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ACTION GROUPOID IN PROTOMODULAR CATEGORIES

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Abstract. We give here some examples of non pointed protomodular categories $\mathbb{C}$ satisfying a property similar to the property of representation of actions which holds for the pointed protomodular category Gp of groups: any slice category of Gp, any category of groupoids with a fixed set of objects, any essentially affine category. This property gives rise to an internal construction of the center of any object X, and consequently to a specific characterization of the abelian objects in $\mathbb{C}$.

1. Introduction.

It is well known that, a group X being given, the group $\mathrm{Aut}\, X$ of automorphisms of X has the following property: given any other group G, the set $\mathrm{Gp}(G, \mathrm{Aut}\, X)$ of group homomorphisms between G and $\mathrm{Aut}\, X$ is in bijection with the set of actions of the group G on the group X, which is itself, via the semidirect product, in bijection with the set of isomorphisms classes $SExt(G, X)$ of split exact sequences:

$$1 \longrightarrow X \overset{k}{\rightarrowtail} H \underset{s}{\overset{g}{\rightleftarrows}} G \longrightarrow 1$$

This is a universal property which can be described in a very general way: consider any category $\mathbb{C}$ which is pointed (i.e. finitely complete with a zero object) and protomodular (see the precise definition below). *An object X of $\mathbb{C}$ is said to have a* split extension classifier *when there is a split extension:*

$$X \overset{\gamma}{\rightarrowtail} D_1(X) \underset{s_0}{\overset{d_0}{\rightleftarrows}} D(X)$$

which is universal in the sense that any other split extension with kernel X as above determines a unique morphism χ such that, in the following diagram, the right hand side squares are pullbacks and the left hand side square commutes:

$$\begin{array}{ccccc} X & \overset{k}{\rightarrowtail} & H & \underset{s}{\overset{g}{\rightleftarrows}} & G \\ {\scriptstyle 1_X}\downarrow & & {\scriptstyle \chi_1}\downarrow & & \downarrow{\scriptstyle \chi} \\ X & \underset{\gamma}{\rightarrowtail} & D_1(X) & \underset{s_0}{\overset{d_0}{\rightleftarrows}} & D(X) \end{array}$$

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This implies that the map χ_1 is uniquely determined by the map χ, and this is the reason of the assumption of protomodularity. Indeed a category $\mathbb{C}$ is *protomodular* [5] when it is finitely complete [12] and when, given any split epimorphism (g, s) and any pullback diagram:

$$\begin{array}{ccc} U & \xrightarrow{\bar{h}} & X \\ {\scriptstyle \bar{g}}\downarrow & & {\scriptstyle g}\downarrow\uparrow{\scriptstyle s} \\ V & \xrightarrow[h]{} & Y \end{array}$$

the pair $(s, \bar{h})$ is jointly strongly epic (of course, the category Gp is the leading and guiding example of a pointed protomodular category). This insures that, in the diagram defining the universal property of the split extension classifier, the pair (k, s) is jointly epimorphic and the map χ_1 uniquely determined by the map χ and the equations $\chi_1.k = \gamma$ and $\chi_1.s = s_0.\chi$. In the category Gp, the split extension classifier associated with the group X is the split exact sequence determined by the group homomorphism $Id : \mathrm{Aut}\, X \to \mathrm{Aut}\, X$, namely:

$$1 \longrightarrow X \overset{\gamma}{\rightarrowtail} \mathrm{Aut}\, X \ltimes X \underset{s_0}{\overset{d_0}{\rightleftarrows}} \mathrm{Aut}\, X \longrightarrow 1$$

with $\gamma(x) = (Id_X, x)$, $d_0(f, x) = f$ and $s_0(f) = (f, 1)$.

A pointed protomodular category $\mathbb{C}$ is said to be *action representative* when such a split extension classifier does exist for any object X [2]. We just observed that this is the case for the category Gp. This is also classically true for the category R-Lie of Lie algebras on a ring R, with $D(X) = \mathrm{Der}(X)$ the Lie algebra of derivations of X. Many other examples and counterexamples are given in [3] in the more restricted context of semi-abelian categories [10].

Let X be an object with split extension classifier. Then it is shown in [2] that this classifier is underlying a groupoid structure $D_\bullet(X)$, endowed with a canonical discrete fibration $j_\bullet : \nabla X \to D_\bullet(X)$ (where ∇X is the indiscrete equivalence relation associated with X). For that, first denote by $j_X : X \to D(X)$ the classifying map of the upper split extension which makes the following diagram commute:

$$\begin{array}{ccccc} X & \overset{r_X}{\rightarrowtail} & X \times X & \underset{s_0}{\overset{p_0}{\rightleftarrows}} & X \\ {\scriptstyle 1_X}\downarrow & & {\scriptstyle \tilde{j}_X}\downarrow & & \downarrow{\scriptstyle j_X} \\ X & \underset{\gamma}{\rightarrowtail} & D_1(X) & \underset{s_0}{\overset{d_0}{\rightleftarrows}} & D(X) \end{array}$$

Then consider the following split extension, with $R[d_0]$ the kernel equivalence relation of $d_0 : D_1(X) \to D(X)$, and s_1 (according to the simplicial notations) the unique map such that $p_0.s_1 = s_0.d_0$ and $p_1.s_1 = 1_{D_1(X)}$:

$$X \overset{s_1.\gamma}{\rightarrowtail} R[d_0] \underset{s_0}{\overset{d_0}{\rightleftarrows}} D_1(X)$$

It determines a unique pair (d_1, δ_2) of arrows making the following commutative square a pullback:

$$\begin{array}{ccc} R[d_0] & \xrightarrow{\delta_2} & D_1(X) \\ {\scriptstyle d_0}\downarrow\uparrow{\scriptstyle s_0} & & {\scriptstyle d_0}\downarrow\uparrow{\scriptstyle s_0} \\ D_1(X) & \xrightarrow[d_1]{} & D(X) \end{array}$$

Since any protomodular category $\mathbb{C}$ is Mal'cev, this is sufficient (see [9]) to produce the following internal groupoid $D_\bullet(X)$:

$$R[d_0] \;\overset{\delta_2}{\underset{d_0}{\overset{\xrightarrow{d_1}}{\rightrightarrows}}}\; D_1(X) \;\overset{d_1}{\underset{d_0}{\overset{\xleftarrow{s_0}}{\rightrightarrows}}}\; D(X)$$

We call this $D_\bullet(X)$ the *action groupoid* of the object X. Moreover we have $d_1.\tilde{j}_X = j_X.p_1$ since the two maps clearly classify the same split extension. This produces an internal functor $j_\bullet : \nabla X \to D_\bullet(X)$ which is actually a discrete fibration. When $\mathbb{C} = \mathrm{Gp}$, this action groupoid is just the internal groupoid associated with the canonical crossed module $X \to \mathrm{Aut}\, X$, where the map $d_1 : \mathrm{Aut}\, X \ltimes X \to X$ is defined by $d_1(f, x) = \iota_x \circ f$, and where ι_x is the inner automorphism associated with x. In any pointed protomodular category $\mathbb{C}$, this discrete fibration $j_\bullet$ has still a universal property, and the existence of split extension classifiers is equivalent to the existence of action groupoids (again see [2]).

Actually this universal property still makes sense even if the category $\mathbb{C}$ is no longer pointed, and even no longer protomodular. Consequently the notion of action groupoid (and consequently of action representative category) is still valid in any finitely complete context. The point of this work is mainly to show that there are actual examples of such objects in the non pointed case, more precisely: 1) when the pointed protomodular category $\mathbb{C}$ is action representative, the non pointed protomodular slice category $\mathbb{C}/X$ is still action representative in this new sense, 2) the fibration $()_0 : \mathrm{Grd} \to \mathrm{Set}$ which associates with any groupoid its object of objects (and whose fibre above 1 is precisely Gp) has any of its (non pointed and protomodular, see [5]) fibres action representative, 3) when $\mathbb{C}$ is essentially affine [5], it is not only naturally Mal'cev [11], but also action representative.

On the model of what happens in the category Gp of groups, the existence of action groupoids in $\mathbb{C}$ gives rise to an internal construction of the center of any object X, and consequently to a specific characterization of the abelian objects in $\mathbb{C}$. This article gives the details of the abstract published for the announcement of the Charles Ehresmann's centennial birthday Meeting [7].

2. Action groupoid.

From now on we shall consider a protomodular category $\mathbb{C}$ [5], see also [1] where the fundamentals on this notion are collected. An internal groupoid $Z_\bullet$ in $\mathbb{C}$ will be presented (see [4]) as a reflexive graph endowed with an operation ζ_2:

$$R^2[z_0] \underset{z_0}{\overset{z_2,\, z_1}{\rightrightarrows\!\!\to}} R[z_0] \underset{z_0}{\overset{z_1}{\rightrightarrows}} Z_1 \underset{z_0}{\overset{z_1,\, s_0}{\leftrightarrows\!\!\to}} Z_0 \qquad R(\zeta_2) : R^2[z_0] \to R[z_0], \quad \zeta_2 : R[z_0] \to Z_1$$

making the diagram above satisfy all the simplicial identities (including the ones involving the degeneracies), where $R[z_0]$ is the kernel equivalence relation of the map z_0. In the set theoretical context, this operation ζ_2 associates the composite $g.f^{-1}$ with any pair (f, g) of arrows with same domain. Actually, when the category $\mathbb{C}$ is protomodular, and thus Mal'cev, we can even truncate this diagram at level 2, see [9].

2.1. Definition. *An object X in $\mathbb{C}$ is said to be action representative, or to have an action groupoid, when there is an internal groupoid $D_\bullet(X)$:*

$$R[d_0] \underset{d_0}{\overset{\delta_2,\, d_1}{\rightrightarrows\!\!\to}} D_1(X) \underset{d_0}{\overset{d_1,\, s_0}{\leftrightarrows\!\!\to}} D(X)$$

endowed with a discrete fibration $j : \nabla X \to D_\bullet(X)$:

$$\begin{array}{ccccc} R[p_0] & \underset{p_0}{\overset{p_2,\, p_1}{\rightrightarrows\!\!\to}} & X \times X & \underset{p_0}{\overset{p_1,\, s_0}{\leftrightarrows\!\!\to}} & X \\ \downarrow{\scriptstyle R(\tilde{j})} & & \downarrow{\scriptstyle \tilde{j}} & & \downarrow{\scriptstyle j} \\ R[d_0] & \underset{d_0}{\overset{\delta_2,\, d_1}{\rightrightarrows\!\!\to}} & D_1(X) & \underset{d_0}{\overset{d_1,\, s_0}{\leftrightarrows\!\!\to}} & D(X) \end{array}$$

where ∇X is the indiscrete equivalence relation associated with X (i.e. the upper internal groupoid in the diagram above), satisfying the following property: given any other internal groupoid $Z_\bullet$ endowed with a discrete fibration $k_\bullet = (k_0, k_1) : \nabla X \to Z_\bullet$, there is a unique internal functor $\check{k}_\bullet = (\check{k}_0, \check{k}_1) : Z_\bullet \to D_\bullet$ such that $\check{k}_\bullet.k_\bullet = j_\bullet$. The category $\mathbb{C}$ is said to be action representative *when any object X has an action groupoid.*

If $\mathbb{C}$ is also protomodular, this implies that $\check{k}_\bullet$ is itself a discrete fibration. When $\mathbb{C} = \mathrm{Gp}$, this action groupoid is just the internal groupoid associated with the canonical crossed module $X \to \mathrm{Aut}\, X$. In the pointed protomodular case, thanks to the results of [2], many examples are available in [3]. We have moreover:

2.2. Proposition. *Let $\mathbb{C}$ be an action representative protomodular category. Then any coslice category $Y\backslash\mathbb{C}$ is still action representative.*

Proof. The category $Y\backslash\mathbb{C}$ has the same underlying products and pullbacks as $\mathbb{C}$. So it is straitforward that if $(X, e), e : Y \to X$ is an object in $Y\backslash\mathbb{C}$, we have $D(X, e) = (D(X), j.e)$ and $D_1(X, e) = (D_1(X), s_0.j.e)$. ■

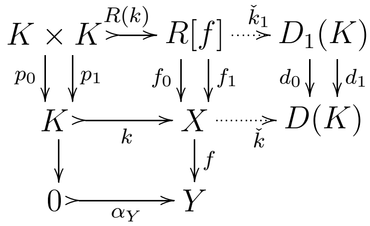
2.3. Remark. In particular the category $\mathbb{C}_*$ of pointed objects in $\mathbb{C}$, which is nothing but the coslice category $1\backslash\mathbb{C}$, is still action representative, and then any object in $\mathbb{C}_*$ (which is clearly a pointed category) has a split extension classifier, as in [2].

3. Slice categories in pointed case.

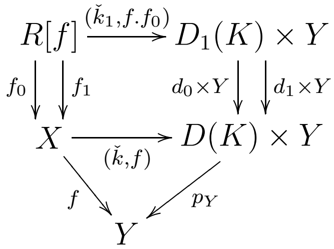
Suppose now $\mathbb{C}$ is an action representative pointed protomodular category. The slice categories $\mathbb{C}/Y$ are no longer pointed.

3.1. Proposition. *When $\mathbb{C}$ is an action representative pointed protomodular category, the slice categories $\mathbb{C}/Y$ are still action representative. For any $h : Y' \to Y$, the change of base functor $h^* : \mathbb{C}/Y \to \mathbb{C}/Y'$ preserves the action groupoids.*

Proof. Let $f : X \to Y$ be an object of $\mathbb{C}/Y$. Consider the following diagram where K is the kernel of f:

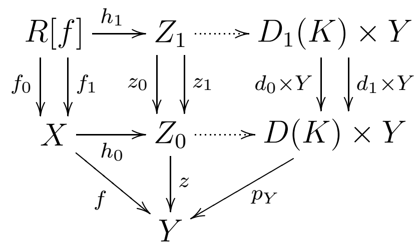


Then the upper left hand side part of this diagram is underlying a discrete fibration, and thus produces the internal functorial factorization $\check{k}_\bullet$ (actually a discrete fibration, since $\mathbb{C}$ is protomodular). Whence the following diagram in $\mathbb{C}/Y$:

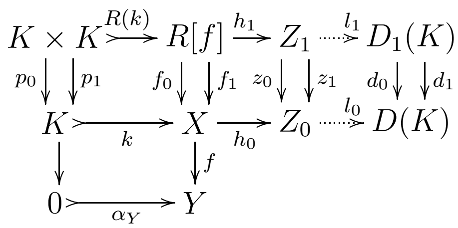


The upper part of this diagram is a discrete fibration since the composite with the projection towards $D_\bullet(K)$ is the discrete fibration $\check{k}_\bullet$. We claim that this discrete fibration in $\mathbb{C}/Y$ makes the internal groupoid on the right be the action groupoid of the object f in $\mathbb{C}/Y$, the equivalence relation $R[f]$ being, in the slice category $\mathbb{C}/Y$, the indiscrete equivalence relation associated with this object f. So, consider an internal discrete fibration

$h_\bullet : R[f] \to Z_\bullet$ in $\mathbb{C}/Y$:



We must find the dotted factorization. The projection of this factorization towards Y is forced by the diagram. The projection towards $D_\bullet(K)$ is given by the following diagram:



where the functor $l_\bullet : Z_\bullet \to D_\bullet(K)$ is the factorization produced by the discrete fibration $h_\bullet.k_\bullet : \nabla K \to Z_\bullet$ in $\mathbb{C}$.

The second point of the statement comes from the fact that the image $h^*(f)$ in $\mathbb{C}/Y'$ of the object f in $\mathbb{C}/Y$ by the change of base functor h^* has the same kernel K as f in $\mathbb{C}$. ■

We thus get the following:

3.2. Corollary. *Let $\mathbb{C}$ be an action representative pointed protomodular category; and $\pi : \mathrm{Pt}\mathbb{C} \to \mathbb{C}$ [5] the associated fibration of pointed objects. Then its fibres are action representative and its change of base functors preserve the action groupoids and the split extension classifiers.*

Proof. The fibre $\mathrm{Pt}_Y\mathbb{C}$ above Y is nothing but the category $(\mathbb{C}/Y)_*$ of points of $\mathbb{C}/Y$ (in other words: split epimorphisms with codomain Y). Then by Propositions 2.1 and 1.1, it is action representative. The change of base functors are given by pullbacks, so, by Proposition 2.1, they preserve the action groupoids, and consequently the split extension classifiers which are part of this structure, see [2]. ■

4. The fibres of the fibration $()_0 : \mathrm{Grd} \to \mathrm{Set}$.

Let Set and Grd be respectively the categories of sets and groupoids, and $()_0 : \mathrm{Grd} \to \mathrm{Set}$ the forgetful functor associating the object of objects Z_0 with any groupoid $Z_\bullet$. This functor is a fibration whose fibre above the singleton 1 is nothing but the category Gp of groups which is action representative. On the other hand, any fibre Grd_X above a set X is protomodular [5] and clearly non pointed unless $X \simeq 1$. We are going to show that it is still action representative. This fibre has an initial object ΔX, namely the discrete equivalence relation on X, and a final object ∇X the indiscrete equivalence relation on X. For any groupoid $Z_\bullet$ in Grd_X, we shall need the subgroupoid defined by the following pullback in Grd_X (namely the subgroupoid of endomaps of $Z_\bullet$):

$$\begin{array}{ccc} \mathrm{Aut}\, Z_\bullet & \rightarrowtail & Z_\bullet \\ \downarrow & & \downarrow \\ \Delta X & \rightarrowtail & \nabla X \end{array}$$

Let $Z_\bullet$ be a groupoid such that $Z_0 = X$. We shall denote by $Z(x, x')$ the set of arrows from x to x'. Its action groupoid in Grd_X, if ever it exists, must be an internal groupoid in Grd_X, which is nothing but a 2-groupoid with X as object of objects. Let us denote by $D(Z_\bullet)$ the groupoid whose object of objects is X, and arrows $\phi : x \to x'$ are the group isomorphisms $\phi : Z(x, x) \to Z(x', x')$. There is a canonical bijective on objects functor $j_\bullet : Z_\bullet \to D(Z_\bullet)$: given a map $f : x \to x'$ in $Z_\bullet$, its image $j(f) : x \to x'$ in $D(Z_\bullet)$ is the group isomorphism $j(f) : Z(x, x) \to Z(x', x')$ given by $j(f)(\alpha) = f.\alpha.f^{-1}$, $\forall \alpha \in Z(x, x)$, i.e. such that the following diagram commutes in the groupoid $Z_\bullet$:

$$\begin{array}{ccc} x & \xrightarrow{f} & x' \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle j(f)(\alpha)} \\ x & \xrightarrow[f]{} & x' \end{array}$$

This groupoid $D(Z_\bullet)$ is actually underlying a 2-groupoid. A 2-cell $\nu : \phi \Rightarrow \psi$ is given by a map $\nu \in Z(x', x')$ such that $\forall \alpha \in Z(x, x)$ we have $\psi(\alpha) = \nu.\phi(\alpha).\nu^{-1}$. The "vertical" composition is induced by the composition in $Z_\bullet$, the "horizontal" one:

$$x \; \underset{\psi}{\overset{\phi}{\rightrightarrows}} \; x' \; \underset{\psi'}{\overset{\phi'}{\rightrightarrows}} \; x'' \quad (\Downarrow \nu,\ \Downarrow \nu') \longmapsto x \; \underset{\psi'.\psi}{\overset{\phi'.\phi}{\rightrightarrows}} \; x'' \quad (\Downarrow \nu' \circ \nu)$$

is defined by $\nu' \circ \nu = \nu'.\phi'(\nu)(= \psi'(\nu).\nu')$.

Accordingly, we define the groupoid $D_1(Z_\bullet)$ as the groupoid whose object of objects is X, and arrows $x \to x'$ are the pairs (ϕ, ν), with $\phi : x \to x'$ an arrow of $D(Z_\bullet)$ and $\nu \in Z(x', x')$. The composition is defined by $(\phi', \nu') \circ (\phi, \nu) = (\phi'.\phi, \nu'.\phi'(\nu))$. The bijective on objects functors $d_i : D_1(Z_\bullet) \to D(Z_\bullet)$ are defined by $d_0(\phi, \nu) = \phi$ and $d_1(\phi, \nu) = \psi$ with $\psi(\alpha) = \nu.\phi(\alpha).\nu^{-1}$. We have also a bijective on objects functor:

$$\tilde{j}_\bullet : Z_\bullet \times_X Z_\bullet \to D_1(Z_\bullet)$$

(where $Z_\bullet \times_X Z_\bullet$ denotes the product in the fibre Grd_X), which is defined by $\tilde{j}(f, g) = (j(f), g.f^{-1})$ for any arrow in $Z_\bullet \times_X Z_\bullet$, i.e. for any parallel pair of arrows $(f, g) : x \rightrightarrows x'$ in $Z_\bullet$. Whence the following commutative diagram in Grd_X which is actually underlying a discrete fibration:

$$\begin{array}{ccccc}
R[p_0] & \overset{p_2,\,p_1}{\underset{p_0}{\rightrightarrows\!\!\!\to}} & Z_\bullet \times_X Z_\bullet & \overset{p_1}{\underset{p_0}{\underset{\xleftarrow{s_0}}{\rightrightarrows}}} & Z_\bullet \\
{\scriptstyle R(\tilde{j}_\bullet)}\downarrow & & \downarrow{\scriptstyle \tilde{j}_\bullet} & & \downarrow{\scriptstyle j_\bullet} \\
R[d_0] & \overset{\delta_2,\,d_1}{\underset{d_0}{\rightrightarrows\!\!\!\to}} & D_1(Z_\bullet) & \overset{d_1}{\underset{d_0}{\underset{\xleftarrow{s_0}}{\rightrightarrows}}} & D(Z_\bullet)
\end{array}$$

4.1. Remark. Actually the definition of the lower groupoid depends uniquely on $\operatorname{Aut} Z_\bullet$. Only the comparison $j_\bullet$ depends on $Z_\bullet$. This observation will be essential in the proof of the uniqueness in the following proposition:

4.2. Proposition. *The diagram above determines the lower groupoid as the action 2-groupoid $D_\bullet(Z_\bullet)$ associated with the groupoid $Z_\bullet$ in the protomodular fibre* Grd_X*. Accordingly the fibres* Grd_X *are action representative.*

Proof. Suppose we are given a discrete fibration in Grd_X:

$$\begin{array}{ccccc}
R[p_0] & \overset{p_2,\,p_1}{\underset{p_0}{\rightrightarrows\!\!\!\to}} & Z_\bullet \times_X Z_\bullet & \overset{p_1}{\underset{p_0}{\underset{\xleftarrow{s_0}}{\rightrightarrows}}} & Z_\bullet \\
{\scriptstyle R(k_{\bullet,1})}\downarrow & & \downarrow{\scriptstyle k_{\bullet,1}} & & \downarrow{\scriptstyle k_{\bullet,0}} \\
R[w_0] & \overset{w_2,\,w_1}{\underset{w_0}{\rightrightarrows\!\!\!\to}} & W_{\bullet,1} & \overset{w_1}{\underset{w_0}{\underset{\xleftarrow{s_0}}{\rightrightarrows}}} & W_{\bullet,0}
\end{array}$$

The fact that this is a discrete fibration means that any 2-cell in the 2-groupoid $W_\bullet$:

$$x \;\underset{h}{\overset{k(f)}{\Downarrow \nu}}\; x'$$

determines a unique arrow $g : x \to x'$ in the groupoid $Z_\bullet$ such that $k(g) = h$. In particular any 2-cell in $W_\bullet$:

$$x \;\underset{h}{\overset{1_x}{\Downarrow \nu}}\; x$$

produces a unique arrow $g : x \to x$ in the groupoid $Z_\bullet$ such that $k(g) = h$. We must now define a 2-functor:

$$\begin{array}{ccccc}
R[w_0] & \overset{w_2,\ w_1}{\underset{w_0}{\rightrightarrows\!\!\rightarrow}} & W_{\bullet,1} & \overset{w_1,\ s_0}{\underset{w_0}{\rightleftarrows\!\!\rightarrow}} & W_{\bullet,0} \\
\downarrow{\scriptstyle R(\check{k}_{\bullet,1})} & & \downarrow{\scriptstyle \check{k}_{\bullet,1}} & & \downarrow{\scriptstyle \check{k}_{\bullet,0}} \\
R[d_0] & \overset{\delta_2,\ d_1}{\underset{d_0}{\rightrightarrows\!\!\rightarrow}} & D_1(Z_\bullet) & \overset{d_1,\ s_0}{\underset{d_0}{\rightleftarrows\!\!\rightarrow}} & D(Z_\bullet)
\end{array}$$

Let $h : x \to x'$ be an arrow in $W_\bullet$; let us define $\check{k}(h) : Z(x, x) \to Z(x', x')$. So consider $\alpha : x \to x$ in $Z_\bullet$. We have a 2-cell $k(1_x, \alpha)$ in $W_\bullet$. Consider the following diagram:

$$\begin{array}{ccc}
x & \overset{1_x}{\underset{k(\alpha)}{\rightrightarrows}}\ \Downarrow k(1_x, \alpha) & x \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\
x' & \overset{1_{x'}}{\underset{h.k(\alpha).h^{-1}}{\rightrightarrows}}\ \Downarrow h.k(1_x, \alpha).h^{-1} & x'
\end{array}$$

Then the lower 2-cell in the diagram above produces a unique arrow $\check{k}(h)(\alpha) : x' \to x'$ in $Z_\bullet$ such that $k(\check{k}(h)(\alpha)) = h.k(\alpha).h^{-1}$. The unicity of this map assures that $\check{k}(h)$ is a group homomorphism, and the last equation that we have $\check{k}_{\bullet,0}.k_{\bullet,0} = j_\bullet$. It is easy to check that the construction $\check{k}_\bullet : W_{\bullet,0} \to D(Z_\bullet)$ is functorial.

We must now extend $\check{k}$ to the 2-cells of the 2-groupoid $W_\bullet$. So let $\nu : h \Rightarrow h'$ be a 2-cell in $W_\bullet$. Then $\nu.h^{-1} : 1_{x'} \Rightarrow h'.h^{-1}$ is a 2-cell in $W_\bullet$ which determines a unique map $\check{\nu} : x' \to x'$ in $Z_\bullet$ such that $k(\check{\nu}) = h'.h^{-1}$. We define $\check{k}(\nu)$ as the 2-cell $(\check{k}(h), \check{\nu})$ in $D_\bullet(Z_\bullet)$. This completes the 2-functor $\check{k}_{\bullet,\bullet}$ we were looking for.

To prove the unicity of this factorization, let us look at the following diagram:

$$\begin{array}{ccccc}
R[p_0] & \overset{p_2,\ p_1}{\underset{p_0}{\rightrightarrows\!\!\rightarrow}} & \mathrm{Aut}\, Z_\bullet \times_X \mathrm{Aut}\, Z_\bullet & \overset{p_1,\ s_0}{\underset{p_0}{\rightleftarrows\!\!\rightarrow}} & \mathrm{Aut}\, Z_\bullet \\
\downarrow & & \downarrow & & \downarrow \\
R[p_0] & \overset{p_2,\ p_1}{\underset{p_0}{\rightrightarrows\!\!\rightarrow}} & Z_\bullet \times_X Z_\bullet & \overset{p_1,\ s_0}{\underset{p_0}{\rightleftarrows\!\!\rightarrow}} & Z_\bullet \\
\downarrow{\scriptstyle R(k_{\bullet,1})} & & \downarrow{\scriptstyle k_{\bullet,1}} & & \downarrow{\scriptstyle k_{\bullet,0}} \\
R[w_0] & \overset{w_2,\ w_1}{\underset{w_0}{\rightrightarrows\!\!\rightarrow}} & W_{\bullet,1} & \overset{w_1,\ s_0}{\underset{w_0}{\rightleftarrows\!\!\rightarrow}} & W_{\bullet,0} \\
\downarrow{\scriptstyle R(\check{k}_{\bullet,1})} & & \downarrow{\scriptstyle \check{k}_{\bullet,1}} & & \downarrow{\scriptstyle \check{k}_{\bullet,0}} \\
R[d_0] & \overset{\delta_2,\ d_1}{\underset{d_0}{\rightrightarrows\!\!\rightarrow}} & D_1(Z_\bullet) & \overset{d_1,\ s_0}{\underset{d_0}{\rightleftarrows\!\!\rightarrow}} & D(Z_\bullet)
\end{array}$$

The upper part of this diagram, induced by the inclusion $\mathrm{Aut}\, Z_\bullet \rightarrowtail Z_\bullet$, is actually a discrete fibration. Moreover, as we noticed in the previous remark the lower 2-groupoid is also the action 2-groupoid associated with Aut $Z_\bullet$. So the unicity of the factorization $\check{k}_\bullet$ can be equally checked from Aut $Z_\bullet$. This is relatively easy by using the similar result holding in the category Gp of groups, since the groupoid Aut $Z_\bullet$ is nothing but a family of ordinary groups. ■

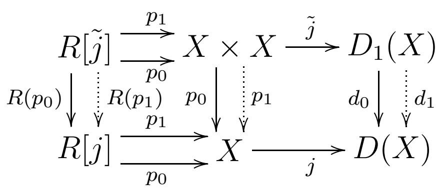
4.3. Remark. As in any (even not pointed) protomodular category, there is, in the fibres Grd_X, an intrinsic notion of normal subobject which is explicited in [6] (Theorem 3). The normal subobjects are closely related to the action groupoids, see [2]. Let us quickly mention here, that a subgroupoid $V_\bullet \rightarrowtail Z_\bullet$ in Grd_X is normal if and only if, given any map $f : x \to x'$ in $Z_\bullet$, the restriction of the isomorphism $j(f) : Z(x,x) \to Z(x',x')$ to $V(x,x)$ takes its values in $V(x',x')$.

5. Action groupoid and centrality.

In any protomodular category $\mathbb{C}$, there is an intrinsic notion of abelian objects, see for instance [8] or [1]. The existence of action groupoids allows us to measure the obstruction to abelianness:

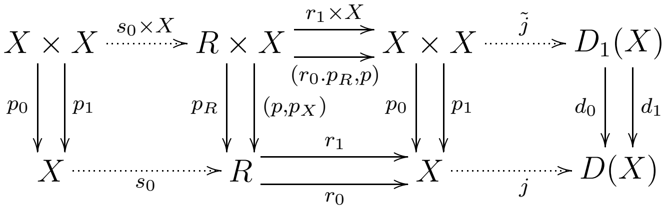
5.1. Proposition. *Suppose the object X in $\mathbb{C}$ admits an action groupoid. The kernel relation $R[j]$ of the map $j : X \to D(X)$ is the centre of X, i.e. the greatest central equivalence relation on X.*

Proof. Recall that an equivalence relation $(r_0, r_1) : R \rightrightarrows X$ on X is central when there is a "connector" between the equivalence relations R and ∇X, which is a map $p : R \times X \to X$, satisfying internally the Mal'cev equations $p(x,y,y) = x$ and $p(x,x,y) = y$, see [8]. Now let us consider the following diagram:



Since the downward right hand side square is a pullback (as a part of a discrete fibration), the downward left hand side squares are pullbacks and $R[\tilde{j}]$ is then isomorphic to $R \times X$. Consequently the map $p_0.R(p_1) : R[\tilde{j}] \to X$ is a connector which makes $R[j]$ central.

If R is another central relation, the connector p between R and ∇X allows us to complete the following diagram in a way which makes the central squares commute and determine internal functors, see [8]:



On the other hand, the dotted arrows on the left hand side determine a discrete fibration, while the composition by s_0 (resp. by $s_0 \times X$) equalizes the horizontal arrows. Accordingly both maps $j.r_0$ and $j.r_1$ are the classifier of the left hand side dotted discrete fibration, and are consequently equal. Accordingly we get $R \leq R[j]$. ∎

In this way we obtain a characterization of abelian objects in $\mathbb{C}$, where, classically an object X is called abelian when the indiscrete equivalence relation ∇X is central:

5.2. Corollary. *Let the object X have an action groupoid. Then the following conditions are equivalent:*
1) the object X is abelian in $\mathbb{C}$
2) $d_0 = d_1$ (in other words the action groupoid $D_\bullet(X)$ is absolutely disconnected)

Proof. Suppose X abelian. Then its centre is the indiscrete relation ∇X and, according to the previous proposition, $R[j] = \nabla X$. It is the case if and only if $j.p_0 = j.p_1 : X \times X \rightrightarrows X \to D(X)$. Now, considering the following diagram:

$$\begin{array}{ccc} X \times X & \xrightarrow{\tilde{j}} & D_1(X) \\ {\scriptstyle p_0}\downarrow\;\downarrow{\scriptstyle p_1} & & {\scriptstyle d_0}\downarrow\;\downarrow{\scriptstyle d_1} \\ X & \xrightarrow[j]{} & D(X) \end{array}$$

we have always $d_i.\tilde{j} = p_i.j$. So that X is abelian if and only if $d_0.\tilde{j} = d_1.\tilde{j}$. But the pair $(s_0, \tilde{j})$ is jointly strongly epic. Since d_0 and d_1 are clearly equalized by s_0, this last equality holds if and only if $d_0 = d_1$. ∎

6. Essentially affine categories.

A category $\mathbb{C}$ is essentially affine [5] when it admits pullbacks of split epimorphisms, pushouts of split monomorphisms and is such that, given any commutative square of split epimorphisms:

$$\begin{array}{ccc} X & \xrightarrow{u} & X' \\ {\scriptstyle f}\downarrow\uparrow{\scriptstyle s} & & {\scriptstyle f'}\downarrow\uparrow{\scriptstyle s'} \\ Y & \xrightarrow[v]{} & Y' \end{array}$$

the downward square is a pullback if and only if the upward square is a pushout. A pointed finitely complete category $\mathbb{C}$ is essentially affine if and only if it is additive. The slice categories of any finitely complete additive category are essentially affine. On the other hand, any finitely complete essentially affine category is necessarily protomodular and naturally Mal'cev in the sense of [11]. Inside the protomodular context, this naturally Mal'cev condition exactly means that any object is abelian.

We showed in [2] that when the category $\mathbb{C}$ is additive, then, for each object X, the action groupoid structure is nothing but the canonical (internal) abelian group structure on X, namely:

$$X \times X \overset{\xrightarrow{d}}{\underset{\xrightarrow[p_0]{}}{\xrightarrow{p_1}}} X \underset{\tau_X}{\overset{\alpha_X}{\rightleftarrows}} 0$$

where $d = p_1 - p_0$. In the same order of ideas, we have:

6.1. Proposition. *Any finitely complete essentially affine category $\mathbb{C}$ is action representative.*

Proof. Given any object X let us consider the following pushout of the split monomorphism s_0 along the terminal map:

$$\begin{array}{ccc} X \times X & \xrightarrow{d} & \bar{X} \\ {\scriptstyle p_0}\downarrow \uparrow{\scriptstyle s_0} \downarrow{\scriptstyle p_1} & & \downarrow\uparrow{\scriptstyle e} \\ X & \longrightarrow & 1 \end{array}$$

The object $\bar{X}$, being pointed by e and abelian, is canonically endowed with an internal (abelian) group structure, and the map d determines an internal functor $d_\bullet : \nabla X \to \bar{X}$. Moreover the downward squares are pullbacks and this functor is a dicrete fibration. Let us show that this dicrete fibration makes the group structure on $\bar{X}$ be the action groupoid of X. So consider any other discrete fibration $k_\bullet : \nabla X \to W_\bullet$:

$$\begin{array}{ccccc} X \times X & \xrightarrow[k_1]{} & W_1 & \xrightarrow[\check{k}_1]{} & \bar{X} \\ {\scriptstyle p_0}\downarrow\uparrow\downarrow{\scriptstyle p_1} & & {\scriptstyle w_0}\downarrow\uparrow\downarrow{\scriptstyle w_1} & & \downarrow\uparrow{\scriptstyle e} \\ X & \xrightarrow[k_0]{} & W_0 & \xrightarrow[\tau]{} & 1 \end{array} \qquad (\check{k}_1 . k_1 = d)$$

We must explicit a unique dotted factorization. Clearly $\check{k}_0$ is the terminal map τ. Since $k_\bullet$ is a discrete fibration, the downward left hand side squares are pullbacks, and consequently the upward left hand side square is a pushout. Accordingly there is a unique map $\check{k}_1$ such that $\check{k}_1 . k_1 = d$ and $\check{k}_1 . s_0 = e . \tau$. It is easy to check that this diagram is actually underlying an internal functor $\check{k}_\bullet : W_\bullet \to \bar{X}$. ∎

6.2. Remark. As we emphasized it in the introduction, the definition of action groupoid is still valid in any finitely complete category. Would there be examples of action groupoids in a non protomodular context?

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