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UNIVERSAL PROPERTIES OF SPAN

Dedicated to Aurelio Carboni on the occasion of his 60th birthday

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ABSTRACT. We give two related universal properties of the span construction. The first involves sinister morphisms out of the base category and sinister transformations. The second involves oplax morphisms out of the bicategory of spans having an extra property; we call these "jointed" oplax morphisms.

Introduction

Even before they were formally introduced in [Ka], the importance of adjoint functors was recognized in many individual cases, *e.g.*, free groups, fraction fields, Stone-Čech compactifications, and adjunctions on linear spaces. Any functor that has an adjoint has many important properties; for instance, a functor with a right adjoint preserves colimits [M2]. Of course, most functors (and even many important ones) have neither left nor right adjoints. This motivates the introduction of profunctors [Bé2]. These are generalizations of functors and in the bicategory **Prof** of categories with profunctors every functor (viewed as a profunctor) has a right adjoint.

The reader will note that these adjunctions exist in a bicategory different from the 2-category **Cat** of categories in which adjunctions were originally defined. Indeed, the usual characterization (or definition) of adjunction in terms of unit and counit satisfying the triangle equalities is a 2-categorical concept, and adding the appropriate isomorphisms makes it into a bicategorical one. The idea of an adjunction is fruitful enough to motivate adding a 2-cell structure to an ordinary category, just so that one can consider adjunctions. This has been done in several ways.

Historically, the first instance of this was probably the generalization of the concept of "function" to that of "relation" (before categories were even invented). The idea of relations being ordered by inclusion (*i.e.*, as subsets of the product) also goes back long before the invention of categories, but can be considered as providing a 2-cell structure for the category **Rel** of sets and relations. In this bicategory, every function determines a relation and as such has a right adjoint, namely the reverse relation. Furthermore, every relation is the composite of one coming from a function with the reverse of one of these,

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and a relation is that of a function if and only if it has a right adjoint. The notion of relation has been extended to other categories such as groups, to generalize the element level calculation of homology to arbitrary exact categories (cf. [Br, Hi]) and has numerous applications [Lam, M1, P].

The two constructions which concern us here are rather more general than the above. The first concerns the notion of *span*, also introduced by Bénabou [Bé1]. (The definition and basic properties will be reviewed at the beginning of Section 1.) Bicategories of spans have been studied in contexts such as input-feedback-output systems [KSW] and the theory of databases [JRW]. For any category $\mathbf{A}$, there is a canonical embedding $\mathbf{A} \to \mathbf{Span}(\mathbf{A})$. It has been known for some time that $\mathbf{Span}(\ )$ is a free construction subject to the Beck condition. This was made precise by Hermida in [He1, Thm A.2]. In the notation of the present paper, Hermida's theorem states that there is an equivalence of categories

$$\mathbf{Hom}(\mathbf{Span}(\mathbf{A}), \mathcal{B}) \simeq \mathbf{Beck}(\mathbf{A}, \mathcal{B}). \tag{1}$$

The category $\mathbf{Hom}(\mathbf{Span}(\mathbf{A}), \mathcal{B})$ has homomorphisms as objects and oplax transformations as morphisms. The category $\mathbf{Beck}(\mathbf{A}, \mathcal{B})$ has all sinister morphisms $A \to B$ that satisfy the Beck condition as objects, and strong transformations as arrows. (Recall that sinister morphisms were defined in [DPP] as those sending every arrow to a left adjoint.) Moreover, under this equivalence strong transformations in $\mathbf{Hom}(\mathbf{Span}(\mathbf{A}), \mathcal{B})$ correspond to sinister transformations (as in Definition 1.2) in $\mathbf{Beck}(\mathbf{A}, \mathcal{B})$.

This result can also be viewed as a restriction of the universal property of $\Pi_2\mathbf{A}$, the 2-category obtained by freely adding right adjoints to all arrows in $\mathbf{A}$ as defined in [DPP], *i.e.*, there is an equivalence of categories

$$\mathbf{Hom}(\Pi_2\mathbf{A}, \mathcal{B}) \simeq \mathbf{Sin}(\mathbf{A}, \mathcal{B}), \tag{2}$$

where $\mathbf{Hom}(\Pi_2\mathbf{A}, \mathcal{B})$ is the category of 2-functors and oplax transformations, and $\mathbf{Sin}(\mathbf{A}, \mathcal{B})$ is the category of sinister functors, and strong transformations. Moreover, under this equivalence strong transformations in $\mathbf{Hom}(\Pi_2\mathbf{A}, \mathcal{B})$ correspond to sinister transformations in $\mathbf{Sin}(\mathbf{A}, \mathcal{B})$.

Note that (1) above can be obtained from (2) through restriction of the notion of morphism on the righthand side of the equation. The purpose of this paper is to describe a suitable generalization for the notion of morphism on the lefthand side of the equation, to obtain

$$\mathrm{JOL}(\mathbf{Span}(\mathbf{A}), \mathcal{B}) \simeq \mathbf{Sin}(\mathbf{A}, \mathcal{B}), \tag{3}$$

where $\mathrm{JOL}(\mathbf{Span}(\mathbf{A}), \mathcal{B})$ consists of *jointed* oplax morphisms. These are normal oplax morphisms that preserve certain composites involving adjoints (cf. Definitions 2.4 and 2.14). In future work the universal property (3) will be used to generalize the $\mathbf{Span}(\ )$ construction to categories that may not have pullbacks, and even to bicategories.

1. Span

Spans were introduced by Bénabou [Bé1] as an example of a bicategory. Let us recall the relevant definitions. Let **A** be a category with pullbacks. A span $A \nrightarrow B$ in **A**, *i.e.*, an arrow in $\mathbf{Span}(\mathbf{A})$, is a diagram $A \leftarrow S \to B$ in **A**. A 2-cell $\begin{tikzcd} A \arrow[r, bend left, "S"] \arrow[r, bend right, "S'"'] \arrow[r, phantom, "s\Downarrow"] & B \end{tikzcd}$ is a commutative diagram

$$\begin{tikzcd} A \arrow[d, equal] & S \arrow[l] \arrow[r] \arrow[d, "s"'] & B \arrow[d, equal] \\ A & S' \arrow[l] \arrow[r] & B \end{tikzcd}$$

Clearly, spans from A to B with 2-cells form a category. The composition $T \otimes S$ of two spans

$$A \overset{S}{\nrightarrow} B \overset{T}{\nrightarrow} C$$

is defined using the pullback

$$T \otimes S = \begin{tikzcd} & & S \times_B T \arrow[dl] \arrow[dr] & & \\ & S \arrow[dl] \arrow[dr] & & T \arrow[dl] \arrow[dr] & \\ A & & B & & C \end{tikzcd}$$

and $\otimes$ extends to 2-cells in the usual way, using the universal property of pullback. For each A, the span $A \xleftarrow{1_A} A \xrightarrow{1_A} A$ is the identity span I_A. This defines a bicategory $\mathbf{Span}(\mathbf{A})$.

The canonical embedding $(\ )_* : \mathbf{A} \to \mathbf{Span}(\mathbf{A})$ is a homomorphism of bicategories where **A** is considered as a locally discrete bicategory. It is defined as follows: each arrow $f\colon A \to B$ of **A** gives a span $f_* = \left(A \xleftarrow{1_A} A \xrightarrow{f} B \right)$.

The only 2-cells $f_* \to g_*$ are identities so $(\ )_*$ is locally full and faithful. The span $f^* = B \xleftarrow{f} A \xrightarrow{1_A} A$ is right adjoint to f_*. Indeed, the adjunctions are

$$\eta\colon I_A \to f^* f_* \quad \text{and} \quad \varepsilon\colon f_* f^* \to I_B$$

$$\begin{tikzcd} A \arrow[d, equal] & A \arrow[l, "1_A"'] \arrow[r, "1_A"] \arrow[d, "\delta"'] & A \arrow[d, equal] \\ A & A \times_B A \arrow[l, "p_1"] \arrow[r, "p_2"'] & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[d, equal] & A \arrow[l, "f"'] \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, equal] \\ B & B \arrow[l, "1_B"] \arrow[r, "1_B"'] & B, \end{tikzcd}$$

where $A \times_B A$ is the pullback of f along itself. The triangle equalities are easily verified. Moreover, every span with a right adjoint is isomorphic to some f_* (*cf.* [BCSW, CKS]).

Every span $A \xleftarrow{p} S \xrightarrow{q} B$ is isomorphic to $q_* \otimes p^*$; so **Span**(**A**) is generated by spans coming from **A** and their right adjoints, but not freely. In fact, a Beck condition is satisfied: if

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "k"'] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

is a pullback, then the canonical morphism

$$\begin{tikzcd} D \arrow[d, "k_*"', "|" marking] \arrow[dr, phantom, "\Rightarrow"] & C \arrow[l, "h^*"', "|" marking] \arrow[d, "g_*", "|" marking] \\ B & A \arrow[l, "f^*", "|" marking] \end{tikzcd}$$

is an isomorphism. Indeed $k_* \otimes h^*$ can be computed as follows

$$\begin{tikzcd}[column sep=small] & & D \arrow[dl, "1"'] \arrow[dr, "1"] & & \\ & D \arrow[dl, "h"'] \arrow[dr, "1"] & & D \arrow[dl, "1"'] \arrow[dr, "k"] & \\ C & & D & & B \end{tikzcd}$$

and $f^* \otimes g_*$ as

$$\begin{tikzcd}[column sep=small] & & D \arrow[dl, "h"'] \arrow[dr, "k"] & & \\ & C \arrow[dl, "1"'] \arrow[dr, "g"] & & B \arrow[dl, "f"'] \arrow[dr, "1"] & \\ C & & A & & B. \end{tikzcd}$$

To pave the way for our main theorem, we give a universal property for **Span**(**A**), which was stated, and a proof sketched, in [He1, Thm A.2] (see also [He2, Thm 2.2]). For our purposes a somewhat different notation will be appropriate, which we shall set out and then use to state the universal property. We shall also give a complete proof, details of which we shall use for our main result. First we recall (*cf.* [KS]):

1.1. Definition. Let $\eta, \varepsilon \colon f \dashv u \colon A \to B$ and $\eta', \varepsilon' \colon f' \dashv u' \colon A' \to B'$ be two pairs of adjoint arrows. Two 2-cells

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & A' \arrow[d, "f'"] \\ B \arrow[r, "h"'] & B' \end{tikzcd} \qquad \qquad \begin{tikzcd} A \arrow[r, "g"] \arrow[dr, phantom, "\Downarrow\beta"] & A' \\ B \arrow[u, "u"] \arrow[r, "h"'] & B' \arrow[u, "u'"'] \end{tikzcd}$$

are called *mates under the adjunctions $f \dashv u$ and $f' \dashv u'$* if β is the composite

$$
\begin{tikzcd}
 & A \arrow[rr, "g"] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\varepsilon"] & & A' \arrow[rr, equals] \arrow[dr, "f'"'] \arrow[d, phantom, "\Downarrow\alpha"] & & A' \arrow[dl, phantom, "\Downarrow\eta'"] \\
B \arrow[ur, "u"] \arrow[rr, equals] & & B \arrow[rr, "h"'] & & B' \arrow[ur, "u'"'] &
\end{tikzcd}
$$

and (consequently) α is the composite

$$
\begin{tikzcd}
A \arrow[rr, equals] \arrow[dr, "f"'] & & A \arrow[rr, "g"] \arrow[dl, phantom, "\Downarrow\eta"] \arrow[dr, phantom, "\Downarrow\beta"] & & A' \arrow[dr, "f'"] \arrow[d, phantom, "\Downarrow\varepsilon'"] & \\
 & B \arrow[ur, "u"'] \arrow[rr, "h"'] & & B' \arrow[ur, "u'"] \arrow[rr, equals] & & B'.
\end{tikzcd}
$$

Note that the notion of mates defines a bijection between 2-cells of the form α and β as in the definition above. In this paper, we will write α^* for β, and β_* for α, when appropriate.

Recall that an oplax transformation $t\colon F \to G$ provides, for each arrow $f\colon A \to B$, a 2-cell

$$
\begin{tikzcd}
FA \arrow[r, "tA"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Uparrow tf"] & GA \arrow[d, "Gf"] \\
FB \arrow[r, "tB"'] & GB.
\end{tikzcd}
$$

satisfying certain conditions described, for instance, in [Ke]. We will see that all the transformations involved in the universal properties of Span which are not strong are at least oplax. So we will write the strong transformations as oplax ones for which the components are isomorphisms.

1.2. Definition. A homomorphism of bicategories $F : \mathcal{X} \to \mathcal{Y}$ is called *sinister* if for every arrow $x\colon X \to X'$ in $\mathcal{X}$, the image $F(x)\colon F(X) \to F(X')$ is a left adjoint (*i.e.*, it has a right adjoint $F(x)^*$). Given two sinister homomorphisms $F, G : \mathcal{X} \to \mathcal{Y}$, a *sinister transformation* $t : F \to G$ is a strong transformation such that for each arrow $x\colon X \to X'$ the mates

$$
\begin{tikzcd}
FX \arrow[r, "tX"] \arrow[dr, phantom, "\Downarrow ((tx)^{-1})^*"] & GX \\
FX' \arrow[r, "tX'"'] \arrow[u, "(Fx)^*"] & GX' \arrow[u, "(Gx)^*"']
\end{tikzcd}
$$

of the naturality isomorphisms of t,

$$
\begin{tikzcd}
FX \arrow[r, "tX"] \arrow[d, "Fx"'] \arrow[dr, phantom, "\cong \Downarrow (tx)^{-1}"] & GX \arrow[d, "Gx"] \\
FX' \arrow[r, "tX'"'] & GX',
\end{tikzcd}
$$

are also invertible.

The definition of sinister morphism may seem inadequate in that it might be thought that a functoriality condition on the right adjoints and the adjunctions would be desirable. In fact, as will be shown in Proposition 1.4, this is automatic.

1.3. DEFINITION. For any bicategory $\mathcal{B}$ let $\mathcal{M}\mathbf{ap}(\mathcal{B})$ be the bicategory of *maps* in $\mathcal{B}$, *i.e.*, the objects are the same as those of $\mathcal{B}$ and an arrow $B \to B'$ in $\mathcal{M}\mathbf{ap}(\mathcal{B})$ is an adjunction $(f,\ u,\ \varepsilon,\ \eta)$, with $f\colon B \to B'$, $u\colon B' \to B$, $\varepsilon\colon fu \to 1_{B'}$, $\eta\colon 1_B \to uf$ satisfying the usual triangle equalities. A 2-cell $(\alpha, \beta)\colon (f, u, \varepsilon, \eta) \to (\bar{f}, \bar{u}, \bar{\varepsilon}, \bar{\eta})$ is a pair of transformations $\alpha\colon f \to \bar{f}$, $\beta\colon \bar{u} \to u$ which are mates of each other. Composition of arrows is given by

$$(f', u', \varepsilon', \eta')(f, u, \varepsilon, \eta) = (f'f, uu', \varepsilon' \cdot f'\varepsilon u', u\eta' f \cdot \eta).$$

There is an obvious 2-functor $\Theta\colon \mathcal{M}\mathbf{ap}(\mathcal{B}) \to \mathcal{B}$ which is the identity on objects and which is locally fully faithful, so basically an inclusion.

1.4. PROPOSITION. *A homomorphism $F : \mathcal{A} \to \mathcal{B}$ is sinister if and only if it factors as*

$$\mathcal{A} \xrightarrow{G} \mathcal{M}\mathbf{ap}(\mathcal{B}) \xrightarrow{\Theta} \mathcal{B}$$

for some homomorphism G.

PROOF. The "if" part is obvious. So assume that F is sinister. For every $f\colon A \to A'$, pick a right adjoint $U(f)\colon FA' \to FA$ for $F(f)$ and adjunctions $\varepsilon_f\colon F(f)U(f) \to 1_{FA'}$ and $\eta_f\colon 1_{FA} \to U(f)F(f)$. We let $G(f) = (Ff, Uf, \varepsilon_f, \eta_f)$. On 2-cells there is no problem because Θ is locally fully faithful, *i.e.*, mates are uniquely determined.

As Θ is locally fully faithful, for any $A \xrightarrow{f} A' \xrightarrow{f'} A''$ there is a unique isomorphism $\psi_{f',f}\colon U(f'f) \to U(f')U(f)$ making

$$(\varphi_{f',f}, \psi^{-1}_{f',f})\colon (Ff', Uf', \varepsilon_{f'}, \eta_{f'}) \cdot (Ff, Uf, \varepsilon_f, \eta_f) \to (F(f'f), U(f'f), \varepsilon_{f'f}, \eta_{f'f})$$

into an isomorphism

$$Gf' \cdot Gf \to G(f'f)$$

in $\mathcal{M}\mathbf{ap}(\mathcal{B})$. Similarly, there is a unique isomorphism $1_{G(A)} \to G(1_A)$ projecting to $\varphi_A : 1_{FA} \to F(1_A)$. Because each component of these isomorphisms uniquely determines the other, the coherence conditions needed for G to be a homomorphism follow from those for F. ∎

Although this proposition presents sinister morphisms in a suitably functorial fashion, sinister transformations $t\colon F \to F'$ do not lift to transformations $\mu\colon G \to G'$ between the lifted $G, G'\colon \mathcal{A} \to \mathcal{M}\mathbf{ap}(\mathcal{B})$. This is because the components $t(A)$ of t need not be left adjoints. To put sinister transformations in the right perspective one needs double categories. We will introduce the double category $\mathbb{M}\mathrm{ap}(\mathcal{B})$ in a forthcoming paper.

1.5. Remark. In [DPP] the definition of sinister transformation was wrongly given. What we called sinister there is no condition at all; it follows from strong naturality for 2-functors. What we called strongly sinister there is here called sinister.

Since a sinister morphism sends arrows to left adjoints, the image of a comma object

$$\begin{tikzcd} X \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\Downarrow\xi"] & X_1 \arrow[d, "x_1"] \\ X_2 \arrow[r, "x_2"'] & X_0 \end{tikzcd}$$

will have a mate

$$\begin{tikzcd} FX \arrow[r, "Fp_1"] \arrow[dr, phantom, "\Downarrow F(\xi)^*"] & FX_1 \\ FX_2 \arrow[u, "Fp_2^*"] \arrow[r, "Fx_2"'] & FX_0. \arrow[u, "Fx_1^*"'] \end{tikzcd}$$

Explicitly, the cell $(F\xi)^*$ is given by the composite

$$Fp_1 \cdot Fp_2^* \xrightarrow{\eta_{x'_1} \cdot Fp_1 \cdot Fp_2^*} Fx_1{}^* \cdot Fx_1 \cdot Fp_1 \cdot Fp_2^* \xrightarrow[\cong]{- \cdot \phi^{-1} \cdot -} Fx_1^* \cdot F(x_1 p_1) \cdot Fp_2^*$$

$$\xrightarrow{- \cdot F(\xi) \cdot -} Fx_1^* \cdot F(x_2 p_2) \cdot Fp_2^* \xrightarrow[\cong]{- \cdot \phi \cdot -} Fx_1{}^* \cdot Fx_2 \cdot Fp_2 \cdot Fp_2{}^* \xrightarrow{- \cdot \varepsilon_{x_2}} Fx_1^* \cdot Fx_2.$$

1.6. Definition. Let $F\colon \mathcal{X} \to \mathcal{Y}$ be a sinister morphism and assume that $\mathcal{X}$ has comma objects. We say that F satisfies the *Beck condition* if, for every comma object

$$\begin{tikzcd} X \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\Downarrow\xi"] & X_1 \arrow[d, "x_1"] \\ X_2 \arrow[r, "x_2"'] & X_0, \end{tikzcd}$$

its mate

$$\begin{tikzcd} FX \arrow[r, "Fp_1"] \arrow[dr, phantom, "\Downarrow F(\xi)^*"] & FX_1 \\ FX_2 \arrow[u, "Fp_2^*"] \arrow[r, "Fx_2"'] & FX_0 \arrow[u, "Fx_1^*"'] \end{tikzcd}$$

is invertible. We call $F(\xi)^*$ the *Beck morphism* of ξ.

1.7. PROPOSITION. *Let*

$$\begin{tikzcd} X \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\Downarrow\xi"] & X_1 \arrow[d, "x_1"] \\ X_2 \arrow[r, "x_2"'] & X_0 \end{tikzcd}$$

be a comma square in $\mathcal{X}$ and assume that x_1 has a right adjoint r; then p_2 also has a right adjoint s and the square

$$\begin{tikzcd} X \arrow[r, "p_1"] & X_1 \\ X_2 \arrow[r, "x_2"'] \arrow[u, "s"] & X_0 \arrow[u, "r"'] \end{tikzcd}$$

commutes. The Beck condition for such a comma square is automatically satisfied by any sinister F.

PROOF. Take s to be the unique arrow such that $\xi s = \varepsilon x_2$

$$\begin{tikzcd} X_2 \arrow[r, "x_2"] \arrow[dr, dotted, "s"] \arrow[ddr, bend right, "1_{X_2}"'] & X_0 \arrow[dr, "r"] & \\ & X \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\Downarrow\xi"] & X_1 \arrow[d, "x_1"] \\ & X_2 \arrow[r, "x_2"'] & X_0. \end{tikzcd}$$

We see right away that $p_1 s = r x_2$. Also, $p_2 s = 1_{X_2}$; and this will be the counit ε_{p_2} for the adjunction.

Consider the morphisms $p_1 \xrightarrow{\eta p_1} r x_1 p_1 \xrightarrow{r\xi} r_2 x_2 p_2 = p_1 s p_2$ and $1_{p_2}\colon p_2 \to p_2 s p_2$. x_1 times the first and x_2 times the second commute with ξ in the sense that the outside of the following diagram commutes

$$\begin{tikzcd} x_1 p_1 \arrow[r, "x_1 \eta p_1"] \arrow[dr, equal] \arrow[dd, "\xi"'] & x_1 r x_1 p_1 \arrow[r, "x_1 r \xi"] \arrow[d, "\varepsilon x_1 p_1"] & x_1 r x_2 p_2 \arrow[r, equal] \arrow[d, "\varepsilon x_2 p_2"] & x_1 p_1 s p_2 \arrow[dd, "\xi s p_2"] \\ & x_1 p_1 \arrow[r, "\xi"] & x_2 p_2 \arrow[dr, equal] & \\ x_2 p_2 \arrow[rrr, "1"'] & & & x_2 p_2 s p_2 \end{tikzcd}$$

Thus, by the two-dimensional universal property of comma squares, there exists a unique $\bar{\eta}\colon 1_X \to s p_2$ such that $p_1 \bar{\eta} = r\xi \cdot \eta p_1$ and $p_2 \bar{\eta} = 1_{p_2}$. This last equality is one of the triangle equalities. The other, $\bar{\eta} s = 1_s$, follows from the fact that

$$(r\xi \cdot \eta p_1) s = r\xi s \cdot \eta p_1 s = r\varepsilon x_2 \cdot \eta r x_2 = (r\varepsilon \cdot \eta r) x_2 = 1_{p_1 s}.$$

As homomorphisms preserve adjoints we can choose $F(x_1)^*$ to be $F(r)$ and $F(p_2)^*$ to be $F(s)$ together with F of the corresponding adjunctions. Then the Beck morphism is

$$\begin{tikzcd}
& FX \arrow[rr, "Fp_1"] \arrow[dr, "Fp_2"] \arrow[d, phantom, "\Downarrow F\varepsilon_{p_2}"] & & FX_1 \arrow[rr, equal] \arrow[dr, "Fx_1"'] & & FX_1 \\
FX_2 \arrow[ur, "Fs"] \arrow[rr, equal] & & FX_2 \arrow[rr, "Fx_2"'] \arrow[ur, phantom, "\Downarrow F\xi"] & & FX_0 \arrow[ur, "Fr"'] \arrow[u, phantom, "\Downarrow F\eta_{x_1}"] & .
\end{tikzcd}$$

which by definition of ε_{p_2} is

$$\begin{tikzcd}
& & FX_1 \arrow[rr, equal] \arrow[dr, "Fx_1"'] \arrow[d, phantom, "\Downarrow F\varepsilon_{x_1}"] & & FX_1 \\
FX_2 \arrow[r, "Fx_2"'] & FX_0 \arrow[ur, "Fr"] \arrow[rr, equal] & & FX_0 \arrow[ur, "Fr"'] \arrow[u, phantom, "\Downarrow F\eta_{x_1}"] & .
\end{tikzcd}$$

which is the identity. Other choices for the adjunctions would give an isomorphism. ■

1.8. Remark. In this paper, we will only use Definition 1.6 for locally discrete $\mathcal{X}$, in which case comma objects are pullbacks.

1.9. Lemma. *Let $F, G\colon \mathcal{A} \to \mathcal{B}$ be homomorphisms of bicategories and $t\colon F \to G$ an oplax transformation. For each adjoint pair of arrows $\eta, \varepsilon\colon f \dashv u\colon A \to B$ in $\mathcal{A}$, the mate $(tu)_*$ of tu is the inverse of tf.*

Proof. Since $f \dashv u$, we also have $F(f) \dashv F(u)$ and $G(f) \dashv G(u)$, with adjunctions

$$F(f)F(u) \xrightarrow{\varphi^{-1}} F(fu) \xrightarrow{F(\varepsilon)} F(1_B) \xrightarrow{\varphi_B} 1_{FB},$$

and

$$1_{FA} \xrightarrow{\varphi_A^{-1}} F(1_A) \xrightarrow{F(\eta)} F(uf) \xrightarrow{\varphi} F(u)F(f)\ ,$$

and similarly for G. The composite $(tu)_* \cdot tf$ is given by the following pasting diagram:

$$\begin{tikzcd}
FA \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Uparrow\varphi_A^{-1}"] & FA \arrow[r, equal] \arrow[d, "F(1_A)"] \arrow[dr, phantom, "\Uparrow F\eta"] & FA \arrow[r, "Ff"] \arrow[d, "F(uf)"] \arrow[dr, phantom, "\Uparrow\varphi"] & FB \arrow[r, "t_B"] \arrow[d, "Fu"] \arrow[dr, phantom, "\Uparrow tu"] & GB \arrow[r, equal] \arrow[d, "Gu"] \arrow[dr, phantom, "\Uparrow\psi^{-1}"] & GB \arrow[r, equal] \arrow[d, "G(fu)"] \arrow[dr, phantom, "\Uparrow G\varepsilon"] & GB \arrow[r, equal] \arrow[d, "G(1_B)"] \arrow[dr, phantom, "\Uparrow\psi_B"] & GB \arrow[d, equal] \\
FA \arrow[r, equal] \arrow[d, equal] & FA \arrow[r, equal] \arrow[d, equal] & FA \arrow[r, equal] \arrow[d, equal] & FA \arrow[r, "t_A"'] \arrow[d, equal] & GA \arrow[r, "Gf"'] \arrow[d, phantom, "\Uparrow tf"] & GB \arrow[r, equal] \arrow[d, equal] & GB \arrow[r, equal] \arrow[d, equal] & GB \arrow[d, equal] \\
FA \arrow[r, equal] & FA \arrow[r, equal] & FA \arrow[r, equal] & FA \arrow[r, "Ff"'] & FB \arrow[r, "t_B"'] & GB \arrow[r, equal] & GB \arrow[r, equal] & GB.
\end{tikzcd}$$

By functoriality of t, this pasting is equal to

$$\begin{tikzcd}
FA \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Uparrow\varphi_A^{-1}"] & FA \arrow[r, equal] \arrow[d, "F(1_A)"] \arrow[dr, phantom, "\Uparrow F\eta"] & FA \arrow[r, "Ff"] \arrow[d, "F(uf)"] \arrow[dr, phantom, "\Uparrow\varphi"] & FB \arrow[r, equal] \arrow[d, "Fu"] \arrow[dr, phantom, "\Uparrow\varphi^{-1}"] & FB \arrow[r, "t_B"] \arrow[d, "F(fu)"] \arrow[dr, phantom, "\Uparrow t(fu)"] & GB \arrow[r, equal] \arrow[d, "G(fu)"] \arrow[dr, phantom, "\Uparrow G\varepsilon"] & GB \arrow[r, equal] \arrow[d, "G(1_B)"] \arrow[dr, phantom, "\Uparrow\psi_B"] & GB \arrow[d, equal] \\
FA \arrow[r, equal] & FA \arrow[r, equal] & FA \arrow[r, equal] & FA \arrow[r, "Ff"'] & FB \arrow[r, "t_B"'] & GB \arrow[r, equal] & GB \arrow[r, equal] & GB,
\end{tikzcd}$$

and by naturality of t, this equals

$$\begin{tikzcd}
FA \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Uparrow\varphi_A^{-1}"] & FA \arrow[r, equal] \arrow[d, "F(1_A)"] \arrow[dr, phantom, "\Uparrow F\eta"] & FA \arrow[r, "Ff"] \arrow[d, "F(uf)"] \arrow[dr, phantom, "\Uparrow\varphi"] & FB \arrow[r, equal] \arrow[d, "Fu"] \arrow[dr, phantom, "\Uparrow\varphi^{-1}"] & FB \arrow[r, equal] \arrow[d, "F(fu)"] \arrow[dr, phantom, "\Uparrow F\varepsilon"] & FB \arrow[r, "t_B"] \arrow[d, "F(1_B)"] \arrow[dr, phantom, "\Uparrow t(1_B)"] & GB \arrow[r, equal] \arrow[d, "G(1_B)"] \arrow[dr, phantom, "\Uparrow\psi_B"] & GB \arrow[d, equal] \\
FA \arrow[r, equal] & FA \arrow[r, equal] & FA \arrow[r, equal] & FA \arrow[r, "Ff"'] & FB \arrow[r, equal] & FB \arrow[r, "t_B"'] & GB \arrow[r, equal] & GB.
\end{tikzcd}$$

By the triangle equality, this is equal to

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, equal] & FB \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Uparrow\varphi_B^{-1}"] & FB \arrow[r, "t_B"] \arrow[d, "F(1_B)"] \arrow[dr, phantom, "\Uparrow t(1_B)"] & GB \arrow[r, equal] \arrow[d, "G(1_B)"] \arrow[dr, phantom, "\Uparrow\psi_B"] & GB \arrow[d, equal] \\
FA \arrow[r, "Ff"'] & FB \arrow[r, equal] & FB \arrow[r, "t_B"'] & GB \arrow[r, equal] & GB
\end{tikzcd}$$

which equals $\mathrm{id}_{t_B \circ Ff}$. The fact that the other composite $tf \cdot (tu)_*$ is equal to $\mathrm{id}_{t_B \circ Ff}$ can be proved in a similar way. ∎

1.10. PROPOSITION. *Let* $\mathbf{A}$ *be a category with pullbacks and* $\mathcal{B}$ *any bicategory. Then composition with* $(\ )_*\colon \mathbf{A} \to \mathbf{Span}(\mathbf{A})$ *induces an equivalence of categories between the category* $\mathbf{Hom}(\mathbf{Span}(\mathbf{A}), \mathcal{B})$ *of homomorphisms* $\mathbf{Span}(\mathbf{A}) \to \mathcal{B}$ *with oplax transformations, and the category* $\mathbf{Beck}(\mathbf{A}, \mathcal{B})$ *of sinister morphisms* $\mathbf{A} \to \mathcal{B}$ *satisfying the Beck condition with strong transformations as morphisms. Under this equivalence, strong transformations in* $\mathbf{Hom}(\mathbf{Span}(\mathbf{A}), \mathcal{B})$ *correspond to sinister transformations in* $\mathbf{Beck}(\mathbf{A}, \mathcal{B})$.

PROOF. Given a homomorphism $F : \mathbf{Span}(\mathbf{A}) \to \mathcal{B}$, the composite

$$\mathbf{A} \xrightarrow{(\ )_*} \mathbf{Span}(\mathbf{A}) \xrightarrow{F} \mathcal{B}$$

is sinister and satisfies the Beck condition (because $(\ )_*$ does and F is a homomorphism). If $t\colon F \to G$ is an oplax transformation, Lemma 1.9 gives that for any arrow f in $\mathbf{A}$, the mate of $t(f^*)$ is the inverse of $t(f_*)$, so $t \circ (\ )_*$ becomes a strong transformation. Moreover, if $t\colon F \to G$ is a strong transformation, Lemma 1.9 implies that the mate of $t(f_*)^{-1}$ is $t(f^*)$ and therefore is invertible; so $t \circ (\ )_*$ is sinister.

This gives a functor

$$\mathbf{Hom}(\mathbf{Span}(\mathbf{A}), \mathcal{B}) \to \mathbf{Beck}(\mathbf{A}, \mathcal{B})$$

which we now show to be full and faithful. Let $u\colon F \circ (\ )_* \to G \circ (\ )_*$ be a strong transformation. We wish to extend it to an oplax $t\colon F \to G$. There is no choice for t on objects nor on morphisms of the form f_*. But $t(f^*)$ must be the inverse of the mate of $t(f_*)^{-1}$ and so is also uniquely determined. Finally $t(f_* \otimes g^*)$ is uniquely determined by pasting $t(f_*)$ with $t(g^*)$, thus

$$t(f_* \otimes g^*) = \begin{tikzcd} {} \arrow[r, equal] \arrow[dd] \arrow[ddr, phantom, "\cong"] & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Uparrow((ug)^{-1})^*"] & {} \arrow[r, equal] \arrow[d] \arrow[ddr, phantom, "\cong"] & {} \arrow[dd] \\ & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Uparrow uf"] & {} \arrow[d] & \\ {} \arrow[r, equal] & {} \arrow[r] & {} \arrow[r, equal] & {} \end{tikzcd} .$$

Thus t is defined on all of $\mathbf{Span}(\mathbf{A})$. Checking that t is an oplax transformation is a straightforward computation which we omit. If u is sinister, all the 2-cells in this calculation become isomorphisms and consequently t thus defined is a strong transformation.

Finally we must show that every sinister $H\colon \mathbf{A} \to \mathbf{B}$ satisfying the Beck condition is isomorphic to some $F \circ (\ )_*$. Define $F\colon \mathbf{Span}(\mathbf{A}) \to \mathcal{B}$ by $F(A) = H(A)$ and $F(\ A \xleftarrow{p} S \xrightarrow{q} B\ ) = H(q)H(p)^*$, where $H(p)^*$ is a chosen right adjoint for $H(p)$. Given a 2-cell

$$\begin{tikzcd} & S \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, "s"] & \\ A & & B \\ & S' \arrow[ul, "p'"] \arrow[ur, "q'"'] & \end{tikzcd} ,$$

$F(s)$ is

$$H(q)H(p)^* = H(q's)H(p's)^* \cong H(q')H(s)H(s)^*H(p')^* \xrightarrow{H(q')\varepsilon_s H(p')} H(q')H(p').$$

That F is functorial on 2-cells is another straightforward calculation involving all the coherence conditions on H. Again, we omit the details. The Beck condition ensures that F is a homomorphism of bicategories. ∎

1.11. Remark. As an application of this proposition we get the fact that a cocomplete $\mathbf{S}$-indexed category can be represented by a homomorphism $\mathbf{Span}(\mathbf{S}) \to \mathbf{Cat}$ (cf. [BW]).

2. Jointed Oplax Morphisms

As defined in [DPP], $\Pi_2\mathbf{A}$ is the free 2-category generated by $\mathbf{A}$ and right adjoints for all morphisms of $\mathbf{A}$. A special instance of this construction for the category $\mathbb{2} = (\ A \xrightarrow{f} B\ )$ was presented by Schanuel and Street in [SS]. This example shows very clearly that in general $\Pi_2\mathbf{A}$ and $\mathbf{Span}(\mathbf{A})$ are not equivalent. The bicategory $\mathbf{Span}\,(\mathbb{2})$ has only three nontrivial arrows, f_*, f^*, and f_*f^* as shown; and one nontrivial 2-cell $\varepsilon\colon f_*f^* \Rightarrow 1_B$.

$$\begin{tikzcd} A \arrow[r, bend left, "f_*"] & B \arrow[l, bend left, "f^*"] \arrow[loop right, "\Leftarrow\ f_*f^*"] \end{tikzcd} \tag{4}$$

However, $\Pi_2(\mathbb{2})$ (which Schanuel and Street call $\mathbf{Adj}$), has, for instance, $f^*f_* \ncong 1_A$, and in fact there are infinitely many non-isomorphic arrows and infinitely many 2-cells.

In this paper we are mainly interested in bicategories and only need to work up to the level of bicategorical equivalence. So, while $\Pi_2\mathbf{A}$, like any 2-category, is a bicategory, we may represent it in this context by any equivalent bicategory. Many of the technicalities in the original definition of $\Pi_2\mathbf{A}$ were only there to make horizontal composition associative and unitary "on the nose" and make the inclusion functor $(\ )_*\colon \mathbf{A} \to \Pi_2\mathbf{A}$ a 2-functor. Here we may simplify calculations considerably by choosing a bicategory from the class of those equivalent to $\Pi_2\mathbf{A}$ which makes the relationship with $\mathbf{Span}(\mathbf{A})$ more transparent.

The bicategory we use, which we shall by abuse of notation also refer to as $\Pi_2\mathbf{A}$, is defined as follows:

Its objects are those of $\mathbf{A}$.

An arrow $A \to B$ is a nonempty path of spans in $\mathbf{A}$,

$$A = A_0 \leftarrow S_1 \rightarrow A_1 \leftarrow S_2 \rightarrow A_2 \cdots A_{n-1} \leftarrow S_n \rightarrow A_n = B \tag{5}$$

We say that this is a path of length n.

The 2-cells are represented by equivalence classes of certain commutative diagrams called fences which look like

$$\begin{tikzcd} A \arrow[d, equal] & A_0 \arrow[d, "y_0"'] & S_1 \arrow[l] \arrow[r] \arrow[d, "x_1"'] & A_1 \arrow[d, "y_1"'] & S_2 \arrow[l] \arrow[r] \arrow[d, "x_2"'] \arrow[drr, "x_3"'] & A_2 = B \arrow[drr, "y_2"] \arrow[drrr, equal] & & & \\ A & B_0 & T_1 \arrow[l] \arrow[r] & B_1 & T_2 \arrow[l] \arrow[r] & B_2 & T_3 \arrow[l] \arrow[r] & B_3 & B \arrow[l, equal] \end{tikzcd} \tag{6}$$

The y_i and x_j are indexed as follows. A fence from a path of length m to one of length n is a triple $(\phi, \langle y_i \rangle, \langle x_j \rangle)$, where

$$\phi : \{0, \ldots, m\} \to \{0, \ldots, n\}$$

is an order-preserving map such that $\phi(0) = 0$ and $\phi(m) = n$. For each $0 \le i \le m$, $y_i\colon A_i \to B_{\phi(i)}$, with $y_0 = 1_A$ and $y_m = 1_B$. As ϕ preserves the top element, it has a left adjoint ψ and for each $0 \le j \le n$, $x_j\colon S_{\psi(j)} \to T_j$. The equivalence relation is generated by identifying two fences if they differ only by

$$\begin{tikzcd} S_i \arrow[r] & A_i \arrow[dl, "h_i"'] \arrow[d, dotted, "l"] & S_{i+1} \arrow[l] \arrow[dl, "x_j"] \\ B_{j-1} & T_j \arrow[l] \arrow[r] & B_j \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} S_i \arrow[r] \arrow[dr, "x'_j"'] & A_i \arrow[d, dotted, "l"] \arrow[dr, "h'_i"] & S_{i+1} \arrow[l] \\ B_{j-1} & T_j \arrow[l] \arrow[r] & B_j \end{tikzcd}$$

and there exists an l factoring both parallelograms.

Horizontal composition is by concatenation and vertical composition by composing the individual components of fences

$$(\phi', \langle y'_j \rangle, \langle x'_k \rangle)(\phi, \langle y_i \rangle, \langle x_j \rangle) = (\phi'\phi, \langle y'_{\phi(i)} y_i \rangle, \langle x'_k x_{\psi(k)} \rangle).$$

In a forthcoming paper we will discuss the properties of this bicategory in more detail. For now we only need that there is a homomorphism of bicategories $\Upsilon \colon \Pi_2\mathbf{A} \to \mathbf{Span}(\mathbf{A})$, for which we will give a direct construction.

On objects Υ is the identity. Υ takes an arrow, represented by a path as in (5) to the inverse limit, or generalized pullback, of the path:

$$S_1 \times_{A_1} S_2 \times \cdots \times S_{n-1} \times_{A_{n-1}} S_n \colon A \longrightarrow\!\!\!\!\!\!\!\!+\;\; B.$$

It can be computed using pullbacks. Given another path

$$A = B_0 \leftarrow T_1 \rightarrow B_1 \leftarrow T_2 \rightarrow B_2 \cdots B_{m-1} \leftarrow T_m \rightarrow B_m = B$$

and a fence $(\phi, \langle y_i \rangle, \langle x_j \rangle)$ between them, Υ associates to it the morphism of spans given by the universal property of the limit

$$(x_j p_{\phi^*(j)}) \colon \Pi_{A_i} S_i \to \Pi_{B_j} T_j$$

where $p_i \colon \Pi_{A_i} S_i \to S_i$ is the projection.

For example, for the fence (6) we get

$$S_1 \times_{A_1} S_2 \xrightarrow{(x_1p_1, x_2p_2, x_3p_2)} T_1 \times_{B_1} T_2 \times_{B_2} T_3.$$

If we have a different fence $(\phi', \langle h'_i \rangle, \langle x'_j \rangle)$, differing only by

$$\begin{tikzcd}
S_i \arrow[r] & A_i \arrow[dl, "h_i"'] \arrow[d, dotted, "l"'] & S_{i+1} \arrow[l] \arrow[dl, "x_j"] \\
B_{j-1} & T_j \arrow[l] \arrow[r] & B_j
\end{tikzcd}
\qquad
\begin{tikzcd}
S_i \arrow[r] \arrow[dr, "x'_j"'] & A_i \arrow[d, dotted, "l"'] & S_{i+1} \arrow[l] \\
B_{j-1} & T_j \arrow[l] \arrow[r] & B_j
\end{tikzcd}$$

with the same l factoring both parallelograms, the induced morphism will be the same except possibly in the j^{th} component. But the commutative diagram

$$\begin{tikzcd}
& \prod_{A_i} S_i \arrow[dl, "p_i"'] \arrow[dr, "p_{i+1}"] & \\
S_i \arrow[r] \arrow[dr, "x'_j"'] & A_i \arrow[d, "l"] & S_{i+1} \arrow[l] \arrow[dl, "x_j"] \\
& T_j &
\end{tikzcd}$$

shows that in fact they are equal in the j^{th} component as well, so that Υ is well-defined on 2-cells of $\Pi_2\mathbf{A}$. It is not hard to check that $\Upsilon\colon \Pi_2\mathbf{A} \to \mathbf{Span}(\mathbf{A})$ is a homomorphism of bicategories. $\mathbf{Span}(\mathbf{A})$ is contained in $\Pi_2\mathbf{A}$, but not as a sub-bicategory.

Recall the definition of oplax morphism of bicategories (the vertical dual of Bénabou's morphism of bicategories [Bé1]). An *oplax morphism* $\Phi\colon \mathcal{A} \to \mathcal{B}$ of bicategories takes an object A of $\mathcal{A}$ to an object $\Phi(A)$ of $\mathcal{B}$ and for all pairs A, B of objects we have a functor

$$\Phi_{A,B}\colon \mathcal{A}(A,B) \to \mathcal{B}(\Phi A, \Phi B).$$

For each object A we are given a morphism

$$\varphi_A\colon \Phi(1_A) \to 1_{\Phi A}$$

and for each pair of morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ a morphism

$$\varphi_{g,f}\colon \Phi(gf) \to \Phi(g)\Phi(f).$$

The φ_A and $\varphi_{g,f}$ satisfy the following naturality and coherence conditions:

- φ_A is natural in A.
- $\varphi_{g,f}$ is natural in g and f.
- The diagram

$$\begin{tikzcd} \Phi(hgf) \arrow[r, "\varphi_{hg,f}"] \arrow[d, "\varphi_{h,gf}"'] & \Phi(hg)\Phi(f) \arrow[d, "\varphi_{h,g}\Phi(f)"] \\ \Phi(h)\Phi(gf) \arrow[r, "\Phi(h)\varphi_{g,f}"'] & \Phi(h)\Phi(g)\Phi(f) \end{tikzcd}$$

  commutes.
- The diagram

$$\begin{tikzcd} \Phi(f1_A) \arrow[r, "\varphi_{f,1_A}"] \arrow[dr, "\cong"'] & \Phi(f)\Phi(1_A) \arrow[d, "\Phi(f)\varphi_A"] \\ & \Phi(f)1_{\Phi(A)} \end{tikzcd}$$

  commutes.
- The diagram

$$\begin{tikzcd} \Phi(1_Bf) \arrow[r, "\varphi_{1_B,f}"] \arrow[dr, "\cong"'] & \Phi(1_B)\Phi(f) \arrow[d, "\varphi_B\Phi(f)"] \\ & 1_{\Phi(B)}\Phi(f) \end{tikzcd}$$

  commutes.

We say that Φ is *normal* if all the φ_A are isomorphisms. Note that from now on *all our oplax morphisms will be assumed to be normal.*

The inclusion $\Psi\colon \mathbf{Span}(\mathbf{A}) \to \Pi_2\mathbf{A}$ (which forms a splitting for Υ) is an oplax morphism of bicategories. Again, on the objects it is the identity. It takes a span $A \leftarrow S \to B$ to itself considered as a path of length one. A morphism of spans $s\colon S \to S'$ is sent to the equivalence class of fences determined by

$$\begin{tikzcd}
A \arrow[d, equal] & S \arrow[l] \arrow[r] \arrow[d, "s"'] & B \arrow[d, equal] \\
A & S' \arrow[l] \arrow[r] & B
\end{tikzcd}$$

(which consists only of that one fence).

2.1. Proposition. *Ψ is a locally fully faithful normal oplax morphism* $\mathbf{Span}(\mathbf{A}) \to \Pi_2\mathbf{A}$.

Proof. That the morphism Ψ is locally fully faithful and normal is obvious. The morphisms $\Psi(T \otimes S) \to \Psi(T)\Psi(S)$ making Ψ oplax are

$$\begin{tikzcd}
A \arrow[d, equal] & & S \times_B T \arrow[ll] \arrow[rr] \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & C \arrow[d, equal] \\
A & S \arrow[l] \arrow[r] & B & T \arrow[l] \arrow[r] & C.
\end{tikzcd}$$

The associativity and unit conditions are easily checked. This is routine except for the non-emptiness of identity paths; we illustrate how this is handled for one of the unit laws. We must show that

$$\begin{tikzcd}
\Psi(I_B \otimes S) \arrow[r, "\Psi(\lambda_S)"] \arrow[d] & \Psi(S) \\
\Psi(I_B)\Psi(S) \arrow[r, "="'] & I_B\Psi(S) \arrow[u, "\lambda'_{\Psi(S)}"']
\end{tikzcd}$$

commutes.

When we define $\mathbf{Span}(\mathbf{A})$ we first make an arbitrary choice of pullbacks for all pairs of morphisms but we may as well make life easier by choosing pullbacks of identities to be identities. Then the top map of our diagram is the identity. Going around the long way gives

$$\begin{tikzcd}
A \arrow[d, equal] & & S \arrow[ll] \arrow[rr] \arrow[dl, "1"'] \arrow[dr] & & B \arrow[d, equal] \\
A \arrow[d, equal] & S \arrow[l] \arrow[r] \arrow[dr, "1"'] & B & B \arrow[l] \arrow[r] & B \arrow[d, equal] \\
A & & S \arrow[ll] \arrow[rr] & & B
\arrow[from=2-3, to=3-5, "1"]
\end{tikzcd}$$

which is the identity on S. ∎

2.2. Proposition. *Ψ is a local left adjoint for Υ, i.e. the functors*

$$\Pi_2(\mathbf{A})(A,B) \underset{\Psi_{A,B}}{\overset{\Upsilon_{A,B}}{\rightleftarrows}} \mathbf{Span}(\mathbf{A})(A,B)$$

obtained by applying Υ and Ψ are adjoints

$$\Psi_{A,B} \dashv \Upsilon_{A,B}.$$

Proof. Consider an object $A \leftarrow S \rightarrow B$ of $\mathbf{Span}(A,B)$ and $A \leftarrow T_1 \rightarrow B_1 \leftarrow T_2 \rightarrow B_2 \cdots T_n \rightarrow B$ an object of $\Pi_2(\mathbf{A})(A,B)$. In this case, a morphism $\Psi_{A,B}(S) \rightarrow \langle B_j, T_j \rangle$ is a diagram of the form

$$\begin{tikzcd}
 & & A \arrow[dll, equal] & S \arrow[l] \arrow[r] \arrow[dll] \arrow[d] \arrow[drrr] & B \arrow[drrr, equal] & & & \\
A & T_1 \arrow[l] \arrow[r] & B_1 & T_2 \arrow[l] \arrow[r] & B_2 & \cdots & T_n \arrow[r] & B
\end{tikzcd}$$

as there is only one choice for the indexing function and no possibility of applying the equivalence relation. This is clearly the same as a morphism of spans

$$\begin{tikzcd}
A \arrow[d, equal] & S \arrow[l] \arrow[r] \arrow[d] & B \arrow[d, equal] \\
A & \prod_{B_j} T_j \arrow[l] \arrow[r] & B,
\end{tikzcd}$$

i.e., a morphism $S \rightarrow \Upsilon_{A,B}\langle B_j, T_j \rangle$. ∎

The composite $\Psi\Upsilon$ gives an oplax comonad on $\Pi_2\mathbf{A}$ which is idempotent and the identity on objects. Lax monads on bicategories were considered by Carboni and Rosebrugh [CR] as an extension of Kock's work on tensor products in categories of algebras over a monoidal category [Ko]. They show that given a lax monad on a bicategory which is the identity on objects, a new bicategory can be formed with the same objects, whose arrows are algebras and composition given by a tensor product which classifies "bilinear" cells. This tensor product is defined as a joint coequalizer of 2-cells which is assumed to exist and be preserved by the monad. Our situation is dual with the 2-cells reversed. Local equalizers don't exist in $\Pi_2\mathbf{A}$ but as our comonad is idempotent the arrows to be equalized are already equal (this point was already made in [RSW] in their Proposition 43 and the remark before it), so those equalizers do exist and are preserved by the comonad. Thus we obtain $\mathbf{Span}(\mathbf{A})$ as coalgebras for the oplax comonad $\Psi\Upsilon$ on $\Pi_2\mathbf{A}$. We summarize this in the following.

2.3. Proposition. $\mathbf{Span}(\mathbf{A})$ *is equivalent to the Eilenberg-Moore category* $(\Pi_2\mathbf{A})_{\Psi\Upsilon}$ *for the idempotent oplax comonad $\Psi\Upsilon$ on $\Pi_2\mathbf{A}$.*

The existence of Ψ hints at another possible universal property of $\mathbf{Span}(\mathbf{A})$, namely with respect to normal oplax morphisms. But these don't preserve adjoints in general, so the value of such a morphism on a span would not be determined by its values on arrows coming from $\mathbf{A}$. A closer look at Ψ shows that it preserves some compositions, in particular those of the form $f_* \otimes S$. By this we mean that the canonical morphism $\Psi(f \otimes S) \to \Psi(f_*)\Psi(S)$ is an isomorphism. We show something a bit stronger in Proposition 2.9 below. This leads to the following definition.

2.4. Definition. We say that a (normal) oplax morphism of bicategories $\Phi : \mathcal{A} \to \mathcal{B}$ is *jointed* if for any pair of morphisms $f : A \to B$ and $g : B \to C$ for which g has a right adjoint, the structure cell $\phi_{g,f} \colon \Phi(gf) \to \Phi(g)\Phi(f)$ is an isomorphism.

Note that we do not require Φ to preserve composites gf where f has a left adjoint; we will show below that this follows automatically.

Note also that jointedness can be defined identically for lax morphisms. If the structural cells $\psi_A \colon \Phi(1_A) \to 1_{\Phi A}$ and (when g has a right adjoint) $\psi_{g,f} \colon \Psi(g)\Psi(f) \to \Psi(gf)$ are isomorphisms, then their inverses are isomorphisms exactly as required for a jointed oplax morphism; the difference rests entirely in the direction of the "ordinary" structural cells.

The usefulness of this condition has already been noted by several authors. The authors of [CKW] study morphisms of bicategories that are weaker than lax or oplax. Their "flabby" morphisms have laxity for composition on the left, and oplaxity for composition on the right, by *maps*. They observe that, when there is full oplaxity, they get what we call jointedness. However, they only consider the locally-ordered case. In [CKVW], they develop a very nice general theory (not just locally-ordered) in which they consider all combinations of laxity and oplaxity. They do a detailed study of the functorial properties of the **Span** construction but do not consider its universal property. In [DMS], Day, McCrudden, and Street use jointedness; Clementino, Hofmann, and Tholen have also noted the usefulness of the property (see [CHT]).

2.5. Example. Let R be a commutative ring and consider the contravariant functor $\mathrm{Hom}(-, R) : R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$. There is a canonical morphism for any R-modules A and B,

$$\varphi_{A,B} : \mathrm{Hom}(A, R) \otimes \mathrm{Hom}(B, R) \to \mathrm{Hom}(A \otimes B, R)$$

$$f \otimes g \longmapsto (A \otimes B \xrightarrow{f\otimes g} R \otimes R \xrightarrow{\mu} R).$$

If we make $R - \mathbf{Mod}$ into a one-object bicategory $\mathcal{M}$ we get an oplax morphism

$$(\ )^* \colon \mathcal{M} \to \mathcal{M}^{co}$$

It is normal as $R^* \cong R$. Moreover, it is jointed: A has a right adjoint if and only if it is finitely generated and projective, and it is well known that in that case, $\varphi_{A,B}$ is an isomorphism for all B.

2.6. Example. Closer to our discussion is the following. Let $\mathbf{A}$ and $\mathbf{B}$ be categories with pullbacks and $F : \mathbf{A} \to \mathbf{B}$ an arbitrary functor. We get an oplax morphism

$$
\begin{array}{ccc}
\mathbf{Span}(F)\colon \mathbf{Span}(\mathbf{A}) & \longrightarrow & \mathbf{Span}(\mathbf{B}) \\
\begin{tikzcd}[column sep=small] & S \arrow[dl, "p"'] \arrow[dr, "g"] & \\ A & & A' \end{tikzcd} & \longmapsto & \begin{tikzcd}[column sep=small] & FS \arrow[dl, "Fp"'] \arrow[dr, "Fg"] & \\ FA & & FA'. \end{tikzcd}
\end{array}
$$

The structure morphisms expressing oplaxity are given by the universal property of pullback

$$
\begin{tikzcd}
 & & F(S \times_{A'} S') \arrow[dl] \arrow[dr] \arrow[d, dotted, "\varphi"] & & \\
 & & FS \times_{FA'} FS' \arrow[dl] \arrow[dr] & & \\
 & FS \arrow[dl] \arrow[dr] \arrow[rr, phantom, "\text{p.b.}"] & & FS' \arrow[dl] \arrow[dr] & \\
FA & & FA' & & FA''.
\end{tikzcd}
$$

Clearly, if F preserves pullbacks then all the φ are isomorphisms and $\mathbf{Span}(F)$ is a homomorphism. But $A' \xleftarrow{p'} S' \xrightarrow{q'} A$ has a right adjoint if and only if p' is an isomorphism and any F will preserve the pullback

$$
\begin{tikzcd}[column sep=small]
 & S \times_{A'} S' \arrow[dl] \arrow[dr] & \\
S \arrow[dr, "q"'] & & S' \arrow[dl, "p'"] \\
 & A' &
\end{tikzcd}
$$

when p' is an isomorphism. So $\mathbf{Span}(F)$ is a jointed oplax morphism.

2.7. Proposition. *Let $\Phi\colon \mathcal{A} \to \mathcal{B}$ be oplax (and normal) and suppose that $f \dashv u$, $\varepsilon\colon fu \to 1_B$, $\eta\colon 1_A \to uf$ is an adjunction. If there is a 2-cell $\overline{\varepsilon}\colon \Phi(f)\Phi(u) \to 1_{\Phi(B)}$ such that*

$$
\begin{tikzcd}
\Phi(fu) \arrow[r, "\Phi(\varepsilon)"] \arrow[d, "\varphi_{f,u}"'] & \Phi(1_B) \arrow[d, "\varphi_B"] \\
\Phi(f)\Phi(u) \arrow[r, "\overline{\varepsilon}"'] & 1_{\Phi(B)}
\end{tikzcd}
\tag{7}
$$

commutes, then $\Phi(f) \dashv \Phi(u)$ with counit $\overline{\varepsilon}$.

Proof. Let $\overline{\eta}$ be the unique arrow such that

$$
\begin{tikzcd}
\Phi(1_A) \arrow[r, "\Phi(\eta)"] \arrow[d, "\varphi_A"', "\cong"] & \Phi(uf) \arrow[d, "\varphi_{u,f}"] \\
1_{\Phi(A)} \arrow[r, "\overline{\eta}"'] & \Phi(u)\Phi(f);
\end{tikzcd}
\tag{8}
$$

then

$$
\begin{tikzcd}
\Phi(f) \arrow[rrrrrr, equals] \arrow[ddd, equals] \arrow[dr, "\cong"] & & & & & & \Phi(f) \arrow[ddd, equals] \\
& \Phi(f1_A) \arrow[rr, "\Phi(f\eta)"] \arrow[d, "\varphi_{f,1_A}"'] \arrow[dr, phantom, "\boxed{\text{Nat}}"] & & \Phi(fuf) \arrow[rr, "\Phi(\varepsilon f)"] \arrow[dl, "\varphi_{f,uf}"'] \arrow[dr, "\varphi_{fu,f}"] & & \Phi(1_B f) \arrow[ur, "\cong"'] \arrow[d, "\varphi_{1_B,f}"] \arrow[dl, phantom, "\boxed{\text{Nat}}"] & \\
& \Phi(f)\Phi(1_A) \arrow[r, "\Phi(f)\Phi(\eta)"'] \arrow[d, "\Phi(f)\varphi_A"'] \arrow[dr, phantom, "\boxed{(8)}"] & \Phi(f)\Phi(uf) \arrow[rr, phantom, "\boxed{\text{coh}}"] \arrow[dr, "\Phi(f)\varphi_{u,f}"'] & & \Phi(fu)\Phi(f) \arrow[r, "\Phi(\varepsilon)\Phi(f)"'] \arrow[dl, "\varphi_{f,u}\Phi(f)"] & \Phi(1_B)\Phi(f) \arrow[d, "\varphi_B\Phi(f)"] \arrow[dl, phantom, "\boxed{(7)}"] & \\
\Phi(f) \arrow[r, "\cong"'] & \Phi(f)1_{\Phi A} \arrow[rr, "\Phi(f)\overline{\eta}"'] & & \Phi(f)\Phi(u)\Phi(f) \arrow[rr, "\overline{\varepsilon}\Phi(f)"'] & & 1_{\Phi B}\Phi(f) \arrow[r, "\cong"'] & \Phi(f)
\end{tikzcd}
\tag{9}
$$

shows one of the triangle equalities. The other triangle equality is obtained in a similar fashion. ∎

We say that the morphism Φ *preserves the adjunction* $f \dashv u$ when the condition of this proposition is satisfied, and Φ *preserves adjoints* if it preserves all adjunctions.

2.8. Proposition. *Jointed oplax morphisms of bicategories preserve adjoints.*

Proof. Let the oplax morphism $\Phi\colon \mathcal{A} \to \mathcal{B}$ be jointed, and let $f\colon A \to X$, $u\colon X \to A$, $\varepsilon\colon fu \to 1_X$, $\eta\colon 1_A \to uf$ be the data for an adjunction. We may take

$$
\overline{\varepsilon} = \Phi(f)\Phi(u) \xrightarrow{\varphi_{f,u}^{-1}} \Phi(fu) \xrightarrow{\Phi(\varepsilon)} \Phi(1_X) \xrightarrow{\varphi_X} 1_{\Phi(X)}.
$$

∎

2.9. Proposition. *If Φ preserves adjoints and is normal, then it is jointed.*

Proof. Suppose f has a right adjoint u with adjunctions ε and η as usual. Let g be any morphism composable with f. We will show that the composite

$$
\Phi(f)\Phi(g) \xrightarrow{\Phi(f)\Phi(\eta g)} \Phi(f)\Phi(ufg) \xrightarrow{\Phi(f)\varphi_{u,fg}} \Phi(f)\Phi(u)\Phi(fg) \xrightarrow{\overline{\varepsilon}\Phi(fg)} \Phi(fg)
$$

is inverse to $\varphi_{f,g}$. Indeed

$$
\begin{tikzcd}
\Phi(f)\Phi(g) \arrow[r, "\Phi(f)\Phi(\eta g)"] & \Phi(f)\Phi(ufg) \arrow[r, "\Phi(f)\varphi_{u,fg}"] & \Phi(f)\Phi(u)\Phi(fg) \arrow[r, "\overline{\varepsilon}\Phi(fg)"] & \Phi(fg) \\
\Phi(fg) \arrow[u, "\varphi_{f,g}"] \arrow[r, "\Phi(f\eta g)"'] \arrow[ur, phantom, "\boxed{\text{Nat}}"] \arrow[drr, bend right, "\mathrm{id}_{\Phi(fg)}"'] & \Phi(fufg) \arrow[u, "\varphi_{f,ufg}"] \arrow[r, "\delta_{fu,fg}"'] \arrow[ur, phantom, "\boxed{\text{Assoc}}"] \arrow[dr, "\Phi(\varepsilon fg)"] \arrow[d, phantom, "\boxed{\Delta =}"] & \Phi(fu)\Phi(fg) \arrow[u, "\varphi_{f,u}\Phi(fg)"] \arrow[r, "\Phi(\varepsilon)\Phi(fg)"'] \arrow[ur, phantom, "\boxed{(7)}"] \arrow[d, phantom, "\boxed{\text{Nat}}"] & \Phi(1_B)\Phi(fg) \arrow[u, "\varphi_B\Phi(fg)"'] \\
& & \Phi(1_B fg) \arrow[ur, "\varphi_{1_B,fg}"'] &
\end{tikzcd}
$$

commutes for the indicated reasons and the composite of the φ's on the right is the identity (unit law for oplax morphisms). We also have

$$
\begin{tikzcd}
\Phi(f)\Phi(g) \arrow[r, "\Phi(f)\Phi(\eta g)"] \arrow[d, "\Phi(f)\varphi_{1,g}"'] \arrow[dr, phantom, "\boxed{\text{Nat}}"] & \Phi(f)\Phi(ufg) \arrow[r, "\Phi(f)\varphi_{u,fg}"] \arrow[d, "\Phi(f)\varphi_{uf,g}"'] \arrow[dr, phantom, "\boxed{\text{Assoc}}"] & \Phi(f)\Phi(u)\Phi(fg) \arrow[r, "\bar{\varepsilon}\Phi(fg)"] \arrow[d, "\Phi(f)\Phi(u)\varphi_{f,g}"] \arrow[dr, phantom, "\boxed{\text{Nat}}"] & \Phi(fg) \arrow[d, "\varphi_{f,g}"] \\
\Phi(f)\Phi(1_A)\Phi(g) \arrow[r, "\Phi(f)\Phi(\eta)\Phi(g)"'] \arrow[dr, "\Phi(f)\varphi_A\Phi(g)"'] & \Phi(f)\Phi(uf)\Phi(g) \arrow[r, "\Phi(f)\varphi_{u,f}\Phi(g)"'] \arrow[d, phantom, "\boxed{(8)}"] & \Phi(f)\Phi(u)\Phi(f)\Phi(g) \arrow[r, "\bar{\varepsilon}\Phi(f)\Phi(g)"'] \arrow[d, phantom, "\boxed{(9)}"] & \Phi(f)\Phi(g) \\
& \Phi(f)\Phi(g) \arrow[ur, "\Phi(f)\bar{\eta}\Phi(g)"'] \arrow[urr, bend right, "\mathrm{id}_{\Phi(f)\Phi(g)}"'] & &
\end{tikzcd}
$$

and the composite of φ's on the left is the identity by the unit law for oplax morphisms. This shows the Φ preserves composites when the second arrow (diagrammatically) has a right adjoint. Thus Φ is jointed. ∎

We have shown that, for normal oplax morphisms, jointedness is equivalent to preservation of adjoints, which is a self-dual property (reversing arrows). This gives the following corollary.

2.10. Corollary. *A normal oplax morphism Φ is jointed if and only if for every f and g where f has a left adjoint, $\varphi_{g,f}\colon \Phi(gf) \to \Phi(g)\Phi(f)$ is an isomorphism.*

The following example shows that the condition that Φ be normal is necessary in Proposition 2.9.

2.11. Example. *Let G be a comonad which is adjoint to itself,* e.g.,

$$
\begin{tikzcd}
\mathbf{Ab} \arrow[loop right, "G"]
\end{tikzcd}
\qquad G\colon A \mapsto A \oplus A.
$$

This comonad gives rise to an oplax morphism $\mathbf{1} \to \mathbf{Cat}$ which is not normal and preserves adjoints. However, it doesn't preserve composition with adjoints, since $1 \circ 1 = 1$, but $G \circ G \not\cong G$.

2.12. Proposition. $\Psi\colon \mathbf{Span}(\mathbf{A}) \to \Pi_2\mathbf{A}$ *is jointed.*

Proof. Let $A \overset{S}{\to} B \overset{T}{\to} C$ with T a left adjoint, so that the h in the diagram below is an isomorphism. Then $\varphi_{S,T}$ is represented by the bottom fence in the following diagram, and $\varphi^{-1}_{S,T}$ the top one.

$$
\begin{tikzcd}
A \arrow[d, equal] & S \arrow[l, "f"'] \arrow[r, "g"] \arrow[dr, "1"] & B \arrow[drr, "kh^{-1}"] & T \arrow[l, "h"'] \arrow[r, "k"] & C \arrow[d, equal] \\
A \arrow[d, equal] & & S \arrow[ll, "f"] \arrow[rr, "kh^{-1}g"'] \arrow[dl, "1"'] \arrow[dr, "h^{-1}g"] & & C \arrow[d, equal] \\
A & S \arrow[l, "f"] \arrow[r, "g"'] & B & T \arrow[l, "h"] \arrow[r, "k"'] & C
\end{tikzcd}
$$

The composite is

$$\begin{tikzcd} A \arrow[d, equal] & S \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "1"'] \arrow[drrr, "h^{-1}g"'] & B \arrow[drr, "kh^{-1}"] & T \arrow[l, "h"'] \arrow[r, "k"] & C \arrow[d, equal] \\ A & S \arrow[l, "f"] \arrow[r, "g"'] & B & T \arrow[l, "h"] \arrow[r, "k"'] & C \end{tikzcd}$$

and h^{-1} factors the parallelogram, showing that this fence is equivalent to the identity. The other composite gives the identity immediately. ∎

The following lemma will be useful in the proof of the main theorem of this section.

2.13. LEMMA. *Let $G, H: \mathcal{X} \to \mathcal{B}$ be jointed oplax morphisms and $t: G \to H$ an oplax transformation. If $f: X \to Y$ is left adjoint to u in $\mathcal{X}$, then the cell*

$$\begin{tikzcd} GX \arrow[r, "tX"] \arrow[d, "Gf"'] \arrow[dr, phantom, "\Uparrow tf"] & HX \arrow[d, "Hf"] \\ GY \arrow[r, "tY"'] & HY \end{tikzcd}$$

is an isomorphism whose inverse is the mate of

$$\begin{tikzcd} GY \arrow[r, "tY"] \arrow[d, "Gu"'] \arrow[dr, phantom, "\Uparrow tu"] & HY \arrow[d, "Hu"] \\ GX \arrow[r, "tX"'] & HX \end{tikzcd}$$

PROOF. The proof of this lemma is exactly the same as the proof of Lemma 1.9. Note that in that proof we only use φ^{-1} and ψ^{-1} for composites of arrows where one of the arrows has an adjoint, as in the definition of jointedness. ∎

2.14. DEFINITION. For bicategories $\mathcal{A}$ and $\mathcal{B}$, we will denote the category of jointed oplax morphisms from $\mathcal{A} \to \mathcal{B}$ with oplax transformations by $\mathrm{JOL}\,(\mathcal{A}, \mathcal{B})$.

For a category $\mathbf{A}$, we let $\mathbf{Sin}(\mathbf{A}, \mathcal{B})$ be the category of sinister morphisms from $\mathbf{A}$ to $\mathcal{B}$ with strong transformations.

2.15. THEOREM. *Let $\mathbf{A}$ be a category with pullbacks with canonical inclusion*

$$(\ )_*: \mathbf{A} \to \mathbf{Span}(\mathbf{A}),$$

and let $\mathcal{B}$ be any bicategory. Then:

1. *Composing with $(\ )_*$ gives an equivalence of categories between* $\mathrm{JOL}\,(\mathbf{Span}(\mathbf{A}), \mathcal{B})$ *and* $\mathbf{Sin}(\mathbf{A}, \mathcal{B})$.

2. *An oplax transformation $t: G \to H: \mathbf{Span}(\mathbf{A}) \to \mathcal{B}$ is strong if and only if $t(\ )_*$ is sinister.*

3. *G is a homomorphism if and only if $G(\ )_*$ satisfies the Beck condition.*

PROOF. For jointed oplax $G : \mathbf{Span}(\mathbf{A}) \to \mathcal{B}$, $G(\ )_* : \mathbf{A} \to \mathcal{B}$ is a homomorphism as G preserves left composition by left adjoints. G also preserves adjoints and f_* is a left adjoint for each f in $\mathbf{A}$, so $G(\ )_*$ is sinister.

For an oplax transformation $t\colon G \to H$, $t(\ )_*$ is certainly an oplax transformation, but as f_* is a left adjoint, $t(\ )_*$ is in fact a strong transformation in view of the previous lemma.

Note that t and $t(\ )_*$ have the same values at both the arrow and 2-cell levels, so composing with $(\ )_*$ does indeed give a functor

$$\mathrm{JOL}(\mathbf{Span}(\mathbf{A}), \mathcal{B}) \to \mathbf{Sin}(\mathbf{A}, \mathcal{B}).$$

We shall show that it is full and faithful. Let $G, H : \mathbf{Span}(\mathbf{A}) \to \mathcal{B}$ be jointed oplax morphisms and $t\colon G(\ )_* \to H(\ )_*$ a strong transformation. We must show that there is a unique oplax transformation $u\colon G \to H$ such that $t = u(\ )_*$. As the objects of $\mathbf{A}$ and $\mathbf{Span}(\mathbf{A})$ are the same, there is no problem there. Let $A \xleftarrow{p} S \xrightarrow{q} B$ be a morphism in $\mathbf{Span}(\mathbf{A})$. If there is such a u, then by the previous lemma, the mate of $u(p^*)$ is the inverse of $u(p_*)$, or put another way, $u(p^*)$ is the mate $(u(p_*)^{-1})^*$ of the inverse of $u(p_*)$. Thus $u(S)$ must be given by

$$\begin{tikzcd}
GA \arrow[r, equal] \arrow[dd, "GS"'] & GA \arrow[r, "tA"] \arrow[d, "G(p^*)"'] \arrow[dr, phantom, "\Uparrow(t(p_*)^{-1})^*"] & HA \arrow[r, equal] \arrow[d, "H(p^*)"] & HA \arrow[dd, "HS"] \\
\arrow[r, phantom, "\Uparrow\varphi"] & GS \arrow[r, "tS"'] \arrow[d, "G(q_*)"'] \arrow[dr, phantom, "\Uparrow t(q_*)"] & HS \arrow[d, "H(q_*)"] \arrow[r, phantom, "\Uparrow\varphi^{-1}"] & \\
GB \arrow[r, equal] & GB \arrow[r, "tB"'] & HB \arrow[r, equal] & HB.
\end{tikzcd}$$

This shows the uniqueness of u. Showing that u, thus defined, is a lax transformation is a tedious calculation that we won't reproduce here.

We see immediately from the formula for $u(S)$ that u is strong if and only if t is sinister.

Next we show that for every sinister $F\colon \mathbf{A} \to \mathcal{B}$ there exists a jointed oplax $G\colon \mathbf{Span}(\mathbf{A}) \to \mathcal{B}$ such that $G(\ )_* \cong F$. Of course G is the same as F on objects. As jointed oplax morphisms preserve left composition by left adjoints $G(q_* \otimes p^*)$ must be $G(q_*)G(p^*)$ and as they preserve adjoints as well this must be $F(q)F(p)^*$ where $F(p)^*$ is a chosen right adjoint for $F(p)$. For a morphism of spans

$$\begin{tikzcd}
A \arrow[d, equal] & S \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "x"] & B \arrow[d, equal] \\
A & S' \arrow[l, "r"] \arrow[r, "s"'] & B
\end{tikzcd}$$

$G(x)\colon G(S) \to G(S')$ is given by the composite

$$F(q)\cdot F(p)^* \xrightarrow{\varphi\cdot F(p)^*} F(s)\cdot F(x)\cdot F(p)^* \xrightarrow{F(s)\cdot\eta_r\cdot F(x)\cdot F(p)^*} F(s)\cdot F(r)^*\cdot F(r)\cdot F(x)\cdot F(p)^*$$
$$\xrightarrow{F(s)\cdot F(r)^*\cdot\varphi^{-1}\cdot F(p)^*} F(s)\cdot F(r)^*\cdot F(p)\cdot F(p)^* \xrightarrow{F(s)\cdot F(r)^*\varepsilon_p} F(s)\cdot F(r)^*.$$

Functoriality is easily verified.

To see that G is oplax consider a composite of spans,

$$\begin{tikzcd}
& & S \times_B T \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\
& S \arrow[dl, "p"'] \arrow[dr, "q"] & & T \arrow[dl, "t"'] \arrow[dr, "u"] & \\
A & & B & & C.
\end{tikzcd}$$

We have

$$F(up_2)\cdot F(pp_1)^* \xrightarrow[\cong]{\text{can}} F(u)\cdot F(p_2)\cdot F(p_1)^*\cdot F(p)^* \xrightarrow{\text{Beck}} F(u)\cdot F(t)^*\cdot F(q)\cdot F(p)^*.$$

That G is normal is easily seen. T has a right adjoint if and only if t is an isomorphism. Then by Proposition 1.7, the Beck morphism is an isomorphism; so G is jointed.

It is now clear from the above formula that G is a homomorphism if and only if F satisfies the Beck condition. ∎

2.16. Remark. As the above universal property of **Span**(**A**) does not involve pullbacks or the Beck condition, we could use it to define **Span** for an arbitrary **A** or even a bicategory $\mathcal{A}$; this will be done in a forthcoming paper.

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