Theory and Applications of Categories, Vol. 10, No. 8, 2002, pp. 162–179.

COLOCALIZATIONS AND THEIR REALIZATIONS AS SPECTRA

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ABSTRACT. Every chain functor $\boldsymbol{A}_*$ (cf. appendix B below or [2] §4), admits a $\mathfrak{L}$-colocalization (corollary 1.3., $\mathfrak{L}$ a subcategory of the category on which $\boldsymbol{A}_*$ is defined) $\boldsymbol{A}_*^{\mathfrak{L}}$ which (in contrast to the case of $\mathfrak{L}$-localizations, cf. [3]) in general does not allow a realization as a spectrum (even if $\boldsymbol{A}_*$ stems from a spectrum itself). The $[E,\ ]_*$- colocalization of A. K. Bousfield [6] is retrieved as a special case of a general colocalization process for chain functors.

Introduction

It is well-known that every spectrum A can be localized in the sense of A. K. Bousfield [5] with respect to another spectrum E. This amounts to the existence of a natural exact sequence of spectra

$$ {}_EA \longrightarrow A \xrightarrow{\eta} A_E \tag{1} $$

where ${}_EA$ is E-acyclic, A_E E-local and η an E-isomorphism. E-local means that $[B,\ A_E] = 0$ for any E-acyclic spectrum B.

Every spectrum A gives rise to a chain functor $\boldsymbol{A}_*$ (cf. appendix B or [2] for further references). Let $\mathfrak{L} \subset \mathbf{Top}^2$ or $\mathfrak{L} \subset \mathfrak{B}$ (= the Boardman category [1]) be a full subcategory (e.g. $\mathfrak{L} = \{E\}$, being determined by a single object), then there exists a natural $\mathfrak{L}$-localization sequence

$$ {}_{\mathfrak{L}}\boldsymbol{A}_* \longrightarrow \boldsymbol{A}_* \xrightarrow{\eta} \boldsymbol{A}_{\mathfrak{L}*} \tag{2} $$

which implies (1) as a special case.

The objective of the present paper is to establish a dual $\mathfrak{L}$-colocalization sequence for chain functors

$$ \boldsymbol{A}_*^{\mathfrak{L}} \xrightarrow{\eta} \boldsymbol{A}_* \longrightarrow {}^{\mathfrak{L}}\boldsymbol{A}_* \tag{3} $$

where ${}^{\mathfrak{L}}\boldsymbol{A}_*$ is $\mathfrak{L}$-acyclic, η an $\mathfrak{L}$-isomorphism and $\boldsymbol{A}_*^{\mathfrak{L}}$ $\mathfrak{L}$-*colocal* (i.e. $[\boldsymbol{A}_*^{\mathfrak{L}},\ \boldsymbol{B}_*] = 0$ for any $\mathfrak{L}$-acyclic chain functor $\boldsymbol{B}_*$).

Theorem 1.3 and Corollary 1.4 are existence theorems for $\mathfrak{L}$- colocalizations. So far we have (for the assertions, not for the proofs) full duality with the case of $\mathfrak{L}$-localizations.

Received by the editors 2001 September 6 and, in revised form, 2002 April 3.
Transmitted by James Stasheff. Published on 2002 April 11.
2000 Mathematics Subject Classification: Primary: 55P60, 55N20; Secondary: 55U30, 55P42.
Key words and phrases: Colocalizations, chain functors, realizations of chain functors as spectra.

However, even if the chain functor $\boldsymbol{A}_*$ originates from a spectrum A (i.e. if it can be realized as a spectrum), it turns out that in general $\boldsymbol{A}_*^{\mathfrak{L}}$ can not be realized as a spectrum, because it does not have compact carriers. So we rediscover the fact that in general there are no colocalizations on the level of spectra. In the case $\mathfrak{L} = \{E\}$ we present in §2 an explicit construction of $\boldsymbol{A}_*^E$ by means of the E-colocalization of a specific (highly irregular, i.e. not realizable) chain functor $\mathbb{Z}_*$, which is taken from [3]. By enlarging the given category (e.g. the category of Boardman spectra, the same would also work for pairs of CW-spaces) by some "formal S-duals" DE, we are permitted to deal with homology theories of the form $\{E,\ \}_*$ (which are isomorphic to $DE_*(\ ) = \{S^\bullet, (\ ) \wedge DE\}_*$ whenever a S-dual DE exists). Although $\{E,\ \}_*$ does not have compact carriers, it turns out that (theorem 3.1.) DE-colocalization of a chain functor $\boldsymbol{A}_*$ can be realized by a spectrum. As a result we obtain a $\{E,\ \}_*$-colocalization of a spectrum *as a spectrum* and not only as a chain functor. This coincides with the results of [6].

The proof of the main theorem is accomplished in §4 and prepared by some material on chain functors in an appendix A.

In contrast to [2], [3] we do not require that homology theories and (even regular) chain functors (cf. appendix B)) automatically have compact carriers, so that $\{E,\ \}_*$ now becomes a homology theory. In our present notation a *chain functor* might be an irregular one (cf. Appendix B).

The results of this paper have been recorded in the expository article [4].

1. Colocalizations of chain functors:

Suppose that $\mathfrak{L} \subset \mathfrak{K} \subset \mathbf{Top}^2$ are categories and $\mathfrak{L}$ a full subcategory of $\mathfrak{K}$. Alternatively we may take a full subcategory of the Boardman category [1].

1.1. Definition.

1) A chain functor $\boldsymbol{C}_*$ is $\mathfrak{L}$-acyclic, *whenever for any $(E,F) \in \mathfrak{L}$ there exists a chain homotopy $D:\ C_*(E,F) \longrightarrow C_{*+1}(E,F),\quad D:\ 1 \simeq 0$, i.e.*

$$dD(\zeta) + D(d\zeta) = \zeta, \quad \zeta \in C_*(E,F).$$

We assume that D commutes with $l : C'_* \subset C_*$, i.e. $D(\zeta) \in C'_{*+1},\quad \zeta \in C'_*$.

2) Let $[\cdots,\cdots]$ denote the set of all homotopy classes of chain functor transformations. A chain functor $\boldsymbol{A}_*$ is $\mathfrak{L}$-colocal *whenever $[\boldsymbol{A}_*,\ \boldsymbol{B}_*] = 0$ for any $\mathfrak{L}$-acyclic chain functor $\boldsymbol{B}_*$.*

3) A transformation $\omega : \boldsymbol{A}_* \longrightarrow \boldsymbol{B}_*$ between chain functors is an $\mathfrak{L}$-isomorphism whenever $H_*(\omega) = \omega_*$ is an isomorphism on the category $\mathfrak{L}$.

1.2. Remark. Observe that we do not require the naturality of the chain homotopy in 1).

The main objective of this paper is the verification of Theorem 4.2, which subsumes:

1.3. Theorem. *To any $\mathfrak{L}$ and any (regular) chain functor $\boldsymbol{A}_*$ there exists a natural (regular) $\mathfrak{L}$-colocal $\boldsymbol{A}_*^{\mathfrak{L}}$ and a $\mathfrak{L}$-isomorphism $\eta:\ \boldsymbol{A}_*^{\mathfrak{L}} \longrightarrow \boldsymbol{A}_*$.*

We call $(\boldsymbol{A}_*^{\mathfrak{L}}, \eta)$ the $\mathfrak{L}$-*colocalization* of $\boldsymbol{A}_*$. The analogy with $\mathfrak{L}$-localizations as treated in [3] can be pursued even further by introducing

$$^{\mathfrak{L}}\boldsymbol{A}_* = \boldsymbol{A}_* \cup_\eta \operatorname{cone} \boldsymbol{A}_*^{\mathfrak{L}}$$

which is $\mathfrak{L}$-acyclic: Suppose $z = c + c_1 \in A_*(E) \cup_\eta \operatorname{cone} A_*^{\mathfrak{L}}(E)$, $E \in \mathfrak{L}$, is a cycle, $c \in A_*(E))$, $c_1 \in \operatorname{cone} A_*^{\mathfrak{L}}(E)$. We calculate: $dc_1 = -dc \in (\operatorname{cone} A_*^{\mathfrak{L}}) \cap A_*$, hence $dc_1 \in \eta(A_*^{\mathfrak{L}}(E))$. Because η is an $\mathfrak{L}$- isomorphism, $dc_1 = dc_2$, $c_2 \in \eta(A_*^{\mathfrak{L}}(E))$. We deduce that $z \sim z_1 = c + c_2 \in A_*(E) \cup_\eta A_*^{\mathfrak{L}}(E)$ and therefore $z_1 \sim z_2 \in A_*^{\mathfrak{L}}(E)$. So $z \sim 0 \in {}^{\mathfrak{L}}A_*(E)$. Since all chain complexes involved are free, the acyclicity of ${}^{\mathfrak{L}}\boldsymbol{A}_*$ follows.

We have:

1.4. Corollary. *To any $\mathfrak{L}$ and any chain functor $\boldsymbol{A}_*$ there exists a natural (in the same sense as in theorem 1.3.) exact colocalization sequence*

$$\boldsymbol{A}_*^{\mathfrak{L}} \xrightarrow{\eta} \boldsymbol{A}_* \longrightarrow {}^{\mathfrak{L}}\boldsymbol{A}_*$$

with $\mathfrak{L}$-colocal $\boldsymbol{A}_*^{\mathfrak{L}}$, $\mathfrak{L}$-acyclic ${}^{\mathfrak{L}}\boldsymbol{A}_*$ which are regular, whenever $\boldsymbol{A}_*$ is, and $\mathfrak{L}$-isomorphism η.

Proof. Only naturality has not yet been proven. However the construction of a ${}^{\mathfrak{L}}\gamma:\ {}^{\mathfrak{L}}\boldsymbol{A}_* \longrightarrow {}^{\mathfrak{L}}\boldsymbol{B}_*$ associated with a $\gamma:\ \boldsymbol{A}_* \longrightarrow \boldsymbol{B}_*$, fitting into a homology commutative diagram is immediate. ∎

1.5. Proposition. *A chain functor $\boldsymbol{A}_*$ is $\mathfrak{L}$-colocal whenever the following condition is satisfied:*

2′) Any $\mathfrak{L}$-isomorphism $\gamma:\ \boldsymbol{B}_* \longrightarrow \boldsymbol{C}_*$ induces an isomorphism

$$[\boldsymbol{A}_*,\, \gamma]:\ [\boldsymbol{A}_*,\, \boldsymbol{B}_*] \xrightarrow{\approx} [\boldsymbol{A}_*,\, \boldsymbol{C}_*].$$

Proof. Define

$$\boldsymbol{K}_* = \boldsymbol{C}_* \cup_\gamma \operatorname{cone} \boldsymbol{B}_*,$$

then we have an exact sequence of chain functors

$$\Sigma^{-1}\boldsymbol{K}_* \longrightarrow \boldsymbol{B}_* \xrightarrow{\gamma} \boldsymbol{C}_* \longrightarrow \boldsymbol{K}_*$$

where $\Sigma^{-1}\boldsymbol{K}_*$ is the (formal) desuspension which is defined as for chain complexes. Now $\boldsymbol{K}_*$ turns out to be $\mathfrak{L}$-acyclic and $[\boldsymbol{A}_*,\, \boldsymbol{K}_*] = 0$ is equivalent to the statement that $[\boldsymbol{A}_*,\, \gamma]:$ $[\boldsymbol{A}_*, \boldsymbol{B}_*] \longrightarrow [\boldsymbol{A}_*, \boldsymbol{C}_*]$ is an isomorphism. ∎

1.6. Proposition. *Let ${}^1\boldsymbol{A}_* \xrightarrow{{}^1\eta} \boldsymbol{A}_*$, ${}^2\boldsymbol{A}_* \xrightarrow{{}^2\eta} \boldsymbol{A}_*$ be two $\mathfrak{L}$-colocalizations of $\boldsymbol{A}_*$, then there exists a homotopy equivalence $\gamma:\ {}^1\boldsymbol{A}_* \longrightarrow {}^2\boldsymbol{A}_*$, commuting with ${}^i\eta$, $i = 1, 2$.*

In view of 1.5, the proof is immediate. ∎

1.7. Remarks.

1) The formal suspensions and desuspensions used in the proof of 1.5 are not related to any excision properties of the individual (regular or irregular) chain functors involved.

2) If $\boldsymbol{A}_*$ is a chain functor which allows a realization $\mid \boldsymbol{A}_* \mid$ as a Boardman spectrum or alternatively $\boldsymbol{A}_*$ is the chain functor associated with a spectrum, then it is not necessarily true that $\boldsymbol{A}_*^{\mathfrak{L}}$ can be realized as a spectrum $\mid \boldsymbol{A}_*^{\mathfrak{L}} \mid$ (cf. [2] for further reference). The reason is that $\boldsymbol{A}_*^{\mathfrak{L}}$, while enjoying all other properties of a realizable chain functor, does not necessarily have compact support. In the next section we will encounter an explicit example even for the case that $\mathfrak{L}$ consists of a single object. So $\mathfrak{L}$-colocalizations do not always exist for (and as) *spectra* while they are always available as chain functors.

3) If $\boldsymbol{A}_*$ has compact carriers, then $\boldsymbol{A}_* \mid \mathfrak{L}$ in general does not (because $\mathfrak{L}$ may not contain any compact subsets at all). This is also the reason that, in contrast to $\mathfrak{L}$-localizations in [3] (cf. in particular the proof of 4.1.), we cannot change $\boldsymbol{A}_*^{\mathfrak{L}}$ by simply taking compact carriers. The resulting chain functor would no longer be a $\mathfrak{L}$-colocalization of $\boldsymbol{A}_*$.

2. $\boldsymbol{A}_*^{\mathfrak{L}}$ for $\mathfrak{L} = \{E\}$ a single object

In this case we write $\boldsymbol{A}_*^E$ instead of $\boldsymbol{A}_*^{\mathfrak{L}}$. The purpose of this section is to give an explicit description of E-colocalization in this case. To this end we briefly recall the definition of the (irregular) chain functor $\boldsymbol{\mathbb{Z}}_*$ (cf. [3] §3 or appendix B):

$$\mathbb{Z}_n(X) = \begin{cases} \langle z_X \rangle \cdots dz_X = 0, \; n = 0, \; X \neq \emptyset \\ 0 \ldots \ldots \text{otherwise} \end{cases}$$

$$\mathbb{Z}_n(X, A) = 0, \quad A \neq \emptyset$$

This can easily be endowed with the structure of an irregular chain functor.

In order to describe $\mathbb{Z}_*^E(X)$ explicitly, we refer to the existence proof of an E-colocalization in §4 and in appendix A.

According to the existence proof of an E-colocalization (cf. §4), the generating elements of $\mathbb{Z}_*^E(X)$ are of the form $w[\zeta, f]$, where w denotes a word in the sense of the proof of A.1 and $\zeta = mz_E, \;\; m \in \mathbb{Z}$. Suppose $\boldsymbol{A}_*$ is any chain functor defined on $\mathfrak{K}$, then we establish a new (irregular) chain functor

$$\tilde{A}_n(\;\;) = (A_*(E) \otimes \mathbb{Z}_*^E(\;\;))_n = \bigoplus_{p+q=n} A_p(E) \otimes \mathbb{Z}_q^E(\;\;),$$

with the usual boundary operator. We set

$$\tilde{A}'_*(\;\;) = A_*(E) \otimes (\mathbb{Z}_*^E)'(\;\;)$$

and $\varphi, \; \kappa$ for $\tilde{A}_*$ are the tensor products of the identity of $A_*(E)$ with the corresponding $\varphi, \; \kappa$ for $\mathbb{Z}_*^E$. Moreover $l : \; \tilde{A}'_*(\;\;) \subset \tilde{A}_*(\;\;)$ is the tensor product $1 \otimes l$, correspondingly for $i' = \kappa i$. This furnishes an irregular chain functor (cf. appendix B or [3] §3).

2.1. CLAIM. *For any $\zeta \in A_*(E)$ we have a chain homotopy $\zeta \simeq l\,\kappa(\zeta)$.*

PROOF. The relations $\varphi\kappa \simeq 1$, $j_\#\varphi \simeq l$ imply $l\kappa \simeq j_\# : A_*(E) \longrightarrow A_*(E,\emptyset) = A_*(E)$. Let $\zeta \otimes mw[z_E, f] \in \tilde{A}_*(X,Y)$, $f : E \longrightarrow (X,Y)$, w a word as in the proof of A.1, be given, then we find

$$\tilde{\eta} : \tilde{A}_*(X,Y) \longrightarrow A_*(X,Y)$$

by setting

$$\tilde{\eta}(\zeta \otimes mw[z_E, f]) = mwf_\#(\zeta), \qquad m \in \mathbb{Z}$$

whenever this is defined and $mw\kappa(\zeta)$, when $w(\zeta)$ is not defined, which happens whenever $\zeta \notin A'_*(E)$, $w\,[z_E, 1] \in (\mathbb{Z}^E_*)'(X,Y)$ and w is of the form $w = w_1\,\varphi f$, with continuous f. This furnishes a mapping of irregular chain functors. ■

2.2. PROPOSITION.

1) $\tilde{\eta}$ is an E-isomorphism;

2) let $\boldsymbol{B}_*$ be E-acyclic, then for any $\gamma : \tilde{\boldsymbol{A}}_* \longrightarrow \boldsymbol{B}_*$ one has

$$\gamma \simeq 0.$$

PROOF. .

1) Let $\zeta \otimes mw[z_E, f] \in \tilde{A}_*(E)$ be an element, we can assume that $f = 1_E$, moreover w is a word $w : E \longrightarrow E$. By inspecting all possible words of this kind in A.1, we deduce that

2.3. CLAIM. [(**)] *The assignment*

$$\zeta \otimes mw[z_E, 1] \mapsto w\zeta \otimes m[z_E, 1]$$

induces an E-isomorphism.

PROOF OF (**). The inverse (up to homology) to

$$\tilde{\eta} : \tilde{A}_*(E) \longrightarrow A_*(E)$$

is

$$\bar{\eta}(\zeta) = \zeta \otimes [z_E, 1].$$

We calculate:

$$\bar{\eta}\tilde{\eta}(\zeta \otimes mw[z_E, 1]) = mw\zeta \otimes [z_E, 1]$$

$$\tilde{\eta}\bar{\eta}(\zeta) = \tilde{\eta}(\zeta \otimes [z_E, 1]) = \zeta.$$

Due to (**) this implies that $\tilde{\eta}$ is an E-isomorphism. ■

2) Let $z \in Z_n(\tilde{A}_*(X,Y))$ be a cycle, then there exists because of proposition A.1 B) a cycle $z \sim z' = \sum_{i=1}^{m} \zeta_i \otimes [z_E, f_i]$. We collect all elements with the same f_i, obtaining $z' = \sum f_{i\#} z_i$, observing that all z_i are cycles themselves. Therefore $\gamma z_i = dx_i$ is bounding. Hence $\gamma f_{i\#} z_i = f_{i\#} \gamma z_i = d f_{i\#} x_i$ and consequently

$$\gamma z' = d \sum f_{i\#} x_i$$

is bounding. As a result we confirm that any γz is bounding which completes the proof of 2).

■

2.4. Theorem. *There is a homotopy equivalence of (irregular) chain functors*

$$\gamma : A_*(E) \otimes \mathbb{Z}_*^E(\ \ ) \simeq \boldsymbol{A}_*^E(\ \ ),$$

which is compatible with $\tilde{\eta}, \eta$.

Proof. This follows from 1.6 and 2.2. ■

2.5. Remark. $A_*(E) \otimes \mathbb{Z}_*^E$ does not have compact carriers unless E is compact, because a continuous $f : E \longrightarrow X$ does not necessarily factor through some $f_1 : E \longrightarrow K \subset X$, K compact. As a result $[z_E, 1_E] \in \mathbb{Z}_*^E(E)$ does not have a counterimage in some $\mathbb{Z}_*^E(K)$, $K \subset E$, K compact. This implies that $\mathbb{Z}_*^E(\ )$ and consequently that $A_*(E) \otimes \mathbb{Z}_*^E(\ )$ does not necessarily have compact carriers.

3. $[E,\ \ ]_*$-colocalization:

Among the homology theories defined on the Boardman category $\mathfrak{B}$ (cf. [1]), respectively on a category of pairs of CW-spaces $\mathfrak{K}$, we encounter those of the type $[E,\ \ ]_*$ $(E \in \mathfrak{B})$ respectively

$$\{E,\ \ \}_* = \operatorname*{colim}_k [\Sigma^{*+k} E, \Sigma^k(\ \ )], \quad E \in \mathfrak{K}.$$

They satisfy all conditions of a generalized homology theory with the exception of the compact carrier condition. According to a theorem of A. Neeman ([7], theorem 2.1.) a homology theory $h_*(\ )$ is isomorphic to a homology theory $\{E,\ \ \}_*$, for some $E \in \mathfrak{B}$ if and only if h_* preserves products, i. e. whenever

$$h_*(\prod_{\iota \in J} X_\iota) \approx \prod_{\iota \in J} h_*(X_\iota).$$

If E is an object which admits an S-dual (e.g. a finite CW-spectrum), we have

$$\{E,\ \ \}_* = DE_*(\ \ ).$$

So, by an abuse of notation, we call such a homology theory DE_* also, even when the object DE is not defined within the original category.

The natural transformations $\gamma : \ DE_* \longrightarrow DE'_*$ are in one-to-one correspondence with stable homotopy classes of mappings $\hat{\gamma} : \ E' \longrightarrow E$, while the transformations $\gamma : \ [E, \ \ ]_* \longrightarrow F_*$ are in one-to-one correspondence with stable mappings $\hat{\gamma} : \ S^\bullet \longrightarrow E \wedge F$. In the same way we associate $\gamma : \ F_* \longrightarrow [E, \ \ ]_*$ with $\hat{\gamma} : \ F \wedge E \longrightarrow S^\bullet$. If DE is not defined as an object in the original category, we employ this strategy as a definition of new morphisms, so that we can enlarge the original categories $\mathfrak{B}$ or $\mathfrak{K}$ by these new objects and morphisms, denoting the new category by $\overline{\mathfrak{B}}$ respectively $\overline{\mathfrak{K}}$. The $[E, \ \ ]_*$-colocalization is nothing but the DE-colocalization in $\overline{\mathfrak{K}}$.

Let A be a spectrum, $\boldsymbol{A}_*$ the associated chain functor with $A \simeq | \boldsymbol{A}_* |$, then we define

$$A_*(DE) = (DE)_*(A).$$

Using the terminology of §4, the elements of $\tilde{A}_*(X, Y)$ are classes of pairs $[\zeta, \bar{f}]$, $\zeta \in A_*(DE)$, $\bar{f} : \ DE \longrightarrow X \cup CY$. By the previous identification, we discover the associated $f : \ S^\bullet \longrightarrow (X \cup CY) \wedge E$ which factors over a finite subcomplex $K \subset (X \cup CY) \wedge E$, whose cells are $\wedge$-products of finitely many cells $\sigma \wedge \varepsilon$, σ in $X \cup CY$, ε in E. As a result, f factors over $X' \wedge E$, $X' \subset X \cup CY$, a finite, hence compact, subcomplex. This confirms that $\tilde{A}_*$ admits compact carriers.

According to the proof of 4.1 all generating elements of $A_*^{DE}(X, Y)$ are of the form $w[\zeta, \bar{f}]$, with w being a word as in the proof of A.1. Hence $\boldsymbol{A}_*^{DE}$ has compact carriers. All other properties of a chain functor are readily verified as in the appendix. As a result we have:

3.1. Theorem. *Let A be a Boardman spectrum, then the chain functor $\boldsymbol{A}_*^{DE}$ can be realized as a spectrum $| \boldsymbol{A}_*^{DE} | = \boldsymbol{A}^{[E, \ \ ]_*}$.*

3.2. Remark. In [6] A. K. Bousfield introduced a $[E, \ \ ]_*$-colocalization which agrees with the present one where defined.

4. Proof of the colocalization theorem:

Suppose $\mathfrak{L} \subset \mathfrak{K}$ is a subcategory of a category of pairs of topological spaces or of pairs of spectra in the Boardman category. We need the following:

4.1. Theorem. *For each chain functor $\boldsymbol{L}_* : \mathfrak{K} \longrightarrow$ **ch**, there exists a chain functor $\bar{\boldsymbol{L}}_*$ on $\mathfrak{K}$ and a transformation of chain functors $\tau : \ \boldsymbol{L}_* \mid \mathfrak{L} \longrightarrow \bar{\boldsymbol{L}}_* \mid \mathfrak{L}$ (defined on $\mathfrak{L}$) such that for any chain functor $\boldsymbol{K}_*$ on $\mathfrak{K}$ and any transformation $\alpha : \ \boldsymbol{L}_* \mid \mathfrak{L} \longrightarrow \boldsymbol{K}_* \mid \mathfrak{L}$, there exists a $\bar{\alpha} : \ \bar{\boldsymbol{L}}_* \longrightarrow \boldsymbol{K}_*$ satisfying $(\bar{\alpha} \mid \mathfrak{L})\tau = \alpha$. Regularity of $\boldsymbol{L}_*$ implies that of $\bar{\boldsymbol{L}}_*$.*

Proof. In a first step we neglect the specific properties of a chain functor, dealing merely with functors L_*, K_* going into the category of free chain complexes, equipped with subfunctors L'_*, K'_* and natural inclusions $l : \ L'_* \subset L_*$, $i' : \ L_*(A) \subset L'_*(X, A)$ respectively for K_*. Having extended $L_* \mid \mathfrak{L}$ over $\mathfrak{K}$ to a $\tilde{L}_*$, we are able to apply A.1.

I) Let (ζ_i, f_i) be pairs, $\zeta_i \in L_*(E_i, F_i)$, $f_i:\ (E_i, F_i) \longrightarrow (X, A)$ continuous, then we establish an equivalence relation

$$(\zeta_1, f_1) \sim (\zeta_2, f_2)$$

which is generated by the relationship

$$(\zeta_1, f_1 g) \sim (g_\# \zeta_1, f_1), \quad g \in \mathfrak{L}.$$

Let $\tilde{L}_*(X, A)$ $(\tilde{L}'_*(X, A))$ be the free abelian group generated by equivalence classes $[\zeta, f]$ $(\zeta \in L'_*(E, F))$ of such pairs together with the following relations:

$$-[\zeta, f] = [-\zeta, f]$$

$$[\zeta_1, f] + [\zeta_2, f] = [\zeta_1 + \zeta_2, f].$$

The resulting groups are still free abelian.

Suppose $h:\ (X, A) \longrightarrow (Y, B) \in \mathfrak{K}$, then we set

$$h_\#[\zeta, f] = [\zeta,\ hf] \tag{1}$$

$$d[\zeta, f] = [d\zeta,\ f]. \tag{2}$$

This furnishes a functor $\tilde{L}_*:\ \mathfrak{K} \longrightarrow \mathbf{ch}$ $(\tilde{L}'_*:\ \mathfrak{K} \longrightarrow \mathbf{ch})$ together with inclusions $l,\ i'$, defined in the following way:

The inclusion $l[\zeta, f] = [l(\zeta), f]$ is obvious. Suppose $[\zeta, f] \in \tilde{L}_n(F)$, $f: F \longrightarrow A$, then f determines a mapping $\tilde{f}$ as the composition

$$(F, F) \xrightarrow{(f,f)} (A, A) \subset (X, A)$$

allowing us to set $i'[\zeta, f] = [i'(\zeta), \tilde{f}]$. Since i' for $\boldsymbol{L}_*$ is by definition natural (in (X, A)) this equips $\tilde{L}_*$ with the required i'.

Suppose $\zeta \in L_n(X, A)$, then there exists by definition a natural chain homotopy $\chi(\zeta)$, commuting with l and i', satisfying

$$d\chi(\zeta) + \chi(d\zeta) = i_{0\#}(\zeta) - i_{1\#}(\zeta).$$

By setting

$$[\zeta, f] = [\chi(\zeta), f \times 1],$$

we find a chain homotopy for $\tilde{L}_*$

$$d\chi[\zeta, f] = [\chi(\zeta), f] + \chi[d\zeta, f] = i_{0\#}[\zeta, f] - i_{1\#}[\zeta, f].$$

We have a transformation

$$\begin{aligned} \tau :\ L_* \mid \mathfrak{L} &\longrightarrow \tilde{L}_* \mid \mathfrak{L} \\ \zeta \qquad &\longmapsto [\zeta,\ 1_{E,F)}], \quad \zeta \in L_*(E,F). \end{aligned}$$

Suppose α is given, then

$$\begin{aligned} \tilde{\alpha} :\ \tilde{L}_* &\longrightarrow K_* \\ [\zeta, f] &\longmapsto K_*(f)(\alpha(\zeta)) \end{aligned}$$

is defined on $\mathfrak{K}$ and satisfies

$$(\tilde{\alpha} \mid \mathfrak{L})\tau = \alpha.$$

Both τ and $\tilde{\alpha}$ commute with the natural chain homotopies inherent in $\tilde{\boldsymbol{L}}_*$, $\boldsymbol{K}_*$ and $\boldsymbol{L}_*$. Suppose $[\zeta, f] \in \tilde{L}_*(X,X)$, $f :\ (E,F) \longrightarrow (X,X)$, then we find $f' :\ (E,E) \longrightarrow (X,X)$ and a $\xi \in L_{*+1}(E,E)$ such that $d\xi = g_{\#}\zeta$, $g :\ (E,F) \subset [E,E)$. As a result

$$d[\xi, f'] = [\zeta, f].$$

This ensures that $\tilde{L}_*(X,X)$ is always acyclic. All other properties of a *partial chain functor* (cf. appendix A) are immediately verified.

We still have to ensure that for the derived homology of $\tilde{L}_*$ an excision axiom holds: Let (ζ, f), $f : (E,F) \longrightarrow (X,A)$, $\zeta \in Z_*(L_*(E,F))$ be a pair and suppose $\bar{U} \subset Int_X A$, then we define $V = f^{-1}(U)$ and observe that $\bar{V} \subset Int_E F$. By restricting f we obtain a map $f' : (E \setminus V, F \setminus V)) \longrightarrow (X \setminus U, A \setminus U)$. Let $i : (X \setminus U, A \setminus U) \subset (X,A)$, $i' : (E \setminus V, F \setminus V)) \subset (E,F)$ be the inclusions, then there exists a $\zeta' \in Z_*(L_*(E \setminus V, F \setminus V))$ such that $i'_{\#}\zeta' \sim \zeta$. Hence

$$i_{\#}([\zeta', f']) \sim [\zeta, f].$$

So $i_{\#}$ is epic.

Let $[\zeta', f']$ be a given cycle and suppose $d[\xi, f] = [d\xi, f] = i_{\#}[\zeta', f']$, then excision for $\boldsymbol{L}_*$ enables us to find $[\xi', f']$ satisfying $d[\xi', f'] = [\zeta', f']$, so that $i_{\#}$ is monic.

II) This provides us with a $\tilde{L}_* :\ \mathfrak{K} \longrightarrow \mathbf{ch}$ $(\tilde{L}'_* :\ \mathfrak{K} \longrightarrow \mathbf{ch})$ satisfying all prerequisites of A.1. We obtain a chain functor $\bar{\boldsymbol{L}}_* = \bar{\tilde{\boldsymbol{L}}}_*$ by applying A.1.

There is an inclusion $L_* \subset \bar{L}_*$ commuting with the operators φ, κ, δ, δ_i etc., and therefore a transformation of chain functors $\tau :\ \boldsymbol{L}_* \mid \mathfrak{L} \longrightarrow \bar{\boldsymbol{L}}_* \mid \mathfrak{L}$. Let α be a transformation as in the theorem. Assume that $\bar{\zeta} \in \bar{L}_*$ is any element, then either $\bar{\zeta} \in \tilde{L}_*$ allowing us to set $\bar{\alpha}(\bar{\zeta}) = \tilde{\alpha}(\bar{\zeta})$ or $\bar{\zeta} \notin \tilde{L}_*$. In this case $\bar{\zeta} = \Sigma w_i \tilde{\zeta}_i$, $\tilde{\zeta}_i \in \tilde{L}_*$ in a unique way, where w_i is a word in the sense of the proof of A.1, so that we are entitled to set

$$\bar{\alpha}(\bar{\zeta}) = \Sigma w_i \tilde{\alpha}(\tilde{\zeta}_i).$$

This furnishes a transformation $\bar{\alpha} :\ \bar{\boldsymbol{L}}_* \longrightarrow \boldsymbol{K}_*$ satisfying $(\bar{\alpha} \mid \mathfrak{L})\tau = \alpha$.

This completes the proof of the theorem for regular $\boldsymbol{L}_*$. The changes in the irregular case are, in view of A.5, immediate. ■

4.2. Theorem. *Let $\boldsymbol{A}_*$ be any chain functor, $\mathfrak{L}$ as before, then there exists a natural (in $\boldsymbol{A}_*$) $\mathfrak{L}$-colocalization*

$$\eta : \boldsymbol{A}_*^{\mathfrak{L}} \longrightarrow \boldsymbol{A}_*$$

(cf. definition 1.1.2)). More precisely we have:

1) η is an $\mathfrak{L}$-isomorphism;

2) if $\boldsymbol{B}_*$ is $\mathfrak{L}$ acyclic, then $[\boldsymbol{A}_*^{\mathfrak{L}}, \boldsymbol{B}_*] = 0$;

3) Let $\gamma : \boldsymbol{A}_* \longrightarrow \boldsymbol{B}_*$ be a transformation of chain functors then there exists in a canonical way a $\gamma^{\mathfrak{L}} : \boldsymbol{A}_*^{\mathfrak{L}} \longrightarrow \boldsymbol{B}_*^{\mathfrak{L}}$ such that the diagram

$$\begin{tikzcd} \boldsymbol{A}_*^{\mathfrak{L}} \arrow[r, "\eta"] \arrow[d, "\gamma^{\mathfrak{L}}"'] & \boldsymbol{A}_* \arrow[d, "\gamma"] \\ \boldsymbol{B}_*^{\mathfrak{L}} \arrow[r, "\eta"'] & \boldsymbol{B}_* \end{tikzcd} \tag{3}$$

is commutative up to homology. If $\boldsymbol{A}_*$ is regular, then $\boldsymbol{A}_*^{\mathfrak{L}}$ is also regular.

Proof. Let $\boldsymbol{L}_* = \boldsymbol{A}_*$, $\alpha : \boldsymbol{A}_* \mid \mathfrak{L} \longrightarrow \boldsymbol{A}_* \mid \mathfrak{L}$ be the identity, then we have $\bar{\alpha} : \bar{A}_* \longrightarrow \boldsymbol{A}_*$ such that $(\bar{\alpha} \mid \mathfrak{L}_*)$ is an isomorphism. We set

$$\boldsymbol{A}_*^{\mathfrak{L}} = \bar{\boldsymbol{A}}_*$$

and

$$\bar{\alpha} = \eta.$$

Suppose $\boldsymbol{B}_*$ is $\mathfrak{L}$-acyclic, $z \in \bar{A}_*(X, Y)$ a cycle, then A.1 B) implies that $z \sim z' \in Z_*(\tilde{A}_*(X, Y))$. Now $z' = \sum[\zeta_i,\ f_i]$ and, because of (2), we can assume that there is a representation of z' as a sum with all $[\zeta_i,\ f_i]$ being cycles. Let $\omega : \boldsymbol{A}_*^{\mathfrak{L}} \longrightarrow \boldsymbol{B}_*$ be a mapping. We deduce, because $\boldsymbol{B}_*$ is $\mathfrak{L}$-acyclic, that there exists a ξ_i in $\boldsymbol{B}_*$ such that $d\xi_i = \omega[\zeta_i, 1]$, hence $d \sum f_{i\#}\xi_i = \sum f_{i\#}\omega[\zeta_i, 1] = \omega \sum[\zeta_i,\ f_i] = \omega(z')$. Thus $\omega(z)$ is bounding. Since $\boldsymbol{B}_*$ is a free chain complex, this implies $\omega \simeq 0$.

Concerning naturality, we consider $\gamma : \boldsymbol{A}_* \longrightarrow \boldsymbol{B}_*$ and construct in a functorial way $\gamma^{\mathfrak{L}} : \boldsymbol{A}_*^{\mathfrak{L}} \longrightarrow \boldsymbol{B}_*^{\mathfrak{L}}$. First, we set

$$\gamma^{\mathfrak{L}}[\zeta, f] = [\gamma\, \zeta, f].$$

For free generators, we define

$$\gamma^{\mathfrak{L}} w[\zeta, f] = w[\gamma\, \zeta, f].$$

This furnishes a transformation $\gamma^{\mathfrak{L}} : \boldsymbol{A}_*^{\mathfrak{L}} \longrightarrow \boldsymbol{B}_*^{\mathfrak{L}}$ in a canonical way. In order to investigate the homology commutativity of (3), let $z \in Z_n(A_*^{\mathfrak{L}}(X, Y))$ be a cycle, then there exists a cycle $z \sim z' = \sum[\zeta_i, f_i]$. Since

$$\gamma\, \eta(z') = \sum f_{i\#}\gamma\, \zeta_i = \eta\, \gamma^{\mathfrak{L}}(z'),$$

the assertion follows. ∎

A. Appendix: Auxiliary material for the proof of the main theorem

This section is devoted to some material on chain functors which is needed in §4 to establish Theorem 4.1 and especially Theorem 4.2, which is the existence theorem for colocalizations. For further background, e.g. concerning explicit definitions of *chain functors*, regularity, etc., we refer to [2] for further references or to the next section (appendix B), where, for the convenience of the reader, the definitions are recorded.

Let $C_* : \mathfrak{K} \longrightarrow \mathbf{ch}$ be a functor into the category of free chain complexes, $l : C'_* \subset C_*$ a subfunctor, $i' : C_*(A) \subset C'_*(X,A)$ a natural inclusion (all going into direct summands, cf. [2] for further references), then $\boldsymbol{C}_*$ need not be a chain functor: There may be no $\varphi : C'_*(X,A) \longrightarrow C_*(X)$, $\kappa : C_*(X) \longrightarrow C'_*(X,A)$ and no chain homotopies $j_\# \varphi \simeq l$, $\varphi \kappa \simeq 1$ together with all the other relations which are required for a chain functor and which are eventually not available.

We list now all these properties and conditions, by expressing them as specific operations on $\boldsymbol{C}_*$:

1) There are possibly non-natural chain mappings φ, κ.

2) There are (non-natural) homomorphisms

$$\delta_1 : C_*(X) \longrightarrow C_{*+1}(X)$$

$$\delta_2 : C'_*(X,A) \longrightarrow C'_{*+1}(X,A)$$

satisfying

$$\left\{ \begin{array}{lcl} d\delta_1(\zeta) + \delta_1(d\zeta) & = & \varphi\kappa(\zeta) - \zeta \\ d\delta_2(\zeta) + \delta_2(d\zeta) & = & j_\# \varphi(\zeta) - l(\zeta). \end{array} \right. \tag{1}$$

The existence of δ_i takes care of the above mentioned chain homotopies. The existence of δ_i (as well as that of δ) follow for a chain functor, because $C_*(\ )$ is assumed to be free.

3) There is a homomorphism

$$\eta : Z_*(C_*(X,A)) \longrightarrow C'_*(X,A) \oplus C_{*+1}(X,A) \oplus C_{*-1}(A)$$

$$\eta(\zeta) = (\eta_1(\zeta),\ \eta_2(\zeta),\ \eta_3(\zeta))$$

satisfying

$$\zeta + d\,\eta_2(\zeta) = l\,\eta_1(\zeta) + q_\#\delta s_\#\eta_4(\zeta),$$

$$q : (A,A) \subset (X,A), \quad s : A \subset (A,A)$$

where δ stems from 1). This expresses the fact that every cycle $z \in C_*(X,A)$ is homologous to a cycle of the form

$$l\,z' + q_\# a, \quad z' \in C'_*(X,A), \quad a \in C_*(A,A)$$

$dz' \in \operatorname{im} i'$, $dz' = i'\,da$.

4) There exists a homomorphism

$$\beta : \ B_*(X,A) \longrightarrow C_*(A)$$

$B_*(\ \ ) =$ bounding cycles, satisfying

$$i' \ \beta(\zeta) = d\eta_1 \ (\zeta);$$

5)

$$\lambda : \ j_\#^{-1} \ B_*(X,A) \longrightarrow C'_*(X,A) \oplus C_*(A)$$

$$\lambda(\zeta) = (\lambda_1(\zeta), \lambda_2(\zeta)),$$

satisfying

$$\kappa \ \zeta = d\lambda_1(\zeta) + i' \ \lambda_2(\zeta).$$

Here 4), respectively 5), are translations of

$$ker \ \psi \subset ker \ \bar{\partial}$$

and

$$ker \ j_* \subset ker \ p_* \ \kappa_*,$$

$p : \ C'_*(X,A) \longrightarrow C'''_*(X,A) = C'(X,A)/\text{im } i'$, j_*, p_* denoting the induced mappings for homology.

There are no relations between all these operations and continuous mappings except those already mentioned and the fact that $i' = \kappa \ i$ is natural (i.e. commutes with continuous mappings).

By an abuse of notation, we write e.g. $\delta : \ (X,X) \longrightarrow (X,X)$, $\eta_2 : \ (X,A) \longrightarrow (X,A)$ and do not distinguish between these symbols and the original homomorphisms. This allows us to combine them with continuous maps as if they constitute new morphisms which induce the original homomorphisms.

We now form words w by using these operators or continuous mappings, where we assume of course that the compositions (like $\varphi\kappa$, $\kappa\delta_1$, but *not necessarily* $\varphi\delta_1$) make sense.

Suppose that $(C_*, \ C'_*, \ l, \ i')$ is in the following sense *partially* a (free) chain functor (or simply: a *partial-chain-functor*):

p1) $C_*(X,X)$ is always acyclic; all inclusions j induce monomorphisms $C_*(j)$ onto direct summands, the same holds for i' and l; the homology groups of C_* satisfy an excision property; there are natural and chain homotopies D, commuting with i' and l, satisfying, for each $\zeta \in L_*(X,A)$,

$$dD(\zeta) + D(d\zeta) = i_{0\#}(\zeta) - i_{1\#}(\zeta);$$

p2) if $\zeta \in C_*$ and v is any of the operators 1) - 6) (i.e. $\varphi,\ \kappa,\ \delta_i,\ \eta_i,\ \beta,\ \lambda,$), then either none or all those $v\zeta$ which formally make sense, are defined;

p3) if $\kappa\zeta$ is defined, then $\varphi\,\kappa\,\zeta$ is also defined;

p4) let $\zeta = \sum a_i\,\zeta_i$, $a_i \neq 0$, ζ_i a base element (with respect to a basis of C_*) be given and $v\zeta$ is defined, then all $v\zeta_i$ are defined.

A.1. Proposition. *There exists a chain functor* $\bar{\boldsymbol{C}}_* = \{\bar{C}_*,\ \bar{C}'_*\ , i',\ l,\ \varphi,\ \kappa\}$, $C_* \subset \bar{C}_*$, $C'_* \subset \bar{C}'_*$ *such that*

A) all operations 2) - 7) of $\bar{C}_*$ *agree with those of* C_* *whenever both are defined.*

B) The inclusion $C_*(X,A) \subset \bar{C}_*(X,A)$ *induces an isomorphism of homology groups.*

Proof. Let ${}^1C_* \supset C_*$ be a partial chain functor satisfying p1) - p4) and in addition the following condition:

p5) ${}^1\zeta \in {}^1C_* \Longrightarrow {}^1\zeta = \sum w_i\zeta_i$, where w_i denotes an operation v as in p2) or a word of the form $\varphi\ \kappa$ and $\zeta_i \in C_*$. Such an extension is called *strongly admissible* and denoted by ${}^1C_* \geq C_*$. More generally we have for $C_* \leq {}^1C_* \subset {}^2C_*$ the possibility to detect ${}^2C_*$ as a *strongly admissible extension* ${}^1C_* \leq {}^2C_*$ by the same procedure. We observe:

A.2. Claim. [(*)] *If* ${}^1C_* \geq {}^2C_*$ *is a strongly admissible extension, then the inclusion induces an isomorphism of homology groups:*

$$H_*({}^1C_*(X,A)) \approx H_*(({}^2C_*(X,A)).$$

Proof. The conditions p1) - p5) guarantee that the extension of ${}^2C_*$ to ${}^1C_*$ neither produces new cycles nor does it convert non-bounding into bounding cycles.

A chain of strongly admissible extensions

$${}^1C_* \leq \cdots \leq {}^2C_*$$

is called an *admissible extension*. Let $\mathfrak{S}$ be the set of all admissible extensions with the $\leq$ relation as partially ordering.

A.3. Claim. [(**)] $\mathfrak{S}$ *is inductive.*

Proof. Let $\mathfrak{T} \subset \mathfrak{S}$ be a tower, then

$$\bigcup_{\tilde{C}_* \in \mathfrak{T}} \tilde{C}_*$$

is an upper bound for all $\tilde{C}_* \in \mathfrak{T}$.

The proposition is now reduced to the verification of:

A.4. Claim. [(***)] *A maximal* $\bar{C}_*$ *is a chain functor satisfying B).*

Proof. Suppose there exists a $\zeta \in \bar{C}_*$ such that $v\zeta$ for some v as before is not defined. Without loss of generality, we can assume that ζ is a base element. Considering the free abelian group generated by $v\zeta$ for all v as in p2), p3), we achieve an admissible extension of $\bar{C}_*$ contrary to the assumption that $\bar{C}_*$ is maximal. So, with every $\zeta \in \bar{C}_*$ and every letter v, we conclude that $v\zeta \in \bar{C}_*$, guaranteeing that $\bar{C}_*$ is a chain functor. By construction of $\mathfrak{S}$ and in view of (*), the homologies of $\bar{C}_*$ and C_* are isomorphic. ∎

This completes the proof of A.1. ∎

The following corollary is immediate:

A.5. Corollary. *Suppose that* $i' : C_*(A) \longrightarrow C'_*(X, A)$, $l : C'_* \longrightarrow C_*$ *as well as* $j_\# = C_*(j)$ *for inclusions* j *are not necessarily inclusions, then there still exists an irregular* $\mathbf{C}_*$ *satisfying A) and B).*

B. Appendix: Chain functors and associated homology theories

In this appendix, we present without proofs for the convenience of the reader some material about the definition of and the motivation for chain functors. Concerning details we refer to [2].

It would be advantageous to define a homology theory $h_*(\ )$ as the derived homology of a functor

$$C_* : \mathfrak{K} \longrightarrow \boldsymbol{ch},$$

$\mathfrak{K}$ = the category on which h_* is defined (e.g. a subcategory of the category of all pairs of topological spaces, or pairs of spectra or pairs of CW spaces or CW spectra together with the appropriate morphisms), $\boldsymbol{ch}$ = category of chain complexes (i.e. $C_* = \{C_n,\ d_n,\ n \in \mathbb{Z},\ d^2 = 0\} \in \boldsymbol{ch}$). Let $(X, A) \in \mathfrak{K}$ be a pair, then one would like to have an exact sequence

$$0 \longrightarrow C_*(A) \xrightarrow{i_\#} C_*(X) \xrightarrow{j_\#} C_*(X, A) \longrightarrow 0 \qquad (1)$$

such that the associated boundary $\bar{\partial} : H_n(C_*(X, A)) \longrightarrow H_{n-1}(C_*(A))$ corresponds to the boundary $\partial : h_n(X, A) \longrightarrow h_{n-1}(A)$ via the isomorphism $h_*(\ ) \approx H_*(C_*(\ ))$. In accordance with [2] we call a homology with this property *flat.* Due to a result of R. O. Burdick, P. E. Conner and E. E. Floyd (cf. [4] for further reference), this implies for $\mathfrak{K}$ = category of CW pairs that $h_*(\ )$ is a sum of ordinary homology theories, i.e. of those satisfying a dimension axiom, although not necessarily in dimension 0.

We call a functor C_* together with a short exact sequence (1) determining the boundary operator, a *chain theory* for h_*. The non-existence of such a chain theory gives rise to the theory of chain functors.

A chain functor $\boldsymbol{C}_* = \{C_*,\ C'_*,\ l,\ i',\ \kappa,\ \varphi\}$ is

1) a pair of functors $C_*,\ C'_* : \mathfrak{K} \longrightarrow \boldsymbol{ch}$, natural inclusions $i' :\ C_*(A) \subset C'_*(X, A)$, and $l : C'_*(X, A) \subset C_*(X, A,)$

2) possibly non-natural chain mappings

$$\varphi :\ C'_*(X, A) \longrightarrow C_*(X)$$

$$\kappa :\ C_*(X) \longrightarrow C'_*(X, A),$$

chain homotopies $\varphi\kappa \simeq 1$, $\quad j_\#\ \varphi \simeq l \quad (j :\ X \subset (X, A))$, as well as an identity

$$\kappa\ i_\# = i' \qquad i :\ A \subset X.$$

3) All inclusions $k:\ (X,A)\subset(Y,B)$ are supposed to induce monomorphisms. All $C_*(X,X)$ are acyclic.

The exact sequence (1) is replaced by the sequence

$$0 \longrightarrow C_*(A) \xrightarrow{i'} C'_*(X,A) \xrightarrow{p} C'_*(X,A)/\mathrm{im}\ i' \longrightarrow 0 \tag{2}$$

and there exists a homomorphism

$$\psi:\ H_*(C'_*(X,A)/\mathrm{im}\ i') \longrightarrow H_*(C_*(X,A)) \tag{3}$$

$$[z'] \longmapsto [l(z')+q_{\#}\bar{a}]$$

where $z'\in C'_*(X,A),\ dz'\in \mathrm{im}\ i',\ q:\ (A,A)\subset(X,A),\ \bar{a}\in C_*(A,A),\ d\bar{a}=-dz'$. It is assumed that ψ is epic.

Since $C_*(A,A)$ is acyclic, $dz'\in\mathrm{im}\ i'$, such that an $\bar{a}$ exists and $[l(z')+q_{\#}(\bar{a})]$ is independent of the choice of $\bar{a}$. This assumption implies that each cycle $z\in C_*(X,A)$ is homologous to a cycle of the form $l(z')+q_{\#}(\bar{a})$, with z' being a *relative* cycle, the analogue of a classical relative cycle $z\in C_*(X)$ with $dz\in\mathrm{im}\ i_{\#}$, whenever (1) holds, i.e. whenever we are dealing with a chain theory.

4) We assume

$$\mathit{ker}\ \psi\subset \mathit{ker}\ \bar{\partial}, \tag{4}$$

$\bar{\partial}:\ H_n(C'_*(X,A)/\mathrm{im}\ i')\longrightarrow H_{n-1}(C_*(A))$ being the boundary induced by the exact sequence (2). Moreover, we assume

$$\mathit{ker}\ j_*\subset \mathit{ker}\ p_*\ \kappa_*, \tag{5}$$

with e.g. κ_* denoting the mapping induced by κ for the homology groups.

5) Homotopies $H:\ (X,A)\times I\longrightarrow(Y,B)$ induce chain homotopies $D(H):\ C_*(X,A)\longrightarrow C_{*+1}(Y,B)$ naturally and compatible with i' and l.

These are almost all the ingredients of a chain functor we need. The derived (or associated) homology of a chain functor

$$h_*(X,A)=H_*(C_*(X,A))$$

respectively for the induced mappings, is endowed with a boundary operator

$$\partial:\ H_n(C_*(X,A))\longrightarrow H_{n-1}(C_*(A)),$$

determined by $\bar{\partial}$ as follows: We seek to $\zeta\in H_n(C_*(X,A))$ a representative $l(z')+\ q_{\#}(\bar{a})$ and set

$$\partial\zeta:=\bar{\partial}[z']=[i'^{-1}\ d\ z']. \tag{6}$$

This turns out to be independent of the choices involved.

This $h_*(\ )$ satisfies all properties of a homology theory with the exception of an excision axiom. Therefore it is convenient to add:

6) Let $p:\ (X,A) \longrightarrow (X',A')$ be an excision map (of some kind, e.g. $p:\ (X,A) \longrightarrow (X/A,\star)$,) then $p_* = H_*(C_*(p))$ is required to be an isomorphism.

Then it turns out that $H_*(C_*(\ )) = h_*(\ )$ becomes a homology theory. Moreover it turns out that under very general conditions on $\mathfrak{K}$ every homology theory $h_*(\ )$ is isomorphic to the derived homology theory of some chain functor.

Let $\lambda : C_* \longrightarrow L_*,\quad \lambda' : C'_* \longrightarrow L'_*$ be natural transformations, where $\boldsymbol{C}_*,\ \boldsymbol{L}_*$ are chain functors, compatible with i', l and the natural homotopies of 5), then we call $\lambda :\ \boldsymbol{C_*} \longrightarrow \boldsymbol{L_*}$ a *transformation of chain functors.* Such a transformation induces obviously a transformation $\lambda_* :\ H_*(\boldsymbol{C_*}) \longrightarrow H_*(\boldsymbol{L_*})$ of the derived homology (i.e. λ_* commutes also with the boundary ∂ as defined in (6)). This furnishes a category $\mathfrak{Ch}$ of chain functors. A*weak equivalence* in $\mathfrak{Ch}$ is a $\lambda :\ \boldsymbol{C_*} \longrightarrow \boldsymbol{L_*}$ which has a homotopy inverse. Incidentally there are also fibrations and cofibrations, endowing $\mathfrak{Ch}$ with the essential features of a closed model category .

One could require that λ also commutes with φ and κ, however it turns out that up to homotopy this does not matter ([2] proposition 4.5.). Furthermore we can introduce the homotopy category $\mathfrak{Ch}_h$ with chain homotopy classes of transformations of chain functors $\mathfrak{Ch}_h(\cdots,\cdots) = [\cdots,\cdots]$ as morphisms.

In order to establish all this, it becomes sometimes necessary to assume that a chain functor $\boldsymbol{C_*}$ satisfies:

7) All chain complexes $C_*(X,A)$ are free. However this is not a severe restriction as the following lemma ensures:

B.1. Lemma. *To any chain functor $\boldsymbol{C_*}$ there exists a canonically defined chain functor $\boldsymbol{L_*}$ and a transformation of chain functors $\lambda :\ \boldsymbol{L_*} \longrightarrow \boldsymbol{C_*}$ compatible with φ and κ, inducing an isomorphism of homology, such that:*

L1) All $L_*(X,A)$ have natural bases $\boldsymbol{b}$ in all dimensions;

L2) $b \in \boldsymbol{b} \Longrightarrow db \in \boldsymbol{b}$; $b \in \boldsymbol{b} \Longrightarrow i'(b) \in \boldsymbol{b}$, $l(b) \in \boldsymbol{b}$, whenever this is defined and makes sense;

L3) For every homology class $\zeta \in H_*(C_*(X,A))$ there exists a basic $b \in (\lambda_*)^{-1}\zeta$.

Proof. Consider the free abelian group $F(C_n(X,A))$ determined by the elements of $C_*(X,A)$ and convert this into a chain complex $F_*(X,A)$ in an obvious way. To each $a \in C_n(X,A)$ corresponds a basic $\bar{a} \in F(C_n(X,A))$. Let $i :\ M_* \subset F_*$ be the subcomplex generated by all elements of the form $\sum m_i\,\bar{a}_i\ -\ \overline{\sum m_i\,a_i}$ and define

$$L_*(X,A)\ =\ F_*(X,A) \cup_i\ \text{cone}\ M_*(X,A).$$

This furnishes evidently a functor into the category of chain complexes. We set

$$\lambda(\sum m_i\,\bar{a}_i)\ =\ \sum m_i\,a_i,$$

and $\lambda \mid M_*\ =\ 0.$

Moreover $\sum m_i\, \bar{a}_i \in L'_*$ whenever all $a_i \in C'_*$, respectively for the elements of cone M_*. This implies that L2) holds. One can immediately equip $\boldsymbol{L_*}$ and λ with the structure of a chain functor, respectively of a transformation between chain functors.

Every cycle $z \in Z_n(C_*(X,A))$ is of the form $\lambda(\bar{z}) = z$, hence λ_* is epic. Any cycle $\tilde{z} \in Z_n(L_*(X,A))$ is homologous to a $\bar{z}$, for some $z \in Z_n(C_*(X,A))$: Suppose $\tilde{z} = \Sigma m_i \overline{a}_i +$ c where $c \in$ cone M_*, then we have $\tilde{z} = \overline{a} + c_1$ where $c_1 \in$ cone M, hence $d\overline{a} = \overline{da} \in$ cone M_*, implying that $d\overline{a} = \overline{d}a = 0$. So $\overline{a}$ and c_1 are cycles, and since c_1 is bounding in cone M_*, we conclude that $\tilde{z} \sim \overline{a}$. If $z = dx$, then $\bar{z} = d\bar{x}$ and λ_* is therefore monic.

This completes the proof of the lemma. ∎

Finally we repeat the definition of an *irregular chain functor* (cf. [2] definition 4.1.) for more details or [3] §3 for an example): $\{C_*, C'_*, \varphi, \kappa, i', l\}$ satisfies all conditions of a chain functor, but we do no longer require a) that all inclusions induce isomorphisms; b) nor that i', l are necessarily monomorphisms; c) nor any excision properties. Whenever we talk about a *regular* chain functor, we mean that it is not irregular.

The role of the unnatural mappings φ and κ seems at the first glance to be a little mysterious. A chain functor $\boldsymbol{K}_*$ is called *flat* whenever φ, κ and the chain homotopies $\varphi\kappa \simeq 1, \quad j_\# \, \varphi \simeq l$ are natural. In the beginning, we introduced the concept of a flat homology theory.

B.2. Theorem. *The following conditions for a homology theory are equivalent:*

1) h_* is flat;

2) there exists a flat chain functor associated with h_*.

B.3. Corollary. *For a homology theory defined on the category of CW spaces, conditions 1), 2) are equivalent to*

3) h_* is the direct sum of ordinary homology theories.

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