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BAER INVARIANTS IN SEMI-ABELIAN CATEGORIES I: GENERAL THEORY

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ABSTRACT. Extending the work of Fröhlich, Lue and Furtado-Coelho, we consider the theory of Baer invariants in the context of semi-abelian categories. Several exact sequences, relative to a subfunctor of the identity functor, are obtained. We consider a notion of commutator which, in the case of abelianization, corresponds to Smith's. The resulting notion of centrality fits into Janelidze and Kelly's theory of central extensions. Finally we propose a notion of nilpotency, relative to a Birkhoff subcategory of a semi-abelian category.

Contents

1. Introduction

1.1. It is classical to present a group G as a quotient F/R of a free group F and a "group of relations" R. The philosophy is that F is easier to understand than G. Working with these presentations of G, it is relevant to ask which expressions of the datum $R \lhd F$ are independent of the chosen presentation. A first answer to this was given by Hopf in [23], where he showed that

$$\frac{[F,F]}{[R,F]} \qquad \text{and} \qquad \frac{R\cap[F,F]}{[R,F]}$$

are such expressions. In [1], Baer further investigated this matter, constructing several of these invariants: he constructed expressions of presentations such that "similar presenta-

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tions" induce isomorphic groups—different presentations of a group by a free group and a group of relations being always similar. Whence the name *Baer invariant* to denote such an expression. As in the two examples above, he constructed Baer invariants using commutator subgroups.

The work of Baer was followed up by Fröhlich [18], Lue [30] and Furtado-Coelho [19], who generalized the theory to the case of Higgins's Ω-groups [22]. Whereas Baer constructs invariants using commutator subgroups, these authors obtain, in a similar fashion, generalized Baer invariants from certain subfunctors of the identity functor of the variety of Ω-groups considered. Fröhlich and Lue use subfunctors associated with subvarieties of the given variety, and Furtado-Coelho extends this to arbitrary subfunctors of the identity functor. By considering the variety of groups and its subvariety of abelian groups, one recovers the invariants obtained by Baer.

That the context of Ω-groups, however, could still be further enlarged, was already hinted at by Furtado-Coelho, when he pointed out that

> ... all one needs, besides such fundamental concepts as those of kernel, image, etc., is the basic lemma on connecting homomorphisms.

1.2. We will work in the context of pointed exact protomodular categories. A category with pullbacks $\mathcal{A}$ is *Bourn protomodular* if the fibration of points is conservative [6]. In a pointed context, this amounts to the validity of the Split Short Five Lemma—see Bourn [6]. A category $\mathcal{A}$ is *regular* [2] when it has finite limits and coequalizers of kernel pairs (i.e. the two projections $k_0, k_1 : R[f] \longrightarrow A$ of the pullback of an arrow $f : A \longrightarrow B$ along itself), and when a pullback of a regular epimorphism (a coequalizer) along any morphism is again a regular epimorphism. In this case, every regular epimorphism is the coequalizer of its kernel pair, and every morphism $f : A \longrightarrow B$ has an *image factorization* $f = \mathsf{Im}\, f \circ p$, unique up to isomorphism, where $p : A \longrightarrow I[f]$ is regular epi and the image $\mathsf{Im}\, f : I[f] \longrightarrow B$ of f is mono. Moreover, in a regular category, regular epimorphisms are stable under composition, and if a composition $f \circ g$ is regular epi, then so is f. A regular category in which every equivalence relation is a kernel pair is called *Barr exact*.

Note that, if, in addition to being pointed, exact and protomodular, we assume that a category $\mathcal{A}$ has binary coproducts—an assumption which makes $\mathcal{A}$ a finitely cocomplete category, see Borceux [3]—$\mathcal{A}$ is called *semi-abelian*. This notion was introduced by Janelidze, Márki and Tholen in [28]. Some examples of semi-abelian categories: any variety of Ω-groups [22]—in particular groups, rings and crossed modules; any abelian category; the dual of the category of pointed sets. Semi-abelian varieties were characterized by Bourn and Janelidze in [11].

An important implication of these axioms is the validity of the basic diagram lemmas of homological algebra, such as the 3×3 Lemma and the Snake Lemma. Thus, keeping in mind Furtado-Coelho's remark, one could expect this context to be suitable for a general description of Fröhlich's, Lue's and Furtado-Coelho's theory of Baer invariants. Sections 4 and 5 of this text are a confirmation of that thesis.

1.3. In Section 3 we give a definition of Baer invariants. We call *presentation* of an object A of a category $\mathcal{A}$ any regular epimorphism $p : A_0 \longrightarrow A$. $\mathsf{Pr}\mathcal{A}$ will denote the category of presentations in $\mathcal{A}$ and commutative squares between them, and $\mathrm{pr} : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ the forgetful functor which maps a presentation to the object presented. Two morphisms of presentations $\boldsymbol{f}, \boldsymbol{g} : p \longrightarrow q$ are called *isomorphic*, notation $\boldsymbol{f} \simeq \boldsymbol{g}$, if $\mathrm{pr}\, \boldsymbol{f} = \mathrm{pr}\, \boldsymbol{g}$. A functor $B : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ is called a *Baer invariant* if $\boldsymbol{f} \simeq \boldsymbol{g}$ implies that $B\boldsymbol{f} = B\boldsymbol{g}$.

We prove that any functor $L_0 : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ can be turned into a Baer invariant by dividing out a subfunctor S of L_0 that is "large enough". In case L_0 arises from a functor $L : \mathcal{A} \longrightarrow \mathcal{A}$, the class $\mathcal{F}_{L_0}$ of such subfunctors S of L_0 is seen to have a minimum $L_1 : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$. (A different interpretation of the functor L_1 is given in Section 6.) Finally we show that, given an appropriate subcategory of $\mathsf{Pr}\mathcal{A}$, a Baer invariant can be turned into a functor $\mathcal{A} \longrightarrow \mathcal{A}$.

In Section 4, the context is reduced to pointed, exact and protomodular categories. Next to L_1, a second canonical functor in $\mathcal{F}_{L_0}$ is obtained. Using Noether's Third Isomorphism Theorem, we construct two exact sequences of Baer invariants. Finally, as an application of the Snake Lemma, we find a six-term exact sequence of functors $\mathcal{A} \longrightarrow \mathcal{A}$.

We call *Birkhoff subfunctor of* $\mathcal{A}$ any normal subfunctor V of $1_{\mathcal{A}}$ (i.e. a kernel $V \Longrightarrow 1_{\mathcal{A}}$) which preserves regular epimorphisms. In this way, we capture Fröhlich's notion of *variety subfunctor*. As introduced by Janelidze and Kelly in [25], a full and reflective subcategory $\mathcal{B}$ of an exact category $\mathcal{A}$ is called *Birkhoff* when it is closed in $\mathcal{A}$ under subobjects and quotient objects. A Birkhoff subcategory of a variety of universal algebra [15] (thus, in particular, any variety of Ω-groups) is nothing but a subvariety. In Section 5, we see that Birkhoff subfunctors correspond bijectively to the Birkhoff subcategories of $\mathcal{A}$: assuming that the sequence

$$0 \Longrightarrow V \overset{\mu}{\Longrightarrow} 1_{\mathcal{A}} \overset{\eta}{\Longrightarrow} U \Longrightarrow 0$$

is exact, V is a Birkhoff subfunctor if and only if U reflects $\mathcal{A}$ onto a Birkhoff subcategory. It follows that Baer invariants can be obtained by considering suitable subcategories of a pointed, exact and protomodular category. This allows us to refine the six-term exact sequence from Section 4.

In Section 6, we show that the functor $V_1 : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ associated with a Birkhoff subcategory $\mathcal{B}$ of $\mathcal{A}$ may be interpreted as a commutator. The resulting notion of *centrality* fits into Janelidze and Kelly's theory of central extensions [25]. In case $\mathcal{B}$ is the Birkhoff subcategory $\mathcal{A}_{\mathsf{Ab}}$ of all abelian objects in $\mathcal{A}$, our commutator corresponds to the one introduced by Smith [33] and generalized by Pedicchio in [32].

Finally, in Section 7, we propose a notion of *nilpotency*, relative to a Birkhoff subfunctor V of $\mathcal{A}$. We prove that an object is V-nilpotent if and only if its *V-lower central series* reaches 0. The nilpotent objects of class n form a Birkhoff subcategory of $\mathcal{A}$.

1.4. In our forthcoming paper [16], we apply our theory of Baer invariants to obtain a generalization of Hopf's formula [23] in integral homology of groups. As a corollary we find that sequence **O** is a version of the Stallings-Stammbach sequence [34], [35].

1.5. For the basic theory of semi-abelian categories we refer to the Borceux's survey [3] and Janelidze, Márki and Tholen's founding paper [28]. For general category theory we used Borceux [4] and Mac Lane [31].

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2. The context

2.1. Notation. Given a morphism $f : A \longrightarrow B$ in $\mathcal{A}$, (if it exists) its kernel is denoted by $\mathsf{Ker}\, f : K[f] \longrightarrow A$, its image by $\mathsf{Im}\, f : I[f] \longrightarrow B$ and its cokernel by $\mathsf{Coker}\, f : B \longrightarrow \mathrm{Cok}[f]$. In a diagram, the forms $A \rightarrowtail B$, $A \triangleright\!\!\longrightarrow B$ and $A \longrightarrow\!\!\triangleright B$ signify that the arrow is, respectively, a monomorphism, a normal monomorphism and a regular epimorphism.

The main results in this paper are proven in categories that are pointed, Barr exact and Bourn protomodular—thus, semi-abelian, but for the existence of binary coproducts. The reason is that such categories form a natural context for the classical theorems of homological algebra—as the Snake Lemma and Noether's Isomorphism Theorems—to hold: see, e.g. Bourn [8] or Borceux and Bourn [5]. We think it useful to recall some definitions and basic properties, and start with the crucial notion of protomodularity.

2.2. Definition. [6] *A pointed category with pullbacks* $\mathcal{A}$ *is* protomodular *as soon as the* Split Short Five-Lemma *holds. This means that for any commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r, "k'"] \arrow[d, "u"'] & A' \arrow[r, shift left, "f'"] \arrow[d, "v"] & B' \arrow[l, shift left, "s'"] \arrow[d, "w"] \\
0 \arrow[r] & K \arrow[r, "k"'] & A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "s"]
\end{tikzcd}$$

such that f *and* f' *are split epimorphisms (with resp. splittings* s *and* s'*) and such that* $k = \mathsf{Ker}\, f$ *and* $k' = \mathsf{Ker}\, f'$*,* u *and* w *being isomorphisms implies that* v *is an isomorphism.*

In a protomodular category $\mathcal{A}$, an intrinsic notion of normal monomorphism exists (see Bourn [7]). We will, however, not introduce this notion here. It will be sufficient to note that, if $\mathcal{A}$ is moreover exact, the normal monorphisms are just the kernels. The following property is very important and will be needed throughout the paper. Note that we will only apply it in the case that $\mathcal{A}$ is exact.

2.3. Proposition. [Non-Effective Trace of the 3×3 Lemma [9, Theorem 4.1]] *Consider, in a regular and protomodular category, a commutative square with horizontal regular epimorphisms*

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d, "v"'] & B' \arrow[d, "w"] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

When w is a monomorphism and v a normal monomorphism, then w is normal. ∎

A morphism $f : A \longrightarrow B$ in a pointed regular protomodular category is *proper* [8] when its image is a kernel.

In a pointed and regular context, the notion of protomodularity is strong enough to imply the basic lemma's of homological algebra, such as the 3×3 Lemma and the Snake Lemma. In a pointed, exact and protomodular category also Noether's Isomorphism Theorems hold.

We shall call a sequence

$$K \xrightarrow{k} A \xrightarrow{f} B \qquad (\mathbf{A})$$

in a pointed category *short exact*, if $k = \mathsf{Ker}\, f$ and $f = \mathsf{Coker}\, k$. We denote this situation

$$0 \longrightarrow K \overset{k}{\rightarrowtail} A \overset{f}{\twoheadrightarrow} B \longrightarrow 0.$$

If we wish to emphasize the object K instead of the arrow k, we denote the cokernel f by $\eta_K : A \longrightarrow A/K$. In a regular and protomodular category the exactness of sequence $\mathbf{A}$ is equivalent to demanding that $k = \mathsf{Ker}\, f$ and f is a regular epimorphism. Thus, a pointed, regular and protomodular category has all cokernels of kernels. A sequence of morphisms

$$\dots \longrightarrow A_{i+1} \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} A_{i-1} \longrightarrow \dots$$

in pointed, regular and protomodular category is called *exact* if $\mathsf{Im}\, f_{i+1} = \mathsf{Ker}\, f_i$, for any i.

2.4. Proposition. [Noether's Third Isomorphism Theorem, [5]] *Let $A \subseteq B \subseteq C$ be objects of a pointed, exact and protomodular category $\mathcal{A}$, such that A and B are normal in C (i.e. the inclusions are kernels). Then*

$$0 \longrightarrow \frac{B}{A} \rightarrowtail \frac{C}{A} \twoheadrightarrow \frac{C}{B} \longrightarrow 0$$

is a short exact sequence in $\mathcal{A}$. ∎

2.5. Proposition. [Snake Lemma [8, Theorem 14]] *Let $\mathcal{A}$ be a pointed, regular and protomodular category. Any commutative diagram with exact rows as below such that u, v and w are proper, can be completed to the following diagram, where all squares commute,*

$$\begin{tikzcd}
 & K[u] \arrow[r, dotted] \arrow[d, dotted, "\mathsf{Ker}\,u"'] & K[v] \arrow[r, dotted] \arrow[d, dotted, "\mathsf{Ker}\,v"'] & K[w] \arrow[d, dotted, "\mathsf{Ker}\,w"] \arrow[llddd, dotted, "\delta"] & \\
 & K' \arrow[r, "k'"'] \arrow[d, "u"'] & A' \arrow[r, "f'"'] \arrow[d, "v"] & B' \arrow[r] \arrow[d, "w"] & 0 \\
0 \arrow[r] & K \arrow[r, "k"] \arrow[d, dotted, "\mathsf{Coker}\,u"'] & A \arrow[r, "f"] \arrow[d, dotted, "\mathsf{Coker}\,v"] & B \arrow[d, dotted, "\mathsf{Coker}\,w"] & \\
 & \mathrm{Cok}[u] \arrow[r, dotted] & \mathrm{Cok}[v] \arrow[r, dotted] & \mathrm{Cok}[w] &
\end{tikzcd}$$

in such a way that

$$K[u] \longrightarrow K[v] \longrightarrow K[w] \xrightarrow{\delta} \mathrm{Cok}[u] \longrightarrow \mathrm{Cok}[v] \longrightarrow \mathrm{Cok}[w]$$

is exact. Moreover, this can be done in a natural way, i.e. defining a functor $\mathsf{pAr}(\mathsf{Pr}\mathcal{A}) \longrightarrow \mathsf{6tE}(\mathcal{A})$*, where* $\mathsf{pAr}(\mathsf{Pr}\mathcal{A})$ *is the category of proper arrows of* $\mathsf{Pr}\mathcal{A}$ *and* $\mathsf{6tE}(\mathcal{A})$ *the category of six-term exact sequences in* $\mathcal{A}$*.* ■

In fact, in [8] only the exactness of the sequence is proven. However, it is quite clear from the construction of the connecting morphism δ that the sequence is, moreover, natural.

The converse of the following property is well known to hold in any pointed category. In fact, the condition that f' be regular epi, vanishes.

2.6. Proposition. [8, 5] *Let $\mathcal{A}$ be a pointed, regular and protomodular category. Consider the following commutative diagram, where $k = \mathsf{Ker}\, f$, f' is regular epi and the left hand square a pullback:*

$$\begin{tikzcd}
 & K' \arrow[r, "k'"] \arrow[d, "u"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A' \arrow[r, "f'"] \arrow[d, "v"] & B' \arrow[d, "w"] \\
0 \arrow[r] & K \arrow[r, "k"'] & A \arrow[r, "f"'] & B.
\end{tikzcd}$$

If $k' = \mathsf{Ker}\, f'$, then w is a monomorphism. ■

The converse of the following property is well known to hold in any pointed category. In fact, to conclude the converse, it is enough for u to be an epimorphism.

2.7. Proposition. [10, Lemma 1.1] *Consider, in a pointed, exact and protomodular category, a commutative diagram with exact rows*

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r, tail, "k'"] \arrow[d, "u"'] & A' \arrow[r, two heads, "f'"] \arrow[d, two heads, "v"] \arrow[dr, phantom, "\mathrm{(I)}"] & B' \arrow[r] \arrow[d, two heads, "w"] & 0 \\
0 \arrow[r] & K \arrow[r, tail, "k"'] & A \arrow[r, two heads, "f"'] & B \arrow[r] & 0
\end{tikzcd}$$

such that v and w are regular epimorphisms. If (I) *is a pushout, then u is a regular epimorphism.* ■

We will also need the following concept, introduced by Carboni, Lambek and Pedicchio in [13].

2.8. Definition. *A category $\mathcal{A}$ is a Mal'cev category if every reflexive relation in $\mathcal{A}$ is an equivalence relation.*

As regular categories constitute a natural context to work with relations, regular Mal'cev categories constitute a natural context to work with equivalence relations. If $\mathcal{A}$ has finite limits, then $\mathcal{A}$ protomodular implies $\mathcal{A}$ Mal'cev.

It is well known that when, in a regular category, a commutative square of regular epimorphisms

$$\begin{tikzcd}
A' \arrow[r, two heads, "f'"] \arrow[d, two heads, "v"'] \arrow[dr, phantom, "\lrcorner", very near start] & B' \arrow[d, two heads, "w"] \\
A \arrow[r, two heads, "f"'] & B
\end{tikzcd}$$

is a pullback, it is a pushout. (When the category is protomodular, one gets the same property for pullbacks of any map along a regular epimorphism: this is Proposition 14 in Bourn [6].) In a regular category, a commutative square of regular epimorphisms is said to be a *regular pushout* when the comparison map $r : A' \longrightarrow P$ to the pullback of f along w is a regular epimorphism (see Carboni, Kelly and Pedicchio [12]).

$$\begin{tikzcd}
A' \arrow[dr, dotted, "r"] \arrow[drr, bend left, two heads, "f'"] \arrow[ddr, bend right, two heads, "v"'] & & \\
& P \arrow[r, dotted, two heads] \arrow[d, dotted, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & B' \arrow[d, two heads, "w"] \\
& A \arrow[r, two heads, "f"'] & B
\end{tikzcd} \qquad \textbf{(B)}$$

The following characterizes exact Mal'cev categories among the regular ones.

2.9. Proposition. [12, Theorem 5.7] *A regular category is exact Mal'cev if and only if, given regular epimorphisms $v : A' \longrightarrow A$ and $f' : A' \longrightarrow B'$ such as in $\mathbf{B}$, their pushout (the diagram of solid arrows in $\mathbf{B}$) exists, and moreover the comparison map r—where the square is a pullback—is a regular epimorphism.* ∎

It follows that in any exact Mal'cev category, a square of regular epimorphisms is a regular pushout if and only if it is a pushout. Thus the following can be viewed as a denormalized version of Proposition 2.7.

2.10. Proposition. [9, Proposition 3.3] *Consider, in a regular Mal'cev category, a commutative diagram of augmented kernel pairs, such that p, p', q and r are regular epimorphisms:*

$$\begin{tikzcd} R[p'] \arrow[r, shift left, "k'_0"] \arrow[r, shift right, "k'_1"'] \arrow[d, "s"'] & A' \arrow[r, "p'"] \arrow[d, "q"] & B' \arrow[d, "r"] \\ R[p] \arrow[r, shift left, "k_0"] \arrow[r, shift right, "k_1"'] & A \arrow[r, "p"'] & B. \end{tikzcd}$$

The right hand square is a regular pushout if and only if s is a regular epimorphism. ∎

3. The general case

We start this section by giving a definition of Baer invariants. In order to turn a functor $L_0 : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ into a Baer invariant, we consider a class $\mathcal{F}_{L_0}$ of subfunctors of L_0. Proposition 3.7 shows that for any L_0 and any $S \in \mathcal{F}_{L_0}$, $L_0/S : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ is a Baer invariant. We give equivalent descriptions of the class $\mathcal{F}_{L_0}$ in case L_0 arises from a functor $\mathcal{A} \longrightarrow \mathcal{A}$. In that case the class $\mathcal{F}_{L_0}$ is shown to have a minimum L_1 (Proposition 3.9). Finally, in Proposition 3.18, we show that a Baer invariant can be turned into a functor $\mathcal{A} \longrightarrow \mathcal{A}$, given an appropriate subcategory of $\mathsf{Pr}\mathcal{A}$.

3.1. Definition. *Let $\mathcal{A}$ be a category. By a* presentation *of an object A of $\mathcal{A}$ we mean a regular epimorphism $p : A_0 \longrightarrow A$. We denote by $\mathsf{Pr}\mathcal{A}$ the category of presentations of objects of $\mathcal{A}$, a morphism $\boldsymbol{f} = (f_0, f) : p \longrightarrow q$ being a commutative square*

$$\begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[d, "p"'] & B_0 \arrow[d, "q"] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

Let pr : $\mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ *denote the forgetful functor which maps a presentation to the object presented, sending a morphism of presentations $\boldsymbol{f} = (f', f)$ to f. Two morphisms of presentations $\boldsymbol{f}, \boldsymbol{g} : p \longrightarrow q$ are called* isomorphic, *notation $\boldsymbol{f} \simeq \boldsymbol{g}$, if $\mathrm{pr}\, \boldsymbol{f} = \mathrm{pr}\, \boldsymbol{g}$ (or $f = g$).*

3.2. Remark. Note that in any category $\mathcal{A}$, a kernel pair $(R[p], k_0, k_1)$ of a morphism $p : A_0 \longrightarrow A$ is an equivalence relation, hence an internal category. In case $\mathcal{A}$ has kernel pairs of regular epimorphisms, a morphism of presentations $\boldsymbol{f} : p \longrightarrow q$ gives rise to an internal functor

$$\begin{tikzcd} R[p] \arrow[r, "R\boldsymbol{f}"] \arrow[d, shift right, "k_0"'] \arrow[d, shift left, "k_1"] & R[q] \arrow[d, shift right, "l_0"'] \arrow[d, shift left, "l_1"] \\ A_0 \arrow[r, "f_0"'] & B_0. \end{tikzcd}$$

Then $\boldsymbol{f} \simeq \boldsymbol{g} : p \longrightarrow q$ if and only if the corresponding internal functors are naturally isomorphic.

3.3. Definition. *A functor* $B : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ *is called a* Baer invariant *if* $\boldsymbol{f} \simeq \boldsymbol{g}$ *implies that* $B\boldsymbol{f} = B\boldsymbol{g}$.

The following shows that a Baer invariant maps "similar" presentations—in the sense of Baer [1]—to isomorphic objects.

3.4. Proposition. *In* $\mathcal{A}$ *let* p *and* p' *be presentations and* f *and* g *maps*

$$\begin{tikzcd} A_0 \arrow[rr, shift left, "f"] \arrow[dr, "p"'] & & A'_0 \arrow[ll, shift left, "g"] \arrow[dl, "p'"] \\ & A & \end{tikzcd}$$

such that $p' \circ f = p$ *and* $p \circ g = p'$. *If* $B : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ *is a Baer invariant, then* $Bp \cong Bp'$.

Proof. The existence of f and g amounts to p and p' being such that $(g, 1_A) \circ (f, 1_A) \simeq 1_p$ and $(f, 1_A) \circ (g, 1_A) \simeq 1_{p'}$. Obviously then, $B(f, 1_A) : Bp \longrightarrow Bp'$ is an isomorphism with inverse $B(g, 1_A)$. ■

Recall that a *subfunctor* of a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a subobject of F in the functor category $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$. We shall denote such a subfunctor by a representing monic natural transformation $\mu : G \Longrightarrow F$, or simply by $G \subseteq F$ or G. Now let $\mathcal{D}$ be a pointed category. A subfunctor $\mu : G \Longrightarrow F$ is called *normal* if every component $\mu_C : G(C) \longrightarrow F(C)$ is a kernel. This situation will be denoted by $G \lhd F$. In case $\mathcal{D}$ has cokernels of kernels, a normal subfunctor gives rise to a short exact sequence

$$0 \Longrightarrow F \overset{\mu}{\Longrightarrow} G \overset{\eta_F}{\Longrightarrow} \frac{G}{F} \Longrightarrow 0$$

of functors $\mathcal{C} \longrightarrow \mathcal{D}$. Like here, in what follows, exactness of sequences of functors will always be pointwise. If confusion is unlikely we shall omit the index F, and write η for η_F and η_C for $(\eta_F)_C$.

The following, at first, quite surprised us:

3.5. Example. Any subfunctor G of $1_{\mathsf{Gp}} : \mathsf{Gp} \longrightarrow \mathsf{Gp}$ is normal. Indeed, by the naturality of $\mu : G \Longrightarrow 1_{\mathsf{Gp}}$, every $\mu_C : G(C) \longrightarrow C$ is the inclusion of a fully-invariant, hence normal, subgroup $G(C)$ into C—and kernels in Gp and normal subgroups coincide.

This is, of course, not true in general: consider the category ω-Gp of groups with an operator ω. An object of ω-Gp is a pair (G, ω), with G a group and $\omega : G \longrightarrow G$ an endomorphism of G, and an arrow $(G, \omega) \longrightarrow (G', \omega')$ is a group homomorphism $f : G \longrightarrow G'$ satisfying $f \circ \omega = \omega' \circ f$. Then putting $L(G, \omega) = (\omega(G), \omega|_{\omega(G)})$ defines a subfunctor of the identity functor $1_{\omega\text{-}\mathsf{Gp}} : \omega\text{-}\mathsf{Gp} \longrightarrow \omega\text{-}\mathsf{Gp}$. L is, however, not normal: for any group endomorphism $\omega : G \longrightarrow G$ of which the image is not normal in G, the inclusion of $L(G, \omega)$ in (G, ω) is not a kernel.

If $\mathcal{A}$ is pointed and has cokernels of kernels, any functor $\mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ can be turned into a Baer invariant by dividing out a “large enough” subfunctor. In order to make this precise, we make the following

3.6. Definition. [19] *Consider a functor $L_0 : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ and a presentation $q : B_0 \longrightarrow B$. Then $\mathcal{F}^q_{L_0}$ denotes the class of kernel subobjects $S \triangleleft L_0 q$ for which $\boldsymbol{f} \simeq \boldsymbol{g} : p \longrightarrow q$ implies that $\eta_S \circ L_0\boldsymbol{f} = \eta_S \circ L_0\boldsymbol{g}$, i.e. that the two compositions in the diagram*

$$L_0 p \overset{L_0\boldsymbol{f}}{\underset{L_0\boldsymbol{g}}{\rightrightarrows}} L_0 q \xrightarrow{\eta_S} \frac{L_0 q}{S}$$

are equal.

$\mathcal{F}_{L_0}$ is the class of functors $S : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ with $Sq \in \mathcal{F}^q_{L_0}$, for every $q \in \mathsf{Pr}\mathcal{A}$. Hence, $S \in \mathcal{F}_{L_0}$ if and only if the following conditions are satisfied:

(i) $S \triangleleft L_0$;

(ii) $\boldsymbol{f} \simeq \boldsymbol{g} : p \longrightarrow q$ *implies that $\eta_q \circ L_0\boldsymbol{f} = \eta_q \circ L_0\boldsymbol{g}$, i.e. that the two compositions in the diagram*

$$L_0 p \overset{L_0\boldsymbol{f}}{\underset{L_0\boldsymbol{g}}{\rightrightarrows}} L_0 q \xrightarrow{\eta_q} \frac{L_0}{S} q$$

are equal.

The class $\mathcal{F}^q_{L_0}$ (resp. $\mathcal{F}_{L_0}$) may be considered a subclass of $\mathsf{Sub}(L_0 q)$ (resp. $\mathsf{Sub}(L_0)$), and as such carries the inclusion order.

3.7. Proposition. [19, Proposition 1] *Suppose that $\mathcal{A}$ is pointed and has cokernels of kernels. Let $L_0 : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ be a functor and S an element of $\mathcal{F}_{L_0}$. Then $L_0/S : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ is a Baer invariant.*

Proof. Condition (i) ensures that L_0/S exists. The naturality of $\eta : L_0 \Longrightarrow L_0/S$ implies that, for morphisms of presentations $\boldsymbol{f}, \boldsymbol{g} : p \longrightarrow q$, both left and right hand downward pointing squares

$$\begin{tikzcd} L_0p \arrow[r, "\eta_p"] \arrow[d, shift right, "L_0\boldsymbol{f}"'] \arrow[d, shift left, "L_0\boldsymbol{g}"] & \frac{L_0}{S}p \arrow[d, shift right, "\frac{L_0}{S}\boldsymbol{f}"'] \arrow[d, shift left, "\frac{L_0}{S}\boldsymbol{g}"] \\ L_0q \arrow[r, "\eta_q"'] & \frac{L_0}{S}q \end{tikzcd}$$

commute. If $\boldsymbol{f} \simeq \boldsymbol{g}$ then, by (ii), $\eta_q \circ L_0\boldsymbol{f} = \eta_q \circ L_0\boldsymbol{g}$, so $\frac{L_0}{S}\boldsymbol{f} \circ \eta_p = \frac{L_0}{S}\boldsymbol{g} \circ \eta_p$, whence the desired equality $\frac{L_0}{S}\boldsymbol{f} = \frac{L_0}{S}\boldsymbol{g}$. ∎

Proposition 3.7 is particularly useful for functors $L_0 : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ induced by a functor $L : \mathcal{A} \longrightarrow \mathcal{A}$ by putting

$$L_0(p : A_0 \longrightarrow A) = L(A_0) \quad \text{and} \quad L_0((f_0, f) : p \longrightarrow q) = Lf_0.$$

In this case, the class $\mathcal{F}^q_{L_0}$ in Definition 3.6 has some equivalent descriptions.

3.8. Proposition. *Suppose that $\mathcal{A}$ is pointed with cokernels of kernels. Given a functor $L : \mathcal{A} \longrightarrow \mathcal{A}$ and a presentation $q : B_0 \longrightarrow B$, the following are equivalent:*

1. *$S \in \mathcal{F}^q_{L_0}$;*
2. *$\boldsymbol{f} \simeq \boldsymbol{g} : p \longrightarrow q$ implies that $\eta_S \circ L_0\boldsymbol{f} = \eta_S \circ L_0\boldsymbol{g}$;*
3. *for morphisms $f_0, g_0 : A_0 \longrightarrow B_0$ in $\mathcal{A}$ with $q \circ f_0 = q \circ g_0$, $\eta_S \circ Lf_0 = \eta_S \circ Lg_0$, i.e. the two compositions in the diagram*

$$\begin{tikzcd} L(A_0) \arrow[r, shift left, "Lf_0"] \arrow[r, shift right, "Lg_0"'] & L(B_0) \arrow[r, "\eta_S"] & \frac{L(B_0)}{S} \end{tikzcd}$$

are equal.

If, moreover, $\mathcal{A}$ has kernel pairs of regular epimorphisms, these conditions are equivalent to

4. *$L(R[q]) \subseteq R\big[\eta_S : L(B_0) \longrightarrow \frac{L(B_0)}{S}\big]$.*

Proof. We show that 2. implies 3.: a presentation p and maps of presentations $\boldsymbol{f} \simeq \boldsymbol{g} : p \longrightarrow q$ such as needed in condition 2. are given by

$$\begin{tikzcd} A_0 \arrow[r, shift left, "f_0"] \arrow[r, shift right, "g_0"'] \arrow[d, equal] & B_0 \arrow[d, "q"] \\ A_0 \arrow[r, "q\circ f_0"'] & B \end{tikzcd}$$

$p = 1_{A_0}$, $\boldsymbol{f} = (f_0, q \circ f_0)$, $\boldsymbol{g} = (g_0, q \circ f_0)$. ∎

3.9. Proposition. [cf. [19, Proposition 4]] *Suppose that $\mathcal{A}$ is pointed with pullbacks, coequalizers of reflexive graphs and cokernels of kernels. For any $L : \mathcal{A} \longrightarrow \mathcal{A}$, the ordered class $\mathcal{F}_{L_0}$ has a minimum $L_1 : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$.*

Proof. For a morphism of presentations $\boldsymbol{f} : p \longrightarrow q$, $L_1\boldsymbol{f}$ is defined by first taking kernel pairs

$$\begin{tikzcd} R[p] \arrow[r, shift left, "k_0"] \arrow[r, shift right, "k_1"'] \arrow[d, dotted, "R\boldsymbol{f}"'] \arrow[dr, phantom, "\mathrm{(I)}"] & A_0 \arrow[r, twoheadrightarrow, "p"] \arrow[d, "f_0"] \arrow[dr, phantom, "\mathrm{(II)}"] & A \arrow[d, "f"] \\ R[q] \arrow[r, shift left, "l_0"] \arrow[r, shift right, "l_1"'] & B_0 \arrow[r, twoheadrightarrow, "q"'] & B, \end{tikzcd} \tag{C}$$

next applying L and taking coequalizers

$$\begin{tikzcd} L(R[p]) \arrow[r, shift left, "Lk_0"] \arrow[r, shift right, "Lk_1"'] \arrow[d, "LR\boldsymbol{f}"'] & L(A_0) \arrow[r, twoheadrightarrow, "c"] \arrow[d, "Lf_0"] \arrow[dr, phantom, "\mathrm{(III)}"] & Coeq[Lk_0, Lk_1] \arrow[d, dotted, "Coeq(LR\boldsymbol{f}, Lf_0)"] \\ L(R[q]) \arrow[r, shift left, "Ll_0"] \arrow[r, shift right, "Ll_1"'] & L(B_0) \arrow[r, twoheadrightarrow, "d"'] & Coeq[Ll_0, Ll_1] \end{tikzcd} \tag{D}$$

and finally taking kernels

$$\begin{tikzcd} L_1p \arrow[r, tail, "\mathsf{Ker}\, c"] \arrow[d, dotted, "L_1\boldsymbol{f}"'] & L(A_0) \arrow[r, twoheadrightarrow, "c"] \arrow[d, "Lf_0"] \arrow[dr, phantom, "\mathrm{(III)}"] & Coeq[Lk_0, Lk_1] \arrow[d, dotted, "Coeq(LR\boldsymbol{f}, Lf_0)"] \\ L_1q \arrow[r, tail, "\mathsf{Ker}\, d"'] & L(B_0) \arrow[r, twoheadrightarrow, "d"'] & Coeq[Ll_0, Ll_1]. \end{tikzcd} \tag{E}$$

It is easily seen that this defines a functor $L_1 : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ which is a minimum in $\mathcal{F}_{L_0}$ for the inclusion order. ∎

3.10. Remark. Observe that the construction of L_1 above is such that, for every $p \in \mathsf{Pr}\mathcal{A}$, L_1p is a minimum in the class $\mathcal{F}^p_{L_0}$.

3.11. Corollary. *Let $\mathcal{A}$ be a pointed category with pullbacks, coequalizers of reflexive graphs and cokernels of kernels, and $L : \mathcal{A} \longrightarrow \mathcal{A}$ a functor. Then*

1. *$S \in \mathcal{F}_{L_0}$ if and only if $L_1 \leq S \lhd L_0$;*
2. *$S \in \mathcal{F}^p_{L_0}$ if and only if $L_1p \leq S \lhd L_0p$.* ∎

We now show how a Baer invariant gives rise to a functor $\mathcal{A} \longrightarrow \mathcal{A}$, given the following additional datum:

3.12. Definition. *Let $\mathcal{A}$ be a category. We call a subcategory $\mathcal{W}$ of $\mathsf{Pr}\mathcal{A}$ a* web *on $\mathcal{A}$ if*

1. *for every object A of $\mathcal{A}$, a presentation $p : A_0 \longrightarrow A$ in $\mathcal{W}$ exists;*
2. *given presentations $p : A_0 \longrightarrow A$ and $q : B_0 \longrightarrow B$ in $\mathcal{W}$, for every morphism $f : A \longrightarrow B$ of $\mathcal{A}$ there exists a morphism $\boldsymbol{f} : p \longrightarrow q$ in $\mathcal{W}$ such that $\mathrm{pr}\, \boldsymbol{f} = f$.*

3.13. Example. A *split* presentation is a split epimorphism p of $\mathcal{A}$. (Note that an epimorphism $p : A_0 \longrightarrow A$ with splitting $s : A \longrightarrow A_0$ is a coequalizer of the maps 1_{A_0} and $s \circ p : A_0 \longrightarrow A_0$.) The full subcategory of $\mathsf{Pr}\mathcal{A}$ determined by the split presentations of $\mathcal{A}$ is a web $\mathcal{W}_{\mathsf{split}}$, the *web of split presentations*. Indeed, for any $A \in \mathcal{A}$, $1_A : A \longrightarrow A$ is a split presentation of A; given split presentations p and q and a map f such as indicated by the diagram

$$\begin{tikzcd}
A_0 \arrow[r, dotted, "t\circ f\circ p"] \arrow[d, shift right, "p"'] & B_0 \arrow[d, shift right, "q"'] \\
A \arrow[r, "f"'] \arrow[u, shift right, "s"'] & B, \arrow[u, shift right, "t"']
\end{tikzcd}$$

$(t \circ f \circ p, f)$ is the needed morphism of $\mathcal{W}$.

3.14. Example. Let $F : \mathcal{A} \longrightarrow \mathcal{A}$ be a functor and let $\pi : F \Longrightarrow 1_{\mathcal{A}}$ be a natural transformation of which all components are regular epimorphisms. Then the presentations $\pi_A : F(A) \longrightarrow A$, for $A \in \mathcal{A}$, together with the morphisms of presentations $(F(f), f) : \pi_A \longrightarrow \pi_B$, for $f : A \longrightarrow B$ in $\mathcal{A}$, constitute a web $\mathcal{W}_F$ on $\mathcal{A}$ called the *functorial web determined by F*.

3.15. Example. A presentation $p : A_0 \longrightarrow A$ is called *projective* if A_0 is a projective object of $\mathcal{A}$. If $\mathcal{A}$ has sufficiently many projectives every object A obviously has a projective presentation. In this case the full subcategory $\mathcal{W}_{\mathsf{proj}}$ of all projective presentations of objects of $\mathcal{A}$ is a web, called the *web of projective presentations*.

Recall [4] that a *graph morphism* $F : \mathcal{C} \longrightarrow \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ has the structure, but not all the properties, of a functor from $\mathcal{C}$ to $\mathcal{D}$: it need neither preserve identities nor compositions.

3.16. Definition. *Let $\mathcal{W}$ be a web on a category $\mathcal{A}$ and $i : \mathcal{W} \longrightarrow \mathsf{Pr}\mathcal{A}$ the inclusion. By a* choice c of presentations in $\mathcal{W}$, *we mean a graph morphism $c : \mathcal{A} \longrightarrow \mathcal{W}$ such that* $\mathrm{pr} \circ i \circ c = 1_{\mathcal{A}}$.

A functor $B : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ is called a Baer invariant relative to $\mathcal{W}$ *when*

1. *for any choice of presentations $c : \mathcal{A} \longrightarrow \mathcal{W}$, the graph morphism $B \circ i \circ c : \mathcal{A} \longrightarrow \mathcal{A}$ is a functor;*
2. *for any two choices of presentations $c, c' : \mathcal{A} \longrightarrow \mathcal{W}$, the functors $B \circ i \circ c$ and $B \circ i \circ c'$ are naturally isomorphic.*

3.17. Example. Any functor $B : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ is a Baer invariant relative to any functorial web $\mathcal{W}_F$: there is only one choice $c : \mathcal{A} \longrightarrow \mathcal{W}_F$, and it is a functor.

3.18. Proposition. *If $B : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ is a Baer invariant, it is a Baer invariant relative to any web $\mathcal{W}$ on $\mathcal{A}$.*

PROOF. Let $\mathcal{W}$ be a web and $c, c' : \mathcal{A} \longrightarrow \mathcal{W}$ two choices of presentations in $\mathcal{W}$. For the proof of 1., let $f : A \longrightarrow A'$ and $g : A' \longrightarrow A''$ be morphisms of $\mathcal{A}$. Then $1_{ic(A)} \simeq ic1_A$ and $ic(f \circ g) \simeq icf \circ icg$; consequently, $1_{Bic(A)} = Bic1_A$ and $Bic(f \circ g) = Bicf \circ Bicg$.

The second statement is proven by choosing, for every object A of $\mathcal{A}$, a morphism $\tau_A : c(A) \longrightarrow c'(A)$ in the web $\mathcal{W}$. Then $\nu_A = Bi\tau_A : Bic(A) \longrightarrow Bic'(A)$ is independent of the choice of τ_A, and the collection $(\nu_A)_{A \in \mathcal{A}}$ is a natural isomorphism $B \circ i \circ c \Longrightarrow B \circ i \circ c'$. ■

This proposition implies that a Baer invariant $B : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ gives rise to functors $B \circ i \circ c : \mathcal{A} \longrightarrow \mathcal{A}$, independent of the choice c in a web $\mathcal{W}$. But it is important to note that such functors do depend on the chosen web: indeed, for two choices of presentations c and c' in two different webs $\mathcal{W}$ and $\mathcal{W}'$, the functors $B \circ i \circ c$ and $B \circ i' \circ c'$ need not be naturally isomorphic—see Remark 4.8.

Suppose that $\mathcal{A}$ is a pointed category with cokernels of kernels and let c be a choice in a web $\mathcal{W}$ on $\mathcal{A}$. If $L_0 : \mathsf{Pr}\mathcal{A} \longrightarrow A$ is a functor and $S \in \mathcal{F}_{L_0}$, then by Proposition 3.7

$$\tfrac{L_0}{S} \circ i \circ c$$

is a functor $\mathcal{A} \longrightarrow \mathcal{A}$.

If, moreover, $\mathcal{A}$ has pullbacks and coequalizers of reflexive graphs, and if $L : \mathcal{A} \longrightarrow \mathcal{A}$ is a functor, then, by Proposition 3.9, we have the canonical functor

$$DL_{\mathcal{W}} = \tfrac{L_0}{L_1} \circ i \circ c : \mathcal{A} \longrightarrow \mathcal{A}.$$

4. The semi-abelian case

In this section we show that, when working in the stronger context of pointed exact protomodular categories, we have a second canonical functor in $\mathcal{F}_{L_0}$. Then, as an application of Noether's Third Isomorphism Theorem, we construct two exact sequences. As a consequence, we get some new Baer invariants. Finally, applying the Snake Lemma, we find a six-term exact sequence of functors $\mathcal{A} \longrightarrow \mathcal{A}$.

4.1. REMARK. If $\mathcal{A}$ is pointed, regular and protomodular, giving a presentation p is equivalent to giving a short exact sequence

$$0 \longrightarrow K[p] \xrightarrow{\mathsf{Ker}\, p} A_0 \xrightarrow{p} A \longrightarrow 0.$$

$K[\cdot] : \mathsf{Fun}(2, \mathcal{A}) \longrightarrow \mathcal{A}$, and its restriction $K[\cdot] : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$, denote the kernel functor.

4.2. PROPOSITION. [19, Proposition 2] *Let $\mathcal{A}$ be a pointed, regular and protomodular category. If L is a subfunctor of $1_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$ then $K[\cdot] \cap L_0 : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ is in $\mathcal{F}_{L_0}$. Hence*

$$\frac{L_0}{K[\cdot] \cap L_0} : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$$

is a Baer invariant.

Proof. Let $\mu : L \Longrightarrow 1_{\mathcal{A}}$ be the subfunctor L, and $L_0 : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ the functor defined above Proposition 3.8. By Proposition 3.7 it suffices to prove the first statement. We show that, for every presentation $q : B_0 \longrightarrow B$, $K[q] \cap L(B_0) \in \mathcal{F}^q_{L_0}$, by checking the third condition of Proposition 3.8. Since pullbacks preserve kernels—see the diagram below—$K[q] \cap L(B_0) \lhd L(B_0)$. Moreover, applying Proposition 2.6, the map m_q, defined by first pulling back $\mathsf{Ker}\, q$ along μ_{B_0} and then taking cokernels

$$\begin{tikzcd}
0 \arrow[r] & K[q] \cap L(B_0) \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & L(B_0) \arrow[r, two heads] \arrow[d, tail, "\mu_{B_0}"'] & \frac{L(B_0)}{K[q]\cap L(B_0)} \arrow[r] \arrow[d, dotted, "m_q"] & 0 \\
0 \arrow[r] & K[q] \arrow[r, tail, "\mathsf{Ker}\, q"'] & B_0 \arrow[r, two heads, "q"'] & B \arrow[r] & 0,
\end{tikzcd}$$

is a monomorphism. Consider morphisms $f_0, g_0 : A_0 \longrightarrow B_0$ in $\mathcal{A}$ with $q \circ f_0 = q \circ g_0$. We are to prove that in the following diagram, the two compositions above are equal:

$$\begin{tikzcd}
L(A_0) \arrow[r, shift left, "Lf_0"] \arrow[r, shift right, "Lg_0"'] & L(B_0) \arrow[r, two heads] \arrow[d, tail, "\mu_{B_0}"'] & \frac{L(B_0)}{K[q]\cap L(B_0)} \arrow[d, tail, "m_q"] \\
 & B_0 \arrow[r, two heads, "q"'] & B.
\end{tikzcd}$$

Since m_q is a monomorphism, it suffices to prove that $q \circ \mu_{B_0} \circ Lf_0 = q \circ \mu_{B_0} \circ Lg_0$. But, by naturality of μ, $q \circ \mu_{B_0} = \mu_B \circ Lq$, and so

$$q \circ \mu_{B_0} \circ Lf_0 = \mu_B \circ L(q \circ f_0) = \mu_B \circ L(q \circ g_0) = q \circ \mu_{B_0} \circ Lg_0.$$

■

From now on, we will assume $\mathcal{A}$ to be pointed, exact and protomodular.

4.3. Remark. Note that any exact Mal'cev category, hence, *a fortiori*, any exact and protomodular category, has coequalizers of reflexive graphs. We get that Proposition 3.9 is applicable.

4.4. Remark. For $L \subseteq 1_{\mathcal{A}}$ we now have the following inclusion of functors $\mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$:

$$L \circ K[\cdot] \quad \lhd \quad L_1 \quad \lhd \quad K[\cdot] \cap L_0 \quad \lhd \quad L_0.$$

Only the left-most inclusion is not entirely obvious. For a presentation p, let r denote the cokernel of the inclusion $L_1 p \longrightarrow L(A_0)$. Then $p \circ \mathsf{Ker}\, p = p \circ 0$, thus $r \circ L\mathsf{Ker}\, p = r \circ L0 = 0$. This yields the required map.

4.5. Remark. Note that, since $L(A) = L_1(A \longrightarrow 0)$, the functor L may be regained from L_1: indeed, evaluating the above inclusions in $A \longrightarrow 0$, we get

$$L(A) = (L \circ K[\cdot])(A \longrightarrow 0) \subseteq L_1(A \longrightarrow 0) \subseteq L_0(A \longrightarrow 0) = L(A).$$

4.6. Proposition. [cf. [19, Proposition 6]] *Consider a subfunctor L of $1_{\mathcal{A}}$. For $S \in \mathcal{F}_{L_0}$ with $S \subseteq K[\cdot] \cap L_0$, the following sequence of functors $\mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ is exact, and all its terms are Baer invariants.*

$$0 \Longrightarrow \frac{K[\cdot]\cap L_0}{S} \Longrightarrow \frac{L_0}{S} \Longrightarrow \frac{L_0}{K[\cdot]\cap L_0} \Longrightarrow 0 \qquad (\mathbf{F})$$

If, moreover, $L \lhd 1_{\mathcal{A}}$, the sequence

$$0 \Longrightarrow \frac{K[\cdot]\cap L_0}{S} \Longrightarrow \frac{L_0}{S} \Longrightarrow \mathrm{pr} \Longrightarrow \frac{\mathrm{pr}}{L_0/(K[\cdot]\cap L_0)} \Longrightarrow 0 \qquad (\mathbf{G})$$

is exact, and again all terms are Baer invariants.

Proof. The exactness of $\mathbf{F}$ follows from Proposition 2.4, since S and $K[\cdot] \cap L_0$ are normal in L_0, both being in $\mathcal{F}_{L_0}$. The naturality is rather obvious.

Now suppose that $L \lhd 1_{\mathcal{A}}$. To prove the exactness of $\mathbf{G}$, it suffices to show that $L_0/(K[\cdot] \cap L_0)$ is normal in pr. Indeed, if so,

$$0 \Longrightarrow \frac{L_0}{K[\cdot]\cap L_0} \Longrightarrow \mathrm{pr} \Longrightarrow \frac{\mathrm{pr}}{L_0/(K[\cdot]\cap L_0)} \Longrightarrow 0$$

is a short exact sequence, and by pasting it together with $\mathbf{F}$, we get $\mathbf{G}$. Reconsider, therefore, the following commutative diagram from the proof of Proposition 4.2:

$$\begin{tikzcd}
L(A_0) \arrow[r] \arrow[d, "\mu_{A_0}"'] & \frac{L(A_0)}{K[p]\cap L(A_0)} \arrow[d, "m_p"] \\
A_0 \arrow[r, "p"'] & A.
\end{tikzcd}$$

The Non-Effective Trace of the 3×3 Lemma 2.3 implies that m_p is normal, because μ_{A_0} is.

The terms of $\mathbf{F}$ and $\mathbf{G}$ being Baer invariants follows from the exactness of the sequences, and from the fact that L_0/S, $L_0/(K[\cdot] \cap L_0)$ and pr are Baer invariants. ∎

4.7. Corollary. *Let L be a subfunctor of $1_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$ and $S \in \mathcal{F}_{L_0}$ with $S \subseteq K[\cdot] \cap L_0$. Then*

1. *for any $A \in \mathcal{A}$, $S1_A = 0$;*
2. *for any split presentation $p : A_0 \longrightarrow A$ in $\mathcal{A}$, $Sp \cong K[p] \cap L(A_0)$;*
3. *for any presentation $p : A_0 \longrightarrow A$ of a projective object A of $\mathcal{A}$, $Sp \cong K[p] \cap L(A_0)$.*

PROOF. The first statement holds because $S1_A \subseteq (K[\cdot] \cap L_0)(1_A) = 0 \cap L(A) = 0$.

For the second statement, let $s : A \longrightarrow A_0$ be a splitting of p. Then $(s, 1_A) : 1_A \longrightarrow p$ and $(p, 1_A) : p \longrightarrow 1_A$ are morphisms of presentations. By Proposition 4.6,

$$\tfrac{K[\cdot]\cap L_0}{S} : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$$

is a Baer invariant. Hence, Proposition 3.4 implies that

$$0 = \tfrac{K[\cdot]\cap L_0}{S}(1_A) \cong \tfrac{K[\cdot]\cap L_0}{S}(p) = \tfrac{K[p]\cap L(A_0)}{Sp},$$

whence the isomorphism $Sp \cong K[p] \cap L(A_0)$.

The third statement immediately follows from 2. ∎

Let c be a choice in a web $\mathcal{W}$ on $\mathcal{A}$. If $L \subseteq 1_{\mathcal{A}}$, Proposition 4.6 and 3.9 tell us that

$$\Delta L_{\mathcal{W}} = \frac{K[\cdot] \cap L_0}{L_1} \circ i \circ c \qquad \text{and} \qquad \nabla L_{\mathcal{W}} = \frac{L_0}{K[\cdot] \cap L_0} \circ i \circ c$$

are functors $\mathcal{A} \longrightarrow \mathcal{A}$. If, moreover, $L \lhd 1_{\mathcal{A}}$, then

$$M_{\mathcal{W}} = \frac{\mathrm{pr}}{L_0/(K[\cdot] \cap L_0)} \circ i \circ c : \mathcal{A} \longrightarrow \mathcal{A}$$

is a functor as well. We omit the references to c, since any other choice c' gives naturally isomorphic functors. In the following we shall always assume that a particular c has been chosen. When the index $\mathcal{W}$ is omitted, it is understood that $\mathcal{W}$ is the web $\mathcal{W}_{\mathsf{proj}}$ of projective presentations.

4.8. REMARK. Note that these functors *do* depend on the chosen web; for instance, by Corollary 4.7, $\Delta L_{\mathcal{W}_{1_{\mathcal{A}}}} = 0$, for any $L \subseteq 1_{\mathcal{A}}$. But, as will be shown in Example 6.1 and Proposition 6.2, if $\mathcal{A} = \mathsf{Gp}$ and L is the subfunctor associated with the Birkhoff subcategory Ab of abelian groups, then for any $R \lhd F$ with F projective,

$$\Delta(\eta_R : F \longrightarrow \tfrac{F}{R}) = \frac{R \cap [F, F]}{[R, F]}.$$

By Hopf's formula [23], this is the second integral homology group of $G = F/R$—which need not be 0.

From Proposition 4.6 we deduce the following.

4.9. PROPOSITION. [19, Proposition 6,7] *Let $\mathcal{W}$ be a web on a pointed, exact and protomodular category $\mathcal{A}$. If $L \subseteq 1_{\mathcal{A}}$, then*

$$0 \Longrightarrow \Delta L_{\mathcal{W}} \rightarrowtail DL_{\mathcal{W}} \twoheadrightarrow \nabla L_{\mathcal{W}} \Longrightarrow 0 \tag{H}$$

is exact. If, moreover, $L \lhd 1_{\mathcal{A}}$, then also

$$0 \Longrightarrow \Delta L_{\mathcal{W}} \rightarrowtail DL_{\mathcal{W}} \Longrightarrow 1_{\mathcal{A}} \twoheadrightarrow M_{\mathcal{W}} \Longrightarrow 0 \tag{I}$$

is exact. ∎

4.10. Remark. [cf. [19, Proposition 8]] In the important case that the functor $L : \mathcal{A} \longrightarrow \mathcal{A}$ preserves regular epimorphisms, we get that L and $\nabla L_{\mathcal{W}}$ represent the same subfunctor of $1_{\mathcal{A}}$. (Accordingly, in this case, $\nabla L_{\mathcal{W}}$ does not depend on the chosen web.) Of course, the exact sequence **H** then simplifies to

$$0 \Longrightarrow \Delta L_{\mathcal{W}} \rightarrowtail\!\!\Longrightarrow DL_{\mathcal{W}} \Longrightarrow L \Longrightarrow 0. \qquad (\mathbf{J})$$

Let, indeed, A be an object of $\mathcal{A}$ and $p : A_0 \longrightarrow A$ a presentation of A. Then the converse of Proposition 2.6 implies that the upper sequence in the diagram

$$\begin{tikzcd}
0 \arrow[r, dotted] & K[p] \cap L(A_0) \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & L(A_0) \arrow[r, "Lp"] \arrow[d, tail, "\mu_{A_0}"'] & L(A) \arrow[r, dotted] \arrow[d, tail, "\mu_A"] & 0 \\
0 \arrow[r] & K[p] \arrow[r, tail, "\mathsf{Ker}\, p"'] & A_0 \arrow[r, "p"'] & A \arrow[r] & 0.
\end{tikzcd}$$

is short exact, since μ_A is a monomorphism and the left hand square a pullback.

To prove Theorem 4.13, we need the following two technical lemma's concerning the functor L_1. The second one essentially shows that Fröhlich's V_1 [18] and Furtado-Coelho's L_1 [19] coincide, although Fröhlich demands $V_1 p$ to be normal in A_0, and Furtado-Coelho only demands that $L_1 p$ is normal in $L_0 p \subseteq A_0$.

4.11. Lemma. *Let $L : \mathcal{A} \longrightarrow \mathcal{A}$ be a functor. If a morphism of presentations $\boldsymbol{f} : p \longrightarrow q$ is a pullback square*

$$\begin{tikzcd}
A_0 \arrow[r, shift left, "f_0"'] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & B_0 \arrow[l, shift left, "s_0"'] \arrow[d, "q"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

in $\mathcal{A}$ with f_0 split epi, then $L_1 \boldsymbol{f}$ is a regular epimorphism.

Proof. First note that, if square (II) of diagram **C** is a pullback, then so are both squares (I). Consequently, if, moreover, f_0 is split epi, $LR\boldsymbol{f}$ is a split, hence regular, epimorphism—$\mathcal{A}$is a regular category. Because, in this case, also Lf_0 is split epi, the Proposition 2.10 implies that square (III) of Diagram **D** is a regular pushout, so $L_1 \boldsymbol{f}$ is regular epi by Proposition 2.7. ■

4.12. Lemma. *Suppose that $L \lhd 1_{\mathcal{A}}$, and let $p : A_0 \longrightarrow A$ be a presentation in $\mathcal{A}$. Then $L_1 p$ is normal in A_0.*

Proof. Taking the kernel pair of p

$$\begin{tikzcd}
R[p] \arrow[r, "k_0"] \arrow[d, "k_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A_0 \arrow[d, "p"] \\
A_0 \arrow[r, "p"'] & A
\end{tikzcd}$$

yields split epimorphisms k_0 and k_1 and defines a morphism of presentations $(k_0, p) : k_1 \longrightarrow p$. Now, by Lemma 4.11, the arrow $L_1(k_0, p)$ in the square

$$\begin{tikzcd} L_1 k_1 \arrow[r, tail] \arrow[d, "L_1(k_0,p)"'] & R[p] \arrow[d, two heads, "k_0"] \\ L_1 p \arrow[r, tail] & A_0 \end{tikzcd}$$

is regular epi. (In fact it is an isomorphism.) Moreover, $L_1 k_1$ is normal in $R[p]$, since, according to Corollary 4.7, it is the intersection $K[k_1] \cap L(R[p])$ of two normal subobjects of $R[p]$. The Non-Effective Trace of the 3×3 Lemma 2.3 now implies that $L_1 p$ is normal in A_0. ∎

We will now apply the Snake Lemma to obtain a six term exact sequence of Baer invariants.

In what follows, $\mathcal{A}$ will be a pointed, exact and protomodular category with sufficiently many projectives. Let

$$0 \longrightarrow K \rightarrowtail A \xrightarrow{f} B \longrightarrow 0$$

be a short exact sequence in $\mathcal{A}$. By naturality of the sequence **I** we get a commutative square (II) and thus we have a factorisation γ, such that (I) commutes:

$$\begin{tikzcd} 0 \arrow[r] & K[DLf] \arrow[r, tail] \arrow[d, dotted, "\gamma"'] \arrow[dr, phantom, "\mathrm{(I)}"] & DL(A) \arrow[r, "DLf"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "\mathrm{(II)}"] & DL(B) \arrow[r, dotted] \arrow[d, "\alpha_B"] & 0 \\ 0 \arrow[r] & K \arrow[r, tail] & A \arrow[r, two heads, "f"'] & B \arrow[r] & 0. \end{tikzcd} \tag{\textbf{K}}$$

A priori, only the left exactness of the upper row is clear. Nevertheless, it is easily shown that DLf is a regular epimorphism by choosing a projective presentation $p : A_0 \longrightarrow A$ of A and then using the map $(1_{A_0}, f) : p \longrightarrow f \circ p$ of $\mathcal{W}_{\mathsf{proj}}$.

4.13. Theorem. [19, Theorem 9] *Let $\mathcal{A}$ be a pointed, exact and protomodular category with enough projectives. Consider a short exact sequence*

$$0 \longrightarrow K \rightarrowtail A \xrightarrow{f} B \longrightarrow 0$$

in $\mathcal{A}$. If $L \lhd 1_{\mathcal{A}}$ there is an exact sequence

$$0 \longrightarrow K[\gamma] \rightarrowtail \Delta L(A) \xrightarrow{\Delta Lf} \Delta L(B) \longrightarrow \frac{K}{I[\gamma]} \longrightarrow M(A) \xrightarrow{Mf} M(B) \longrightarrow 0. \tag{\textbf{L}}$$

This exact sequence depends naturally on the given short exact sequence.

Proof. It suffices to prove that γ is proper. Indeed, we already know that α_A and α_B are proper, as they can be put into exact sequences, and thus we get $\mathbf{L}$ by applying the Snake Lemma 2.5. The naturality then follows from 2.5 and Proposition 4.9.

Choose projective presentations $p : A_0 \longrightarrow A$ and $f \circ p : A_0 \longrightarrow B$ of A and B as above. Then, by the Noether's Third Isomorphism Theorem 2.4, diagram $\mathbf{K}$ becomes

$$\begin{tikzcd}
0 \arrow[r] & \frac{L_1(f\circ p)}{L_1 p} \arrow[r, tail] \arrow[d, "\gamma"'] & \frac{L(A_0)}{L_1 p} \arrow[r, two heads] \arrow[d, "\alpha_A"'] & \frac{L(A_0)}{L_1(f\circ p)} \arrow[r] \arrow[d, "\alpha_B"] & 0 \\
0 \arrow[r, tail] & \frac{K[f\circ p]}{K[p]} \arrow[r, tail] & \frac{A_0}{K[p]} \arrow[r, two heads] & \frac{A_0}{K[f\circ p]} \arrow[r] & 0.
\end{tikzcd}$$

The map γ is unique for the diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & L_1 p \arrow[r, tail] \arrow[d, tail] & L_1(f\circ p) \arrow[r, two heads] \arrow[d, tail] & \frac{L_1(f\circ p)}{L_1 p} \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & K[p] \arrow[r, tail] & K[f\circ p] \arrow[r, two heads] & \frac{K[f\circ p]}{K[p]} \arrow[r] & 0
\end{tikzcd}$$

to commute. By Lemma 4.12, we get that $L_1(f \circ p)$ is normal in A_0, thus in $K[f \circ p]$. The Non-Effective Trace of the 3×3 Lemma 2.3 now implies that $\mathsf{Im}\,\gamma$ is normal. Hence γ is proper. ∎

5. The case of Birkhoff subfunctors

We will now improve our main result of Section 4—Theorem 5.9—by putting extra conditions on the subfunctor $L \subseteq 1_{\mathcal{A}}$. We therefore introduce the notion of *Birkhoff subfunctor*. We show that these subfunctors correspond to the Birkhoff subcategories of $\mathcal{A}$, as defined by Janelidze and Kelly in [25]. But first we prove that that $L_1 : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ preserves regular epimorphisms if $L : \mathcal{A} \longrightarrow \mathcal{A}$ does so. What are the regular epimorphisms of $\mathsf{Pr}\mathcal{A}$?

5.1. Proposition. *If $\mathcal{A}$ is an exact Mal'cev category, then the regular epimorphisms of $\mathsf{Pr}\mathcal{A}$ are exactly the regular pushout squares in $\mathcal{A}$.*

Proof. $\mathsf{Pr}\mathcal{A}$ is a full subcategory of the "category of arrows" $\mathsf{Fun}(2, \mathcal{A})$, which has the limits and colimits of $\mathcal{A}$, computed pointwise (see Section II.4 of Mac Lane [31] or Section I.2.15 of Borceux [4]). Hence, given a pair $\boldsymbol{f}, \boldsymbol{g} : p \longrightarrow q$ of parallel arrows in $\mathsf{Pr}\mathcal{A}$, their coequalizer in $\mathsf{Fun}(2, \mathcal{A})$ exists:

$$\begin{tikzcd}
A_0 \arrow[r, shift left, "f_0"] \arrow[r, shift right, "g_0"'] \arrow[d, two heads, "p"'] & B_0 \arrow[r, two heads, "c_0"] \arrow[d, two heads, "q"] \arrow[dr, phantom, "\mathrm{(I)}"] & \mathsf{Coeq}\,(f_0, g_0) \arrow[d, two heads, "r"] \\
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, two heads, "c"'] & \mathsf{Coeq}\,(f, g).
\end{tikzcd}$$

Since r is a regular epimorphism, (c_0, c) is an arrow in $\mathsf{Pr}\mathcal{A}$; moreover, by Proposition 2.10, the right hand square (I) is a regular pushout.

Conversely, given a regular pushout square such as (I), Proposition 2.10 ensures that its kernel pair in $\mathsf{Fun}(2, \mathcal{A})$ is a pair of arrows in $\mathsf{Pr}\mathcal{A}$. Their coequalizer is again the regular pushout square (I). ∎

5.2. Proposition. *Let $\mathcal{A}$ be a pointed, exact and protomodular category, and $L : \mathcal{A} \longrightarrow \mathcal{A}$ a functor. If L preserves regular epimorphisms, then so does L_1.*

Proof. Suppose that L preserves regular epis. If $\boldsymbol{f}$ is a regular epimorphism of $\mathsf{Pr}\mathcal{A}$, square (II) of diagram $\mathbf{C}$ is a regular pushout. Proposition 2.10 implies that $R\boldsymbol{f}$ is a regular epimorphism. By assumption, Lf_0 and $LR\boldsymbol{f}$ are regular epis; hence, again using Proposition 2.10, we get that square (III) of Diagram $\mathbf{D}$ is a regular pushout. An application of Proposition 2.7 on $\mathbf{E}$ shows that $L_1\boldsymbol{f}$ is regular epi, which proves the first statement. ∎

Recall the following definition from Janelidze and Kelly [25].

5.3. Definition. *Let $\mathcal{A}$ be an exact category. A reflective subcategory $\mathcal{B}$ of $\mathcal{A}$ which is full and closed in $\mathcal{A}$ under subobjects and quotient objects is said to be a* Birkhoff subcategory *of $\mathcal{A}$. We will denote by $U : \mathcal{A} \longrightarrow \mathcal{B}$ the left adjoint of the inclusion $I : \mathcal{B} \longrightarrow \mathcal{A}$ and by $\eta : 1_{\mathcal{A}} \Longrightarrow I \circ U$ the unit of the adjunction. We shall omit all references to the functor I, writing $\eta_A : A \longrightarrow U(A)$ for the component of η at $A \in \mathcal{A}$.*

5.4. Remark. In [25] it is shown that a Birkhoff subcategory $\mathcal{B}$ of an exact category $\mathcal{A}$ is necessarily exact. We may add that if $\mathcal{A}$ is, moreover, pointed, protomodular and finitely cocomplete, then so is $\mathcal{B}$. Obviously, then, $\mathcal{B}$ is pointed. The Split Short Five Lemma holds in $\mathcal{B}$ because it holds in $\mathcal{A}$ and because I preserves kernels. Finally, a full and reflective subcategory $\mathcal{B}$ of $\mathcal{A}$ has all colimits that exist in $\mathcal{A}$. We conclude that a Birkhoff subcategory of a semi-abelian category is semi-abelian.

If the functor L from Remark 4.10 is a normal subfunctor of $1_{\mathcal{A}}$, L satisfies the conditions Fröhlich used in the article [18] to obtain his Baer invariants. Because of Corollary 5.7, and its importance in what follows, we think this situation merits a name and slightly different notations.

5.5. Definition. *Let $\mathcal{A}$ be a pointed, exact and protomodular category. We call* Birkhoff subfunctor of $\mathcal{A}$ *any normal subfunctor V of $1_{\mathcal{A}}$ which preserves regular epimorphisms.*

The following establishes a bijective correspondence between Birkhoff subfunctors and Birkhoff subcategories of a given pointed, exact and protomodular category $\mathcal{A}$.

5.6. Proposition. [25, Section 3.1] *Let $\mathcal{A}$ be an exact category and $U : \mathcal{A} \longrightarrow \mathcal{B}$ a reflector (with unit η) onto a full replete subcategory $\mathcal{B}$ of $\mathcal{A}$. Then*

1. *$\mathcal{B}$ is closed in $\mathcal{A}$ under subobjects if and only if the component $\eta_A : A \longrightarrow U(A)$ of η at an $A \in \mathcal{A}$ is a regular epimorphism;*

2. *if 1. holds, then $\mathcal{B}$ is closed in $\mathcal{A}$ under quotient objects if and only if, for any regular epimorphism $f : A \longrightarrow B$ in $\mathcal{A}$, the naturality square*

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & U(A) \arrow[d, "Uf"] \\ B \arrow[r, "\eta_B"'] & U(B) \end{tikzcd}$$

is a pushout. ∎

5.7. Corollary. [cf. 18, Theorem 1.2] *Let $\mathcal{A}$ be a pointed, exact and protomodular category.*

1. *If $U : \mathcal{A} \longrightarrow \mathcal{B}$ is the reflector of $\mathcal{A}$ onto a Birkhoff subcategory $\mathcal{B}$, setting $V(A) = K[\eta_A]$ and $\mu_A = \mathsf{Ker}\, \eta_A$ defines a Birkhoff subfunctor $\mu : V \Longrightarrow 1_{\mathcal{A}}$ of $\mathcal{A}$.*
2. *Conversely, if $\mu : V \Longrightarrow 1_{\mathcal{A}}$ is a Birkhoff subfunctor of $\mathcal{A}$, putting $U(A) = \mathrm{Cok}[\mu_A]$, $\eta_A = \mathsf{Coker}\, \mu_A$ defines a full functor $U : \mathcal{A} \longrightarrow \mathcal{A}$ and a natural transformation $\eta : 1_{\mathcal{A}} \Longrightarrow U$. Here we can, and will, always choose $\mathsf{Coker}\,(0 \longrightarrow A) = 1_A$. The image of U is a Birkhoff subcategory $\mathcal{B}$ of $\mathcal{A}$. Furthermore, U, considered as a functor $\mathcal{A} \longrightarrow \mathcal{B}$, is left adjoint to the inclusion $\mathcal{B} \longrightarrow \mathcal{A}$, and η is the unit of this adjunction.*

In both cases, for any $A \in \mathcal{A}$, the sequence

$$0 \longrightarrow V(A) \xrightarrow{\mu_A} A \xrightarrow{\eta_A} U(A) \longrightarrow 0$$

is exact.

Proof. For 1. we only need to prove that V preserves regular epimorphisms. But this follows immediately from Proposition 5.6 and Proposition 2.7, applied to the diagram

$$\begin{tikzcd} 0 \arrow[r] & V(A) \arrow[r, "\mu_A"] \arrow[d, "Vf"'] & A \arrow[r, "\eta_A"] \arrow[d, "f"'] & U(A) \arrow[r] \arrow[d, "Uf"] & 0 \\ 0 \arrow[r] & V(B) \arrow[r, "\mu_B"'] & B \arrow[r, "\eta_B"'] \arrow[ur, phantom, "\ulcorner", very near end] & U(B) \arrow[r] & 0. \end{tikzcd} \qquad \textbf{(M)}$$

To prove 2, we first show that, for any $C \in \mathcal{A}$, $V(U(C)) = 0$. In the diagram

$$\begin{tikzcd} 0 \arrow[r] & V(C) \arrow[r, "\mu_A"] \arrow[d, "V\eta_C"'] & C \arrow[r, "\eta_C"] \arrow[d, "\eta_C"'] & U(C) \arrow[r] \arrow[d, "U\eta_C"] & 0 \\ 0 \arrow[r] & V(U(C)) \arrow[r, "\mu_{U(C)}"'] & U(C) \arrow[r, "\eta_{U(C)}"'] & U(U(C)) \arrow[r] & 0, \end{tikzcd}$$

the right square is a pushout, because $V\eta_C$ is an epimorphism, as V preserves regular epimorphisms. It follows easily that $\eta_{U(C)}$ is a split monomorphism, hence an isomorphism, and $V(U(C)) = 0$.

For the fullness of U, any map $h : U(A) \longrightarrow U(B)$ should be Ug for some g. But h itself is such a g: indeed $\eta_{U(C)} = 1_{U(C)}$, as we just demonstrated that $\mu_{U(C)} = 0$, for any $C \in \mathcal{A}$.

Now let A be any object, and $f : A \longrightarrow U(B)$ any map, of $\mathcal{A}$. To prove that a unique arrow $\overline{f}$ exists in $\mathcal{B}$ such that

$$\begin{tikzcd} 0 \arrow[r] & V(A) \arrow[r, tail, "\mu_A"] & A \arrow[r, two heads, "\eta_A"] \arrow[dr, "f"'] & U(A) \arrow[r] \arrow[d, dotted, "\overline{f}"] & 0 \\ & & & U(B) & \end{tikzcd}$$

commutes, it suffices to show that $f \circ \mu_A = 0$. (The unique $\overline{f}$ in $\mathcal{A}$ such that the diagram commutes is a map of $\mathcal{B}$, because U is full.) But, as V is a subfunctor of the identity functor, $f \circ \mu_A = \mu_{U(B)} \circ Vf = 0$.

This shows that for any object A of $\mathcal{A}$, the morphism $\eta_A : A \longrightarrow U(A)$ is a universal arrow from A to $U(A)$. It follows that U is a reflector with unit η. The subcategory $\mathcal{B}$ is then Birkhoff by 5.6, as in **M** the right square is a pushout, Vf being a regular epimorphism. ∎

5.8. Remark. In case V is a Birkhoff subfunctor of $\mathcal{A}$, note that, by Remark 4.10 and Corollary 5.7, the exact sequence **I** becomes

$$0 \Longrightarrow \Delta V \Longrightarrow DV \Longrightarrow 1_{\mathcal{A}} \Longrightarrow U \Longrightarrow 0. \tag{N}$$

Theorem 4.13 now has the following refinement.

5.9. Theorem. [cf. [18, Theorem 3.2]] *Let $\mathcal{A}$ be a pointed, exact and protomodular category with enough projectives. Consider a short exact sequence*

$$0 \longrightarrow K \rightarrowtail A \xrightarrow{f} B \longrightarrow 0$$

in $\mathcal{A}$. If V is a Birkhoff subfunctor of $\mathcal{A}$, then the sequence

$$0 \longrightarrow K[\gamma] \rightarrowtail \Delta V(A) \xrightarrow{\Delta Lf} \Delta V(B) \longrightarrow \tfrac{K}{V_1 f} \longrightarrow U(A) \xrightarrow{Uf} U(B) \longrightarrow 0 \tag{O}$$

is exact and depends naturally on the given short exact sequence.

Proof. To get **O** from **L**, it suffices to recall Remark 5.8, and to prove that the image $I[\gamma]$ of γ is $V_1 f$. Choosing presentations $p : A_0 \longrightarrow A$ and $f \circ p : A_0 \longrightarrow B$ such as in the proof of Theorem 4.13, γ becomes a map

$$\frac{V_1(f \circ p)}{V_1 p} \longrightarrow \frac{K[f \circ p]}{K[p]}.$$

Hence, to prove that $I[\gamma] = V_1 f$, we are to show that the arrow

$$\frac{V_1(f \circ p)}{V_1 p} \longrightarrow V_1 f$$

is regular epi. But this is equivalent to $V_1(p, 1_B) : V_1(f \circ p) \longrightarrow V_1 f$ being regular epi. This is the case, since by Proposition 5.2, V_1 preserves regular epimorphisms, and

$$\begin{tikzcd} A_0 \arrow[r, "p"] \arrow[d, "f \circ p"'] & A \arrow[d, "f"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

is a regular pushout square, which means that $(p, 1_B) : (f \circ p) \longrightarrow f$ is a regular epimorphism of $\mathsf{Pr}\mathcal{A}$. ■

5.10. Remark. Recalling Corollary 4.7, note that in case A (or B) is projective, the exact sequences $\mathbf{L}$ and $\mathbf{O}$ become much shorter, because then $\Delta L(A)$ and $\Delta V(A)$ (or $\Delta L(B)$ and $\Delta V(B)$) are 0.

6. V_1 as a commutator

In an exact Mal'cev category with coequalizers (hence, in a semi-abelian category) an intrinsic notion of abelian object exists. In fact, an object is abelian if and only if it can be provided with a (necessarily unique) structure of internal abelian group—see Gran [21] or Borceux and Bourn [5]. In [21, Theorem 4.2] it is proven that the full subcategory of abelian objects $\mathcal{A}_{\mathsf{Ab}}$ in any exact Mal'cev category with coequalizers $\mathcal{A}$ is a Birkhoff subcategory of $\mathcal{A}$. In the case of groups we get the following.

6.1. Example. [Abelianization of groups] Consider the category Gp of groups and its Birkhoff subcategory $\mathsf{Ab} = \mathsf{Gp}_{\mathsf{Ab}}$ of abelian groups. The associated Birkhoff subfunctor (cf. 5.7) sends G to $[G, G]$, the commutator subgroup of G. Indeed, it is well known that the abelianization of a group G, i.e. the reflection of G along the inclusion $\mathsf{Ab} \longrightarrow \mathsf{Gp}$, is just $G/[G, G]$. For groups $N \triangleleft G$ we denote by $[N, G]$ the (normal) subgroup of G generated by the elements $ngn^{-1}g^{-1}$, with $g \in G$ and $n \in N$. Proposition 19 of Furtado-Coelho [19] states the following

6.2. Proposition. *If $V : \mathsf{Gp} \longrightarrow \mathsf{Gp}$ is the Birkhoff subfunctor defined by $V(G) = [G, G]$ then*

$$V_1\big(\eta_N : G \longrightarrow \tfrac{G}{N}\big) = [N, G].$$

■

Thus one could ask whether, even in a more general situation, it makes sense to view V_1 as a commutator. The aim of this section to give a positive answer to this question.

The last decade, a lot has research has been done on commutators and the related topic of central extensions, in Mal'cev and protomodular categories. Very important are the papers [25] and [32]. In [25], Janelidze and Kelly discuss central extensions in the context of exact categories, an *extension* being a regular epimorphism. The definition is relative to a so-called *admissible* subcategory $\mathcal{B}$ of an exact category $\mathcal{A}$. If $\mathcal{A}$ is Mal'cev or has the weaker *Goursat* property, then its admissible subcategories are exactly the Birkhoff subcategories of $\mathcal{A}$. In [32] Pedicchio considers Smith's notion of commutator of equivalence relations [33] in the (more general) context of exact Mal'cev categories with coequalizers. Janelidze and Pedicchio show in [29] that it is possible to consider commutator theory in a very general context (that of *finitely well-complete* categories) by basing it on the theory of internal categorical structures. The correspondence between the two notions—centrality and commutators—was made clear, first by Janelidze and Kelly in [27] and [26], next by Bourn and Gran in [10], then, most generally, by Gran, in [20]. In this article, he proves that, in any *factor permutable* category (hence, in any exact Mal'cev category with coequalizers), an extension $p : A \longrightarrow B$ is central relative to the subcategory of "abelian objects", precisely when (a generalization of) the Smith commutator $[R[p], \nabla_A]$ is equal to Δ_A. Here $\nabla_A = (A \times A, p_0, p_1)$ and $\Delta_A = (A, 1_A, 1_A)$ denote, respectively, the largest and the smallest equivalence relation on A.

Now, following Fröhlich [18], we use V_1 to define a notion of central extension relative a Birkhoff subcategory. This notion coincides with Janelidze and Kelly's. Moreover, in the case of abelianization, our V_1 corresponds to the Smith commutator—this is Theorem 6.9. Let us make all this precise.

6.3. Definition. *Let $\mathcal{A}$ be a pointed, exact and protomodular category, V a Birkhoff subfunctor of $\mathcal{A}$ and $V_1 : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$ the associated functor. Then a presentation $p \in \mathsf{Pr}\mathcal{A}$ is called V-*central *if $V_1 p = 0$.*

6.4. Example. An inclusion of groups $N \triangleleft G$ is called central when N lies in the centre of G. This is clearly equivalent to saying that $[N, G] = 0$.

6.5. Proposition. [[10, Theorem 2.1], [25, Theorem 5.2]] *Let $\mathcal{A}$ be a pointed, exact and protomodular category, $\mathcal{B}$ a Birkhoff subcategory of $\mathcal{A}$ and $V : \mathcal{A} \longrightarrow \mathcal{A}$ associated Birkhoff subfunctor. Let $p : A_0 \longrightarrow A$ be a presentation in $\mathcal{A}$. Then the following are equivalent:*

1. *p is central (relative to $\mathcal{B}$) in the sense of Janelidze and Kelly;*
2. *for every $f_0, g_0 : B_0 \longrightarrow A_0$ with $p \circ f_0 = p \circ g_0$, one has $Vf_0 = Vg_0$;*
3. *$V(R[p]) = \Delta_{V(A_0)}$;*
4. *p is V-central (in the sense of Definition 6.3).* ■

Two equivalence relations (R, r_0, r_1) and (S, s_0, s_1) on an object A *admit a centralizing relation* if there exists a double relation [14]

$$\begin{tikzcd} C \arrow[r, shift left, "\sigma_0"] \arrow[r, shift right, "\sigma_1"'] \arrow[d, shift right, "\rho_0"'] \arrow[d, shift left, "\rho_1"] & R \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] \\ S \arrow[r, shift left, "s_0"] \arrow[r, shift right, "s_1"'] & A \end{tikzcd}$$

such that the square $r_0 \circ \sigma_0 = s_0 \circ \rho_0$ is a pullback ([32], Definition 2.7).

For a regular epimorphism $p : A \longrightarrow B$ in a regular category $\mathcal{A}$ and a relation (R, r_0, r_1) on A, $p(R)$ denotes the relation obtained from the regular epi-mono factorization

$$\begin{tikzcd} R \arrow[r, "\pi"] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & p(R) \arrow[d, shift right, "\rho_0"'] \arrow[d, shift left, "\rho_1"] \\ A \arrow[r, "p"'] & B \end{tikzcd}$$

of $p \circ (r_0, r_1) : R \longrightarrow B \times B$. Clearly, when R is reflexive, $p(R)$ is reflexive as well. Hence, when $\mathcal{A}$ is Mal'cev, if R is an equivalence relation, then so is $q(R)$.

If $\mathcal{A}$ is exact and (R, r_0, r_1) is an equivalence relation on an object A of $\mathcal{A}$, then $q_R : A \longrightarrow A/R$ denotes the coequalizer of r_0 and r_1.

Now, the following characterizes the commutator introduced by Pedicchio ([32], Definition 3.1, Theorem 3.9):

6.6. Definition. *Let $\mathcal{A}$ be an exact Mal'cev category with coequalizers and R and S equivalence relations on an object $A \in \mathcal{A}$. The* commutator $[R, S]$ *is the equivalence relation on A defined by the following properties:*

1. *$q_{[R,S]}(R)$ and $q_{[R,S]}(S)$ admit a centralizing relation;*
2. *for any equivalence relation T on A such that $q_T(R)$ and $q_T(S)$ admit a centralizing relation, there exists a unique $\delta : A/[R, S] \longrightarrow X/T$ with $q_T = \delta \circ q_{[R,S]}$.*

When we choose the subcategory $\mathcal{A}_{\mathsf{Ab}}$ of abelian objects as Birkhoff subcategory of $\mathcal{A}$, the list of equivalent statements of Proposition 6.5 can be enlarged:

6.7. Proposition. *Consider an exact Mal'cev category with coequalizers $\mathcal{A}$ and its Birkhoff subcategory of abelian objects $\mathcal{A}_{\mathsf{Ab}}$. If p be a presentation in $\mathcal{A}$, then the following are equivalent:*

1. *$p : A_0 \longrightarrow A$ is central in the sense of Janelidze and Kelly;*
2. *$[R[p], \nabla_{A_0}] = \Delta_{A_0}$;*
3. *$R[p]$ and ∇_{A_0} admit a centralizing relation.*

Proof. The equivalence of 1. and 2. is proven in Gran [20] (in the context of factor permutable categories), and the equivalence of 2. and 3. in Proposition 3.6 of Pedicchio [32] (in the context of exact Mal'cev categories with coequalizers). ■

6.8. PROPOSITION. *Let $\mathcal{A}_{\mathsf{Ab}}$ be the Birkhoff subcategory of abelian objects of a pointed, exact and protomodular category with coequalizers $\mathcal{A}$, V the associated Birkhoff subfunctor and $p : A_0 \longrightarrow A \in \mathsf{Pr}\mathcal{A}$. For a subobject $F \subseteq V(A_0)$ such that $F \lhd A_0$, the following are equivalent:*

1. *$F \in \mathcal{F}^p_{V_0}$;*
2. *$V_1\big(q_{\eta_F R[p]} : \frac{A_0}{F} \longrightarrow \frac{A_0}{F}/\eta_F(R[p])\big) = 0$;*
3. *$\eta_F(R[p])$ and $\eta_F(\nabla_{A_0}) = \nabla_{A_0/F}$ admit a centralizing relation.*

PROOF. The equivalence of 2. and 3. follows from Proposition 6.5 and 6.7. We will prove the equivalence of 1. and 2.

Consider the image $\eta_F R[p]$ of $R[p]$ along η_F.

$$\begin{tikzcd}
 & & R[p] \arrow[r, "\pi"] \arrow[d, shift right, "k_0"'] \arrow[d, shift left, "k_1"] & \eta_F(R[p]) \arrow[d, shift right, "\kappa_0"'] \arrow[d, shift left, "\kappa_1"] & \\
0 \arrow[r] & F \arrow[r] & A_0 \arrow[r, "\eta_F"'] & \frac{A_0}{F} \arrow[r] & 0
\end{tikzcd}$$

Applying V yields the following commutative diagram of $\mathcal{A}$, where the isomorphism exists due to the fact—see Remark 4.10—that ∇_V and V represent the same subfunctor.

$$\begin{tikzcd}
 & R[V\eta_F] \arrow[d, shift right] \arrow[d, shift left] & \\
V(R[p]) \arrow[r, shift left, "Vk_0"] \arrow[r, shift right, "Vk_1"'] \arrow[d, "V\pi"'] & V(A_0) \arrow[d, "V\eta_F"'] \arrow[dr, "\eta'_F"] & \\
V\eta_F(R[p]) \arrow[r, shift left, "V\kappa_0"] \arrow[r, shift right, "V\kappa_1"'] & V(\frac{A_0}{F}) \arrow[r, "\cong"'] & \frac{V(A_0)}{F}
\end{tikzcd}$$

$V\pi$ is a regular epimorphism because V is Birkhoff; it follows that $V(R[p]) \subseteq R[V\eta_F] = R[\eta'_F]$ precisely when $V\eta_F(R[p]) = \Delta_{V(A_0/F)}$.

Now, on one hand, by Proposition 3.8, $F \in \mathcal{F}^p_{V_0}$ if and only if

$$V(R[p]) \subseteq R\big[\eta'_F : V(A_0) \longrightarrow \tfrac{V(A_0)}{F}\big];$$

on the other hand, Proposition 6.5 implies that $V\eta_F(R[p]) = V(R[q_{\eta_F(R[p])}]) = \Delta_{V(A_0/F)}$ if and only if $V_1 q_{\eta_F R[p]} = 0$. This shows that 1. and 2. are equivalent. ∎

Let (R, r_0, r_1) be an equivalence relation on an object A in a pointed and exact category $\mathcal{A}$. If it exists, we denote the kernel $K[q_R]$ of the coequalizer $q_R : A \longrightarrow A/R$ by N_R. This defines a one-one correspondence between (isomorphism classes of) equivalence relations on A and normal subobjects of A. Furthermore, this bijection $\mathsf{Eq}(A) \cong \mathsf{NSub}(A)$ is an order isomorphism, the order on both classes being (induced by) the usual order on subobjects.

6.9. THEOREM. *Consider a pointed, exact and protomodular category $\mathcal{A}$, its Birkhoff subcategory $\mathcal{A}_{\mathsf{Ab}}$ of abelian objects, the associated Birkhoff subfunctor V, and the resulting functor $V_1 : \mathsf{Pr}\mathcal{A} \longrightarrow \mathcal{A}$. For any presentation $p : A_0 \longrightarrow A$ in $\mathcal{A}$, the following equality holds:*

$$V_1 p = N_{[R[p],\nabla_{A_0}]}.$$

PROOF. By the above mentioned order isomorphism and Definition 6.6, $N = N_{[R[p],\nabla_{A_0}]}$ is the smallest normal subobject of A_0 such that $\eta_N(R[p])$ and $\eta_N(\nabla_{A_0}) = \nabla_{A_0/N}$ admit a centralizing relation. Thus, by Proposition 6.8, it is the smallest element in $\mathcal{F}^p_{V_0}$; hence, it is $V_1 p$. ■

7. One more application of V_1: nilpotency

Using our notion of commutator, we now propose a notion of nilpotency. We show that, as in the case of abelianization of groups, an object is nilpotent if and only if its lower central series reaches 0. The nilpotent objects of class n form a Birkhoff subcategory.

Note that in their book [17], Freese and McKenzie study nilpotency in the context of *congruence modular* varieties.

From now on, $\mathcal{A}$ will be a pointed, exact and protomodular category and V a Birkhoff subfunctor of $\mathcal{A}$.

7.1. DEFINITION. *Let A be an object of $\mathcal{A}$. A V-*central series *of $\mathcal{A}$ is a descending sequence*

$$A = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_n \supseteq \ldots$$

of normal subobjects of A, such that, for all $n \in \mathbb{N}$, the natural arrow $\frac{A}{A_{n+1}} \longrightarrow \frac{A}{A_n}$ is V-central.

*A is called V-*nilpotent *when there exists in $\mathcal{A}$ a V-central series that reaches 0, i.e. such that $A_n = 0$ for some $n \in \mathbb{N}$. In that case, A is said to be V-nilpotent* of class *n.*

*The V-*lower central series *of A is the descending sequence*

$$A = V_1^0(A) \supseteq V_1^1(A) \supseteq \cdots \supseteq V_1^n(A) \supseteq \ldots \qquad (\mathbf{P})$$

defined, for $n \in \mathbb{N}$, by putting $V_1^0(A) = A$ and $V_1^{n+1}(A) = V_1(\eta_{V_n^1(A)} : A \longrightarrow \frac{A}{V_1^n(A)})$.

7.2. REMARK. Note that, by Lemma 4.12, $V_1^i(A)$ is a normal subobject of A.

7.3. REMARK. In case $\mathcal{A} = \mathsf{Gp}$ and V the kernel of the abelianization functor—see Example 6.1—by Proposition 6.2, one has $V_1(\eta_B : A \longrightarrow A/B) = [B, A]$. Thus $\mathbf{P}$ becomes the sequence

$$A \supseteq [A, A] \supseteq [[A, A], A] \supseteq [[[A, A], A], A] \supseteq \ldots,$$

whence the name "lower central series".

7.4. Proposition. *Let $A \supseteq B \supseteq C$ be objects in $\mathcal{A}$ with $C \triangleleft A$ and $B \triangleleft A$. The following are equivalent:*

1. $V_1\big(p : \frac{A}{C} \longrightarrow \frac{A}{B}\big) = 0$;
2. $C \supseteq V_1\big(\eta_B : A \longrightarrow \frac{A}{B}\big)$.

Proof. Consider the following diagram.

$$\begin{tikzcd}
R[\eta_B] \arrow[r, shift left] \arrow[r, shift right] \arrow[d, dotted, "\sigma"'] & A \arrow[r, "\eta_B"] \arrow[d, "\eta_C"'] & \frac{A}{B} \\
R[p] \arrow[r, shift left] \arrow[r, shift right] & \frac{A}{C} \arrow[ur, "p"'] &
\end{tikzcd}$$

By Proposition 2.10, the factorization σ is a regular epimorphism. Now, applying V, we get the following diagram; the isomorphism exists due to the fact—see Remark 4.10—that ∇_V and V represent the same subfunctor.

$$\begin{tikzcd}
 & R[V\eta_C] \arrow[d, shift left] \arrow[d, shift right] & \\
V(R[\eta_B]) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "V\sigma"'] & V(A) \arrow[d, "V\eta_C"'] \arrow[dr, "\eta'_{V(A)\cap C}"] & \\
V(R[p]) \arrow[r, shift left] \arrow[r, shift right] & V(\frac{A}{C}) \arrow[r, "\cong"'] & \frac{V(A)}{V(A)\cap C}
\end{tikzcd}$$

V being Birkhoff, $V\sigma$ is an epimorphism, hence $V(R[\eta_B]) \subseteq R[V\eta_C] = R[\eta'_{V(A)\cap C}]$ precisely when $V(R[p]) = \Delta_{V(A/C)}$.

Using Proposition 3.8 and Proposition 6.5, we get that 1. holds if and only if $V(A) \cap C \in \mathcal{F}^{\eta_B}_{V_0}$. But, by Corollary 3.11, $V(A) \cap C$ is an element of $\mathcal{F}^{\eta_B}_{V_0}$ if and only if $V_1\eta_B \subseteq V(A) \cap C$. This last statement is equivalent to condition 2. ∎

7.5. Corollary. *For any object $A \in \mathcal{A}$, the V-lower central series in A*

$$V_1^0(A) \supseteq V_1^1(A) \supseteq V_1^2(A) \supseteq \dots$$

is a V-central series, i.e.

$$V_1\Big(\frac{A}{V_1^{n+1}(A)} \longrightarrow \frac{A}{V_1^n(A)}\Big) = 0$$

for all $n \in \mathbb{N}$.

Proof. Take $A = A$, $B = V_1^n(A)$ and $C = V_1^{n+1}(A)$ in Proposition 7.4; then 2. is trivially fulfilled. ∎

7.6. Corollary. *$A \in \mathcal{A}$ is V-nilpotent of class n if and only if $V_1^n(A) = 0$.*

Proof. By Corollary 7.5, one implication is obvious. For the other, suppose that there exists a descending sequence

$$A = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_n = 0$$

with all objects normal in A, and that for all $i \in \{0, \dots, n-1\}$, $V_1\big(\frac{A}{A_{i+1}} \longrightarrow \frac{A}{A_i}\big) = 0$. We will show by induction that, for all $i \in \{0, \dots, n\}$, $V_1^i(A) \subseteq A_i$. If so, $V_1^n(A) \subseteq A_n = 0$, which proves our claim.

The case $i = 0$ is clear. Now suppose $V_1^i(A) \subseteq A_i$ for a certain $i \in \{0, \dots, n-1\}$. Recall from Remark 4.4 that there is an inclusion of functors $V_1 \subseteq K[\cdot] \cap V_0 \subseteq K[\cdot]$. Hence we get a commutative diagram of inclusions (all normal, since the four objects are normal subobjects of A):

$$\begin{tikzcd}
V_1\big(A \longrightarrow \frac{A}{V_1^i(A)}\big) \arrow[r, tail] \arrow[d, tail] & V_1^i(A) \arrow[d, tail] \\
V_1\big(A \longrightarrow \frac{A}{A_i}\big) \arrow[r, tail] & A_i.
\end{tikzcd}$$

In particular, we have $V_1\big(A \longrightarrow \frac{A}{V_1^i(A)}\big) \subseteq V_1\big(A \longrightarrow \frac{A}{A_i}\big)$ and thus

$$V_1^{i+1}(A) = V_1\big(A \longrightarrow \tfrac{A}{V_1^i(A)}\big) \subseteq V_1\big(A \longrightarrow \tfrac{A}{A_i}\big) \subseteq A_{i+1},$$

where the last inclusion follows from Proposition 7.4, as $V_1\big(\frac{A}{A_{i+1}} \longrightarrow \frac{A}{A_i}\big) = 0$. ∎

7.7. Remark. Since, by Remark 4.5, $V_1^1 = V$, a 1-nilpotent object is nothing but an object in the Birkhoff subcategory associated with V.

The following was inspired by Section 4.3 in Huq [24].

7.8. Proposition. *For $n \in \mathbb{N}$, $V_1^n : \mathcal{A} \longrightarrow \mathcal{A}$ is a Birkhoff subfunctor of $\mathcal{A}$. The corresponding Birkhoff subcategory is the full subcategory of all objects of V-nilpotency class n.*

Proof. Let $p : A \longrightarrow B$ be a regular epimorphism. The first statement is clear in case $n = 0$, so suppose that V_1^n preserves regular epimorphisms. Then, by the converse of Proposition 2.7, the induction hypothesis implies that the right hand square in the diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & V_1^n(A) \arrow[r, tail] \arrow[d, "V_1^n p"', two heads] & A \arrow[r, two heads] \arrow[d, "p", two heads] & \frac{A}{V_1^n(A)} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & V_1^n(B) \arrow[r, tail] & B \arrow[r, two heads] & \frac{B}{V_1^n(B)} \arrow[r] & 0
\end{tikzcd}$$

is a pushout. Hence, the category $\mathcal{A}$ being exact Mal'cev, it is a regular pushout; by Proposition 2.9 we get that it is a regular epimorphism $\boldsymbol{f}$ of $\mathsf{Pr}\mathcal{A}$. We conclude with Proposition 5.2 that $V_1^n p = V_1 \boldsymbol{f}$ is regular epi.

The second statement immediately follows from Corollary 5.7. ∎

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