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$\mathcal{M}$-COMPLETENESS IS SELDOM MONADIC OVER GRAPHS

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Abstract. For a set $\mathcal{M}$ of graphs the category $\mathbf{Cat}_{\mathcal{M}}$ of all $\mathcal{M}$-complete categories and all strictly $\mathcal{M}$-continuous functors is known to be monadic over $\mathbf{Cat}$. The question of monadicity of $\mathbf{Cat}_{\mathcal{M}}$ over the category of graphs is known to have an affirmative answer when $\mathcal{M}$ specifies either (i) all finite limits, or (ii) all finite products, or (iii) equalizers and terminal objects, or (iv) just terminal objects. We prove that, conversely, these four cases are (essentially) the only cases of monadicity of $\mathbf{Cat}_{\mathcal{M}}$ over the category of graphs, provided that $\mathcal{M}$ is a set of finite graphs containing the empty graph.

1. Introduction

For more than twenty years several authors have studied properties of the category $\mathbf{Cat}_{\mathcal{M}}$ of all $\mathcal{M}$-complete categories and all strictly $\mathcal{M}$-continuous functors. Here we might have taken $\mathcal{M}$ to be a set of categories, or more generally a set of weights in the context of weighted limits; but for the present paper we are concerned with the case where $\mathcal{M}$ is a set of graphs (in the usual modern sense of the word "graph" : namely a diagram in $\mathbf{Set}$ of the form $s, t : E \longrightarrow V$). Then an object of $\mathbf{Cat}_{\mathcal{M}}$ is a small category $\mathcal{C}$ together with a *choice* of a limit-object and a limit-cone for each diagram $D : M \longrightarrow \mathcal{C}$ with $M \in \mathcal{M}$ (so that the same $\mathcal{C}$ with different limit-choices would be a different object of $\mathbf{Cat}_{\mathcal{M}}$), and a morphism in $\mathbf{Cat}_{\mathcal{M}}$ is a functor preserving strictly the chosen $\mathcal{M}$-limits; we may often use the alternative phrase "*assigned* $\mathcal{M}$-limits". In 1975, C. Lair [LR] observed that the evident forgetful functor from $\mathbf{Cat}_{\mathcal{M}}$ to the category $\mathbf{Cat}$ of small categories is monadic; a recent proof of this can also be found in [KL]. A deeper question is the monadicity of the forgetful functor $U_{\mathcal{M}} : \mathbf{Cat}_{\mathcal{M}} \longrightarrow \mathbf{Gph}$ where $\mathbf{Gph}$ is the category of (small) graphs and homomorphisms; when this is monadic, we often loosely say that "$\mathbf{Cat}_{\mathcal{M}}$ is monadic over $\mathbf{Gph}$", it being understood that $U_{\mathcal{M}}$ is the functor in question. For example, C. Lair showed in [LR] that

(i) $\mathbf{Cat}$ is monadic over $\mathbf{Gph}$ (the case of empty $\mathcal{M}$)

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and

(ii) $\mathbf{Cat}_{FP}$, the category of all small categories with finite products, is also monadic over $\mathbf{Gph}$ (the case where $\mathcal{M}$ consists of all finite discrete graphs).

The categories $\mathbf{Cat}_{\mathcal{M}}$ were later studied by A. Burroni in [BU] who proved, among other things, that the following categories are monadic over $\mathbf{Gph}$:

(iii) $\mathbf{Cat}_{T}$, the category of all small categories with a terminal object (the case where $\mathcal{M}$ consists of the empty graph $\emptyset$ alone)

and

(iv) $\mathbf{Cat}_{PB}$, the category of all small categories with pullbacks (the case where $\mathcal{M} = \{\mathbf{Pb}\}$ and $\mathbf{Pb}$ is a single co-span).

Burroni's proofs consisted of giving operations whose arities were finite graphs, together with equations between derived operations, in such a way that the algebras for the theory so presented were the categories with chosen limits of the given kind; this provides a complete proof, since the algebras for such a theory are known to be those for a finitary monad — for a very general account of monads so presented on locally finitely presentable (perhaps enriched) categories, see [PW]. Accordingly, by putting together all the operations and all the equations in (iii) and (iv), we may conclude the monadicity over $\mathbf{Gph}$ of

(v) $\mathbf{Cat}_{PB+T}$, the category of all small categories with pullbacks and a terminal object (the case where $\mathcal{M} = \{\mathbf{Pb}, \emptyset\}$).

Burroni interprets this last as the monadicity over $\mathbf{Gph}$ of

(vi) $\mathbf{Cat}_{LEX}$, the category of all finitely-complete small categories (the case where $\mathcal{M}$ consists of all finite graphs);

but in fact this interpretation overlooks a delicate point that we shall discuss more fully below : namely, that the evident forgetful functor $\mathbf{Cat}_{LEX} \longrightarrow \mathbf{Cat}_{PB+T}$ is not an equivalence, failing even to be fully faithful. In a subsequent paper [MS], J. MacDonald and A. Stone provide explicit operations and equations to establish the monadicity over $\mathbf{Gph}$ of (iii) once again, and also of

(vii) $\mathbf{Cat}_{EQ}$, the category of all small categories with equalizers (the case where $\mathcal{M} = \{\mathbf{Eq}\}$ and $\mathbf{Eq}$ is a single parallel pair)

and of

(viii) $\mathbf{Cat}_{BP}$, the category of all small categories with binary products (the case where $\mathcal{M} = \{\mathbf{Pr}\}$ and $\mathbf{Pr}$ is the two-vertex discrete graph).

From (iii), (vii), and (viii) there follows at once the monadicity over $\mathbf{Gph}$ of such combinations as

(ix) $\mathbf{Cat}_{EQ+T}$, the category of all small categories with equalizers and a terminal object (the case where $\mathcal{M} = \{\mathbf{Eq}, \emptyset\}$, which we need to refer to below)

and

(x) $\mathbf{Cat}_{BP+EQ+T}$ (the case where $\mathcal{M} = \{\mathbf{Pr}, \mathbf{Eq}, \emptyset\}$;

once again, overlooking the delicate point above, MacDonald and Stone interpret (x) as the monadicity over **Gph** of $\mathbf{Cat}_{LEX}$.

The methods used by the authors above, and others who investigated the categories $\mathbf{Cat}_{\mathcal{M}}$ (see for example [CL], [DK]), were quite disparate. The second author and I.J. Le Creurer proved in [KL] a single result which subsumes all of the valid results above : recall that a graph without cycles is called *acyclic* and observe that, except in (vi), all the graphs used above ($\mathbf{Pb}, \mathbf{Eq}, \emptyset$, $\mathbf{Pr}$ and other discrete graphs) are acyclic.

1.1 Theorem. ([KL]) *For every class $\mathcal{M}$ of finite acyclic graphs, the functor $U_{\mathcal{M}}$: $\mathbf{Cat}_{\mathcal{M}} \longrightarrow \mathbf{Gph}$ is monadic.*

The paper [KL] also contains the first published case which fails to be monadic: categories having limits of endomorphisms. More precisely, let **Le** be the one-vertex graph $\{s : e \longrightarrow e\}$; then the forgetful functor $U_{\{\mathbf{Le}\}} : \mathbf{Cat}_{\{\mathbf{Le}\}} \longrightarrow \mathbf{Gph}$ (from the category of all small categories with limits of endomorphisms) is not monadic.

Note that Theorem 1.1 does not imply the monadicity over **Gph** of $\mathbf{Cat}_{LEX}$, since not all finite graphs are acyclic. In view of the gaps in the earlier arguments, it is not clear that any valid proof of the monadicity over **Gph** of $\mathbf{Cat}_{LEX}$ has yet been given.

The present paper has two goals. First, we analyze and resolve such difficulties as the failure of $\mathbf{Cat}_{LEX} \longrightarrow \mathbf{Cat}_{PB+T}$ to be an equivalence, proving Theorem 1.4 below, which among other things fills the gaps in the earlier arguments regarding $\mathbf{Cat}_{LEX}$ and restores their validity — although its real use is much wider, and we need it to state our main result. However we do this analysis in an Appendix, since it necessarily involves the more general concept of *weighted limit*, and is as easily carried out for enriched categories as for ordinary ones. The second and more central goal is the rather surprising result that all the monadic cases of $U_{\mathcal{M}}$ are among those listed above – provided that $\mathcal{M}$ is required to contain the empty graph, so that terminal objects are among the $\mathcal{M}$-limits. (If we don't insist on a terminal object, we get rather odd examples, such as categories with *ternary* products and such other limits as are consequences of these, namely the products $A_1 \times A_2 \cdots \times A_n$ with n odd; the reader will easily construct even wilder examples.) Both in order to formulate and prove Theorem 1.4, and to state our central result with precision, we need to introduce the following concept (called "closure" in the Albert-Kelly paper [AK], where it was considered in much greater generality):

1.2 Definition. *The saturation of a class $\mathcal{M}$ of finite graphs is the class $\overline{\mathcal{M}}$ of all those finite graphs A such that every category with $\mathcal{M}$-limits has A-limits. We call $\mathcal{M}$ saturated when $\mathcal{M} = \overline{\mathcal{M}}$; and we call the classes $\mathcal{M}$ and $\mathcal{N}$ equivalent, writing $\mathcal{M} \sim \mathcal{N}$, when they have the same saturation.*

Remark. It is proved in [AK] that every functor between $\mathcal{M}$-complete categories that preserves $\mathcal{M}$-limits (in the usual, non-strict, sense) also preserves $\overline{\mathcal{M}}$-limits. We recall in Theorem 3.1 below the *explicit* condition for A to lie in $\overline{\mathcal{M}}$. Note that the concept of saturation is context-sensitive : here we are speaking about classes of finite graphs, but there are analogous concepts for classes of finite categories, of arbitrary small categories, or of the *weights* involved in weighted limits.

There is of course an evident "forgetful" functor $U_{\overline{\mathcal{M}},\mathcal{M}} : \mathbf{Cat}_{\overline{\mathcal{M}}} \longrightarrow \mathbf{Cat}_{\mathcal{M}}$, which is *over* **Cat** in the sense that it commutes with the forgetful functors $\mathbf{Cat}_{\mathcal{M}} \longrightarrow \mathbf{Cat}$ and $\mathbf{Cat}_{\overline{\mathcal{M}}} \longrightarrow \mathbf{Cat}$; but except when $\mathcal{M} = \overline{\mathcal{M}}$ it fails to be an equivalence or indeed to be fully faithful. The reason is like that for the failure of $\mathbf{Cat}_{LEX} \longrightarrow \mathbf{Cat}_{PB+T}$ to be fully faithful : there is no reason why a functor strictly preserving the chosen pullbacks and the chosen terminal object should strictly preserve the other chosen finite limits, such as (say) a ternary product – for these other choices may be quite arbitrary. The following simple example illustrates the point starkly:

1.3 Example. Certainly $\mathcal{M} \sim \mathcal{N}$ if $\mathcal{M}$ is empty and $\mathcal{N}$ consists of the single graph having one object and no arrows. Here $\mathbf{Cat}_{\mathcal{M}} = \mathbf{Cat}$; while an object of $\mathbf{Cat}_{\mathcal{N}}$ is a category $\mathcal{A}$ along with an isomorphism $\alpha_A : L(A) \longrightarrow A$ in $\mathcal{A}$ for each object $A \in \mathcal{A}$, and a morphism in $\mathbf{Cat}_{\mathcal{N}}$ is to preserve these isomorphisms strictly. So the forgetful $\mathbf{Cat}_{\mathcal{N}} \longrightarrow \mathbf{Cat}_{\mathcal{M}}$ is not fully faithful.

In Example 1.3, however, there is a "natural" functor $\mathbf{Cat}_{\mathcal{M}} \longrightarrow \mathbf{Cat}_{\mathcal{N}}$ over **Cat**, given by choosing all the α_A to be identity maps. In the Appendix we prove that this is no coincidence : whenever $\mathcal{M} \subset \mathcal{N} \subset \overline{\mathcal{M}}$ there is a functor $\Gamma : \mathbf{Cat}_{\mathcal{M}} \longrightarrow \mathbf{Cat}_{\mathcal{N}}$ over **Cat** with $U_{\mathcal{N},\mathcal{M}}\Gamma = 1$. Based on this result (which we actually formulate at the level natural to it, of general weighted limits for enriched categories), together with Lemma 2.2 below, we are able in the Appendix to prove:

1.4 Theorem. *If $\mathcal{M}$ and $\mathcal{N}$ are equivalent classes of finite graphs and $\mathbf{Cat}_{\mathcal{M}}$ is monadic over* **Gph***, so is $\mathbf{Cat}_{\mathcal{N}}$.*

We can now formulate precisely our central result:

1.5 Main Theorem. *For a set $\mathcal{M}$ of finite graphs containing the empty graph, $U_{\mathcal{M}} : \mathbf{Cat}_{\mathcal{M}} \longrightarrow \mathbf{Gph}$ is monadic if and only if $\mathcal{M} \sim \mathcal{N}$ where $\mathcal{N}$ is one of the sets $\{\emptyset\}, \{\mathbf{Eq}, \emptyset\}, \{\mathbf{Pr}, \emptyset\}, \{\mathbf{Pb}, \emptyset\}$; which correspond (for suitable choices of the set $\mathcal{M}$ equivalent to $\mathcal{N}$) to $\mathbf{Cat}_T, \mathbf{Cat}_{EQ+T}, \mathbf{Cat}_{FP}$, and $\mathbf{Cat}_{LEX}$, respectively.*

Our first result needed for the proof of Theorem 1.5 concerns equalizers : we prove in Theorem 2.5 that, for any set $\mathcal{M}$ of finite graphs such that $\mathcal{M}$-completeness implies the existence of equalizers (that is to say, such that $\mathbf{Eq} \in \overline{\mathcal{M}}$), the category $\mathbf{Cat}_{\mathcal{M}}$ is monadic over **Gph**. This result can also be derived from Proposition 47 of [CL]. However that proposition is stated without proof (and the preceding Proposition 46, which essentially states the monadicity of $\mathbf{Cat}_{\{\mathbf{Eq}\}}$ over **Gph**, has a proof completely different from ours below); accordingly we elect to present a full proof in Section 2, and in fact one which extends easily to cover an important class of *weighted* limits – see Remark 2.7. Of course Theorem 2.5 provides yet another proof of the monadicity over **Gph** of $\mathbf{Cat}_{LEX}$.

Section 3 presents a number of rather technical results on the saturation $\overline{\mathcal{M}}$ of $\mathcal{M}$. Using those results we are able, in Section 4, actually to characterize those sets $\mathcal{M}$ of finite graphs with $\emptyset \in \mathcal{M}$ for which $\mathbf{Cat}_{\mathcal{M}}$ is monadic over **Gph**; and then, based on that characterization, we present a proof of Theorem 1.5. Finally, Section 5 comprises the Appendix mentioned above.

1.6 Notational Remarks. Throughout the paper we denote by M^* the *free category* on a graph M : the objects of M^* are the nodes (or vertices, or objects) of M, and the hom-set $M^*(a, b)$ is the set of all paths

$$a = p_0 \xrightarrow{\alpha_1} p_1 \xrightarrow{\alpha_2} p_2 \cdots \xrightarrow{\alpha_n} p_n = b$$

in M, including when $a = b$ the empty path, giving the identity morphism; here, of course, the α_i are elements of what are variously called the edges, or arrows, or morphisms, of M. We often write $a \in M$, rather than $a \in$ ob M, to mean that a is an object of M; similarly we often use $A \in \mathcal{C}$ to mean that A is an object of the category $\mathcal{C}$.

Note: When dealing with the following common graphs, we may sometimes loosely confuse M with M^* — as when we write $[\mathbf{Eq}, \mathbf{Set}]$ for the more correct $[\mathbf{Eq}^*, \mathbf{Set}]$, to denote the functor-category.

Eq, the equalizer domain $\quad e_1 \underset{t}{\overset{s}{\rightrightarrows}} e_2$

Pb, the pullback domain $\quad e_1 \xrightarrow[s]{} e_3 \xleftarrow[t]{} e_2$

Pr, the binary-product domain $\quad e_1 \quad e_2$

Le, the limit-of-endomorphism domain $\quad e \xrightarrow[s]{} e$

$\emptyset$, the empty domain.

We moreover use henceforth this (e_i, s, t) notation for the graphs above, without further explanation.

1.7 Foundational remarks. We need to be precise about matters of size. We suppose chosen once for all an inaccessible cardinal ∞, whereupon a set is said to be *small* when its cardinal is less than ∞; we write **Set** for the category of small sets. A category $\mathcal{A}$ is *small* when its set of arrows is small, and is *locally small* when each hom-set $\mathcal{A}(A, B)$ is small; we write **Cat** for the category (or the 2-category, in some contexts) of small categories, and **CAT** for the category (or 2-category) of locally-small categories. In order that **CAT** be an honest category, we must also put some restriction on the size of ob $\mathcal{A}$ for a locally-small $\mathcal{A}$ — by supposing that ob $\mathcal{A}$ has cardinal less than or equal to some inaccessible cardinal $\infty' \geqslant \infty$, which is usually taken to be ∞ itself. (When, however, we speak of *locally-finite* categories, which is the case $\infty = \omega$, it is usual to take ∞' to be a larger inaccessible.)

2. Equalizers Imply Monadicity

2.1 We start with a general necessary and sufficient condition for the monadicity over **Gph** of $\mathbf{Cat}_\mathcal{M}$ (where $\mathcal{M}$ is a set of graphs). The forgetful functor $U_\mathcal{M} : \mathbf{Cat}_\mathcal{M} \longrightarrow$ **Gph** under study is the composite of the forgetful functor $E_\mathcal{M} : \mathbf{Cat}_\mathcal{M} \longrightarrow \mathbf{Cat}$ and the evident forgetful functor $V : \mathbf{Cat} \longrightarrow \mathbf{Gph}$. (We use the usual convention of ignoring V notationally : a category is denoted by the same letter as the underlying graph, and similarly for functors.) Since both $E_{\mathcal{M}}$ and V are monadic functors (see [LR]), the functor $U_{\mathcal{M}} = V \cdot E_{\mathcal{M}}$ is always a right adjoint. We conclude from Beck's Theorem, in the form given by Mac Lane as Theorem 1 on page 147 of [ML], that $U_{\mathcal{M}}$ is monadic if and only if creates coequalizers of $U_{\mathcal{M}}$-split pairs.

Let us therefore consider a pair of morphisms $P_1, P_2 : \mathcal{K} \longrightarrow \mathcal{L}$ in $\mathbf{Cat}_{\mathcal{M}}$ and a split coequalizer

$$\begin{tikzcd} U_{\mathcal{M}}(\mathcal{K}) \arrow[r, shift left=2, "{U_{\mathcal{M}}(P_1), U_{\mathcal{M}}(P_2)}"] & U_{\mathcal{M}}(\mathcal{L}) \arrow[l, "J"] \arrow[r, shift left=2, "Q"] & \mathcal{C} \arrow[l, "I"] \end{tikzcd}$$

in $\mathbf{Gph}$; that is to say :

$$Q \cdot U_{\mathcal{M}}(P_1) = Q \cdot U_{\mathcal{M}}(P_2) \ ; \ Q \cdot I = id \ ; \ U_{\mathcal{M}}(P_1) \cdot J = id \ ; \ U_{\mathcal{M}}(P_2) \cdot J = I \cdot Q \ .$$

Since V is a monadic functor, there is a unique category-structure on the graph $\mathcal{C}$ for which Q is the underlying homomorphism of a functor (called Q); and moreover

$$\begin{tikzcd} \mathcal{K} \arrow[r, shift left, "P_1"] \arrow[r, shift right, "P_2"'] & \mathcal{L} \arrow[r, "Q"] & \mathcal{C} \end{tikzcd}$$

is a coequalizer in $\mathbf{Cat}$. So $U_{\mathcal{M}}$ creates the $U_{\mathcal{M}}$-split coequalizer above if and only if (i) there is a unique way of so assigning a limit to each diagram $D : M \longrightarrow \mathcal{C}$ with $M \in \mathcal{M}$ that the functor $Q : \mathcal{L} \longrightarrow \mathcal{C}$ strictly preserves $\mathcal{M}$-limits, and (ii) Q is then the coequalizer of P_1 and P_2 not only in $\mathbf{Cat}$ but also in $\mathbf{Cat}_{\mathcal{M}}$. When this is so, each $D : M \longrightarrow \mathcal{C}$ must have as its assigned limit the cone $(Q\alpha_d : QL \longrightarrow QIDd = Dd)_{d \in M}$, where $(\alpha_d : L \longrightarrow IDM)$ is the assigned limit-cone for the diagram $ID : M \longrightarrow \mathcal{L}$; in these circumstances, let us write $\lim D = Q \lim ID$, it being understood that such equations assert the equality not only of the objects but of the cones as well. Until further notice, lim denotes the *assigned* limit.

2.2 Lemma. *For any class $\mathcal{M}$ of graphs, $U_{\mathcal{M}} : \mathbf{Cat}_{\mathcal{M}} \longrightarrow \mathbf{Gph}$ is monadic if and only if, given morphisms $P_1, P_2 : \mathcal{K} \longrightarrow \mathcal{L}$ in $\mathbf{Cat}_{\mathcal{M}}$ whose coequalizer*

$$\begin{tikzcd} \mathcal{K} \arrow[r, shift left, "P_1"] \arrow[r, shift right, "P_2"'] & \mathcal{L} \arrow[r, "Q"] & \mathcal{C} \end{tikzcd}$$

in $\mathbf{Cat}$ has a splitting in $\mathbf{Gph}$, given by homomorphisms $I : \mathcal{C} \longrightarrow \mathcal{L}$ and $J : \mathcal{L} \longrightarrow \mathcal{K}$ with

$$QI \ = \ id \, , \ P_1 J \ = \ id \, , \ \textit{and} \ \ P_2 J = IQ \ ,$$

then $\mathcal{C}$ has and Q preserves $\mathcal{M}$-limits (in the usual sense, that Q takes any $\mathcal{M}$-limit-cone in $\mathcal{L}$ to a limit-cone in $\mathcal{C}$). In fact, it suffices that Q preserve, in this usual sense, the limit of $ID : M \longrightarrow \mathcal{L}$ for each diagram $D : M \longrightarrow \mathcal{C}$ with $M \in \mathcal{M}$.

Proof. The "only if" part is clear from the above, and we turn to "if". If Q is to preserve $\mathcal{M}$-limits strictly, we are *forced* to assign as the limit of $D : M \longrightarrow \mathcal{C}$ the cone $(Q\alpha_d)$, where $(\alpha_d : L \longrightarrow IDd)$ is lim ID; and we *can* do this when $(Q\alpha_d)$ is a limit-cone. So now lim $D = Q$ lim ID. We must verify next that Q strictly preserves the assigned $\mathcal{M}$-limits. In fact, for $E : M \longrightarrow \mathcal{L}$ with $M \in \mathcal{M}$ we have

$$\begin{aligned} Q \lim E &= Q \lim P_1 JE \quad \text{since } P_1 J = 1 \\ &= QP_1 \lim JE \quad \text{since } P_1 \text{ preserves } \mathcal{M}\text{-limits strictly} \\ &= QP_2 \lim JE \quad \text{since } QP_2 = QP_1 \\ &= Q \lim P_2 JE \quad \text{since } P_2 \text{ preserves } \mathcal{M}\text{-limits strictly} \\ &= Q \lim IQE \quad \text{since } IQ = P_2 J \\ &= \lim QE \quad \text{by our choice of } \mathcal{M}\text{-limits in } \mathcal{C}\,. \end{aligned}$$

Finally we must verify that the coequalizer Q of P_1 and P_2 in **Cat** is also their coequalizer in $\mathbf{Cat}_{\mathcal{M}}$. This will follow if a functor $R : \mathcal{C} \longrightarrow \mathcal{D}$, where $\mathcal{D} \in \mathbf{Cat}_{\mathcal{M}}$, preserves $\mathcal{M}$-limits strictly whenever RQ does so; and this is so because

$$R \lim D = RQ \lim ID = \lim RQID = \lim RD.$$

2.3 Remark. In the following proof we work with finite diagrams $D : M \longrightarrow \mathcal{X}$ in categories $\mathcal{X}$ having equalizers. Let us call a collection of morphisms $(\alpha_d : A \longrightarrow Dd)$ for $d \in M$ (with no side conditions) a *pre-cone* over D, writing it as $\alpha : A \longrightarrow D$. There is a universal map $e_\alpha : E_\alpha \longrightarrow A$ having the property that $\alpha \cdot e_\alpha$ is a cone over D (where $\alpha \cdot e_\alpha$ is the collection of all $\alpha_d e_\alpha$ for $d \in M$). In fact, e_α is a joint equalizer of the pairs $Dm \cdot \alpha_d, \alpha_{d'} : A \longrightarrow Dd'$ for all morphisms $m : d \longrightarrow d'$ of M. We call e_α a *conifier* of the pre-cone α. Of course, α is a cone precisely when e_α is an isomorphism. Moreover functors preserving equalizers preserve (in the obvious non-strict sense) conifiers of finite diagrams.

2.4 Notation. Recall that **Eq** denotes the graph consisting of a parallel pair of morphisms between two objects; see 1.6.

2.5 Theorem. *Let $\mathcal{M}$ be a set of finite graphs whose saturation $\overline{\mathcal{M}}$ contains* **Eq**. *Then* $\mathbf{Cat}_{\mathcal{M}}$ *is monadic over* **Gph**.

Proof. We are going to verify the condition of Lemma 2.2, that Q preserves the limit of ID for each $D : M \longrightarrow \mathcal{C}$ with $M \in \mathcal{M}$; we retain the notation of that lemma. The proof consists of the five parts (a) - (e) below. First, a property of Q used in the proof will be established. Recall that, since $\mathbf{Eq} \in \overline{\mathcal{M}}$, the categories $\mathcal{K}$ and $\mathcal{L}$ have equalizers, and the functors P_1 and P_2 preserve equalizers. We say that a parallel pair $p', p'' : A \longrightarrow B$ in $\mathcal{L}$ is *Q-equalized* if and only if $Qp' = Qp''$; then

(a) *Q maps the equalizer of each Q-equalized pair p', p'' to an isomorphism in $\mathcal{C}$.*

Proof. Let $e : E \longrightarrow JA$ be an equalizer of Jp', Jp'' in $\mathcal{K}$. Then since P_1 preserves equalizers and $P_1 J = id$, we see that $P_1 e$ is an equalizer of p', p'' in $\mathcal{L}$. It is sufficient to show that $QP_1 e$ is an isomorphism in $\mathcal{C}$. Since P_2 preserves equalizers and $P_2 J = IQ$, we see that $P_2 e$ is an equalizer of IQp', IQp''. But by assumption, the last two morphisms are equal, so that $P_2 e$ is an isomorphism in $\mathcal{L}$. It follows that $QP_2 e = QP_1 e$ is an isomorphism in $\mathcal{C}$.

(b) *Given a finite collection of parallel pairs in $\mathcal{L}$ with a common domain, each of which is Q-equalized, then Q maps their joint equalizer to an isomorphism in $\mathcal{C}$.*

Proof. This is an easy induction based on (a) : suppose $p'_i, p''_i : A \longrightarrow B_i$ are Q-equalized pairs ($i = 1, \cdots, n$) and let e be a joint equalizer of these n pairs. If $n = 1$, then Qe is an isomorphism by (a). For $n > 1$ let $\overline{e}$ be a joint equalizer of the pairs p'_i, p''_i, for $i = 1, \cdots, n-1$; then $Q\overline{e}$ is an isomorphism by the induction hypothesis. Let e be an equalizer of $p'_n\overline{e}, p''_n\overline{e}$; since this last pair is Q-equalized, Qe is an isomorphism by (a). And $\overline{e}e$ is clearly a joint equalizer of the given pairs.

(c) $Q(\lim ID)$ *is a weak limit of* D.

Proof. Let us denote the cone $\lim ID$ by $\alpha : A \longrightarrow ID$. Then, since α is a cone over ID (and Q is a functor), we see that $Q\alpha$ is a cone over $QID = D$. Now let $\beta : B \longrightarrow D$ be a cone over D. Then $I\beta$ is a pre-cone over ID, and we form as in 2.3 a conifier $e_{I\beta} : E_{I\beta} \longrightarrow IB$ of $I\beta$. Then $e_{I\beta}$ is a joint equalizer of Q-equalized pairs : indeed, for each pair $IDm \cdot I\beta_d, I\beta_{d'}$, where $m : d \longrightarrow d'$ is a morphism of M, we have, since $QI = id$, the equality

$$Q(IDm \cdot I\beta_d) = Dm \cdot \beta_d = \beta_{d'} = Q(I\beta_{d'}) \ .$$

By (b), we conclude that

$$Qe_{I\beta} \ \text{ is an isomorphism .}$$

Next, since $I\beta \cdot e_{I\beta}$ is a cone over ID, it factorizes uniquely through the limit-cone $\alpha = \lim ID$; that is, there is a unique morphism f_β such that the diagram

$$\text{(3)} \qquad \begin{tikzcd} & E_{I\beta} \arrow[dl, "e_{I\beta}"'] \arrow[dr, "f_\beta"] & \\ IB \arrow[dr, "I\beta"'] & & A \arrow[dl, "\alpha"] \\ & ID & \end{tikzcd}$$

commutes. Consequently, $QI\beta \cdot Qe_{I\beta} = Q\alpha \cdot Qf_\beta$, and since $\beta = QI\beta$, while $Qe_{I\beta}$ is invertible, this yields the desired factorization of β through the cone $Q\alpha$; in fact we have a *canonical* factorization

$$\beta = Q\alpha \cdot (Qf_\beta \cdot [Qe_{I\beta}]^{-1}) \ .$$

(d) *When we take for β above the cone $Q\alpha$ itself, the corresponding $f_{Q\alpha}$ in (3) has $Qf_{Q\alpha}$ invertible.*

Proof. Let $\alpha' : A' \longrightarrow JID$ be the chosen limit-cone of JID in $\mathcal{K}$. Let us form a conifier $e_{J\alpha}$ of $J\alpha$ in $\mathcal{K}$; then we have a unique morphism g such that the following square

$$\text{(4)} \qquad \begin{tikzcd} & E_{J\alpha} \arrow[dl, "e_{J\alpha}"'] \arrow[dr, "g"] & \\ JA \arrow[dr, "J\alpha"'] & & A' \arrow[dl, "\alpha'"] \\ & JID & \end{tikzcd}$$

commutes in $\mathcal{K}$. Consider the image of this square under P_1 and P_2. Since both are morphisms of $\mathbf{Cat}_{\mathcal{M}}$ (that is, strictly $\mathcal{M}$-continuous), we have $P_i\alpha' = \alpha$ for $i = 1, 2$. Moreover both preserve equalizers, and thus preserve conifiers. For P_1 this means that $P_1 e_{J\alpha}$ is a conifier of $P_1 J\alpha = \alpha$; and since α is a cone, $P_1 e_{J\alpha}$ is an isomorphism. The square above is mapped by P_1 to the following square:

$$\begin{tikzcd} & P_1E_{J\alpha} \arrow[dl, "P_1e_{J\alpha}"'] \arrow[dr, "P_1g"] & \\ A \arrow[dr, "\alpha"'] & & A \arrow[dl, "\alpha"] \\ & ID & \end{tikzcd}$$

Since α is a limit cone, $P_1g = P_1e_{J\alpha}$; so that P_1g too is an isomorphism. Next, P_2 maps (4) to

$$\begin{tikzcd} & P_2E_{J\alpha} \arrow[dl, "P_2e_{J\alpha}"'] \arrow[dr, "P_2g"] & \\ IQA \arrow[dr, "IQ\alpha"'] & & A \arrow[dl, "\alpha"] \\ & ID & \end{tikzcd}$$

Since $P_2e_{J\alpha}$ is, like $e_{IQ\alpha}$, a conifier of $IQ\alpha$, there is an isomorphism $u : E_{IQ\alpha} \longrightarrow P_2E_{J\alpha}$ with $e_{IQ\alpha} = (P_2e_{J\alpha})\cdot u$. Now we conclude that $f_{Q\alpha} = (P_2g)\cdot u$ because α is a limit-cone with (see (3)) $\alpha \cdot f_{Q\alpha} = IQ\alpha \cdot e_{IQ\alpha} = IQ\alpha \cdot (P_2e_{J\alpha}) \cdot u = \alpha \cdot (P_2g) \cdot u$. Applying Q, we see that $Qf_{Q\alpha} = QP_2g \cdot Qu$; and since $QP_2g = QP_1g$ is an isomorphism and Qu is an isomorphism, this proves (d). To complete the proof of the theorem, it remains only to prove :

(e) *$Q\alpha$ is a jointly monomorphic cone.*

Proof. Let $u_1, u_2 : B \longrightarrow QA$ be morphisms with $Q\alpha \cdot u_1 = Q\alpha \cdot u_2$; we show that $u_1 = u_2$. Let R be the family of parallel pairs in $\mathcal{L}$, with the domain IB, consisting of the following pairs:

(α) all the pairs

$$\begin{tikzcd} IB \arrow[r, shift left, "Iu_1"] \arrow[r, shift right, "Iu_2"'] & IQA \arrow[r, "IQ\alpha_d"'] & IDd \end{tikzcd}$$

where $d \in M$;

(β) all the pairs

$$\begin{tikzcd}
IB \arrow[r, "Iu_t"] & IQA \arrow[rr, "IQ\alpha_{d'}"] \arrow[dr, "IQ\alpha_d"'] & & IDd' \\
& & IDd \arrow[ur, "IDm"'] &
\end{tikzcd}$$

where $m : d \longrightarrow d'$ is a morphism in M and $t = 1, 2$.

Observe that all these pairs are Q-equalized : for (α) we have

$$Q(IQ\alpha_d \cdot Iu_1) \;=\; Q\alpha_d \cdot u_1 = Q\alpha_d \cdot u_2 \;=\; Q(IQ\alpha_d \cdot Iu_2)\ ,$$

and for (β) this follows from the fact that $Q\alpha$ like α is a cone, whence

$$Q(IDm \cdot IQ\alpha_d) \;=\; Dm \cdot Q\alpha_d \;=\; Q\alpha_{d'} \;=\; Q(IQ\alpha_{d'})\ .$$

By (b) above, if $h \;:\; H \longrightarrow IB$ is the joint equalizer of all the pairs in R, then

(5) $$Qh \text{ is an isomorphism in } \mathcal{C}\ .$$

It is clear, from the pairs (β) of R, that $IQ\alpha \cdot Iu_t \cdot h$ is a cone over ID (for $t = 1, 2$); and the universal property of the conifier $e_{IQ\alpha}$ implies that there exists a unique morphism $v_t : H \longrightarrow E_{IQ\alpha}$ with

(6) $$Iu_t \cdot h \;=\; e_{IQ\alpha} \cdot v_t \quad \text{for } \ t = 1, 2\ .$$

Thus, we get a diagram as follows:

$$\begin{tikzcd}
& H \arrow[dl, "h"'] \arrow[dr, shift left, "v_1"] \arrow[dr, shift right, "v_2"'] & & \\
IB \arrow[dr, shift left, "Iu_1"] \arrow[dr, shift right, "Iu_2"'] & & E_{IQ\alpha} \arrow[dl, "e_{IQ\alpha}"] \arrow[dr, "f_{Q\alpha}"] & \\
& IQA \arrow[dr, "IQ\alpha"'] & & A \arrow[dl, "\alpha"] \\
& & ID &
\end{tikzcd}$$

Because the pairs (α) lie in R, it follows from the commutativity above that

$$\alpha \cdot (f_{Q\alpha} \cdot v_1) \; = \; \alpha \cdot (f_{Q\alpha} \cdot v_2) \; .$$

Now $\alpha = \lim ID$ is a jointly monomorphic cone, so that

$$f_{Q\alpha} \cdot v_1 \; = \; f_{Q\alpha} \cdot v_2 \; .$$

Next (d) gives

$$Qv_1 \; = \; Qv_2 \; ,$$

whence (6) yields $Q(Iu_1 \cdot h) = Q(Iu_2 \cdot h)$; that is,

$$u_1 \cdot Qh \; = \; u_2 \cdot Qh \; ;$$

which, together with (5), gives $u_1 = u_2$, as desired

2.6 Remark. On the one hand, the theorem above implies Proposition 47 of [CL]; on the other, we could of course have greatly simplified the proof of our theorem by an appeal to the result of [CL]. We gave in the Introduction our reason for providing an independent complete proof.

2.7 Remark. In Section 5 below we recall the notion of the limit $\{\phi, T\}$ of $T : \mathcal{K} \longrightarrow \mathcal{C}$ weighted by $\phi : \mathcal{K} \longrightarrow \mathbf{Set}$, where $\mathcal{K}$ is a small category and ϕ, T are functors. This has the special case where $\mathcal{K}$ is the free category M^* on a graph M, so that to give ϕ and T is just to give graph-homomorphisms $\theta : M \longrightarrow \mathbf{Set}$ and $D : M \longrightarrow \mathcal{C}$; then we may as well write $\{\theta, D\}$ in place of $\{\phi, T\}$ for the limit. Of course the classical limit, lim D, is obtained by taking for θ the homomorphism constant at the set 1. It is easy to see that the proof of Theorem 2.5 extends, with only minor changes, to the case where the set $\mathcal{M}$ of finite categories is replaced by a set of "finite weights" $\theta : M \longrightarrow \mathbf{Set}$ with *graphs* as domains, such a weight being called *finite* when M is a finite graph and θd is a finite set for each $d \in M$. For example, when M is the graph $b \longleftarrow a \longrightarrow c$ and $\theta a = \emptyset$ (the empty set) while $\theta b = \theta c = 1$ (the singleton), a category $\mathcal{C}$ admits (M, θ)-limits precisely when it has binary products $B \times C$, not for *all* pairs B, C, but for such pairs as belong to *some* diagram of the form $B \longleftarrow A \longrightarrow C$; this is quite an important kind of completeness. As for the changes needed in the proof, the cone $\beta : B \longrightarrow D$ is now replaced by a natural transformation $\beta : \theta \longrightarrow \mathcal{C}(B, D-)$, and here $e_{I\beta} : E_{I\beta} \longrightarrow IB$, for instance, is replaced by the joint equalizer of the finite set of pairs

$$IDm \cdot I\beta_d(x), \quad I(Dm \cdot \beta_d(x)) \; : \; IB \longrightarrow IDd' \; ,$$

where $m : d \longrightarrow d'$ is an edge in M and $x \in \theta d$; and similarly for the other steps in the proof.

3. Some properties of the saturation

The saturation $\overline{\mathcal{M}}$ has been fully characterized in [AK] as follows (see the Appendix for the more general case of weighted limits) : given a small graph A we denote by $Y : A^* \longrightarrow [A^*, \mathbf{Set}]^{\mathrm{op}}$ the usual Yoneda embedding sending a to $A^*(a, -)$. Let $\mathcal{M}(A)$ denote the closure of the set of all these representables $A^*(a, -)$ under $\mathcal{M}$-limits in $[A^*, \mathbf{Set}]^{\mathrm{op}}$. That is, $\mathcal{M}(A)$ is the smallest full replete subcategory of $[A^*, \mathbf{Set}]^{\mathrm{op}}$ which contains $Y(A^*)$ and is closed in $[A^*, \mathbf{Set}]^{\mathrm{op}}$ under the formation of M-limits for all $M \in \mathcal{M}$.

3.1 Theorem. [AK]. *A graph A belongs to the saturation of a class $\mathcal{M}$ of graphs if and only if the functor $\Delta 1 : A^* \longrightarrow \mathbf{Set}$ (the functor constant at $1 \in \mathbf{Set}$) lies in $\mathcal{M}(A)$.*

3.2 Remark. In order to avoid variance confusion, we work with the dual description : $A \in \overline{\mathcal{M}}$ if and only if $\Delta 1$ lies in the closure $\mathcal{M}(A)$ of the representables $A^*(a, -)$ under $\mathcal{M}^{\mathrm{op}}$-colimits in $[A^*, \mathbf{Set}]$, where $\mathcal{M}^{\mathrm{op}} = \{M^{\mathrm{op}} | M \in \mathcal{M}\}$. We will thus be concerned with diagrams $D : \mathcal{D} \longrightarrow [A^*, \mathbf{Set}]$ having $\Delta 1 \cong$ colimit D, or equivalently, diagrams such that, for each object $a \in A$, if $D(a)$ denotes D evaluated at a — that is, the composite of D with the functor $\mathrm{eval}_a : [A^*, \mathbf{Set}] \longrightarrow \mathbf{Set}$ — then $1 \cong \operatorname{colim} D(a)$ in $\mathbf{Set}$ for all $a \in A$.

3.3 Definition. *A graph is said to be poor if it has no pair of objects a, b such that*

(i) *there exists a pair of disjoint paths from a to b (that is, two paths with no common arrow);*

and

(ii) *there exists no path from b to a.*

3.4 Proposition. *For every non-poor graph A the saturation of $\{A, \emptyset\}$ contains* $\mathbf{Eq}$.

Proof. By Theorem 3.1 we are to prove that $\Delta 1 : \mathbf{Eq} \longrightarrow \mathbf{Set}$ lies in the closure of $Y(\mathbf{Eq}^{\mathrm{op}})$ under A^{op}-colimits and the initial object $\Delta\emptyset$ (the constant functor with value the empty set $\emptyset$) in $[\mathbf{Eq}, \mathbf{Set}]$. To this end, we shall find a diagram $D : A^{\mathrm{op}} \longrightarrow [\mathbf{Eq}, \mathbf{Set}]$ which maps A^{op} into the full subcategory $\{Ye_1, Ye_2, \Delta\emptyset\}$ of $[\mathbf{Eq}, \mathbf{Set}]$, and satisfies $\Delta 1 \cong \operatorname{colim} D$. Recall from 1.6 that $\mathbf{Eq}$ has edges $s, t : e_1 \longrightarrow e_2$.

Since A is not poor, it has objects a and b satisfying (i) and (ii) in Definition 3.3. Denote by A_1 the full subgraph of A determined by all objects x with the hom-set $A^*(x, a)$ non-empty and by A_2 the full subgraph of A determined by all objects $x \notin A_1$ with $A^*(x, b)$ non-empty. We define D on objects x of $A^{\mathbf{op}}$ by

$$Dx = \begin{cases} Ye_1 & \text{if } x \in A_1, \\ Ye_2 & \text{if } x \in A_2, \\ \Delta\emptyset & \text{otherwise.} \end{cases}$$

To define D on morphisms, let the disjoint paths of Definition 3.3 be

$$a = p_0 \xrightarrow{\alpha_0} p_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} p_n = b$$

and

$$a = q_0 \xrightarrow{\beta_0} q_1 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_m} q_m = b\ .$$

Since $A^*(b,a) = \emptyset$, there exists $i^* \in \{1, \cdots, n\}$ with

$$p_{i^*-1} \in A_1 \ \text{ and } \ p_{i^*} \in A_2$$

as well as $j^* \in \{1, \cdots, m\}$ with

$$q_{j^*-1} \in A_1 \ \text{ and } \ q_{j^*} \in A_2\ .$$

We define $Dh : Dy \longrightarrow Dx$ for a morphism $h : x \longrightarrow y$ of A as follows. First, $Dh = 1$ whenever $Dx = Dy$; next, the case of $y \notin A_1 \cup A_2$ is clear (since Dy is the initial object), and so is the case $x = y$ (for then $Dh = id$). The only remaining case has $x \in A_1$ and $y \in A_2$. We have just two morphisms from $Dy = Ye_2$ to $Dx = Ye_1$, namely Ys and Yt. For $h : x \longrightarrow y$ with $x \in A_1$ and $y \in A_2$, we set

$$Dh = Ys \ \text{ if } \ h \neq \alpha_i* \ , \quad \text{and } \ D\alpha_i* = Yt\ .$$

We claim that $\Delta 1 \cong \operatorname{colim} D$ in [**Eq**, **Set**]; that is, the diagrams $D(e_1)$ and $D(e_2)$: $A^{\text{op}} \longrightarrow$ **Set** (see 3.2) have each a singleton colimit.

(i) $D(e_1)$ has the value $\{id\}$ on objects of A_1 and $\emptyset$ elsewhere. Since A_1 is a connected graph, we conclude that $1 \cong \operatorname{colim} D(e_1)$.

(ii) $D(e_2)$ has the value $\{s,t\}$ on objects of A_1, the value $\{id\}$ on objects of A_2, and the value $\emptyset$ elsewhere. Observe that A_1 and A_2 are connected graphs, and that $D(e_2)$ maps each arrow of A_1 to the identity map of $\{s,t\}$. Moreover, since $D\alpha_i* = Yt$ and $D\beta_{j*} = Ys$, the diagram $D(e_2)$ has $D(e_2)\alpha_i*$ mapping id to t and $D(e_2)\beta_{j*}$ mapping id to s; from which we immediately conclude that colim $D(e_2)$ is a singleton set : in fact, all the elements of type id are identified in the colimit (since A_2 is connected), as are all the elements of type s (since A_1 is connected), and all the elements of type t (for the same reason); while the final remark shows that id and t and s are also identified.

3.5 Definition. *A graph A is said to be rooted if each component of A^* has a weakly-initial object. Note that the empty graph $\emptyset$, having no components, is rooted.*

3.6 Proposition. *For every finite non-rooted graph A, the saturation of $\{A, \emptyset\}$ contains all finite graphs.*

Proof. It is clearly sufficient to prove that the saturation of $\{A, \emptyset\}$ contains **Pb**. That is, that $\Delta 1$: **Pb** $\longrightarrow$ **Set** lies in the closure under A^{op}-colimits of $Y(\mathbf{Pb}^{\text{op}})$ and the initial object $\Delta\emptyset$ in [**Pb**, **Set**]. Recall from 1.6 that **Pb** has edges $s : e_1 \longrightarrow e_3$ and $t : e_2 \longrightarrow e_3$. We shall find a diagram $D : A^{\text{op}} \longrightarrow$ [**Pb**, **Set**] mapping A^{op} into the full subcategory $\{Ye_1, Ye_2, Ye_3, \Delta\emptyset\}$ of $[\mathbf{Pb}, \mathbf{Set}]$ and satisfying $\Delta 1 \cong \operatorname{colim} D$. The diagram D is specified on objects by defining full subgraphs A_1, A_2, A_3 of A and putting

$$Dx = \begin{cases} Ye_i & \text{if } x \in A_i \text{ for some } i = 1, 2, 3, \\ \Delta\emptyset & \text{otherwise .} \end{cases}$$

To define A_1, A_2, A_3, choose a component of A^* without any weakly initial object, and denote by $\widetilde{A}$ the preordered reflection of that component. Then $\widetilde{A}$ is a finite, connected, preordered set with no least element. Consequently, $\widetilde{A}$ has two minimal elements that are incomparable. Choose $a, b \in \widetilde{A}$ as two incomparable minimal elements with a shortest zig-zag from a to b; such a zig-zag exists since $\widetilde{A}$ is connected. It is obvious that this zig-zag has the form

$$a \leqslant c \;\text{ and }\; c \geqslant b \;\text{ for some }\; c \in \widetilde{A}\,.$$

Thus in A^* we have paths

$$a = p_0 \xrightarrow{\alpha_1} p_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} p_n = c$$

and

$$b = q_0 \xrightarrow{\beta_1} q_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_m} q_m = c\,.$$

We now take the A_i to be the following full subgraphs of A:

$$\begin{aligned} A_1 &= \{x \in A \mid A^*(x, a) \neq \emptyset\}\,, \\ A_2 &= \{x \in A \mid A^*(x, b) \neq \emptyset\}\,, \end{aligned}$$

and

$$A_3 = \{x \in A - (A_1 \cup A_2) \mid A^*(x, c) \neq \emptyset\}\,.$$

Since a and b are minimal and incomparable, A_1 and A_2 are disjoint, and moreover $A^*(c, a) = \emptyset = A^*(c, b)$; thus there exist $i^* \in \{1, \cdots, n\}$ with

$$p_{i^*-1} \in A_1 \;\text{ and }\; p_{i^*} \in A_3$$

and $j^* \in \{1, \cdots, m\}$ with

$$q_{j^*-1} \in A_2 \;\text{ and }\; q_{j^*} \in A_3\,.$$

Define $Dh : Dy \longrightarrow Dx$ for a morphism $h : x \longrightarrow y$ of A as follows. The only cases to be considered are $x \in A_1 \cup A_2$ and $y \in A_3$ (since the Yp_i, for $i \neq 1, 2, 3$, have only trivial endomorphisms id, and $\Delta\emptyset$ is the initial object, while $A^*(x, y) = \emptyset$ if $x \in A_3$ and $y \in A_1 \cup A_2$). We put, as we must,

$$Dh = Ys \text{ if } x \in A_1, y \in A_3$$

and

$$Dh = Yt \text{ if } x \in A_2, y \in A_3 \ .$$

We claim that $1 \cong \operatorname{colim} D(e_i)$ in **Set** for $i = 1, 2, 3$.

(i) $1 \cong \operatorname{colim} D(e_1)$. In fact, $D(e_1)$ takes the value $\{id\}$ for objects of A_i and $\emptyset$ elsewhere, and A_1 is a connected graph.

(ii) $1 \cong \operatorname{colim} D(e_2)$ — this is analogous to (i).

(iii) $1 \cong \operatorname{colim} D(e_3)$. In fact, $D(e_3)$ takes the values $\{s\}, \{t\}, \{id\}$ for objects of A_1, A_2 and A_3 respectively, and $\emptyset$ elsewhere. Each of the graphs A_1, A_2, A_3 is connected, and $D(e_2)\alpha_i^*$ takes $\{id\}$ to $\{s\}$ while $D(e_3)\beta_j^*$ takes $\{id\}$ to $\{t\}$. It follows that $1 \cong \operatorname{colim} D(e_3)$ in **Set**.

3.7 Definition. *A graph A is said to be smooth if every cycle C in A contains every arrow whose codomain lies on C. That is, if C is*

$$a = p_0 \xrightarrow{\alpha_1} p_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} p_n = a \ ,$$

then the only arrow with codomain a in A is α_n.

3.8 Proposition. *For every non-smooth graph A the saturation of $\{A, \mathbf{Pr}, \emptyset\}$ contains all finite graphs.*

Proof. Since A is not smooth it has a cycle

$$a = p_0 \xrightarrow{\alpha_1} p_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} p_n = a$$

and a morphism $\beta : b \longrightarrow a$ with $\beta \neq \alpha_n$.

Case I. Suppose $A^*(a, b) = \emptyset$.

We shall prove that **Eq** lies in the closure of $\{A, \mathbf{Pr}, \emptyset\}$; this obviously implies that all finite graphs lie in that closure. We are going to find a diagram $D : A^{\mathrm{op}} \longrightarrow$ [**Eq**, **Set**] whose values lie in the full subcategory $\{Ye_1, Ye_2 + Ye_2, \Delta\emptyset\}$ of [**Eq**, **Set**], and for which $\Delta 1 \cong \operatorname{colim} D$ in [**Eq**, **Set**]. This proves that $\Delta 1$ lies in the closure of $Y(\mathbf{Eq})$ under finite coproducts and A^{op}-colimits, whereupon we apply Theorem 3.1.

Denote by A_1 the full subgraph of A determined by all objects x with $A^*(x, b) \neq \emptyset$ and A_2 the full subgraph of A determined by all objects $x \notin A_1$ with $A^*(x, a) \neq \emptyset$. The definition of D on objects x of A^{op} is:

$$Dx = \begin{cases} Ye_1 & \text{if } x \in A_1 \ , \\ Ye_2 + Ye_2 & \text{if } x \in A_2 \ , \\ \Delta\,\emptyset & \text{otherwise.} \end{cases}$$

Denote by $r : Ye_2 + Ye_2 \longrightarrow Ye_1$ the morphism with components Ys and Yt, and by $q : Ye_2 + Ye_2 \longrightarrow Ye_2 + Ye_2$ the canonical isomorphism (interchanging the two copies of Ye_2). We define $Dh : Dy \longrightarrow Dx$ for a morphism $h : x \longrightarrow y$ of A as follows:

(a) if $x \in A_1$ and $y \in A_2$, then $Dh = r$;
(b) if $x, y \in A_2$, then $Dh = 1$ if $h \neq \alpha_1$ and $D\alpha_1 = q$.

In the remaining cases ($x, y \in A_1$ or $y \notin A_1 \cup A_2$), the value of Dh is clear. We claim that $\Delta 1 \cong \operatorname{colim} D$ in $[\mathbf{Eq}, \mathbf{Set}]$:

(i) $D(e_1)$ has the value $\{id\}$ on objects of A_1 and $\emptyset$ elsewhere. Since A_1 is a connected graph, it follows that $1 \cong \operatorname{colim} D(e_1)$ in $\mathbf{Set}$.

(ii) $D(e_2)$ has the value $\{s, t\}$ on objects of A_1, the value $\{id\} + \{id\}$ on objects of A_2, and the value $\emptyset$ elsewhere. Denote, for simplicity, the two copies of id in $\{id\}$ and $\{id\}$ by id^1 and id^2. Observe that A_1 is connected and $D(e_2)$ maps arrows of A_1 to the identity map. Therefore, all the elements of the form s are identified in $\operatorname{colim} D(e_2)$, as are all those of the form t. Further, the graph A_2 is connected, and remains connected when α_1 is deleted from the arrows of A_2; since D maps all arrows of A_2 except α_1 to the identity map, we see that all elements of the form id^1 are identified in $\operatorname{colim} D(e_2)$, as are all those of the form id^2. Finally, from $D\alpha_1 = q$ and $D\beta = r$ we see that $(De_2)\alpha_1$ sends id^1 to id^2 and id^2 to id^1, while $(De_2)\beta$ sends id^1 to s and id^2 to t. Consequently, $1 \cong \operatorname{colim} D(e_2)$.

Case II. Suppose $A^*(a, b) \neq 0$.

Here, again, we show that $\mathbf{Eq}$ lies in the closure of $\{A, \mathbf{Pr}, \emptyset\}$ by presenting a diagram $D : A^{\mathrm{op}} \longrightarrow [\mathbf{Eq}, \mathbf{Set}]$ whose values lie now in the full subcategory $\{Ye_1 + Ye_2, \Delta\emptyset\}$ and which is such that $\Delta 1 \cong \operatorname{colim} D$.

Let A_0 be the full subgraph of A determined by all objects x with $A^*(x, a) \neq \emptyset$. We define D on objects x of A^{op} by $Dx = Ye_1 + Ye_2$ if $x \in A_0$, $Dx = \Delta\emptyset$ otherwise. Denote by $\overline{Ys}, \overline{Yt} : Ye_1 + Ye_2 \longrightarrow Ye_1 + Ye_2$ the morphisms whose first components are equal to the first injection of $Ye_1 + Ye_2$ and whose second components are Ys or Yt respectively, composed with this first injection. We define $Dh : Dy \longrightarrow Dx$ for a morphism $h : x \longrightarrow y$ of A as follows: the only case to consider is $x, y \in A_0$, and then $Dh = id$ if h is neither β nor α_n, while $D\beta = \overline{Yt}$ and $D\alpha_n = \overline{Ys}$. We shall show that $\Delta 1 \cong \operatorname{colim} D$.

(i) $D(e_1)$ has the value $\{id\}$ on objects of A_0 and $\emptyset$ elsewhere. Since A_0 is connected, we conclude that $1 \cong \operatorname{colim} D(e_1)$.

(ii) $D(e_2)$ has the value $\{s, t, id\}$ on objects of A_0 and $\emptyset$ elsewhere. First observe that A_0 is a connected graph and remains connected even if α_n and β are removed from the arrows of A_0. (In fact, we have a path from a to b. Take the shortest path possible; then certainly α_n is not among the arrows of that path. Thus after removing α_n we still have a path from a to b, which allows us to remove β without disconnecting the graph). All arrows in A_0 except α_n and β are mapped by $D(e_2)$ to the identity mapping; thus all elements of the same type (s, t or id) are identified in $\operatorname{colim} D(e_2)$. Since $D(e_2)$ maps α_n and β respectively to the mappings f and g, where $f(id) = f(s) = s$ and $f(t) = t$, while $g(id) = g(t) = t$ and $g(s) = s$, we conclude that $1 \cong \text{colim } D(e_2)$.

3.9 Proposition. *For every graph A with a cycle the saturation of $\{A, \emptyset\}$ contains* **Le**.

Proof. Recall from 1.6 that **Le** has a single edge $e : s \longrightarrow s$. We shall find a diagram $D : A^{\text{op}} \longrightarrow [\mathbf{Le}, \mathbf{Set}]$, mapping A^{op} into the full subcategory $\{Ye, \Delta\emptyset\}$ of $[\mathbf{Le}, \mathbf{Set}]$, for which $\Delta 1 \cong \text{colim } D$.

Let $\alpha : a \longrightarrow b$ be an arrow lying on a cycle of A. Denote by A_0 the full subgraph of A determined by all objects x with $A^*(x, a) \neq \emptyset$. Observe that A_0 remains a connected graph after the arrow α has been deleted from it. Define D on objects x by $Dx = Ye$ for $x \in A_0, Dx = \Delta\emptyset$ otherwise. For morphisms $h : x \longrightarrow y$ of A with $x, y \in A_0$ put $Dh = id$ if $h \neq \alpha$ and $D\alpha = Ys$. Then $\Delta 1 \cong \text{colim } D$.

In fact, after evaluating at e we get the diagram $D(e)$ in **Set** whose value is the set $\{s^n\}_{n \in \omega}$ on objects of A_0 and is the set $\emptyset$ elsewhere. Since each arrow in the connected graph obtained by deleting α from A_0 is mapped by $D(e)$ to an identity map, and since $D(e)\alpha$ is the mapping $s^n \longmapsto s^{n+1}$ for $n \in \omega$, it is clear that $1 \cong \text{colim } D(e)$ in **Set**.

3.10 Proposition. *The saturation of $\{\mathbf{Eq}\}$ contains all finite, connected, rooted graphs; that is, it contains every finite graph A for which A^* has a weakly-initial object.*

Proof. Let r be a weakly-initial object in A^*. Thus for every object x we can choose a morphism $\phi_x : r \longrightarrow x$ in A^*; in particular we choose $\phi_r = id$.

It is sufficient to prove that $\Delta 1 : A^* \longrightarrow \mathbf{Set}$ lies in the closure of $Y(A^{\text{op}})$ under coequalizers in $[A^*, \mathbf{Set}]$. To do this, consider for each arrow $\psi : x \longrightarrow y$ in A the pair $\phi_y, \psi\phi_x : r \longrightarrow y$ in A^* and its image under Yoneda, namely $Y(\phi_y), Y(\psi\phi_x) : Yy \longrightarrow Yr$; and write $t : Yr \longrightarrow T$ for the joint coequalizer in $[A^*, \mathbf{Set}]$ of these latter pairs. Since A is finite, T certainly lies in the closure of $Y(A^{\text{op}})$ under coequalizers; and we shall now show that $T \cong \Delta 1$. This is equivalent to saying that, for each object a of A, the joint coequalizer Tz of the pairs

$$A^*(y, z) \underset{- \cdot (\psi\phi_x)}{\overset{- \cdot \phi_y}{\rightrightarrows}} A^*(r, z) \qquad \text{for all } \psi : x \longrightarrow y \text{ in } A$$

is a singleton set. In fact, Tz is isomorphic to $A^*(r, z)/\sim$, where $\sim$ is the smallest equivalence relation such that

$$\alpha\phi_y \sim \alpha\psi\phi_x \text{ for all } \psi : x \longrightarrow y \text{ in } A \text{ and all } \alpha : y \longrightarrow z \text{ in } A^* .$$

By the transitivity of $\sim$ we conclude that the property above holds, in fact, for all $\psi : x \longrightarrow y$ in A^*. In particular, if we set $y = z, \alpha = id$, and $x = r$ we thus obtain

$$\phi_z \sim \psi \text{ for all } \psi : r \longrightarrow z \text{ in } A^* .$$

This means that $\sim$ has precisely one equivalence class, which concludes the proof.

3.11. We next want to prove that the saturation of $\{\mathbf{Le}\}$ contains all connected, rooted, poor, smooth graphs. We first establish a result about the nature of such graphs. Recall that a non-identity path in a graph is said to be *simple* when its nodes are pairwise distinct; and similarly for a simple cycle.

PROPOSITION. *Let A be a connected, rooted, poor, smooth graph. Then if A is acyclic, the category A^* has an initial object r, so that for each $x \in A$ there is in A^* a unique morphism $\gamma_x : r \longrightarrow x$. When A is not acyclic, we can find a simple cycle $\beta : r \longrightarrow r$ in A^* and associate to each object x of A a morphism $\gamma_x : r \longrightarrow x$ in A^*, which is the identity id_r if $x = r$ and a simple path otherwise, such that every morphism $r \longrightarrow x$ in A^* has the form $\gamma_x \beta^i$ for a unique natural number $i \geqslant 0$.*

Proof. (1) A^* has a weakly-initial object r because A is rooted and connected. We shall prove that every cycle C of A contains r.

In fact, choose a node $x \in C$. If $x \neq r$, there exists a non-identity path $\gamma : r \longrightarrow x$. The last arrow of γ lies on the cycle C, since A is smooth and the codomain of that arrow lies on C. Analogously, the last-but-one arrow of r lies on the cycle C, and so on. Thus C contains the whole path γ; in particular, it contains r.

(2) If A is acyclic, put $\beta = id_r$; otherwise, let $\beta : r \longrightarrow r$ be a cycle of the smallest possible length. Then β is a simple cycle, and we shall prove that A has no cycles other than the β^i, for $i = 1, 2, 3, \cdots$.

In fact, let $\gamma : r \longrightarrow r$ be an arbitrary cycle of A. Arguing as in (1), we see that each arrow of γ lies on the cycle β; and conversely, each arrow of β lies on γ. Consequently, since β is simple, we conclude that $\gamma = \beta^i$ for some $i \geqslant 1$. Therefore $A^*(r, r) = \{\beta^i | \, i = 0, 1, 2, \cdots\}$.

(3) For every object $x \neq r$ choose a path $\gamma_x : r \longrightarrow x$ of the smallest length; this is clearly simple. We shall prove that every path $\delta : r \longrightarrow x$ has the form $\delta = \gamma_x \beta^i$ for some $i \geqslant 0$; in particular, if δ is simple, it follows that $\delta = \gamma_x$.

The statement is true if x lies on the cycle β. In fact, here γ_x is the section of the cycle β from r to x (because, arguing as above, γ_x lies on the cycle β and γ_x is simple). That is, we have $\beta = \beta' \gamma_x$ for some $\beta' : x \longrightarrow r$ in A^*. Now the composite $\beta' \delta : r \longrightarrow r$ has the form β^i, by (2) above, and $i > 0$ since $x \neq r$; therefore

$$\beta' \delta \; = \; \beta \beta^{i-1} \; = \; \beta' \gamma_x \beta^{i-1} \; ,$$

giving $\delta = \gamma_x \beta^{i-1}$.

Let us prove that statement for x lying outside of the cycle β. If the paths γ_x and δ do not share the last arrow, put $\delta' = \delta$; if they do, denote by δ'' the maximum section at the end of the path δ which is shared with γ_x. That is, we have a decomposition $\delta = \delta'' \delta'$, with $\delta' : r \longrightarrow y$ and $\delta'' : y \longrightarrow x$, such that the last arrow of δ' does not lie on the path γ_x, and we have $\gamma_x = \delta'' \gamma'_x$ with $\gamma'_x : r \longrightarrow y$. We shall prove that y lies on the cycle β. This will conclude the proof : from the last paragraph we then know (since γ'_x is simple) that $\gamma'_x = \gamma_y$ and $\delta' = \gamma_y \beta^i$ for some i, so that $\delta = \delta'' \delta' = \delta'' \gamma_y \beta^i = \delta'' \gamma'_x \beta^i = \gamma_x \beta^i$.

To prove that y lies on the cycle β, denote by z the last node of the path γ'_x which lies on the path δ. Then the sections of the paths γ'_x and δ from z to y are disjoint.

Since A is poor, we conclude that a path exists from y to z. Composing that path with the section of γ'_x from z to y, we obtain a cycle containing y. But by (2) the only cycles of A are the β^i; and so y lies on the cycle β.

3.12 Corollary. *The saturation of* $\{\mathbf{Le}\}$ *contains all connected, rooted, poor, smooth graphs.*

Proof. Let A be such a graph as in Lemma 3.11. We shall prove that the closure of $Y(A^{\mathrm{op}})$ under colimits of endomorphisms in $[A^*, \mathbf{Set}]$ contains $\Delta 1 : A^* \longrightarrow \mathbf{Set}$; in other words that $A \in \overline{\{\mathbf{Le}\}}$. If A^* has an initial object r, then the diagram $D : \mathbf{Le} \longrightarrow [A^*, \mathbf{Set}]$ with $De = Yr$ and $Ds = id$ satisfies $\Delta 1 \cong \operatorname{colim} D$; in fact, for every object x of A, $D(x)$ has a single element. If A^* does not have an initial object, define $D : \mathbf{Le} \longrightarrow [A^*, \mathbf{Set}]$ by $De = Yr$ and $Ds = Y\beta$. Then $\Delta 1 \cong \operatorname{colim} D$. In fact, for every object x the diagram $D(x)$ is the endomorphism of the set $A^*(r, x) = \{\gamma_x\beta^i |\ i = 0, 1, 2, \cdots\}$ defined by composition with β; that is, $\gamma_x\beta^i \longmapsto \gamma_x\beta^{i+1}$. So all the $\gamma_x\beta^i$ get identified, and $\operatorname{colim} D(x) \cong 1$.

Corollary 3.12 has the following simple generalization:

3.13 Corollary. *The saturation of* $\{\mathbf{Le}, \mathbf{Pr}\}$ *contains all non-empty rooted smooth poor graphs.*

Proof. Let the connected components of the rooted simple poor graph A be $A_1, \cdots, A_n$, so that the graph A is the coproduct $A_1 + \cdots + A_n$, and the category A^* is the coproduct $A_1^* + \cdots + A_n^*$. Each A_j is a connected rooted smooth poor graph, and has the structure described in 3.11, with a corresponding $\beta_j : r_j \longrightarrow r_j$ which is either the identity or a simple cycle. We define a diagram $D_j : \mathbf{Le} \longrightarrow [A^*, \mathbf{Set}]$ by taking for D_je the representable Yr_j and for D_js the endomorphism β_j. Then $D = \sum_{j=1}^{n} D_j : \mathbf{Le} \longrightarrow [A^*, \mathbf{Set}]$ is a coproduct of representables, and $\operatorname{colim} D \cong \Delta 1$; for $\operatorname{colim} D(x) = \sum \operatorname{colim} D_j(x)$, and, as in the proof of 3.12, $\operatorname{colim} D_j(x)$ is a singleton for $x \in A_j$ and empty for $x \notin A_j$. So the result follows from 3.1.

3.14 Proposition. *The saturation of* $\{\mathbf{Pr}\}$ *consists of these non-empty finite graphs* A *for which each component of* A^* *has an initial object; so by 3.11 it contains all non-empty finite, acyclic, rooted poor graphs.*

Proof. By 3.1, A lies in the saturation of $\{\mathbf{Pr}\}$ if and only if $\Delta 1 \in [A^*, \mathbf{Set}]$ lies in the closure under binary coproducts of the representables $A^*(a, -)$; that is, if and only if we have $a_1, \cdots a_n$ in A with $n > 0$ such that the coproduct $A^*(a_1, b) + \cdots A^*(a_n, b)$ is a singleton for each $b \in A$. This, however, is just to say that each a_i is initial in its component of A^*.

3.15 Proposition. *The saturation of the empty class of graphs consists of those non-empty finite graphs* A *for which* A^* *has an initial object; so by 3.11 it includes all connected finite acyclic rooted poor graphs.*

Proof. By 3.1, A lies in the saturation of the empty class of graphs if and only if $\Delta 1 \in [A^*, \mathbf{Set}]$ is representable — that is, isomorphic to $A^*(a, -)$ for some $a \in A$. But this is to say that each $A^*(a, b)$ is a singleton, or that a is initial in A^*.

3.16 Proposition. *If the graph A has at least two components A_1 and A_2, $\mathbf{Pr}$ lies in the saturation of $\{A, \emptyset\}$.*

Proof. A is a coproduct $A_1 + A_2 + B$; define $D : A \longrightarrow [\mathbf{Pr}^*, \mathbf{Set}]$ to have the constant values Ye_1 on A_1, Ye_2 on A_2, and $\Delta\emptyset$ on B; then colim $D \cong \Delta 1$.

3.17 Proposition. *The saturation of $\{\mathbf{Eq}, \emptyset\}$ consists of the empty graph and all graphs A for which A^* has a weakly-initial object.*

Proof. By Proposition 3.10, A lies in the saturation of $\{\mathbf{Eq}\}$, and hence of $\{\mathbf{Eq}, \emptyset\}$, if A^* has a weakly-initial object. For the converse, suppose that A is a non-empty graph in the saturation of $\{\mathbf{Eq}, \emptyset\}$. Since $\mathbf{Pr}$ does not lie in the saturation of $\{\mathbf{Eq}, \emptyset\}$, it does not lie in the saturation of $\{A, \emptyset\}$; accordingly A is connected by 3.16 and rooted by 3.6.

4. A characterization of the monadicity of $\mathbf{Cat}_{\mathcal{M}}$

4.1 Theorem. *Let $\mathcal{M}$ be a set of finite graphs containing the empty graph $\emptyset$. Then $\mathbf{Cat}_{\mathcal{M}}$ is monadic over $\mathbf{Gph}$ if and only if the set $\mathcal{M}$*

(i) *contains a non-rooted graph, or*
(ii) *contains a non-poor graph, or*
(iii) *contains both a non-smooth graph and a non-empty disconnected one, or*
(iv) *consists of acyclic graphs.*

Remark. As mentioned in the Introduction, Case (iv) has been proved, in a completely different way, in [KL] (where the result is more general since the requirement $\emptyset \in \mathcal{M}$ has not been made). We use, throughout the proof, Theorem 1.4, asserting that the monadicity over $\mathbf{Gph}$ of $\mathbf{Cat}_{\mathcal{M}}$ depends only on $\overline{\mathcal{M}}$; recall that it is to be proved in the Appendix below.

Proof of sufficiency.

Cases (i) and (ii) are sufficient for monadicity by 2.5 because $\overline{\mathcal{M}}$ contains $\mathbf{Eq}$; see 3.6 and 3.4, respectively.

In Case (iii), $\overline{\mathcal{M}}$ contains $\mathbf{Pr}$ by 3.16 and thus it contains all finite graphs by 3.8; once again, 2.5 gives the monadicity.

In Case (iv) we may assume by (i) and (ii) that all the non-empty graphs M of $\mathcal{M}$, besides being acyclic and therefore smooth, are poor and rooted. So by 3.11 each component of each M^* has an initial object. Thus, if all the non-empty graphs M of $\mathcal{M}$ are connected, we have $\mathcal{M} \sim \{\emptyset\}$ by 3.15. If $\mathcal{M}$ contains a disconnected graph, then $\mathcal{M} \sim \{\mathbf{Pr}, \emptyset\}$ by 3.14. And of course each of $\mathbf{Cat}_{\{\emptyset\}}$ and $\mathbf{Cat}_{\{\mathbf{Pr}, \emptyset\}}$ is monadic over $\mathbf{Gph}$, as observed in the Introduction.

PROOF OF NECESSITY.

Supposing each of (i) - (iv) to be false, we are to prove that $\mathbf{Cat}_{\mathcal{M}}$ is not monadic over $\mathbf{Gph}$. Thus we are supposing that $\mathcal{M}$ contains a graph with a cycle, so that $\mathbf{Le} \in \overline{\mathcal{M}}$ by 3.9; that all the non-empty graphs of $\mathcal{M}$ are poor and rooted; and that either all of them are smooth or all of them are connected. In these circumstances we shall construct

(1) an object $\mathcal{L}$ of $\mathbf{Cat}_{\mathcal{M}}$, the chosen limit-cone of $D : M \longrightarrow \mathcal{L}$ with $M \in \mathcal{M}$ being written as $\alpha_D : \lim D \longrightarrow D$, with components $\alpha_{D,x} : \lim D \longrightarrow Dx$,

and

(2) a congruence $\sim$ on (the morphisms of) $\mathcal{L}$, giving the quotient category $\mathcal{C} = \mathcal{L}/\sim$, with the canonical quotient functor $Q : \mathcal{L} \longrightarrow \mathcal{C}$,

having the properties

(3) whenever $D, E : M \longrightarrow \mathcal{L}$ with $M \in \mathcal{M}$ have $QD = QE : M \longrightarrow \mathcal{C}$, then we also have $\lim D = \lim E$ and $Q\alpha_D = Q\alpha_E$ (or equivalently $\alpha_D \sim \alpha_E$);

(4) for each $D : M \longrightarrow \mathcal{L}$ with $M \in \mathcal{M}$, the cone $Q\alpha_D : Q \lim D \longrightarrow QD$ is jointly monomorphic; that is, $f \sim g$ whenever $f, g : A \longrightarrow \lim D$ in $\mathcal{L}$ have $\alpha_D f \sim \alpha_D g$;

and

(5) $\mathcal{C}$ fails to have limits of endomorphisms.

Once this is done, we deduce the non-monadicity of $U_{\mathcal{M}} : \mathbf{Cat}_{\mathcal{M}} \longrightarrow \mathbf{Gph}$ from Lemma 2.2 as follows.

We write $P_1, P_2 : \mathcal{K} \longrightarrow \mathcal{L}$ for the kernel-pair in $\mathbf{Cat}$ of $Q : \mathcal{L} \longrightarrow \mathcal{C}$. We can regard $\mathcal{K}$ as the subcategory of $\mathcal{L} \times \mathcal{L}$ whose objects are those of the form (A, A) and whose morphisms $(A, A) \longrightarrow (B, B)$ are those $(f, g) : (A, A) \longrightarrow (B, B)$ in $\mathcal{L} \times \mathcal{L}$ having $f \sim g$; then P_1, P_2 are the restrictions to $\mathcal{K}$ of the projections from $\mathcal{L} \times \mathcal{L}$ to $\mathcal{L}$. (Of course we could identify the object (A, A) of $\mathcal{K}$ with the object A of $\mathcal{L}$, and for that matter with the object QA of $\mathcal{C}$, treating each of P_1, P_2, Q as being the identity on objects; but it may be clearer to keep the three different names for the objects (A, A), A, and QA.) Clearly $Q : \mathcal{L} \longrightarrow \mathcal{C}$ is the coequalizer in $\mathbf{Cat}$ of $P_1, P_2 : \mathcal{K} \longrightarrow \mathcal{L}$. Moreover this coequalizer admits a splitting in $\mathbf{Gph}$: we get a graph-morphism $I : \mathcal{C} \longrightarrow \mathcal{L}$ with $QI = 1$ by setting $IQA = A$ on objects and by choosing for each arrow $QA \longrightarrow QB$ in $\mathcal{C}$ some representative $A \longrightarrow B$ in $\mathcal{L}$. Then $QIQ = Q = Q1$; and since $P_1, P_2 : \mathcal{K} \longrightarrow \mathcal{L}$ is also the kernel-pair of Q in $\mathbf{Gph}$, there is a graph-morphism $J : \mathcal{L} \longrightarrow \mathcal{K}$ with $P_1 J = 1$ and $P_2 J = IQ$. Next, $\mathcal{L} \times \mathcal{L}$ has the pointwise structure of an object of $\mathbf{Cat}_{\mathcal{M}}$, the chosen limit of $(D, E) : M \longrightarrow \mathcal{L} \times \mathcal{L}$ being $(\alpha_D, \alpha_E) : (\lim D, \lim E) \longrightarrow (D, E)$. We claim that the subcategory $\mathcal{K}$ of $\mathcal{L} \times \mathcal{L}$ is closed under these limits, so that $\mathcal{K}$ too becomes an object of $\mathbf{Cat}_{\mathcal{M}}$, with $P_1, P_2 : \mathcal{K} \longrightarrow \mathcal{L}$ morphisms of $\mathbf{Cat}_{\mathcal{M}}$. For let $QD = QE$. First, (3) gives $\lim D = \lim E$ and $\alpha_D \sim \alpha_E$, so that the object $(\lim D, \lim E) = (\lim D, \lim D)$ lies in $\mathcal{K}$ and the morphisms constituting the cone (α_D, α_E) lie in $\mathcal{K}$. Next, given any cone $(\beta, \gamma) : (B, B) \longrightarrow (D, E)$ in $\mathcal{K}$, we have $\beta = \alpha_D b$ and $\gamma = \alpha_E c$ for unique morphisms $b : B \longrightarrow \lim D$ and $c : B \longrightarrow \lim E = \lim D$ in $\mathcal{L}$, so that $(\beta, \gamma) = (\alpha_D, \alpha_E)(b, c)$ for a unique $(b, c) : (B, B) \longrightarrow (\lim D, \lim D)$ in $\mathcal{L} \times \mathcal{L}$. Finally, (b, c) lies in fact in $\mathcal{K}$; for we have $\beta \sim \gamma$ and so $\alpha_D b \sim \alpha_E c \sim \alpha_D c$, whence $b \sim c$ by

(4). So $P_2, P_2 : \mathcal{K} \longrightarrow \mathcal{L}$ do indeed lie in $\mathbf{Cat}_{\mathcal{M}}$; and now, if $U_{\mathcal{M}}$ were monadic, $\mathcal{C}$ would admit $\mathcal{M}$-limits by Lemma 2.2, and hence limits of endomorphisms since $\mathbf{Le} \in \overline{\mathcal{M}}$; but this is false by (5).

Suppose, then, than $\emptyset \in \mathcal{M}$, that $\mathbf{Le} \in \mathcal{M}$, and that the graphs in $\mathcal{M} - \{\emptyset\}$ are either

(A) poor, rooted, and connected

or

(B) poor, rooted, and smooth.

We treat these two cases in order.

Analysis of Case A

In order to construct our category $\mathcal{L}$, we begin with the graph having three objects a, b, c and three arrows $p : a \longrightarrow b$, $i : c \longrightarrow b$, and $r : b \longrightarrow b$; and we form the category generated by this, subject to the single relation $ri = i$. Then we obtain our category $\mathcal{L}$ from this category by freely adding to it a terminal object 1 and an initial object 0. The congruence $\sim$ on $\mathcal{L}$ is that generated by the single relation $rp \sim p$. The quotient category $\mathcal{C} = \mathcal{L}/\sim$ certainly satisfies (5) above : for in $\mathcal{C}$ the endomorphism $r : b \longrightarrow b$ admits (besides the cone $0 \longrightarrow b$) the two cones $p : a \longrightarrow b$ and $i : c \longrightarrow b$, neither of which is a limit-cone. It remains to examine the M-limits in $\mathcal{L}$ for $M \in \mathcal{M}$, and to make choices of them satisfying (3) and (4).

In the terminal-object case $M = \emptyset$, (3) is trivially satisfied, and so is (4). Turning to non-empty M, write j for a weekly-initial object in the rooted and connected M. For a diagram $D : M \longrightarrow \mathcal{L}$, we shall examine the cones over D, and the limit-cones among them, according to the various values of the object Dj of $\mathcal{L}$.

When Dj is $0, c$, or 1, there is clearly a unique cone over D with vertex Dj, and this is a limit-cone, which of course we choose as lim D; clearly (3) and (4) are then satisfied.

When $Dj = b$, write M_b for the full sub-graph of M given by the objects x having $Dx = b$. It may be the case that D has the property

$(*) Df = Dg$ for each parallel pair $f, g : x \longrightarrow y$ in M_b^*.

When this is the case, we get a cone $\alpha_D : b \longrightarrow D$ by setting $\alpha_{D,x} = Df_x$ for any choice of path $f_x : j \longrightarrow x$ in M^*; this cone has $\alpha_{D,j} = id_b$ and is clearly a limit-cone, and we choose it. Of course we have (3) and (4), since Q is an isomorphism when restricted to the full subcategory determined by $0, c, b$, and 1. When $(*)$ is not the case, D admits no cone of vertex b; the only cones are the unique one of vertex 0 and the unique one of vertex c; the latter is a limit-cone, and we choose it. Once again, for the same reason, we have (3) and (4).

The final case is that where $Dj = a$. Write M_a for the full subgraph of M given by the objects x with $Dx = a$, and M_b for that given by the objects x with $Dx = b$. Once again, if $(*)$ is not satisfied, there is no cone over D of vertex b; the only cone over D is that of vertex 0, which is a limit-cone and which we choose. This time (3) and (4) are trivially satisfied; note that E satisfies $(*)$ when D does so, if $QE = QD$. So we may as well suppose that $(*)$ is satisfied. We claim that, in these circumstances, we have

$Df = Dg$ for each pair $f, g : j \longrightarrow x$ in M^*. Once we prove this we shall be done : we get a cone $\alpha_D : a \longrightarrow D$ by setting $\alpha_{D,x} = Df_x$ for any choice of a path $f_x : j \longrightarrow x$ in M^*; this cone has $\alpha_{D,j} = id_a$ and is clearly a limit-cone, and we choose it. Now (3) is satisfied; for if $QE = QD$, then E too has $Ej = a$ and satisfies $(*)$, so certainly as objects we have $\lim E = \lim D = a$, while $Q\alpha_{E,x} = QEf_x = QDf_x = \alpha_{D,x}$. And (4) is satisfied because $Q\alpha_{D,j} = Q(id_a) = id_{Qa}$.

It remains to show that $Df = Dg$ for any $f, g : j \longrightarrow x$ in M^*. Let f be the path $\beta_n\beta_{n-1}\cdots\beta_1$ where the $\beta_i : z_{i-1} \longrightarrow z_i$ are edges of M, with $z_0 = j$ and $z_n = x$; and similarly let g be the path $\gamma_m\gamma_{m-1}\cdots\gamma_1$ where $\gamma_i : w_{i-1} \longrightarrow w_i$, with $w_0 = j$ and $w_m = x$. If $x \in M_a$ there is nothing to prove, since $Df = Dg = 1$; so suppose that $x \notin M_a$. Let i' be the greatest i for which (a) $z_i \in M_a$ and (b) z_i is one of the w_j. Moreover let j' be the greatest j for which $w_j = z_{i'}$; and write s for the common value $w_{j'} = z_{i'}$. We may call *s the last common node of f and g that lies in M_a*. Next, let i'' be the first $i > i'$ for which z_i is one of the w_j; and let j'' be the least j for which $w_j = z_{i''}$. We may call the common value $t = z_{i''} = w_{j''}$ *the first common node of f and g after s*. Now we can write $f : j \longrightarrow x$ as the composite of $f_1 : j \longrightarrow s$, $f_2 : s \longrightarrow t$, and $f_3 : t \longrightarrow x$, where f_1 is the part of the path up to $z_{i'}$, while f_2 is the part from $z_{i'}$ to $z_{i''}$, and f_3 is the rest; and similarly $g : z \longrightarrow x$ is the composite of $g_1 : j \longrightarrow s$, $g_2 : s \longrightarrow t$, and $g_3 : t \longrightarrow x$. Now of course $Df_1 = Dg_1 = id_a$; while $Df_3 = Dg_3$ by $(*)$ if $t \in M_b$, and trivially if $Dt = 1$. It remains to examine $f_2, g_2 : s \longrightarrow t$. Since $Ds = a$ and Dt is b or 1, there is no morphism $Dt \longrightarrow Ds$, and hence no path $t \longrightarrow s$ in M^*. Since M is poor, it follows from Definition 3.3 that f_2 and g_2 share a common arrow. Since t is the first common node of f and g after s, it must be the case that $i'' = i' + 1$ and $j'' = j' + 1$, so that $f_2 = \beta_{i''}$ and $g_2 = \gamma_{j''}$, and that in fact $\beta_{i''} = \gamma_{j''}$; which is to say that $f_2 = g_2$, giving $Df_2 = Dg_2$, and finally $Df = Dg$ as required — completing the proof of necessity in Case A.

Analysis of Case B.

We may as well suppose that some graph in $\mathcal{M}$ is disconnected, since otherwise we are in Case A. So $\overline{\mathcal{M}}$, besides containing $\emptyset$ and containing $\mathbf{Le}$ by hypothesis, contains $\mathbf{Pr}$ by (3.16). On the other hand, $\mathcal{M}$ lies in the saturation of $\{\mathbf{Le}, \mathbf{Pr}, \emptyset\}$ by 3.13. Accordingly it suffices by Theorem 4.1 to prove the non-monadicity of $U_{\mathcal{M}}$ where $\mathcal{M} = \{\mathbf{Le}, \mathbf{Pr}, \emptyset\}$: which we do by constructing as above an $\mathcal{L}$ in $\mathbf{Cat}_{\mathcal{M}}$ and a congruence $\sim$ on $\mathcal{L}$ satisfying (3), (4), and (5).

We begin with the graph having two objects a, b and two arrows $p : a \longrightarrow b$ and $r : b \longrightarrow b$; and we write $\mathcal{L}_0$ for the free category on this. Then we write $\mathcal{L}_1$ for the free completion of $\mathcal{L}_0$ under finite products, which we may describe as follows. An object (I, x) of $\mathcal{L}_1$ is a finite set I, which we can see as a discrete category, together with a functor $x : I \longrightarrow \mathcal{L}_0$; and a morphism $(f, \phi) : (I, x) \longrightarrow (J, y)$ in $\mathcal{L}_0$ is a functor $f : J \longrightarrow I$ together with a natural transformation $\phi : xf \longrightarrow y$. The composite of $(f, \phi) : (I, x) \longrightarrow (J, y)$ and $(g, \psi) : (J, y) \longrightarrow (K, z)$ is (fg, θ) where $\theta : xfg \longrightarrow z$ is the composite of $\phi g : xfg \longrightarrow yg$ and $\psi : yg \longrightarrow z$; and the identity of (I, x) is $(1_I, id)$. In elementary terms, x is a finite family $x : I \longrightarrow \{a, b\}$ with components x_i, and ϕ has components $\phi_j : x_{fj} \longrightarrow y_j$ which are morphisms in $\mathcal{L}_0$. In these terms, the components of θ in the composite (fg, θ) above are given by $\theta_k = \psi_k \phi_{gk}$. When $x : I \longrightarrow \{a, b\}$ is constant at b, we may call (I, x) the power b^I of b, and similarly for a^I; while the unique (I, x) with $I = \emptyset$ is the terminal object 1 of $\mathcal{L}_1$. (To ensure smallness of $\mathcal{L}_1$, we should restrict our class of finite sets I — for instance, to subsets of the natural numbers.)

Our next task is to examine endomorphisms in $\mathcal{L}_1$, and cones over these, including limit-cones when such exist. So we now consider an endomorphism $(e, \phi) : (I, x) \longrightarrow (I, x)$ in $\mathcal{L}_1$, where $e : I \longrightarrow I$ and $\phi_i : x_{ei} \longrightarrow x_i$. Let us write $g : I \longrightarrow K$ for the colimit in **Set** of the endomorphism e; here K is the quotient of I by the equivalence relation generated by identifying ei with i, and g is the canonical surjection whose fibres are the *orbits* of e in I, which we identify with the elements k of K, so that gi denotes the orbit of i. Write R for $\{i \in I |\ e^n i = i \text{ for some } n > 0\}$; for each orbit k we may call $k \cap R$ its *core*, which has the cyclic form $i, ei, \cdots, e^{n-1}i, e^n i = i$ for some $i \in k$, and has the property that, for every $j \in k$, $e^m j$ lies in the core of k for some $m \geqslant 0$. Since there is no morphism $b \longrightarrow a$ in $\mathcal{L}_0$, the existence of $\phi_i : x_{ei} \longrightarrow x_1$ implies that $x_{ei} = a$ if $x_i = a$, or equally that $x_i = b$ if $x_{ei} = b$. It follows that the elements $\{i, ei, \cdots, e^n i = i\}$ of the core of an orbit k are either all equal to a or all equal to b.

Consider now the possibility of a cone $(r, \rho) : (J, z) \longrightarrow (I, x)$ in $\mathcal{L}_1$ over the endomorphism $(e, \phi) : (I, x) \longrightarrow (I, x)$; that is, a function $r : I \longrightarrow J$ with $re = r$ along with morphisms $\rho_i : z_{ri} \longrightarrow x_i$ for which $\rho_i = \phi_i \rho_{ei}$. Iterating this last equation gives $\rho_i = \phi_i \rho_{ei} = \phi_i \phi_{ei} \rho_{e^2 i}$ and so on, or more generally

$$\rho_i = \phi_i \phi_{ei} \phi_{e^2 i} \cdots \phi_{(e^{n-1} i)} \rho_{e^n i} \tag{6}$$

for $n \geqslant 0$. Suppose now that $i \in R$, so that i is in the core of its orbit gi, and chose n so that $e^n i = i$; now (6) becomes

$$\rho_i = \phi_i \phi_{ei} \cdots \phi_{(e^{n-1} i)} \rho_i\ . \tag{7}$$

We have seen above that either each $e^r i$ here is a, or else each is b; in the first case each $\phi_{e^r i}$ here is id_a, while in the second case it is a power of r. In the latter case, since the category $\mathcal{L}_0$ is *free* on the original graph, it follows from (7) that each $\phi_{e^r i}$, including ϕ_i itself, must be the identity id_b. In other words, there can be no cone over the endomorphism $(e, \phi) : (I, x) \longrightarrow (I, x)$ unless

$$\phi_i \text{ is an identity for each } i \in R\ . \tag{8}$$

When (8) *is* satisfied, consider what it is to give a cone (r, ρ) over (e, ϕ) with vertex (J, z). To give $r : I \longrightarrow J$ with $re = r$ is of course just to give some $s : K \longrightarrow J$, whereupon $r = sg$. Then it remains to give the morphisms $\rho_i : z_{ri} = z_{sgi} \longrightarrow x_i$

satisfying $\rho_i = \phi_i \rho_{ei}$. By (6), the value of ρ_i is forced once we choose the value of $\rho_{e^n i}$ for some n with $e^n i$ in the core of the weight gi. Indeed, it suffices to choose morphisms $\rho_i : z_{sgi} \longrightarrow x_i$ for $i \in R$, satisfying $\rho_i = \phi_i \rho_{ei}$ (which by (8) reduces for $i \in R$ to $\rho_i = \rho_{ei}$) and then to define ρ_i for a general i by (6); for now ρ_i is, by (8), well-defined by (6), independently our choice of an n with $e^n i \in R$, and the ρ_i so defined clearly satisfy $\rho_i = \phi_i \rho_{ei}$. Thus, if we write y_k for the common value of the x_i with i in the core of the orbit k, choosing such a cone comes down to choosing for each $k \in K$ a morphism $\sigma_k : z_{sk} \longrightarrow y_k$, and defining ρ_i for i in the core of k by setting $\rho_i = \sigma_{gi}$.

In fact we find such a cone with vertex (K, y) on taking s to be 1_K and taking $\sigma_k : z_{sk} \longrightarrow y_k$ to be the identity of y_k; call this cone $(g, \psi) : (K, y) \longrightarrow (I, x)$, noting that $\psi_i : y_{gi} \longrightarrow x_i$ is the identity for $i \in R$. Now the description above of the general cone (r, ρ) when (8) is satisfied shows that (g, ψ) *is then the limit-cone*; for the general cone (r, ρ) factorizes uniquely through (g, ψ) as $(r, \rho) = (g, \psi)(s, \sigma)$, where $\sigma : zs \longrightarrow y$ has the components $\sigma_k : z_{sk} \longrightarrow y_k$ above. This completes our analysis of endomorphisms in $\mathcal{L}_1$ and of the cones over them.

We now complete our description of the category $\mathcal{L}$. First we add to $\mathcal{L}_1$ a new object c, along with new morphisms given by the identity of c and, in addition, one new morphism $j_I : c \longrightarrow b^I$ for each power b^I of b; the definition of composition in the resulting category $\mathcal{L}_2$ is forced, and c is an initial object in the full subcategory of $\mathcal{L}_2$ determined by c and the powers b^I. Finally we form $\mathcal{L}$ by freely adding an initial object 0 to $\mathcal{L}_2$.

To define the congruence $\sim$ on $\mathcal{L}$, we first consider the congruence $\equiv$ on $\mathcal{L}_0$ generated by the single relation $rp \equiv p$; then $\sim$ is the congruence on $\mathcal{L}$ generated by setting $(f, \phi) \sim (f, \psi) : (I, x) \longrightarrow (I, y)$ for morphisms of $\mathcal{L}_1$ satisfying $\phi_i \equiv \psi_i : x_{fi} \longrightarrow y_i$ for each $i \in I$. If $\mathcal{C}$ is the quotient category $\mathcal{L}/\sim$, we note that (I, x) is the product of the x_i not only in $\mathcal{L}_1$ but also in $\mathcal{L}$ and moreover in $\mathcal{C}$.

Certainly $\mathcal{C}$ satisfies (5) above : in fact the endomorphism $r : b \longrightarrow b$ of $\mathcal{C}$ admits no limit in $\mathcal{C}$. For it admits the cones $p : a \longrightarrow b$ and $j_1 : c \longrightarrow b$, and any cone through which each of these factorizes must be of the form $h : b^I \longrightarrow b$ for some power b^I of b; but such an h, being a projection $b^I \longrightarrow b$ composed with some $r^n : b \longrightarrow b$, is not a cone over r. So it remains only to examine the M-limits in $\mathcal{L}$ where $M \in \{\mathbf{Le}, \mathbf{Pr}, \emptyset\}$, and to make choices of them satisfying (3) and (4) above.

In the terminal-object case $\mathcal{M} = \emptyset$, we choose the limit 1 given by (I, x) with $I = \emptyset$, whereupon (3) and (4) are trivially satisfied. Next is the case $M = \mathbf{Pr}$ of binary products; here (3) is automatically satisfied since $\mathbf{Pr}$ is a discrete graph. For objects (I, x) and (J, y) of $\mathcal{L}_1$, the product $(I + J, (x, y))$ in $\mathcal{L}_1$ is, as we said, also a product in $\mathcal{L}$, and we choose it; moreover (4) is satisfied because this is also a product in $\mathcal{C}$. Finally a product of c with a power b^I is given by c, as is a product of c with itself; and any other product is given by 0 — moreover (4) is trivially satisfied when we make these choices.

It remains to consider the limit in $\mathcal{L}$ of an endomorphism. If we are speaking of an endomorphism of c or of 0, the limit is trivially c or 0 respectively, and (3) and (4) are trivially satisfied. If we are speaking of an endomorphism $(e, \phi) : (I, x) \longrightarrow (I, x)$ in $\mathcal{L}_1$, and if (8) is not satisfied, the limit is c if (I, x) is a power of b, and is 0 otherwise; moreover (3) and (4) are trivially satisfied once we observe that, if $(e', \phi') : (I, x) \longrightarrow (I, x)$ is another endomorphism with $(e', \phi') \sim (e, \phi)$, and if (e, ϕ) satisfies (8), so does (e', ϕ'); for then $e' = e$ and $\phi'_i \equiv \phi_i : x_{ei} \longrightarrow x_i$ for each $i \in I$.

So there remains only the case where $(e, \phi) : (I, x) \longrightarrow (I, x)$ satisfies (8); here the $(g, \psi) : (K, y) \longrightarrow (I, x)$ above is a limit of (e, ϕ) not only in $\mathcal{L}_1$, but also in $\mathcal{L}$; and we choose it.

To verify that the limit (g, ψ) satisfies (4), suppose that $(s, \sigma), (s', \sigma') : (J, z) \longrightarrow (K, y)$ have $(sg, \psi \cdot \sigma g) \sim (s'g, \psi \cdot \sigma' g)$. Then $sg = s'g$, giving $s' = s$, and $\psi_i \cdot \sigma_{gi} \equiv \psi_i \cdot \sigma'_{gi}$ for each $i \in I$. Since ψ_i is an identity when i lies in the core of the orbit $k = gi$, this gives $\sigma_k \equiv \sigma'_k$, and hence $(s, \sigma) \sim (s', \sigma')$, as desired.

Finally we must show that (3) is satisfied in this case. Suppose then that the endomorphisms $(e, \phi), (e', \phi') : (I, x) \longrightarrow (I, x)$ have $(e', \phi') \sim (e, \phi)$, or equivalently $e' = e$ and $\phi'_i \equiv \phi_i : x_{ei} \longrightarrow x_i$ for each $i \in I$; as we saw above, (e', ϕ') satisfies (8) when (e, ϕ) does so, which is the case under consideration. Because $e' = e$, the limits of (e, ϕ) and of (e', ϕ') coincide as objects, each being (K, y) as before; and the limit-cones (y, ψ) and (y', ψ') have $y' = y$. We have seen however that $\psi_i : y_{gi} \longrightarrow x_i$ has the form

$$\psi_i = \phi_i \phi_{ei} \cdots \phi_{e^{n-1}i} \psi_{e^n i} ,$$

where $\psi_{e^n i}$ is an identity when n is such that $e^n i$ lies in the core of the orbit gi; and now from $\psi_i = \phi_i \phi_{ei} \cdots \phi_{e^{n-1}i}$ and $\psi'_i = \phi'_i \phi'_{ei} \cdots \phi'_{e^{n-1}i}$, along with $\phi_j \equiv \phi'_j$ for each j, we get $\psi'_i \equiv \psi_i$, giving $(y, \psi') \sim (y, \psi)$, as desired. This completes the proof of necessity in Case B, and hence completes the proof of Theorem 4.1.

4.2 Proof of Theorem 1.5

It remains to show that, if a set $\mathcal{M}$ of finite graphs containing $\emptyset$ satisfies the conditions of Theorem 4.1, we have $\mathcal{M} \sim \mathcal{N}$ where $\mathcal{N}$ is one of the sets $\{\emptyset\}, \{\mathbf{Eq}, \emptyset\}, \{\mathbf{Pr}, \emptyset\}$, and $\{\mathbf{Pb}, \emptyset\}$. We look in order at (i) - (iv) of Theorem 4.1.

Case (i): $\mathcal{M}$ contains a non-empty, non-rooted graph. Here $\overline{\mathcal{M}}$ contains all finite graphs by 3.6, so that $\mathcal{M} \sim \{\mathbf{Pb}, \emptyset\}$.

For the remaining cases we may suppose that each non-empty graph in $\mathcal{M}$ is rooted.

Case (ii)a: $\mathcal{M}$ contains a non-poor graph with at least two components. Here $\overline{\mathcal{M}}$ contains $\mathbf{Eq}$ by 3.4 and contains $\mathbf{Pr}$ by 3.16, so that again $\mathcal{M} \sim \{\mathbf{Pb}, \emptyset\}$.

Case (ii)b: $\mathcal{M}$ contains a non-poor graph, and each non-empty graph in $\mathcal{M}$ is connected. Then $\mathbf{Eq} \in \overline{\mathcal{M}}$ by 3.4, and $\mathcal{M}$ is contained in the saturation of $\{\mathbf{Eq}, \emptyset\}$ by (3.17) : for each non-empty graph in $\mathcal{M}$, being rooted and connected, has a weakly-initial object. Thus $\mathcal{M} \sim \{\mathbf{Eq}, \emptyset\}$.

Case (iii): $\mathcal{M}$ contains a non-smooth graph and one with at least two components. Then $\overline{\mathcal{M}}$ contains $\mathbf{Pr}$ by 3.16, whence it contains all finite graphs by 3.8. So again $\mathcal{M} \sim \{\mathbf{Pb}, \emptyset\}$.

Case (iv): Here the graphs in $\mathcal{M}$ are acyclic, and hence smooth, and we may suppose them rooted and poor. Then, as we already saw in the proof of sufficiency for Theorem

4.1, we have $\mathcal{M} \sim \{\emptyset\}$ by 3.15 if all the non-empty graphs in $\mathcal{M}$ are connected, and $\mathcal{M} \sim \{\mathbf{Pr}, \emptyset\}$ otherwise.

5. Appendix on Saturation

5.1 Even when $\mathcal{M}$ is only a class of graphs, the Albert-Kelly criterion for a graph A to lie in the saturation $\overline{\mathcal{M}}$, given in Theorem 3.1 above, refers to $\mathcal{M}(A)$, which is a set of presheaves in $[A^*, \mathbf{Set}]$. Accordingly it is much more natural to discuss saturation and its properties, as was done in [AK], at the level of *weighted* limits, since weights are nothing but presheaves. Moreover, by the nature of the theory, it is less complicated if we work with categories admitting certain weighted *colimits*, rather than certain weighted limits; we observed this already in Remark 3.2. Once we are at this level, it is just as easy to deal with *enriched* categories as with ordinary ones. Because some readers may be less familiar with enriched categories, we shall in fact write our discussion for ordinary ones — noting, however, that the results remain true for enriched ones, with exactly the same proofs.

Recall now (for instance from [KE], wherein weighted colimits were called indexed colimits) that a *weight* is a functor $\phi : \mathcal{K}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ with $\mathcal{K}$ a small category. A functor $T : \mathcal{K} \longrightarrow \mathcal{A}$, where $\mathcal{A}$ is locally small, is said to admit a ϕ-weighted colimit, or just a ϕ-colimit, if the functor $\mathcal{A} \longrightarrow \mathbf{Set}$ sending A to $[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}](\phi, \mathcal{A}(T-, A))$ is representable; that is, if there is an object $\phi * T$ in $\mathcal{A}$ and an isomorphism

$$\pi : \mathcal{A}(\phi * T, A) \;\cong\; [\mathcal{K}^{\mathrm{op}}, \mathbf{Set}](\phi, \mathcal{A}(T-, A))$$

natural in $A \in \mathcal{A}$. Then we say that $(\phi * T, \pi)$ is *a choice of ϕ-colimit for T*. If another such choice is given by

$$\pi' : \mathcal{A}(\phi *' T, A) \;\cong\; [\mathcal{K}^{\mathrm{op}}, \mathbf{Set}](\phi, \mathcal{A}(T-, A))\,,$$

then there is a unique isomorphism $\lambda : \phi * T \longrightarrow \phi *' T$ for which $\pi' = \pi \cdot \mathcal{A}(\lambda, A)$. Of course we often speak loosely of $\phi * T$, which is determined to within a unique isomorphism, as *the ϕ-colimit of T*; but, even in such phrases, $\phi * T$ is really a shorthand for the pair $(\phi * T, \pi)$. When ϕ is $\Delta 1 : \mathcal{K}^{\mathrm{op}} \longrightarrow \mathbf{Set}$, the functor constant at the singleton set 1, we note that $[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}](\Delta 1, \mathcal{A}(T-, A))$ is in effect the set of inductive cones over T with vertex A, so that $\Delta 1 * T$ is the classical colimit, colim T, of T. This in turn has the special case where $\mathcal{K}$ is the free category M^* on a graph M, so that $T : M^* \longrightarrow \mathcal{A}$ corresponds to a graph-morphism $P : M \longrightarrow \mathcal{A}$; here it is common to write colim P for colim T. Of course ϕ-weighted *limits* in $\mathcal{A}$ are just ϕ-weighted colimits in $\mathcal{A}^{\mathrm{op}}$: more precisely, if $\phi : \mathcal{K}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is a weight as above, and $T : \mathcal{K}^{\mathrm{op}} \longrightarrow \mathcal{A}$ is a functor, giving rise to the dual functor $T^{\mathrm{op}} : \mathcal{K} \longrightarrow \mathcal{A}^{\mathrm{op}}$, the ϕ-weighted colimit $\phi * T^{\mathrm{op}}$ is also called the *ϕ-weighted limit* $\{\phi, T\}$ of $T : \mathcal{K}^{\mathrm{op}} \longrightarrow \mathcal{A}$; in particular, $\{\Delta 1, T\}$ is the classical lim T.

If $G : \mathcal{A} \longrightarrow \mathcal{B}$ is a functor for which both $\phi * T$ and $\phi * GT$ exist, we can form the composite

$$\begin{tikzcd}
\mathcal{A}(\phi * T, A) \arrow[rr, "\pi"] & & {[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}](\phi, \mathcal{A}(T-, A))} \arrow[d, "{[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}](\phi, G_{T-,A})}"] \\
\mathcal{B}(\phi * GT, GA) & & {[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}](\phi, \mathcal{B}(GT-, GA))} \arrow[ll, "\pi^{-1}"']
\end{tikzcd} ;$$

and by the Yoneda lemma, this composite has the form

$$\mathcal{A}(\phi * T, A) \xrightarrow[G_{\phi * T, A}]{} \mathcal{B}(G(\phi * T), GA) \xrightarrow[\mathcal{B}(\tau, GA)]{} \mathcal{B}(\phi * GT, GA)$$

for a unique

$$\tau : \phi * GT \longrightarrow G(\phi * T)$$

which we call the *canonical comparison map.* We say that G *preserves the colimit* $\phi * T$ when τ is invertible, and that G *strictly preserves the colimit* $\phi * T$ when τ is an identity.
5.2 A category $\mathcal{A}$ is said to *admit* Φ*-colimits*, or to *be* Φ*-cocomplete*, where Φ is a class of weights, if it admits the colimit $\phi * T$ for each pair (ϕ, T) where $\phi : \mathcal{K}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is a weight in Φ and $T : \mathcal{K} \longrightarrow \mathcal{A}$ is a functor with domain $\mathcal{K}$. We write Φ-**COCOM** for the 2-category of Φ-cocomplete locally-small categories, functors preserving Φ-colimits (such functors are also said to be Φ-*cocontinuous*), and all natural transformations between these. There is an evident forgetful 2-functor $W_\Phi : \Phi\text{-}\mathbf{COCOM} \longrightarrow \mathbf{CAT}$.

There is also a 2-category Φ-**COLIM**, an object of which is a category $\mathcal{A}$ in Φ-**COCOM** together with, for each pair $(\phi : \mathcal{K}^{\mathrm{op}} \longrightarrow \mathbf{Set}, T : \mathcal{K} \longrightarrow \mathcal{A})$ having $\phi \in \Phi$, a *choice* $(\phi * T, \pi)$ of a ϕ-colimit of T; more briefly, we call such an object a *category* $\mathcal{A}$ *with chosen* Φ*-colimits.* A morphism in Φ-**COLIM** is a functor which *strictly* preserves the chosen Φ-colimits; and a 2-cell is again an arbitrary natural transformation. So the forgetful 2-functor $J_\Phi : \Phi\text{-}\mathbf{COLIM} \longrightarrow \Phi\text{-}\mathbf{COCOM}$, which is surjective on objects, is not fully faithful. Let us write $U_\Phi : \Phi\text{-}\mathbf{COLIM} \longrightarrow \mathbf{CAT}$ for the composite to $W_\Phi J_\Phi$.

When we want to restrict to *small* Φ-cocomplete categories, or to *small* categories with chosen Φ-colimits, we replace the notation

$$\Phi\text{-}\mathbf{COLIM} \xrightarrow[J_\Phi]{} \Phi\text{-}\mathbf{COCOM} \xrightarrow[W_\Phi]{} \mathbf{CAT}$$

along with the composite $U_\Phi = W_\Phi J_\Phi$, by

$$\Phi\text{-}\mathbf{Colim} \xrightarrow[j_\Phi]{} \Phi\text{-}\mathbf{Cocom} \xrightarrow[w_\Phi]{} \mathbf{Cat}$$

along with the composite $u_\Phi = w_\Phi j_\Phi$.

When $\mathcal{M}$ is a class of finite graphs, and Φ consists of the weights $\Delta 1 : M^* \longrightarrow \mathbf{Set}$ (that is, $((M^*)^{\mathrm{op}})^{\mathrm{op}} \longrightarrow \mathbf{Set}$) for all $M \in \mathcal{M}$, Φ-**Colim** is the 2-category of small categories $\mathcal{A}$ with chosen colimits (in the classical sense) of each diagram $D : M^{\mathrm{op}} \longrightarrow \mathcal{A}$ with $M \in \mathcal{M}$. When we consider this only as a category, by ignoring the 2-cells, it is isomorphic to the category $\mathbf{Cat}_{\mathcal{M}}$ of the Introduction, by the isomorphism $(\ )^{\mathrm{op}}$ sending $\mathcal{A}$ to $\mathcal{A}^{\mathrm{op}}$. Then the forgetful functor $U_{\mathcal{M}} : \mathbf{Cat}_{\mathcal{M}} \longrightarrow \mathbf{Cat}$ of the Introduction is the composite

$$\mathbf{Cat}_{\mathcal{M}} \xrightarrow[(\ )^{\mathrm{op}}]{\cong} \Phi\text{-}\mathbf{Colim} \xrightarrow[u_\Phi]{} \mathbf{Cat} \xrightarrow[(\ )^{\mathrm{op}}]{\cong} \mathbf{Cat} \xrightarrow[V]{} \mathbf{Gph}.$$

For classes $\Phi \subset \Psi$ of weights we have evident forgetful 2-functors $U_{\Psi,\Phi} : \Psi\text{-}\mathbf{COLIM} \longrightarrow \Phi\text{-}\mathbf{COLIM}$ and $W_{\Psi,\Phi} : \Psi\text{-}\mathbf{COCOM} \longrightarrow \Phi\text{-}\mathbf{COCOM}$, connected by the equality $J_\Phi U_{\Psi,\Phi} = W_{\Psi,\Phi} J_\Psi$, and further satisfying $W_\Phi W_{\Psi,\Phi} = W_\Psi$ and $U_\Phi U_{\Psi,\Phi} = U_\Psi$; recall from the Introduction that we express these last two equations by saying that $W_{\Psi,\Phi}$ and $U_{\Psi,\Phi}$ are 2-functors *over* $\mathbf{Cat}$.

It may be the case that $W_{\Psi,\Phi}$ is an *equality* of 2-categories : that is, every Φ-cocomplete category is Ψ-cocomplete and every Φ-cocontinuous functor between Φ-cocomplete categories is Ψ-cocontinuous. For a given Φ there is clearly a greatest Ψ with this property, namely the class $\overline{\Phi}$ consisting of those weights $\psi : \mathcal{K}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ such that every Φ-cocomplete category is ψ-cocomplete and every Φ-cocontinuous functor between such categories is ψ-cocontinuous. (The second clause here, about cocontinuity, is known to be a consequence of the first one, about cocompleteness, when ψ is of the form $\Delta 1 : \mathcal{L}^{\mathrm{op}} \longrightarrow \mathbf{Set}$, so that ψ-colimits are classical $\mathcal{K}$-colimits; but not for a general ψ — see [AK] and [PR].) We call $\overline{\Phi}$ the *saturation* of Φ, and call the class Φ *saturated* when $\overline{\Phi} = \Phi$. (Note that, since limits in $\mathcal{A}$ are colimits in $\mathcal{A}^{\mathrm{op}}$, it also follows from $\psi \in \overline{\Phi}$ that Φ-complete categories are ψ-complete, and so on.) The saturation of Φ was determined by Albert and Kelly [AK], as follows.

For any locally-small category $\mathcal{K}$ we have the fully-faithful Yoneda embedding $Y : \mathcal{K} \longrightarrow [\mathcal{K}^{\mathrm{op}}, \mathbf{Set}]$ sending the object K of $\mathcal{K}$ to $\mathcal{K}(-, K)$. The functor-category $[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}]$ admits, like $\mathbf{Set}$, all small colimits and thus all weighted ones; write $\Phi(\mathcal{K})$ for the closure of the representables in $[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}]$ under Φ-colimits — that is, the intersection of those replete full subcategories of $[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}]$ which contain the representables $\mathcal{K}(-, K)$ and admit Φ-colimits. It is well known that $\Phi(\mathcal{K})$ is again locally small, and is a "free" Φ-cocomplete category on $\mathcal{K}$, in the sense that composition with the embedding $Z : \mathcal{K} \longrightarrow \Phi(\mathcal{K})$ induces an equivalence of categories $\Phi\text{-}\mathbf{COCOM}\ (\Phi(\mathcal{K}), \mathcal{A}) \simeq \mathbf{CAT}\ (\mathcal{K}, \mathcal{A})$; but this property of $\Phi(K)$ does not explicitly concern us here. The following formulation of the Albert-Kelly result needs only the definition of $\Phi(\mathcal{K})$ for a small $\mathcal{K}$:

5.3 Theorem. [AK] *Given a class Φ of weights, the weight $\psi : \mathcal{K}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ lies in the saturation $\overline{\Phi}$ of Φ if and only if the object ψ of $[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}]$ lies in the closure $\Phi(\mathcal{K})$ of the representables under Φ-colimits.*

5.4. We can build up $\Phi(\mathcal{K})$ from $\mathcal{K}$ by transfinite induction. We describe inductively full replete subcategories $\mathcal{K}_\alpha$ of $[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}]$, where α is an ordinal $\leqslant \infty$. First, $\mathcal{K}_0$ consists of all the isomorphs of the representables $\mathcal{K}(-, K)$; that is, it consists of what, in a wider sense, are called the "representables". Next, $\mathcal{K}_{\alpha+1}$ consists of $\mathcal{K}_\alpha$ together with all those objects of $[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}]$ which appear as colimits $\phi * S$ where $\phi : \mathcal{L}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is a weight in Φ and $S : \mathcal{L} \longrightarrow [\mathcal{K}^{\mathrm{op}}, \mathbf{Set}]$ is a functor taking its values in $\mathcal{K}_\alpha$. (In this context we are not speaking of *chosen* colimits : an object ψ of $[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}]$ is in $\mathcal{K}_{\alpha+1}$ if, for some ϕ and some S as above, there is a natural (in θ) isomorphism $[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}](\psi, \theta) \cong [\mathcal{L}^{\mathrm{op}}, \mathbf{Set}](\phi, [\mathcal{K}^{\mathrm{op}}, \mathbf{Set}](S-, \theta))$, expressing ψ as a value of $\phi * S$.) Finally, for a limit-ordinal α, we set $\mathcal{K}_\alpha = \bigcup_{\beta<\alpha} \mathcal{K}_\beta$. Clearly each $\mathcal{K}_\alpha$ is contained in $\Phi(\mathcal{K})$; and in fact $\mathcal{K}_\infty = \Phi(\mathcal{K})$, since $\mathcal{K}_\infty$ is closed in $[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}]$ under Φ-colimits. For if $\phi : \mathcal{L}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ lies in Φ and $S : \mathcal{L} \longrightarrow [\mathcal{K}^{\mathrm{op}}, \mathbf{Set}]$ takes its values in $\mathcal{K}_\infty$, the smallness of $\mathcal{L}$ ensures that S in fact takes its values in $\mathcal{K}_\alpha$ for some small α, so that $\phi * S$ lies $\mathcal{K}_{\alpha+1}$, and hence in $\mathcal{K}_\infty$.

A similar argument shows that, when Φ is a *small* class of weights, $\Phi(\mathcal{K})$ is equal to $\mathcal{K}_\alpha$ for some *small* α; whence $\Phi(\mathcal{K})$ is (equivalent to) a small category. When each $\Phi(\mathcal{K})$ is small (to within equivalence) we say that Φ is *locally small*; this may well be the case when Φ is not small. Indeed, since Φ and $\overline{\Phi}$ have the same saturation, it follows from Theorem 5.3 that $\overline{\Phi}(\mathcal{K}) = \Phi(\mathcal{K})$ for each $\mathcal{K}$; and even when Φ is small, $\overline{\Phi}$ is not — since it certainly contains all the representables $\mathcal{L}(-, L)$ for every small category $\mathcal{L}$.

Of course there are also *relative* notions of saturation : it may be that we are interested purely in weights belonging to some class Θ, and then, by the saturation of a subclass Φ of Θ, we should mean $\overline{\Phi} \cap \Theta$. For instance, Θ might be the class of all weights of the form $\Delta 1 : \mathcal{L}^{\mathrm{op}} \longrightarrow \mathbf{Set}$, corresponding to the *classical* (or "conical") colimits. Or again, Θ might be the class of weights of the form $\Delta 1 : M^* \longrightarrow \mathbf{Set}$, where M^* is the free category on a graph M; or perhaps just a *finite* graph M. It is this last that corresponds to the notion of saturation in the body of this paper, given in Definition 1.2; a class $\mathcal{M}$ of finite graphs corresponds to the class Φ of weights of the form $\Delta 1 : M^* \longrightarrow \mathbf{Set}$ with $M \in \mathcal{M}$, and then the $\overline{\mathcal{M}}$ of Definition 1.2 corresponds to the class $\overline{\Phi} \cap \Theta$, when Θ has the last sense above. Here, of course, $\overline{\mathcal{M}}$ is indeed small, since Θ is small.

We now prove the result that will ultimately give us Theorem 1.4. Although the forgetful 2-functor $W_{\overline{\Phi},\Phi} : \overline{\Phi}\text{-}\mathbf{COCOM} \longrightarrow \Phi\text{-}\mathbf{COCOM}$ is an equality of 2-categories, the forgetful 2-functor $U_{\overline{\Phi},\Phi} : \overline{\Phi}\text{-}\mathbf{COLIM} \longrightarrow \Phi\text{-}\mathbf{COLIM}$ is not even an equivalence at the level of the underlying categories : indeed it fails to be fully faithful for the same reason that $\mathbf{Cat}_{LEX} \longrightarrow \mathbf{Cat}_{PB+T}$ failed to be so in the Introduction, namely that a functor preserving the chosen Φ-colimits has no reason to preserve the chosen ψ-colimits for $\psi \in \overline{\Phi} - \Phi$. However we can prove the following, a special case of which we foreshadowed in the Introduction.

5.5 Theorem. *For any class Φ of weights, there is a 2-functor $\Gamma : \Phi\text{-}\mathbf{COLIM} \longrightarrow \overline{\Phi}\text{-}\mathbf{COLIM}$ over $\mathbf{CAT}$ for which $U_{\overline{\Phi},\Phi}\Gamma$ is the identity of $\Phi\text{-}\mathbf{COLIM}$.*

Proof. To give a 2-functor $\Gamma : \Phi\text{-}\mathbf{COLIM} \longrightarrow \overline{\Phi}\text{-}\mathbf{COLIM}$ over $\mathbf{CAT}$ is just to assign a choice of $\overline{\Phi}$-colimits to every (locally small) category with chosen Φ-colimits, in such a way that each functor strongly preserving the original Φ-colimits also strongly preserves the newly-chosen $\overline{\Phi}$-colimits. We shall shortly construct such a Γ, but without making any attempt to meet the further requirement that $U_{\overline{\Phi},\Phi}\Gamma = 1$, which asks the new $\overline{\Phi}$-colimit $\psi * T$ to be the old Φ-colimit $\psi * T$ (with its representing isomorphism) when

$\psi \in \Phi$; for such an attempt would complicate our induction below. Once we have such a Γ, we can modify it trivially to meet the requirement that $U_{\overline{\Phi},\Phi}\Gamma = 1$: for $\psi \in \Phi$, we just throw away the new $\overline{\Phi}$-colimit $\psi * T$, and replace it by the original Φ-colimit.

Suppose, then, that $\mathcal{A}$ is a category with chosen Φ-colimits, and that $\mathcal{K}$ is a small category. We shall now assign to $\mathcal{A}$ inductively all those $\overline{\Phi}$-colimits $\psi * T$ for which $T : \mathcal{K} \longrightarrow \mathcal{A}$ is a functor with domain $\mathcal{K}$ and $\psi : \mathcal{K}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is an element of $\overline{\Phi}$ with domain $\mathcal{K}^{\mathrm{op}}$, each with its representing isomorphism $\pi : \mathcal{A}(\psi * T, A) \cong [\mathcal{K}^{\mathrm{op}}, \mathbf{Set}](\psi, \mathcal{A}(T-, A))$; and we shall check at each stage that $\psi * T$ is strictly preserved by all those functors $G : \mathcal{A} \longrightarrow \mathcal{B}$ that strictly preserve the chosen Φ-colimits.

The ψ in $\overline{\Phi}$ with domain $\mathcal{K}^{\mathrm{op}}$ are the objects of $\Phi(\mathcal{K})$, which as in Section 5.4 is $\bigcup_{\alpha<\infty} \mathcal{K}_\alpha$. For each ψ in $\mathcal{K}_0$ we *choose* a pair (K, ρ) where $K \in \mathcal{K}$ and $\rho : \mathcal{K}(-, K) \cong \psi$ is an isomorphism in $[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}]$. We have the composite natural (in A) isomorphism

$$\begin{tikzcd}
{[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}](\psi, \mathcal{A}(T-, A))} \arrow[r, "{[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}](\rho, 1)}"'] & {[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}](\mathcal{K}(-, K), \mathcal{A}(T-, A))} \arrow[d, "\kappa"] \\
 & \mathcal{A}(TK, A)
\end{tikzcd}$$

wherein π is the Yoneda isomorphism. Accordingly we may assign as the colimit $\psi * T$ the object TK of $\mathcal{A}$, with the inverse of the composite isomorphism above as the representing isomorphism π for $\psi * T$. It is immediate that *any* functor $G : \mathcal{A} \longrightarrow \mathcal{B}$ strictly preserves this colimit $\psi * T$.

We now suppose inductively that the colimits $\psi * T$, along with this representing isomorphisms π, have been chosen for all $\psi \in \mathcal{K}_\alpha$, and are strictly preserved by such functors as strictly preserve the (original) chosen Φ-colimits; and we now consider a weight $\psi : \mathcal{K}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ lying in $\mathcal{K}_{\alpha+1}$ but not in $\mathcal{K}_\alpha$. For each such ψ we *choose* a triple (ϕ, S, λ) where $\phi : \mathcal{L}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is a weight in Φ, while $S : \mathcal{L} \longrightarrow [\mathcal{K}^{\mathrm{op}}, \mathbf{Set}]$ is a functor taking its values in $\mathcal{K}_\alpha$, and λ is a natural (in θ) isomorphism $[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}](\psi, \theta) \cong [\mathcal{L}^{\mathrm{op}}, \mathbf{Set}](\phi, [\mathcal{K}^{\mathrm{op}}, \mathbf{Set}](S-, \theta))$ expressing ψ as a value of the colimit $\phi * S$. By the inductive hypothesis we have for each L in $\mathcal{L}$ the Φ-colimit $SL * T$, with its representing isomorphism which we re-name

$$\mu_L : \mathcal{A}(SL * T, A) \;\cong\; [\mathcal{K}^{\mathrm{op}}, \mathbf{Set}](SL, \mathcal{A}(T-, A)) \;.$$

Again, we have the original Φ-colimit $\phi * (S? * T)$, where we are now using ? to denote a variable object of $\mathcal{L}$, and – to denote one of $\mathcal{K}$; along with its representing isomorphism

$$\nu : \mathcal{A}(\phi * (S? * T), A) \;\cong\; [\mathcal{L}^{\mathrm{op}}, \;\mathbf{Set}](\phi, \mathcal{A}(S? * T, A)) \;.$$

Putting $\theta = \mathcal{A}(A, T-)$ in the isomorphism λ, we have the composite isomorphism

$$\begin{tikzcd}
{[\mathcal{K}^{\mathrm{op}},\mathbf{Set}](\psi,\mathcal{A}(T-,A))} \arrow[r, "\lambda"] & {[\mathcal{L}^{\mathrm{op}},\mathbf{Set}](\phi?,[\mathcal{K}^{\mathrm{op}},\mathbf{Set}]((S-)?,\mathcal{A}(T-,A))} \arrow[d, "{[\mathcal{L}^{\mathrm{op}},\mathbf{Set}](\phi?,\mu^{-1})}"] \\
\mathcal{A}(\phi*(S?*T),A) & {[\mathcal{L}^{\mathrm{op}},\mathbf{Set}](\phi?,\mathcal{A}(S?*T,A)).} \arrow[l, "\nu^{-1}"]
\end{tikzcd}$$

Accordingly we may assign as the colimit $\psi * T$ the object $\phi * (S? * T)$ of $\mathcal{A}$, with the inverse of the composite isomorphism above as its representing isomorphism π. Moreover $G : \mathcal{A} \longrightarrow \mathcal{B}$ strictly preserves this colimit if it strictly preserves Φ-colimits; for in the relevant diagram from Section 5.1, the λ-part commutes by the naturality of λ in θ, the μ-part commutes by the inductive hypothesis, and the ν-part commutes because G strictly preserves Φ-colimits. The truth of the theorem now follows by induction.

5.6 Corollary. *Whenever Ψ is a class of weights with $\Phi \subset \Psi \subset \overline{\Phi}$, there is a 2-functor $\Gamma' : \Phi$-**COLIM** $\longrightarrow \Psi$-**COLIM** over **CAT** for which $U_{\Psi,\Phi}\Gamma'$ is the identity of Φ-**COLIM**.*

Proof. With $\Gamma : \Phi$-**COLIM** $\longrightarrow \overline{\Phi}$-**COLIM** as in Theorem 5.5, we have only to set $\Gamma' = U_{\overline{\Phi},\Psi}\Gamma$.

Restricting to small categories and to classical colimits — or rather limits — of diagrams with finite graphs as domains, and now writing Γ rather than Γ' in Corollary 5.6, we get what we need for Theorem 1.4:

5.7 Corollary. *Given classes $\mathcal{M}$ and $\mathcal{N}$ of finite graphs with $\mathcal{M} \subset \mathcal{N} \subset \overline{\mathcal{M}}$, write (as in the Introduction) $U_{\mathcal{N},\mathcal{M}} : \mathbf{Cat}_{\mathcal{N}} \longrightarrow \mathbf{Cat}_{\mathcal{M}}$ for the forgetful functor. Then there is a functor $\Gamma : \mathbf{Cat}_{\mathcal{M}} \longrightarrow \mathbf{Cat}_{\mathcal{N}}$ over **Cat** having $U_{\mathcal{N},\mathcal{M}}\Gamma = 1$.*

5.8 The proof of Theorem 1.4 We use Corollary 5.7 and Lemma 2.2. It suffices to treat the case where $\mathcal{M} \subset \mathcal{N} \subset \overline{\mathcal{M}}$, as in the corollary above. Suppose first that $\mathbf{Cat}_{\mathcal{M}}$ is monadic over **Gph**, and consider as in Lemma 2.2 a coequalizer

$$\begin{tikzcd}
\mathcal{K} \arrow[r, shift left, "P_1"] \arrow[r, shift right, "P_2"'] & \mathcal{L} \arrow[r, "Q"'] & \mathcal{C}
\end{tikzcd}$$

in **Cat** which is split in **Gph**, the functors P_1 and P_2 being morphisms in $\mathrm{Cat}_{\mathcal{N}}$. Then they are *a fortiori* morphisms in $\mathbf{Cat}_{\mathcal{M}}$; and since this is monadic over **Gph** it follows that $\mathcal{C}$ has and Q preserves $\mathcal{M}$-limits. Because $\mathcal{N} \subset \overline{\mathcal{M}}$ it is also the case that $\mathcal{C}$ has and Q preserves $\mathcal{N}$-limits; so that $\mathrm{Cat}_{\mathcal{N}}$ is monadic over **Gph** by Lemma 2.2.

Now suppose that $\mathbf{Cat}_{\mathcal{N}}$ is monadic over **Gph**, and consider again a coequalizer as above in **Cat** which is split in **Gph**, but now with $P_1, P_2 : \mathcal{K} \longrightarrow \mathcal{L}$ being morphisms in $\mathbf{Cat}_{\mathcal{M}}$. Then $\Gamma P_1, \Gamma P_2 : \Gamma\mathcal{K} \longrightarrow \Gamma\mathcal{L}$ (with the Γ of 5.7) are morphisms in $\mathbf{Cat}_{\mathcal{N}}$, which as functors are just $P_1, P_2 : \mathcal{K} \longrightarrow \mathcal{L}$, since $U_{\mathcal{N},\mathcal{M}}\Gamma = 1$. By the monadicity over **Gph** of $\mathbf{Cat}_{\mathcal{N}}$, it follows that $\mathcal{C}$ has and Q preserves $\mathcal{N}$-limits; so certainly $\mathcal{C}$ has and Q preserves $\mathcal{M}$-limits, so that $\mathbf{Cat}_{\mathcal{M}}$ is monadic over **Gph** by Lemma 2.2.

5.9 Remark. We have not used above that part of Theorem 5.5 and Corollary 5.6 stating that Γ and Γ' are not merely *functors* but in fact *2-functors* over **CAT**. However this extra piece of knowledge has important consequences. Kelly and Lack show in the article [LK] that Φ-**Colim** is 2-monadic over **Cat**, and that Φ-**COLIM** is 2-monadic over **CAT**, even for large classes Φ and even for enriched categories (with **CAT** replaced by $\mathcal{V}$-**CAT**). If T is the 2-monad on **CAT** for Φ-**COLIM**, and $\overline{T}$ that for $\overline{\Phi}$-**COLIM**, there is a canonical morphism $\rho : \overline{T} \longrightarrow T$ of 2-monads corresponding to the forgetful $U_{\overline{\Phi},\Phi}$. Now Theorem 5.5 gives the existence of a morphism $\sigma : T \longrightarrow \overline{T}$ of 2-monads with $\sigma\rho = 1$; whereupon $\rho\sigma \cong 1$ follows from the equality $\overline{\Phi}$-**COCOM** = Φ-**COCOM**. Thus the 2-monads T and $\overline{T}$ are exhibited as equivalent, although not isomorphic; see Section 4 of [LK] for the details.

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