Theory and Applications of Categories, Vol. 13, No. 15, 2004, pp. 235–251.

INTERNAL MONOTONE-LIGHT FACTORIZATION FOR CATEGORIES VIA PREORDERS

Dedicated to Aurelio Carboni on the occasion of his sixtieth birthday

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ABSTRACT. It is shown that, for a finitely-complete category $\mathcal{C}$ with coequalizers of kernel pairs, if every product-regular epi is also stably-regular then there exist the reflections $(\mathbf{R})\mathbf{Grphs}(\mathcal{C}) \to (\mathbf{R})\mathbf{Rel}(\mathcal{C})$, from (reflexive) graphs into (reflexive) relations in $\mathcal{C}$, and $\mathbf{Cat}(\mathcal{C}) \to \mathbf{Preord}(\mathcal{C})$, from categories into preorders in $\mathcal{C}$. Furthermore, such a sufficient condition ensures as well that these reflections do have stable units. This last property is equivalent to the existence of a monotone-light factorization system, provided there are *sufficiently many* effective descent morphisms with domain in the respective full subcategory. In this way, we have internalized the monotone-light factorization for small categories via preordered sets, associated with the reflection $\mathbf{Cat} \to \mathbf{Preord}$, which is now just the special case $\mathcal{C} = \mathbf{Set}$.

1. Introduction

Monotone-light factorization of morphisms in an abstract category $\mathcal{C}$, with respect to a full reflective subcategory $\mathcal{X}$, was studied by A. Carboni, G. Janelidze, G.M. Kelly, and R. Paré in [2].

According to [2], the existence of such factorization requires strong additional conditions on the reflection $\mathcal{C} \to \mathcal{X}$, which hold in the (Galois theory of the) adjunction between compact Hausdorff and Stone spaces, needed to make the classical monotone-light factorization of S. Eilenberg (cf. [4]) and G.T. Whyburn (cf. [9]) a special case of the categorical one.

In fact, A. Carboni and R. Paré studied in categorical terms the classical monotone-light factorization, relating this factorization system with the reflection $\mathbf{CompHaus} \to \mathbf{Stone}$ of compact Hausdorff spaces into Stone spaces. But the connection between adjunctions and factorization systems was already known by M. Kelly (from [3]), and surprisingly the (reflective) factorization system associated to $\mathbf{CompHaus} \to \mathbf{Stone}$ did not coincide with the one of Carboni and Paré. The connection between the two distinct factorization systems was to be made by the categorical Galois theory of G. Janelidze: the

The author acknowledges partial financial assistance by Unidade de Investigação e Desenvolvimento Matemática e Aplicações da Universidade de Aveiro/FCT

Received by the editors 2004-05-18 and, in revised form, 2004-10-07.

Published on 2004-12-05.

2000 Mathematics Subject Classification: 18A32, 12F10.

Key words and phrases: (reflexive) graph, (reflexive) relation, category, preorder, factorization system, localization, stabilization, descent theory, Galois theory, monotone-light factorization.

right-hand class of the former factorization system $(\mathrm{E}', \mathrm{M}^*)$ is the class of all *coverings*, while the right-hand class of the latter (E, M) only includes the *trivial coverings*.

In our Ph.D. thesis we showed that monotone-light factorization also does exist for $\mathcal{C} = \mathbf{Cat}$, the category of all categories, $\mathcal{X}$ being the category **Preord** of preordered sets (cf. [10]). A crucial observation here is that the reflection $\mathbf{Cat} \to \mathbf{Preord}$ has stable units in the sense of [2].

What we do in our present work is to internalize this reflection and associated monotone-light factorization system. For a given finitely-complete category $\mathcal{C}$ with coequalizers of kernel pairs, we start with the categories of internal categories $\mathbf{Cat}(\mathcal{C})$ and internal preorders $\mathbf{Preord}(\mathcal{C})$. Then, we give sufficient conditions for the existence of a full reflection $\mathbf{Cat}(\mathcal{C}) \to \mathbf{Preord}(\mathcal{C})$ with stable units, and for the existence of a monotone-light factorization associated.

Hence, the reflection $\mathbf{Cat} \to \mathbf{Preord}$ turns out to be a special case of our construction, when $\mathcal{C} = \mathbf{Set}$. In the process, we also found that having stable units is necessary for the existence of a monotone-light factorization, in case a certain very weak condition holds, as it happens with both **Cat** and **CompHaus** (see Corollary 6.2).

Notice that there are few known similar situations where there is a (categorical) monotone-light factorization. This work provides an helpful frame for the search of new ones.

In fact, our results are also valid for internal (reflexive) graphs, which may be reflected onto the internal (reflexive) relations. This corresponds to the reflection $(\mathbf{R})\mathbf{Grphs} \to (\mathbf{R})\mathbf{Rel}$ for the case $\mathcal{C} = \mathbf{Set}$, presented below in Example 6.5.

2. The reflection $\mathbf{Cat}(\mathcal{C}) \to \mathbf{CEqRel}(\mathcal{C})$ and related ones

For any finitely-complete category $\mathcal{C}$, there is the category $\mathbf{Cat}(\mathcal{C})$ of categories in $\mathcal{C}$. That is, the category whose objects C are the diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} C = C_1 \times_{C_0} C_1 \arrow[r, shift left=2, "\pi_2"] \arrow[r, "\gamma" description] \arrow[r, shift right=2, "\pi_1"'] & C_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & C_0 \arrow[l, "i" description] \end{tikzcd} \tag{2.1}$$

satisfying the conditions

$$d_0 i = 1_{C_0} = d_1 i,\ d_0\pi_1 = d_1\pi_2,\ d_0\gamma = d_0\pi_2,\ \textit{and } d_1\gamma = d_1\pi_1,$$

where the square represented by the second equation is a pullback and the *composition operation* γ satisfies the associative and unit laws. The obvious morphisms $f = (f_1, f_0) :$

$C \to D$ are called internal functors and are displayed in the commutative diagram

$$\begin{tikzcd}
C_1 \times_{C_0} C_1 \arrow[r, shift left=2, "\pi_2"] \arrow[r, "\gamma"] \arrow[r, shift right=2, "\pi_1"'] \arrow[d, "f_1 \times f_1"] & C_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] \arrow[d, "f_1"] & C_0 \arrow[l, "i"'] \arrow[d, "f_0"] \\
B_1 \times_{B_0} B_1 \arrow[r, shift left=2, "\pi_2"] \arrow[r, "\gamma"] \arrow[r, shift right=2, "\pi_1"'] & B_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & B_0 \arrow[l, "i"']
\end{tikzcd} .$$

There is also the adjunction

$$D \dashv H : \mathcal{C} \to \mathbf{Cat}(\mathcal{C}), \tag{1}$$

whose left adjoint takes C in $\mathbf{Cat}(\mathcal{C})$ to $D(C) = C_0$ in $\mathcal{C}$, forgetting all the other structure of C (see diagram 2.1). The right adjoint takes C_0 in $\mathcal{C}$ to the object

$$\begin{tikzcd}
H(C_0) = C_0 \times C_0 \times C_0 \arrow[r, shift left=2, "\pi_2"] \arrow[r, "\gamma"] \arrow[r, shift right=2, "\pi_1"'] & C_0 \times C_0 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & C_0 \arrow[l, "i"']
\end{tikzcd}$$

in $\mathbf{Cat}(\mathcal{C})$, wherein:

- the morphisms d_0, d_1 and $i =< 1_{C_0}, 1_{C_0} >$ are respectively the product projections and the diagonal map in the product diagram

$$\begin{tikzcd}
 & C_0 \arrow[dl, "1_{C_0}"'] \arrow[d, "i"] \arrow[dr, "1_{C_0}"] & \\
C_0 & C_0 \times C_0 \arrow[l, "d_0"'] \arrow[r, "d_1"] & C_0
\end{tikzcd} ;$$

- the morphisms π_1 and π_2 are the projections in the pullback diagram

$$\begin{tikzcd}
C_0 \times C_0 \times C_0 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & C_0 \times C_0 \arrow[d, "d_1"] \\
C_0 \times C_0 \arrow[r, "d_0"] & C_0
\end{tikzcd} ;$$

- the morphism $\gamma = d_0 \times d_1$ is the one in the product diagram

$$
\begin{tikzcd}
C_0 \times C_0 \arrow[d, "d_0"] & C_0 \times C_0 \times C_0 \arrow[l, "\pi_2"'] \arrow[r, "\pi_1"] \arrow[d, "\gamma"] & C_0 \times C_0 \arrow[d, "d_1"] \\
C_0 & C_0 \times C_0 \arrow[l, "d_0"'] \arrow[r, "d_1"] & C_0 \quad .
\end{tikzcd}
$$

Hence, the adjunction 1 can be thought as the full reflection

$$\mathbf{Cat}(\mathcal{C}) \to \mathbf{CEqRel}(\mathcal{C}), \tag{2}$$

from categories in $\mathcal{C}$ into *connected equivalence relations* in $\mathcal{C}$, such that its unit is displayed by

$$\delta_C = (d_C, 1_{C_0}) : C \to D(C), \tag{3}$$

for each category C in $\mathcal{C}$, where $d_C =< d_0, d_1 >$ is the morphism pictured in the product diagram

$$
\begin{tikzcd}
& C_1 \arrow[dl, "d_0"'] \arrow[d, "d_C"] \arrow[dr, "d_1"] & \\
C_0 & C_0 \times C_0 \arrow[l] \arrow[r] & C_0 \quad .
\end{tikzcd} \tag{2.2}
$$

Moreover, it is easy to check that its reflector $D : \mathbf{Cat}(\mathcal{C}) \to \mathbf{CEqRel}(\mathcal{C})$ is a left exact functor.

Therefore, one knows from [3] and [2] that the factorization system (E, M) associated with the full reflection 2 is stable. Meaning that, the class E of vertical morphisms coincides with the class E' of stably-vertical morphisms in $\mathbf{Cat}(\mathcal{C})$; and so also the class M of trivial coverings coincides with the class M^* of coverings in $\mathbf{Cat}(\mathcal{C})$. In other words, the reflective factorization system coincides with the *monotone-light* factorization system $(\mathrm{E}', \mathrm{M}^*)$, where both classes are stable under pullbacks in $\mathbf{Cat}(\mathcal{C})$.

The class $\mathrm{E} = \mathrm{E}'$ consists of those morphisms $f : C \to B$ in $\mathbf{Cat}(\mathcal{C})$ such that $D(f) = (f_0 \times f_0, f_0)$ is an isomorphism in $\mathbf{CEqRel}(\mathcal{C})$, i.e., of those morphisms $f = (f_1, f_0)$ in $\mathbf{Cat}(\mathcal{C})$ such that $f_0 : C_0 \to B_0$ is an isomorphism in $\mathcal{C}$.

And $\mathrm{M} = \mathrm{M}^*$ consists of those morphisms $f = (f_1, f_0) : C \to B$ in $\mathbf{Cat}(\mathcal{C})$ such that the commutative diagram

$$
\begin{tikzcd}
C \arrow[r, "\delta_C"] \arrow[d, "f"] & D(C) \arrow[d, "D(f)"] \\
B \arrow[r, "\delta_B"] & D(B)
\end{tikzcd}
$$

is a pullback in $\mathbf{Cat}(\mathcal{C})$. Hence, since pullbacks are calculated componentwise in $\mathbf{Cat}(\mathcal{C})$, a morphism $f = (f_1, f_0)$ in $\mathbf{Cat}(\mathcal{C})$ belongs to $\mathrm{M} = \mathrm{M}^*$ if and only if the commutative diagram

$$\begin{tikzcd} C_1 \arrow[r, "d_C"] \arrow[d, "f_1"] & C_0 \times C_0 \arrow[d, "f_0 \times f_0"] \\ B_1 \arrow[r, "d_B"] & B_0 \times B_0 \end{tikzcd}$$

is a pullback in $\mathcal{C}$.

The functor $D : \mathbf{Cat}(\mathcal{C}) \to \mathbf{CEqRel}(\mathcal{C})$ and the adjunction unit $\delta : 1_{\mathbf{Cat}(\mathcal{C})} \to HD$ can be thought respectively as an endofunctor on $\mathbf{Cat}(\mathcal{C})$ and a natural transformation from $1_{\mathbf{Cat}(\mathcal{C})}$ to that endofunctor D.

In this way, the fact that $\mathbf{CEqRel}(\mathcal{C})$ is a reflective full subcategory of $\mathbf{Cat}(\mathcal{C})$ is expressed equivalently by stating that the pair (D, δ) is an *idempotent pointed endofunctor* of $\mathbf{Cat}(\mathcal{C})$, in the sense of [7], i.e., (D, δ) is *well-pointed* ($D\delta = \delta D$, see [5]) and δD is an iso. The same reasoning is also valid for the simpler cases of graphs in $\mathcal{C}$ and reflexive graphs in $\mathcal{C}$, when diagram 2.1 is reduced respectively to

$$\begin{tikzcd} C = C_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & C_0 \end{tikzcd}$$

and

$$\begin{tikzcd} C = C_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & C_0 \arrow[l, "i"'] \end{tikzcd} \quad ,$$

with $d_0 i = 1_{C_0} = d_1 i$.

For these two simpler cases we have therefore the full reflections

$$D \dashv H : \mathcal{C} \to \mathbf{Grphs}(\mathcal{C}) \tag{4}$$

and

$$D \dashv H : \mathcal{C} \to \mathbf{RGrphs}(\mathcal{C}), \tag{5}$$

from the functor categories of graphs and reflexive graphs in $\mathcal{C}$, respectively.

In the next, we shall show that, for the last two reflections 4 and 5, the natural transformation $\delta : 1 \to D$ can be factorized as

$$\delta = \mu\eta : 1 \to I \to D,$$

with (I, η) a well-pointed endofunctor, i.e., $I\eta = \eta I$.

The same happens to be true for the reflection 1 under a certain sufficient condition, which implies (for all the three reflections) that (I, η) is idempotent and gives rise to monotone-light factorization systems.

3. The well-pointed endofunctors (I, η) of $(\mathbf{R})\mathbf{Grphs}(\mathcal{C})$ and $\mathbf{Cat}(\mathcal{C})$

Throughout the rest of the paper $\mathcal{C}$ denotes a finitely-complete category with coequalizers of kernel pairs.

We shall also use the notation $(\mathbf{R})\mathbf{Grphs}(\mathcal{C})$ when stating results that are valid both for the category $\mathbf{Grphs}(\mathcal{C})$ of graphs in $\mathcal{C}$ and for the category $\mathbf{RGraphs}(\mathcal{C})$ of reflexive graphs in $\mathcal{C}$. The translation from one realm to another shall always be trivial.

Let

$$\ker(d_C) = (p_1, p_2)$$

be the kernel pair of the morphism $d_C : C_1 \to C_0 \times C_0$ in diagram 2.2.

And let

$$e_C = coker(p_1, p_2) : C_1 \to I(C_1)$$

be the coequalizer of p_1 and p_2, as in the diagram

$$\begin{tikzcd}
C_1 \times_{C_0 \times C_0} C_1 \arrow[rr, "p_2"] \arrow[dd, "p_1"'] & & C_1 \arrow[dl, "e_C"] \arrow[dd, "d_C"] \\
& I(C_1) \arrow[dr, "m_C"] & \\
C_1 \arrow[ur, "e_C"] \arrow[rr, "d_C"] & & C_0 \times C_0
\end{tikzcd} \quad , \tag{3.1}$$

where $m_C : I(C_1) \to C_0 \times C_0$ is the unique morphism such that $d_C = m_C e_C$.

We shall call product-regular epi of the graph C in $\mathcal{C}$ such a morphism $e_C : C_1 \to I(C_1)$ in $\mathcal{C}$.

We will denote by $I(d_0)$ and $I(d_1)$ the components of $m_C =< I(d_0), I(d_1) >$ in the product diagram

$$\begin{tikzcd}
& I(C_1) \arrow[dl, "I(d_0)"'] \arrow[d, "m_C"] \arrow[dr, "I(d_1)"] & \\
C_0 & C_0 \times C_0 \arrow[l] \arrow[r] & C_0
\end{tikzcd} \quad .$$

Next Lemma 3.1 is just an easy consequence of the fact that any kernel pair (p_1, p_2) is always the kernel pair of its coequalizer $e = coker(p_1, p_2)$. It helps to prove the following Proposition 3.2, which introduces a certain endofunctor I on the category of (reflexive) graphs in $\mathcal{C}$ (and such that it may induce a reflection into the full subcategory of (reflexive) relations in $\mathcal{C}$, as we shall see in next section 4).

3.1. Lemma. *If $m_C e_C f = m_C e_C g$, with m_C and e_C the morphisms in diagram* 3.1*, then $e_C f = e_C g$, for any morphisms f and g in $\mathcal{C}$.*

3.2. Proposition. *There is a well-pointed endofunctor (I, η) of $(\mathbf{R})\mathbf{Grphs}(\mathcal{C})$, in the sense of [7] (i.e., a functor $I : (\mathbf{R})\mathbf{Grphs}(\mathcal{C}) \to (\mathbf{R})\mathbf{Grphs}(\mathcal{C})$ and a natural transformation $\eta : 1_{(\mathbf{R})\mathbf{Grphs}(\mathcal{C})} \to I$ with $\eta I = I\eta$), such that:*

(a) the functor $I : (\mathbf{R})\mathbf{Grphs}(\mathcal{C}) \to (\mathbf{R})\mathbf{Grphs}(\mathcal{C})$ takes a general morphism of (reflexive) graphs $f = (f_1, f_0) : C \to B$ to $I(f) = (I(f_1), f_0) : I(C) \to I(B)$;

(b)

$$I(C) = I(C_1) \begin{array}{c} \xrightarrow{I(d_0)} \\ \xleftarrow{e_C i} \\ \xrightarrow[I(d_1)]{} \end{array} C_0 \quad \text{and} \quad I(B) = I(B_1) \begin{array}{c} \xrightarrow{I(d_0)} \\ \xleftarrow{e_B i} \\ \xrightarrow[I(d_1)]{} \end{array} B_0$$

are obtained through the process described above;

(c) the morphism $I(f_1)$ in $\mathcal{C}$ is the one uniquely determined in the diagram

$$\begin{tikzcd} C_1 \times_{C_0 \times C_0} C_1 \arrow[r, shift left, "p_2"] \arrow[r, shift right, "p_1"'] & C_1 \arrow[r, "e_C"] \arrow[d, "f_1"] & I(C_1) \arrow[r, "m_C"] \arrow[d, "I(f_1)"] & C_0 \times C_0 \arrow[d, "f_0 \times f_0"] \\ & B_1 \arrow[r, "e_B"] & I(B_1) \arrow[r, "m_B"] & B_0 \times B_0 \end{tikzcd} ; \qquad (3.2)$$

(d) the natural transformation $\eta : 1_{(\mathbf{R})\mathbf{Grphs}(\mathcal{C})} \to I$ is displayed by

$$\eta_C = (e_C, 1_{C_0}) : C \to I(C),$$

for each (reflexive) graph

$$C = C_1 \begin{array}{c} \xrightarrow{d_0} \\ \xleftarrow{i} \\ \xrightarrow[d_1]{} \end{array} C_0$$

in $\mathcal{C}$.

Proof. We only have to show that:

(i) $I(f_1)$ is indeed uniquely determined in the diagram 3.2;

(ii) $\eta_{I(C)} = I(\eta_C)$, for every (reflexive) graph C in $\mathcal{C}$.

(i) As $m_B e_B f_1 = (f_0 \times f_0) m_C e_C$, we have that $(m_B e_B f_1)p_1 = (m_B e_B f_1)p_2$.
Which implies, by the previous Lemma 3.1, that $(e_B f_1)p_1 = (e_B f_1)p_2$.

(ii) First remark that the diagram

$$\begin{tikzcd} C_1 \arrow[r, "e_C"] \arrow[d, "e_C"] & I(C_1) \arrow[d, "I(e_C)"] \\ I(C_1) \arrow[r, "e_{I(C)}"] & I(I(C_1)) \end{tikzcd}$$

is commutative, since it is an instance of the left square in diagram 3.2.

Hence, $I(e_C) = e_{I(C)}$ because e_C is an epimorphism.

Therefore,

$$I(\eta_C) = I(e_C, 1_{C_0}) = (I(e_C), 1_{C_0}) = (e_{I(C)}, 1_{C_0}) = \eta_{I(C)},$$

for every (reflexive) graph C in $\mathcal{C}$.

■

3.3. Proposition. *If, for every product-regular epi e_C in $\mathcal{C}$, the obvious morphisms $e_C \times_{C_0} e_C : C_1 \times_{C_0} C_1 \to I(C_1) \times_{C_0} I(C_1)$ and $e_C \times_{C_0} e_C \times_{C_0} e_C : C_1 \times_{C_0} C_1 \times_{C_0} C_1 \to I(C_1) \times_{C_0} I(C_1) \times_{C_0} I(C_1)$ are all regular epis in $\mathcal{C}$, then there is a well-pointed endofunctor (I, η) of $\mathbf{Cat}(\mathcal{C})$ such that conditions (a), (b), (c) and (d), at the statement of the previous Proposition 3.2, are valid (when one considers of course only the reflexive graph part of internal categories in $\mathcal{C}$).*

In particular, if we just demand that every product-regular epi $e_C : C_1 \to I(C_1)$ is a stably-regular epi in $\mathcal{C}$ (i.e., that any pullback $p^*(e_C)$ of it along any morphism $p : E \to I(C_1)$ in $\mathcal{C}$ is a regular epi), there is also such a well-pointed endofunctor (I, η) of $\mathbf{Cat}(\mathcal{C})$.

Proof. We still need to show that:

(1) for every category C in $\mathcal{C}$ (see diagram 2.1), there exists a unique morphism $I(\gamma) : I(C_1) \times_{C_0} I(C_1) \to I(C_1)$ making the square in the following diagram

$$\begin{tikzcd} \cdot \arrow[r, shift left, "p'_2"] \arrow[r, shift right, "p'_1"'] & C_1 \times_{C_0} C_1 \arrow[r, "e_C \times_{C_0} e_C"] \arrow[d, "\gamma"] & I(C_1) \times_{C_0} I(C_1) \arrow[d, "I(\gamma)"] \\ & C_1 \arrow[r, "e_C"] \arrow[d, "d_C"] & I(C_1) \arrow[dl, "m_C"] \\ & C_0 \times C_0 & \end{tikzcd}$$

commute, where $(p'_1, p'_2) = \ker(e_C \times_{C_0} e_C)$ is the kernel pair of $e_C \times_{C_0} e_C$;

(2) the diagram

$$\begin{tikzcd} I(C)=I(C_1)\times_{C_0} I(C_1) \arrow[r, shift left=3, "I(\pi_2)"] \arrow[r, "I(\gamma)" description] \arrow[r, shift right=3, "I(\pi_1)"'] & I(C_1) \arrow[r, shift left=3, "I(d_0)"] \arrow[r, shift right=3, "I(d_1)"'] & C_0 \arrow[l, "e_C i" description] \end{tikzcd}$$

is a category in $\mathbf{Cat}(\mathcal{C})$, wherein $I(\pi_1)$ and $I(\pi_2)$ denote the obvious pullback projections.

(1) We observe then: $d_C\gamma =< d_0, d_1 > \gamma$

$=< d_0\gamma, d_1\gamma >$

$=< d_0\pi_2, d_1\pi_1 >$ (since C is a category in $\mathcal{C}$)

$=< I(d_0)e_C\pi_2, I(d_1)e_C\pi_1 >$

$=< I(d_0)I(\pi_2)e_C \times_{C_0} e_C, I(d_1)I(\pi_1)e_C \times_{C_0} e_C >$

$=< I(d_0)I(\pi_2), I(d_1)I(\pi_1) > e_C \times_{C_0} e_C$

$\Rightarrow d_C\gamma p_1' = d_C\gamma p_2'$

$\Rightarrow e_C\gamma p_1' = e_C\gamma p_2'$ (by Lemma 3.1)

$\Rightarrow$ there exists one and only one morphism $I(\gamma)$ such that

$$I(\gamma)e_C \times_{C_0} e_C = e_C\gamma,$$

since $e_C \times_{C_0} e_C$ is a regular epi by hypothesis.

(2) We have by definition that

$$I(d_0)(e_Ci) = 1_{C_0} = I(d_1)(e_Ci) \text{ and } I(d_0)I(\pi_1) = I(d_1)I(\pi_2).$$

Then, we observe: $m_CI(\gamma)e_C \times_{C_0} e_C = d_C\gamma$

$=< d_0\pi_2, d_1\pi_1 >$ (since C is a category in $\mathcal{C}$)

$=< I(d_0)I(\pi_2), I(d_1)I(\pi_1) > e_C \times_{C_0} e_C$

$\Rightarrow m_CI(\gamma) =< I(d_0)I(\pi_2), I(d_1)I(\pi_1) >$, since by hypothesis $e_C \times_{C_0} e_C$ is an epi.

So, there is only left to prove that $I(\gamma)$ satisfies the associative and unit laws.

The unit law holds for $I(\gamma)$ if in the following obvious commutative diagrams

$$
\begin{tikzcd}
C_0 \times_{C_0} C_1 \arrow[r, "1 \times_{C_0} e_C"] \arrow[d, "i \times 1"] & C_0 \times_{C_0} I(C_1) \arrow[d, "I(i) \times 1"] \\
C_1 \times_{C_0} C_1 \arrow[r, "e_C \times_{C_0} e_C"] \arrow[d, "\gamma"] & I(C_1) \times_{C_0} I(C_1) \arrow[d, "I(\gamma)"] \\
C_1 \arrow[r, "e_C"] & I(C_1)
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \times_{C_0} C_0 \arrow[r, "e_C \times_{C_0} 1"] \arrow[d, "1 \times i"] & I(C_1) \times_{C_0} C_0 \arrow[d, "1 \times I(i)"] \\
C_1 \times_{C_0} C_1 \arrow[r, "e_C \times_{C_0} e_C"] \arrow[d, "\gamma"] & I(C_1) \times_{C_0} I(C_1) \arrow[d, "I(\gamma)"] \\
C_1 \arrow[r, "e_C"] & I(C_1)
\end{tikzcd}
$$

where $I(i)$ stands for $e_C i$, the arrows $1 \times_{C_0} e_C$ and $e_C \times_{C_0} 1$ are regular epis.

But this is indeed so because $1 \times_{C_0} e_C$ and $e_C \times_{C_0} 1$ are isomorphic to e_C, as it is easy to check by drawing the respective pullback diagrams.

The associative law holds for $I(\gamma)$ if, in the obvious commutative diagram

$$
\begin{tikzcd}
C_1 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[r, "e_C \times_{C_0} e_C \times_{C_0} e_C"] \arrow[d, shift right, "1 \times \gamma"'] \arrow[d, shift left, "\gamma \times 1"] & I(C_1) \times_{C_0} I(C_1) \times_{C_0} I(C_1) \arrow[d, shift right, "1 \times I(\gamma)"'] \arrow[d, shift left, "I(\gamma) \times 1"] \\
C_1 \times_{C_0} C_1 \arrow[r, "e_C \times_{C_0} e_C"] \arrow[d, "\gamma"] & I(C_1) \times_{C_0} I(C_1) \arrow[d, "I(\gamma)"] \\
C_1 \arrow[r, "e_C"] & I(C_1)
\end{tikzcd}
,
$$

where $\gamma(1\times\gamma) = \gamma(\gamma\times 1)$ because C is a category in $\mathcal{C}$, the morphism $e_C \times_{C_0} e_C \times_{C_0} e_C$ is a regular epi.

At last, it is easy to check that if e_C is a stably-regular epi then $e_C \times_{C_0} e_C$ and $e_C \times_{C_0} e_C \times_{C_0} e_C$ are also stably-regular epis (one should use the well-known fact that the class of stably-regular epis is closed under composition).

■

4. The reflections $(\mathbf{R})\mathbf{Grphs}(\mathcal{C}) \to (\mathbf{R})\mathbf{Rel}(\mathcal{C})$ and $\mathbf{Cat}(\mathcal{C}) \to \mathbf{Preord}(\mathcal{C})$

In this section we are going to investigate under which conditions the well-pointed endofunctors of the last section become reflections into the relations and preorders in $\mathcal{C}$, respectively.

4.1. PROPOSITION. *Consider the pullback diagram*

$$\begin{tikzcd}
C_1 \times_{C_0\times C_0} C_1 \arrow[rr, "p_2"] \arrow[dr, "e_C \times_{C_0\times C_0} e_C"] \arrow[dd, "p_1"'] & & C_1 \arrow[d, "e_C"] \\
& I(C_1) \times_{C_0\times C_0} I(C_1) \arrow[r, "p_2'"] \arrow[d, "p_1'"'] & I(C_1) \arrow[d, "m_C"] \\
C_1 \arrow[r, "e_C"] & I(C_1) \arrow[r, "m_C"] & C_0 \times C_0
\end{tikzcd} \quad , \tag{4.1}$$

where the morphisms p_1, p_2, e_C and m_C are those in diagram 3.1.

Then, the following three conditions are equivalent:

(a) the morphism $m_C : I(C_1) \to C_0 \times C_0$ is monic in $\mathcal{C}$;

(b) the morphism $e_C \times_{C_0\times C_0} e_C : C_1 \times_{C_0\times C_0} C_1 \to I(C_1) \times_{C_0\times C_0} I(C_1)$ is epi in $\mathcal{C}$;

(c) the morphism $e_C \times_{C_0\times C_0} e_C : C_1 \times_{C_0\times C_0} C_1 \to I(C_1) \times_{C_0\times C_0} I(C_1)$ is a regular epi in $\mathcal{C}$.

In particular, if the product-regular epi $e_C : C_1 \to I(C_1)$ is a stably-regular epi in $\mathcal{C}$ then the morphism $m_C : I(C_1) \to C_0 \times C_0$ is monic in $\mathcal{C}$.

PROOF. If m_C is monic in $\mathcal{C}$ then the pullback projections p_1' and p_2' in diagram 4.1 are isos, since they constitute its kernel pair. Therefore, we have that $e_C \times_{C_0\times C_0} e_C$ is for instance isomorphic to $e_C p_2$. It follows that $e_C \times_{C_0\times C_0} e_C$ is a regular epi in $\mathcal{C}$, since e_C is a regular epi and p_2 is a split epi.

On the other way, let us assume that $e_C \times_{C_0\times C_0} e_C$ is an epimorphism. By Lemma 3.1 we know that $e_C p_2 = e_C p_1$, so that the pullback projections p_1' and p_2' are identical. It follows that m_C is monic in $\mathcal{C}$.

Finally, we are going to suppose that $e_C : C_1 \to I(C_1)$ is a stably-regular epi. Observe then that $e_C \times_{C_0\times C_0} e_C$ is the composite of the morphisms $1 \times_{C_0\times C_0} e_C : I(C_1) \times_{C_0\times C_0} C_1 \to I(C_1) \times_{C_0\times C_0} I(C_1)$ and $e_C \times_{C_0\times C_0} 1 : C_1 \times_{C_0\times C_0} C_1 \to I(C_1) \times_{C_0\times C_0} C_1$. Each of these morphisms is a pullback of e_C, and therefore stably-regular in case e_C is so. ∎

The next theorem follows trivially from the above considerations.

4.2. THEOREM. *The well-pointed endofunctors given at Proposition 3.2 and Proposition 3.3 induce respectively the reflections* $(\mathbf{R})\mathbf{Grphs}(\mathcal{C}) \to (\mathbf{R})\mathbf{Rel}(\mathcal{C})$, *from the category of (reflexive) graphs in $\mathcal{C}$ to its full subcategory of (reflexive) relations in $\mathcal{C}$, and* $\mathbf{Cat}(\mathcal{C}) \to \mathbf{Preord}(\mathcal{C})$, *from the category of categories in $\mathcal{C}$ to its full subcategory of preorders in $\mathcal{C}$, if and only if for every product-regular epi $e_C : C_1 \to I(C_1)$ in $\mathcal{C}$ the associated morphism $e_C \times_{C_0\times C_0} e_C$ in diagram* 4.1 *is epi in $\mathcal{C}$.*

In particular, for the existence of these reflections it is sufficient that all product-regular epis are stably-regular epis in $\mathcal{C}$.

5. The stabilization of $(\mathbf{R})\mathbf{Grphs}(\mathcal{C}) \to (\mathbf{R})\mathbf{Rel}(\mathcal{C})$ and $\mathbf{Cat}(\mathcal{C}) \to \mathbf{Preord}(\mathcal{C})$

The following Proposition 5.1 states that if every product-regular is stably-regular then the reflections $(\mathbf{R})\mathbf{Grphs}(\mathcal{C}) \to (\mathbf{R})\mathbf{Rel}(\mathcal{C})$ and $\mathbf{Cat}(\mathcal{C}) \to \mathbf{Preord}(\mathcal{C})$ do have stable units in the sense of [3] and [2].

5.1. Proposition. *Every unit morphism $\eta_C = (e_C, 1_{C_0}) : C \to I(C)$ is stably-vertical in $(\mathbf{R})\mathbf{Grphs}(\mathcal{C})$ or $\mathbf{Cat}(\mathcal{C})$ (in the sense of [7]; see section 2), provided that all product-regular epis $e_B : B_1 \to I(B_1)$ are stably-regular in $\mathcal{C}$.*

Proof. We have to show that any pullback $g^*(\eta_C) = (g_1^*(e_C), g_0^*(1_{C_0})) : P \to B$ of $\eta_C = (e_C, 1_{C_0}) : C \to I(C)$, along any morphism $g = (g_1, g_0) : B \to I(C)$, is vertical.

As pullbacks in $(\mathbf{R})\mathbf{Grphs}(\mathcal{C})$ and $\mathbf{Cat}(\mathcal{C})$ are calculated pointwise, the question amounts to ask if the morphism $I(g_1^*(e_C))$ in the following diagram

$$\begin{tikzcd} P_1 \arrow[r, "e_P"] \arrow[d, "g_1^*(e_C)"] & I(P_1) \arrow[r, "m_P"] \arrow[d, "I(g_1^*(e_C))"] & P_0 \times P_0 \arrow[d, "g_0^*(1_{C_0}) \times g_0^*(1_{C_0})"] \\ B_1 \arrow[r, "e_B"] & I(B_1) \arrow[r, "m_B"] & B_0 \times B_0 \end{tikzcd} \tag{5.1}$$

is invertible in $\mathcal{C}$.

Remark that diagram 5.1 is analogous to diagram 3.2 and that within it:

- $g_1^*(e_C)$ stands for the pullback of e_C along $g_1 : B_1 \to I(C_1)$ in $\mathcal{C}$;
- $g_0^*(1_{C_0})$ is the pullback of the identity morphism 1_{C_0} along g_0 in $\mathcal{C}$, and so is an iso in $\mathcal{C}$ as well as $g_0^*(1_{C_0}) \times g_0^*(1_{C_0})$;
- $P_1 = B_1 \times_{I(C_1)} C_1$ and $P_0 \cong B_0$.

We then observe:

the morphisms $g_1^*(e_C)$, e_B and e_P are stably-regular epis in $\mathcal{C} \Rightarrow$

(since the class of stably-regular epis is known to be closed under composition) the morphisms $e_B g_1^*(e_C)$ and e_P are stably-regular epis in $\mathcal{C} \Rightarrow$

(since $e_B g_1^*(e_C) = I(g_1^*(e_C))e_P$) the morphisms $I(g_1^*(e_C))e_P$ and e_P are stably-regular epis in $\mathcal{C} \Rightarrow$

(since the stably-regular epis are known to have the strong right cancellation property) the morphism $I(g_1^*(e_C))$ is a stably-regular epi.

Furthermore, $I(g_1^*(e_C))$ is also monic in $\mathcal{C}$, since $m_B I(g_1^*(e_C)) \cong m_P$ and $m_P : I(P_1) \to P_0 \times P_0$ is monic in $\mathcal{C}$ by Proposition 4.1. Being simultaneously monic and regular epi, the morphism $I(g_1^*(e_C))$ must be invertible in $\mathcal{C}$. ∎

The fact that both reflections $(\mathbf{R})\mathbf{Grphs}(\mathcal{C}) \to (\mathbf{R})\mathbf{Rel}(\mathcal{C})$ and $\mathbf{Cat}(\mathcal{C}) \to \mathbf{Preord}(\mathcal{C})$ have stable units, is known to induce respectively a reflective factorization system on $(\mathbf{R})\mathbf{Grphs}(\mathcal{C}) \to (\mathbf{R})\mathbf{Rel}(\mathcal{C})$ and $\mathbf{Cat}(\mathcal{C}) \to \mathbf{Preord}(\mathcal{C})$ (cf. [2]).

Corollary 5.2 singles out the already stated consequences of product-regular epis being as well stably-regular.

5.2. COROLLARY. *If all product-regular epis are as well stably-regular epis in $\mathcal{C}$, then there are reflections* $(\mathbf{R})\mathbf{Grphs}(\mathcal{C}) \to (\mathbf{R})\mathbf{Rel}(\mathcal{C})$ *and* $\mathbf{Cat}(\mathcal{C}) \to \mathbf{Preord}(\mathcal{C})$*, from the category of (reflexive) graphs to the category of relations in $\mathcal{C}$ and from the category of categories to the category of preorders in $\mathcal{C}$, which do have stable units (in the sense of [3] and [2]). Therefore these reflections induce reflective factorization systems* (E, M) *on* $(\mathbf{R})\mathbf{Grphs}(\mathcal{C})$ *and* $\mathbf{Cat}(\mathcal{C})$ *such that (cf. [2]):*

- E *is the class of vertical morphisms, in the sense of [7];*
- M *is the class of trivial coverings, in the sense of categorical Galois theory.*

5.3. EXAMPLES. If $\mathcal{C}$ is a topos (for instance, $\mathcal{C} = \mathbf{Set}$), an abelian category ($\mathcal{C} = \mathbf{Ab}$), Barr-exact like $\mathcal{C} = \mathbf{Grps}$, or only regular (with finite limits!), then $\mathcal{C}$ is in the conditions of last Corollary 5.2. Therefore, in all these examples, there are reflections $(\mathbf{R})\mathbf{Grphs}(\mathcal{C}) \to (\mathbf{R})\mathbf{Rel}(\mathcal{C})$ and $\mathbf{Cat}(\mathcal{C}) \to \mathbf{Preord}(\mathcal{C})$ with stable units: indeed, in all these cases, regular epimorphisms are also stably-regular.

6. The monotone-light factorization systems for the reflections $(\mathbf{R})\mathbf{Grphs}(\mathcal{C}) \to (\mathbf{R})\mathbf{Rel}(\mathcal{C})$ and $\mathbf{Cat}(\mathcal{C}) \to \mathbf{Preord}(\mathcal{C})$

In this section we shall be interested in sufficient conditions for the existence of monotone-light factorization systems $(\mathrm{E}', \mathrm{M}^*)$ associated with the reflections of last section.

E' denotes the class of stably-vertical morphisms, and M^* the class of coverings, either in $(\mathbf{R})\mathbf{Grphs}(\mathcal{C})$ or in $\mathbf{Cat}(\mathcal{C})$.

A morphism $f = (f_1, f_0) : C \to B$ belongs to M^* if and only if there exists an effective descent morphism (abbr. e.d.m.) $p = (p_1, p_0) : E \to B$ such that the pullback $p^*(f)$ of f along p is in the class of trivial coverings M.

We start by recalling the main result of [2], and then derive from it a corollary giving some conditions under which the existence of a monotone-light factorization system can be identified with the stable units property.

6.1. THEOREM. *[2] Let $\mathcal{A}$ be a finitely-complete category and* (E, M) *a factorization system on $\mathcal{A}$. Then, the pair* $(\mathrm{E}', \mathrm{M}^*)$*, obtained by simultaneously stabilizing* E *and localizing* M *is a factorization system if and only if every morphism $\alpha : A \to B$ in $\mathcal{A}$ is locally stable (i.e., there is an e.d.m. $p : E \to B$ in $\mathcal{A}$ such that, in the* (E, M)*-factorization $p^*(\alpha) = me$, of the pullback $p^*(\alpha) : E \times_B A \to E$ of α along p, the morphism e belongs to* E'*).*

6.2. Corollary. *Let $\mathcal{A}$ be a finitely-complete category, $\mathcal{X}$ a full reflective subcategory of $\mathcal{A}$, and* (E, M) *the prefactorization system on $\mathcal{A}$ associated to $\mathcal{X}$ (cf. [2]).*

Consider now the following two hypothesis (1), (2) and the two statements (a), (b):

(1) the unique morphism $1 : 1 \to 1$ between terminal objects in $\mathcal{A}$ is projective with respect to effective descent morphisms (i.e., for every e.d.m. $p : E \to 1$ there is a morphism $q : 1 \to E$ in $\mathcal{A}$ such that $1 = pq$);

(2) for each object B in $\mathcal{A}$, there is an object E in $\mathcal{X}$ and an e.d.m. $p : E \to B$ from E to B in $\mathcal{A}$;

(a) (E, M) *and* $(\mathrm{E}', \mathrm{M}^*)$ *are factorization systems;*

(b) the reflection $\mathcal{A} \to \mathcal{X}$ has stable units (in the sense of [3] and [2]).

If (1) holds then (a) implies (b).
If (2) holds then (b) implies (a).
Remark also that:

- *under hypotheses (1) and (2) the statements (a) and (b) are equivalent;*
- *the monotone-light factorization system* $(\mathrm{E}', \mathrm{M}^*)$ *may coincide with the reflective one* (E, M)*, being therefore crucial to establish whether the class of morphisms* E *is stable under pullbacks or not.*

Proof. We shall first prove that under hypothesis *(1)* the statement *(a)* implies *(b)*.

The reader should refer to the following pullback diagram in $\mathcal{A}$:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\eta_A"] & E \times A \arrow[r, "\pi_2"] \arrow[dd, "\pi_1"] & \mathrm{A} \arrow[dd, "!_A"] \\
HI(A) \arrow[d, "!_{HI(A)}"] & & \\
1 \arrow[r, "f"] & E \arrow[r, "p"] & 1
\end{tikzcd} .$$

The unique morphism $!_A$, from any object A to the terminal object 1 in $\mathcal{A}$, can be factorized through the unit morphism η_A as

$$!_A = !_{HI(A)} \eta_A : A \to HI(A) \to 1,$$

where H and I denote respectively the inclusion and the reflector functor.

We know that η_A is vertical, i.e., it is in E; and that $!_{HI(A)}$ is a trivial covering, i.e., it is in M, since

$$!_{HI(A)} = H(!_{I(A)}) : HI(A) \to H(1) = 1.$$

By the preceding Theorem 6.1, there is an e.d.m. $p : E \to 1$ such that the (E, M)-factorization of the product projection $\pi_1 : E \times A \to A$ is stable, i.e., $\pi_1 = me$ with e a stably-vertical morphism ($e \in \mathrm{E}'$). But then, the pullback $f^*(\pi_1)$ of π_1 along the morphism $f : 1 \to E$ such that $1 = pf$ (f exists by hypothesis *(1)*), also has a stable (E, M)-factorization (M and E$'$ are both pullback stable classes).

Finally, remark that the morphism $f^*(\pi_1)$ is isomorphic to $!_A : A \to 1$. Therefore, as the (E, M)-factorization is unique up to isomorphism, we conclude that η_A is a stably-vertical morphism.

In order to prove that under hypothesis *(2)* the statement *(b)* implies *(a)*, we consider for any morphism $\alpha : A \to B$ in $\mathcal{A}$ the pullback diagram

$$\begin{tikzcd} E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & A \arrow[d, "\alpha"] \\ E \arrow[r, "p"] & B \end{tikzcd},$$

where p is an e.d.m. such that E is in $\mathcal{X}$ (it exists by hypothesis *(2)*).

The projection π_1 in the previous diagram can be factorized through the unit morphism $\eta_{E \times_B A}$ as

$$\pi_1 = \pi_1' \eta_{E \times_B A} : E \times_B A \to HI(E \times_B A) \to E.$$

Hence, as π_1' belongs to M and $\eta_{E \times_B A}$ is stably-vertical by statement (b), we just proved that every morphism $\alpha : A \to B$ in $\mathcal{A}$ is locally stable (cf. Theorem 6.1). Since the reflection has stable units, it follows that (E, M) is a factorization system (cf. [2]). ∎

Now, we can state, in the next Corollary 6.3, sufficient conditions for the existence of monotone-light factorization systems on $(\mathbf{R})\mathbf{Grphs}(\mathcal{C})$ and $\mathbf{Cat}(\mathcal{C})$.

Parentheses will be used to state simultaneously the three cases.

6.3. Corollary. *There is a full reflection* $(\mathbf{R})\mathbf{Grphs}(\mathcal{C}) \to (\mathbf{R})\mathbf{Rel}(\mathcal{C})$ *(*$\mathbf{Cat}(\mathcal{C}) \to \mathbf{Preord}(\mathcal{C})$*) with stable units, and giving rise to a monotone-light factorization system on* $(\mathbf{R})\mathbf{Grphs}(\mathcal{C})$ *(*$\mathbf{Cat}(\mathcal{C})$*), provided the following two conditions hold:*

- *(a) all product-regular epis are stably-regular epis in* $\mathcal{C}$*;*
- *(b) for each (reflexive) graph (category)* B *in* $\mathcal{C}$ *there is respectively a (reflexive) relation (category)* E *in* $\mathcal{C}$ *and an e.d.m.* $p = (p_1, p_0) : E \to C$ *in* $(\mathbf{R})\mathbf{Grphs}(\mathcal{C})$ *(*$\mathbf{Cat}(\mathcal{C})$*) from* E *to* C*.*

6.4. Remark. The morphism $p = (p_1, p_0)$ is an e.d.m. in $(\mathbf{R})\mathbf{Graphs}(\mathcal{C})$ in case p_0 and p_1 are effective descent morphisms in $\mathcal{C}$.

For the case of reflexive graphs demanding that p_0 and p_1 are effective descent morphisms in $\mathcal{C}$ amounts to ask the first component p_1 of p to be an e.d.m. in $\mathcal{C}$.

In effect, being p_1, d_0' and d_0 effective descent morphisms, $p_0 d_0 = d_0' p_1$ is an e.d.m.. Hence, p_0 must also be an e.d.m., by the strong right cancellation property for effective descent morphisms.

6.5. Example. For $\mathcal{C} = \mathbf{Set}$ we have that:

- $\mathbf{Cat}(\mathbf{Set}) = \mathbf{Cat}$, the category of all small categories;
- $\mathbf{Preord}(\mathbf{Set}) = \mathbf{Preord}$, the category of preordered sets;
- $(\mathbf{R})\mathbf{Grphs}(\mathbf{Set}) = (\mathbf{R})\mathbf{Grphs}$, the category of (reflexive) graphs;
- $(\mathbf{R})\mathbf{Rel}(\mathbf{Set}) = (\mathbf{R})\mathbf{Rel}$, the category of (reflexive) relations.

The reflection $\mathbf{Cat} \to \mathbf{Preord}$ was studied in [10]. There, we concluded that it verifies hypotheses *(1)* and *(2)* of Corollary 6.2, and that it has stable units. The monotone-light factorization system associated was also shown to be non-trivial, i.e., $(\mathrm{E}', \mathrm{M}^*) \neq (\mathrm{E}, \mathrm{M})$.

Being $(\mathbf{R})\mathbf{Grphs}$ a topos, its effective descent morphisms coincide with the epimorphisms, i.e., with *surjections on arrows*. Then, it is easy to verify that hypotheses *(1)* and *(2)* of Corollary 6.2 hold for the reflection $(\mathbf{R})\mathbf{Grphs} \to (\mathbf{R})\mathbf{Rel}$. Using the same techniques of [10] for $\mathbf{Cat} \to \mathbf{Preord}$, one concludes that the reflection $(\mathbf{R})\mathbf{Graphs} \to (\mathbf{R})\mathbf{Rel}$ also has stable units, and one easily characterizes the classes E' and M^* of morphisms in $(\mathbf{R})\mathbf{Graphs}$ with respect to this reflection:

- a morphism $\alpha : A \to B$ in $(\mathbf{R})\mathbf{Graphs}$ is stably-vertical ($\alpha \in \mathrm{E}'$) if and only if it is *bijective on objects* (the vertices!) and *surjective on arrows*;
- a morphism $\alpha : A \to B$ in $(\mathbf{R})\mathbf{Graphs}$ is a covering ($\alpha \in \mathrm{M}^*$) if and only if it is *faithful* (on arrows!).

In fact, the reflection $(\mathbf{R})\mathbf{Grphs} \to (\mathbf{R})\mathbf{Rel}$ is in every aspect an extension of the studied reflection $\mathbf{Cat} \to \mathbf{Preord}$, and so we shall give no further details.

We want also to mention the counter-example corresponding to the reflection $\mathbf{RGrphs} \to \mathbf{Preord}$. Which although *admissible* (also called *semi-left-exact*), and satisfying hypotheses *(1)* and *(2)* of Corollary 6.2, has not stable units, and therefore it does not have a monotone-light factorization system.

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