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HOMOLOGY OF LIE ALGEBRAS WITH $\Lambda/q\Lambda$ COEFFICIENTS AND EXACT SEQUENCES

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ABSTRACT. Using the long exact sequence of nonabelian derived functors, an eight term exact sequence of Lie algebra homology with $\Lambda/q\Lambda$ coefficients is obtained, where Λ is a ground ring and q is a nonnegative integer. Hopf formulas for the second and third homology of a Lie algebra are proved. The condition for the existence and the description of the universal q-central relative extension of a Lie epimorphism in terms of relative homologies are given.

1. Introduction

Using results of [BaRo], Ellis and Rodriguez-Fernandez in [ElRo] have generalized Brown and Loday's eight term exact sequence in integral group homology [BrLo] to an eight term exact sequence in group homology with $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$ coefficients, where q is a nonnegative integer. For any group G and its normal subgroup N, they obtained the following natural exact sequence

$$\begin{aligned} &H_3(G,\mathbb{Z}_q) \to H_3(G/N,\mathbb{Z}_q) \to \mathrm{Ker}(N \wedge^q G \to G) \to H_2(G,\mathbb{Z}_q) \\ &\qquad \to H_2(G/N,\mathbb{Z}_q) \to N/N\#_q G \to H_1(G,\mathbb{Z}_q) \to H_1(G/N,\mathbb{Z}_q) \to 0 \ , \end{aligned}$$

where $H_i(G,\mathbb{Z}_q)$ (i=1,2,3) denotes the i-th homology group of G with coefficients in the trivial G-module $\mathbb{Z}_q$, $N\#_q G$ denotes the subgroup of N generated by the commutators $[n,g]$ and the elements of the form n^q for $n \in N$, $g \in G$. Tensor versions of the exterior product $N \wedge^q G$ have subsequently been studied in [Br] and in [CoRo].

For an ideal M of a Lie algebra P over a commutative ring Λ, Ellis [El2] has obtained the exact sequence

$$\mathrm{Ker}(M \wedge P \to P) \to H_2(P) \to H_2(P/M) \to M/[M,P] \to H_1(P) \to H_1(P/M) \to 0 \ ,$$

where $H_n(P)$ denotes the n-th homology of P with coefficients in the trivial P-module Λ and $M \wedge P$ denotes the nonabelian exterior product of Lie algebras M and P [El1].

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In [Kh] we introduced and studied the nonabelian tensor and exterior products of Lie algebras modulo q, these being mod q analogues of the tensor and exterior products in [El1].

The aim of this paper is to obtain the Lie algebra analogue of the eight term exact sequence of [ElRo], which will generalize the six term exact sequence above to the case of coefficients in $\Lambda/q\Lambda$ and will extend this sequence to the left by two terms.

As an application, Hopf formulas for the second and the third homologies of a Lie algebra with $\Lambda/q\Lambda$ coefficients are proved. The condition for the existence of the universal q-central relative extension of a Lie epimorphism [Kh] and the description of the kernel of such extension in terms of relative homologies are given.

Notations. Throughout the paper q denotes a nonnegative integer and Λ a commutative ring with identity. We write Λ_q instead of $\Lambda/q\Lambda$. All Lie algebras are Λ-Lie algebras and [,] denotes the Lie bracket.

2. Nonabelian derived functors of the exterior square modulo q

In this section we investigate derived functors of the nonabelian exterior square modulo q, establishing their relationship with the homology groups of a Lie algebra with coefficients in Λ_q.

First we give the definition of the nonabelian derived functors to the category of Lie algebras, denoted by $\mathcal{LIE}$ (see also [El1]).

Let $\mathcal{G} = (G, \epsilon, \delta)$ be a cotriple on a category $\mathcal{A}$ and $T : \mathcal{A} \to \mathcal{LIE}$ be a functor. For an object A of $\mathcal{A}$ let us consider the $\mathcal{G}$ cotriple resolution of A [BaBe2, Ke1]

$$\mathcal{G}(A)_* \equiv \cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} G^2(A) \underset{d_1^1}{\overset{d_0^1}{\rightrightarrows}} G^1(A) \xrightarrow{d_0^0} A \ ,$$

where $G^n(A) = G(G^{n-1}(A))$, $d_i^n = G^i \epsilon G^{n-i}$, $s_i^n = G^i \delta G^{n-i}$. Applying T dimension-wise to $\mathcal{G}(A)_*$ yields a simplicial Lie algebra

$$T\mathcal{G}(A)_* \equiv \cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} TG^2(A) \rightrightarrows TG^1(A) \ .$$

The n-th homotopy group of $T\mathcal{G}(-)_*$ is called the n-th nonabelian derived functor of T with respect to the cotriple $\mathcal{G} = (G, \epsilon, \delta)$ and it is denoted by $\mathcal{L}_n^{\mathcal{G}} T(-)$. Recall from [Cu] that the homotopy groups of $T\mathcal{G}(A)_*$ are the homology groups of the associated Moore complex

$$M_* \equiv \cdots M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} M_0 \xrightarrow{d_0} 0 \ ,$$

where $M_0 = TG(A)$, $M_n = \bigcap_{i=1}^{n-1} \mathrm{Ker} T(d_i^n)$ and d_n is the restriction of $T(d_n^n)$. Hence

$$\mathcal{L}_n^{\mathcal{G}} T(A) = \mathrm{Ker} d_n / \mathrm{Im} d_{n+1} \ , \quad n \geq 0 \ .$$

Let $\mathcal{F} = (F, \epsilon, \delta)$ be the cotriple on $\mathcal{LIE}$ generated by the adjoint pair [BaBe2, Ke1]

$$\begin{array}{ccc} \mathcal{LIE} & \xrightarrow{F} & \mathcal{LIE} \\ & {\scriptstyle U}\searrow & \uparrow{\scriptstyle F'} \\ & & \mathcal{SET} \end{array} \quad ,$$

where U is the forgetful functor sending a Lie algebra to its underlying set, F' is the functor sending a set to the free Lie algebra generated by this set.

Let M and N be two ideals of a Lie algebra P. We denote by $M\#_q N$ the submodule of $M \cap N$ generated by the elements $[m, n]$ and qk for $m \in M$, $n \in N$, $k \in M \cap N$. Then $M\#_q N$ is an ideal of $M \cap N$. In particular, $P\#_q P$ is an ideal of P. Let us consider the endofunctors V, $\mathcal{V} : \mathcal{LIE} \to \mathcal{LIE}$ defined by

$$V(P) = P\#_q P \text{ and } \mathcal{V}(P) = P/V(P) \ .$$

2.1. Lemma. *There is a natural isomorphism*

$$\mathcal{L}_n^{\mathcal{F}} \mathcal{V}(P) \approx H_{n+1}(P, \Lambda_q) \ \ (n \geq 0),$$

where $H_n(P, \Lambda_q)$ denotes the n-th homology of a Lie algebra P with coefficients in the trivial P-module Λ_q.

Proof. As pointed out in [Qu, Chapter II, Section 5], the cotriple description of group cohomology [BaBe1] carries over to the case of Lie algebra cohomology. Hence the cotriple description of group homology [BaBe2] carries over to the description of Lie algebra homology. Now if U_P and IP denote respectively the universal enveloping algebra and the augmentation ideal of a Lie algebra P, then the isomorphism $\Lambda_q \otimes_{U_P} IP \approx P/P\#_q P$ completes the proof. ∎

Let P be a Lie algebra with an ideal M. The exterior product modulo q of M and P [Kh] is the Lie algebra $M \wedge^q P$ generated by the symbols $m \wedge p$ and $\{m\}$ with $m \in M$, $p \in P$ subject to the relations

$$\lambda(m \wedge p) = \lambda m \wedge p = m \wedge \lambda p, \tag{1}$$

$$\begin{aligned} (m + m') \wedge p &= m \wedge p + m' \wedge p, \\ m \wedge (p + p') &= m \wedge p + m \wedge p', \end{aligned} \tag{2}$$

$$\begin{aligned} [m, m'] \wedge p &= m \wedge [m', p] - m' \wedge [m, p], \\ m \wedge [p, p'] &= [p', m] \wedge p - [p, m] \wedge p', \end{aligned} \tag{3}$$

$$[m \wedge p, m' \wedge p'] = [m, p] \wedge [m', p'], \tag{4}$$

$$[\{m'\}, m \wedge p] = [qm', m] \wedge p + m \wedge [qm', p], \tag{5}$$

$$\{\lambda m + \lambda' m'\} = \lambda\{m\} + \lambda'\{m'\}, \tag{6}$$

$$[\{m\}, \{m'\}] = qm \wedge qm', \tag{7}$$

$$\{[m, p]\} = q(m \wedge p), \tag{8}$$

$$m \wedge m = 1 \tag{9}$$

for all $m, m' \in M, \quad p, p' \in P, \quad \lambda, \lambda' \in \Lambda.$

2.2. Lemma. *If M is an ideal of a Lie algebra P then there is an exact sequence of Lie algebras*

$$(M \wedge^q P) \rtimes (M \wedge^q P) \stackrel{\alpha}{\longrightarrow} P \wedge^q P \stackrel{\beta}{\longrightarrow} P/M \wedge^q P/M \longrightarrow 0 \ ,$$

where $\rtimes$ denotes the semidirect product and the action of $M \wedge^q P$ on itself is given by Lie multiplication.

Proof. β is the functorial homomorphism induced by the projection $P \to P/M$ and it is surjective [Kh, Proposition 1.8]. Let $\alpha' : M \wedge^q P \to P \wedge^q P$ be the functorial homomorphism induced by the inclusion $M \to P$ and by the identity map $P \to P$. We set $\alpha(x, y) = \alpha'(x) + \alpha'(y)$ for $x, y \in M \wedge^q P$. It is easy to check that α is a Lie homomorphism. The image of α is generated by the elements $m \wedge p$ and $\{m\}$ for $m \in M$, $p \in P$. Clearly $\beta\alpha$ is the trivial homomorphism. By the formulas (4), (5) $\mathrm{Im}(\alpha)$ is an ideal of $P \wedge^q P$. Let us define a homomorphism $\beta' : P/M \wedge^q P/M \longrightarrow (P \wedge^q P)/\mathrm{Im}(\alpha)$ as follows: $\beta'(\overline{p}_1 \wedge \overline{p}_2) = \overline{p_1 \wedge p_2}$, $\beta(\{\overline{p}\}) = \overline{\{p\}}$, $p, p_1, p_2 \in P$. It is easy to see that β' is correctly defined and there is an inverse homomorphism of β' induced by β. ∎

Note that there is a Lie homomorphism $\partial : M \wedge^q P \to P$ defined by $\partial(m \wedge p) = [m, p]$, $\partial(\{m\}) = qm$ [Kh, Proposition 1.3] and the image of ∂ is $M \#_q P$.

2.3. Lemma. *If $q \geq 1$, Λ is a q-torsion-free ground ring and F is a free Lie algebra, then the homomorphism $\partial : F \wedge^q F \to F$ induces an isomorphism $F \wedge^q F \approx F \#_q F$.*

Proof. Let $F \wedge F$ be the nonabelian exterior square (for the definition see [El1]). By [Kh, Proposition 1.6] one has the following commutative diagram of Lie algebras with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F \wedge F & \stackrel{\varphi}{\longrightarrow} & F \wedge^q F & \longrightarrow & F^{ab} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial'} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial''} & & \\ 0 & \longrightarrow & [F, F] & \longrightarrow & F \#_q F & \longrightarrow & qF^{ab} & \longrightarrow & 0 \end{array} \ ,$$

where $F^{ab} = F/[F, F]$, ∂' is an isomorphism [El2, Proposition 1.2] and hence φ is injective. ∂'' is induced by ∂ and clearly it is surjective. F^{ab} is a free Λ-module. Since Λ is a q-torsion-free, ∂'' is an isomorphism and so is ∂. ∎

Consider the endofunctor $\wedge^q : \mathcal{LIE} \to \mathcal{LIE}$, which we call nonabelian exterior square modulo q, defined by

$$\wedge^q(P) = P \wedge^q P \ .$$

One has the following

2.4. Proposition. *There is a natural isomorphism*

$$\mathcal{L}_0^{\mathcal{F}} \wedge^q (P) \approx P \wedge^q P \ .$$

Moreover, if $q \geq 1$ and Λ is a q-torsion-free ring, then there is a natural isomorphism

$$\mathcal{L}_n^{\mathcal{F}} \wedge^q (P) \approx H_{n+2}(P, \Lambda_q)$$

for every $n \geq 1$.

Proof. Consider the diagram of Lie algebras

$$F^2(P) \wedge^q F^2(P) \underset{d_1^1 \wedge d_1^1}{\overset{d_0^1 \wedge d_0^1}{\rightrightarrows}} F(P) \wedge^q F(P) \overset{d_0^0 \wedge d_0^0}{\longrightarrow} P \wedge^q P \longrightarrow 0 \ .$$

We have to show $(d_1^1 \wedge d_1^1)(\mathrm{Ker}(d_0^0 \wedge d_0^0)) = \mathrm{Ker}(d_0^0 \wedge d_0^0)$. By Lemma 2.2 we get that $\mathrm{Ker}(d_0^1 \wedge d_0^1)$ is generated by the elements $x \wedge k$ and $\{k\}$ with $x \in F^2(P)$, $k \in \mathrm{Ker} d_0^1$. Thus $(d_1^1 \wedge d_1^1)(\mathrm{Ker}(d_0^0 \wedge d_0^0))$ is generated by the elements $x' \wedge k'$ and $\{k'\}$ with $x' \in F(P)$, $k' \in d_1^1(\mathrm{Ker} d_0^1)$. On the other hand it follows from Lemma 2.2 that $\mathrm{Ker}(d_0^0 \wedge d_0^0)$ is generated by the elements $x'' \wedge k''$ and $\{k''\}$ with $x'' \in F(P)$, $k'' \in \mathrm{Ker} d_0^0$. Then the identity $d_1^1(\mathrm{Ker} d_0^1) = \mathrm{Ker} d_0^0$ proves the first isomorphism.

Consider the $\mathcal{F}$ cotriple resolution of P

$$\mathcal{F}(P)_* \equiv \cdots \begin{array}{c}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{array} F^2(P) \underset{d_1^1}{\overset{d_0^1}{\rightrightarrows}} F^1(P) \overset{d_0^0}{\longrightarrow} P \ .$$

By Lemma 2.3 there is a simplicial isomorphism

$$\mathcal{F}(P)_* \#_q \mathcal{F}(P)_* \approx \wedge^q \mathcal{F}(P)_* \ .$$

Thus one has the following short exact sequence of simplicial Lie algebras

$$0 \to \wedge^q \mathcal{F}(P)_* \to \mathcal{F}(P)_* \to \mathcal{V}\mathcal{F}(P)_* \to 0 \ .$$

Then by Lemma 2.1 the respective long exact homotopy sequence is of the form

$$\begin{aligned} \cdots \to 0 \to H_{n+2}(P, \Lambda_q) \to \mathcal{L}_n^{\mathcal{F}} \wedge^q (P) \to 0 \\ \to H_{n+1}(P, \Lambda_q) \to \cdots \to \mathcal{L}_0^{\mathcal{F}} \wedge^q (P) \to P \to P/P\#_q P \ , \end{aligned}$$

which gives the second isomorphism. ∎

3. Eight term exact sequence of Lie algebra homology with Λ_q coefficients

Let $\mathcal{LIE}_1$ denotes the category whose objects are surjective morphisms of $\mathcal{LIE}$ and a morphism from $P \overset{\alpha}{\longrightarrow} Q$ to $P' \overset{\alpha'}{\longrightarrow} Q'$ is a commutative square in $\mathcal{LIE}$

$$\begin{array}{ccc} P & \overset{\alpha}{\longrightarrow} & Q \\ {\scriptstyle h_0}\downarrow & & \downarrow{\scriptstyle h_1} \\ P' & \underset{\alpha'}{\longrightarrow} & Q' \end{array} \ .$$

The cotriple $\mathcal{F} = (F, \epsilon, \delta)$ on $\mathcal{LIE}$ extends to a cotriple $\mathcal{F}_1 = (F_1, \epsilon_1, \delta_1)$ on $\mathcal{LIE}_1$ which is generated by the adjoint pair

$$\begin{array}{ccc} \mathcal{LIE}_1 & \xrightarrow{F_1} & \mathcal{LIE}_1 \\ & \searrow^{U_1} & \uparrow {F'}_1 \\ & & \mathcal{SET}_1 \end{array} \quad ,$$

where $\mathcal{SET}_1$ is the category whose objects are surjective maps of sets and whose morphisms are commutative squares in $\mathcal{SET}$, U_1 and F'_1 are induced respectively by U and F'.

We say that $(h_0, h_1) : \alpha \to \alpha'$ is a surjective morphism of $\mathcal{LIE}_1$ if $U_1(h_0, h_1)$ has a splitting in $\mathcal{SET}_1$.

Inductively we define a category $\mathcal{LIE}_m$, a cotriple $\mathcal{F}_m = (F_m, \epsilon_m, \delta_m)$ on $\mathcal{LIE}_m$ and surjective morphisms of $\mathcal{LIE}_m$ for $m \geq 0$:

$$\mathcal{LIE}_{m+1} = (\mathcal{LIE}_m)_1 \ , \ \ \mathcal{LIE}_0 = \mathcal{LIE}$$

and

$$\mathcal{F}_{m+1} = (\mathcal{F}_m)_1 \ , \ \ \mathcal{F}_0 = \mathcal{F} \ .$$

Moreover, if $T : \mathcal{LIE} \to \mathcal{LIE}$ is an endofunctor, we define $T_m : \mathcal{LIE}_m \to \mathcal{LIE}$, $m \geq 0$, as follows: if α, α' are objects of $\mathcal{LIE}_1$ and $(h_0, h_1) : \alpha \to \alpha'$ is a morphism of $\mathcal{LIE}_1$ then

$$T_1(\alpha) = \mathrm{Ker} T(\alpha) \ , \ \ T_1(h_0, h_1) = T(h_0)|_{T_1(\alpha)};$$

and

$$T_{m+1} = (T_m)_1 \ , \ \ T_0 = T \ .$$

It is easy to see that a surjective morphism $f : X \to Y$ of $\mathcal{LIE}_m$ induces a surjection of simplicial Lie algebras $f_* : T_m\mathcal{F}_m(X)_* \to T_m\mathcal{F}_m(Y)_*$, which yields a long exact sequence of homotopy groups. Thus we have immediately

3.1. Proposition. *A surjective morphism $f : X \to Y$ of $\mathcal{LIE}_m$ $(m \geq 0)$ yields a natural long exact sequence*

$$\cdots \to \mathcal{L}_n^{\mathcal{F}_{m+1}} T_{m+1}(f) \to \mathcal{L}_n^{\mathcal{F}_m} T_m(X) \to \mathcal{L}_n^{\mathcal{F}_m} T_m(Y) \to \cdots \to \mathcal{L}_0^{\mathcal{F}_m} T_m(Y) \to 0 \ .$$

Further for a functor $T : \mathcal{LIE} \to \mathcal{LIE}$ we shall write $\mathcal{L}_n T_m(-)$ to mean the n-th derived functor with respect to the cotriple $\mathcal{F}_m$.

Let V, $\mathcal{V} : \mathcal{LIE} \to \mathcal{LIE}$ be the endofunctors defined in the previous section. Then one has the following

3.2. Proposition. *Let M and N be two ideals of a Lie algebra P such that $M+N=P$. Consider the following object (α,γ) in the category $\mathcal{LIE}_2$*

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & N/M\cap N \\ \downarrow & & \downarrow \\ N/M\cap N & \xrightarrow[\gamma]{} & 0 \end{array} .$$

Then there is a natural long exact sequence

$$\begin{aligned} \cdots \to H_{n+1}(P,\Lambda_q) \to H_{n+1}(M/M\cap N,\Lambda_q) \oplus H_{n+1}(N/M\cap N,\Lambda_q) \\ \to \mathcal{L}_{n-1}\mathcal{V}_2(\alpha,\gamma) \to \cdots \to H_2(P,\Lambda_q) \\ \to H_2(M/M\cap N,\Lambda_q) \oplus H_2(N/M\cap N,\Lambda_q) \to \mathcal{L}_0\mathcal{V}_2(\alpha,\gamma) \\ \to H_1(P,\Lambda_q) \to H_1(M/M\cap N,\Lambda_q) \oplus H_1(N/M\cap N,\Lambda_q) \to 0 \ . \end{aligned}$$

Proof. First note that $\mathcal{L}_n\mathcal{V}_2(\alpha,\gamma) = \mathcal{L}_n\mathcal{V}_2(h_0,h_1)$, $n \geq 0$. Then using Proposition 3.1 it is easy to get the following natural long exact sequence (compare [El1, Lemma 31])

$$\begin{aligned} \cdots \to \mathcal{L}_n\mathcal{V}(P) \to \mathcal{L}_n\mathcal{V}(M/M\cap N) \oplus \mathcal{L}_n\mathcal{V}(N/M\cap N) \\ \to \mathcal{L}_{n-1}\mathcal{V}_2(\alpha,\gamma) \to \cdots \to \mathcal{L}_0\mathcal{V}_2(\alpha,\gamma) \to \mathcal{L}_0\mathcal{V}(P) \\ \to \mathcal{L}_0\mathcal{V}(M/M\cap N) \oplus \mathcal{L}_0\mathcal{V}(N/M\cap N) \to 0 \ . \end{aligned}$$

Then the isomorphism of Lemma 2.1 gives the result. ∎

3.3. Corollary. *Let M be an ideal of a Lie algebra P and $\alpha : P \to P/M$ the natural epimorphism. One has the following exact sequence*

$$\begin{aligned} \cdots \to H_{n+1}(P,\Lambda_q) \to H_{n+1}(P/M,\Lambda_q) \to \mathcal{L}_{n-1}\mathcal{V}_1(\alpha) \\ \to \cdots \to H_3(P,\Lambda_q) \to H_3(P/M,\Lambda_q) \to \mathcal{L}_1\mathcal{V}_1(\alpha) \to H_2(P,\Lambda_q) \\ \to H_2(P/M,\Lambda_q) \to \mathcal{L}_0\mathcal{V}_1(\alpha) \to H_1(P,\Lambda_q) \to H_1(P/M,\Lambda_q) \to 0 \ . \end{aligned}$$

Proof. The result follows from the previous proposition by considering $N=P$ and the object (α,γ) in the category $\mathcal{LIE}_2$

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & P/M \\ \downarrow & & \downarrow \\ 0 & \xrightarrow[\gamma]{} & 0 \end{array} ,$$

for which we have $\mathcal{L}_n\mathcal{V}_2(\alpha,\gamma) = \mathcal{L}_n\mathcal{V}_1(\alpha)$, $n \geq 0$. ∎

Now we compute $\mathcal{L}_0\mathcal{V}_1(-)$ and $\mathcal{L}_1\mathcal{V}_1(-)$ to give an interpretation of the last eight term of the long exact sequence of Corollary 3.3.

First we recall the following fact from [El1]. (See [Ke2] for the group case). For any ideal M of a Lie algebra P let us consider the diagram

$$(M \oplus M) \rtimes P \overset{l_1}{\underset{l_3}{\overset{\xrightarrow{l_2}}{\longrightarrow}}} M \rtimes P \underset{p_2}{\overset{p_1}{\rightrightarrows}} P \ ,$$

where $\rtimes$ denotes a semi-direct product; the action of P on M is given by Lie multiplication; the action of P on $M \oplus M$ is ${}^p(m, m') = ([p, m], [p, m'])$; the homomorphisms are defined by

$$p_1(m, p) = m + p, \quad p_2(m, p) = p \ ;$$

$$l_1(m', m, p) = (m' - m, m + p), \quad l_2(m', m, p) = (m', p), \quad l_3(m', m, p) = (m, p) \ .$$

If $T : \mathcal{LIE} \to \mathcal{LIE}$ is any endofunctor, on applying $\mathcal{L}_0T$ to the above diagram we obtain a diagram

$$\mathcal{L}_0T((M \oplus M) \rtimes P) \overset{l'_1}{\underset{l'_3}{\overset{\xrightarrow{l'_2}}{\longrightarrow}}} \mathcal{L}_0T(M \rtimes P) \underset{p'_2}{\overset{p'_1}{\rightrightarrows}} \mathcal{L}_0T(P) \ ,$$

where we write p'_i and l'_i instead of $\mathcal{L}_0T(p_i)$ and $\mathcal{L}_0T(l_i)$. Suppose $\alpha : P \to P/M$ is the natural epimorphism. Then we have

3.4. Lemma. [E1] *There is an isomorphism*

$$\mathcal{L}_0T_1(\alpha) = \{\mathrm{Ker} p'_2\}/\{l'_1(\mathrm{Ker} l'_2 \cap \mathrm{Ker} l'_3)\} \ .$$

3.5. Proposition. *Let* $0 \to M \to P \to P/M \to 0$ *be a short exact sequence of Lie algebras, then*

(i) $\mathcal{L}_0\mathcal{V}_1(\alpha) \approx M/M\#_qP$,

(ii) $\mathcal{L}_1\mathcal{V}_1(\alpha) \approx \mathrm{Ker}(\partial : M \wedge^q P \to P)$, *if* $q \geq 1$ *and* Λ *is a* q*-torsion-free ring.*

Proof. (i) Consider $l'_i : \mathcal{V}((M \oplus M) \rtimes P) \to \mathcal{V}(M \rtimes P)$. It is easy to check that $\mathrm{Ker} l'_2 \cap \mathrm{Ker} l'_3 = 0$, then by Lemma 3.4 one has

$$\mathcal{L}_0\mathcal{V}_1(\alpha) \approx \mathrm{Ker}\{\mathcal{V}(M \rtimes P) \xrightarrow{p'_2} \mathcal{V}(P)\} \approx M/M\#_qP \ .$$

(ii) Let $I : \mathcal{LIE} \to \mathcal{LIE}$ be the identity functor. Then

$$0 \to V_1 \to I_1 \to \mathcal{V}_1 \to 0$$

is an exact sequence of functors from $\mathcal{LIE}_1$ to $\mathcal{LIE}$. Since $\mathcal{L}_0I_1 \approx I_1$ and $\mathcal{L}_nI_1 = 0$ for $n \geq 1$, the resulting long exact homotopy sequence provides an isomorphism

$$\mathcal{L}_1\mathcal{V}_1(\alpha) \approx \mathrm{Ker}(\mathcal{L}_0V_1(\alpha) \to I_1(\alpha)) \ .$$

Since $I_1(\alpha) = M$, by Lemma 2.3 we get an isomorphism

$$\mathcal{L}_1\mathcal{V}_1(\alpha) \approx \operatorname{Ker}(\mathcal{L}_0 \wedge_1^q (\alpha) \to M) \ .$$

Thus to prove the isomorphism (ii) we need to show that there is an isomorphism $M \wedge^q P \xrightarrow{\approx} \mathcal{L}_0 \wedge_1^q (\alpha)$ such that the diagram

$$\begin{array}{ccc} M \wedge^q P & \xrightarrow{\partial} & M \\ \approx\downarrow & & \| \\ \mathcal{L}_0 \wedge_1^q (\alpha) & \longrightarrow & M \end{array}$$

commutes. Consider the diagram

$$((M \oplus M) \rtimes P) \wedge^q ((M \oplus M) \rtimes P) \underset{l'_3}{\overset{l'_1}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xrightarrow{l'_2}}}}} (M \rtimes P) \wedge^q (M \rtimes P) \ \underset{p'_2}{\overset{p'_1}{\rightrightarrows}} P \wedge^q P \ ,$$

where l'_i, p'_i are the homomorphisms of Lemma 3.4. By Lemma 2.2 $\operatorname{Ker}l'_2$ is generated by the elements $(0, m, 0) \wedge (m_1, m_2, p)$ and $\{(0, m, 0)\}$, $\operatorname{Ker}l'_3$ is generated by the elements $(m, 0, 0) \wedge (m_1, m_2, p)$ and $\{(m, 0, 0)\}$. Thus $\operatorname{Ker}l'_2 \cap \operatorname{Ker}l'_3$ is generated by the elements $(m, 0, 0) \wedge (0, m', 0)$ and then $l'_1(\operatorname{Ker}l'_2 \cap \operatorname{Ker}l'_3)$ is generated by elements of the form $(m, 0) \wedge (-m', m')$. It is easy to check that $\operatorname{Ker}p'_2 = (M \rtimes 0) \wedge^q (M \rtimes P)$. Then by Lemma 3.4 one has

$$\mathcal{L}_0 \wedge_1^q (\alpha) \approx (M \rtimes 0) \wedge^q (M \rtimes P)/l'_1(\operatorname{Ker}l'_2 \cap \operatorname{Ker}l'_3) \approx M \wedge^q P \ ,$$

where the last isomorphism is defined by $\overline{(m, 0) \wedge (m', p)} \mapsto m \wedge (m' + p)$, $\{\overline{(m, 0)}\} \mapsto \{m\}$. It is readily seen that the above diagram commutes. ■

The previous results give immediately the following

3.6. Theorem. *Let $q \geq 1$, Λ is a q-torsion-free ground ring and P be a Lie algebra with an ideal M. There is a natural exact sequence*

$$\begin{aligned} &H_3(P, \Lambda_q) \to H_3(P/M, \Lambda_q) \to \operatorname{Ker}(M \wedge^q P \xrightarrow{\partial} P) \to H_2(P, \Lambda_q) \\ &\qquad \to H_2(P/M, \Lambda_q) \to M/M\#_q P \to H_1(P, \Lambda_q) \to H_1(P/M, \Lambda_q) \to 0 \ . \end{aligned}$$

Observe that the exact sequence of Theorem 3.6 generalizes the six term exact sequence in [El2] to eight term and to the case of coefficients in Λ_q. The group theoretic version of this sequence is obtained in [ElRo].

3.7. Corollary. *Let $q \geq 1$ and P be a Lie algebra over a q-torsion-free ground ring Λ. There is an isomorphism*

$$H_2(P, \Lambda_q) \approx \operatorname{Ker}(P \wedge^q P \xrightarrow{\partial} P).$$

Furthermore, for any free presentation

$$0 \to R \to F \to P \to 0$$

of P, there is an isomorphism

$$H_3(P, \Lambda_q) \approx \mathrm{Ker}(R \wedge^q F \xrightarrow{\partial} F).$$

In the rest of this section, as an application of the previous results, we prove Hopf formulas for the second and the third homology groups of a Lie algebra with Λ_q coefficients. Also we give the condition for the existence and the description of the universal q-central relative extension [Kh] in terms of relative homologies.

3.8. Theorem. *Let P be a Lie algebra and*

$$0 \to R \to F \xrightarrow{\alpha} P \to 0$$

be a free presentation of P. Then there is an isomorphism

$$H_2(P, \Lambda_q) \approx (R \cap (F \#_q F))/(R \#_q F) \ .$$

Proof. Since $H_2(F, \Lambda_q) = 0$, by Corollary 3.3 and Proposition 3.5(i) we get

$$\begin{aligned} H_2(P, \Lambda_q) &\approx \mathrm{Ker}(R/R\#_q F \to H_1(F, \Lambda_q)) \\ &\approx \mathrm{Ker}(R/R\#_q F \to F/F\#_q F) \approx (R \cap (F\#_q F))/(R\#_q F) \ . \end{aligned}$$

■

Note that the isomorphism of Theorem 3.8 is the mod q version of the well known Hopf formula for the second homology of a Lie algebra (see for example [HiSt]). Now we prove the mod q version of the Hopf formula for the third homology (see [El1]). In order to do this we need the following lemma which can be proved in a similar way as Theorem 35(ii) of [El1].

3.9. Lemma. *For the following object (α, γ) in the category $\mathcal{LIE}_2$*

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & P/M \\ \downarrow & & \downarrow \\ P/N & \xrightarrow[\gamma]{} & P/(M+N) \end{array} ,$$

where M and N are two ideals of a Lie algebra P, there is an isomorphism

$$\mathcal{L}_0\mathcal{V}_2(\alpha, \gamma) = (M \cap N)/(P\#_q(M \cap N) + M\#_q N) \ .$$

3.10. THEOREM. *Let F be a Lie algebra and $H_2(F,\Lambda_q)=0$ (for example, F is a free Lie algebra). Let R and S be two ideals of F such that $H_i(F/R,\Lambda_q)=H_i(F/S,\Lambda_q)=0$ for $i=2,3$ (for example, the Lie algebras F/R and F/S are free). Then there is an isomorphism*

$$H_3(F/(R+S),\Lambda_q)\approx (R\cap S\cap F\#_qF)/((R\cap S)\#_qF+R\#_qS)\ \ .$$

PROOF. Consider the object (α,γ) in $\mathcal{LIE}_2$

$$\begin{array}{ccc} F & \xrightarrow{\ \alpha\ } & F/R \\ {\scriptstyle h_0}\downarrow & & \downarrow{\scriptstyle h_1} \\ F/S & \xrightarrow[\ \gamma\ ]{} & F/(R+S) \end{array}\ \ .$$

By Proposition 3.1 and by Lemma 2.1 there are the following three long exact sequences

$$\cdots\to\mathcal{L}_1\mathcal{V}_2(\alpha,\gamma)\to\mathcal{L}_1\mathcal{V}_1(h_0)\to\mathcal{L}_1\mathcal{V}_1(h_1)$$
$$\to\mathcal{L}_0\mathcal{V}_2(\alpha,\gamma)\to\mathcal{L}_0\mathcal{V}_1(h_0)\to\mathcal{L}_0\mathcal{V}_1(h_1)\to 0\ \ ;\quad (*)$$

$$\cdots\to H_3(F,\Lambda_q)\to H_3(F/S,\Lambda_q)\to\mathcal{L}_1\mathcal{V}_1(h_0)\to H_2(F,\Lambda_q)$$
$$\to H_2(F/S,\Lambda_q)\to\mathcal{L}_0\mathcal{V}_1(h_0)\to H_1(F,\Lambda_q)\to H_1(F/S,\Lambda_q)\to 0\ \ ;\quad (**)$$

$$\cdots\to H_3(F/R,\Lambda_q)\to H_3(F/(R+S),\Lambda_q)\to\mathcal{L}_1\mathcal{V}_1(h_1)$$
$$\to H_2(F/R,\Lambda_q)\to H_2(F/(R+S),\Lambda_q)\to\mathcal{L}_0\mathcal{V}_1(h_1)$$
$$\to H_1(F/R,\Lambda_q)\to H_1(F/(R+S),\Lambda_q)\to 0\ \ .\quad (***)$$

$(***)$ gives us an isomorphism $H_3(F/(R+S),\Lambda_q)\approx\mathcal{L}_1\mathcal{V}_1(h_1)$ since $H_i(F/R,\Lambda_q)=0$ for $i=2,3$. From $(**)$ we have $\mathcal{L}_1\mathcal{V}_1(h_0)=0$ since $H_2(F,\Lambda_q)=0$ and $H_i(F/S,\Lambda_q)=0$ for $i=2,3$. Thus from $(*)$ we get

$$H_3(F/(R+S),\Lambda_q)\approx \mathrm{Ker}(\mathcal{L}_0\mathcal{V}_2(\alpha,\gamma)\to\mathcal{L}_0\mathcal{V}_1(h_0)).$$

By Theorem 2.8 $S\#_qF=S\cap(F\#_qF)$ since $0\to S\to F\to F/S\to 0$ is a free presentation of F/S. Then by Lemma 3.9 and Proposition 3.5(i) we have

$$H_3(F/(R+S),\Lambda_q)\approx\mathrm{Ker}((R\cap S)/((R\cap S)\#_qF+R\#_qS)\to S/S\#_qF)$$
$$\approx (R\cap S\cap F\#_qF)/((R\cap S)\#_qF+R\#_qS)\ \ .$$

■

Let $\alpha : P \to Q$ be a Lie epimorphism and A be a Q-module. Recall from [KaLo] that a relative extension of α by A is an exact sequence of Lie algebras

$$0 \to A \to E \xrightarrow{\mu} P \xrightarrow{\alpha} Q \to 0 \ ,$$

where μ is a crossed module. Such extension is called a q-central relative extension [Kh] if Q acts trivially on A and $qa = 0$ for any $a \in A$. q-central relative extension of α is called universal if there exists a unique morphism of relative extensions [KaLo] from it to any q-central relative extension of α.

Let

$$0 \to M \to P \xrightarrow{\alpha} Q \to 0$$

be a short exact sequence of Lie algebras. The Lie epimorphism α has a universal q-central relative extension if and only if $M = M\#_q P$ and such extension is given by the following exact sequence [Kh, Theorem 2.8]

$$0 \to \mathrm{Ker}\partial \to M \wedge^q P \xrightarrow{\partial} P \xrightarrow{\alpha} Q \to 0 \ .$$

Using notations of [KaLo] let us denote by $H_n(\alpha, \Lambda_q)$, $n \geq 0$, the n-th relative homology group of a Lie epimorphism $\alpha : P \to Q$ with coefficients in the trivial Q-module Λ_q. Clearly $H_{n+2}(\alpha, \Lambda_q) \approx \mathcal{L}_n \mathcal{V}_1(\alpha)$, $n \geq 0$. Then from Proposition 3.5 we get

$$H_2(\alpha, \Lambda_q) \approx M/M\#_q P$$

and if $q \geq 1$ and Λ is a q-torsion-free ring then

$$H_3(\alpha, \Lambda_q) \approx \mathrm{Ker}(M \wedge^q P \xrightarrow{\partial} P) \ .$$

So the description of the universal q-central relative extension can be expressed in terms of relative homologies as follows:

3.11. Theorem. *The Lie epimorphism α has a universal q-central relative extension if and only if $H_2(\alpha, \Lambda_q) = 0$. Moreover, if $q \geq 1$ and Λ is a q-torsion-free ring, then the sequence*

$$0 \to H_3(\alpha, \Lambda_q) \to M \wedge^q P \xrightarrow{\partial} P \xrightarrow{\alpha} Q \to 0 \ ,$$

is the universal q-central relative extension of α.

This result is mod q version of [KaLo, Theorem A.4], or alternatively, it is the Lie algebra version of [CoRo, Corollary 2.16].

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