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ON BRANCHED COVERS IN TOPOS THEORY

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ABSTRACT. We present some new findings concerning branched covers in topos theory. Our discussion involves a particular subtopos of a given topos that can be described as the smallest subtopos closed under small coproducts in the including topos. Our main result is a description of the covers of this subtopos as a category of fractions of branched covers, in the sense of Fox [10], of the including topos. We also have some new results concerning the general theory of KZ-doctrines, such as the closure under composition of discrete fibrations for a KZ-doctrine, in the sense of Bunge and Funk [6].

Introduction

The notion of a *branched cover* is essentially due to Riemann. Since his time, branched covers have found applications in knot theory [9, 21], in the study of 3-manifolds [17], and in algebraic geometry [22], for example. A development that is of particular interest to this paper came when Fox [10] introduced *complete spreads*, which he used to provide a topological formulation of branched covers. More recently, it has been shown [4] that complete spreads have a natural definition in topos theory. In particular, the theory of complete spreads in topos theory ought to provide a basis for an investigation of branched covers in topos theory, which is the purpose of this paper.

We shall adopt Lawvere's proposal and refer to a cocontinuous functor from one topos into another as a *distribution* [15, 16]. The canonical correspondence of distributions with complete spreads [4] has a direct influence on our investigation of branched covers. Indeed, our study of branched covers will involve a certain subtopos of a given topos that has an interesting interpretation in terms of distributions; we shall show that every locally connected topos has a smallest subtopos containing those sheaves that are a *density of a distribution*, in the terminology of [5]. A sheaf that is such a density might be thought of as a 'Lebesgue-measurable' sheaf so we have nicknamed this topos the Lebesgue subtopos. This subtopos can also be described as the smallest for which (the direct image functor of) its inclusion preserves small coproducts.

Our main goal will be to explain a relationship that exists between the branched covers of a given locally connected topos and the covers of its Lebesgue subtopos. Specifically, we shall show that the category of covers of the Lebesgue subtopos is equivalent to a category of fractions of a certain category of *non-surjective* branched covers of the given topos. We also have other new results that we shall explain in the course of our investigation, such as the closure under composition of discrete fibrations for a KZ-doctrine. We are thus

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$$\begin{tikzcd} \mathcal{D} \arrow[r, tail, "\nu"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\cong"] & \mathcal{S}^{\mathbf{D}^{\mathrm{op}}} \arrow[d] \\ \mathcal{E} \arrow[r, tail] & \mathcal{S}^{\mathbf{C}^{\mathrm{op}}} \end{tikzcd}$$

Figure 1: Complete spread as a 'display topos'.

able to clarify the closure of complete spreads under composition in terms of the general theory of KZ-doctrines, since from [6] we know that a complete spread may be equivalently regarded as a discrete fibration for the symmetric monad [2].

We shall begin in the next section with a brief introduction to complete spreads in topos theory, based on the results of [4, 5, 6]. For details concerning locally connected toposes we refer the reader to [1].

1. Review of complete spreads

We recall [10] that a continuous mapping $D \xrightarrow{\varphi} E$ of topological spaces, where D is locally connected, is a *spread* if the connected components of the open sets $\varphi^{-1}U$, for U open in E, are a base for D. Given a point x of E, if a *consistent* selection of components $\{c_U \subseteq \varphi^{-1}U \mid x \in U\}$ is one for which $U \subseteq U' \Rightarrow c_U \subseteq c_{U'}$, then a spread is *complete* if for every x and every such consistent choice of components $\{c_U\}$, the set $\bigcap_U c_U$ is non-vacuous (and hence, if D satisfies the T_1 separation axiom, equal to a singleton).

Let us next recall [4, 5] two equivalent formulations of complete spread for geometric morphisms that we shall use in this paper. We shall express the first formulation as a definition, and the second as a theorem.

1.1. DEFINITION. [4]. *Consider a geometric morphism $\mathcal{D} \xrightarrow{\varphi} \mathcal{E}$ with locally connected domain, and consider Figure 1. The small category $\mathbf{D}$ has as its objects all pairs (c, α), where c is an object of a site $\mathbf{C}$ for $\mathcal{E}$ and α is a component of $\varphi^* c$. The geometric morphism ν satisfies $\nu^*(c, \alpha) = \alpha$, and the presheaf geometric morphism, depicted vertically in Figure 1, is induced by the functor $(c, \alpha) \mapsto c$. Then φ is a* complete spread *if this square is a topos pullback.*

The second formulation of complete spread (Theorem 1.2 below) that we shall use is expressed in terms of the theory of *KZ-doctrines* [13] and the *symmetric monad* [2]. (Without going into details, a KZ-doctrine is a certain kind of 2-dimensional monad.) We recall [3, 4] that the symmetric monad is a KZ-doctrine in toposes that are bounded over a base topos $\mathcal{S}$. In accordance with the notation of [4], we shall denote this KZ-doctrine by $< M, \delta, \mu >$. For any topos $\mathcal{A}$, the unit $\mathcal{A} \xrightarrow{\delta} M\mathcal{A}$ is an $\mathcal{S}$-essential geometric morphism, i.e., there is a left adjoint $\delta_! \dashv \delta^*$ over $\mathcal{S}$. The symmetric topos classifies distributions in the sense that for a topos $\mathcal{A}$ over $\mathcal{S}$, the functor that carries a geometric morphism $\mathcal{X} \xrightarrow{p} M\mathcal{A}$ to the distribution $\mathcal{A} \xrightarrow{p^* \cdot \delta_!} \mathcal{X}$ is a natural equivalence. Consider that the bicomma object in Figure 2 defines *a discrete fibration φ for the symmetric monad*, in the terminology of [6].

1.2. Theorem. [4]. *A geometric morphism $\mathcal{D} \xrightarrow{\varphi} \mathcal{E}$ over $\mathcal{S}$ is a complete spread if and only if it is a discrete fibration for the symmetric monad. I.e., φ is a complete spread if and only if there is a point p of $M\mathcal{E}$ so that φ appears in a bicomma object in toposes over $\mathcal{S}$, as in Figure 2.*

$$\begin{tikzcd} \mathcal{D} \arrow[r, "d"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{S} \arrow[d, "p"] \\ \mathcal{E} \arrow[r, tail, "\delta"'] & M\mathcal{E} \end{tikzcd}$$

Figure 2: Complete spread as a discrete fibration.

It is appropriate to refer to a point of the symmetric topos as the point associated with, or as the point corresponding to a given complete spread, for these categories of points and of complete spreads are equivalent [4]. Also, it follows [6] that the domain topos $\mathcal{D}$ of a complete spread φ is locally connected, i.e., that the structure geometric morphism $\mathcal{D} \xrightarrow{d} \mathcal{S}$ is $\mathcal{S}$-essential (we write $d_! \dashv d^*$).

We conclude this section by indicating a connection between complete spreads in the sense of geometric morphisms and Fox's complete spreads, which we described in the first paragraph of this section. Consider that the universal property of the bicomma object for a complete spread $\mathcal{D} \xrightarrow{\varphi} \mathcal{E}$ associated with a point p of the symmetric topos $M\mathcal{E}$, or equivalently, with a distribution λ in $\mathcal{E}$, equates with a point of $\mathcal{D}$ a point q of $\mathcal{E}$ and a natural transformation $q^* \cdot \delta^* \to p^*$. This transposes to $q^* \xrightarrow{t} p^* \cdot \delta_!$. The given λ is isomorphic to $p^* \cdot \delta_! \cong d_! \cdot \varphi^*$. Thus, we may think of $\mathcal{D}$ as the 'space' of pairs $(q, q^* \xrightarrow{t} \lambda)$. When $\mathcal{E}$ is a localic topos $Sh(X)$, so is $\mathcal{D} = Sh(D)$. In particular, if X is a topological space, then the points of D correspond (if X is sober) to pairs consisting of an element x of X and a natural transformation $x^* \xrightarrow{t} \lambda$, where $Sh(X) \xrightarrow{x^*} \mathcal{S}$ is the point-distribution determined by x. The points of a locale carry in a natural way a topology. In the case of D, the open sets of this topology,

$$(U, \alpha) = \{(x, t) \mid x \in U \text{ and } t_U = \alpha\};\ U \text{ open in } X,\ \alpha \in \lambda U\ ,$$

describe the *display space* [11] of λ. The display space illustrates a link with the topological notion of complete spread since a natural transformation $x^* \xrightarrow{t} \lambda$ amounts to a consistent choice $\{t_U \in \lambda U \cong d_! \varphi^* U\}$ of components of inverse images under φ of neighbourhoods U of x. When X is a complete metric space, D is always spatial. It has been shown [11] that over a complete metric space X, the category of complete spreads in Fox's sense (with locally connected and T_1 domain) is equivalent to the category of complete spreads over $Sh(X)$ in the sense of Definition 1.1.

2. The Lebesgue subtopos

Fox requires that the non-singular part of a branched covering space be "locally connected in" the base space. This means that the intersection of a connected open with the non-singular part is still connected. (We shall insist that connected opens are non-empty, so that a subspace that is locally connected in the whole space is dense.) More generally, let us refer to a map of locally connected topological spaces as *pure* if the inverse image (under the map) of a connected open set remains connected. According to our convention, such maps are dense. The terminology persists for geometric morphisms; a *pure* (*dense*) [4] geometric morphism $\mathcal{G} \xrightarrow{p} \mathcal{D}$ between locally connected toposes over $\mathcal{S}$ is one for which $g_! \cdot p^* \cong d_!$, where $g_!$ and $d_!$ are the connected components functors. (We have no particular need to consider dense maps separately, so we can safely drop "dense" from the terminology, and refer to such a geometric morphism just as *pure*, as in [12].)

Let us recall some definitions and facts from [6]. Let $< M, \delta, \mu >$ denote a KZ-doctrine in a 2-category. We shall say that a 1-cell f *admits an M-adjoint* ("is admissible for M" in [6]) if Mf admits a right adjoint. If f admits an M-adjoint, then we denote the right adjoint by r_f. In the case of the symmetric monad, a geometric morphism over a base topos $\mathcal{S}$ admits an M-adjoint if and only if it is $\mathcal{S}$-essential. We also recall that a 1-cell $G \xrightarrow{p} D$ is *final for a KZ-doctrine M* if the canonical 2-cell $Mp \cdot r_g \Rightarrow r_d$ is an isomorphism, where $G \xrightarrow{g} T$ is the essentially unique 1-cell to the terminal object T, and where $Mg \dashv r_g$. Here it is assumed that g and d admit an M-adjoint. Then a geometric morphism between locally connected toposes is pure if and only if it is final for the symmetric monad.

For the following proposition, we shall assume that a KZ-doctrine $< M, \delta, \mu >$ is, in the terminology of [6], *locally fully faithful*. I.e., we shall assume that the functor M is locally fully faithful, which is equivalent to assuming that the units of the KZ-doctrine are fully faithful 1-cells.

2.1. Proposition. *Let $< M, \delta, \mu >$ be a locally fully faithful KZ-doctrine. Let $G \xrightarrow{p} D$ be a 1-cell, and assume that $G \xrightarrow{g} T$ and $D \xrightarrow{d} T$ admit an M-adjoint. Then the following two conditions are equivalent.*

1. *The 1-cell p is final, i.e., the canonical 2-cell $Mp \cdot r_g \Rightarrow r_d$ is an isomorphism.*

2. *For every 1-cell $D \xrightarrow{b} X$ and every 'constant' $T \xrightarrow{a} X$, composition with p gives a bijection*

$$\frac{b \Rightarrow a \cdot d}{b \cdot p \Rightarrow a \cdot d \cdot p}$$

of 2-cells, where $D \xrightarrow{d} T$ is the unique 1-cell. (To paraphrase: every constant map perceives every 1-cell to be the left extension of that 1-cell's restriction along p.)

Proof. If *1* holds, then using that M is locally fully faithful, we have the following natural bijections.

$$\frac{\frac{\frac{\frac{\frac{b \Rightarrow a \cdot d}{Mb \Rightarrow Ma \cdot Md}}{Mb \cdot \mathrm{r}_d \Rightarrow Ma}}{Mb \cdot Mp \cdot \mathrm{r}_g \Rightarrow Ma}}{Mb \cdot Mp \Rightarrow Ma \cdot Mg}}{b \cdot p \Rightarrow a \cdot g \cong a \cdot d \cdot p}$$

Conversely, if *2* holds, then there are the following natural bijections. Here, h and k denote arbitrary *M-homomorphisms* (with the appropriate domain and codomain).

$$\begin{array}{c} h \cdot Mp \cdot \mathrm{r}_g \Rightarrow k \\ \hline h \cdot Mp \Rightarrow k \cdot Mg \cong k \cdot Md \cdot Mp \\ \hline h \cdot Mp \cdot \delta_G \Rightarrow k \cdot Md \cdot Mp \cdot \delta_G \\ \hline h \cdot \delta_D \cdot p \Rightarrow k \cdot \delta_T \cdot d \cdot p \\ \hline h \cdot \delta_D \Rightarrow k \cdot \delta_T \cdot d \quad (\text{by } 2) \\ \hline h \cdot \delta_D \Rightarrow k \cdot Md \cdot \delta_D \\ \hline h \Rightarrow k \cdot Md \end{array}$$

This shows that $Md \dashv Mp \cdot \mathrm{r}_g$ as M-homomorphisms, and hence as ordinary 1-cells. I.e., this shows that $Mp \cdot \mathrm{r}_g \cong \mathrm{r}_d$. Note: r_g is an M-homomorphism ([6], 1.3). ∎

Proposition 2.1 reveals that the notion of final for a KZ-doctrine has less to do with the KZ-doctrine than perhaps one might first suppose, for notice that condition *2* makes no mention of a KZ-doctrine. (But the KZ-doctrine enters in the determination of the 'kind' of domain and codomain of a final 1-cell, since for condition *1* they must admit an M-adjoint.) Let us see what condition *2* amounts to for geometric morphisms. Let $\mathcal{G} \xrightarrow{p} \mathcal{D}$ denote a geometric morphism over $\mathcal{S}$, and let us take for X in condition *2* the object classifier $M\mathcal{S}$. Then the stated condition immediately translates to the condition that for every A of $\mathcal{S}$, and every B of $\mathcal{D}$, application of p^* gives a bijection $\mathcal{D}(B, A) \cong \mathcal{G}(p^*B, p^*A)$. This is equivalent to the unit $A \to p_*p^*A$ being an isomorphism for every A of $\mathcal{S}$, and that says that p_* preserves $\mathcal{S}$-coproducts, a condition already known to be equivalent to pure when $\mathcal{G}$ and $\mathcal{D}$ are locally connected [4]. A further refinement that we will need later is that when $\mathcal{G}$ and $\mathcal{D}$ are locally connected, p_* preserves $\mathcal{S}$-coproducts if and only if $\Omega_\mathcal{S} \to p_*p^*\Omega_\mathcal{S}$ is an isomorphism, where $\Omega_\mathcal{S}$ denotes the subobject classifier of the base topos $\mathcal{S}$. In general, this last condition is the definition for pure (dense) [4, 12].

Proposition 2.1 will be used in the proof of Theorem 3.3. Final 1-cells are immediately seen to be closed under composition. Furthermore, if $p \cdot q$ and q are final, then p is final.

We are ready to define and characterize what we shall refer to as the *Lebesgue subtopos* of a given topos. This subtopos is related to branched covers, as we shall see later. We shall be considering *pure morphisms of a topos*, i.e., morphisms $X \to Y$ of a topos $\mathcal{E}$ such that $\mathcal{E}/X \longrightarrow \mathcal{E}/Y$ is a pure geometric morphism. The formulation of pure that stems from "locally connected in," which is given in terms of connected components functors, says that a morphism $X \to Y$ in a locally connected topos $\mathcal{E}$ is pure if and only if for every $Z \to Y$, the projection $X \times_Y Z \to Z$ induces an isomorphism $e_!(X \times_Y Z) \cong e_!(Z)$. On the other hand, a morphism $X \xrightarrow{m} Y$ satisfies condition *2* of Proposition 2.1 if and only if the canonical morphism $Y^*A \to (Y^*A)^m$ is an isomorphism for every constant sheaf A (Y^*A denotes $Y \times A \to Y$). When $\mathcal{E}$ is locally connected, this holds if and only if $Y^*\Omega_{\mathcal{S}} \to (Y^*\Omega_{\mathcal{S}})^m$ is an isomorphism.

The following more concrete site description of a pure morphism in a locally connected topos is available.

2.2. Proposition. *Let* $\mathbf{C} \longrightarrow \mathcal{E}$ *denote a small site for a locally connected topos* $\mathcal{E}$ *such that every object* c *of* $\mathbf{C}$ *is connected* (i.e., such that $e_!c \cong 1$). *Then a morphism* $X \xrightarrow{m} Y$ *is pure if and only if for every 'section'* $c \xrightarrow{s} Y$, *the pullback* m^*c *in* $\mathcal{E}$ *is connected.*

$$\begin{tikzcd} m^*c \arrow[r] \arrow[d] & c \arrow[d, "s"] \\ X \arrow[r, "m"'] & Y \end{tikzcd}$$

Proof. By the remarks above, pure morphisms have the property stated in the proposition. Conversely, if the stated property holds, then for every $c \xrightarrow{s} Y$ and every constant A we have the following natural bijections:

$$\begin{array}{c} Y^*A(c,s) \\ \hline s \to Y^*A \text{ in } \mathcal{E}/Y \\ \hline c \to A \text{ in } \mathcal{E} \\ \hline 1 \cong e_!(c) \to A \\ \hline 1 \cong e_!(m^*c) \to A \\ \hline m^*c \to A \\ \hline m \times s \to Y^*A \\ \hline (Y^*A)^m(c,s)\ . \end{array}$$

We have shown that $Y^*A \to (Y^*A)^m$ is an isomorphism at stage $c \xrightarrow{s} Y$. ∎

Now consider that a monomorphism $S \overset{m}{\hookrightarrow} T$ of a topos $\mathcal{E}$ over $\mathcal{S}$ is pure if and only if $T^*\Omega_{\mathcal{S}} \to (T^*\Omega_{\mathcal{S}})^m$ is an isomorphism. Thus the pure monomorphisms are just, in Paré's terminology ([20], Prop. 18), the monomorphisms that are dense for $\Omega_{\mathcal{S}}$, in the sense of Grothendieck topologies. (Or in other words, $S \hookrightarrow T$ is pure if and only if $\Omega_{\mathcal{S}}$ has the sheaf property for every pullback of $S \hookrightarrow T$.) The class of pure monomorphisms in $\mathcal{E}$ is therefore a topology in $\mathcal{E}$ ([20], Thm. 21); they comprise the largest topology for which the constant $\Omega_{\mathcal{S}}$ is a sheaf. Let us refer to this topology as *the pure topology*. We shall refer to the corresponding subtopos of sheaves as the Lebesgue subtopos, which we shall denote by $\mathcal{L}$. The following is immediate.

2.3. Proposition. *A morphism in a topos that is bidense for the pure topology is pure.*

Pure morphisms are dense in the topological sense: $X \xrightarrow{m} Y$ is dense if and only if $Y^*\Omega_{\mathcal{S}} \to (Y^*\Omega_{\mathcal{S}})^m$ is a monomorphism ([4], 2.2). Thus the Lebesgue subtopos always contains the smallest dense subtopos [18, 14].

We shall need a preliminary lemma before proceeding to the main result of this section, Theorem 2.5. If $\mathcal{D} \xrightarrow{\tau} \mathcal{E}$ is a localic geometric morphism, such as a complete spread, then we may consider the object of $\mathcal{E}$ that consists of the $\mathcal{E}$-points of τ. As in [5] we shall denote this object by $\mathbf{d}\tau$ ($\mathbf{d}$ is for density - see below). We shall sometimes refer to $\mathbf{d}\tau$ as the interior of τ. On the other hand, if X is an object of a locally connected $\mathcal{E}$, then we shall denote the spread completion of $\mathcal{E}/X \longrightarrow \mathcal{E}$ by $\mathcal{D} \xrightarrow{\bar{X}} \mathcal{E}$, referring to $\bar{X}$ as *the closure of* X. Of course, the closure of an object or morphism of a locally connected topos is not necessarily again an object or morphism of the topos. Interior is right adjoint to closure [5].

Lawvere calls a distribution μ on a locally connected topos $\mathcal{E}$ *absolutely continuous* if there is an X in $\mathcal{E}$ such that $\mu(Y) \cong e_!(X \times Y)$ naturally in Y, where $e_! \dashv e^*$. If μ is absolutely continuous, we shall write $\mu \cong X.e_!$. From [5], the distributions that correspond to the complete spreads $\bar{X}$ are precisely the absolutely continuous ones. We have the following.

2.4. Lemma. *The closure of a pure morphism in a locally connected topos is an equivalence geometric morphism.*

Proof. Let $X \overset{\alpha}{\to} Y$ be a pure morphism of $\mathcal{E}$. Then for every $Z \to Y$, there is a canonical isomorphism $e_!(Z) \cong e_!(X \times_Y Z)$. In particular, this holds when $Z \to Y$ is of the form $W \times Y \to Y$, so that $X \times_Y Z \to X$ is $W \times X \to X$. Hence, the distributions of $\bar{X}$ and $\bar{Y}$ are isomorphic, so that $\bar{X} \xrightarrow{\bar{\alpha}} \bar{Y}$ is an equivalence. ∎

Let us adopt the term *regular* for an object X of a locally connected topos for which the canonical morphism $X \to \mathbf{d}\bar{X}$ is an isomorphism.

2.5. Theorem. *Every topos (over a base topos $\mathcal{S}$) has a smallest pure subtopos, the Lebesgue subtopos $\mathcal{L}$. When the including topos is locally connected, we also have the following: $\mathcal{L}$ is the smallest subtopos for which (the direct image functor of) its inclusion preserves $\mathcal{S}$-coproducts, $\mathcal{L}$ is locally connected, an object of the given topos that is the interior of a complete spread* (such as a regular object) *is a sheaf for the pure topology, and moreover, the pure topology is the largest for which such interiors are sheaves.*

Proof. The first assertion is clear from the definitions. Now let $\mathcal{E}$ denote a locally connected topos over $\mathcal{S}$, with inclusion $\mathcal{L} \xrightarrow{i} \mathcal{E}$. In this case, the pure topology can be described as the largest topology in $\mathcal{E}$ for which all constants $A \in \mathcal{S}$ are sheaves. Thus, for every $A \in \mathcal{S}$, the unit $A \to i_* i^* A$ is an isomorphism. This says that i_* preserves $\mathcal{S}$-coproducts, and $\mathcal{L}$ is clearly the smallest such subtopos with this property. By [4], 2.9, $\mathcal{L}$ is locally connected.

Now consider an object $\mathbf{d}\tau$ of $\mathcal{E}$, where τ is a complete spread over $\mathcal{E}$. We wish to show that for a given pure monomorphism $S \hookrightarrow T$, a morphism $S \to \mathbf{d}\tau$ lifts uniquely to one $T \to \mathbf{d}\tau$. Morphisms $T \to \mathbf{d}\tau$ are in bijection with those $\bar{T} \to \tau$, so the result follows from Lemma 2.4. Constant objects are regular, and hence they are interiors of a complete spread. Thus, the last statement of the theorem holds. ∎

$$\mathcal{L} \supseteq \text{densities (=interiors)} \supseteq \text{regular objects} \supseteq \text{complete spread objects} \supseteq \text{locally constant objects}$$

Figure 3: The Lebesgue subtopos $\mathcal{L}$.

We have adopted the name Lebesgue for the subtopos $\mathcal{L}$ since the interior of a complete spread has an interpretation [5] as the *density* of the distribution that is associated with the complete spread, analogous to the Radon-Nikodym derivative of a measure. Thus, a density is a 'Lebesgue-measurable' sheaf, and by Theorem 2.5, these sheaves are in $\mathcal{L}$. Furthermore, the topology for $\mathcal{L}$ is the largest for which these densities are sheaves. Figure 3 displays a chain of inclusions describing $\mathcal{L}$. From [5], a locally constant object X has the property that $\mathcal{E}/X \longrightarrow \mathcal{E}$ is a complete spread, i.e., a locally constant object is a *complete spread object*. Of course a complete spread object is a regular object. The locally constant objects include the constant objects, and in particular, the constant object $\Omega_{\mathcal{S}}$ with which we started.

Lemma 2.6 below is a fact about complete spreads that we will use in Proposition 2.8 and also in the last section on branched covers. For Lemma 2.6 and Proposition 2.8 notice that if $\mathcal{F} \xrightarrow{i} \mathcal{E}$ is an inclusion of toposes, then distributions in $\mathcal{F}$ can be regarded as a full subcategory of distributions in $\mathcal{E}$ by associating with a distribution $\mathcal{F} \xrightarrow{\lambda} \mathcal{S}$ the distribution $\mathcal{E} \xrightarrow{i^*} \mathcal{F} \xrightarrow{\lambda} \mathcal{S}$.

2.6. Lemma. *Let $\mathcal{F} \xrightarrow{i} \mathcal{E}$ be an inclusion over $\mathcal{S}$. Let $\mathcal{D} \xrightarrow{\psi} \mathcal{F}$ be a complete spread and let $\mathcal{D}' \xrightarrow{\psi'} \mathcal{E}$ denote the spread completion of $\mathcal{D} \xrightarrow{\psi} \mathcal{F} \xrightarrow{i} \mathcal{E}$, as in the following diagram* (in which p is a pure geometric morphism).

$$\begin{tikzcd}
\mathcal{D} \arrow[r, tail, "p"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\cong"] & \mathcal{D}' \arrow[d, "\psi'"] \\
\mathcal{F} \arrow[r, tail, "i"'] & \mathcal{E}
\end{tikzcd}$$

Then this square is a topos pullback (so that p is an inclusion)*.*

Proof. We may construct the spread completion ψ' as follows. Let λ $(= d_! \cdot \psi^*)$ denote the distribution for ψ, so that the distribution of ψ' is $\lambda \cdot i^*$. We choose a site for $\mathcal{E}$ so that there is the following pullback, as in Definition 1.1.

$$\begin{tikzcd}
\mathcal{D}' \arrow[r, tail] \arrow[d, "\psi'"'] \arrow[dr, phantom, "\cong"] & \mathcal{S}^{\mathbf{D}^{\mathrm{op}}} \arrow[d] \\
\mathcal{E} \arrow[r, tail] & \mathcal{S}^{\mathbf{C}^{\mathrm{op}}}
\end{tikzcd}$$

A typical object of $\mathbf{D}$ is a pair (c, α), where $\alpha \in \lambda i^*(\epsilon c)$, and where $\mathbf{C} \xrightarrow{\epsilon} \mathcal{E}$ denotes the canonical functor. But we may also regard $\mathbf{C}$ as (the underlying category of) a site for $\mathcal{F}$, where now we have $\mathbf{C} \xrightarrow{\epsilon'} \mathcal{F}$ such that $\epsilon' \cong i^* \cdot \epsilon$. Then $\mathbf{D}$ is isomorphic to the site for

ψ, as described in Definition 1.1, since $\lambda \cdot \epsilon' \cong \lambda \cdot i^* \cdot \epsilon$. Therefore, there must be $\mathcal{D} \xrightarrow{p} \mathcal{D}'$ such that the diagram

$$\begin{tikzcd} \mathcal{D} \arrow[r, tail, "p"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\cong"] & \mathcal{D}' \arrow[r, tail] \arrow[d, "\psi'"] \arrow[dr, phantom, "\cong"] & \mathcal{S}^{\mathbf{D}^{\mathrm{op}}} \arrow[d] \\ \mathcal{F} \arrow[r, tail, "i"'] & \mathcal{E} \arrow[r, tail] & \mathcal{S}^{\mathbf{C}^{\mathrm{op}}} \end{tikzcd}$$

commutes, where the outer and right squares are pullbacks. Since $\mathcal{D} \longrightarrow \mathcal{S}^{\mathbf{D}^{\mathrm{op}}}$ is pure and $\mathcal{D}' \longrightarrow \mathcal{S}^{\mathbf{D}^{\mathrm{op}}}$ is an inclusion, p must be pure ([5], 1.2), so that ψ' is the spread completion of $i \cdot \psi$. To conclude, the left square must be a pullback. ∎

2.7. Lemma. *Let $\mathcal{F} \xrightarrow{i} \mathcal{E}$ be an inclusion of toposes, with $\mathcal{E}$ locally connected. If i is pure* (i.e., if $\Omega_{\mathcal{S}} \to i_* i^* \Omega_{\mathcal{S}}$ is an isomorphism)*, then i_* preserves $\mathcal{S}$-coproducts.*

Proof. For any $A \in \mathcal{S}$, we have the following pullback in $\mathcal{S}$.

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "\{\cdot\}"'] & 1 \arrow[d, tail, "\top"] \\ \Omega^A \arrow[r] & \Omega \end{tikzcd}$$

We may lift this pullback to $\mathcal{E}$ under inverse image noting that, since we are assuming that $\mathcal{E}$ is locally connected, exponentials are preserved. Our hypothesis says that $\Omega_{\mathcal{S}} = \Omega$ is an $\mathcal{F}$-sheaf. Therefore Ω^A is an $\mathcal{F}$-sheaf, and hence the pullback A is an $\mathcal{F}$-sheaf. Hence, i_* preserves $\mathcal{S}$-coproducts. ∎

2.8. Proposition. *Let $\mathcal{F} \xrightarrow{i} \mathcal{E}$ be a pure subtopos, with $\mathcal{E}$ locally connected. A distribution μ in $\mathcal{E}$ is absolutely continuous if and only if there is an X in $\mathcal{F}$ such that $\mu \cong X.e_!$. $\mathcal{F}$ is locally connected, and the categories of absolutely continuous distributions in $\mathcal{E}$ and in $\mathcal{F}$ are equivalent. The equivalence is given in one direction by composition with i^*, and in the other with i_*. If $X \in \mathcal{F}$, then the distributions $(X.f_!) \cdot i^*$ and $(i_* X).e_!$ are isomorphic. In terms of complete spreads, the equivalence is given by spread completion and by pullback along i.*

Proof. Since $\mathcal{F}$ is pure in $\mathcal{E}$ it contains the Lebesgue subtopos. Therefore, the bidense morphisms for $\mathcal{F}$ must be pure. In particular, for any X, the bidense unit $X \to i_* i^* X$ is a pure morphism. Therefore, by Lemma 2.4, any X and its associated $\mathcal{F}$-sheaf have equivalent closures. The first assertion of the proposition follows.

By Lemma 2.7 and by [4], 2.9, $\mathcal{F}$ is locally connected, and we have $f_! \cdot i^* \cong e_!$ and $f_! \cong e_! \cdot i_*$. Suppose that μ is an absolutely continuous distribution in $\mathcal{E}$, with $\mu \cong X.e_!$. We may assume that X is in $\mathcal{F}$. Then $\mu \cdot i_* \cong X.f_!$, and for any Y in $\mathcal{E}$ we have

$$\mu \cdot i_* \cdot i^*(Y) \cong f_!(X \times i^* Y) \cong f_!(i^* i_* X \times i^* Y) \cong f_! i^*(i_* X \times Y) \cong e_!(i_* X \times Y) \cong \mu(Y) \; .$$

Conversely, if λ is an absolutely continuous distribution in $\mathcal{F}$, so that there is X in $\mathcal{F}$ such that $\lambda \cong X.f_!$, then for any Y in $\mathcal{E}$, we have

$$\lambda \cdot i^*(Y) \cong X.f_!(i^* Y) \cong f_!(X \times i^* Y) \cong f_! i^*(i_* X \times Y) \cong e_!(i_* X \times Y) \; .$$

Thus, $\lambda \cdot i^* \cong (i_*X).e_!$, so $\lambda \cdot i^*$ is absolutely continuous in $\mathcal{E}$. We have $\lambda \cdot i^* \cdot i_* \cong \lambda$. The last assertion of the proposition follows from Lemma 2.6. ■

In particular, Proposition 2.8 holds for the Lebesgue subtopos.

2.9. Corollary. *Under the hypothesis of 2.8, let X be any object of $\mathcal{F}$. Then the spread completion of $\mathcal{F}/X \longrightarrow \mathcal{F} \xrightarrow{i} \mathcal{E}$ is isomorphic to the closure $\overline{i_*X}$.*

Proof. The spread completion of $\mathcal{F}/X \longrightarrow \mathcal{F} \xrightarrow{i} \mathcal{E}$ corresponds to the distribution $(X.f_!) \cdot i^*$. The complete spread that corresponds to the distribution $(i_*X).e_!$ is the closure $\overline{i_*X}$. ■

3. Composition of complete spreads and of discrete fibrations for a KZ-doctrine

The composite of two covers (i.e., of two locally constant objects) in a topos is not always again a cover. That is, there can be covers $A \to B$ and $B \to 1$ for which $A \to 1$ is not a cover. (Even if B is a constant object of the topos, it can happen that A is not locally constant. I.e., covers are not even closed under coproducts.) It has been established [11] that over a localic topos, complete spreads are closed under composition. No explanation of this fundamental property of complete spreads for geometric morphisms appears in the literature, so we shall include here a direct proof starting from Definition 1.1, which is similar to the localic argument. We shall also provide an explanation in terms of KZ-doctrines that depends on Theorem 1.2.

The following lemma will prepare us for Theorem 3.2.

3.1. Lemma. *Let $\mathcal{G} \xrightarrow{\psi} \mathcal{D}$ be a geometric morphism between locally connected toposes. Suppose that $\alpha \hookrightarrow X$ is a component of an object in $\mathcal{D}$, and that $\beta \hookrightarrow \psi^*\alpha$ is a component of $\psi^*\alpha$ in $\mathcal{G}$. Then β is a component of ψ^*X.*

Proof. Let $\hat{\beta}$ denote the component determined by β under the composite inclusion

$$\beta \hookrightarrow \psi^*\alpha \hookrightarrow \psi^*X \ .$$

We want to show that $\beta = \hat{\beta}$. Let $\bar{\hat{\beta}}$ denote the component of X that contains the image under ψ of $\hat{\beta}$. Then $\bar{\hat{\beta}}$ and α have the image of β under ψ in common, so that $\alpha = \bar{\hat{\beta}}$. Therefore, $\hat{\beta} \hookrightarrow \psi^*\bar{\hat{\beta}} = \psi^*\alpha$. Thus, $\hat{\beta}$ must be a component also of $\psi^*\alpha$. Since $\hat{\beta}$ contains β, we must have $\beta = \hat{\beta}$. ■

3.2. Theorem. *The composite of two complete spreads is a complete spread. If geometric morphisms φ and $\varphi \cdot \psi$ are complete spreads, then ψ is one as well.*

Proof. Suppose that $\mathcal{D} \xrightarrow{\varphi} \mathcal{E}$ and $\mathcal{G} \xrightarrow{\psi} \mathcal{D}$ are two complete spreads over a base topos $\mathcal{S}$, with defining pullbacks below, left (Definition 1.1). The toposes $\mathcal{D}$ and $\mathcal{G}$ are locally connected. We must show that the far right square is also a pullback.

$$
\begin{tikzcd}
\mathcal{D} \arrow[r, "i"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\cong"] & \mathcal{S}^{\mathbf{D}^{\mathrm{op}}} \arrow[d] \\
\mathcal{E} \arrow[r] & \mathcal{S}^{\mathbf{C}^{\mathrm{op}}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{G} \arrow[r] \arrow[d, "\psi"'] \arrow[dr, phantom, "\cong"] & \mathcal{S}^{\mathbf{E}^{\mathrm{op}}} \arrow[d] \\
\mathcal{D} \arrow[r, "i"'] & \mathcal{S}^{\mathbf{D}^{\mathrm{op}}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{G} \arrow[r] \arrow[d, "\varphi\cdot\psi"'] \arrow[dr, phantom, "\cong"] & \mathcal{S}^{\mathbf{F}^{\mathrm{op}}} \arrow[d] \\
\mathcal{E} \arrow[r] & \mathcal{S}^{\mathbf{C}^{\mathrm{op}}}
\end{tikzcd}
$$

The objects of the small categories $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{F}$ are, respectively:

$\mathbf{D}$: (c,α); $c\in\mathbf{C}$ and $\alpha\in d_!\varphi^*c$,

$\mathbf{E}$: $((c,\alpha),\beta)$; $(c,\alpha)\in\mathbf{D}$ and $\beta\in g_!\psi^*\alpha$ (i satisfies $i^*(c,\alpha)=\alpha$),

$\mathbf{F}$: (c,β); $c\in\mathbf{C}$ and $\beta\in g_!\psi^*\varphi^*c$.

We will show that the square involving the composite $\varphi\cdot\psi$ is a pullback by showing that $\mathbf{E}$ and $\mathbf{F}$ are equivalent categories over $\mathbf{C}$. There is a unique natural transformation $g_!\psi^*\to d_!$ since $d_!$ is the terminal distribution $\mathcal{D}\longrightarrow\mathcal{S}$. By composing with φ^* we obtain a natural transformation $g_!\psi^*\varphi^*\to d_!\varphi^*$ that we shall denote by $\beta\mapsto\bar\beta$. Intuitively, this map sends a component β of $\psi^*\varphi^*X$ to the component $\bar\beta$ of φ^*X that contains the image of β under ψ. Using this map we may define a functor from $\mathbf{F}$ to $\mathbf{E}$.

$$(c,\beta)\ \mapsto\ ((c,\bar\beta),\beta)$$

A functor from $\mathbf{E}$ to $\mathbf{F}$,

$$((c,\alpha),\beta)\ \mapsto\ (c,\beta)\ ,$$

arises when we consider that a component $\alpha\hookrightarrow\varphi^*c$ in $\mathcal{D}$ may be lifted to $\mathcal{G}$, where we have the component β of $\psi^*\alpha$: $\beta\hookrightarrow\psi^*\alpha\hookrightarrow\psi^*\varphi^*c$. Then by Lemma 3.1, β is a component of $\psi^*\varphi^*c$. It is almost clear that these two functors are mutual inverses. What remains to be seen is that for a given $((c,\alpha),\beta)$, we have $\alpha=\bar\beta$. This is so because the two components α and $\bar\beta$ of φ^*c have the image of β under ψ in common.

The second statement of the theorem follows in a similar way; if the left and right squares above are pullbacks, then so is the middle one. ∎

Theorem 3.2, when combined with the canonical correspondence of distributions with complete spreads, shows that the comma category of a category of distributions is again a category of distributions.

What point of the symmetric topos corresponds to the composite of two complete spreads? Suppose that $\mathcal{D}\xrightarrow{\varphi}\mathcal{E}$ and $\mathcal{G}\xrightarrow{\psi}\mathcal{D}$ are complete spreads, and that $\mathcal{S}\xrightarrow{q}M\mathcal{D}$ corresponds to ψ, so that $q\cong\Sigma_g(\delta_{\mathcal{D}}\cdot\psi)\cong M\psi\cdot r_g\cdot\delta_{\mathcal{S}}$. ($\Sigma_f(m)$ denotes the left extension of m along f. We refer the reader to [6] for further details concerning left extension and KZ-doctrines, and in particular, for details concerning the formula for left extension that we have just used.) Then the point of $M\mathcal{E}$ corresponding to $\varphi\cdot\psi$ is

$$\Sigma_g(\delta_{\mathcal{E}}\cdot\varphi\cdot\psi)\cong M(\varphi\cdot\psi)\cdot r_g\cdot\delta_{\mathcal{S}}\cong M\varphi\cdot M\psi\cdot r_g\cdot\delta_{\mathcal{S}}\cong M\varphi\cdot q\ .$$

$$\begin{tikzcd} f \Downarrow g \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\Rightarrow"] & B \arrow[d, "g"] \\ C \arrow[r, "f"'] & D \end{tikzcd} \qquad \begin{array}{l} Mf \dashv r_f,\ Mp \dashv r_p \\ Mq \cdot r_p \cong r_f \cdot Mg \end{array}$$

Figure 4: Colimit-completion KZ-doctrine.

We will use this observation when we establish Theorem 3.3 below.

Theorem 3.3 involves what in [6] is introduced as an "admissible KZ-doctrine." We recall that a KZ-doctrine $< M, \delta, \mu >$ is said to be admissible if there exist bicomma objects $f \Downarrow g$ of opspans in which f has an M-adjoint, and if also the 1-cell opposite f in such a bicomma object admits an M-adjoint. Furthermore, it is required that the canonical 'commuting' 2-cell $Mq \cdot r_p \Rightarrow r_f \cdot Mg$ be an isomorphism (Figure 4). Instead of the term "admissible" we shall use *(colimit)-completion KZ-doctrine*.[1]

Next we recall that pure geometric morphisms are orthogonal to complete spreads in the sense that the two classes comprise a factorization system of geometric morphisms with locally connected domain [4]. It is shown in [6] how this system is a special case of a final, discrete fibration, or *comprehensive*, factorization that is associated with any completion KZ-doctrine (provided that a certain isomorphism reflecting property is satisfied, which we state in Theorem 3.3). We will use comprehensive factorization in order to obtain the following result concerning the composition of discrete fibrations.

3.3. Theorem. *Let M denote a locally fully faithful completion KZ-doctrine in a 2-category, and assume that for every discrete fibration φ, $M\varphi$ reflects isomorphisms. Then the composite of two discrete fibrations is a discrete fibration. If a composite and its second factor are discrete fibrations, then its first factor is also.*

Proof. Let $G \xrightarrow{\psi} D$ and $D \xrightarrow{\varphi} E$ be two discrete fibrations for M. First, we write the bicomma object for φ as the following composite diagram, where T denotes the terminal object.

$$\begin{tikzcd} D \arrow[r, "\delta_D"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\cong"] & MD \arrow[r, "z"] \arrow[d, "M\varphi"'] \arrow[r, phantom, shift right=3, "\Rightarrow"] & T \arrow[dl, "p"] \\ E \arrow[r, "\delta_E"'] & ME & \end{tikzcd}$$

That this can be done follows from the fact that there uniquely corresponds a 2-cell $M\varphi \Rightarrow p \cdot z$ to the 2-cell

$$M\varphi \cdot \delta_D \cong \delta_E \cdot \varphi \Rightarrow p \cdot d \cong p \cdot z \cdot \delta_D \ .$$

[1]An example of a KZ-doctrine that is not a colimit-completion one would be of interest. The author knows of none.

The reason for this correspondence is that an M-homomorphism with domain MD, in this case $M\varphi$, is the left extension of its restriction to D ([6], 1.6). Next, by a previous remark if the composite $\varphi \cdot \psi$ is a discrete fibration, then its corresponding point $T \longrightarrow ME$ must be $M\varphi \cdot q$, where $T \xrightarrow{q} MD$ is the point of ψ. Thus we should consider the bicomma object $\delta_E \Downarrow M\varphi \cdot q$,

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "\kappa"'] \arrow[dr, phantom, "\Rightarrow"] & T \arrow[d, "M\varphi\cdot q"] \\ E \arrow[r, "\delta_E"'] & ME \end{tikzcd}$$

and the intervening final 1-cell $G \xrightarrow{h} K$. (We have constructed the comprehensive factorization of $\varphi \cdot \psi$. It is only here that the isomorphism reflecting property assumed of the KZ-doctrine is used; it is used to show that h satisfies condition *1* of Proposition 2.1.) Now regard the bicomma object for φ. Since there is a 2-cell

$$\delta_E \cdot \kappa \Rightarrow M\varphi \cdot q \cdot k \Rightarrow p \cdot z \cdot q \cdot k \cong p \cdot k \ ,$$

we can factor κ through φ by a 1-cell $K \xrightarrow{\gamma} D$. By the universal property of this bicomma object we also have $\gamma \cdot h \cong \psi$. Using this isomorphism we obtain a 2-cell

$$\delta_D \cdot \gamma \cdot h \cong \delta_D \cdot \psi \Rightarrow q \cdot g \cong q \cdot k \cdot h \ ,$$

which, by Proposition 2.1, corresponds to a 2-cell

$$\delta_D \cdot \gamma \Rightarrow q \cdot k \ .$$

The bicomma object for ψ now gives a 1-cell $K \xrightarrow{j} G$ such that $\psi \cdot j \cong \gamma$, and it is a routine matter to show that h is an equivalence with pseudo-inverse j. This shows that $\varphi \cdot \psi$ is a discrete fibration, witnessed by $M\varphi \cdot q$.

The second assertion of the theorem is a formal consequence of the first and the comprehensive factorization. ∎

Theorem 3.3 together with the fact ([6], 4.3, 4.7) that the symmetric monad is a locally fully faithful completion KZ-doctrine that has the isomorphism reflecting property provide a new proof of Theorem 3.2.

4. Branched covers in topos theory

In this section we investigate branched covering spaces in topos theory. The theory of complete spreads (in topos theory) provides us with a natural context in which to work. Our findings feature *non-surjective covers with pure support*, and an involvement of the Lebesgue subtopos. The spread completions of these non-surjective covers can be regarded as a kind of non-surjective branched cover. We shall show that not only do the covers of the Lebesgue subtopos include in a natural way the branched covers of the

$$
\begin{tikzcd}
\mathcal{E}/\mathbf{d}\varphi \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & \mathcal{D} \arrow[d, "\varphi"] \\
\mathcal{E}/S \arrow[r, tail] & \mathcal{E}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{D}/\varphi^*S \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\cong"] & \mathcal{D} \arrow[d, "\varphi"] \\
\mathcal{E}/S \arrow[r, tail] & \mathcal{E}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{D}/\varphi^*X \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & \mathcal{D} \arrow[d, "\varphi"] \\
\mathcal{E}/X \arrow[r] & \mathcal{E}
\end{tikzcd}
$$

Figure 5: Three kinds of branched covers: $bc_{\mathcal{E}}$, $Bc_{\mathcal{E}}$, and $BC_{\mathcal{E}}$.

given topos (Theorem 4.9), but furthermore, that every cover of the Lebesgue subtopos arises as the pullback of a non-surjective cover with pure support of the topos (but not uniquely). Ultimately, we shall see that the category of covers of the Lebesgue subtopos is equivalent to a category of fractions of non-surjective branched covers of the the given topos (Theorem 4.10).

We shall first introduce two categories, one a full sub-category of the other. The objects of the smaller category will be the branched covers in the sense of Fox, or what we shall refer to as 'traditional' branched covers. As mentioned above, we will find it natural to consider a slightly larger category consisting of certain non-surjective branched covers. Then we shall show how this category of non-surjective branched covers is a full reflective subcategory of a third category of generalized branched covers. A reason for introducing this third category is that, unlike the first two, its definition makes sense for any colimit-completion KZ-doctrine.

Definition 4.1 below is a translation of Fox's original defining conditions into the present context. This definition was arrived at jointly in discussions with M. Bunge, and it is also considered in [8]. Throughout this section, $\mathcal{E}$ will denote a locally connected topos over a base topos $\mathcal{S}$. By a cover of a topos, we mean a locally constant object of the topos.

4.1. Definition. *Let $\mathcal{D} \xrightarrow{\varphi} \mathcal{E}$ denote a complete spread with interior $\mathbf{d}\varphi$ whose support is S, as in Figure 5 (left). We shall refer to φ as a* (traditional) branched cover of $\mathcal{E}$ *if:*

1. *$\mathbf{d}\varphi \twoheadrightarrow S$ is a cover (which we call the cover associated with φ),*

2. *S is a pure object, and*

3. *$\mathcal{E}/\mathbf{d}\varphi \longrightarrow \mathcal{D}$ is pure (so that φ is the spread completion of $\mathbf{d}\varphi$).*

We shall denote by $bc_{\mathcal{E}}$ the full subcategory of complete spreads over $\mathcal{E}$ determined by these branched covers.

Fox observes that condition 3 in Definition 4.1 rules out such phenomena as folding, while condition 2 excludes complete spreads that are cuts [19].

Condition 3 above guarantees that the distribution of a branched cover is absolutely continuous (in Lawvere's sense, as in Prop. 2.8). But not every absolutely continuous distribution corresponds to a branched cover.

Our next observation we state as the following lemma. It is a special case of [8], 8.2, but we shall deduce it from the more general Lemma 2.6.

4.2. Lemma. *Conditions 1 and 3 of Definition 4.1 together ensure that the square in Figure 5 (left) is a topos pullback. In particular, $\mathcal{E}/\mathbf{d}\varphi \longrightarrow \mathcal{D}$ is a local homeomorphism.*

Proof. Covers are complete spreads ([5], 7.3), so Lemma 2.6 applies. ∎

We may paraphrase Lemma 4.2 by saying that if φ is a branched cover with interior $\mathbf{d}\varphi$ whose support is S, then $\mathbf{d}\varphi \cong \varphi^* S$. We have not in Definition 4.1 adopted Fox's requirement that $\varphi^* S$ be connected.

Before proceeding to non-surjective and generalized branched covers, and to their relationship with the previous traditional ones, we observe that the square in Figure 5 (left) is not a pullback for every complete spread φ. Picture a sphere that has been squeezed horizontally in the middle, and projected vertically onto a disk of the same radius. One can think of (the domain space of) this complete spread as having the shape of a typical hourglass. The sheaf space associated with, or the interior of, the hourglass consists of four pieces: two disks and two annuli, all missing their boundaries. The support of the interior is the open disk. But the pullback of the hourglass along this support consists of *three* pieces: two disks and the middle part of the hourglass still intact, all three pieces again missing their boundaries. Thus, for the hourglass, the pullback does not coincide with the interior; in the pullback, the two annuli are joined by a circle along their inside rims producing the middle part of the hourglass. In this particular example, conditions 1 and 3 both do not hold; the interior is not a cover of its support (though it is the coproduct of a cover and a non-surjective cover), and the inclusion of the interior into the complete spread is not pure.

We now consider certain non-surjective branched covers. The middle square in Figure 5 describes a typical object of $Bc_{\mathcal{E}}$; it is a pair consisting of a pure subobject S of 1, and a complete spread $\mathcal{D} \xrightarrow{\varphi} \mathcal{E}$ such that $\varphi^* S$ is a pure object of $\mathcal{D}$, and such that $\mathcal{D}/\varphi^* S \longrightarrow \mathcal{E}/S$ is a cover. A morphism $(S, \varphi) \longrightarrow (T, \psi)$ in this category is a pair consisting of a morphism $S \to T$ in $\mathcal{E}$, which must be a pure monomorphism, and a geometric morphism $\mathcal{D} \longrightarrow \mathcal{G}$ over $\mathcal{E}$, where $\mathcal{G}$ is the domain of ψ. The distribution corresponding to such a complete spread φ is absolutely continuous. Indeed, there is a Y such that $\mathcal{D}/\varphi^* S \simeq \mathcal{E}/Y$, so that $\varphi \cong \bar{Y}$.

4.3. Proposition. *$Bc_{\mathcal{E}}$ is equivalent to the category whose objects are pairs (S, π) such that π is a cover of a pure subobject S of 1, and whose morphisms are the obvious commuting squares in $\mathcal{E}$. The category $bc_{\mathcal{E}}$ is a full subcategory of $Bc_{\mathcal{E}}$, by an inclusion that associates with a branched cover φ the pair (S, φ), where S is the support of the interior of φ.*

Proof. The stated equivalence is given in one direction by pullback: $(S, \varphi) \mapsto (S, S^* \varphi)$, and in the other by associating with (S, π) the spread completion of $\mathcal{E}/C \xrightarrow{\pi} \mathcal{E}/S \longrightarrow \mathcal{E}$, together with S. By the argument used for Lemma 4.2, these functors give an equivalence of categories. Regarding the other statement of the proposition, by Lemma 4.2, the functor $\varphi \rightsquigarrow (S, \varphi)$ from $bc_{\mathcal{E}}$ to $Bc_{\mathcal{E}}$ does indeed produce an object of $Bc_{\mathcal{E}}$. ∎

Recall that a complete spread object of a locally connected topos $\mathcal{T}$ is an object X such that $\mathcal{T}/X \longrightarrow \mathcal{T}$ is a complete spread.

4.4. Lemma. *Let Y be a complete spread object of a pure subtopos $\mathcal{F} \xrightarrow{i} \mathcal{E}$* ($\mathcal{F}$ is locally connected by Lemma 2.7 and [4], 2.9). *Let $\mathcal{D} \xrightarrow{\varphi} \mathcal{E}$ denote the spread completion of $\mathcal{F}/Y \longrightarrow \mathcal{F} \xrightarrow{i} \mathcal{E}$. Then there is a canonical isomorphism $i_*Y \cong \mathbf{d}\varphi$. Consequently, i_*Y is a regular object of $\mathcal{E}$ and $\mathcal{E}/\mathbf{d}\varphi \longrightarrow \mathcal{D}$ is pure* (so that $\varphi \cong \overline{\mathbf{d}\varphi}$).

Proof. By Corollary 2.9, $\overline{i_*Y} \cong \varphi$. By adjointness, there is a morphism $i_*Y \to \mathbf{d}\varphi$ in $\mathcal{E}$. On the other hand, there is a commutative square

$$\begin{tikzcd} \mathcal{F}/i^*\mathbf{d}\varphi \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & \mathcal{D} \arrow[d, "\varphi"] \\ \mathcal{F} \arrow[r, tail, "i"'] & \mathcal{E} \end{tikzcd}$$

arising from $\mathcal{E}/\mathbf{d}\varphi \longrightarrow \mathcal{D}$. Since $\mathcal{F}/Y \longrightarrow \mathcal{F}$ is the pullback of φ along i (Prop. 2.8), there is a morphism $i^*\mathbf{d}\varphi \to Y$ in $\mathcal{F}$ that transposes to one $\mathbf{d}\varphi \to i_*Y$. It follows that the two morphisms we have obtained give an isomorphism $i_*Y \cong \mathbf{d}\varphi$. The object i_*Y is regular since $i_*Y \cong \mathbf{d}\varphi \cong \mathbf{d}(\overline{i_*Y})$. Finally, $\mathcal{E}/\mathbf{d}\varphi \longrightarrow \mathcal{D}$ is pure since i_*Y is pure in its closure $\overline{i_*Y} \cong \varphi$. ■

Consider a cover $Y \xrightarrow{\pi} S$ of a pure subobject S of 1. Let φ denote the spread completion of $\mathcal{E}/Y \xrightarrow{\pi} \mathcal{E}/S \longrightarrow \mathcal{E}$. If π is actually an epimorphism (as in the case when the fiber of π has global support in the base topos $\mathcal{S}$), then by examining the following square we see that the support of $\mathbf{d}\varphi$ must contain S.

$$\begin{tikzcd} Y \arrow[r] \arrow[d, two heads, "\pi"'] & \mathbf{d}\varphi \arrow[d] \\ S \arrow[r, tail] & 1 \end{tikzcd}$$

(Incidentally, by Lemma 4.4, $\mathbf{d}\varphi \cong Y^S$.) We have referred to the branched cover (S, φ) as non-surjective since this containment may be proper.

If we drop the requirement of a branched cover (S, φ) that the pure object S be a subobject of 1, then we arrive at a category whose objects have the same description as those of $Bc_{\mathcal{E}}$ except that the codomain of the cover, denoted by X in Figure 5, is simply a pure object of $\mathcal{E}$. We denote this category by $BC_{\mathcal{E}}$. We remark that $BC_{\mathcal{E}}$ is a full subcategory of the category of 'locally closed' toposes over $\mathcal{E}$, i.e., of those toposes over $\mathcal{E}$ that are products (over $\mathcal{E}$) of an 'open' topos $\mathcal{E}/X \longrightarrow \mathcal{E}$ and a 'closed' one $\mathcal{D} \xrightarrow{\varphi} \mathcal{E}$, meaning a complete spread. ('Locally closed' can also mean "open in its closure," as in [8].) But the objects of $BC_{\mathcal{E}}$ are special locally closed toposes since, referring to Figure 5, $\mathcal{D}/\varphi^*X \longrightarrow \mathcal{E}/X$ is an étale morphism. Thus $\mathcal{D}/\varphi^*X \longrightarrow \mathcal{E}$ is étale, so that an object of $BC_{\mathcal{E}}$ may be regarded as a *twisted map between bifibrations* [6], §5. We also note that for such an (X, φ), φ^*X is required to be pure in $\mathcal{D}$, so that just like the previous branched covers, the distribution corresponding to the complete spread φ is absolutely continuous.

4.5. Proposition. *$Bc_{\mathcal{E}}$ is a full reflective subcategory of $BC_{\mathcal{E}}$. The reflection of a pair (X, φ) is the pair (S, φ), where S is the support of X.*

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\cong"] & D \arrow[d, "\varphi"] \\ X \arrow[r, "f"'] & E \end{tikzcd}$$

φ: discrete fibration
f: final discrete opfibration
g: *final* discrete opfibration
ψ: discrete fibration *and opfibration*

Figure 6: A typical object of BC_E. The square is a bipullback.

PROOF. Let (X, φ) be an object of $BC_{\mathcal{E}}$, and consider the following topos pullbacks.

$$\begin{tikzcd} \mathcal{D}/\varphi^*X \arrow[r, two heads] \arrow[d, "\pi"'] \arrow[dr, phantom, "\cong"] & \mathcal{D}/\varphi^*S \arrow[r, tail] \arrow[d, "\pi'"] \arrow[dr, phantom, "\cong"] & \mathcal{D} \arrow[d, "\varphi"] \\ \mathcal{E}/X \arrow[r, two heads] & \mathcal{E}/S \arrow[r, tail] & \mathcal{E} \end{tikzcd}$$

π' is a cover since by hypothesis π is one, and since $X \to S$ is an epimorphism. By hypothesis, X and φ^*X are pure, so that S and φ^*S are pure for as follows from [5], 1.2, the support of a pure object is pure. ∎

It ought to be mentioned that thus far the definitions in this section make sense, and the assertions remain valid, if in them complete spread objects are used in place of covers.[2] Also, we emphasize that BC can be defined for any (completion) KZ-doctrine in a 2-category. If E is an object of the 2-category, then Figure 6 shows a typical object of BC_E. In Figure 6, the square is a (bi)pullback, φ is a given discrete fibration, and f is a given final discrete opfibration. Discrete fibrations are stable under pullback along a 1-cell that admits an M-adjoint, and discrete opfibrations are arbitrarily pullback stable ([6], 2.3). Thus, the pullback ψ must be a discrete fibration since a discrete opfibration for a completion KZ-doctrine admits an M-adjoint. We require that ψ also be a discrete opfibration. The 1-cell g is a discrete opfibration, and it is required to be final.

We conclude this section by involving the smallest pure subtopos, the Lebesgue subtopos. For Lemma 4.6 and Proposition 4.7 below, let X be an arbitrary object of a locally connected topos $\mathcal{B}$ over $\mathcal{S}$. We have $\Sigma_X \dashv X^*$.

4.6. LEMMA. *Σ_X applied to a pure monomorphism of $\mathcal{B}/X$ is pure in $\mathcal{B}$.*

PROOF. Let $S \hookrightarrow T$ be a pure monomorphism of $\mathcal{B}/X$. We must show that every pullback of $S \hookrightarrow T$ in $\mathcal{B}$ is dense for the constant object $\Omega_{\mathcal{S}}$. By using $\Sigma_X \dashv X^*$, this follows easily from the fact that the same is true of $S \hookrightarrow T$ in $\mathcal{B}/X$. ∎

4.7. PROPOSITION. *Let $\mathcal{L}$ denote the Lebesgue subtopos of $\mathcal{B}$. Then the pullback $\mathcal{L}/X$ is the Lebesgue subtopos of $\mathcal{B}/X$.*

[2]There are complete spread objects that are not covers. For example, a topos that is not locally simply connected can have two composable covers such that their composite, which must be a complete spread object, is not a cover of the topos.

PROOF. Pure morphisms are stable under pullback along an étale morphism ([5], 1.3). Therefore, the Lebesgue subtopos of $\mathcal{B}/X$ is a subtopos of the pullback $\mathcal{L}/X$. But every pure monomorphism $S \hookrightarrow T$ of $\mathcal{B}/X$ is dense for the topology for $\mathcal{L}/X$. Indeed, by Lemma 4.6 such a monomorphism is pure in $\mathcal{B}$, i.e., such a monomorphism is dense for $\mathcal{L}$. Now consider the pullback

$$\begin{tikzcd} S \arrow[r, tail] \arrow[d] & T \arrow[d] \\ X^*\Sigma_X S \arrow[r, tail] & X^*\Sigma_X T \end{tikzcd}$$

in $\mathcal{B}/X$. I.e., $S \hookrightarrow T$ is the pullback of the inverse image of a dense monomorphism. Thus $S \hookrightarrow T$ is dense for the topology for $\mathcal{L}/X$. Note: we are using Thm. 25 of [20]. ∎

4.8. PROPOSITION. *Let $(S, \mathcal{D} \xrightarrow{\varphi} \mathcal{E})$ be a non-surjective branched cover, i.e., let (S,φ) denote an object of $Bc_{\mathcal{E}}$. Then in the following pullback, π is a cover of $\mathcal{L}$, and $\mathcal{P}$ is the Lebesgue subtopos of $\mathcal{D}$.*

$$\begin{tikzcd} \mathcal{P} \arrow[r, tail] \arrow[d, "\pi"'] \arrow[dr, phantom, "\cong"] & \mathcal{D} \arrow[d, "\varphi"] \\ \mathcal{L} \arrow[r, tail] & \mathcal{E} \end{tikzcd}$$

In particular, we may recover φ as the spread completion of $\mathcal{P} \xrightarrow{\pi} \mathcal{L} \longrightarrow \mathcal{E}$. Every non-surjective branched cover of $\mathcal{L}$ is a cover of $\mathcal{L}$, i.e., $Bc_{\mathcal{L}} = Cov_{\mathcal{L}}$.

PROOF. Since S is a pure subobject of 1, $\mathcal{L}$ must be the smallest pure subtopos of $\mathcal{E}/S$. Then the first statement of the theorem can be easily established by considering the given pullback as a composite of the following two pullbacks.

$$\begin{tikzcd} \mathcal{P} \arrow[r, tail] \arrow[d, "\pi"'] \arrow[dr, phantom, "\cong"] & \mathcal{E}/C \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\cong"] & \mathcal{D} \arrow[d, "\varphi"] \\ \mathcal{L} \arrow[r, tail] & \mathcal{E}/S \arrow[r, tail] & \mathcal{E} \end{tikzcd}$$

By Prop. 4.7, $\mathcal{P}$ is the smallest pure subtopos of $\mathcal{E}/C$. But then $\mathcal{P}$ must be the smallest pure subtopos of $\mathcal{D}$ since $\mathcal{E}/C$ is pure in $\mathcal{D}$. π is a cover since it is the pullback of a cover. Every member of $Bc_{\mathcal{L}}$ is a cover because $\mathcal{L}$ has no pure subtoposes other than itself. ∎

Although we can recover from the restriction to $\mathcal{L}$ of a non-surjective branched cover (S,φ) the complete spread φ, we cannot always recover its support S. However, since a traditional branched cover is a complete spread with special properties (unlike a non-surjective one, which can be regarded as a complete spread φ with the additional structure S), a consequence of Proposition 4.8 is the following.

4.9. THEOREM. *The category of traditional branched covers of $\mathcal{E}$ is equivalent to a full subcategory of the covers of the Lebesgue subtopos of $\mathcal{E}$. Figure 7 describes a chain of full sub-categories* ($Cov_{\mathcal{E}}$ denotes the covers of $\mathcal{E}$, $Cs_{\mathcal{E}}$ the complete spreads over $\mathcal{E}$).

PROOF. See Props. 4.8 and 2.8. Recall that the closures $\bar{X}$ are the complete spreads that correspond to the absolutely continuous distributions. ∎

$$Cov_{\mathcal{E}} \subseteq bc_{\mathcal{E}} \subseteq Cov_{\mathcal{L}} = Bc_{\mathcal{L}} \subseteq \{\bar{X} \mid X \in \mathcal{L}\} \simeq \{\bar{X} \mid X \in \mathcal{E}\} \subseteq Cs_{\mathcal{L}} \subseteq Cs_{\mathcal{E}}$$

Figure 7: Some full sub-categories of complete spreads over $\mathcal{E}$.

We further our analysis of the relationship existing between the covers of $\mathcal{L}$ and the branched covers of $\mathcal{E}$. Recall that a non-surjective branched cover of $\mathcal{E}$ can be equivalently regarded as a cover $Y \xrightarrow{\pi} S$ of a pure subobject S of 1. The morphisms in this category are the obvious commutative squares in $\mathcal{E}$.

4.10. Theorem. *Consider the class Ξ of morphisms*

$$\begin{tikzcd} Y \arrow[r, tail] \arrow[d, "\pi"'] & Z \arrow[d, "\kappa"] \\ S \arrow[r, tail] & T \end{tikzcd}$$

in the category $Bc_{\mathcal{E}}$ for which this square is a pullback in $\mathcal{E}$ (equivalently, for which $Y \to Z$ is pure). *Then Ξ admits a calculus of right fractions, and $Bc_{\mathcal{E}}(\Xi^{-1}) \simeq Cov_{\mathcal{L}}$ by an equivalence that identifies the canonical functor $Bc_{\mathcal{E}} \xrightarrow{p} Bc_{\mathcal{E}}(\Xi^{-1})$ with the pullback functor $Bc_{\mathcal{E}} \longrightarrow Cov_{\mathcal{L}}$.*

Proof. It is not difficult to verify that Ξ admits a calculus of right fractions. The pullback functor $Bc_{\mathcal{E}} \longrightarrow Cov_{\mathcal{L}}$ carries the morphisms in Ξ to isomorphisms, so that there is a commutative diagram as follows.

$$\begin{tikzcd} Bc_{\mathcal{E}} \arrow[r, "p"] \arrow[dr] & Bc_{\mathcal{E}}(\Xi^{-1}) \arrow[d, "q"] \arrow[dl, phantom, "\cong"] \\ & Cov_{\mathcal{L}} \end{tikzcd}$$

The functor q is essentially surjective: let Y denote an arbitrary cover of $\mathcal{L}$, split by U, so that there is a pullback

$$\begin{tikzcd} U \times A \arrow[r, two heads] \arrow[d] & Y \arrow[d, "\pi"] \\ U \arrow[r, two heads] & 1 \end{tikzcd}$$

in $\mathcal{L}$, where A is a constant sheaf. When included in $\mathcal{E}$, this square remains a pullback, but note that because this inclusion $i : \mathcal{L} \longrightarrow \mathcal{E}$ has the property that i_* preserves $\mathcal{S}$-coproducts, we have

$$i_*(U \times A \to U) \cong i_*(U) \times A \to i_* U \ .$$

We now factor the pullback square in $\mathcal{E}$ as the following two pullbacks.

$$\begin{tikzcd}
i_*U \times A \arrow[r, two heads] \arrow[d] & P \arrow[r, tail] \arrow[d, "\pi"] & i_*Y \arrow[d] \\
i_*U \arrow[r, two heads] & T \arrow[r, tail] & 1
\end{tikzcd}$$

Thus, π is a cover of the subobject T. But the object T is necessarily a pure subobject of 1, so that $\mathcal{L}$ must be included in $\mathcal{E}/T$. Therefore we have the following topos pullbacks.

$$\begin{tikzcd}
\mathcal{L}/Y \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\cong"] & \mathcal{E}/P \arrow[r, tail] \arrow[d, "\pi"] \arrow[dr, phantom, "\cong"] & \mathcal{E}/i_*Y \arrow[d] \\
\mathcal{L} \arrow[r, tail] & \mathcal{E}/T \arrow[r, tail] & \mathcal{E}
\end{tikzcd}$$

This shows that Y can be obtained as the pullback of a non-surjective branched cover of $\mathcal{E}$. (Incidentally, by Lemma 4.4 we have $i_*Y \cong \mathbf{d}\varphi$, where φ is the spread completion of $\mathcal{L}/Y \longrightarrow \mathcal{L} \xrightarrow{i} \mathcal{E}$.) That q is fully faithful can be established by a straightforward examination of the definitions. ■

Further investigation of branched covers along these lines would naturally involve an examination of the relationship between *the fundamental group of a topos* [7] and the fundamental group of its Lebesgue subtopos.

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References

[1] M. Barr and R. Paré. Molecular toposes. *J. Pure Appl. Alg.*, 17:127–152, 1980.

[2] M. Bunge. Cosheaves and distributions on toposes. *Algebra Universalis*, 34:469–484, 1995.

[3] M. Bunge and A. Carboni. The symmetric topos. *J. Pure Appl. Alg.*, 105:233–249, 1995.

[4] M. Bunge and J. Funk. Spreads and the symmetric topos. *J. Pure Appl. Alg.*, 113:1–38, 1996.

[5] M. Bunge and J. Funk. Spreads and the symmetric topos II. *J. Pure Appl. Alg.*, 130(1):49–84, 1998.

[6] M. Bunge and J. Funk. On a bicomma object condition for KZ-doctrines. *J. Pure Appl. Alg.*, 143:69–105, 1999.

[7] M. Bunge and I. Moerdijk. On the construction of the Grothendieck fundamental group of a topos by paths. *J. Pure Appl. Alg.*, 116:99–113, 1997.

[8] M. Bunge and S. Niefield. Exponentiality and single universes. *J. Pure Appl. Alg.* To appear.

[9] R. H. Crowell and R. H. Fox. *Introduction to Knot Theory*, volume 57 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Heidelberg-Berlin, 1977.

[10] R. H. Fox. Covering spaces with singularities. In R. H. Fox et al., editors, *Algebraic Geometry and Topology: A Symposium in Honor of S. Lefschetz*, pages 243–257. Princeton University Press, Princeton, 1957.

[11] J. Funk. The display locale of a cosheaf. *Cahiers de Top. et Géom. Diff. Catégoriques*, 36(1):53–93, 1995.

[12] P. T. Johnstone. Factorization theorems for geometric morphisms II. In *Categorical aspects of topology and analysis, Proc. Carleton University, Ottawa 1980, LNM 915*, pages 216–233. Springer, Berlin, 1982.

[13] A. Kock. Monads for which structure are adjoints to units. *J. Pure Appl. Alg.*, 104:41–59, 1995.

[14] A. Kock and G. E. Reyes. A note on frame distributions. *Cahiers de Top. et Géom. Diff. Catégoriques*. To appear.

[15] F. W. Lawvere. Intensive and extensive quantities. Notes for the lectures given at the workshop on Categorical Methods in Geometry, Aarhus, 1983.

[16] F. W. Lawvere. Categories of space and of quantity. In J. Echeverria et al., editors, *The Space of Mathematics*, pages 14–30. W. de Gruyter, Berlin-New York, 1992.

[17] W. B. R. Lickorish. A representation of orientable combinatorial 3-manifolds. *Ann. of Math.*, 76(3):531–540, 1962.

[18] S. Mac Lane and I. Moerdijk. *Sheaves in Geometry and Logic*. Springer-Verlag, Berlin-Heidelberg-New York, 1992.

[19] E. Michael. Cuts. *Acta Math.*, 111:1–36, 1964.

[20] R. Paré. Indexed categories and generated topologies. *J. Pure Appl. Alg.*, 19:385–400, 1980.

[21] D. Rolfsen. *Knots and Links.* Publish or Perish, Inc., Berkeley, CA, U.S.A., 1976.

[22] L. Schneps. *Dessins d'enfants* on the Riemann sphere. In L. Schneps, editor, *The Grothendieck Theory of* Dessins d'Enfants, pages 47–77. London Mathematical Society Lecture Notes Series 200, Cambridge University Press, 1994.

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