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OPMONOIDAL MONADS

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ABSTRACT. Hopf monads are identified with monads in the 2-category OpMon of monoidal categories, opmonoidal functors and transformations. Using Eilenberg-Moore objects, it is shown that for a Hopf monad S, the categories $\mathrm{Alg}(\mathrm{Coalg}(S))$ and $\mathrm{Coalg}(\mathrm{Alg}(S))$ are canonically isomorphic. The monadic arrows OpMon are then characterized. Finally, the theory of multicategories and a generalization of structure and semantics are used to identify the categories of algebras of Hopf monads.

1. Hopf Monads

The purpose of this note is to put the results of [Moerdijk (1999)] into a 2-categorical framework, highlight their universal nature, and so extend those results.

Propositions 1.1 and 1.2 of this section are from [Moerdijk (1999)]. Let $\mathcal{C}$ be a monoidal category, and (S, η, μ) a monad on the underlying category of $\mathcal{C}$. A *Hopf monad* structure on S consists of natural transformations

$$\begin{array}{ccc} \mathcal{C}\times\mathcal{C} & \xrightarrow{\otimes} & \mathcal{C} \\ {\scriptstyle S\times S}\downarrow & \swArrow\chi & \downarrow{\scriptstyle S} \\ \mathcal{C}\times\mathcal{C} & \xrightarrow[\otimes]{} & \mathcal{C} \end{array} \qquad\qquad \begin{array}{ccc} 1 & \xrightarrow{I} & \mathcal{C} \\ {\scriptstyle 1}\downarrow & \swArrow\iota & \downarrow{\scriptstyle S} \\ 1 & \xrightarrow[I]{} & \mathcal{C} \end{array}$$

satisfying various axioms; we shall provide these axioms in Example 2.5. Here, the functor $I\colon 1 \to \mathcal{C}$ is the functor whose value on the only object of 1 is the unit I of $\mathcal{C}$. Given S-algebras $(X, x\colon SX \to X)$ and $(Y, y\colon SY \to Y)$ the tensor product of the objects X and Y becomes an S-algebra with the follow action.

$$S(X\otimes Y) \xrightarrow{\chi} SX \otimes SY \xrightarrow{x\otimes y} X\otimes Y$$

Similarly, the arrow $\iota\colon SI \to I$ makes I into an S-algebra. The associativity and unit isomorphisms for $\mathcal{C}$ are then S-algebra morphisms making the category $\mathrm{Alg}(S)$ into a monoidal category in such a way that the forgetful functor $U\colon \mathrm{Alg}(S) \to \mathcal{C}$ is a strict monoidal functor. The converse of this is also true and we now summarize these remarks as a proposition.

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1.1. Proposition. *There is a bijection between Hopf monad structures on a monad S and monoidal structures on* $\mathrm{Alg}(S)$ *such that the forgetful functor* $U\colon \mathrm{Alg}(S) \to \mathcal{C}$ *is a strict monoidal functor.*

Suppose (C, δ, ε) is a coalgebra in $\mathcal{C}$. Observe that the arrows

$$SC \xrightarrow{S\delta} S(C \otimes C) \xrightarrow{\chi} SC \otimes SC \qquad\qquad SC \xrightarrow{S\varepsilon} SI \xrightarrow{\iota} I$$

make SC into a coalgebra in $\mathcal{C}$. In fact, there is a monad $\mathrm{Coalg}(S)$ on the category $\mathrm{Coalg}(\mathcal{C})$ of coalgebras in $\mathcal{C}$ lifting the monad S.

1.2. Proposition. *The categories* $\mathrm{Alg}(\mathrm{Coalg}(S))$ *and* $\mathrm{Coalg}(\mathrm{Alg}(S))$ *are canonically isomorphic.*

The aim of this note is to analyze and generalize Propositions 1.1 and 1.2. We shall first consider Proposition 1.2.

2. Monads in a 2-category

In this section we recall the notion of a monad in a 2-category and then provide some examples. We then consider monadicity for arrows in a 2-category, and characterize the monadic arrows in the 2-category OpMon. We shall then show that Proposition 1.2 is a consequence of the fact that representable 2-functors preserve Eilenberg-Moore objects.

Let $\mathcal{K}$ be a bicategory. A *monad* [Street (1972)] in $\mathcal{K}$ is a lax functor $1 \to \mathcal{K}$ from the terminal 2-category 1. It amounts to an object X of $\mathcal{K}$ equipped with an arrow $T\colon X \to X$ and two 2-cells,

called the *multiplication* and *unit* respectively, satisfying three axioms that express that μ is associative and unital. We also say that T is a monad *on* the object X. Given monads S and S' on X, a *monad morphism* $f\colon S \to S'$ is a 2-cell $\alpha\colon S' \Rightarrow S$ respecting the multiplication and unit. Note the change of direction. With the evident compositions there is a category $\mathrm{Mnd}(X)$ of monads on X. This category is in fact a subcategory of the underlying category of the 2-category of monads *in* $\mathcal{K}$ [Street (1972)].

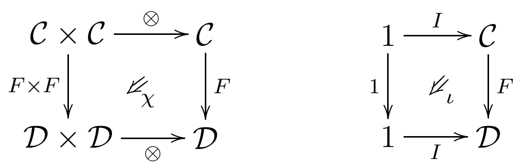
2.1. Example. Ordinary Monads. A monad in the 2-category Cat of categories is of course a monad in the usual sense.

2.2. Example. Monoids. For a monoidal category $\mathcal{C}$, there is a bicategory $\Sigma\mathcal{C}$ called the *suspension* of $\mathcal{C}$. The bicategory $\Sigma\mathcal{C}$ has one object 0 and $\Sigma\mathcal{C}(0,0) = \mathcal{C}$, and with composition given by the monoidal structure. A monad in $\Sigma\mathcal{C}$ is exactly a monoid in $\mathcal{C}$.

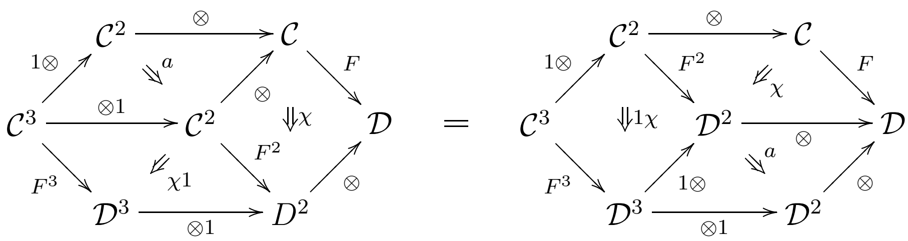
2.3. Example. Categories. Recall the bicategory Span(Set) of spans of sets. An object is a set, an arrow is a span $A \leftarrow B \rightarrow C$ of sets. The set B is called the *head* of the span, and the arrows are called the *left* and *right legs*. A 2-cell is a function between the heads commuting with the legs. Composition is given by pullback, and the identity span is given by the span of identity functions. A monad in Span(Set) is exactly a category [Bénabou (1967)].

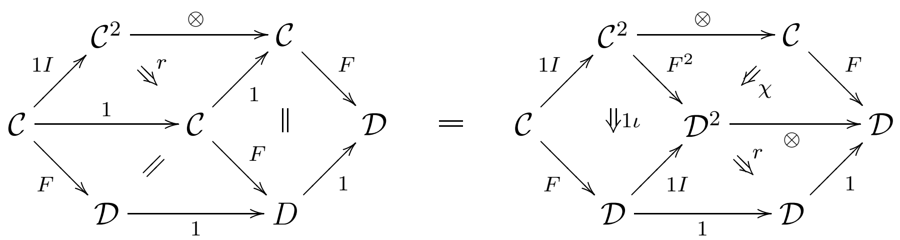
2.4. Example. Multicategories. Multicategories were introduced in [Lambek (1969)] and have experienced a resurgence of interest of late. A *multicategory* is a monad in the Kleisli bicategory $\mathrm{Span}_T(\mathrm{Set})$ of spans, where T is the cartesian free monoid monad on the category Set of sets [Leinster (1997), Hermida (1999)]. We shall return to multicategories in Section 3.

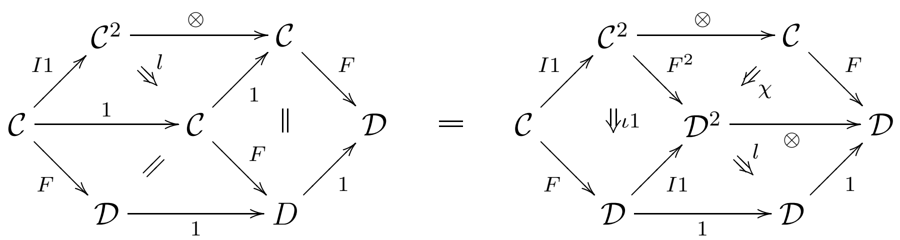
For monoidal categories $\mathcal{C}$ and $\mathcal{D}$, an *opmonoidal functor* $F\colon \mathcal{C} \to \mathcal{D}$ is a functor $F\colon \mathcal{C} \to \mathcal{D}$ along with natural transformations



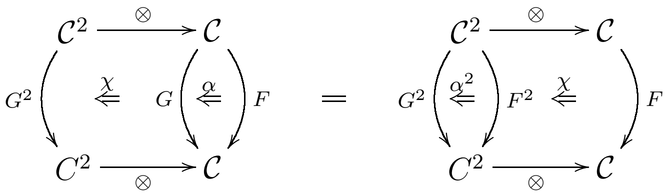
satisfying the following three axioms.

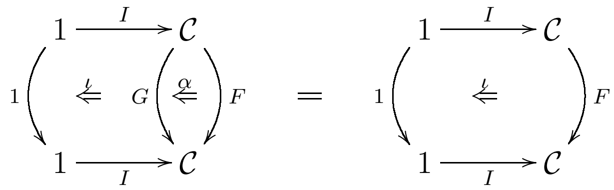






Given parallel opmonoidal functors F and G, a *transformation* of opmonoidal functors from F to G is a natural transformation $\alpha\colon F \Rightarrow G$ satisfying the following two axioms.



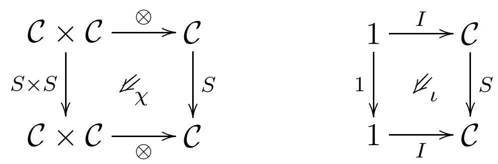


With the evident compositions there is a 2-category OpMon whose objects are monoidal categories, whose arrows are opmonoidal functors and whose 2-cells are transformations of opmonoidal functors. By [Kelly (1974b), Section 10.8], the 2-category OpMon is isomorphic to the 2-category $\mathrm{Alg}_c(T)$ of algebras, colax morphisms and transformations for some 2-monad T on Cat.

2.5. Example. Hopf monads. A Hopf Monad is exactly a monad in the 2-category OpMon.

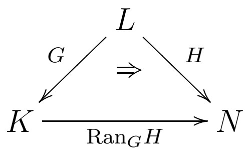
Following Example 2.5, we suggest the more descriptive term *opmonoidal monad* for Hopf monad, which we shall use for the remainder of this paper.

2.6. Example. Bialgebras. Let B be a bialgebra in a symmetric monoidal category $\mathcal{C}$. Let $S\colon \mathcal{C} \to \mathcal{C}$ denote the endofunctor of $\mathcal{C}$ whose value on an object X is $X \otimes B$. The multiplication and unit of B induce natural transformations $\mu\colon SS \Rightarrow S$ and $\eta\colon 1 \Rightarrow S$ making S into a monad on $\mathcal{C}$. The comultiplication, counit and symmetry induce natural transformations



making S into an opmonoidal monad.

Recall that adjunctions and extensions may be be defined in any 2-category [Kelly & Street (1974), Street & Walters (1978)]. Let $\mathcal{K}$ be a 2-category and $G\colon L \to K$ an arrow in $\mathcal{K}$. A *left adjoint* of G is an arrow $F\colon K \to L$ and two 2-cells $\eta\colon 1 \Rightarrow GF$ and $\varepsilon\colon FG \Rightarrow 1$, called the *unit* and *counit* respectively, satisfying $\varepsilon F \circ F\eta = 1$ and $G\varepsilon \circ \eta G = 1$. One calls F the *left adjoint* of G. If $H\colon L \to N$ is an arrow with the same domain as G then the *right* *extension* (also called the *right Kan extension*) of H along G is an arrow $\mathrm{Ran}_G H\colon K \to N$ and a 2-cell



such that for any $J\colon K \to N$, the function

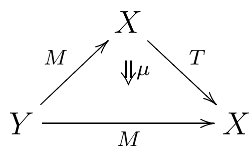
$$\mathcal{K}(K,N)(J,\mathrm{Ran}_F H) \to \mathcal{K}(L,N)(JG,H)$$

given by composing with this 2-cell is a bijection. Left adjoints and right extensions may not always exist, but if they do, they are unique up to canonical isomorphism.

2.7. Lemma. [Street (1972), Theorem 4] *If $G\colon L \to K$ is an arrow in a 2-category $\mathcal{K}$ with a left adjoint F, and if $\varepsilon\colon FG \Rightarrow 1$ is the counit of the adjunction, then any arrow $H\colon L \to N$ has a right extension along G given by the arrow $HF\colon K \to L$ and the 2-cell $H\varepsilon\colon HFG \Rightarrow H$. This extension is preserved by any 2-functor.* □

Let $G\colon L \to K$ be an arrow such that the right extension $R = \mathrm{Ran}_G G$ of G along itself exists. There are 2-cells $\mu\colon RR \Rightarrow R$ and $\eta\colon 1 \Rightarrow R$ making R a monad in $\mathcal{K}$ [Street (1972), Section 2]. We call R the *monad generated by G*. When G has a left adjoint F we also say that R is generated by the adjunction $F \dashv G$.

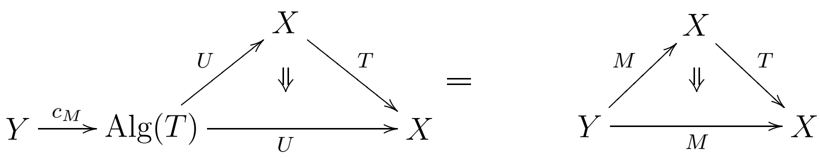
We now turn to the notion of algebras for a monad. Let T be a monad in $\mathcal{K}$ and let Y be an object of $\mathcal{K}$. A *Y-based T-algebra* consists of an arrow $M\colon Y \to X$ equipped with a 2-cell



satisfying two axioms that express that $\mu\colon T \circ M \Rightarrow M$ is associative and unital. If M and N are Y-based T-algebras, then a *Y-based T-algebra morphism* $f : M \to N$ is a 2-cell $f\colon M \Rightarrow N$ in $\mathcal{K}$ satisfying one axiom, expressing that f respects the action of T. With the evident compositions there is a category $\mathrm{Alg}(Y,T)$ whose objects are Y-based T-coalgebras, and whose arrows are Y-based T-algebra morphisms, and $\mathrm{Alg}(Y,T)$ is the value at the object Y of a 2-functor $\mathrm{Alg}(-,T) : \mathcal{K}^{\mathrm{op}} \to \mathrm{Cat}$. An *Eilenberg-Moore object* of T is a representation

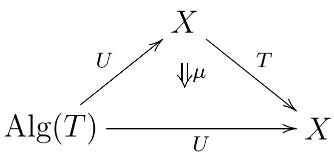
$$\mathcal{K}(-,\mathrm{Alg}(T)) \cong \mathrm{Alg}(-,T)$$

of this 2-functor. This means that $\mathrm{Alg}(T)$ is a T-algebra $(U\colon \mathrm{Alg}(T) \to X, \mu)$ with a 1- and a 2-dimensional universal property. The 1-dimensional property states that if M is a Y-based T-algebra then there exists a unique arrow $c_M\colon Y \to \mathrm{Alg}(T)$, called the *comparison*, such that the following equation holds.



The 2-dimensional property states that if $f\colon M \to N$ is a morphism of Y-based T-algebras, then there exists a unique 2-cell $c_f\colon c_M \Rightarrow c_N$ such that $f = U \circ c_f$. In general, Eilenberg-Moore objects need not exist; one says that $\mathcal{K}$ *admits the construction of Eilenberg-Moore objects* if for all monads T in $\mathcal{K}$ there exists a representation of $\mathrm{Coalg}(-, T)$. The existence of Eilenberg-Moore objects is a completeness property of $\mathcal{K}$. Eilenberg-Moore objects are unique up to isomorphism, not just equivalence.

2.8. Example. Eilenberg-Moore Objects in Cat. It is well known that Eilenberg-Moore objects exist in Cat. Explicitly, the category $\mathrm{Alg}(T)$ has algebras $(X, x : TX \to X)$ as objects, and algebra morphisms as arrows. The arrow $U\colon \mathrm{Alg}(T) \to X$ is the forgetful functor, and the action



has component x at the object (X, x). Of course this category of algebras was known before its universal property.

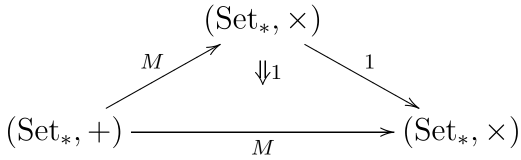
Suppose S is a monad in a 2-category $\mathcal{K}$ and $\mathcal{F}\colon \mathcal{K} \to \mathcal{L}$ is a 2-functor. Then the arrow $\mathcal{F}S$ is canonically a monad in $\mathcal{L}$, and furthermore, $\mathcal{F}$ takes S-algebras to $\mathcal{F}S$-algebras. In particular, if the Eilenberg-Moore object $\mathrm{Alg}(S)$ of S exists, then $\mathcal{F}\mathrm{Alg}(S)$ is canonically an $\mathcal{F}S$ algebra. We say that $\mathcal{F}$ *preserves the Eilenberg-Moore object of* S if $\mathcal{F}\mathrm{Alg}(S)$ is an Eilenberg-Moore object of $\mathcal{F}S$, and we say that $\mathcal{F}$ *preserves Eilenberg-Moore objects* if it preserves all Eilenberg-Moore objects that exist. Any finite limit preserving functor preserves Eilenberg-Moore objects.

Recall that an opmonoidal functor is called *strong* if the natural transformations χ and ι are isomorphisms.

2.9. Proposition. *The 2-category* OpMon *admits Eilenberg-Moore objects and the forgetful 2-functor* $\mathcal{U}\colon \mathrm{OpMon} \to \mathrm{Cat}$ *preserves them. Moreover, for any monad* S *on* $\mathcal{C}$ *in* OpMon*, the universal arrow* $U\colon \mathrm{Alg}(S) \to \mathcal{C}$ *is strong monoidal.*

Proof. The 2-category OpMon is isomorphic to the 2-category $\mathrm{Alg}_c(D)$ of strict D-algebras, colax morphism, and appropriate 2-cells for some 2-monad D on Cat. Lack [Lack (1998)] shows that such 2-categories admit Eilenberg-Moore objects, that the forgetful 2-functor preserves them, and that the universal arrow $U\colon \mathrm{Alg}(S) \to \mathcal{C}$ is strong monoidal. ∎

2.10. Example. Reflection of Eilenberg-Moore objects. We show that the 2-functor $\mathcal{U}\colon \mathrm{OpMon} \to \mathrm{Cat}$ does not reflect Eilenberg-Moore objects. Let $(\mathrm{Set}_*, +)$ and $(\mathrm{Set}_*, \times)$ be the category of pointed sets equipped with the monoidal structures deriving from coproduct and product respectively. The terminal object is also initial, providing canonical arrows $X + Y \to X \times Y$ and $0 \to 1$. These arrows equip the identity functor with the structure of an opmonoidal functor $M : (\mathrm{Set}_*, +) \to (\mathrm{Set}_*, \times)$. The identity natural transformation



makes M into an algebra for the identity monad on $(\mathrm{Set}_*, \times)$. Clearly this is an Eilenberg-Moore object in Cat. It is not, however, an Eilenberg-Moore object in OpMon since M is not strong monoidal.

We now consider the characterization of *monadic* arrows in OpMon. First we recall the definition of a monadic arrow in a 2-category. Let $G\colon L \to K$ be an arrow in a 2-category $\mathcal{K}$ such that the right extension of G along itself exists, and let R be the monad generated by G. The universal 2-cell $R \circ G \Rightarrow G$ is an action of this monad on G. We say that G is *monadic* if R exists, the Eilenberg-Moore object of R exists, and the comparison $c_G : L \to \mathrm{Alg}(R)$ is an *equivalence.*

2.11. Lemma. *Let $G\colon L \to K$ be a monadic arrow. Then G has a left adjoint and the monad generated by the adjunction is canonically isomorphic to the monad generated by G.* □.

2.12. Lemma. *An opmonoidal functor $G\colon \mathcal{D} \to \mathcal{C}$ has a left adjoint if and only if it is strong and the functor G has a left adjoint in* Cat. *Furthermore, G is an equivalence in* OpMon *if an only if it is strong and G is an equivalence in* Cat.

Proof. See [Kelly (1974a), Theorem 1.5]. ■

2.13. Proposition. *An opmonoidal functor $G\colon \mathcal{D} \to \mathcal{C}$ is monadic in* OpMon *if and only if*

(i) The functor G is monadic in Cat*;*

(ii) The opmonoidal functor G is strong.

Proof. Suppose G is monadic. Then by Lemma 2.11, G has a left adjoint in OpMon, and by Lemma 2.12, G is strong. By Proposition 2.9, the functor G is monadic in Cat.

Conversely, suppose the conditions hold. Then by (ii) and Lemma 2.12, G has a left adjoint in OpMon. Thus by Lemma 2.7, the monad in Cat generated by the functor G is equal to the underlying functor of the monad R generated by the opmonoidal functor G. It follows that the underlying functor of the comparison $c\colon \mathcal{D} \to \mathrm{Alg}(R)$ in OpMon is the comparison in Cat. By (i) this functor is an equivalence. Since the universal $\mathrm{Alg}(R) \to \mathcal{C}$

in Cat reflects isomorphisms and G is strong, the comparison is strong. Thus Lemma 2.12, the comparison is an equivalence in OpMon. ∎

Of course the monadic arrows in Cat are characterized in elementary terms by Beck's Theorem [Mac Lane (1971), Chapter VI]. We now return to the motivating proposition.

2.14. Proposition. *Suppose S is an opmonoidal monad. Then the categories* Coalg(Alg(S)) *and* Alg(Coalg(S)) *are canonically isomorphic.*

Proof. This follows immediately from the fact that representable 2-functors preserve Eilenberg-Moore and the observation that for any monoidal category $\mathcal{D}$, the category Coalg($\mathcal{D}$) is canonically isomorphic to the category OpMon$(1, \mathcal{D})$ of opmonoidal functors from the terminal monoidal category. ∎

We now consider the corresponding proposition in the braided case. Recall that if $F\colon \mathcal{C} \to \mathcal{D}$ is an opmonoidal functor between braided monoidal categories, then F is said to be *braided* if for all pairs of objects X and Y of $\mathcal{C}$ the following diagram commutes.

$$\begin{array}{ccc} F(X \otimes Y) & \xrightarrow{\chi} & FX \otimes FY \\ {\scriptstyle Fc}\downarrow & & \downarrow{\scriptstyle c} \\ F(Y \otimes X) & \xrightarrow[\chi]{} & FY \otimes FX \end{array}$$

A *braided transformation* of braided opmonoidal functors is a transformation of opmonoidal functors. With the evident compositions there is a 2-category BrOpMon whose objects are braided monoidal categories, whose arrows are braided opmonoidal functors and whose 2-cells are braided transformations. Again, by [Kelly (1974b), Section 10.8], the 2-category BrOpMon is isomorphic to the 2-category $\mathrm{Alg}_c(T)$ of algebras, colax morphisms and transformations for some 2-monad T on Cat.

2.15. Example. Braided Hopf monads. A monad in the 2-category BrOpMon is exactly a braided Hopf monad.

Following Example 2.15, we suggest the more descriptive term *braided opmonoidal monad* for braided Hopf monad.

2.16. Proposition. *The 2-category* BrOpMon *admits Eilenberg-Moore objects and the forgetful 2-functor $\mathcal{U}$*: BrOpMon → Cat *preserves them.* □

2.17. Proposition. *A braided opmonoidal functor $G\colon \mathcal{D} \to \mathcal{C}$ is monadic in* BrOpMon *if and only if G is monadic as an opmonoidal functor.* □

Suppose C is a coalgebra in a braided monoidal category $\mathcal{D}$. We say that C is *cocommutative* if the following diagram commutes.

$$\begin{array}{ccccc} & & C & & \\ & {\scriptstyle \delta}\swarrow & & \searrow{\scriptstyle \delta} & \\ C \otimes C & & \xrightarrow{\quad c \quad} & & C \otimes C \end{array}$$

Define CocomCoalg($\mathcal{D}$) to be the full sub-category of Coalg($\mathcal{D}$) consisting of the cocommutative coalgebras.

2.18. Proposition. *Suppose S is a braided opmonoidal monad. Then the categories* CocomCoalg(Alg(S)) *and* Alg(CocomCoalg(S)) *are canonically isomorphic.*

Proof. This follows immediately from the fact that representable 2-functors preserve Eilenberg-Moore objects and the observation that for any braided monoidal category D the category CocomCoalg($\mathcal{D}$) is canonically isomorphic to the category BrOpMon$(1, \mathcal{D})$ of opmonoidal functors from the terminal braided monoidal category. ∎

3. Multicategories

In this section we analyze Proposition 1.1. The main tool used in this section will be multicategories and their morphism. The symbol $\mathcal{C}$ will always denote a *strict* monoidal category.

Recall from Example 2.4 that a multicategory is a monad in the bicategory $\mathrm{Span}_T(\mathrm{Set})$ where T is the cartesian free monoid monad on the category Set of sets. We now make explicit this definition. A multicategory M has a set of objects, and for each sequence $\langle X_1, \ldots, X_n\rangle$ of objects and each object X there is a set $M(\langle X_1, \ldots, X_n\rangle, X)$ of arrows. Here n may be 0, and in this case we write $\emptyset$ for the sequence $\langle X_1, \ldots, X_n\rangle$. We write elements of $M(\langle X_1, \ldots, X_n\rangle, X)$ as $f\colon \langle X_1, \ldots, X_n\rangle \to X$. There is a specified arrow $1\colon \langle X\rangle \to X$, and a composition operation taking an arrow $f\colon \langle X_1, \ldots, X_n\rangle \to X$ and a sequence

$$f_1\colon \langle X_{11}, \ldots, X_{1m_1}\rangle \to X_1, \ldots, f_n\colon \langle X_{n1}, \ldots, X_{nm_n}\rangle \to X_n$$

of arrows, to an arrow

$$f\langle f_1, \ldots, f_n\rangle\colon \langle X_{11}, \ldots, X_{nm_n}\rangle \to X$$

subject to axioms expressing that composition is associative and unital.

3.1. Example. Monoidal categories. It is well known that any monoidal category $\mathcal{D}$ gives rise to a multicategory $M_{\mathcal{D}}$. The objects of $M_{\mathcal{D}}$ are those of $\mathcal{D}$, for a sequence $\langle X_1, \ldots, X_n\rangle$ of objects and an object X the set $M_{\mathcal{D}}(\langle X_1, \ldots, X_n\rangle, X)$ is the set $\mathcal{D}(X_1 \otimes \cdots \otimes X_n, X)$, and the set $M_{\mathcal{D}}(\emptyset, X)$ is the set $\mathcal{D}(I, X)$. Here, the object $X_1 \otimes \cdots \otimes X_n$ denotes the tensor product of these objects with all the brackets to the left. Of course, other bracketings give rise to canonically isomorphic objects. The compositions and units are evident.

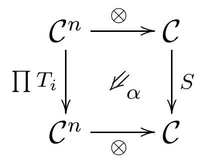
3.2. Example. Categories over $\mathcal{C}$. Let $\mathcal{D}$ be any monoidal category, and A a monoid in $\mathcal{D}$. Recall that the slice category $\mathcal{D}/A$ is again monoidal. The objects are pairs $(X, f\colon X \to A)$, and an arrow $g\colon (X, f) \to (X', f')$ is an arrow $g\colon X \to X'$ in $\mathcal{D}$ such that $f'g = f$. Given objects (X, f) and (X', f') of $\mathcal{D}/M$, define $(X, f) \otimes (X', f')$ to be $(X \otimes X', \mu \circ (f \otimes f'))$. This is the value at the pair $((X, f), (X', f'))$ of a functor

$\otimes\colon \mathcal{D}/A \times \mathcal{D}/A \to \mathcal{D}/A$ which makes $\mathcal{D}/A$ a monoidal category; the unit object is the pair $(I, \eta\colon I \to M)$ and the associativity and unit isomorphism are those of $\mathcal{D}$.

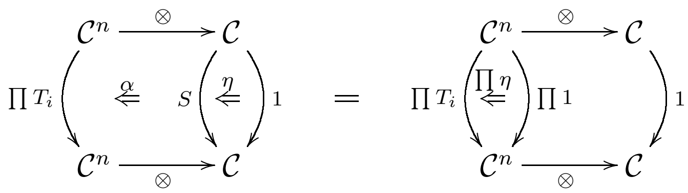
Now suppose $\mathcal{C}$ is a *strict* monoidal category. Thus $\mathcal{C}$ is a monoid in Cat, and so by the above paragraph there is a monoidal category $\mathrm{Cat}/\mathcal{C}$. Let $\mathrm{Cat}/\mathcal{C}$ also denote the multicategory $M_{\mathrm{Cat}/\mathcal{C}}$ as described in Example 3.1.

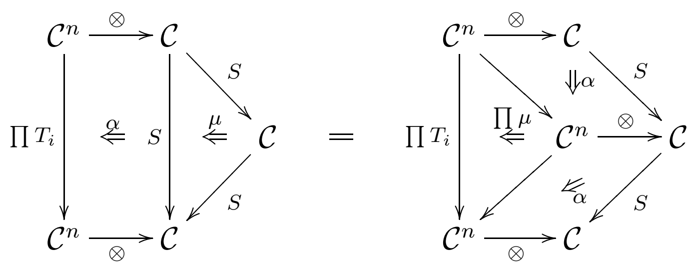
Given a multicategory M, there is a category M_0 which has the same set of objects as M and for each pair X and Y of objects of M_0 the homset $M_0(X,Y)$ is $M(\langle X\rangle, Y)$. Compositions and units are inherited from M. We call M_0 the *underlying category* of M. Note that the underlying category of a multicategory of the form $M_{\mathcal{D}}$ for a monoidal category $\mathcal{D}$, is the underlying category of $\mathcal{D}$.

3.3. Example. Monads on $\mathcal{C}$. Let $\mathcal{C}$ be a strict monoidal category and write $\otimes\colon \mathcal{C}^n \to \mathcal{C}$ for the functor whose value on an n-tuple $X_1, \ldots, X_n$ of objects of $\mathcal{C}$ is the object $X_1 \otimes \cdots \otimes X_n$. We interpret $\otimes\colon \mathcal{C}^0 \to \mathcal{C}$ as the functor $I\colon 1 \to \mathcal{C}$, and we interpret $\otimes\colon \mathcal{C}^1 \to \mathcal{C}$ as the identity functor. We shall now define an multicategory $\mathrm{Mnd}(\mathcal{C})$ whose underlying category is the usual category $\mathrm{Mnd}(\mathcal{C})$. The objects of $\mathrm{Mnd}(\mathcal{C})$ are monads on $\mathcal{C}$. For a sequence $\langle T_1, \ldots, T_n\rangle$ of monads and a monad S, the set $\mathrm{Mnd}(\langle T_1, \ldots, T_n\rangle, S)$ is the set of natural transformations



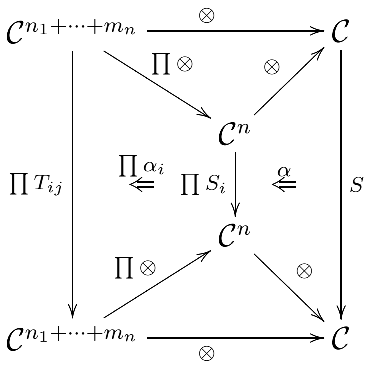
satisfying the following two equations.





The unit arrow $1\colon \langle S\rangle \to S$ is the identity natural transformation. If $\alpha_1\colon \langle T_{11}, \ldots, T_{1m_1}\rangle \to S_1, \ldots, \alpha_n\colon \langle T_{n1}, \ldots, T_{nm_n}\rangle \to S_n$, and $\alpha\colon \langle S_1, \ldots, S_n\rangle \to S$ are arrows then $\alpha\langle\alpha_1, \ldots, \alpha_n\rangle$

is the following pasted composite.



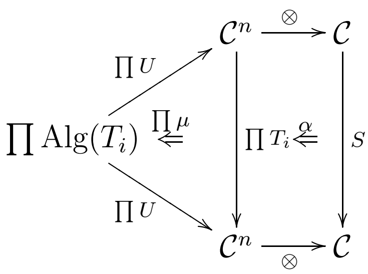
One may show that this is an arrow in $\mathrm{Mnd}(\mathcal{C})$ and that this composition is associative and unital.

For multicategories M and N, a *multicategory morphism* $F\colon M \to N$ consists of a function F from the set of objects of M to the set of objects of N along with a family of functions

$$F\colon M(\langle X_1, \ldots, X_n\rangle, X) \to N(\langle FX_1, \ldots, FX_n\rangle, FX)$$

which respect the composition and units. Given a multicategory morphism $F\colon M \to N$ there is an evident underlying functor $F_0\colon M_0 \to N_0$ between the corresponding underlying categories.

3.4. Example. Semantics. In this example we shall define a multicategory morphism $\mathrm{Alg}\colon \mathrm{Mnd}(\mathcal{C}) \to \mathrm{Cat}/\mathcal{C}$ whose underlying functor is the usual *semantics* functor [Street (1972), Section 2]. For an object S of $\mathrm{Mnd}(\mathcal{C})$, the object $\mathrm{Alg}(S)$ is defined to be the pair $(\mathrm{Alg}(S), U\colon \mathrm{Alg}(S) \to \mathcal{C})$. Suppose $\alpha\colon \langle T_1, \ldots, T_n\rangle \to S$ is an arrow in $\mathrm{Mnd}(\mathcal{C})$. It is not difficult to show that the 2-cell



makes

$$\prod \mathrm{Alg}(T_i) \xrightarrow{\prod U} \mathcal{C}^n \xrightarrow{\otimes} \mathcal{C}$$

into an $\prod \mathrm{Alg}(T_i)$-based S-algebra. It follows by the universal property of $\mathrm{Alg}(S)$ that there is a unique functor $\mathrm{Alg}(\alpha)\colon \prod \mathrm{Alg}(T_i) \to \mathrm{Alg}(S)$ such that the above natural transformation is equal to the following diagram.

$$\begin{array}{ccccc} & & & \mathcal{C} & \\ & & {\scriptstyle U}\nearrow & \Downarrow & \searrow{\scriptstyle S} \\ \prod \mathrm{Alg}(T_i) & \xrightarrow{\mathrm{Alg}(\alpha)} & \mathrm{Alg}(S) & \xrightarrow[U]{\qquad\qquad} & \mathcal{C} \end{array}$$

In particular,

$$\begin{array}{ccc} \prod \mathrm{Alg}(T_i) & \xrightarrow{\mathrm{Alg}(\alpha)} & \mathrm{Alg}(S) \\ {\scriptstyle \prod U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathcal{C}^n & \xrightarrow[\otimes]{\qquad} & \mathcal{C} \end{array}$$

commutes, so that $\mathrm{Alg}(\alpha)\colon \langle \mathrm{Alg}(T_1), \ldots, \mathrm{Alg}(T_n)\rangle \to \mathrm{Alg}(S)$ is a morphism in $\mathrm{Cat}/\mathcal{C}$. This assignment respects compositions and units, and so indeed defines a multicategory morphism $\mathrm{Alg}\colon \mathrm{Mnd}(\mathcal{C}) \to \mathrm{Cat}/\mathcal{C}$. Clearly the underlying functor of this morphism is the usual semantics functor.

Suppose M is any multicategory. A *monoid* in M is an object A of M equipped with arrows $\mu\colon \langle A, A\rangle \to A$ and $\eta\colon \emptyset \to A$, called the *multiplication* and *unit* respectively, such that the following three equations hold.

$$\begin{array}{rcl} \mu\langle 1, \mu\rangle & = & \mu\langle \mu, 1\rangle \\ \mu\langle 1, \eta\rangle & = & 1 \\ \mu\langle \eta, 1\rangle & = & 1 \end{array}$$

If A and B are monoids in M, then a monoid morphism from A to B is an arrow $f\colon \langle A\rangle \to B$ such that $f\langle\mu\rangle = \mu\langle f, f\rangle$. With the evident compositions there is a category $\mathrm{Mon}(M)$ and a forgetful functor $U\colon \mathrm{Mon}(M) \to M_0$ into the underlying category of M.

3.5. Example. Monoids in a monoidal category. For a monoidal category $\mathcal{D}$, the category of monoids $\mathrm{Mon}(M_{\mathcal{D}})$ in the multicategory $M_{\mathcal{D}}$ is the usual category of monoids $\mathrm{Mon}(\mathcal{D})$ in $\mathcal{D}$ and the forgetful functor $U\colon \mathrm{Mon}(M_{\mathcal{D}}) \to (M_{\mathcal{D}})_0$ is the usual forgetful functor $U\colon \mathrm{Mon}(\mathcal{D}) \to \mathcal{D}$.

3.6. Example. Strict monoidal categories over $\mathcal{C}$. By Example 3.5, a monoid in the multicategory $\mathrm{Cat}/\mathcal{C}$ is exactly a monoid in the monoidal category $\mathrm{Cat}/\mathcal{C}$. This is exactly a strict monoidal category A with a strict monoidal functor $U\colon A \to \mathcal{C}$. A monoid morphism is a strict monoidal morphism. When A is of the form $\mathrm{Alg}(T)$ for some monad on $\mathcal{C}$ this amounts to a *lifting* of the monoidal structure of $\mathcal{C}$ to $\mathrm{Alg}(T)$.

3.7. Example. Opmonoidal monads. A monoid in the multicategory $\mathrm{Mnd}(\mathcal{C})$ is exactly an opmonoidal monad on $\mathcal{C}$ and a monoid morphism is exactly a morphism of opmonoidal monads. In order to see this, observe that for a monad S on $\mathcal{C}$, in order to give a multiplication $\mu\colon \langle S,S\rangle \to S$ and a unit $\eta\colon \emptyset \to S$ is to give natural transformations as in the following diagrams.

$$\begin{array}{ccc} \mathcal{C}\times\mathcal{C} & \xrightarrow{\otimes} & \mathcal{C} \\ {\scriptstyle S\times S}\downarrow & \swarrow_{\chi} & \downarrow{\scriptstyle S} \\ \mathcal{D}\times\mathcal{D} & \xrightarrow[\otimes]{} & \mathcal{D} \end{array} \qquad\qquad \begin{array}{ccc} 1 & \xrightarrow{I} & \mathcal{C} \\ {\scriptstyle 1}\downarrow & \swarrow_{\iota} & \downarrow{\scriptstyle S} \\ 1 & \xrightarrow[I]{} & \mathcal{D} \end{array}$$

The axioms that express that these natural transformations make S an opmonoidal functor are exactly the associativity and unit axioms for the monoid. The axioms that express that the natural transformations $\mu\colon SS \Rightarrow S$ and $\eta\colon 1 \Rightarrow S$ are opmonoidal transformations are exactly the axioms for $\mu\colon \langle S,S\rangle \to S$ and $\eta\colon \emptyset \to S$ to be arrows in the multicategory $\mathrm{Mnd}(\mathcal{C})$. A similar argument shows that an opmonoidal monad morphism is exactly a monoid morphism.

Suppose $F\colon M \to N$ is a multicategory morphism and A is a monoid in M. Then clearly the object FA of N is canonically a monoid in N, and this is the value at the monoid A of a functor $\mathrm{Mon}(F)\colon \mathrm{Mon}(M) \to \mathrm{Mon}(N)$, and the following diagram commutes.

$$\begin{array}{ccc} \mathrm{Mon}(M) & \xrightarrow{\mathrm{Mon}(F)} & \mathrm{Mon}(N) \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ M_0 & \xrightarrow[F_0]{} & N_0 \end{array}$$

3.8. Example. Liftings of monoidal structures to $\mathrm{Alg}(S)$. By Examples 3.6 and 3.7 and the above comments, we have the following commutative diagram in Cat.

$$\begin{array}{ccc} \mathrm{OpMonMnd}(\mathcal{C}) & \xrightarrow{\mathrm{Alg}} & \mathrm{Mon}(\mathrm{Cat}/\mathcal{C}) \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathrm{Mnd}(\mathcal{C})^{\mathrm{op}} & \xrightarrow[\mathrm{Alg}]{} & \mathrm{Cat}/\mathcal{C} \end{array}$$

This implies that given an opmonoidal structure on a monad S, the category $\mathrm{Alg}(S)$ inherits a monoidal structure from $\mathcal{C}$. Of course, we already stated this fact in Section 1 and in Proposition 2.9. We are interested in the converse of this statement.

We say that a multicategory morphism $F\colon M \to N$ is *fully faithful* when the functions

$$F\colon M(\langle X_1,\ldots,X_n\rangle, X) \to N(\langle FX_1,\ldots,FX_n\rangle, FX)$$

are isomorphisms. We shall show that fully faithful morphisms are *representably fully faithful.*

There is a 2-category Mlt of multicategories [Hermida (1999)]. The objects and arrows of this 2-category have been described above, and we now recall the 2-cells. Suppose F, G :

$M \to N$ are parallel multicategory morphisms. A *transformation* α from F to G consists of a family of arrows $\alpha_X \colon \langle FX \rangle \to GX$ such that for all arrows $h \colon \langle X_1, \ldots, X_n \rangle \to X$ the following equation holds.

$$\alpha_X \langle Fh \rangle = Gh \langle \alpha_{X_1}, \ldots, \alpha_{X_n} \rangle$$

Recall that an arrow $F \colon K \to L$ in a 2-category $\mathcal{K}$ is said to be *representably fully faithful* when for all objects P of $\mathcal{K}$, the functor

$$\mathcal{K}(P, F) \colon \mathcal{K}(P, M) \to \mathcal{K}(P, N)$$

is fully faithful.

3.9. Proposition. *If a multicategory morphism is fully faithful, then it is representably fully faithful.*

Proof. Suppose $H, G \colon P \to M$ are multicategory morphisms and $\alpha \colon FH \to FG$ is a multicategory transformation. Since the function $F \colon M(\langle HX \rangle, GX) \to N(\langle FHX \rangle, FGX)$ is an isomorphism for all X, there exists a unique arrow $\beta_X \colon \langle HX \rangle \to GX$ such that $F\beta_X = \alpha_X$. This is the component at X of the unique multicategory transformation β such that $F\beta = \alpha$. ■

We shall now show that *monoidally fully faithful* functors induce fully faithful morphisms of multicategories. Suppose $F \colon \mathcal{D} \to \mathcal{D}'$ is a monoidal functor. For each sequence $\langle X_1, \ldots, X_n \rangle$ of objects of D inductively define an arrow $\psi \colon FX_1 \otimes \cdots \otimes FX_n \to F(X_1 \otimes \cdots \otimes X_n)$ as follows. For $n = 0, 1, 2$, define the arrow ψ to be $\iota \colon I \to FI$ and $1 \colon FX_1 \to FX_1$ and $\chi \colon FX_1 \otimes FX_2 \to F(X_1 \otimes X_2)$ respectively. For $n > 2$, define ψ to be the following composite.

$$FX_1 \otimes \cdots \otimes FX_{n-1} \otimes FX_n \xrightarrow{\psi \otimes 1} F(X_1 \otimes \cdots \otimes X_{n-1}) \otimes FX_n \xrightarrow{\chi} F(X_1 \otimes \cdots \otimes X_n)$$

Composition with this arrow and F induces a function

$$M_F \colon M_{\mathcal{D}}(\langle X_1, \ldots, X_n \rangle, X) \to M_{\mathcal{D}'}(\langle FX_1, \ldots, FX_n \rangle, FX)$$

making M_F a morphism of multicategories. Similarly, If $\alpha \colon F \Rightarrow G : \mathcal{D} \to \mathcal{D}'$ is a monoidal natural transformation, then for all objects X of $\mathcal{D}$, the arrow $\alpha \colon FX \to GX$ in $\mathcal{D}'$ is an arrow $\alpha \colon \langle FX \rangle \to GX$ in $M_{\mathcal{D}'}$ which is the component at X of a multicategory transformation $M_\alpha \colon M_F \Rightarrow M_G$. This assignment preserves compositions and so defines a 2-functor $M_{(\ )} \colon \mathrm{MonCat} \to \mathrm{Mlt}$.

A monoidal functor $F \colon \mathcal{D} \to \mathcal{D}'$ is said to be *monoidally fully faithful* when it is fully faithful, and the functions

$$\mathcal{D}'(\chi, FX) \colon \mathcal{D}'(F(X \otimes Y), FZ) \to \mathcal{D}'(FX \otimes FY, FZ)$$

$$\mathcal{D}'(\iota, FX) \colon \mathcal{D}'(FI, FZ) \to \mathcal{D}'(I, FZ)$$

are isomorphisms for all objects X, Y and Z of $\mathcal{D}$ [McCrudden (1999), Section 3.2]. Monoidally fully faithful functors are representably fully faithful [McCrudden (1999), Proposition 3.2.1].

3.10. Proposition. *A monoidal functor $F \colon \mathcal{D} \to \mathcal{D}'$ is monoidally fully faithful, if and only if the multicategory morphism $M_F \colon M_{\mathcal{D}} \to M_{\mathcal{D}'}$ is fully faithful.* □

3.11. Proposition. *Suppose a multicategory morphism $F \colon M \to N$ is fully faithful. The the following diagram is a pullback in the category of categories and functors.*

$$\begin{array}{ccc} \mathrm{Mon}(M) & \xrightarrow{\mathrm{Mon}(F)} & \mathrm{Mon}(N) \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ M_0 & \xrightarrow[F_0]{} & N_0 \end{array}$$

Proof. Suppose $P \colon A \to M_0$ and $Q \colon A \to \mathrm{Mon}(N)$ are functors such that $F_0 P = UQ$. Then for all objects X of A, the object F_0PX is equipped with a multiplication $\mu \colon \langle F_0PX, F_0PX \rangle \to F_0PX$ and a unit $\emptyset \to F_0PX$, making F_0PX a monoid in N. Since the functions

$$F_0 \colon M(\langle PX, PX \rangle, PX) \to N(\langle F_0PX, F_0PX \rangle, F_0PX)$$

$$F_0 \colon M(\emptyset, PX) \to N(\emptyset, F_0PX)$$

are isomorphisms, there are arrows $\mu \colon \langle PX, PX \rangle \to PX$ and $\eta \colon \emptyset \to PX$ which are unique with the property that $F_0\mu = \mu$ and $F_0\eta = \eta$ respectively. This defines the object function of a functor $R \colon A \to \mathrm{Mon}(M)$ which is unique with the desired property. ■

We defer the proof of the following theorem.

3.12. Theorem. *The multicategory morphism* $\mathrm{Alg} \colon \mathrm{Mnd}(\mathcal{C}) \to \mathrm{Cat}/\mathcal{C}$ *is fully faithful.*

3.13. Corollary. *There is a bijection between opmonoidal structures on a monad S on $\mathcal{C}$ and liftings of the monoidal structure on $\mathcal{C}$ to* $\mathrm{Alg}(S)$.

Proof. This follows immediately from Theorems 3.12 and 3.11, and Example 3.8. ■

We now extend this bijection to an equivalence between the category of opmonoidal monads and certain Eilenberg-Moore objects. Underlying this is the equivalence of structure and semantics [Street (1972), Section 2]. Let $\mathrm{EM}(\mathrm{Cat}/\mathcal{C})$ denote the full sub-multicategory of $\mathrm{Cat}/\mathcal{C}$ consisting of the Eilenberg-Moore objects. Observe that the tensor product in $\mathrm{Cat}/\mathcal{C}$ of Eilenberg-Moore objects is not necessarily and Eilenberg-Moore object.

3.14. Theorem. *The morphism* $\mathrm{Alg} \colon \mathrm{Mnd}(\mathcal{C}) \to \mathrm{Cat}/\mathcal{C}$ *of multicategories factors through the sub-multicategory* $\mathrm{EM}(\mathrm{Cat}/\mathcal{C})$ *and the morphism*

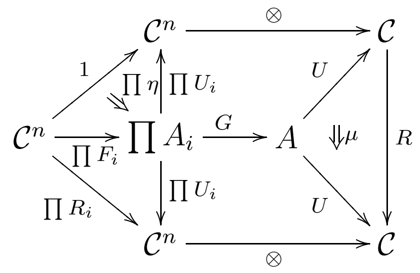
$$\mathrm{Alg} \colon \mathrm{Mnd}(\mathcal{C}) \to \mathrm{EM}(\mathrm{Cat}/\mathcal{C})$$

of multicategories is an equivalence.

Proof. For an object (A, U) of $\mathrm{EM}(\mathrm{Cat}/\mathcal{C})$, define a monad $\mathrm{Str}(A, U)$ to be the monad generated by U and its left adjoint. Now suppose

$$G\colon \langle (A_1, U_1), \ldots, (A_n, U_n) \rangle \to (A, U)$$

is an arrow in $\mathrm{EM}(\mathrm{Cat}/\mathcal{C})$. Let F denote the left adjoint of U and F_i denote the left adjoint of U_i, and let R and R_i denote the monads so induced. Let $\mathrm{Str}(G)$ denote the following 2-cell.



This is an arrow $\langle R_1, \ldots, R_n \rangle \to R$ in $\mathrm{Mnd}(\mathcal{C})$. This assignment preserves composition and so defines a morphism of multicategories $\mathrm{Str}\colon \mathrm{EM}(\mathrm{Cat}/\mathcal{C}) \to \mathrm{Mnd}(\mathcal{C})$. For any object S of $\mathrm{Mnd}(\mathcal{C})$, there is a canonical invertible arrow $\mathrm{Str}(\mathrm{Alg}(S)) \to S$ in $\mathrm{Mnd}(\mathcal{C})$ which is the component of an invertible transformation of multicategories $\varepsilon\colon \mathrm{StrAlg} \Rightarrow 1$. Also, for any object (A, U) of $\mathrm{EM}(\mathrm{Cat}/\mathcal{C})$, the comparison functor $A \Rightarrow \mathrm{Alg}(\mathrm{Str}(A, U))$ is an arrow in $\mathrm{EM}(\mathrm{Cat}/\mathcal{C})$ which is the component of an invertible transformation of multicategories $\eta\colon 1 \to \mathrm{AlgStr}$. Finally, the triangular identities hold, so that Alg has a left adjoint with invertible unit and counit. ■

It is now straightforward to prove Theorem 3.12 using Theorem 3.14.

Recall that given a morphism of multicategories $F\colon M \to N$, there is an induced functor $\mathrm{Mon}(F)\colon \mathrm{Mon}(M) \to \mathrm{Mon}(N)$. Similarly, any multicategory transformation $\alpha\colon F \Rightarrow G$ induces a natural transformation $\mathrm{Mon}(\alpha)\colon \mathrm{Mon}(F) \Rightarrow \mathrm{Mon}(G)$. This assignment preserves compositions and identities, and so defines a 2-functor $\mathrm{Mon}\colon \mathrm{Mlt} \to \mathrm{Cat}$. Since 2-functors preserve equivalences, we have the following corollary as an immediate consequence of Theorem 3.14.

3.15. Corollary. *Formation of the categories of algebras induces a equivalence between the category* $\mathrm{OpMonMnd}(\mathcal{C})$ *of opmonoidal monads on* $\mathcal{C}$ *and the category* $\mathrm{Mon}(\mathrm{EM}(\mathrm{Cat}/\mathcal{C}))$. □

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