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CARTESIAN CLOSED TOPOLOGICAL HULL OF THE CONSTRUCT OF CLOSURE SPACES

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ABSTRACT. A cartesian closed topological hull of the construct Cls of closure spaces and continuous maps is constructed. The construction is performed in two steps. First a cartesian closed extension L of Cls is obtained. We apply a method worked out by J. Adámek and J. Reiterman [3] for constructing extensions of constructs that in some sense "resemble" the construct of uniform spaces. Secondly, within this extension L the cartesian closed topological hull L^* of Cls is characterized as a full subconstruct. In order to find the internal characterization of the objects of L^* we produce a concrete functor to the category of power closed collections based on Cls as introduced by J. Adámek, J. Reiterman and G.E. Strecker in [4].

1. Introduction

Cls is the construct of closure spaces and continuous maps. A closure space (X, α) is a pair, where X is a set and α is a subset of the powerset $\mathcal{P}(X)$ and satisfies the conditions that X and $\emptyset$ belong to α and that α is closed under arbitrary unions. The sets in α are called the open sets. A function $f : (X, \alpha) \to (Y, \beta)$ between closure spaces (X, α) and (Y, β) is said to be continuous if $f^{-1}(B) \in \alpha$ whenever $B \in \beta$. Sometimes we denote the closure space (X, α) simply by its underlying set X. $C(X, Y)$ is the set of all continuous functions from X to Y.

Some isomorphic descriptions of Cls are often used f.i. by giving the collection of all closed sets (the so called Moore family [8]) where as usual the closed sets are the complements of the opens and continuity is defined accordingly. Another isomorphic description is obtained by means of a closure operation and by defining continuity also accordingly. Both isomorphic descriptions were considered by G. Birkhoff in [8]. The closure operation $cl : \mathcal{P}(X) \to \mathcal{P}(X)$ associated with a closure space (X, α) is as usual defined by $x \in clZ \Leftrightarrow (\forall A \in \alpha, x \in A \Rightarrow A \cap Z \neq \emptyset)$ where $Z \subseteq X$ and $x \in X$. The closure operation is allowed to be non-additive, but it does satisfy the conditions $cl\emptyset = \emptyset, (A \subseteq B \Rightarrow clA \subseteq clB), A \subseteq clA$ and $cl(clA) = clA$, whenever A and B are subsets of X.

There are many examples of non-additive closures in Mathematics, in particular in Algebra, Geometry and Analysis. Perhaps the best known example is the convex hull in vector spaces. Other examples are listed f.i. in the introductory chapter of [12].

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In 1940 G. Birkhoff's motivation for considering closures came from the fact that the collection of closed sets of a topological space forms a complete lattice. The interrelation between closures and complete lattices has been investigated thoroughly over the years by many authors. A general treatment of this subject has been presented by M. Erné in [11]. We also refer to that paper for a long list of references on that part of the subject.

Another motivation for considering closures can be found in G. Birkhoff's book. He associates closures with certain Galois connections. Starting with a binary relation R between sets M and N, a Galois connection (called "polarity" in [8]) between the powerset lattices is associated with this relation in a natural way. The Galois connection in turn induces closures on M and N. Similar ideas also appear in G. Aumann's work from 1970 on "contact relations". [7] G. Aumann's work was inspired by applications in social sciences.

Applications on data analysis and knowledge representation were the motivation for B. Ganter and R. Wille for developing a theory on formal contexts which is founded on the same basic mathematical tools: binary relations and lattices and closures associated with them. We refer to the recent book "Formal concept analysis" [12] by B. Ganter and R. Wille for these applications.

In recent years closures are also used in connection with quantum logic and in the representation theory of physical systems. The role of closures in this respect is explained f.i. in [16], [17] and [18]. In this connection D. Aerts in [5] constructed the category SP of "State property systems". In [6] it was proved that the amnestic modification of SP in fact is isomorphic to Cls.

Cls is known to be a well-fibred topological construct. In 1988 D. Dikranjan, E. Giuli and A. Tozzi [10] gave the explicit formulation of initial and final structures in Cls. If $(f_i : (X_i, \alpha_i) \to X)_{i\in I}$ is a structured sink in Cls, then the final structure α on X is defined by $G \in \alpha \Leftrightarrow f_i^{-1}(G) \in \alpha_i \quad \forall i \in I$. If $(f_i : X \to (X_i, \alpha_i))_{i\in I}$ is a structured source in Cls, then the initial structure α on X is defined by first considering $\{f_i^{-1}(G) | G \in \alpha_i, i \in I\}$ and then taking all possible unions of subcollections.
Clearly Top is embedded in Cls as a full bicoreflective subconstruct. Remark that, as in Top, the Sierpinski two point space S_2 is an initially dense object in Cls.

Categorical terminology follows J. Adámek, H. Herrlich and G. Strecker [1].

2. Function Spaces in Cls

An object X in a category with finite products is *exponential* if $X \times -$ has a right adjoint. In a well-fibred topological construct D this notion can be characterized as follows: X is exponential in D iff for each D-object Y the set $\mathrm{Hom}_{\mathsf{D}}(X, Y)$ can be supplied with the structure of a D-object - a function space or a power object Y^X- such that for any D-object Z and any function $f : X \times Z \to Y$ the following conditions are equivalent:
(i) $f : X \times Z \to Y$ is a D-morphism
(ii) $f^* : Z \to Y^X$ defined by $f^*(z)(x) = f(x, z)$ is a D-morphism
It is well known [13], [14] that in the setting of a topological construct D, an object

X is exponential in D iff $X \times -$ preserves final episinks. Moreover small fibredness of D ensures that this is equivalent to the condition that $X \times -$ preserves quotients and coproducts. A well fibred topological construct D is said to be *cartesian closed* (or to have function spaces) if every object is exponential. In view of the previous characterization, we investigate the interaction of products and final episinks in the setting of Cls. An even better compatibility of initial and final constructions is achieved in a quasitopos extension. Such an extension, i.p. a quasitopos hull will be constructed in the forthcoming paper [9].

2.1. Proposition. *In* Cls *arbitrary products of quotients are quotients.*

Proof. Let $X_i \xrightarrow{f_i} Y_i$ be a quotient in Cls for any $i \in I$.
Let $f = \prod_{i\in I} f_i : \prod_{i\in I} X_i \to \prod_{i\in I} Y_i$ and $G = f(H)$ where H is open and f-saturated and $H \neq \prod_{i\in I} X_i$.
If $K \subseteq I$ is such that $H = \bigcup_{k\in K} pr_k^{-1}(H_k)$ $\quad (H_k \subseteq X_k$ open and pr_k the k-th projection) then each H_k is f_k-saturated and so

$$G = \bigcup_{k\in K} pr_k^{-1}(G_k)$$

with $G_k = f_k(H_k)$ for $k \in K$, is open in $\prod_{i\in I} Y_i$ ∎

However, even finite products do not distribute over coproducts, as follows from the next observation.

2.2. Proposition. *If X is not indiscrete and D_2 is the two point discrete space then $D_2 \times X$ is not isomorphic to $X + X$.*

Proof. If $A \neq X, A \neq \emptyset$ and A is open then $\{0\} \times A$ is not open in $D_2 \times X$. ∎

2.3. Corollary. *The class of exponential objects in* Cls *coincides with the class of indiscrete spaces.*

2.4. Corollary. *If* D *is a topological subconstruct of* Cls *which is finitely productive in* Cls *and differs from the class of indiscrete spaces, then* D *is not cartesian closed.*

Because of the previous negative result we will investigate cartesian closed superconstructs of Cls in which Cls is finitely productive. In order to construct such extensions the next result on function spaces in Cls will be very useful.

2.5. Definition. *For closure spaces (X, α) and (Y, β) we consider the closure structure η on $C(X, Y)$ generated by the basic open sets*

$$\{\Gamma_V | V \in \beta\}$$

where $\Gamma_V = \{f \in C(X, Y) | f(X) \subseteq V\}$

2.6. Proposition. *If $M \subseteq C(X, Y)$ is a subset endowed with a closure structure μ such that the evaluation map* ev*: $X \times (M, \mu) \to Y$ is continuous, then the following conditions hold:*
(i) $\eta|_M \leqslant \mu$ $\quad$ *(i.e.* $1_M : (M, \mu) \to (M, \eta|_M)$ *is continuous)*
(ii) ev*:* $X \times (M, \eta|_M) \to Y$ *is continuous*

PROOF. (i) Let $V \in \beta$ then

$$\mathrm{ev}^{-1}(V) = A \times M \cup X \times G$$

with $A \in \alpha$ and $G \in \mu$.
Either $A = X$ and then $\Gamma_V \cap M = M$ or $A \neq X$ and then $\Gamma_V \cap M = G$.
(ii) Analogous. ■

Propositions 2.1 and 2.6 show that Cls is a type of category as considered in [3]. In some sense it resembles Unif, the construct of uniform spaces and uniformly continuous maps. It follows that the general construction as presented in 4.3 of [3] is applicable to Cls. This is developed in the next section.

3. A Cartesian Closed Extension of Cls

In this paragraph a cartesian closed topological construct L is constructed in which Cls is a finitely productive full subconstruct. L is a so called CCT extension.

3.1. CONSTRUCTION. *[3] Objects of* L *are triples* $(X, \mathcal{A}, \alpha)$ *where* X *is a set,* $\mathcal{A}$ *is a cover of* X *such that*

$$A' \subseteq A, A \in \mathcal{A} \Rightarrow A' \in \mathcal{A}$$

and α *is a closure on* X *which is* $\mathcal{A}$*-final in the sense that* $((A, \alpha|_A) \to (X, \alpha))_{A \in \mathcal{A}}$ *is final in* Cls.
The members of $\mathcal{A}$ *are called generating sets. A morphism in* L,

$$f : (X, \mathcal{A}, \alpha) \to (Y, \mathcal{B}, \beta)$$

is a function that is continuous (with respect to (X, α) *and* (Y, β)*) and preserves the generating sets:* $A \in \mathcal{A} \Rightarrow f(A) \in \mathcal{B}$.
Sometimes we denote a triple $(X, \mathcal{A}, \alpha)$ *simply by its underlying set* X. Cls *is fully embedded in* L *by identifying* (X, α) *with* $(X, \mathcal{P}(X), \alpha)$.
Clearly L *is a topological construct. If* $(X \xrightarrow{f_i} (X_i, \mathcal{A}_i, \alpha_i))_{i \in I}$ *is a structured source then the unique initial lift in* L *is* $(X, \mathcal{A}, \alpha)$ *where* $\mathcal{A} = \{P \subseteq X | f_i(P) \in \mathcal{A}_i \quad \forall i \in I\}$ *is the collection of generating sets and* α *is final in* Cls *for* $((A, \alpha'|_A) \to X)_{A \in \mathcal{A}}$ *with* α' *the initial* Cls*-structure for the source* $(X \xrightarrow{f_i} (X_i, \alpha_i))_{i \in I}$.
If $((X_i, \mathcal{A}_i, \alpha_i) \xrightarrow{f_i} X)_{i \in I}$ *is a structured sink then the unique final lift is* $(X, \mathcal{A}, \alpha)$ *where* $\mathcal{A}$ *consists of all possible subsets of* $f_i(A)$ *with* $i \in I$ *and* $A \in \mathcal{A}_i$ *and all singletons and* α *is the final structure in* Cls *for* $((X_i, \alpha_i) \xrightarrow{f_i} X)_{i \in I}$.
It is easily seen that Cls *is finally dense and therefore bireflectively embedded in* L.
By the general theorem in [3] L *is cartesian closed. Next we give the explicit description of the Hom-objects.*

Let $(X, \mathcal{A}, \alpha)$ *and* $(Y, \mathcal{B}, \beta)$ *be* L*-objects.*
Let ρ *be the closure generated by the basic open sets* $\{\Gamma(A, V) | A \in \mathcal{A}, V \in \beta\}$ *where*

$\Gamma(A, V) = \{f \in \mathrm{Hom}(X, Y) | f(A) \subseteq V\}$.
Let $\mathcal{M} = \{M \subseteq \mathrm{Hom}(X, Y) | \mathrm{ev}(A \times M) \in \mathcal{B}$ *and ev:*$(A, \alpha) \times (M, \rho|_M) \to (Y, \beta)$ *continuous for every* $A \in \mathcal{A}\}$.
Finally let σ *be the closure structure on* $\mathrm{Hom}(X, Y)$ *which is final in* Cls *for* $((M, \rho|_M) \to \mathrm{Hom}(X, Y))_{M \in \mathcal{M}}$.
Then $(\mathrm{Hom}(X, Y), \mathcal{M}, \sigma)$ *is the powerobject of* X *and* Y *in* L.

4. The Cartesian Closed Hull of Cls

We first recall the definitions of CCT hull, multimorphism, strictly dense subcategory, power-closed collection and the construction of the CCT- hull presented by J. Adámek, J. Reiterman and G.E. Strecker. Then we use this method to construct the CCT hull of Cls.

4.1. Definition. *[15] A cartesian closed topological construct* B *is called a cartesian closed topological hull (CCT hull) of a construct* A *if* B *is a finally dense extension of* A *with the property that any finally dense embedding of A into a cartesian closed topological construct can be uniquely extended to* B.

4.2. Definition. *[4] Let* K *be a construct and let* H, K *be* K*-objects and* X *a set. A function* $h : X \times H \to K$ *is called a multimorphism if for each* $x \in X$, $h(x, -) : H \to K$ *defined by* $h(x, -)(y) = h(x, y)$ *is a morphism.*

4.3. Definition. *[4] Let* K *be a construct with quotients and finite products. A full subcategory* H *of* K *is said to be strictly dense in* K *provided that :*

1. *for each object* $K \in \mathsf{K}$ *there exists a productively final sink* $(H_i \overset{h_i}{\to} K)_{i \in I}$ *with* $H_i \in \mathsf{H}$, *i.e., a final sink such that for each* $L \in \mathsf{K}$ *the sink* $(H_i \times L \overset{h_i \times 1_L}{\to} K \times L)_{i \in I}$ *is final as well.*

2. H *is well-fibred, closed under quotients, and has productive quotients (i.e., for each quotient* $e : A \to B$ *with* $A \in \mathsf{H}$, *we have* $B \in \mathsf{H}$ *and* $e \times 1_H : A \times H \to B \times H$ *is a quotient for each* $H \in \mathsf{H}$*).*

Since the category Cls has productive quotients (2.1), it is strictly dense in itself.

4.4. Definition. *[4] Let* K *be a construct with quotients and finite products and let* H *be strictly dense in* K. *A collection* A *of* H*-objects* (A, α) *with* $A \subseteq X$ *is said to be power-closed in* X *provided that* A *contains each* H*-object* (A_0, α_0) *with* $A_0 \subseteq X$ *with the following property:*
Given a multimorphism $h : X \times H \to K$ *with* $H \in \mathsf{H}$ *and* $K \in \mathsf{K}$ *such that for each* $(A, \alpha) \in \mathsf{A}$ *the restriction* $h|_A : (A, \alpha) \times H \to K$ *is a morphism, then the restriction* $h|_{A_0} : (A_0, \alpha_0) \times H \to K$ *is also a morphism.*
We denote by $\mathsf{PC}_{\mathsf{H}}(\mathsf{K})$ *the category of power-closed collections in* H. *Objects are pairs* (X, A), *where* X *is a set and* A *is a power-closed collection of* H*-objects in* X. *Morphisms*

$f : (X, \mathsf{A}) \to (Y, \mathsf{B})$ *are functions from* X *to* Y *such that for each* $(A, \alpha) \in \mathsf{A}$ *the final object of the restriction* $f_A : (A, \alpha) \to f(A)$ *is in* B.
If H=K *then we simply write* $\mathsf{PC}(\mathsf{K})$.

4.5. Theorem. *[4] Any concrete category* K *which has quotients and finite products that are preserved by the forgetful functor, and which has a strictly dense subcategory* H*, has a CCT hull. Moreover, this hull is precisely the category of power-closed collections in* H.

Later in our main theorem 4.10 we will apply the previous result and in order to do so we will now define a functor on a suitable full subconstruct L^* of L towards $\mathsf{PC}(\mathsf{Cls})$.

4.6. Definition. *Let* L^* *be the full subconstruct of* L *whose objects are the* L*-objects* $(X, \mathcal{A}, \alpha)$ *that satisfy the following condition:*
If $B \subseteq X \notin \mathcal{A}$*, then there exists a set* $Z \subseteq X$ *and* $U \in \alpha$ *with:* $Z \cap U = \emptyset, B \cap Z \neq \emptyset$ *and* $B \setminus Z \not\subseteq U$*, such that:* $\forall A \in A : A \cap Z \neq \emptyset \Rightarrow A \setminus Z \subseteq U$

First we define the correspondence for the objects.

4.7. Proposition. *For each object* $(X, \mathcal{A}, \alpha)$ *of* L^* *the collection of closure spaces* $\mathsf{C}_X = \{(A, \beta) | A \in \mathcal{A}, \beta \geqslant \alpha|_A\}$ *is power-closed.*

Proof. Let (X_0, α_0) be a closure space with $X_0 \subseteq X$ and $(X_0, \alpha_0) \notin \mathsf{C}_X$, then we have to prove that there exists a multimorphism $h : X \times H \to K$ with $H, K \in \mathsf{Cls}$, such that the restrictions $h|_A : (A, \beta) \times H \to K$ are continuous for all (A, β) in C_X and such that $h|_{X_0} : (X_0, \alpha_0) \times H \to K$ is not continuous.
$(X_0, \alpha_0) \notin \mathsf{C}_X$ means: $\alpha_0 \ngeqslant \alpha|_{X_0}$ or $X_0 \notin \mathcal{A}$.
1. If α_0 is not finer than $\alpha|_{X_0}$, there exists a $B \in \alpha$ such that $B \cap X_0 \notin \alpha_0$. Set

$$h : X \times H \to S_2 : (x, h) \to \begin{cases} 1 & \text{if } (x, h) \in B \times H \\ 0 & \text{if } (x, h) \notin B \times H \end{cases}$$

with S_2 the Sierpinski space and H an arbitrary closure space. It follows immediately that for $(A, \beta) \in \mathsf{C}_X$ the restriction $h|_A : (A, \beta) \times H \to S_2$ is continuous and the restriction $h|_{X_0} : (X_0, \alpha_0) \times H \to S_2$ is not continuous.
2. If $X_0 \notin \mathcal{A}$, then since $(X, \mathcal{A}, \alpha) \in \mathsf{L}^*$, the following holds: $\exists Z \subseteq X$, $\exists U \in \alpha$ with $Z \cap U = \emptyset, X_0 \cap Z \neq \emptyset$ and $X_0 \setminus Z \not\subseteq U$ such that $\forall A \in \mathcal{A} : A \cap Z \neq \emptyset$ implies $A \setminus Z \subseteq U$.
Take a closure space H that has a non-trivial open set V. Set $h = 1_{Z \times V \cup U \times H} : X \times H \to S_2$. Then h is clearly a multimorphism and $h|_A^{-1}(\{1\}) = (A \cap Z) \times V \cup (U \cap A) \times H$. We know that for $(A, \beta) \in \mathsf{C}_X$ either $A \cap Z = \emptyset$, or $A \setminus Z \subseteq U$.
If $A \cap Z = \emptyset$, then $h|_A^{-1}(\{1\}) = (U \cap A) \times H$ and this is clearly open in $(A, \beta) \times H$.
If $A \setminus Z \subseteq U$, then $h|_A^{-1}(\{1\}) = A \times V \cup (U \cap A) \times H$ and this is open in $(A, \beta) \times H$. Hence, for all $(A, \beta) \in \mathsf{C}_X$ the restriction $h|_A : (A, \beta) \times H \to S_2$ is continuous. Since $X_0 \cap Z \neq \emptyset$, and $X_0 \setminus Z \not\subseteq U$ we have that $h|_{X_0}^{-1}(\{1\}) = (X_0 \cap Z) \times V \cup (U \cap X_0) \times H$ is not open in $(X_0, \alpha_0) \times H$. Therefore the restriction $h|_{X_0} : (X_0, \alpha_0) \times H \to S_2$ is not continuous. ∎

Next we prove that the correspondence on the objects is bijective.

4.8. Proposition. *If* C *is a power-closed collection of* Cls*-objects in a set* X *then* $\mathsf{C} = \mathsf{C}_X$ *for a unique* $(X, \mathcal{A}, \alpha) \in \mathsf{L}^*$.

PROOF. Let C be a power-closed collection of Cls-objects in X. Consider $(X, \mathcal{A}, \alpha)$ where $\mathcal{A} = \{A \subseteq X : (A, \beta) \in \mathsf{C} \text{ for some } \beta\}$ and α the final structure determined by the sink of inclusion maps $((A, \beta) \xrightarrow{i} X)_{(A,\beta)\in\mathsf{C}}$.
We will first prove that $(A, \alpha|_A) \in \mathsf{C}$ for each $A \in \mathcal{A}$.
If $A \in \mathcal{A}$, then by definition of $\mathcal{A}$ there exists a closure structure β on A such that $(A, \beta) \in \mathsf{C}$. Let $h : X \times H \to K$ be a multimorphism such that for each (A', β') the restriction $h|_{A'} : (A', \beta') \times H \to K$ is continuous. For $V \subseteq K$ open, set $U = \bigcup\{x \in X : \{x\} \times H \subseteq h^{-1}(V)\}$. Since for each $(A', \beta') \in \mathsf{C}$ the restriction of h is continuous, $h|_{A'}^{-1}(V)$ has the form $(U \cap A') \times H \cup A' \times W_{A'}$ with $U \cap A' \in \beta'$ and $W_{A'} \subseteq H$ open. Since α is final for the sink $((A', \beta') \xrightarrow{i} X)_{(A',\beta')\in\mathsf{C}}$, this implies that $U \in \alpha$.
Since $(A, \beta) \in \mathsf{C}$ we have that $h|_A^{-1}(V)$ has the form $(U \cap A) \times H \cup A \times W_A$ with $W_A \subseteq H$ open. From the preceding, it follows that $h|_A^{-1}(V)$ is open in $(A, \alpha_A) \times H$, and thus the restriction $h|_A : (A, \alpha_A) \times H \to K$ is continuous too. Since h and V were arbitrary, it follows that $(A, \alpha_A) \in \mathsf{C}$. This implies that $\mathsf{C} = \mathsf{C}_X$.
It remains to be proved that $(X, \mathcal{A}, \alpha)$ is an object of L^*.

1. $\{x\} \in \mathcal{A}$ because each power-closed collection contains all singleton objects.

2. Given $A \in \mathcal{A}$ and $A' \subseteq A$, then there exists a closure structure β such that $(A, \beta) \in \mathsf{C}$. C is a power-closed collection, thus $(A', \beta|_{A'}) \in \mathsf{C}$ and hence $A' \in \mathcal{A}$.

3. It is clear from the definition of α that α is $\mathcal{A}$-final.

4. We have to prove that for each $B \notin \mathcal{A}$, there exists a set $Z \subseteq X$ and $U \in \alpha$ with $Z \cap U = \emptyset, B \cap Z \neq \emptyset, B \setminus Z \not\subseteq U$ such that $\forall A \in \mathcal{A} : A \cap Z \neq \emptyset \Rightarrow A \setminus Z \subseteq U$.
   We know that there exists a multimorphism $h : X \times H \to K$ whose restrictions $h|_A : (A, \beta) \times H \to K$ are continuous for all $(A, \beta) \in \mathsf{C}$ and such that $h|_B : (B, \mathcal{P}(B)) \times H \to K$ is not continuous. Therefore, there exists an open set $V \in K$ such that $h|_A^{-1}(V)$ is open in $(A, \beta) \times H$ for each $(A, \beta) \in \mathsf{C}$ and such that $h|_B^{-1}(V)$ is not open in $(B, \mathcal{P}(B)) \times H$.
   As before set $U = \bigcup\{x \in X : \{x\} \times H \subseteq h^{-1}(V)\}$. For each $(A, \beta) \in \mathsf{C}$ we have $h|_A^{-1}(V) = (U \cap A) \times H \cup A \times W_A$ with $U \cap A \in \beta$ and $W_A \subseteq H$ open. Since α is final for the sink $((A, \beta) \xrightarrow{i} X)_{(A,\beta)\in\mathsf{C}}$, this implies that $U \in \alpha$. Since h is a multimorphism, we have: $h|_{\{x\}}^{-1}(V) = \{x\} \times W_x$ with $W_x \subseteq H$ open. Now $h|_B^{-1}(U)$ is not open, hence there exists a $b \in B$ with $W_b \neq H, W_b \neq \emptyset$. Put $Z = \{z \in X : W_z = W_b\}$. Then we have: $Z \cap U = \emptyset, B \cap Z \neq \emptyset$. Suppose that $B \setminus Z \subseteq U$, then: $h|_B^{-1}(V) = (U \cap B) \times H \cup B \times W_b$ is open in $(B, \mathcal{P}(B)) \times H$. This is a contradiction. Thus we can conclude that $B \setminus Z \not\subseteq U$.
   Because $h|_A^{-1}(V)$ has the form $(U \cap A) \times H \cup A \times W_A$ for each $A \in \mathcal{A}$, it's clear that $A \cap Z \neq \emptyset$ implies $A \setminus Z \subseteq U$.

The uniqueness of $(X, \mathcal{A}, \alpha)$ follows immediately. ∎

4.9. Corollary. *The category* L^* *is isomorphic to the category* $\mathsf{PC}(\mathsf{Cls})$ *of power-closed collections in* Cls*.*

Proof. Let $(X, \mathcal{A}, \alpha)$ and $(X', \mathcal{A}', \alpha')$ be L^*-objects and $f : X \to X'$ be an L-morphism. For $(A, \beta) \in \mathsf{C}_X$, let γ be the final Cls-structure for the sink $f_A : (A, \beta) \to f(A)$. It follows from the definition of an L-morphism that $f(A) \in \mathcal{A}'$. Since $(A, \beta) \in \mathsf{C}_X$ we have that $f \circ i : (A, \beta) \to (X', \alpha')$ with i the inclusion from (A, β) to (X, α) is continuous. The sink $f_A : (A, \beta) \to (f(A), \gamma)$ is final and $i \circ f_A = f \circ i$ is continuous, so we have that the inclusion $i : (f(A), \gamma) \to (X, \alpha)$ is continuous. This implies that $(f(A), \gamma) \in \mathsf{C}_{X'}$. Therefore the correspondence $F : \mathsf{L}^* \to \mathsf{PC}(\mathsf{CLS})$ defined by $F(X \xrightarrow{f} X') = (X, \mathsf{C}_X) \xrightarrow{f} (X', \mathsf{C}_{X'})$ is a functor. It follows from the two previous propositions that the functor F is bijective on objects. It is a functor between constructs and thus it is faithful.
Take objects $(X, \mathcal{A}, \alpha)$ and $(X', \mathcal{A}', \alpha')$ in L^* and let $f : (X, \mathsf{C}_X) \to (X', \mathsf{C}_{X'})$ be a $\mathsf{PC}(\mathsf{CLS})$-morphism. It is easy to verify that $f(A) \in \mathcal{A}'$ for each $A \in \mathcal{A}$.
For each $A \in \mathcal{A}$, set γ_A the final Cls-structure for the sink $f_A : (A, \alpha|_A) \to f(A)$. Then, since f is a $\mathsf{PC}(\mathsf{CLS})$-morphism we have $(f(A), \gamma_A) \in \mathsf{C}_{X'}$. Thus the inclusion $i : (f(A), \gamma_A) \to (X', \alpha')$ is continuous for each $A \in \mathcal{A}$. Since $((A, \alpha|_A) \xrightarrow{i} (X, \alpha))_{A \in \mathcal{A}}$ is final and $f \circ i = i \circ f_A$ is continuous for each $A \in \mathcal{A}$, it follows that $f : (X, \alpha) \to (X', \alpha')$ is continuous. Consequently, the functor F is an isomorphism. ∎

From 4.9 and 4.5 we have the following final result:

4.10. Theorem. L^* *is the CCT hull of* Cls*.*

References

[1] J. Adámek, H. Herrlich and G.E. Strecker, Abstract and concrete categories, Wiley, New York 1990.

[2] J. Adámek, V. Koubek, What to embed into a cartesian closed topological category, Comment. Math. Univ. Carolinae 21 (1980), 817-821.

[3] J. Adámek and J. Reiterman, Cartesian closed hull of the category of uniform spaces, Topology Appl. 19 (1985), 261-276.

[4] J. Adámek, J. Reiterman and G.E. Strecker, Realization of cartesian closed topological hulls, Manuscripta Math. 53 (1985),1-33.

[5] D. Aerts, Foundations of quantum physics: a general realistic and operational approach, Int. J. Theor. Phys. 38 (1999), 289-358.

[6] D. Aerts, E. Colebunders, A. Van der Voorde and B. Van Steirteghem, On the amnestic modification of state property systems, Appl. Cat. Struct. (to appear).

[7] G. Aumann, Kontaktrelationen, Sitz. Ber. Bayer Ak. Wiss., Math.-Nat. Kl. (1970), 67-77.

[8] G. Birkhoff, Lattice Theory, American Mathematical Society, Providence, Rhode Island 1940.

[9] V. Claes, The topological quasitopos hull of the category of closure spaces, preprint.

[10] D. Dikranjan, E. Giuli and A. Tozzi, Topological categories and closure operators, Quaest. Math. 11 (1988), 323-337.

[11] M. Erné, Lattice representations for categories of closure spaces, Categorical Topology, Sigma Series in Pure Mathematics 5, Heldermann Verlag Berlin (1984), 197-222.

[12] B. Ganter, R. Wille, Formal Concept Analysis, Springer, Berlin 1998.

[13] H. Herrlich, Cartesian closed topological categories, Math. Coll. Univ. Cape Town 9 (1974), 1-16.

[14] H. Herrlich, Categorical topology 1971-1981, in Proc. Fifth Prague Topol. Sympos. 1981, Heldermann Verlag, 1982.

[15] H. Herrlich and L.D. Nel, Cartesian closed topological hulls, Proc. Amer. Math. Soc. 62 (1977),215-222.

[16] D.J. Moore, Categories of representations of physical systems, Helvetica Physica Acta 68 (1995).

[17] D.J. Moore, Closure categories, Int. J. Theor. Phys. 36, No.12 (1997).

[18] C. Piron, Mécanique quantique, Bases et applications, Presses polytechniques et universitaires romandes, Lausanne. Second Edition 1998.

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