Theory and Applications of Categories, Vol. 13, No. 12, 2004, pp. 172–183.

SPLIT STRUCTURES

To our friend Aurelio Carboni for his 60th birthday

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Abstract. In the early 1990's the authors proved that the full subcategory of 'sup-lattices' determined by the constructively completely distributive (CCD) lattices is equivalent to the idempotent splitting completion of the bicategory of sets and relations. Having many corollaries, this was an extremely useful result. Moreover, as the authors soon suspected, it specializes a much more general result.

Let D be a monad on a category $\mathcal{C}$ in which idempotents split. Write $\mathbf{kar}(\mathcal{C}_D)$ for the idempotent splitting completion of the Kleisli category. Write $\mathbf{spl}(\mathcal{C}^D)$ for the category whose objects are pairs $((L,s),t)$, where (L,s) is an object of $\mathcal{C}^D$, the Eilenberg-Moore category, and $t:(L,s)\longrightarrow(DL,mL)$ is a *homomorphism* that splits $s:(DL,mL)\longrightarrow(L,s)$, with $\mathbf{spl}(\mathcal{C}^D)(((L,s),t),((L',s'),t'))=\mathcal{C}^D((L,s)(L',s'))$.

The main result is that $\mathbf{kar}(\mathcal{C}_D)\cong\mathbf{spl}(C^D)$. We also show how this implies the CCD lattice characterization theorem and consider a more general context.

1. Introduction

1.1 Raney [Ran53] first characterized completely distributive lattices in terms of what has been called Raney's anonymous relation. A variant of this relation became important in the study of continuous lattices and earned a notation, $\ll$, and a name: the way below relation. In [R&W94b] we used $\ll$ for Raney's anonymous relation and, following modern terminology in the theory of categories, called it the totally below relation. Thus in a complete lattice L, $a\ll b$ if and only if, for every downset (or order ideal) S of L, $b\leq\bigvee S$ implies $a\in S$. (The way below relation differs only in that the universally quantified downsets S are required to be also updirected.) Raney showed that, for completely distributive L, $\ll$ is idempotent which is equivalent to saying that $\ll$ is transitive and interpolative, where the latter means that if $a\ll b$ then $(\exists c)(a\ll c\ll b)$. For an arbitrary idempotent binary relation, $<$ on a set X, Raney defined $L(<)$ to be those subsets S of X for which $a\in S$ if and only if $(\exists b)(a<b\in S)$, ordered by inclusion, and showed it to be a completely distributive lattice. In fact, he showed that every completely distributive lattice is isomorphic to one of the form $L(<)$.

The authors gratefully acknowledge financial support from the Canadian NSERC. Diagrams typeset using M. Barr's diagram package, diagxy.tex.

Received by the editors 2004-06-09 and, in revised form, 2004-10-28.

Published on 2004-12-05.

2000 Mathematics Subject Classification: 18C15, 18D20, 06D10.

Key words and phrases: complete distributivity, idempotent splitting completion, K-Z doctrine.

1.2 Without being fully aware of Raney's work, we showed in [R&W94b] that the 2-category of (constructively) completely distributive lattices and supremum-preserving functions $\mathbf{ccd_{sup}}$ is bi-equivalent to the idempotent splitting completion $\mathbf{kar}(\mathbf{rel})$ of the 2-category of sets and relations. We stressed 2-categorical ideas throughout, including the definition of *constructively* completely distributive lattice itself. In a sense, we took Raney's result from dimension 0 to dimension 2 without due consideration of purely 1-dimensional categorical ideas. Carboni, in his inimitable way, repeatedly told us that we were not fully exploiting the mere splitting of idempotents. We hope here to make amends.

1.3 In praise of Raney, and somewhat immodestly with respect to our own contributions, the theorems alluded to above are good ones in that a great deal follows easily from them. For example, **rel** has many nice properties and structures (studied by Carboni and Walters [C&W87]) that are inherited by splitting of idempotents and they transfer to $\mathbf{ccd_{sup}}$ via the equivalence with $\mathbf{kar}(\mathbf{rel})$. Since the real category of interest for completely distributive lattices, **ccd**, has for arrows those functions that preserve both suprema and infima and this is the locally full sub 2-category of $\mathbf{ccd_{sup}}$ determined by those arrows which have right adjoints, $\mathbf{map}(\mathbf{ccd_{sup}})$ as Carboni would call it, we continue to think that 2-categorical ideas are important here. But we digress.

1.4 It has seemed to us for some time that these theorems characterizing completely distributive lattices are not really about lattices. If we regard the 2-category **ord** of ordered sets as Ω-**cat**, where we treat Ω, the category of truth values, as a monoidal category via conjunction, then $\mathbf{ccd_{sup}}$ is the full sub 2-category of the totally cocomplete Ω-categories for which the colimit structure functor has a left adjoint. On the other hand, the objects of $\mathbf{kar}(\mathbf{rel})$ are the Ω-taxons, where for monoidal $\mathcal{V}$, a $\mathcal{V}$-taxon with objects $|X|$ consists of a $\mathcal{V}$-valued matrix $X\colon |X| \twoheadrightarrow |X|$ with a composition $XX \xrightarrow{\cdot} X$ that is a coequalizer of $XXX \overset{X\cdot}{\underset{\cdot X}{\rightrightarrows}} XX$. We refer the reader to Koslowski [Kos97] for details. The arrows of $\mathbf{kar}(\mathbf{rel})$ are the Ω-distributors, where a $\mathcal{V}$-distributor between $\mathcal{V}$-taxons X and A is a matrix $P\colon |X| \twoheadrightarrow |A|$ together with actions $AP \xrightarrow{@} P$ and $PX \xrightarrow{@} P$ that are, respectively, coequalizers of $AAP \overset{A@}{\underset{\cdot P}{\rightrightarrows}} AP$ and $PXX \overset{P\cdot}{\underset{@X}{\rightrightarrows}} PX$. Again, we refer the reader to [Kos97] where distributors are called i-modules. These observations, and some preliminary calculations with the monoidal category $\mathcal{V}$ replaced by a small bicategory $\mathcal{W}$, led us to conjecture that the full sub 2-category of $\mathcal{W}$-total $\mathcal{W}$-categories determined by the completely distributive objects is biequivalent to the bicategory of $\mathcal{W}$-distributors.

1.5 A rather burdensome requirement for this project is that in order to work on it one must redo much of enriched category theory for enriched taxons. The amount of space required to do this, even with quite broad brush strokes, dwarfs the theorem itself. Fortunately, there is a simple theorem about a mere monad D on a mere category $\mathcal{C}$, in which idempotents split, that covers what we want to say now. We leave the development of enriched taxon theory for [R&W05].

1.6 In the next section we establish our basic result showing equivalence of the idempotent splitting completion $\mathbf{kar}(\mathcal{C}_D)$ of the Kleisli algebras for a monad D and the Eilenberg-Moore algebras for D with (a specified) homomorphic splitting, denoted $\mathbf{spl}(\mathcal{C}^D)$. These latter are seen to be the projectives for the homomorphisms with underlying arrow a split epi. In Section 3 we show how the basic result gives $\mathbf{ccd_{sup}}$ equivalent to $\mathbf{kar}(\mathbf{rel})$. After noting that a homomorphic splitting need not be unique, we consider the case of a KZ-doctrine D on a 2-category $\mathcal{C}$. For a D-algebra (L, s), any homomorphic splitting (to within isomorphism) of s, satisfying a mild coherence condition, is actually a left adjoint of s. Finally, in Section 4 we sketch the extension to the bicategory enriched context.

2. The Basic Theorem

2.1 Suppose that idempotents split in the category $\mathcal{C}$ and that $D = (D, d, m)$ is a monad on $\mathcal{C}$. Forgetful functors create all limits, so it is clear that idempotents split in $\mathcal{C}^D$ the category of Eilenberg-Moore algebras. Explicitly, let (L, s) be a D-algebra and $e\colon (L, s) \to (L, s)$ be an idempotent in $\mathcal{C}^D$ and $L \xrightarrow{p} S \xrightarrow{i} L$ be a splitting of $e\colon L \to L$ in $\mathcal{C}$. It is easy to verify that the idempotent homomorphism e is split by $(S, p.s.Di)$ and p and i become homomorphisms. Of course any structure arrow $s\colon DL \to L$ admits dL as a section in $\mathcal{C}$.

2.2 We will write $\mathbf{spl}(\mathcal{C}^D)$ for the category whose objects are triples (L, s, t) where (L, s) is a D-algebra and $t\colon (L, s) \to (DL, mL)$ is a section for s in $\mathcal{C}^D$ and whose arrows $h\colon (L, s, t) \to (L', s', t')$ are D-homomorphisms $h\colon (L, s,) \to (L', s')$. It follows of course that if t and t' are both splittings for a structure s then (L, s, t) and (L, s, t') are isomorphic objects in $\mathbf{spl}(\mathcal{C}^D)$. We write $\mathbf{kar}(\mathcal{C}_D)$ for the idempotent splitting completion of the Kleisli category $\mathcal{C}_D$. We find it convenient to treat $\mathcal{C}_D$ as the full subcategory of $\mathcal{C}^D$ determined by the free D-algebras (DX, mX) rather than employing the syntactic description found, for example, in Mac Lane [MAC71].

2.3 Let $e\colon DX \to DX$ be an idempotent in $\mathcal{C}^D$ and thus an object of $\mathbf{kar}(\mathcal{C}_D)$ split by $DX \xrightarrow{p} S \xrightarrow{i} DX$. Consider $(S, s := (p.mX.Di))$, as noted above the splitting of e in $\mathcal{C}^D$. The arrow $t := (Dp.DdX.i)\colon S \to DS$ is a homomorphism and easily $st = 1_S$. We define $\mathcal{S}(DX, e) = (S, s, t)$, an object of $\mathbf{spl}(\mathcal{C}^D)$. If $f\colon (DX, e) \to (DX', e')$ is an arrow in $\mathbf{kar}(\mathcal{C}_D)$ consider $p'.f.i\colon S \to S'$, where S' with p' and i' provides a splitting for e'. Calculating that $p'.f.i.(p.mX.Di) = (p'.mX.Di').Dp'.Df.Di$ shows that defining $\mathcal{S}(f) = p'.f.i$ provides an arrow from $\mathcal{S}(DX, e)$ to $\mathcal{S}(DX', e')$ in $\mathbf{spl}(\mathcal{C}^D)$ and further routine calculations show that we have a functor $\mathcal{S}\colon \mathbf{kar}(\mathcal{C}_D) \to \mathbf{spl}(\mathcal{C}^D)$.

2.4 For (L, s, t) in $\mathbf{spl}(\mathcal{C}^D)$, define $\mathcal{I}(L, s, t) = (DL, ts\colon DL \to DL)$ which is obviously an object of $\mathbf{kar}(\mathcal{C}_D)$. For $h\colon (L, s, t) \to (L', s', t')$ in $\mathbf{spl}(\mathcal{C}^D)$ define

$$\mathcal{I}h = t's'.Dh.ts\colon (DL, ts) \to (DL', t's')$$

which is a morphism of idempotents and hence an arrow of $\mathbf{kar}(\mathcal{C}_D)$ by its very definition. It is easy to see that the definitions provide a functor $\mathcal{I}: \mathbf{spl}(\mathcal{C}^D) \longrightarrow \mathbf{kar}(\mathcal{C}_D)$. In particular, for $1_L: (L, s, t) \longrightarrow (L, s, t)$, $\mathcal{I}(1_L: (L, s, t) \longrightarrow (L, s, t)) = ts.ts = ts$, which is the identity on (DL, ts). It is worth noting too that if t and t' are both splittings of s then $\mathcal{I}(1_L: (L, s, t) \longrightarrow (L, s, t')) = t's.ts = t's: (DL, ts) \longrightarrow (DL, t's)$ is an isomorphism (as it must be, since $1_L: (L, s, t) \longrightarrow (L, s, t')$ is an isomorphism in $\mathbf{spl}(\mathcal{C}^D)$.

2.5. Theorem. *For a category $\mathcal{C}$ in which idempotents split and a monad D on $\mathcal{C}$, the functors $\mathcal{S}$ and $\mathcal{I}$ defined above provide inverse equivalences*

$$\mathbf{kar}(\mathcal{C}_D) \underset{\mathcal{I}}{\overset{\mathcal{S}}{\rightleftarrows}} \mathbf{spl}(\mathcal{C}^D)$$

Proof. Let (DX, e) be an object of $\mathbf{kar}(\mathcal{C}_D)$. With the notation of 2.3 and 2.4, $\mathcal{IS}(DX, e) = (DS, ts)$. Define $\eta_{(DX,e)} = tp: DX \longrightarrow DS$. We have $ts.tp = tp = tp.e$ because t is a section of s and p coequalizes e and 1_{DX}. Thus $tp: (DX, e) \longrightarrow (DS, ts)$ is an arrow in $\mathcal{C}_D$. Now for $f: (DX, e) \longrightarrow (DX', e')$ a calculation shows that $t's'.D(p'fi).ts.tp = t'p'.f$ and so that $\eta: 1_{\mathbf{kar}(\mathcal{C}_D)} \longrightarrow \mathcal{IS}$ is a natural transformation. For $is: DS \longrightarrow DX$ we have $e.is = is = is.ts$ so that $is: (DS, ts) \longrightarrow (DX, e)$ is an arrow of $\mathbf{kar}(\mathcal{C}_D)$. Moreover $is.tp = ip = e$ and $tp.is = ts$ shows that $is = (tp)^{-1}$ in $\mathbf{kar}(\mathcal{C}_D)$, making $\eta: 1_{\mathbf{kar}(\mathcal{C}_D)} \longrightarrow \mathcal{IS}$ is invertible.

For (L, s, t) in $\mathbf{spl}(\mathcal{C}^D)$, we may as well take $\mathcal{SI}(L, s, t) = (L, s, t)$ since L with s and t already provides a splitting for the idempotent $\mathcal{I}(L, s, t) = (DL, ts)$ and following the description of $\mathcal{S}(DL, ts)$ in 2.3 we get $s.mL.Dt = sts = s$ (since t is a D-homomorphism) and $Ds.DdL.t = t$ (since s is a D-structure). In fact, if we use these obvious splittings for the idempotents (DL, ts) then $\mathcal{SI}$ is the identity on arrows as follows, by using the descriptions of 2.3 and 2.4 in calculations applied to a homomorphism $h: (L, s) \longrightarrow (L', s')$ from $s'.t's'.Dh.ts.t = s'.Dh.t = h.s.t = h$. ∎

While any equivalence in any bicategory can be 'adjusted' to give an adjoint equivalence, we note that here we have $\eta\mathcal{I} = 1_{\mathcal{I}}$ and $\mathcal{S}\eta = 1_{\mathcal{S}}$ so that the equivalence described is already an adjoint equivalence. The following corollary is an immediate observation that in general, for objects of $\mathcal{C}^D$, structure arrows are homomorphisms that admit splittings in the base category $\mathcal{C}$:

2.6. Corollary. *The objects (L, s) of $\mathcal{C}^D$ determined by the (L, s, t) of $\mathbf{spl}(\mathcal{C}^D)$ are precisely the projectives in $\mathcal{C}^D$, with respect to those homomorphisms whose underlying $\mathcal{C}$-arrows are split epimorphisms.* ∎

3. Uniqueness of Splittings

3.1 There is no reason why a homomorphic splitting $t: L \longrightarrow DL$ of a D-structure $s: DL \longrightarrow L$ should be unique. In fact, Steve Lack drew to our attention the following interesting example:

3.2. Example. Let $\mathcal{E}$ be a category with (say) finite limits, and let I be an object for which the unique $I \longrightarrow 1$ is effective for descent. Then $I^* = \Delta_I : \mathcal{E} \longrightarrow \mathcal{E}/I$ is monadic, and a splitting for an 'algebra' E is just an arbitrary $E \longrightarrow I$.

On the other hand:

3.3. Example. If $\mathcal{C} = \mathbf{ord}$ and D is the downset monad then $\mathcal{C}^D$ is the category **sup** of complete semi-lattices and supremum-preserving functions. Banaschewski and Niefield [B&N91] pointed out that $\bigvee: DL \longrightarrow L$ has a section in **sup** if and only if it has a left adjoint so that in this case splittings are essentially unique (and unique if ordered sets are taken to be antisymmetric). Thus we have $\mathbf{spl}(\mathbf{ord}^D) \simeq \mathbf{ccd}_{\mathbf{sup}}$, so $\mathbf{kar}(\mathbf{ord}_D) \simeq \mathbf{ccd}_{\mathbf{sup}}$ by Theorem 2.5.

Now we can recover $\mathbf{kar}(\mathbf{rel}) \simeq \mathbf{ccd}_{\mathbf{sup}}$ as follows. First, $\mathbf{ord}_D$ is the category **idl** of ordered objects and order ideals and **rel** (which may also be viewed as the Kleisli category for the powerset monad P on **set**) embeds in **idl** via discrete orders, so $\mathbf{kar}(\mathbf{rel})$ embeds in $\mathbf{kar}(\mathbf{idl})$. An order relation is transitive and reflexive, hence interpolative, so objects of **idl** are in $\mathbf{kar}(\mathbf{rel})$ and an ideal is an arrow of $\mathbf{kar}(\mathbf{rel})$. The universal property of **kar** extends this embedding of **idl** in $\mathbf{kar}(\mathbf{rel})$ to $\mathbf{kar}(\mathbf{idl})$, and it is now easy to verify that $\mathbf{kar}(\mathbf{rel})$ is equivalent to $\mathbf{kar}(\mathbf{idl})$. Summing up, $\mathbf{kar}(\mathbf{rel}) \simeq \mathbf{kar}(\mathbf{ord}_D) \simeq \mathbf{ccd}_{\mathbf{sup}}$.

3.4 Somewhat surprisingly, we can generalize the result of [B&N91] to show that for a KZ-doctrine D on a 2-category $\mathcal{C}$ and a D-algebra (L, s), any homomorphic splitting (to within isomorphism) of s, satisfying a mild coherence condition, is actually a left adjoint of s. We say "Somewhat surprisingly" because in the case of complete semi-lattices homomorphisms *are* left adjoints and the result of [B&N91] could be rephrased to say that if t is a homomorphic section of $\bigvee$ then the right adjoint of t *is* $\bigvee$. We first state our theorems on this matter and then prove them using an idea of Steve Lack that makes use of published work of Kock [Ko95].

3.5. Theorem. *If D is a KZ-doctrine on a 2-category $\mathcal{C}$ in which idempotent transformations (2-cells) split, and (L, s) is a D-algebra, and $t: (L, s) \longrightarrow (DL, mL)$ is a D-homomorphism for which there is an isomorphism $\eta: 1_L \xrightarrow{\simeq} st$ then there is an adjunction $f \dashv s$ with f a local retract of t.*

If $D = (D, d, m)$ is a KZ-doctrine on a 2-category $\mathcal{C}$ then we follow the approach of Marmolejo [Mar99] (although as noted below we work in somewhat less generality) in that the defining data is taken to provide a fully faithful adjoint string $Dd \dashv m \dashv dD$. As pointed out in both [Mar99] and [Ko95], of fundamental importance is the transformation $\delta: Dd \longrightarrow dD$ that arises unambiguously from the data of the defining adjunctions. If (L, s) is a D-algebra then $s \dashv dL$ and, because we will assume that D is a 2-functor and d is 2-natural, $Ds \dashv DdL$ so that we have the adjoint string $Ds \dashv DdL \dashv mL \dashv dDL$. Taking left adjoints of $\delta L: DdL \longrightarrow dDL$ gives a transformation $\gamma: mL \longrightarrow Ds$ about which we have more to say later.

For $t: L \longrightarrow DL$ in $\mathcal{C}$ there is a canonical transformation $\tau: mL.Dt \longrightarrow ts$, the mate of the equality $Dt.dL \xrightarrow{=} dDL.t$, and invertibility of τ is precisely the statement that t

provides a D-homomorphism $(L, s) \longrightarrow (DL, mL)$.

3.6. THEOREM. *If D is a KZ-doctrine on a 2-category $\mathcal{C}$, and (L, s) is a D-algebra, and $t\colon (L, s) \longrightarrow (DL, mL)$ is a D-homomorphism for which there is an isomorphism $\eta\colon 1_L \xrightarrow{\simeq} st$ that satisfies*

$$\begin{tikzcd}
Ds.Dt.t & t \arrow[l, "D\eta.t"'] \arrow[d, "t.\eta"] \\
mL.Dt.t \arrow[u, "\gamma.Dt.t"] & t.s.t \arrow[l, "\tau^{-1}.t"]
\end{tikzcd}$$

then $t \dashv s$ with unit η.

Given an isomorphism $\eta\colon 1_L \xrightarrow{\simeq} st$, its inverse $\eta^{-1}\colon st \xrightarrow{\simeq} 1_L$ corresponds via the adjunction $s \dashv dL$ to a transformation $\sigma\colon t \longrightarrow dL$ with $s\sigma$ invertible. In fact the adjunction provides a bijective correspondence between invertible transformations $st \xrightarrow{\simeq} 1_L$ and transformations $\sigma\colon t \longrightarrow dL$ with $s\sigma$ invertible. It follows that the coherence constraint of Theorem 3.6 can be expressed in terms of σ, which provides a more resonant condition:

3.7. THEOREM. *If D is a KZ-doctrine on a 2-category $\mathcal{C}$, and (L, s) is a D-algebra, and $t\colon (L, s) \longrightarrow (DL, mL)$ is a D-homomorphism for which there is an isomorphism $\eta\colon 1_L \xrightarrow{\simeq} st$ that satisfies*

$$\begin{tikzcd}
mL.Dt.t \arrow[r, "mL.Dt.\sigma"] \arrow[d, "mL.D\sigma.t"'] & mL.Dt.dL \arrow[d, "|"] \\
mL.DdL.t \arrow[r, "mL.\delta L.t"'] & mL.dDL.t
\end{tikzcd}$$

then $t \dashv s$ with unit η.

Commutativity of the diagram above is equivalent to commutativity of that obtained by composing its two paths with $\vee.t\colon mL.dDL.t \xrightarrow{\simeq} t$, where $\vee\colon mL.dDL \xrightarrow{\simeq} 1_{DL}$ is the invertible counit for $mL \dashv dDL$. (In the proof of Theorem 3.7 we will denote units [counits] for adjunctions, generically, by $\wedge$ [$\vee$].) We explain that the coherence condition of Theorem 3.7 is more 'resonant' than that of Theorem 3.6 by considering $t\colon L \longrightarrow DL$ to be an arrow $T\colon L \dashrightarrow L$ in $\mathcal{C}_D$, which in many situations can be seen as a proarrow in the sense of the second author [Wd82], [Wd85]. In this event, $\sigma\colon t \longrightarrow dL\colon L \longrightarrow DL$ provides $\Sigma\colon T \longrightarrow 1_L\colon L \dashrightarrow L$ and $mL.Dt.t\colon L \longrightarrow DL$ provides $TT\colon L \dashrightarrow L$. With this notation the coherence condition of Theorem 3.7 reads $T\Sigma = \Sigma T$ so that T becomes a co-well-pointed endo proarrow of L. In fact T admits a comultiplication so that it underlies an idempotent comonad structure in $\mathcal{C}_D$. Such considerations were important to the authors in [R&W94a] and revisited more generally in [R&W95]. It should be said that derivation of the idempotent comonad structure on T is known as the 'Interpolation Lemma' in the very special case of this situation in which $\mathbf{spl}(\mathcal{C}^D)$ is the category of continuous ordered sets and directed-sup-preserving functions.

3.8 To approach proofs of the theorems of this section, consider the biadjunction $F \dashv U : \mathcal{C}^D \longrightarrow \mathcal{C}$. The generality of the approach of Marmolejo [Mar99] to KZ-doctrines is sufficient to ensure that, for $G = FU : \mathcal{C}^D \longrightarrow \mathcal{C}^D$ and $g : G \longrightarrow 1_{\mathcal{C}^D}$ defined by $g(L,s) = s : (DL, mL) = FU(L,s) \longrightarrow (L,s)$, there is a co-fully faithful adjoint string $Gg \dashv c \dashv gG$. In fact $c(L,s) = DdL : (DL, mL) \longrightarrow (DDL, mDL)$ provides the comultiplication and in terms of data in $\mathcal{C}$ the defining adjoint string is $Ds \dashv DdL \dashv mL$. We have $\gamma : gG \longrightarrow Gg$ given by $\gamma(L,s) = \gamma : mL \longrightarrow Ds$, the transformation defined prior to the statement of Theorem 3.6. The following result is in Kock [Ko95], but restricted to the case where $\mathcal{C}$ is locally ordered, as Theorem 4.1 there. He points out that the theorem is undoubtedly valid without the locally ordered assumption. This, as sketched above, is the case via Marmolejo's approach.

3.9. PROPOSITION. *$G = (G, g, c)$ is a KZ-doctrine on $(\mathcal{C}^D)^{\mathrm{coop}}$.* ∎

3.10 Of course, in the usual way, we prefer to think of (G, g, c) as a KZ-co-doctrine on $\mathcal{C}^D$ but the point here is that no new definitions are needed and results for KZ-doctrines yield results for KZ-co-doctrines. In particular, a D-algebra (L,s) supports a G-coalgebra structure if and only if s has a left adjoint in $\mathcal{C}^D$. It suffices to ask that s have a left adjoint in $\mathcal{C}$ since all left adjoints between D-algebras are D-homomorphisms. (See [Ko95], Proposition 2.5.) Our theorems 3.5, 3.6, and 3.7 all deal with the question of finding left adjoints to a structure arrow s given a homomorphic 'splitting' for s. Thus such theorems follow, by Proposition 3.9, from simpler theorems concerning D-structure on an object L in $\mathcal{C}$ via a mere 'retraction' for dL. Such a theorem is already in [Ko95], as part of Theorem 3.5 there. We present it in a less strict form that suits our purposes.

3.11. PROPOSITION. *If D is a KZ-doctrine on a 2-category $\mathcal{C}$ in which idempotent transformations split, and L is an object of $\mathcal{C}$, and $r : DL \longrightarrow L$ is a $\mathcal{C}$ arrow for which there is an isomorphism $\epsilon : r.dL \xrightarrow{\simeq} 1_L$ then there is an adjunction $f \dashv dL$ with f a local retract of r.*

PROOF. Define $\eta : 1_{DL} \longrightarrow dL.r$ to be the paste composite:

$$
\begin{tikzcd}
DL \arrow[rr, bend left=60, "1_{DL}"] \arrow[rr, phantom, bend left=35, "D\epsilon^{-1}\Downarrow"] \arrow[r, bend left, "DdL"] \arrow[r, phantom, "\delta L\Downarrow"] \arrow[r, bend right, "dDL"'] \arrow[dr, bend right, "r"'] & DDL \arrow[r, "Dr"'] \arrow[d, phantom, "||"] & DL \\
& L \arrow[ur, bend right, "dL"'] &
\end{tikzcd}
$$

and now

$$
\begin{tikzcd}
L \arrow[r, "dL"] \arrow[drr, bend right, "1_L"'] & DL \arrow[r, bend left, "DdL"] \arrow[r, bend right, "dDL"'] \arrow[r, phantom, "\delta L\Downarrow"] \arrow[rr, bend left=60, "1_{DL}"] \arrow[dr, "r"'] \arrow[d, phantom, "\epsilon\Downarrow"] & DDL \arrow[r, "Dr"] \arrow[d, phantom, "||"] \arrow[u, phantom, "D\epsilon^{-1}\Downarrow"] & DL \\
& & L \arrow[ur, "dL"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& DL \arrow[dr, "DdL"] \arrow[d, phantom, "||"] \arrow[drr, bend left=50, "1_{DL}"] & & \\
L \arrow[ur, "dL"] \arrow[dr, "dL"'] \arrow[ddr, bend right=50, "1_L"'] & {} & DDL \arrow[r, "Dr"] \arrow[d, phantom, "||"] \arrow[u, phantom, "D\epsilon^{-1}\Downarrow"] & DL \\
& DL \arrow[ur, "dDL"] \arrow[dr, "r"] \arrow[d, phantom, "\epsilon\Downarrow"] & {} & \\
& {} & L \arrow[uur, bend right, "dL"'] &
\end{tikzcd}
$$

since dL identifies δL when, as we are assuming, d is 2-natural. By 2-naturality of d the right paste composite above is 1_{dL} and this verifies one of the triangular equations, were we to be proving $r \dashv dL$. As pointed out in [Ko95], there is no reason for the composite $\epsilon r.r\eta$ to be the identity but from the famous Paré exercise, given in Mac lane [MAC71] as Exercise 4 of IV.1, it is an idempotent on r which is split in $\mathcal{C}(DL, L)$ by $f\colon DL \longrightarrow L$ if and only if $f \dashv dL$. ∎

3.12 Proof of Theorem 3.5: Apply Proposition 3.11 to the case provided by Proposition 3.9. ∎

3.13. Proposition. *If D is a KZ-doctrine on a 2-category $\mathcal{C}$, and L is an object of $\mathcal{C}$, and $r\colon DL \longrightarrow L$ is a $\mathcal{C}$ arrow for which there is an isomorphism $\epsilon\colon r.dL \xrightarrow{\simeq} 1_L$ that satisfies*

$$
\begin{tikzcd}
r.Dr.DdL \arrow[r, "r.D\epsilon"] \arrow[d, "r.Dr.\delta L"'] & r \\
r.Dr.dDL \arrow[r, "-"'] & r.dL.r \arrow[u, "\epsilon.r"']
\end{tikzcd}
$$

then $r \dashv dL$ with counit η.

Proof. Define $\eta\colon 1_{DL} \longrightarrow dL.r$ as in the proof of Proposition 3.11 and obtain the triangular equation on dL as before. Observe that the assumed coherence condition is the triangular equation on r. ∎

3.14 Proof of Theorem 3.6: Apply Proposition 3.13 to the case provided by Proposition 3.9. ∎

3.15 Proof of Theorem 3.7: It remains to be shown that the coherence condition of Theorem 3.7 is equivalent to that of Theorem 3.6. In the latter it is convenient to reverse the directions of some invertible arrows so that the condition becomes:

$$
\begin{tikzcd}
 & & DDL \arrow[dr, "mL"] \arrow[d, phantom, "\Downarrow\tau"] & \\
L \arrow[r, "t"] \arrow[drr, bend right=40, "1_L"'] & DL \arrow[ur, "Dt"] \arrow[dr, "s"'] \arrow[d, phantom, "\Downarrow\eta^{-1}"] & & DL \\
 & {} & L \arrow[ur, "t"'] &
\end{tikzcd}
\overset{3.6}{=}
\begin{tikzcd}
L \arrow[r, "t"] & DL \arrow[r, "Dt"] \arrow[rr, bend right=60, "1_{DL}"'] \arrow[r, phantom, bend right=40, "\Downarrow D\eta^{-1}"] & DDL \arrow[r, bend left, "mL"] \arrow[r, bend right, "Ds"'] \arrow[r, phantom, "\Downarrow\gamma"] & DL
\end{tikzcd}
$$

while in the former it is convenient to compose with the invertible $\vee.t\colon mL.dDL \xrightarrow{\simeq} t$, as in the discussion after the theorem statement, so that the condition is

$$
\begin{tikzcd}
L \arrow[r, bend left, "t"] \arrow[r, bend right, "dL"'] \arrow[r, phantom, "\Downarrow\sigma"] \arrow[dr, "t"'] & DL \arrow[r, "Dt"] \arrow[d, phantom, "||"] & DDL \arrow[r, "mL"] \arrow[d, phantom, "\Downarrow\vee"] & DL \\
 & DL \arrow[ur, "dDL"] \arrow[urr, "1_{DL}"'] & {} &
\end{tikzcd}
\overset{3.7}{=}
\begin{tikzcd}
L \arrow[r, "t"] & DL \arrow[r, bend left=60, "Dt"] \arrow[r, dashed, "DdL" description] \arrow[r, bend right=60, "dDL"'] \arrow[r, phantom, shift left=3, "\Downarrow D\sigma"] \arrow[r, phantom, shift right=3, "\Downarrow\delta L"] \arrow[rr, bend right=70, "1_{DL}"'] & DDL \arrow[r, "mL"] \arrow[d, phantom, "\vee\Downarrow"] & DL \\
 & & {} &
\end{tikzcd}
$$

Now referring first to Equation 3.6 we have

$$
\eta^{-1} \quad = \quad
\begin{tikzcd}
L \arrow[r, bend left=60, "t"] \arrow[r, "dL"'] \arrow[r, phantom, shift left=3, "\sigma\Downarrow"] \arrow[rr, bend right=50, "1_L"'] & DL \arrow[r, "s"] \arrow[d, phantom, "\vee\Downarrow"] & L \\
 & {} &
\end{tikzcd}
$$

and hence

$$
D\eta^{-1} \quad = \quad
\begin{tikzcd}
DL \arrow[r, bend left=60, "Dt"] \arrow[r, "DdL"'] \arrow[r, phantom, shift left=3, "D\sigma\Downarrow"] \arrow[rr, bend right=50, "1_{DL}"'] & DDL \arrow[r, "Ds"] \arrow[d, phantom, "\vee\Downarrow"] & DL \\
 & {} &
\end{tikzcd}
$$

while

$$
\gamma \quad = \quad
\begin{tikzcd}
 & {} \arrow[d, phantom, "\wedge\Downarrow"] & & \\
DDL \arrow[r, "Ds"'] \arrow[rr, bend left=60, "1_{DDL}"] & DL \arrow[r, bend left, "DdL"] \arrow[r, bend right, "dDL"'] \arrow[r, phantom, "\delta L\Downarrow"] \arrow[rr, bend right=60, "1_{DL}"'] & DDL \arrow[r, "mL"] \arrow[d, phantom, "\Downarrow\vee"] & DL \\
 & & {} &
\end{tikzcd}
$$

and τ is the mate of the equality $Dt.dL \xrightarrow{=} dDL.t$. With these substituted into Equation 3.6 we see that it is equivalent to

$$
\begin{tikzcd}
 & & & DL \arrow[rr, "Dt"] & & DDL \arrow[dr, "mL"] & \\
L \arrow[r, "dL"'] \arrow[r, bend left=60, "t"] \arrow[r, phantom, bend left=30, "\Downarrow\sigma"] \arrow[rr, bend right=50, "1_L"'] \arrow[rr, phantom, bend right=25, "\Downarrow\vee"] & DL \arrow[urr, "1_L"] \arrow[r, "s"'] \arrow[ur, phantom, "\wedge\Downarrow"] & L \arrow[ur, "dL"] \arrow[rr, "t"'] \arrow[rr, phantom, shift left=3, "||"] & & DL \arrow[ur, "dDL"] \arrow[rr, "1_{DL}"'] \arrow[r, phantom, shift left=3, "\Downarrow\vee"] & & DL
\end{tikzcd}
\quad =
$$

$$
\begin{tikzcd}
L \arrow[r, "t"] & DL \arrow[r, "DdL"'] \arrow[r, bend left=60, "Dt"] \arrow[r, phantom, bend left=30, "D\sigma\Downarrow"] \arrow[rr, bend right=60, "1_{DL}"'] \arrow[rr, phantom, bend right=30, "\Downarrow\vee"] & DDL \arrow[r, "Ds"'] \arrow[rr, bend left=60, "1_{DDL}"] \arrow[rr, phantom, bend left=30, "\wedge\Downarrow"] & DL \arrow[r, bend left=30, "DdL"] \arrow[r, bend right=30, "dDL"'] \arrow[r, phantom, "\delta L\Downarrow"] \arrow[rr, bend right=60, "1_{DL}"'] \arrow[rr, phantom, bend right=30, "\Downarrow\vee"] & DDL \arrow[r, "mL"] & DL
\end{tikzcd}
$$

which is the same as Equation 3.7 after ‘cancelling’ as provided by the two evident adjunction identities. ∎

4. Enriched Categories

4.1 Let $\mathcal{W}$ be a small bicategory (meaning that the set of objects of $\mathcal{W}$ is small and all its hom categories are small). We sketch here a development whose details will be given in [R&W05]. For W an object of $\mathcal{W}$, write $\widehat{W}$ for the canonical one-object $\mathcal{W}$-category with extent $\widehat{W}(*) = W$. We will assume also that $\mathcal{W}$ is biclosed (meaning that $\mathcal{W}$ has all right liftings and all right extensions) and that $\mathcal{W}$ is locally totally cocomplete. The hypotheses make $\mathcal{W}$ locally ordered (by the well-known Freyd result) so that subsequently we do not have to face the coherence conditions of Section 3. Let L be a $\mathcal{W}$-category. We obtain a $\mathcal{W}$-category DL as follows. The objects of DL are pairs (W, P), where W is an object of $\mathcal{W}$ and $P\colon \widehat{W} \dashrightarrow L$ is a $\mathcal{W}$-profunctor, with extent $DL(W, P) = W$. Note that the bicategory $\mathcal{W}$-**prof** of $\mathcal{W}$-categories and $\mathcal{W}$-profunctors inherits a biclosed structure from that of $\mathcal{W}$. The $\mathcal{W}$-valued hom arrows of DL are given by $DL((W, P), (X, Q)) = P \Rightarrow Q\colon X \longrightarrow W$, where the right lifting $P \Rightarrow Q\colon \widehat{W} \dashrightarrow \widehat{X}$ can be regarded as an arrow $W \longrightarrow X$ in $\mathcal{W}$ since the homomorphism $\widehat{(-)} : \mathcal{W} \longrightarrow \mathcal{W}\text{-}\mathbf{prof}$ has each $\mathcal{W}(W, X) \longrightarrow \mathcal{W}\text{-}\mathbf{prof}(\widehat{W}, \widehat{X})$ an isomorphism of categories. These aspects can also be found in Walters [Wal80, Wal82] and Street [St81].

D underlies a KZ-doctrine on $\mathcal{W}$-**cat** for which the algebras are the $\mathcal{W}$-totally-cocomplete $\mathcal{W}$-categories. By Theorem 3.5 a homomorphic splitting of a D-structure $s\colon DL \longrightarrow L$ is a left adjoint for s so that the 2-category $\mathbf{spl}(\mathcal{W}\text{-}\mathbf{cat}^D)$, as in Theorem 2.5, is the full sub 2-category of $\mathcal{W}\text{-}\mathbf{cat}^D$ determined by what might be called the constructively completely distributive objects of $\mathcal{W}$-**cat**. On the other hand $\mathcal{W}\text{-}\mathbf{cat}_D$ is $\mathcal{W}$-**prof** so from Theorem 2.5 we have:

4.2. Theorem. *For $\mathcal{W}$ a small, cocomplete, biclosed bicategory, there is a biequivalence of bicategories:*

$$\mathbf{kar}(\mathcal{W}\text{-}\mathbf{prof}) \rightleftarrows \mathcal{W}\text{-}\mathbf{ccd}_{\mathcal{W}\text{-}\mathbf{cocts}}$$

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