), as $AF = A \otimes D$ where D is the graded ring generated by an element $\partial \in D_{-1}$ with $\partial\partial = 0$ in D_{-2}.

(d) We refer to [Eilenberg & Moore (1965a)] for the examples of co- and contra-modules over a coalgebra (as well as an account of the situation when both categories, base and algebras, are additive).

(e) $\underline{A}$ = graded, connected, commutative K-coalgebras, $\mathscr{C}$ = graded, connected, bicommutative Hopf algebras. Then $\mathscr{C} = \mathrm{Ab}\underline{A}$ [Milnor & Moore (1965)]. The graded tensor algebra is left adjoint to the underlying $\mathscr{C} \longrightarrow \underline{A}$ [Moore (1961)], and tripleably. $H^n(X,Y)$ classifies sequences of Hopf algebras which on the coalgebra level decompose into short exact sequences of the form $A \longrightarrow A \otimes B \longrightarrow B$ (cartesian products in $\underline{A}$).

Example (e) also goes through in the ungraded case.

There is a graded dual (e*) in which one takes $\underline{A}$ = commutative algebras, $\mathscr{C}$ as above is then abelian cogroups in $\underline{A}$ and the underlying $\mathscr{C} \longrightarrow \underline{A}$ is cotripleable. The sequences classified by the *co*homology are split as algebras, into tensor products.

I do not know whether the ungraded dual (e*) works; the underlying $\mathscr{C} \longrightarrow K$-algebras may lack a right adjoint.

Obviously (with notation as in (e)), Hopf algebras which are only cocommutative are group objects in $\underline{A}$. Examples 3, 4, 5 on groups and monoids can all be reworked in this context, replacing the category of sets by that of commutative coalgebras; and dually.

In additive categories where a notion of exactness is defined, or in abelian categories, Yoneda's theory of long extensions supplies a cohomology theory $\mathrm{Ext}^n(X,Y)$ without the intervention of projective or injective resolutions of any kind. The triple theory cannot match this feat, for constructing free algebras usually requires infinite direct limits. However, for the restricted class of abelian categories to be discussed next, triple cohomology does make a natural appearance and coincides with the Yoneda Ext. The reader will note considerable contact between our treatment and [Huber (1962)].

Let $\underline{B}$ be an abelian category with direct (inductive) limits, $P \in \underline{B}$ any object. We treat the representable functor $(P,\ ): \underline{B} \longrightarrow \mathrm{Ab}$ as an underlying object functor, and recall that it has a left adjoint $A \Longrightarrow A \otimes P$ (see [Freyd (1964)], [Mitchell (1965)], or assume $\underline{B}$ is a category of modules). As usual, we have

$$\begin{tikzcd} \mathrm{Ab}^{\mathbf{T}} \arrow[dr] & & \underline{B} \arrow[ll, "\Phi"'] \arrow[dl, "{(P,\ )}"] \\ & \mathrm{Ab} & \end{tikzcd}$$

Here a $\mathbf{T}$-structure on an abelian group A is a unitary, associative abelian group map $\theta: AT = (P, A \otimes P) \longrightarrow A$. By adjointness every (P, B) has such a structure, which defines the functor Φ.

Now let $R = (P, P)$, the endomorphism ring of P, and let $\mathbf{R}^0$ be the triple in Ab defined by

$$A \xrightarrow{a \otimes 1} A \otimes R, \qquad A \otimes R \otimes R \xrightarrow{a \otimes r_2 r_1} A \otimes R \cdot$$

$r_2 r_1$ is the composition, in that order, of endomorphisms $P \longrightarrow P$; $\mathrm{Ab}^{\mathbf{R}^0}$ is the category of left R-modules.

A natural transformation $A\varphi: A \otimes R \longrightarrow AT$ is defined if we let $(a \otimes r)(A\varphi)$ be the composition

$$P \xrightarrow{\simeq} \mathbf{Z} \otimes P \xrightarrow{a \otimes r} A \otimes P,$$

thinking of $a \in A$ as a map $\mathbf{Z} \longrightarrow A$. One can verify that $\varphi: \mathbf{R}^0 \longrightarrow \mathbf{T}$ is a *map of triples*, that is, the natural transformation φ commutes with units and multiplications:

$$\begin{tikzcd} & A \arrow[dl, "a \otimes 1"'] \arrow[dr, "A\eta"] & \\ A \otimes R \arrow[rr, "A\varphi"'] & & AT \end{tikzcd} \qquad \begin{tikzcd} A \otimes R \otimes R \arrow[r, "A\varphi\varphi"] \arrow[d, "a \otimes r_2 r_1"'] & ATT \arrow[d, "A\mu"] \\ A \otimes R \arrow[r, "A\varphi"'] & AT \end{tikzcd}$$

(μ results from adjointness, $f \cdot \mu$ being the composition of an f with $(A \otimes P)\epsilon: P \longrightarrow (P, A \otimes P) \otimes P \longrightarrow A \otimes P$.) A map of triples induces a functor between the corresponding algebra categories. In this case, if $\theta: AT \longrightarrow A$ is a $\mathbf{T}$-structure on A, then the composition

$$A \otimes R \xrightarrow{A\varphi} AT \xrightarrow{\theta} A$$

is an $\mathbf{R}^0$-structure on A. Thus we have the following commutative diagram of functors.

$$\begin{tikzcd}
\mathrm{Ab}^{\mathbf{R}^0} \arrow[dr] & \mathrm{Ab}^{\mathbf{T}} \arrow[l, "\mathrm{Ab}^{\varphi}"'] \arrow[d] & \underline{B} \arrow[l, "\Phi"'] \arrow[dl, "(P,\ )"] \\
& \mathrm{Ab} &
\end{tikzcd}$$

We now have

PROPOSITION. *If P is a projective generator in $\underline{B}$, then Φ is an equivalence.*
If P is a small projective, then Ab^{φ} is an isomorphism of categories.

Approximate definitions for the terms used in the proposition can be found in [Freyd (1964)] and [Mitchell (1965)]. The two statements follow from the (Tripleableness) Theorem 1, and the fact that if P is small, $\varphi\colon \mathbf{R}^0 \longrightarrow \mathbf{T}$ is an isomorphism of triples. The first statement can be paraphrased, $\underline{B}$ is tripleable over Ab. Were P also small, we would be able to conclude the familiar corollary below. Thus a triple in Ab can be considered as a sort of "large" generalization of a ring.

COROLLARY. *If P is a small projective generator in $\underline{B}$, then $\underline{B}$ is equivalent to the category of left $R = (P, P) = \mathrm{End}(P)$-modules.*

Similar proofs can be given for the characterization of cocomplete abelian categories with generating sets of small projectives [Freyd (1964)], as well as for M. Bunge's recent characterization of functor categories $\mathscr{S}^{\underline{C}}$ [Bunge (1966)]. (This has been carried out by F.E.J. Linton and the writer.)

This result yields subexample (b) above, which arises when $P = \Gamma \in \mathscr{M}_\Gamma$, $(\Lambda = \mathbf{Z})$.

The following can be proved by an elaboration of the proof of Theorem 1 (hinting the more delicate tripleableness theorem referred to):

PROPOSITION. *Consider a composition of adjoint pairs*

$$\left.\begin{array}{l} \underline{A} \xrightarrow{F} \underline{B} \xrightarrow{U} \underline{A} \\ \underline{A}_0 \xrightarrow{F_0} \underline{A} \xrightarrow{U_0} \underline{A}_0 \end{array}\right\} \implies \underline{A}_0 \xrightarrow{F_0F} \underline{B} \xrightarrow{UU_0} \underline{A}_0$$

If $F \dashv U$ satisfies all the hypotheses of Theorem 1 (hence in particular is tripleable), and if $F_0 \dashv U_0$ is tripleable, then $F_0F \dashv UU_0$ is tripleable.

Let $\underline{B}$ be a cocomplete abelian category (*i.e.*, direct limits) with a projective generator P. Apply this proposition to $(P,\ )\colon \underline{B} \longrightarrow \mathrm{Ab}$, $U_0\colon \mathrm{Ab} \longrightarrow \mathrm{Sets}$, which we denote by $U\colon \underline{B} \longrightarrow \mathrm{Sets}$. We find that every such abelian category is tripleable over sets. Notice that U-exactness in $\underline{B}$ is the same as abelian-category exactness. Hence $\mathrm{Ext}^n(X, Y)$ defined by long abelian-exact sequences $0 \longrightarrow Y \longrightarrow \cdots \longrightarrow X \longrightarrow 0$ coincides with the triple cohomology $H^n(X, Y)$, relative to U.

Note that without the extra, coequalizer-preserving, property of the functor $(P,\ )$, we would not be able to conclude that the composition $B \longrightarrow$ Sets is tripleable. The composition of tripleable underlying object functors is not tripleable in general, for example, Torsion-free abelian groups $\longrightarrow$ Ab $\longrightarrow$ Sets; tripleableness is trivially true of coreflective subcategories ([Mitchell (1965)], but fullness should be added to the definition).

If X is a set, the explicit composite triple above is $X \longrightarrow (P, X \cdot P)$. In case P is also small, let $R^0 = (P, P)$ with backwards multiplication and identify the triple as $X \longrightarrow X \cdot R^0$, the underlying set of the free right R^0-module generated by the set X. Thus we again find $\underline{B} \simeq$ left R-modules.

This result yields subexample (a) above, which arises when the projective generator $P \in \mathscr{M}_\Lambda$ chosen is Λ itself.

References

M. Artin (1962), Grothendieck Topologies. Harvard University (mimeographed).

M. Barr (1965), A cohomology theory for commutative algebras, I, II. Proc. Amer. Math. Soc. **16**, 1379–1384, 1385-1391.

M. Barr (1967), Shukla cohomology and triples. J. Algebra **5**, 222–231.

M. Barr and J. Beck (1966), Acyclic models and triples. Proceedings of the Conference on Categorical Algebra, Springer-Verlag, Berlin, Heidelberg, New York, 336–343.

M. Barr, and G.S. Rinehart (1966), Cohomology as the derived functor of derivations. Trans. Amer. Math. Soc. **122**, 416–426.

J. Beck, The tripleableness theorems. To appear.[8]

H.-B. Brinkmann, D. Puppe (1965), Kategorien und Funktoren. Lecture Notes in Mathematics **18**, Springer-Verlag, Berlin, Heidelberg, New York.

M. Bunge (1966), Characterization of diagrammatic categories. Dissertation, University of Pennsylvania.

H. Cartan and S. Eilenberg (1956), Homological Algebra. Princeton University Press.

J. Dixmier (1957), Homologies des anneaux de Lie. Ann. Sci. École Norm. Sup. **74**, 25–83.

B. Eckmann and P. Hilton (1962), Group-like structures in general categories, I. Math. Ann. **145**, 227–255.

S. Eilenberg (1948), Extensions of general algebras. Ann. Sci. Polon. Math. **21**, 125–134.

S. Eilenberg and S. Mac Lane (1947), Cohomology theory in abstract groups, I. Ann. of Math. **48**, 51–78.

S. Eilenberg and J. Moore (1965a), Adjoint functors and triples. Ill. J. Math., **58**, 381–398.

[8]Editors' note: To our knowledge, this has not appeared. Beck's tripleableness theorems have been exposed in M. Barr and C. Wells, Toposes, Triples and Theories. Springer-Verlag, Berlin, Heidelberg, New York, 1984 as well as other places.

S. Eilenberg and J. Moore (1965b), Foundations of relative homological algebra. Memoirs Amer. Math. Soc. **55**.

P.J. Freyd (1964), Abelian Categories. Harper and Row, New York.

M. Gerstenhaber (1964), A uniform cohomology theory for algebras. Proc. Nat. Acad. Sci. USA **51**, 626–629.

R. Godement (1958), Théorie des faisceaux. Hermann, Paris.

B. Harris (1961), Cohomology of Lie triple systems and Lie algebras with involution. Trans. Amer. Math. Soc. **98**, 148–162.

A. Heller (1958), Homological algebra in abelian categories. Ann. of Math. **68**, 484–525.

G. Hochschild (1945), On the cohomology groups of an associative algebra. Ann. of Math. (2), **46**, 58–67.

G. Hochschild (1956), Relative homological algebra. Trans. Amer. Math. Soc. **82**, 246–269.

P.J. Huber (1961), Homotopy theory in general categories. Math. Ann. **144**, 361–385.

P.J. Huber (1962), Standard constructions in abelian categories. Math. Ann. **146**, 321–325.

D. Kan (1958), Adjoint functors. Trans. Amer. Math. Soc., **87**, 294–329.

F.W. Lawvere (1963), Functorial semantics of algebraic theories. Proc. Nat. Acad. Sci. USA **50**, 869–872.

F.W. Lawvere (1966), The category of categories as a foundation for mathematics. Proceedings of the Conference on Categorical Algebra, Springer-Verlag, Berlin, Heidelberg, New York,

F.E.J. Linton (1966), Some aspects of equational categories. Proceedings of the Conference on Categorical Algebra, Springer-Verlag, Berlin, Heidelberg, New York,

S. Mac Lane (1963), Homology. Springer-Verlag, Berlin, Heidelberg, New York.

K. McCrimmon (1966), A general theory of Jordan rings. Proc. Nat. Acad. Sci. USA **56**, 1072–1079.

J. Milnor and J.C. Moore (1965), Structure of Hopf algebras. Ann. of Math. **81**, 211–264.

B. Mitchell (1965), Theory of Categories. Academic Press, New York.

J.C. Moore (1961), Algèbres de Hopf universelles. Séminaire H. Cartan 1959/60, Exp. 10, École Norm. Sup.

J.-P. Serre (1965), Cohomologie galoisienne. Lecture Notes in Mathematics **5**, Springer-Verlag, Berlin, Heidelberg, New York.

U. Shukla (1961), Cohomologie des algèbres associatives. Ann. Sci. École Norm. Sup. **78**, 163–209.

This article may be accessed from `http://www.tac.mta.ca/tac/reprints` or by anonymous ftp at
`ftp://ftp.tac.mta.ca/pub/tac/html/reprints/articles/2/tr2.{dvi,ps,pdf}`