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MAPPING SPACES OF Gray-CATEGORIES

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Abstract. We define a mapping space for Gray-enriched categories adapted to higher gauge theory. Our construction differs significantly from the canonical mapping space of enriched categories in that it is much less rigid. The two essential ingredients are a path space construction for Gray-categories and a kind of comonadic resolution of the 1-dimensional structure of a given Gray-category obtained by lifting the resolution of ordinary categories along the canonical fibration of GrayCat over Cat.

Contents

1. Introduction

It is well known that among algebraic models for homotopy n-types Gray-groupoids model 3-types; Lack [2011] gives us a proof using model category methods. Wanting to study the homotopy 3-type of the moduli space of 3-connections on a manifold, we thought it apt to define a mapping space $[\mathcal{S}_3(M), \mathcal{C}(\mathcal{H})]$ of Gray-groupoids that could model that moduli space, where $\mathcal{S}_3(M)$ is the fundamental Gray-groupoid and $\mathcal{C}(\mathcal{H})$ is the Gray-groupoid ultimately derived from a 2-crossed Lie-algebra where the triconnections take their values; see for example Schreiber and Waldorf [2011] for 2-connections, to which

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this is an obvious next step. See [Martins and Picken 2011] for the background on the smooth fundamental Gray-groupoid and triconnections. Wang [2013] shows how to obtain the gauge 3-connection from a 3-holonomy, and the 3-gauge transformation from the lax-transformation between holonomy Gray-functors.

The original definition of the Gray-tensor can be found in [Gray 1974]; Gordon et al. [1995] give us the definition of tricategories and show that every tricategory strictifies to a triequivalent Gray-category. Crans [1999] gives an explicit, elementwise definition of Gray-categories.

In 1999 Crans gave a partial solution the mapping space problem; however, the absence of an interchange law in Gray-categories prevents lax transformations between Gray-functors from being composable in general. The slightly unsatisfactory solution is to restrict to those transformations and higher cells that can in fact be composed; this does give a mapping space Gray-category, but a mere stopgap not sufficient for our purposes.

Instead, we enlarge the repertoire of maps, and thereby transformations, in a way that will permit forming all composites of transformations; specifically we introduce a 2-cocycle that intermediates coherently between the two possible evaluations of arrangements of squares shown in (36) and (37). In analogy with Garner [2010] we introduce a co-monadic weakening of strict Gray-functors in section 2. The comonad Q^1 then yields a co-Kleisli category $\mathsf{GrayCat}_{\mathrm{Q}^1}$. We use in an essential way that GrayCat is fibered over Cat.

Inspired by [Bénabou 1967] we axiomatise lax transformations as maps into a path-space. In section 3 we introduce a functorial path-space construction for Gray-categories; subsequently in section 4 we show that this yields an internal category $\overrightarrow{\mathbb{H}} \rightrightarrows \mathbb{H}$ in $\mathsf{GrayCat}_{\mathrm{Q}^1}$ for a given $\mathbb{H}$ in GrayCat.

The n-th iterate of $\overrightarrow{(\_)}$ yields an n-truncated internal cubical object in GrayCat. In section 5 we construct an internal Gray-category

$$\overline{\overline{\overline{\mathbb{H}}}} \rightrightarrows \overline{\overline{\mathbb{H}}} \rightrightarrows \overrightarrow{\mathbb{H}} \rightrightarrows \mathbb{H}$$

in $\mathsf{GrayCat}_{\mathrm{Q}^1}$ as a subobject of the third iterated path-space. It is then a trivial consequence in section 6 that we obtain a mapping space Gray-category by applying the hom functor

$$[\mathbb{G}, \mathbb{H}] := \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, \overline{\overline{\overline{\mathbb{H}}}} \rightrightarrows \overline{\overline{\mathbb{H}}} \rightrightarrows \overrightarrow{\mathbb{H}} \rightrightarrows \mathbb{H}).$$

Furthermore we obtain a restricted mapping space $\{\mathbb{G}, \mathbb{H}\}$, where everything is as before, except only strict Gray-functors are permitted between $\mathbb{G}$ and $\mathbb{H}$. This leads to a natural sesquicategory structure on GrayCat.

We hope to be able to prove in a later paper that this internal hom is part of a monoidal closed structure on $\mathsf{GrayCat}_{\mathrm{Q}^1}$ involving a suitable extension of Crans' tensor product.

Finally, in section 7 we give explicit details of functors, transformations and so on in terms of components. Lastly, we remark that if $\mathbb{H}$ is a Gray-groupoid then $\overrightarrow{\mathbb{H}}$ as well as $[\mathbb{G}, \mathbb{H}]$ will be Gray-groupoids.

Similar work was done by Gohla and Martins [2013] concerning 2-crossed modules, which are equivalent to Gray-groupoids with a single vertex, that is, Gray-groups.

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2. Resolution in Dimension One

We define a resolution of the 1-dimensional structure of a Gray-category using a comonad, by lifting the free category comonad (called "path" in [Dawson et al. 2006]) to Gray-categories; but note that we use the term in a different way in this paper.

The resulting co-Kleisli category can be seen as the category of Gray-categories with an enlarged repertoire of maps, that is flexible enough to carry out our path space construction. After giving an abstract construction of this category of pseudo maps we proceed to characterize them explicitly.

2.1. Basic Fibrations. There are obvious functors

$$\mathsf{GrayCat} \xrightarrow{(\_)_2} \mathsf{SesquiCat} \xrightarrow{(\_)_1} \mathsf{Cat} \xrightarrow{(\_)_0} \mathsf{Set}$$

that forget the 3-cells, the 2-cells and 1-cells respectively. By a slight abuse of language we will denote the composite $(\_)_1(\_)_2$ by $(\_)_1$ also, it is of course a fibration as well; we will use it in section 2.12 to construct the monad Q^1. We will use the fibration $(\_)_2(\_)_1(\_)_0 = (\_)_0$ in section 6 to construct the restricted mapping space $\{\mathbb{G}, \mathbb{H}\}$.

Let $\mathfrak{S}$ be a sesquicategory, $\mathbb{G}$ a Gray-category, and $F\colon \mathfrak{S} \longrightarrow \mathbb{G}_2$ a sesquifunctor. We define $\overline{F}\colon F^*\mathfrak{S} \longrightarrow \mathbb{G}$ as follows:

$$\begin{aligned}
(F^*\mathfrak{S})_0 &= \mathfrak{S}_0\\
(F^*\mathfrak{S})_1 &= \mathfrak{S}_1\\
(F^*\mathfrak{S})_2 &= \mathfrak{S}_2\\
(F^*\mathfrak{S})_3 &= \{(\Gamma;\alpha,\beta)\,|\,\Gamma\colon F\alpha \longrightarrow F\beta\}
\end{aligned}$$

Note that the interchange of two 2-cells α, β in $F^*\mathfrak{S}$ incident on a 0-cell is given essentially by the interchange of their images under F:

$$\beta \otimes \alpha = (F\beta \otimes F\beta; \beta \triangleright \alpha, \beta \triangleleft \alpha)\,.$$

Let us take note of the following useful fact that helps to characterize the Cartesian maps:

2.2. Remark. *For a functor* $p\colon \mathsf{E} \longrightarrow \mathsf{B}$ *that preserves co-limits, let* $D\colon \mathsf{D} \longrightarrow \mathsf{E}$ *a diagram in* E *with co-limit* (C, k_i)

$$\begin{tikzcd}
D_i \arrow[r, "k_i"] & C \arrow[rrd, "g"] & & \\
& & A \arrow[r, "f"'] & B
\end{tikzcd},$$

assume $p(g)$ *factors below as* $p(f)u = p(g)$*. Furthermore, assume that the induced sink* $(u_i) = up(k_i)$ *has fillers* $\langle u_i \rangle$ *above with* $f\langle u_i \rangle = gk_i$*, then the co-universally induced map* $\langle u \rangle : C \longrightarrow A$ *is a filler over* u*.*

This means that to check whether a map f *is Cartesian we don't need to give the filler* u *directly, but we can define it on presumably simpler parts of* C*. These then combine into a valid filler.*

2.3. Remark. *Maps Cartesian with respect to* $(\_)_2$ *are exactly the* Gray*-functors, that are 2-locally isomorphisms of sets. That is, given two parallel 2-cells on the intervening 3-cells, the map is bijective.*

2.4. Lemma. $F^*\mathfrak{S}$ *is a* Gray*-category,* $\overline{F}$ *is a* Gray*-functor and Cartesian with respect to* $(\_)_2$*.* □

Similarly, let $\mathfrak{S}$ be a sesquicategory, C a category, and $F\colon \mathsf{C} \longrightarrow \mathfrak{S}_1$ a functor, then we define a sesquicategory:

$$\begin{aligned}
(F^*\mathsf{C})_0 &= \mathsf{C}_0\\
(F^*\mathsf{C})_1 &= \mathsf{C}_1\\
(F^*\mathsf{C})_2 &= \{(\alpha; f, g) | \alpha\colon Ff \longrightarrow Fg\}
\end{aligned}$$

2.5. Lemma. $F^*\mathsf{C}$ *is a sesquicategory,* $\overline{F}$ *is a sesquifunctor, and Cartesian with respect to* $(\_)_1$*.* □

2.6. Remark. *Maps Cartesian with respect to* $(\_)_1$ *are exactly the sesquifunctors, that are 1-locally isomorphisms of sets. That is, given two parallel 1-cells on the intervening 2-cells, the map is bijective.*

For later reference we describe the Cartesian liftings of $(\_)_1$ explicitly as well. Let $\mathbb{G}$ be a Gray-category, $\mathbb{G}_1$ its underlying category. Let C be an ordinary category and $F\colon \mathsf{C} \longrightarrow \mathbb{G}_1$ a functor. Then $F^*\mathbb{G}$ is given by:

$$\begin{aligned}
(F^*\mathbb{G})_0 &= \mathsf{C}_0\\
(F^*\mathbb{G})_1 &= \mathsf{C}_1\\
(F^*\mathbb{G})_2 &= \{(\alpha; f, g) | f, g\colon x \longrightarrow y,\, \alpha\colon Ff \longrightarrow Fg\}\\
(F^*\mathbb{G})_3 &= \{(\Gamma; \alpha, \beta; f, g) | f, g\colon x \longrightarrow y,\, \Gamma\colon F\alpha \longrightarrow F\beta\}
\end{aligned}$$

Source and target maps are as follows:

$$s_2(\Gamma;\alpha,\beta;f,g) = (\alpha;f,g) \qquad t_2(\Gamma;\alpha,\beta;f,g) = (\beta;f,g)$$

$$s_1(\alpha;f,g) = f \qquad t_1(\alpha;f,g) = g\,.$$

and s_0, t_0 are as given by C. As identities we take:

$$i_1(f) = (\mathrm{id}_{Ff};f,f) \quad i_2(\alpha;f,g) = (\mathrm{id}_\alpha;\alpha,\alpha,f,g)\,.$$

The tensor in $F^*\mathbb{G}$ of two 2-cells is

$$(\beta;g,g')\otimes(\alpha;f,f') = (\beta\otimes\alpha;\beta\triangleleft\alpha,\beta\triangleright\alpha;g\#_0f,g'\#_0f') \tag{1}$$

where

$$\beta\triangleleft\alpha = (\beta\#_0Ff')\#_1(Fg\#_0\alpha), \quad \beta\triangleright\alpha = (Fg'\#_0\alpha)\#_1(\beta\#_1Ff)\,.$$

There is an obvious map $\overline{F}\colon F^*\mathbb{G} \longrightarrow \mathbb{G}$ over F that acts like F on 0- and 1-cells, and on 2- and 3-cells as a projection to $\mathbb{G}$.

2.7. Remark. *The globular set $F^*\mathbb{G}$ is a Gray-category. The composition operations of $F^*\mathbb{G}$ are given by those of C and $\mathbb{G}$ and it is easy to see that they fulfill the axioms of a Gray-category.*

Obviously $G^*F^*\mathbb{G} \cong (FG)^*\mathbb{G}$ and $\mathrm{id}_{\mathsf{C}}^* \cong \mathrm{id}_{\mathsf{GrayCat}_\mathsf{C}}$ coherently. Also, we can always choose $\mathrm{id}_{\mathsf{C}}^* = \mathrm{id}_{\mathsf{GrayCat}_\mathsf{C}}$, but this is not necessary in what follows.

2.8. Lemma. *A map of Gray-categories is Cartesian with respect to $\mathbb{G} \mapsto \mathbb{G}_1$ iff it is 1-locally an isomorphism of categories, i.e. given two parallel 1-cells the map is bijective on the intervening 2-cells and in turn bijective on the 3-cells between parallel such.* ☐

2.9. Definition. *We define a map of Gray-categories to be an n-**isomorphism** if it is Cartesian with respect to $(\_)_n$. It is n-**faithful** if fillers of factorizations under $(\_)_n$ are unique, and n-**full** is there (not necessarily unique) fillers for all factorizations under $(\_)_n$.*

By this definition 0-fidelity is ordinary fidelity of functors, 1-fidelity is local fidelity, and so on.

2.10. Remark. *One property of Cartesian maps in a fibration p that we are going to exploit in the proof of the following theorem is that for three arrows upstairs,*

$$\begin{tikzcd} \cdot \arrow[r, shift left, "r"] \arrow[r, shift right, "s"'] & \cdot \arrow[r, "f"] & \cdot \end{tikzcd}$$

with f Cartesian, $p(r) = p(s)$ downstairs and $fr = fs$ upstairs imply $r = s$, on account of f being p-faithful.

2.11. Lemma. *If fg is Cartesian with respect to a given fibration p and f is p-faithful, then g is p-Cartesian.*

Proof Take k and u such that $p(g)u = p(k)$, then $p(fg)u = p(fk)$ and hence by fg being p-full there is a filler $\langle u \rangle$ such that $fg \langle u \rangle = fk$. Then by f being p-faithful $g \langle u \rangle = k$.

By fg being p-faithful $\langle u \rangle$ is the unique such filler. □

2.12. Comonad Liftings. In this section we show that comonads can be lifted along fibrations of categories.

2.13. Definition. *In an arbitrary 2-category a* **comonad** *on an object A is given by an endomorphism*

$$A \xrightarrow{T} A$$

and 2-cells

$$\begin{tikzcd} A \arrow[r, bend left=50, "T", ""{name=U, below}] \arrow[r, bend right=50, "A"', ""{name=D, above}] \arrow[Rightarrow, from=U, to=D, "\varepsilon"] & A \end{tikzcd}$$

and

$$\begin{tikzcd} A \arrow[r, "T"'] \arrow[rr, bend left=50, "T", ""{name=U, below}] & A \arrow[r, "T"'] \arrow[Rightarrow, from=U, "\delta"] & A \end{tikzcd}$$

such that

$$\begin{tikzcd} A \arrow[r, "T"] \arrow[rr, bend left=70, "T", ""{name=U, below}] & A \arrow[r, bend left=50, "T", ""{name=U2, below}] \arrow[r, bend right=50, "A"', ""{name=D2, above}] \arrow[Rightarrow, from=U, "\delta"] \arrow[Rightarrow, from=U2, to=D2, "\varepsilon"] & A \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left=50, "T", ""{name=U, below}] \arrow[r, bend right=50, "T"', ""{name=D, above}] \arrow[Rightarrow, from=U, to=D, "T"] & A \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left=50, "T", ""{name=U2, below}] \arrow[r, bend right=50, "A"', ""{name=D2, above}] \arrow[Rightarrow, from=U2, to=D2, "\varepsilon"] \arrow[rr, bend left=70, "T", ""{name=U, below}] & A \arrow[r, "T"] \arrow[Rightarrow, from=U, "\delta"] & A \end{tikzcd}$$

and

$$\begin{tikzcd} A \arrow[r, "T"'] \arrow[rrr, bend left=60, "T", ""{name=U, below}] & A \arrow[r, "T"'] \arrow[rr, bend left=50, "T", ""{name=V, below}] \arrow[Rightarrow, from=U, "\delta"] & A \arrow[r, "T"'] \arrow[Rightarrow, from=V, "\delta"] & A \end{tikzcd} = \begin{tikzcd} A \arrow[r, "T"'] \arrow[rrr, bend left=60, "T", ""{name=U, below}] \arrow[rr, bend left=50, "T", ""{name=V, below}] & A \arrow[r, "T"'] \arrow[Rightarrow, from=V, "\delta"] & A \arrow[r, "T"'] \arrow[Rightarrow, from=U, "\delta"] & A \end{tikzcd} .$$

See, for example, Mac Lane [1998].

If A is a category, T a functor and ε and δ natural transformations, then these equations of course amount to the usual equations objectwise in A:

$$\begin{tikzcd} & Tx \arrow[dl, "Tx"'] \arrow[d, "\delta_x"] \arrow[dr, "Tx"] & \\ Tx & TTx \arrow[l, "\varepsilon_{Tx}"] \arrow[r, "T\varepsilon_x"'] & Tx \end{tikzcd}$$

and

$$\begin{tikzcd}
Tx \arrow[r, "\delta_x"] \arrow[d, "\delta_x"'] & TTx \arrow[d, "T\delta_x"] \\
TTx \arrow[r, "\delta_{Tx}"'] & TTTx
\end{tikzcd} .$$

2.14. Theorem. *Given a fibration of categories $p\colon \mathsf{E} \longrightarrow \mathsf{B}$, a comonad (Q, δ, ε) on B can be lifted to a comonad (K, d, e) on E such that $(K, Q)\colon p \longrightarrow p$ is a comonad in the 2-category of all fibrations.*

Proof Let $(\_)^*\colon \mathsf{B}^{\mathrm{op}} \longrightarrow \mathsf{Cat}$ be a chosen cleavage. For every $A \in \mathsf{E}_x$ we let $e_A\colon (KA = \varepsilon_x^* A) \longrightarrow A$ be the chosen Cartesian lift of $\varepsilon_x\colon Qx \longrightarrow x$. For a morphism f over j in

$$\begin{tikzcd}
KA \arrow[r, "e_A"] \arrow[dr, dotted, "Kf"'] & A \arrow[dr, "f"] & \\
& KB \arrow[r, "e_B"'] & B
\end{tikzcd}$$

$$\begin{tikzcd}
Qx \arrow[r, "\varepsilon_x"] \arrow[dr, "Qj"'] & x \arrow[dr, "j"] & \\
& Qy \arrow[r, "\varepsilon_y"'] & y
\end{tikzcd}$$

the dotted arrow is the unique filler induced by the factorization below. This makes K a functor and $e\colon K \longrightarrow \mathrm{id}_{\mathsf{E}}$ a natural transformation.

We define a family of co-multiplication maps d_A as the unique fillers in

$$\begin{tikzcd}
KA \arrow[drr, "KA"] \arrow[dr, dotted, "d_A"'] & & \\
& KKA \arrow[r, "e_{KA}"'] & KA
\end{tikzcd}$$

$$\begin{tikzcd}
Qx \arrow[drr, "Qx"] \arrow[dr, "\delta_x"'] & & \\
& QQx \arrow[r, "\varepsilon_{Qx}"'] & Qx
\end{tikzcd}$$

where the triangle below commutes because Q is co-unital.

In the diagram

$$\begin{tikzcd}
KA \arrow[dr, "d_A"'] \arrow[drr, dotted] \arrow[drr, bend left, "KA"] \arrow[drrr, bend left, "e_A"] & & & \\
& KKA \arrow[r, shift left, "Ke_A"] \arrow[r, shift right, "e_{KA}"'] & KA \arrow[r, "e_A"'] & A
\end{tikzcd}$$

$$\begin{tikzcd}
Qx \arrow[dr, "\delta_x"'] \arrow[drr, bend left, "Qx"] \arrow[drrr, bend left, "\varepsilon_x"] & & & \\
& QQx \arrow[r, shift left, "Q\varepsilon_x"] \arrow[r, shift right, "\varepsilon_{Qx}"'] & Qx \arrow[r, "\varepsilon_x"'] & x
\end{tikzcd}$$

we see that $e_A e_{KA} d_A = e_A K e_A d_A$ by the naturality of e, and $p(e_{KA} d_A) = p(K e_A d_A)$ by Q being a comonad. Hence by remark 2.10 the three endomorphisms of KA above have to coincide, meaning d is co-unital component wise.

The naturality of d, that is, that $d_B K f = KKf d_A$ is the unique filler making the left-hand upstairs square commute

$$\begin{tikzcd}
KA \arrow[r, "d_A"] \arrow[dr, "Kf"'] \arrow[drr, dotted] & KKA \arrow[dr, "KKf"] & & \\
& KB \arrow[r, "d_B"'] & KKB \arrow[r, "e_{KB}"'] & KB
\end{tikzcd}$$

$$\begin{tikzcd}
Qx \arrow[r, "\delta_x"] \arrow[dr, "Qj"'] & QQx \arrow[dr, "QQj"] & & \\
& Qy \arrow[r, "\delta_y"'] & QQy \arrow[r, "\varepsilon_{Qy}"'] & Qy
\end{tikzcd}$$

is obtained by observing that $e_{KB} d_B K f = KF = K f e_{KA} d_A = e_{KB} KKf d_A$, from e being natural and a retraction. Also, $p(d_B K f) = p(KKf d_a)$ by naturality of δ. We apply 2.10 again.

Finally, we show that d is co-associative: Consider the diagram

$$\begin{tikzcd}
KA \arrow[r, "d_A"] \arrow[dr, "d_A"'] \arrow[drr, dotted] & KKA \arrow[dr, "d_{KA}"] & & \\
& KKA \arrow[r, "Kd_A"'] & KKKA \arrow[r, "e_{KKA}"'] & KKA
\end{tikzcd}$$

$$\begin{tikzcd}
Qx \arrow[r, "\delta_x"] \arrow[dr, "\delta_x"'] & QQx \arrow[dr, "\delta_{Qx}"] & & \\
& QQx \arrow[r, "Q\delta_x"'] & QQQx \arrow[r, "\varepsilon_{QQx}"'] & QQx\,.
\end{tikzcd}$$

We calculate that $e_{KKA}Kd_Ad_A = d_Ae_{KA}d_A = d_A = e_{KKA}d_{KA}d_A$, again by naturality of e and its retractiveness. Moreover, δ is co-associative, hence we can apply remark 2.10 once more. ☐

We observe that K preserves Cartesianness of maps, thus in particular Ke is Cartesian component wise.

Finally we can define our resolution comonad. Let $(Q, \delta, \varepsilon) = (FU, F\eta U, \varepsilon)$ be the comonad that arises from the adjunction

$$\begin{tikzcd} \mathsf{RGrph} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{Cat} \arrow[l, bend left, "U"] \end{tikzcd} .$$

Then, according to theorem 2.14, we obtain the comonad (Q^1, d, e) on GrayCat induced by lifting Q along $(\_)_1$. The exponent reminds us that this provides a resolution of the 1-dimensional structure of Gray-categories. See section A for a more abstract point of view on this construction. In section 2.22 we will show explicitly how this comonad acts.

2.15. Corollary. *By the above theorem there is a comonad* Q^1 *on* GrayCat *that pulls back the* Gray*-structure onto the free category on the underlying 1-graph.*

If a category C is already the free category $\mathsf{C} = F\mathfrak{g}$ over a reflexive graph with injection of generators $\eta\colon \mathfrak{g} \longrightarrow U\mathsf{C}$, then by adjointness the counit is split

$$\begin{tikzcd} \mathsf{C} \arrow[r, "F\eta"] \arrow[rr, bend right, "\mathsf{C}"'] & Q\mathsf{C} \arrow[r, "\varepsilon"] & \mathsf{C} \end{tikzcd} .$$

2.16. Definition. *If a* Gray*-category* $\mathbb{G}$ *has an underlying category* $\mathbb{G}_1$ *of the form* $F\mathfrak{g}$ *for some reflexive graph* $\mathfrak{g}$ *we say that* $\mathbb{G}$ *is* **free up to order 1** *with generating 1-cells* $\mathfrak{g}$*.*

Let $k\colon \mathbb{G} \longrightarrow \mathrm{Q}^1\mathbb{G}$ be the filler along $(\_)_1$ for the factorization $e_1F\eta = (\mathrm{id}_{\mathbb{G}})_1$ for the given generating reflexive graph. This of course gives a splitting

$$\begin{tikzcd} \mathbb{G} \arrow[r, "k"] \arrow[rr, bend right, "\mathbb{G}"'] & \mathrm{Q}^1\mathbb{G} \arrow[r, "e"] & \mathbb{G} \end{tikzcd} . \qquad (2)$$

If a Gray-category is free up to order 1 we may look at the 1-cells as follows: every 1-cell f can be written as $[f_1, \ldots, f_n]$, where the $[f_i]$ are generating 1-cells unique up to insertion and deletion of units. Now, the action of $k\colon \mathbb{G} \longrightarrow \mathrm{Q}^1\mathbb{G}$ can be described as follows:

1. 0-cells: $k\colon x \mapsto x$
2. 1-cells: $k\colon f = [f_1, \ldots, f_n] \mapsto [[f_1], \ldots, [f_n]]$

3. 2-cells: $k\colon (\alpha\colon f \Longrightarrow f') \mapsto (\alpha; [[f_1], \dots, [f_n]], [[f'_1], \dots, [f'_{n'}]])$
4. 3-cells: $k\colon (\Gamma\colon \alpha \Rrightarrow \alpha') \mapsto (\Gamma; \alpha, \alpha'; [[f_1], \dots, [f_n]], [[f'_1], \dots, [f'_{n'}]])$

This is obviously a section of $e_{\mathbb{G}}$.

2.17. Definition. *The category of* Gray*-categories and* **pseudo** Gray**-maps** *is the co-Kleisli-category* $\mathsf{GrayCat}_{\mathrm{Q}^1}$ *of the comonad* Q^1*.*

2.18. Lemma. *The map k for a $\mathbb{G}$ free up to order 1 has the following nice behaviour with respect to* Q^1*:*

$$\begin{tikzcd}
\mathbb{G} \arrow[r, "k"] \arrow[d, "k"'] & \mathrm{Q}^1\mathbb{G} \arrow[d, "d"] \\
\mathrm{Q}^1\mathbb{G} \arrow[r, "\mathrm{Q}^1 k"'] & \mathrm{Q}^1\mathrm{Q}^1\mathbb{G}
\end{tikzcd} . \tag{3}$$

commutes.

Proof We apply remark 2.10: The diagram

$$\begin{tikzcd}
\mathbb{G} \arrow[r, "k"] \arrow[d, "k"'] & \mathrm{Q}^1\mathbb{G} \arrow[d, "d"] \\
\mathrm{Q}^1\mathbb{G} \arrow[r, "\mathrm{Q}^1 k"'] \arrow[d, "e"'] & \mathrm{Q}^1\mathrm{Q}^1\mathbb{G} \arrow[d, "e"] \\
\mathbb{G} \arrow[r, "k"'] & \mathrm{Q}^1\mathbb{G}
\end{tikzcd}$$

commutes by co-unitality and the definition of k. Also under $(\_)_1$ the diagram (3) becomes

$$\begin{tikzcd}
F\mathfrak{g} \arrow[r, "F\eta"] \arrow[d, "F\eta"'] & FUF\mathfrak{g} \arrow[d, "F\eta UF"] \\
FUF\mathfrak{g} \arrow[r, "FUFU\eta"'] & FUFUF\mathfrak{g}
\end{tikzcd}$$

which commutes by naturality of η. □

This category has Gray-categories as objects, and morphisms

$$\begin{tikzcd} \mathbb{G} \arrow[r, "f", "/" marking] & \mathbb{H} \end{tikzcd} \qquad \text{are morphisms} \qquad \begin{tikzcd} \mathrm{Q}^1\mathbb{G} \arrow[r, "f"] & \mathbb{H} \end{tikzcd}$$

in GrayCat. Composition of two maps

$$\begin{tikzcd} \mathbb{G} \arrow[r, "f", "/" marking] & \mathbb{H} \arrow[r, "g", "/" marking] & \mathbb{K} \end{tikzcd}$$

is defined by

$$\begin{tikzcd} \mathrm{Q}^1\mathbb{G} \arrow[r, "d_{\mathbb{G}}"] & \mathrm{Q}^1\mathrm{Q}^1\mathbb{G} \arrow[r, "\mathrm{Q}^1 f"] & \mathrm{Q}^1\mathbb{H} \arrow[r, "g"] & \mathbb{K} \end{tikzcd} .$$

Identities are of the form

$$\begin{tikzcd} \mathbb{G} \arrow[r, "\mathrm{id}_{\mathbb{G}}", "/" marking] & \mathbb{G} \end{tikzcd} \;=\; \begin{tikzcd} \mathrm{Q}^1\mathbb{G} \arrow[r, "e_{\mathbb{G}}"] & \mathbb{G} \, . \end{tikzcd}$$

By way of notational convenience in diagrams in $\mathsf{GrayCat}_{\mathrm{Q}^1}$ we use unslashed arrows $f\colon \mathbb{G} \longrightarrow \mathbb{H}$ to denote a strict arrow that is included in $\mathsf{GrayCat}_{\mathrm{Q}^1}$ as $fe\colon \mathbb{G} \nrightarrow \mathbb{H}$.

The comonad axioms make sure this is a category; c. f. e. g. [Mac Lane 1998].

There is an adjunction

$$\begin{tikzcd} \mathsf{GrayCat} \arrow[r, bend left, "R"] \arrow[r, phantom, "\top"] & \mathsf{GrayCat}_{\mathrm{Q}^1} \arrow[l, bend left, "L"] \end{tikzcd}$$

The functor R takes a strict map $f\colon \mathbb{G} \longrightarrow \mathbb{H}$ to a pseudo map $fe\colon \mathbb{G} \nrightarrow \mathbb{H}$ where e is the co-unit of Q^1. Moreover, since e is an epimorphism, R is faithful, and it is bijective on objects, hence R is actually an inclusion; in particular, we have injective maps

$$\mathsf{GrayCat}(\mathbb{G}, \mathbb{H}) \xrightarrow{e^*} \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, \mathbb{H}) \tag{4}$$

for all $\mathbb{G}$ and $\mathbb{H}$.

We note that the composite of a strict map after a pseudo map is particularly simple:

$$\begin{tikzcd} \mathbb{G} \arrow[r, "f", "/" marking] & \mathbb{H} \arrow[r, "ge", "/" marking] & \mathbb{K} & = & \mathrm{Q}^1\mathbb{G} \arrow[r, "d_{\mathrm{Q}^1\mathbb{G}}"] \arrow[dr, equal] & \mathrm{Q}^1\mathrm{Q}^1\mathbb{G} \arrow[r, "\mathrm{Q}^1 f"] \arrow[d, "e_{\mathrm{Q}^1\mathbb{G}}"] & \mathrm{Q}^1\mathbb{H} \arrow[r, "ge"] \arrow[d, "e_{\mathbb{H}}"] & \mathbb{K} \, . \\ & & & & & \mathrm{Q}^1\mathbb{G} \arrow[r, "f"'] & \mathbb{H} \arrow[ur, "g"'] & \end{tikzcd} \tag{5}$$

If $\mathbb{G}$ is free up to order 1 we also get an idempotent function

$$\mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, \mathbb{H}) \xrightarrow{(ke)^*} \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, \mathbb{H}) \tag{6}$$

from (2) we might call strictification (note the reverse order of k and e). It preserves the image of the functor R, that is, strict Gray-functors are preserved.

2.19. LEMMA. *The category* $\mathsf{GrayCat}_{\mathrm{Q}^1}$ *has all limits of diagrams of strict maps, that is, those in the subcategory* $\mathsf{GrayCat}$*, that is,* $\mathsf{GrayCat}$ *is complete and the inclusion* $\mathsf{GrayCat} \longrightarrow \mathsf{GrayCat}_{\mathrm{Q}^1}$ *preserves all limits.*

PROOF Let D be a diagram in $\mathsf{GrayCat}$, let $(\ell_i\colon L \longrightarrow D_i)_i$ be a limiting source in $\mathsf{GrayCat}$, we claim its embedding into $\mathsf{GrayCat}_{\mathrm{Q}^1}$ is a limiting source there as well.

Let $(c_i\colon C \nrightarrow D_i)_i$ be a source over D in $\mathsf{GrayCat}_{\mathrm{Q}^1}$. Thus there is a source $(c_i\colon \mathrm{Q}^1 C \longrightarrow D_i)_i$ in $\mathsf{GrayCat}$, which induces a map $\langle c\rangle\colon \mathrm{Q}^1 C \longrightarrow L$ and this is of course a map $\langle c\rangle\colon C \nrightarrow L$. The diagram

$$\begin{tikzcd} C \arrow[d, "\langle c\rangle"', "/" marking] \arrow[dr, "c_i", "/" marking] & \\ L \arrow[r, "\ell_i"'] & D_i \end{tikzcd}$$

commutes for all i by the co-unit axiom of Q^1 and the naturality of e; c. f. also (5). Because e is an epimorphism $\langle c\rangle$ is the unique filler. $\square$

In particular, the pullback of two strict maps in $\mathsf{GrayCat}_{\mathrm{Q}^1}$ is the same as its pullback in $\mathsf{GrayCat}$. Products are obviously simply the same in both categories since their diagrams do not include any nontrivial morphisms.

2.20. Remark. *For two diagrams* $\{a_k\colon \mathbb{G}_i \longrightarrow \mathbb{G}_j\}$, $\{b_k\colon \mathbb{H}_i \longrightarrow \mathbb{H}_j\}$ *of strict maps of the same type in* $\mathsf{GrayCat}_{Q^1}$ *and a natural transformation* $f_i\colon \mathbb{G}_i \nrightarrow \mathbb{H}_i$ *between them there is an induced map* $\dot{\lim}\{f_i\}$ *such that:*

$$\begin{tikzcd}
\lim\{\mathbb{G}_i, a_k\} \arrow[r, "\dot{\lim} f_i", "/" marking] \arrow[d, "p_i"'] & \lim\{\mathbb{H}_i, b_k\} \arrow[d, "p'_i"] \\
\mathbb{G}_i \arrow[r, "f_i"', "/" marking] & \mathbb{H}_i
\end{tikzcd} \quad . \tag{7}$$

We unravel this diagram in terms of maps in $\mathsf{GrayCat}$ *and obtain*

$$\begin{tikzcd}
\mathrm{Q}^1 \lim\{\mathbb{G}_i, a_k\} \arrow[r, "\langle \mathrm{Q}^1 p_i\rangle"] \arrow[dr, "\mathrm{Q}^1 p_i"'] \arrow[rr, bend left, "\dot{\lim} f_i"] & \lim\{\mathrm{Q}^1\mathbb{G}_i, \mathrm{Q}^1 a_k\} \arrow[r, "\lim f_i"] \arrow[d, "r_i"] & \lim\{\mathbb{H}_i, b_k\} \arrow[d, "p'_i"] \\
& \mathrm{Q}^1\mathbb{G}_i \arrow[r, "f_i"'] & \mathbb{H}_i
\end{tikzcd}$$

where the map $\dot{\lim} f_i$ *is induced by the universal property of the source* $\{f_i\mathrm{Q}^1 p_i\}$ *in* $\mathsf{GrayCat}$, *that is,* $\dot{\lim}\{f_i\} = \langle f_i \mathrm{Q}^1 p_i\rangle$, *which then is the appropriate map in* $\mathsf{GrayCat}_{\mathrm{Q}^1}$. *On the other hand,* $\lim f_i$ *is induced by the cone* $f_i r_i$. *By universality* $\dot{\lim} f_i = \lim f_i \langle \mathrm{Q}^1 p_i\rangle$.

In particular this applies to pullbacks, that is, there is a canonical map

$$f \dot{\times} g\colon \mathbb{G} \times_{\mathbb{K}} \mathbb{H} \nrightarrow \mathbb{G}' \times_{\mathbb{K}'} \mathbb{H}'$$

determined by f, g, h in

$$\begin{tikzcd}
& & \mathbb{H} \arrow[rrr, "g", "/" marking] \arrow[ddl, "a"] & & & \mathbb{H}' \arrow[ddl, "a'"] \\
\mathbb{G} \arrow[rrr, "f", "/" marking] \arrow[dr, "b"'] & & & \mathbb{G}' \arrow[dr, "b'"'] & & \\
& \mathbb{K} \arrow[rrr, "h"', "/" marking] & & & \mathbb{K}' &
\end{tikzcd} \quad . \tag{8}$$

2.21. Remark. *If in (7) the maps f_i are of the form g_ie, i.e. the f_i come from strict maps, then we have*

$$\dot{\lim}(g_ie) = (\lim g_i)e\,.$$

In particular in a situation analogous to (8) we have

$$(fe)\dot{\times}(ge) = (f \times g)e \tag{9}$$

2.22. Special Cells in the Resolved Space. We now take a closer look at the structure of $\mathrm{Q}^1\mathbb{G}$. By definition 1-cells here are non-empty lists $[f_1, \ldots, f_n]$ of composable $\mathbb{G}$-1-cells modulo insertion or removal of identity 1-cells of $\mathbb{G}$; composition is concatenation. For composable 1-cells in $\mathbb{G}$, say, $f_1, \ldots, f_n$ we have several 1-cells in $\mathrm{Q}^1\mathbb{G}$, in particular $[f_1, \ldots, f_n] = [f_1]\#_0 \cdots \#_0[f_n]$ and $[f_1\#_0 \cdots \#_0 f_n]$ and $e_{\mathbb{G}}$ maps all of these to $f_1\#_0 \cdots \#_0 f_n$. Between $[f_1, \ldots, f_n]$ and $[f_1\#_0 \cdots \#_0 f_n]$ we have a 2-cell

$$\kappa_{f_1,\ldots,f_n} = (\mathrm{id}_{f_1\#_0\cdots\#_0 f_n}; [f_1, \ldots, f_n], [f_1\#_0 \cdots \#_0 f_n])$$

that is the pulled back identity 2-cell of $f_1\#_0 \cdots \#_0 f_n$. In particular we have

$$\begin{tikzcd}
\cdot \arrow[r, "{[f_2]}"] \arrow[dr, "{[f_1\#_0 f_2]}"'] & \cdot \arrow[d, "{[f_1]}"] \arrow[dl, phantom, "\overset{\kappa_{f_1,f_2}}{\swarrow}"] \\
& \cdot
\end{tikzcd}$$

for all for all pairs f_1, f_2 of 1-cells of $\mathbb{G}$. Whiskers and composites of higher cells in $\mathrm{Q}^1\mathbb{G}$ are simply carried out in $\mathbb{G}$, hence for example

$$\begin{aligned}
\kappa_{f_1,f_2}\#_0[f_3] &= (\mathrm{id}_{f_1\#_0 f_2}\#_0 f_3; [f_1, f_2]\#_0[f_3], [f_1\#_0 f_2]\#_0[f_3]) \\
&= (\mathrm{id}_{f_1\#_0 f_2\#_0 f_3}; [f_1, f_2, f_3], [f_1\#_0 f_2, f_3])
\end{aligned}$$

and

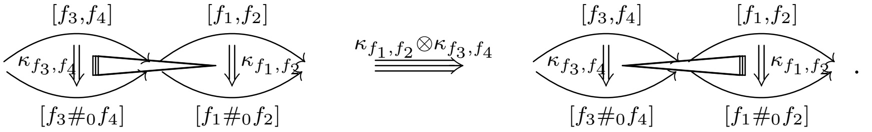
$$\kappa_{f_1\#_0 f_2,f_3}\#_1(\kappa_{f_1,f_2}\#_0[f_3]) = (\mathrm{id}_{f_1\#_0 f_2\#_0 f_3}; [f_1, f_2, f_3], [f_1\#_0 f_2\#_0 f_3]) = \kappa_{f_1,f_2,f_3}\,.$$

Hence we obtain that

$$\begin{tikzcd}
{[f_1]\#_0[f_2]\#_0[f_3]} \arrow[r, Rightarrow, "{[f_1]\#_0\kappa_{f_2,f_3}}"] \arrow[d, Rightarrow, "{\kappa_{f_1,f_2}\#_0[f_3]}"'] \arrow[dr, Rightarrow, "{\kappa_{f_1,f_2,f_3}}"] & {[f_1]\#_0[f_2\#_0 f_3]} \arrow[d, Rightarrow, "{\kappa_{f_1,f_2\#_0 f_3}}"] \\
{[f_1\#_0 f_2]\#_0[f_3]} \arrow[r, Rightarrow, "{\kappa_{f_1\#_0 f_2,f_3}}"'] & {[f_1\#_0 f_2\#_2 f_3]}
\end{tikzcd} \tag{10}$$

commutes.

We consider the possible horizontal composites of κ_{f_1,f_2} and κ_{f_3,f_4} and their tensor:



By (1) we obtain

$$\begin{aligned}
\kappa_{f_1,f_2}\otimes\kappa_{f_3,f_4} &= (\mathrm{id}_{f_1\#_0f_2};[f_1,f_2],[f_1\#_0f_2])\otimes(\mathrm{id}_{f_3\#_0f_4};[f_3,f_4],[f_3\#_0f_4]) \\
&= \begin{pmatrix} \mathrm{id}_{f_1\#_0f_2}\otimes\mathrm{id}_{f_3\#_0f_4}; \\ (\mathrm{id}_{f_1\#_0f_2}\#_0e[f_3\#_0f_4])\#_1(e[f_1,f_2]\#_0\mathrm{id}_{f_3\#_0f_4}), \\ (e[f_1\#_0f_2]\#_0\mathrm{id}_{f_3\#_0f_4})\#_1(\mathrm{id}_{f_1\#_0f_2}\#_0e[f_3,f_4]); \\ [f_1,f_2,f_3,f_4],[f_1\#_0f_2,f_3\#_0f_4] \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{id}_{\mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4}}; \\ (\mathrm{id}_{f_1\#_0f_2}\#_0f_3\#_0f_4)\#_1(f_1\#_0f_2\#_0\mathrm{id}_{f_3\#_0f_4}), \\ (f_1\#_0f_2\#_0\mathrm{id}_{f_3\#_0f_4})\#_1(\mathrm{id}_{f_1\#_0f_2}\#_0f_3\#_0f_4); \\ [f_1,f_2,f_3,f_4],[f_1\#_0f_2,f_3\#_0f_4] \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{id}_{\mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4}}; \\ (\mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4})\#_1(\mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4}), \\ (\mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4})\#_1(\mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4}); \\ [f_1,f_2,f_3,f_4],[f_1\#_0f_2,f_3\#_0f_4] \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{id}_{\mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4}}; \\ \mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4}, \\ \mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4}; \\ [f_1,f_2,f_3,f_4],[f_1\#_0f_2,f_3\#_0f_4] \end{pmatrix},
\end{aligned}$$

meaning that this tensor is the identity of the two possible horizontal composites of κ_{f_1,f_2} and κ_{f_3,f_4}.

Finally, note that by construction the $\kappa_{f_1,\ldots,f_n}$ are all invertible.

2.23. Pseudo Maps Explicitly. We provide an elementary characterization of pseudo Gray-functors.

2.24. Definition. *A* **pseudo Q^1 graph map** $F\colon \mathbb{G}\longrightarrow\mathbb{H}$ *between* Gray*-categories is a map of 3-globular sets, together with a function $F^2\colon \mathbb{G}_1\times_{\mathbb{G}_0}\mathbb{G}_1\longrightarrow\mathbb{H}_2$, such that the following conditions hold:*

1. *the restriction of F to $\mathbb{G}(x,y)$ is a sesquifunctor for all 0-cells x,y of $\mathbb{G}$,*
2. *F^2 is a normalized 2-cocycle, that is, the $F^2_{f_1,f_2}$ are invertible 2-cells $F^2_{f_1,f_2}\colon F(f_1)\#_0F(f_2)\Longrightarrow F(f_1\#_0f_2)$ with*

$$F^2_{f_1,f_2\#_0f_3}\#_1(F(f_1)\#_0F^2_{f_2,f_3}) = F^2_{f_1\#_0f_2,f_3}\#_1(F^2_{f_1,f_2}\#_0F(f_3)), \tag{11}$$

and for f_1 or f_2 an identity 1-cell we have

$$F^2_{f_1,f_2} = \mathrm{id}_{Ff_1\#_0 Ff_2},$$

3. *left and right whiskers of 2-cells by 1-cells along 0-cells are coherently preserved:*

$$\begin{aligned} F(\alpha\#_0 f)\#_1 F^2_{g,f} &= F^2_{g',f}\#_1(F\alpha\#_0 Ff) \\ F(g\#_0\beta)\#_1 F^2_{g,f} &= F^2_{g,f'}\#_1(Fg\#_0 F\beta) \end{aligned} \tag{12}$$

4. *left and right whiskers of 3-cells by 1-cells along 0-cells are coherently preserved:*

$$\begin{aligned} F(\Gamma\#_0 f)\#_1 F^2_{g,f} &= F^2_{g',f}\#_1(F\Gamma\#_0 Ff) \\ F(g\#_0\Delta)\#_1 F^2_{g,f} &= F^2_{g,f'}\#_1(Fg\#_0 F\Delta) \end{aligned} \tag{13}$$

5. *the tensor is coherently preserved:*

$$F(\beta\otimes\alpha)\#_1 F^2_{g,f} = F^2_{g',f'}\#_1(F\beta\otimes F\alpha) \tag{14}$$

6. *the tensors of compositors are trivial:*

$$\left(F^2_{f_1,f_2} \lhd F^2_{f_3,f_4} \xRightarrow{F^2_{f_1,f_2}\otimes F^2_{f_3,f_4}} F^2_{f_1,f_2} \rhd F^2_{fe_3,f_4} \right) = \mathrm{id} \tag{15}$$

7. *tensors of 2-co-cycle elements with images of 2-cells vanish:*

$$\left(F\alpha \lhd F^2_{g,f} \xRightarrow{F\alpha\otimes F^2_{g,f}} F\alpha \rhd F^2_{g,f} \right) = \mathrm{id} \tag{16}$$

$$\left(F^2_{h,g} \lhd F\beta \xRightarrow{F^2_{h,g}\otimes F\beta} F^2_{h,g} \rhd F\beta \right) = \mathrm{id} \tag{17}$$

for all suitably incident cells. Denote the set of all pseudo Q^1*-graph maps from* $\mathbb{G}$ *to* $\mathbb{H}$ *by* $M(\mathbb{G},\mathbb{H})$*.*

Note also how the identity 1-cells of a 0-cells are preserved strictly, this is part of the globularity condition.

Note furthermore how this definition implies that the horizontal composites are also coherently preserved as a consequence of (12):

$$\begin{aligned} F(\alpha\lhd\beta)\#_1 F^2_{g,f} &= F^2_{g',f'}\#_1(F\alpha\lhd F\beta) \\ F(\alpha\rhd\beta)\#_1 F^2_{g,f} &= F^2_{g',f'}\#_1(F\alpha\rhd F\beta)\,. \end{aligned}$$

2.25. Lemma. *There is a canonical correspondence between the set of pseudo* Q^1 *graph maps* $M(\mathbb{G},\mathbb{H})$ *and co-Kleisli maps* $\mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G},\mathbb{H})$.

$$\begin{tikzcd}
M(\mathbb{G},\mathbb{H}) \arrow[r, bend left, "(\_)^{\tilde{}}"] & \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G},\mathbb{H}) \arrow[l, bend left, "(\_)^{\vee}"]
\end{tikzcd}$$

Proof Given a Q^1 graph map $F\colon \mathbb{G} \longrightarrow \mathbb{H}$ we define a Gray-functor $\tilde{F}\colon \mathrm{Q}^1\mathbb{G} \longrightarrow \mathbb{H}$ as follows

1. 0-cells:
$$\tilde{F}(x) = F(x),$$
2. 1-cells:
$$\tilde{F}[f_1,\dots,f_n] = Ff_1\#_0\cdots\#_0Ff_n,$$
3. 2-cells:
$$\tilde{F}(\alpha;[f_1,\dots,f_n],[g_1,\dots,g_m]) = \overline{\tilde{F}\kappa_{g_1,\dots,g_m}}\#_1F\alpha\#_1\tilde{F}\kappa_{f_1,\dots,f_n} \tag{18}$$
where for $n=2$ the 2-cell $\tilde{F}\kappa_{f_1,\dots,f_n}$ is defined as $F^2_{f_1,f_2}$ and for $n\geq 3$ as the unique extension due to (11), (15),
4. 3-cells:
$$\tilde{F}(\Gamma;\alpha,\beta;[f_1,\dots,f_n],[g_1,\dots,g_m]) = \overline{\tilde{F}\kappa_{g_1,\dots,g_m}}\#_1F\Gamma\#_1\tilde{F}\kappa_{f_1,\dots,f_n}\,.$$

To elucidate, we show that 1-2-whiskers are preserved by $\tilde{F}$. For whiskerable cells

$$\begin{tikzcd}
{} \arrow[r, "{[f_1,\dots,f_n]}"] & {} \arrow[r, bend left, "{[g_1,\dots,g_m]}", ""{name=U, below}] \arrow[r, bend right, "{[g'_1,\dots,g'_{m'}]}"', ""{name=D, above}] & {} \arrow[from=U, to=D, Rightarrow, "{(\beta;\dots)}"]
\end{tikzcd}$$

the equation

$$\begin{tikzcd}[column sep=huge]
{} \arrow[r, "{\tilde F[f_1,\dots,f_n]}"] & {} \arrow[r, bend left, "{\tilde F[g_1,\dots,g_m]}"] \arrow[r, bend right, "{\tilde F[g'_1,\dots,g'_{m'}]}"'] \arrow[r, phantom, "{\Downarrow \tilde F(\beta;\dots)}"] & {}
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge]
{} \arrow[r, "Ff_1\#_0\cdots\#_0Ff_n"] & {} \arrow[r, bend left=80, "Fg_1\#_0\cdots\#_0Fg_m"] \arrow[r, bend left=25, "F(g_1\#_0\cdots\#_0g_m)" description] \arrow[r, bend right=25, "F(g'_1\#_0\cdots\#_0g'_{m'})" description] \arrow[r, bend right=80, "Fg'_1\#_0\cdots\#_0Fg'_{m'}"'] \arrow[r, phantom, shift left=6ex, "\Downarrow F\kappa_{g_1,\dots,g_m}"] \arrow[r, phantom, "\Downarrow F\beta"] \arrow[r, phantom, shift right=6ex, "\Downarrow \overline{F\kappa_{g'_1,\dots,g'_{m'}}}"] & {}
\end{tikzcd}$$

$$=
\begin{tikzcd}[column sep=huge]
{} \arrow[r, bend left=80, "Fg_1\#_0\cdots\#_0Fg_m\#_0Ff_1\#_0\cdots\#_0Ff_n"] \arrow[r, bend left=25, "F(g_1\#_0\cdots\#_0g_m\#_0f_1\#_0\cdots\#_0f_n)" description] \arrow[r, bend right=25, "F(g'_1\#_0\cdots\#_0g'_{m'}\#_0f_1\#_0\cdots\#_0f_n)" description] \arrow[r, bend right=80, "Fg'_1\#_0\cdots\#_0Fg'_{m'}\#_0Ff_1\#_0\cdots\#_0Ff_n"'] \arrow[r, phantom, shift left=6ex, "\Downarrow F\kappa_{g_1,\dots,g_m,f_1,\dots,f_n}"] \arrow[r, phantom, "\Downarrow F(\beta\#_0f_1\#_0\cdots\#_0f_n)"] \arrow[r, phantom, shift right=6ex, "\Downarrow \overline{F\kappa_{g'_1,\dots,g'_{m'},f_1,\dots,f_n}}"] & {}
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge]
{} \arrow[r, bend left=50, "{\tilde F([g_1,\dots,g_m]\#_0[f_1,\dots,f_n])}"] \arrow[r, bend right=50, "{\tilde F([g'_1,\dots,g'_{m'}]\#_0[f_1,\dots,f_n]).}"'] \arrow[r, phantom, "{\Downarrow \tilde F((\beta;\dots)\#_0[f_1,\dots,f_n])}"] & {}
\end{tikzcd}$$

is a consequence of (18).

Similarly, we can verify that $\tilde F$ preserves tensors: We calculate

$$\begin{aligned}
&\tilde F\left((\beta;[g_1,\dots,g_m],[g'_1,\dots,g'_{m'}])\otimes(\alpha;[f_1,\dots,f_n],[f'_1,\dots,f'_{n'}])\right)\\
&\quad= \tilde F\left(\beta\otimes\alpha;\beta\lhd\alpha,\beta\rhd\alpha;[g_1,\dots,g_m,f_1,\dots,f_n],[g'_1,\dots,g'_{m'},f'_1,\dots,f'_{n'}]\right)\\
&\quad= \overline{\tilde F\kappa_{g'_1,\dots,g'_{m'},f'_1,\dots,f'_{n'}}}\#_1F(\beta\otimes\alpha)\#_1\tilde F_{g_1,\dots,g_m,f_1,\dots,f_n}\\
&\quad= (\overline{\tilde F\kappa_{g'_1,\dots,g'_{m'}}}\otimes\overline{\tilde F\kappa_{f'_1,\dots,f'_{n'}}})\#_1(F\beta\otimes F\alpha)\#_1(\tilde F_{g_1,\dots,g_m}\otimes\tilde F_{f_1,\dots,f_n})\\
&\quad= (\overline{\tilde F\kappa_{g'_1,\dots,g'_{m'}}}\#_1F\beta\#_1\tilde F_{g_1,\dots,g_m})\otimes(\overline{\tilde F\kappa_{f'_1,\dots,f'_{n'}}}\#_1F\alpha\#_1\tilde F_{f_1,\dots,f_n})\\
&\qquad\tilde F(\beta;[g_1,\dots,g_m],[g'_1,\dots,g'_{m'}])\otimes\tilde F(\alpha;[f_1,\dots,f_n],[f'_1,\dots,f'_{n'}])
\end{aligned}$$

using (14) and (15). Preservation of the remaining operations is equally simple to verify.

Conversely, given a Gray-functor $G\colon \mathrm{Q}^1\mathbb{G}\longrightarrow\mathbb{H}$ we define a pseudo Q^1 graph map $\check G\colon\mathbb{G}\longrightarrow\mathbb{H}$ as follows:

1. 0-cells: $\check G(x)=G(x)$
2. 1-cells: $\check G(f)=G[f]$
3. 2-cells: $\check G(\alpha)=G(\alpha;[f],[f'])$
4. 3-cells: $\check G(\Gamma)=G(\Gamma;\alpha,\beta;[f],[f'])$

5. 2-co-cycle: $\check{G}^2_{f_1,f_2} = G\kappa_{f_1,f_2} = G(\mathrm{id}_{f_1\#_0 f_2}; [f_1\#_0 f_2], [f_1, f_2])$

This is obviously locally a sesquifunctor. We check the co-cycle condition:

$$
\begin{aligned}
&\check{G}^2_{f_1,f_2\#_0 f_3}\#_1(\check{G}f_1\#_0\check{G}^2_{f_2,f_3})\\
&\quad= G(\mathrm{id}_{f_1\#_0 f_2\#_0 f_3}; [f_1, f_2\#_0 f_3], [f_1\#_0 f_2\#_0 f_3])\#_1(G[f_1]\#_0 G(\mathrm{id}_{f_2\#_0 f_3}; [f_2, f_3], [f_2\#_0 f_3]))\\
&\quad= G(\mathrm{id}_{f_1\#_0 f_2\#_0 f_3}; [f_1, f_2\#_0 f_3], [f_1\#_0 f_2\#_0 f_3])\#_1 G(\mathrm{id}_{f_1\#_0 f_2\#_0 f_3}; [f_1, f_2, f_3], [f_1, f_2\#_0 f_3])\\
&\qquad\qquad\qquad\qquad= G(id_{f_1\#_0 f_2\#_0 f_3}; [f_1, f_2, f_3], [f_1\#_0 f_2\#_0 f_3])\\
&\quad= G(\mathrm{id}_{f_1\#_0 f_2\#_0 f_3}; [f_1\#_0 f_2, f_3], [f_1\#_0 f_2\#_0 f_3])\#_1 G(\mathrm{id}_{f_1\#_0 f_2\#_0 f_3}; [f_1, f_2, f_3], [f_1\#_0 f_2, f_3])\\
&\quad= G(\mathrm{id}_{f_1\#_0 f_2\#_0 f_3}; [f_1\#_0 f_2, f_3], [f_1\#_0 f_2\#_0 f_3])\#_1(G(\mathrm{id}_{f_1\#_0 f_2}; [f_1, f_2], [f_1\#_0 f_2])\#_0 G[f_3])\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad= \check{G}^2_{f_1\#_0 f_2,f_3}\#_1(\check{G}^2_{f_1,f_2}\#_0\check{G}f_3)
\end{aligned}
$$

Furthermore, we check the coherent preservation of whiskers:

$$
\begin{aligned}
&\check{G}(\alpha\#_0 f)\#_1\check{G}^2_{g,f}\\
&\qquad\qquad= G(\alpha\#_0 f; [g\#_0 f], [g'\#_0 f])\#_1 G(\mathrm{id}_{g\#_0 f}; [g, f], [g\#_0 f])\\
&\qquad\qquad\qquad\qquad= G(\alpha\#_0 f; [g, f], [g'\#_0 f])\\
&\qquad\qquad= G(\mathrm{id}_{g'\#_0 f}; [g', f], [g'\#_0 f])\#_1 G(\alpha\#_0; [g, f], [g', f])\\
&\qquad\qquad= G(\mathrm{id}_{g'\#_0 f}; [g', f], [g'\#_0 f])\#_1(G(\alpha; [g], [g'])\#_0 G[f])\\
&\qquad\qquad\qquad\qquad\qquad\qquad= \check{G}^2_{g',f}\#_1(\check{G}\alpha\#_0\check{G}f)
\end{aligned}
$$

The remaining axioms are verified just as easily.

We verify briefly that $\check{\check{G}} = G$, for 1-cells we have

$$\check{\check{G}}[f_1, \dots, f_n] = \check{G}f_1\#_0\dots\#_0\check{G}f_n = G[f_1]\#_0\dots\#_0 G[f_n] = G[f_1, \dots, f_n]$$

and for 2-cells:

$$
\begin{aligned}
&\check{\check{G}}(\alpha; [f_1, \dots, f_n], [f'_1, \dots, f'_{n'}]) = \overline{\check{G}\kappa_{f'_1,\dots,f'_{n'}}}\#_1\check{G}\#_1\check{G}\kappa_{f_1,\dots,f_n}\\
&\qquad\qquad= \begin{pmatrix} G(\mathrm{id}_{f'_1\#_0\cdots\#_0 f'_{n'}}; [f'_1\#_0\cdots\#_0 f'_{n'}], [f'_1, \dots, f'_{n'}])\\ \#_1 G(\alpha; [f'_1\#_0\cdots\#_0 f'_{n'}], [f_1\#_0\cdots\#_0 f_n])\\ \#_1 G(\mathrm{id}_{f_1\#_0\cdots\#_0 f_n}; [f_1, \dots, f_n], [f_1\#_0\cdots\#_0 f_n]) \end{pmatrix}\\
&\qquad\qquad\qquad\qquad\qquad\qquad G(\alpha; [f_1, \dots, f_n], [f'_1, \dots, f'_{n'}])
\end{aligned}
$$

Finally, $\check{\check{F}} = F$. ☐

2.26. REMARK. *Given two pseudo* Q^1 *graph maps* $F\colon \mathbb{G} \longrightarrow \mathbb{H}$ *and* $G\colon \mathbb{H} \longrightarrow \mathbb{K}$ *their composite* GF *is simply the composite of the underlying globular maps with cocycle*

$$(GF)^2_{f_1,f_2} = GF^2_{f_1,f_2}\#_1 G^2_{Ff_1,Ff_2}\,.$$

2.27. LEMMA. *Under the correspondence in lemma 2.25 a pseudo* Q^1*-graph map* F *has trivial cocycle* F^2 *iff the corresponding* Gray*-functor* $\tilde{F}$ *is of the form* Ge*.*

PROOF Considering definition 2.24 we see that $F \in M(\mathbb{G}, \mathbb{H})$ is an ordinary Gray-functor iff F^2 is trivial, in which case Fe is the embedding of F in $\mathsf{GrayCat}_{\mathrm{Q}^1}$ with ${(Fe)^{\vee}}^2_{f_1,f_2} = Fe\kappa_{f_1,f_2} = Fe(\mathrm{id}_{f_1\#_0 f_2}; [f_1\#_0 f_2], [f_1, f_2]) = F\mathrm{id}_{f_1\#_0 f_2} = \mathrm{id}_{F(f_1\#_0 f_2)}$. That is actually $G = F$.

In turn, if we are given a co-Kleisli map Ge with G a Gray-functor we obtain ${(Ge)^{\vee}}^2_{f_1,f_2} = Ge\kappa_{f_1,f_2} = \mathrm{id}_{G(f_1\#_0 f_2)}$. ☐

In particular for $\mathbb{G}$ free up to order 1 with section k (6) induces an idempotent map

$$M(\mathbb{G}, \mathbb{H}) \xrightarrow{((\tilde{\_})ke)^{\vee}} M(\mathbb{G}, \mathbb{H}) \tag{19}$$

with image $\mathsf{GrayCat}(\mathbb{G}, \mathbb{H})$.

We spell out the action of this map on an arbitrary pseudo Q^1 graph map $F\colon \mathbb{G} \longrightarrow \mathbb{H}$ for $\mathbb{G}$, free up to order 1, at the level of 1- and 2-cells. Let $f_1 = g_{1,1}\#_0 \cdots \#_0 g_{1,n_1}$ and $f_2 = g_{2,1}\#_0 \cdots \#_0 g_{2,n_2}$ be unique decompositions up to units in $\mathbb{G}$ of the 1-cells f_1, f_2. This means that $k(f_1) = [g_{1,1}, \ldots, g_{1,n_1}]$, $k(f_2) = [g_{2,1}, \ldots, g_{2,n_2}]$. Furthermore, for a 2-cell $\alpha\colon f \Longrightarrow f'$ we have $k(\alpha) = (\alpha; [g_{1,1}, \ldots, g_{1,n}], [g'_{1,1}, \ldots, g'_{1,n'}])$, in particular $k(\mathrm{id}_f) = (\mathrm{id}_f; [g_{1,1}, \ldots, g_{1,n}], [g_{1,1}, \ldots, g_{1,n}])$. Hence for a composite we get

$$\begin{aligned}
&(\tilde{F}ke)^{\vee}(f_1\#_0 f_2)\\
&\qquad = (\tilde{F}ke)[f_1\#_0 f_2] = \tilde{F}k(f_1\#_0 f_2) = \tilde{F}[g_{1,1}, \ldots, g_{1,n_1}, g_{2,1}, \ldots, g_{2,n_2}]\\
&\qquad\qquad = Fg_{1,1}\#_0 \cdots, Fg_{1,n_1}\#_0 Fg_{2,1}\#_0 \cdots \#_0 g_{2,n_2}\\
&\qquad\qquad\qquad = (\tilde{F}ke)^{\vee}(f_1)\#_0(\tilde{F}ke)^{\vee}(f_2)\,.
\end{aligned} \tag{20}$$

For the 2-cocycle we get

$$\begin{aligned}
&{(\tilde{F}ke)^{\vee}}^2_{f_1,f_2} = (\tilde{F}ke)^{\vee}(\kappa_{f_1,f_2}) = \tilde{F}k(\mathrm{id}_{f_1\#_0 f_2})\\
&\qquad = \tilde{F}(\mathrm{id}_{f_1\#_0 f_2}; [g_{1,1}, \ldots, g_{1,n_1}, g_{2,1}, \ldots, g_{2,n_2}], [g_{1,1}, \ldots, g_{1,n_1}, g_{2,1}, \ldots, g_{2,n_2}])\\
&\qquad\qquad = \overline{\tilde{F}\kappa_{g_{1,1},\ldots,g_{1,n_1},g_{2,1},\ldots,g_{2,n_2}}}\#_1 F\mathrm{id}_{f_1\#_0 f_2}\#_1\tilde{F}\kappa_{g_{1,1},\ldots,g_{1,n_1},g_{2,1},\ldots,g_{2,n_2}}\\
&\qquad\qquad \overline{\tilde{F}\kappa_{g_{1,1},\ldots,g_{1,n_1},g_{2,1},\ldots,g_{2,n_2}}}\#_1\mathrm{id}_{F(f_1\#_0 f_2)}\#_1\tilde{F}\kappa_{g_{1,1},\ldots,g_{1,n_1},g_{2,1},\ldots,g_{2,n_2}}\\
&\qquad\qquad\qquad \overline{\tilde{F}\kappa_{g_{1,1},\ldots,g_{1,n_1},g_{2,1},\ldots,g_{2,n_2}}}\#_1\tilde{F}\kappa_{g_{1,1},\ldots,g_{1,n_1},g_{2,1},\ldots,g_{2,n_2}}\\
&\qquad\qquad\qquad\qquad = \mathrm{id}_{\tilde{F}[g_{1,1},\ldots,g_{1,n_1},g_{2,1},\ldots,g_{2,n_2}]}\\
&\qquad\qquad\qquad\qquad\qquad = \mathrm{id}_{Fg_{1,1}\#_0\cdots\#_0 Fg_{1,n_1}\#_0 Fg_{2,1}\#_0\cdots\#_0 Fg_{2,n_2}}\,.
\end{aligned} \tag{21}$$

These equations (20) and (21) make it palpable how the operation (19) yields a strict Gray-functor.

We will see in section 6 how F and it's strictification Fke are related.

3. Path Spaces

We construct a path space for Gray-categories and prove some essential properties. We derived the idea for this construction from Bénabou [1967]. Maps into this space can be viewed as right homotopies between functors and are our axiomatization of transformation for morphisms in $\mathsf{GrayCat}_{\mathbb{Q}^1}$. In section 4 we will introduce an internal category structure for this path space; its composition operation will allows us to compose transformations.

3.1. Definition. *Given a* Gray*-category* $\mathbb{H}$ *we define the* **path space** $\overrightarrow{\mathbb{H}}$ *where the cells in each dimension are diagrams in* $\mathbb{H}$*:*

$$\overrightarrow{\mathbb{H}}_0 = \{ \xrightarrow{f} \} \tag{22}$$

$$\overrightarrow{\mathbb{H}}_1 = \left\{ (g_2; g_0, g_1, f, f') \;\middle|\; \begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "g_0"'] & \cdot \arrow[d, "g_1"] \arrow[dl, Rightarrow, "g_2"] \\ \cdot \arrow[r, "f'"'] & \cdot \end{tikzcd} \right\} \tag{23}$$

$$\overrightarrow{\mathbb{H}}_2 = \left\{ \begin{pmatrix} \alpha_3; \alpha_1, \alpha_2, g_2, h_2; \\ g_0, g_1, h_0, h_1, f, f' \end{pmatrix} \;\middle|\; \begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "g_0"] \arrow[d, bend right=80, "h_0"'] \arrow[d, phantom, bend right=40, "\underset{\alpha_1}{\Leftarrow}"] & \cdot \arrow[d, "g_1"] \arrow[dl, Rightarrow, "g_2"] \arrow[r, phantom, "\overset{\alpha_3}{\Rrightarrow}"] & \cdot \arrow[r, "f"] \arrow[d, "h_0"'] & \cdot \arrow[d, "h_1"'] \arrow[dl, Rightarrow, "h_2"] \arrow[d, bend left=80, "g_1"] \arrow[d, phantom, bend left=40, "\underset{\alpha_2}{\Leftarrow}"] \\ \cdot \arrow[r, "f'"'] & \cdot & \cdot \arrow[r, "f'"'] & \cdot \end{tikzcd} \right\} \tag{24}$$

$$\overrightarrow{\mathbb{H}}_3 = \left\{ \begin{pmatrix} \Gamma_1, \Gamma_2, \alpha_3, \beta_3; g_2, h_2, \\ \alpha_1, \alpha_2, \beta_1, \beta_2; \\ g_0, g_1, h_0, h_1, f, f' \end{pmatrix} \;\middle|\; \begin{pmatrix} \Gamma_1 \colon \alpha_1 \Rrightarrow \beta_1, \\ \Gamma_2 \colon \alpha_2 \Rrightarrow \beta_2 \end{pmatrix} \text{ such that } \begin{matrix} \beta_3 \#_2 ((f' \#_0 \Gamma_1) \#_1 g_2) \\ = (h_2' \#_1 (\Gamma_2 \#_0 f)) \#_2 \alpha_3 \end{matrix} \right\} \tag{25}$$

Compositions and identities arise canonically from pasting of diagrams in $\mathbb{H}$*, as detailed below.*

The condition in (25) on the 3-cells is the commutativity of the following diagram

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "g_0"] \arrow[d, bend right=80, "h_0"'] \arrow[d, phantom, bend right=40, "\Leftarrow\alpha_1"] & \cdot \arrow[d, "g_1"] \arrow[dl, Rightarrow, "g_2"] & \overset{\alpha_3}{\Rrightarrow} & \cdot \arrow[r, "f"] \arrow[d, "h_0"'] & \cdot \arrow[d, "h_1"'] \arrow[dl, Rightarrow, "h_2"] \arrow[d, bend left=80, "g_1"] \arrow[d, phantom, bend left=40, "\Leftarrow\alpha_2"] \\ \cdot \arrow[r, "f'"'] & \cdot & & \cdot \arrow[r, "f'"'] & \cdot \\ (f' \#_0 \Gamma_1) \#_1 g_2 \Downarrow & & & & \Downarrow h_2 \#_1 (\Gamma_2 \#_0 f) \\ \cdot \arrow[r, "f"] \arrow[d, "g_0"] \arrow[d, bend right=80, "h_0"'] \arrow[d, phantom, bend right=40, "\Leftarrow\beta_1"] & \cdot \arrow[d, "g_1"] \arrow[dl, Rightarrow, "g_2"] & \underset{\beta_3}{\Rrightarrow} & \cdot \arrow[r, "f"] \arrow[d, "h_0"'] & \cdot \arrow[d, "h_1"'] \arrow[dl, Rightarrow, "h_2"] \arrow[d, bend left=80, "g_1"] \arrow[d, phantom, bend left=40, "\Leftarrow\beta_2"] \\ \cdot \arrow[r, "f'"'] & \cdot & & \cdot \arrow[r, "f'"'] & \cdot \end{tikzcd} \tag{26}$$

The identities in each dimension are obviously the ones consisting of identity cells.

3.2. Remark. *By construction the map* $(d_0, d_1)\colon \overrightarrow{\mathbb{H}} \longrightarrow \mathbb{H} \times \mathbb{H}$ *is 2-faithful in the sense of definition 2.9, but in general not full.*

3.3. Remark. *The map* $i\colon \mathbb{H} \longrightarrow \overrightarrow{\mathbb{H}}$ *is 2-Cartesian and 1-faithful, but not in general 1-full.*

3.4. Path Spaces and Cartesian Maps.

3.5. Lemma. *The path space construction* $\overrightarrow{(\_)}$ *of* Gray*-categories preserves 1-Cartesianness of maps.*

Proof Let's as assume we have a situation

$$\begin{tikzcd} \overrightarrow{\mathbb{G}} \arrow[r, "\overrightarrow{F}"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & \overrightarrow{\mathbb{H}} \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\ \mathbb{G} \arrow[r, "F"'] & \mathbb{H} \end{tikzcd} ,$$

take a pair of parallel 1-cells in $\overrightarrow{\mathbb{G}}$

$$\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "g_0"'] & {} \arrow[d, "g_1"] \arrow[dl, Rightarrow, shorten=4mm, "g_2"] \\ {} \arrow[r, "f'"'] & {} \end{tikzcd} \qquad\qquad \begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "h_0"'] & {} \arrow[d, "h_1"] \arrow[dl, Rightarrow, shorten=4mm, "h_2"] \\ {} \arrow[r, "f'"'] & {} \end{tikzcd}$$

we need to show that $\overrightarrow{F}$ is bijective on the intervening 2-cells. That means given

$$\beta_1\colon F(g_0) \Longrightarrow f(h_0) \quad \beta_2\colon F(g_1) \Longrightarrow F(h_1) \quad \beta_3\colon F(g_2\#_1(\beta_2\#_0 f)) \Rrightarrow F((f'\#_0\beta_1)\#_1 g_2)$$

there are unique

$$\alpha_1\colon g_0 \Longrightarrow h_0 \qquad \alpha_2\colon g_1 \Longrightarrow h_1 \qquad \alpha_3\colon g_2\#_1(\alpha_2\#_0 f) \Rrightarrow (f'\#_0\alpha_1)\#_1 g_2$$

with $F(\alpha_i) = \beta_i$. But these exist uniquely by the 1-Cartesianness of F.

The same kind of argument can be applied to parallel 2-cells in $\overrightarrow{\mathbb{G}}$. □

3.6. Remark. *The functor* $\overrightarrow{(\_)}$ *preserves 2-Cartesian maps.*

3.7. Lemma. *A pullback of a Cartesian map is Cartesian if* p *preserves pullbacks.*

Proof Let F be p-Cartesian, and G^*F the pullback of F along G.

$$\begin{tikzcd} {} \arrow[dr, dashed] \arrow[drr, "\langle p(F^*G)u\rangle"] \arrow[ddr, "H"'] & & \\ & {} \arrow[r] \arrow[d, "G^*F"] & {} \arrow[d, "F"] \\ & {} \arrow[r, "G"'] & {} \end{tikzcd}$$

Let H factor through G below as $p(H) = p(G^*F)u$, then GH factors through F below as $p(GH) = p(GG^*F)u = p(F)p(F^*G)u$, hence there is a unique lift $\langle p(F^*G)u\rangle$. Hence there is a universally induced $\langle u\rangle$ with $G^*F\langle u\rangle = H$.

The functor p preserving pullbacks ensures that $p\langle u\rangle = u$. □

3.8. VERTICAL COMPOSITION OPERATIONS IN THE PATH SPACE. We need to describe the vertical composition of 1-, 2-, 3-cells along 0-, 1-, 2-cells respectively.

We designate the composition in $\mathbb{H}$ by $\#_i$ and the interchange by $\otimes$, in $\overrightarrow{\mathbb{H}}$ we define the respective operations $\Box_i$ and $\boxtimes$ as follows:

$$h\Box_0 g = (h_2; h_0, h_1, f'', f')\Box_0(g_2; g_0, g_1, f, f') = \begin{pmatrix} (h_2\#_0 g_0)\#_1(h_1\#_0 g_2); \\ h_0\#_0 g_0, h_1\#_0 g_1, f, f'' \end{pmatrix}$$

This is just the vertical pasting

$$\begin{tikzcd}
{} \arrow[r, "f"] \arrow[d, "g_0"'] & {} \arrow[d, "g_1"] \arrow[dl, Rightarrow, shorten=4mm, "g_2" description] \\
{} \arrow[r, "f'" description] \arrow[d, "h_0"'] & {} \arrow[d, "h_1"] \arrow[dl, Rightarrow, shorten=4mm, "h_2" description] \\
{} \arrow[r, "f''"'] & {}
\end{tikzcd} . \tag{27}$$

Obviously this composition is associative and unital.

3.9. REMARK. *Considering (27) we note that if the 1-cells in $\mathbb{H}$ are invertible, with inverse $\overline{(\_)}$, then the 2-cell*

$$(h_2\#_0 g_0)\#_1(h_1\#_0 g_2)$$

in (27) *can also be written as a horizontal composite in two different ways:*

$$(h_2\#_0\overline{f'}) \lhd g_2 = h_2 \lhd (\overline{f'}\#_0 g_2) \tag{28}$$

There is of course also the opposite horizontal composite

$$(h_2\#_0\overline{f'}) \rhd g_2 = h_2 \rhd (\overline{f'}\#_0 g_2) \tag{29}$$

and a 3-cell

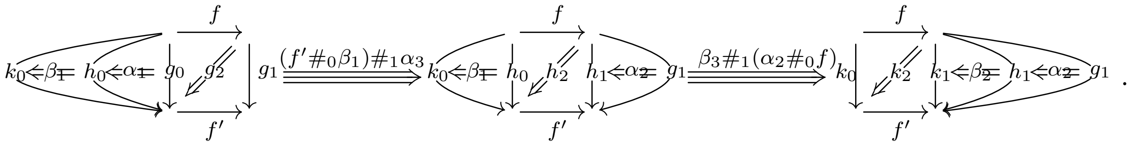
$$(h_2\#_0\overline{f'})\otimes g_2 = h_2\otimes(\overline{f'}\#_0 g_2)$$

going from (28) *to* (29). *The picture* (27)*, however, always means* (28).

The vertical composite of two 2-cells is

$$\beta\Box_1\alpha = \begin{pmatrix} \beta_3; \beta_1, \beta_2, h_2, k_2; \\ h_0, h_1, k_0, k_1, f, f' \end{pmatrix} \Box_1 \begin{pmatrix} \alpha_3; \alpha_1, \alpha_2, g_2, h_2; \\ g_0, g_1, h_0, h_1, f, f' \end{pmatrix}$$
$$= \begin{pmatrix} (\beta_3\#_1(\alpha_2\#_0 f))\#_2((f'\#_0\beta_1)\#_1\alpha_3); \\ \beta_1\#_1\alpha_1, \beta_2\#_1\alpha_2, g_2, h_2; g_0, g_1, k_0, k_1, f, f' \end{pmatrix} \tag{30}$$

which has as its first component the following composite of $\mathbb{H}$-3-cells



We shall henceforth argue mostly diagrammatically in terms of such 3-cell diagrams, as it is fairly obvious what the lower dimensional components are.

Vertical composition of $\overrightarrow{\mathbb{H}}$-3-cells is particularly simple:

$$\Delta\Box_2\Gamma = \begin{pmatrix}\Delta_1\colon \beta_1 \Rrightarrow \gamma_1,\\ \Delta_2\colon \beta_2 \Rrightarrow \gamma_2\end{pmatrix} \Box_2 \begin{pmatrix}\Gamma_1\colon \alpha_1 \Rrightarrow \beta_1,\\ \Gamma_2\colon \alpha_2 \Rrightarrow \beta_2\end{pmatrix} = \begin{pmatrix}\Delta_1\#_2\Gamma_1\colon \alpha_1 \Rrightarrow \gamma_1,\\ \Delta_2\#_2\Gamma_2\colon \alpha_2 \Rrightarrow \gamma_2\end{pmatrix} \tag{31}$$

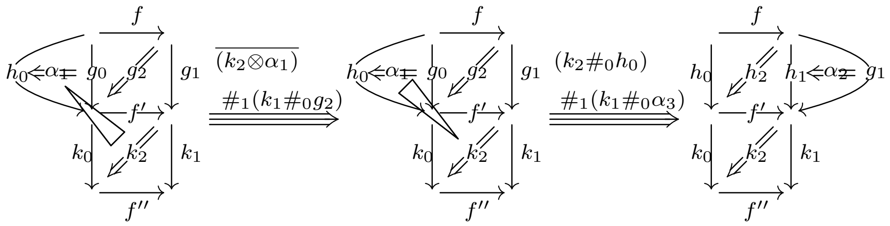
The condition (26) is obviously satisfied, since we just paste two instances of the commuting square vertically.

3.10. Whiskers. We need to define three whiskering operations, ${}^1\Box_0^2$, ${}^1\Box_0^3$, ${}^2\Box_1^3$, where the raised indices indicate the dimension of the operands, the lower one the dimension of the incidence cell. Their symmetry partners are then obvious.

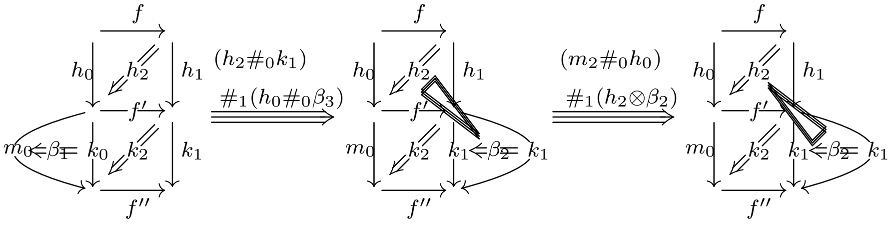
We define right whiskering of a 2-cell by a 1-cell as:

$$\begin{aligned} k\,{}^1\Box_0^2\alpha &= (k_2; k_0, k_1, f', f'')\,{}^1\Box_0^2 \begin{pmatrix}\alpha_3; \alpha_1, \alpha_2;\\ g_0, g_1, h_0, h_1, f, f'\end{pmatrix} \\ &= \begin{pmatrix} ((k_2\#_0 h_0)\#_1(k_1\#_0\alpha_3)) \\ \#_2(\overline{(k_2\otimes\alpha_1)}\#_1(k_1\#_0 g_2)); \\ k_0\#_0\alpha_1, k_1\#_0\alpha_2; \\ k_0\#_0 g_0, k_1\#_1 g_1, k_0\#_0 h_0, k_1\#_0 h_1, f, f'' \end{pmatrix}. \end{aligned} \tag{32}$$

Diagrammatically this is the following composite:



For reference $(\beta_1, \beta_2, \beta_3)\Box_0(h_0, h_1, h_2)$ is

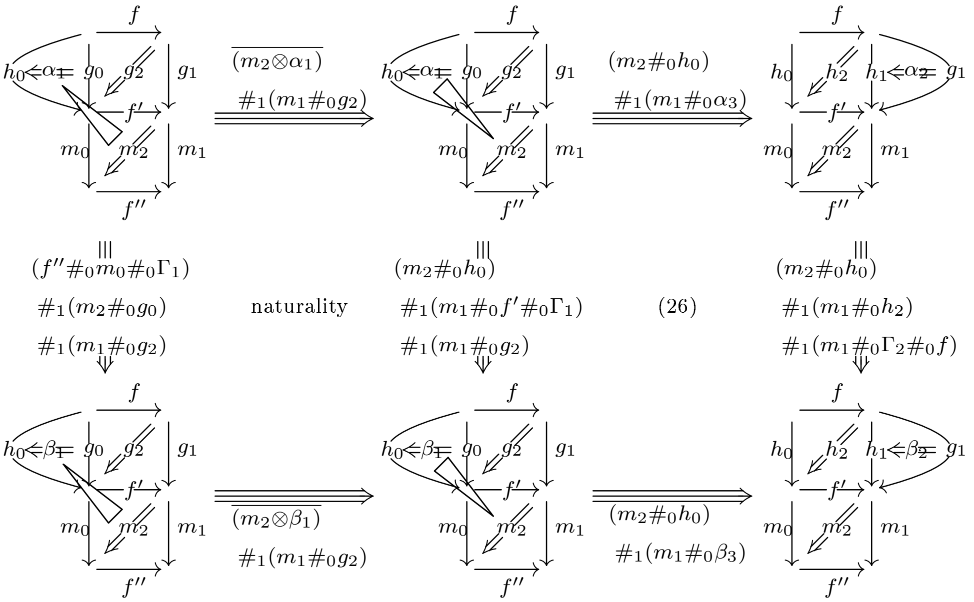


The action of 1-cells on 3-cells is as follows:

$$m^1\Box^3_0\Gamma = (m_2; m_1, m_2, f', f'')^1\Box^3_0 \begin{pmatrix} \Gamma_1, \Gamma_2, \alpha_3, \beta_3;\\ \alpha_1, \alpha_2, \beta_1\beta_2, g_2, h_2;\\ g_0, g_1, h_0, h_1, f, f' \end{pmatrix}$$

$$= \begin{pmatrix} m_0\#_0\Gamma_1, m_1\#_0\Gamma_2,\\ ((m_2\#_0h_0)\#_1(m_1\#_0\alpha_3))\#_2(\overline{(m_2\otimes\alpha_1)}\#_1(m_1\#_0g_2)),\\ ((m_2\#_0h_0)\#_1(m_1\#_0\beta_3))\#_2(\overline{((m_2\otimes\beta_1))}\#_1(m_1\#_0g_2));\\ m_0\#_0\alpha_1, m_0\#_1\alpha_2, m_0\#_0\beta_1, m_1\#_0\beta_2,\\ (m_2\#_0g_0)\#_1(m_1\#_0g_2), (m_2\#_0h_0)\#_1(m_1\#_0h_2);\\ m_0\#_0g_0, m_1\#_0g_1, m_0\#_0h_0, m_1\#_0h_1, f, f'' \end{pmatrix}$$

We claim this is again a proper 3-cell in $\overrightarrow{\mathbb{H}}$, that is, the whisker satisfies (26), as can be easily seen:

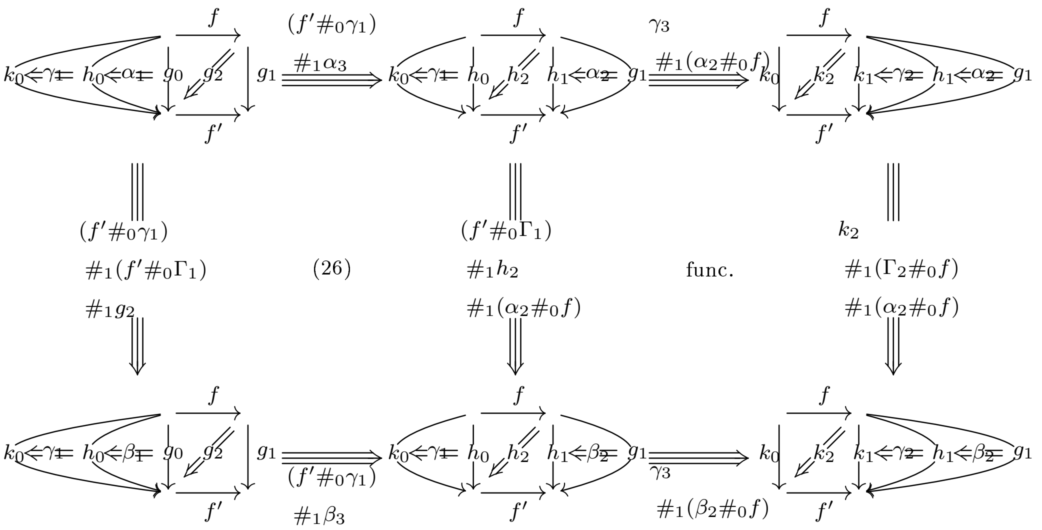


Finally, we define 3-2-whiskering:

$$\gamma^2\Box^3_1\Gamma = \begin{pmatrix} \gamma_3; \gamma_1, \gamma_2, h_2, k_2;\\ h_0, h_1, k_0, k_1, f, f' \end{pmatrix} {}^2\Box^3_1 \begin{pmatrix} \Gamma_1, \Gamma_2, \alpha_3, \beta_3; g_2, h_2,\\ \alpha_1, \alpha_2, \beta_1, \beta_2;\\ g_0, g_1, h_0, h_1, f, f' \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_1\#_1\Gamma_1, \gamma_2\#_1\Gamma_2,\\ (\gamma_3\#_1(\alpha_2\#_0f))\#_2((f'\#_0\gamma_1)\#_1\alpha_3),\\ (\gamma_3\#_1(\beta_2\#_0f))\#_2((f'\#_0\gamma_1)\#_1\beta_3);\\ g_2, k_2, \gamma_1\#_1\alpha_1, \gamma_2\#_1\alpha_2, \gamma_1\beta_1, \gamma_2\beta_2;\\ g_0, g_1, k_0, k_1, f, f' \end{pmatrix} \quad (33)$$

It yields a 3-cell in $\overrightarrow{\mathbb{H}}$:



(34)

3.11. Horizontal Composition of 2-Cells. We shall use the following slightly abbreviated notation for the higher cells of the mapping space, for example writing (32) as:

$$\begin{tikzcd} {} \arrow[r, bend left, "g"] \arrow[r, bend right, "n"'] \arrow[r, phantom, "\Downarrow\alpha"] & {} \arrow[r, "k"] & {} \end{tikzcd}$$

$$= k^1\square_0^2\alpha = (k_2; k_0, k_1, f', f'')^1\square_0^2\left(\alpha_3; \alpha_1, \alpha_2\middle| g, n\right)$$

$$= \begin{pmatrix} ((k_2\#_0 n_0)\#_1(k_1\#_0\alpha_3))\#_2(\overline{(k_2\otimes\alpha_1)}\#_1(k_1\#_0 g_2)); \\ k_0\#_0\alpha_1, k_1\#_0\alpha_2\middle| k\square_0 g, k\square_0 n \end{pmatrix}.$$

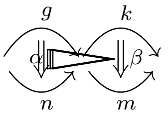
In the same spirit we write the opposite whiskering:

$$\begin{tikzcd} {} \arrow[r, "n"] & {} \arrow[r, bend left, "k"] \arrow[r, bend right, "m"'] \arrow[r, phantom, "\Downarrow\beta"] & {} \end{tikzcd}$$

$$= \beta^2\square_0^1 n = \left(\beta_3; \beta_1, \beta_2\middle| k, m\right)$$

$$= \begin{pmatrix} ((m_2\#_0 n_0)\#_1(\beta_2\otimes n_2))\#_2(\beta_3\#_1(k_1\#_0 n_2)); \\ \beta_1\#_0 n_0, \beta_2\#_0 n_1\middle| k\square_0 n, m\square_0 n \end{pmatrix}.$$

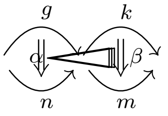
So now we can define the left horizontal composite:



$$= \beta \boxed{\lhd} \alpha = \begin{pmatrix} ((m_2\#_0 n_0)\#_1(\beta_2\otimes n_2)) \\ \#_2(\beta_3\#_1(k_1\#_0 n_2)); \\ \beta_1\#_0 n_0, \beta_2\#_0 n_1 \big| k\square_0 n, m\square_0 n \end{pmatrix} \square_1 \begin{pmatrix} ((k_2\#_0 n_0)\#_1(k_1\#_0\alpha_3)) \\ \#_2(\overline{(k_2\otimes\alpha_1)}\#_1(k_1\#_0 g_2)); \\ k_0\#_0\alpha_1, k_1\#_0\alpha_2 \big| k\square_0 g, k\square_0 n \end{pmatrix}$$

$$= \begin{pmatrix} \left(\begin{pmatrix} ((m_2\#_0 n_0)\#_1(\beta_2\otimes n_2)) \\ \#_2(\beta_3\#_1(k_1\#_0 n_2)) \end{pmatrix} \#_1(k_1\#_0\alpha_2\#_0 f)\right) \\ \#_2 \left((f''\#_0\beta_1\#_0 n_0)\#_1 \begin{pmatrix} ((k_2\#_0 n_0)\#_1(k_1\#_0\alpha_3)) \\ \#_2(\overline{(k_2\otimes\alpha_1)}\#_1(k_1\#_0 g_2)) \end{pmatrix}\right); \\ \alpha_1 \lhd \beta_1, \alpha_2 \lhd \beta_2 \big| k\square_0 g, m\square_0 n \end{pmatrix}$$

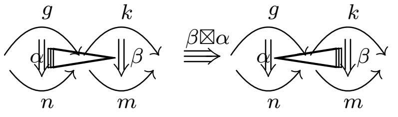
and conversely,



$$= \beta \boxed{\lhd} \alpha = \begin{pmatrix} ((m_2\#_0 n_0)\#_1(\beta_2\otimes n_2)) \\ \#_2(\beta_3\#_1(k_1\#_0 n_2)); \\ \beta_1\#_0 n_0, \beta_2\#_0 n_1 \big| k\square_0 n, m\square_0 n \end{pmatrix} \square_1 \begin{pmatrix} ((k_2\#_0 n_0)\#_1(k_1\#_0\alpha_3)) \\ \#_2(\overline{(k_2\otimes\alpha_1)}\#_1(k_1\#_0 g_2)); \\ k_0\#_0\alpha_1, k_1\#_0\alpha_2 \big| k\square_0 g, k\square_0 n \end{pmatrix}$$

$$= \begin{pmatrix} \left(\begin{pmatrix} ((m_2\#_0 n_0)\#_1(\beta_2\otimes n_2)) \\ \#_2(\beta_3\#_1(k_1\#_0 n_2)) \end{pmatrix} \#_1(k_1\#_0\alpha_2\#_0 f)\right) \\ \#_2 \left((f''\#_0\beta_1\#_0 n_0)\#_1 \begin{pmatrix} ((k_2\#_0 n_0)\#_1(k_1\#_0\alpha_3)) \\ \#_2(\overline{(k_2\otimes\alpha_1)}\#_1(k_1\#_0 g_2)) \end{pmatrix}\right); \\ \alpha_1 \lhd \beta_1, \alpha_2 \lhd \beta_2 \big| k\square_0 g, m\square_0 n \end{pmatrix} .$$

3.12. Tensors. Finally, in



letting $\beta \boxtimes \alpha = (\beta_1\otimes\alpha_1, \beta_2\otimes\alpha_2)$ makes $\overrightarrow{\mathbb{H}}$ a Gray-category. This is a well defined 3-cell.

3.13. Inverses. If $\mathbb{H}$ has invertible 1- and 2-cells the inverse of a 1-cell

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "g_0"'] & \cdot \arrow[d, "g_1"] \arrow[dl, Rightarrow, "g_2"] \\ \cdot \arrow[r, "f'"'] & \cdot \end{tikzcd}$$

in $\overrightarrow{\mathbb{H}}$ is given by

$$\begin{tikzcd} \cdot \arrow[rrr, "f'"] \arrow[ddd, "\overline{g_0}"'] \arrow[dr, "\overline{g_0}"] & & & \cdot \arrow[ddd, "\overline{g_1}"] \\ & \cdot \arrow[r, "g_0"] \arrow[d, "f"'] & \cdot \arrow[d, "f'"] \arrow[dl, Rightarrow, "\overline{g_2}"] & \\ & \cdot \arrow[r, "g_1"'] & \cdot \arrow[dr, "\overline{g_1}"] & \\ \cdot \arrow[rrr, "f"'] & & & \cdot \end{tikzcd} .$$

3.14. AXIOMS. This composition of $\overrightarrow{\mathbb{H}}$-2-cells is associative: Given three 2-cells

$$\alpha = \begin{tikzcd}
\cdot \arrow[r, "f"] \arrow[d, "g_0"{name=A}] \arrow[d, bend right=80, "h_0"'{name=B}] & \cdot \arrow[d, "g_1"] \arrow[dl, Rightarrow, "g_2"] & \overset{\alpha_3}{\Rrightarrow} & \cdot \arrow[r, "f"] \arrow[d, "h_0"] & \cdot \arrow[d, "h_1"'{name=C}] \arrow[dl, Rightarrow, "h_2"'] \arrow[d, bend left=80, "g_1"{name=D}] \\
\cdot \arrow[r, "f'"'] & \cdot & & \cdot \arrow[r, "f'"'] & \cdot
\arrow[from=A, to=B, Rightarrow, "\alpha_1"']
\arrow[from=D, to=C, Rightarrow, "\alpha_2"']
\end{tikzcd}$$

$$\beta = \begin{tikzcd}
\cdot \arrow[r, "f"] \arrow[d, "h_0"{name=A}] \arrow[d, bend right=80, "k_0"'{name=B}] & \cdot \arrow[d, "h_1"] \arrow[dl, Rightarrow, "h_2"] & \overset{\beta_3}{\Rrightarrow} & \cdot \arrow[r, "f"] \arrow[d, "k_0"] & \cdot \arrow[d, "k_1"'{name=C}] \arrow[dl, Rightarrow, "k_2"'] \arrow[d, bend left=80, "h_1"{name=D}] \\
\cdot \arrow[r, "f'"'] & \cdot & & \cdot \arrow[r, "f'"'] & \cdot
\arrow[from=A, to=B, Rightarrow, "\beta_1"']
\arrow[from=D, to=C, Rightarrow, "k_2"']
\end{tikzcd}$$

$$\gamma = \begin{tikzcd}
\cdot \arrow[r, "f"] \arrow[d, "k_0"{name=A}] \arrow[d, bend right=80, "m_0"'{name=B}] & \cdot \arrow[d, "k_1"] \arrow[dl, Rightarrow, "k_2"] & \overset{\gamma_3}{\Rrightarrow} & \cdot \arrow[r, "f"] \arrow[d, "m_0"] & \cdot \arrow[d, "m_1"'{name=C}] \arrow[dl, Rightarrow, "m_2"'] \arrow[d, bend left=80, "k_1"{name=D}] \\
\cdot \arrow[r, "f'"'] & \cdot & & \cdot \arrow[r, "f'"'] & \cdot
\arrow[from=A, to=B, Rightarrow, "\gamma_1"']
\arrow[from=D, to=C, Rightarrow, "\gamma_2"']
\end{tikzcd}$$

we use (30) and the functoriality of the whiskerings in $\mathbb{H}$ to compute:

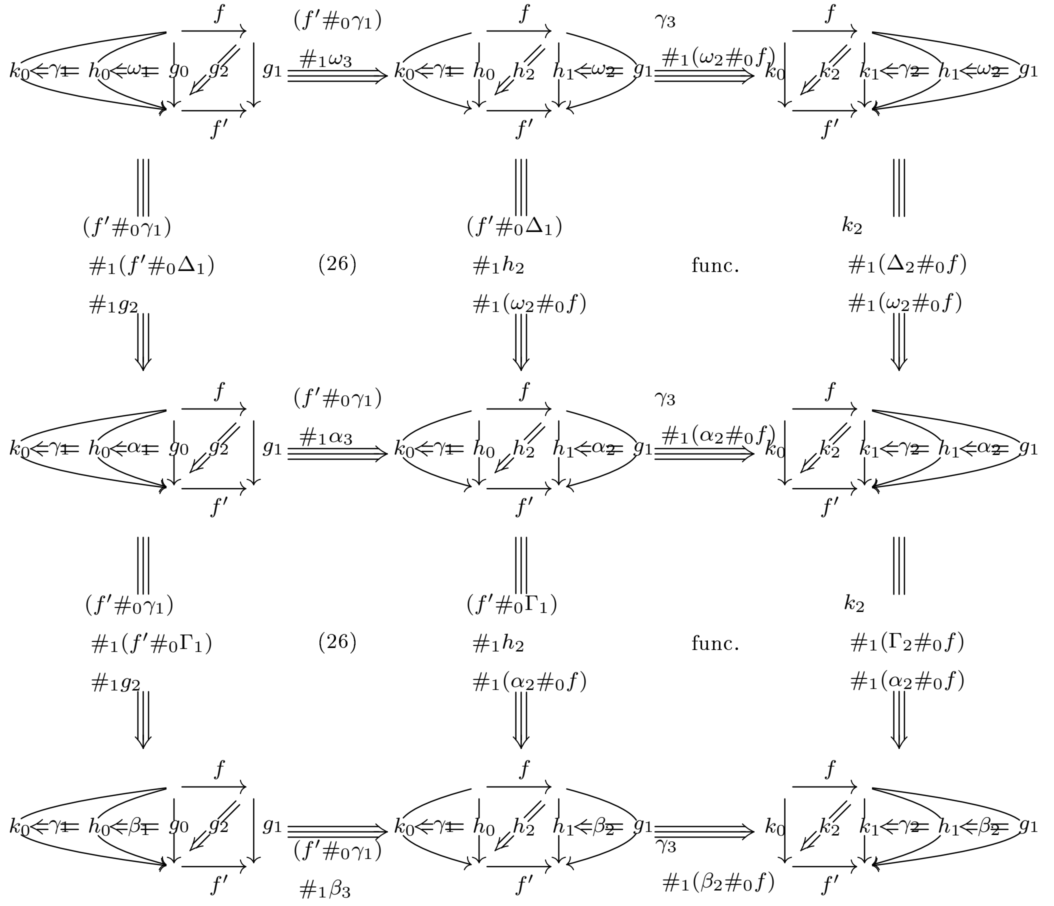
$$\begin{aligned}
(\gamma\square_1\beta)\square_1\alpha &= \begin{pmatrix} \underbrace{(\gamma_3\#_1(\beta_2\#_0f))\#_2((f'\#_0\gamma_1)\#_1\beta_3)}_{\omega_3}; \\ \gamma_1\#_1\beta_1, \gamma_2\#_1\beta_2, h_2, m_2; h_0, h_1, m_0, m_1, f, f' \end{pmatrix} \square_1\alpha \\
&= \begin{pmatrix} (\omega_3\#_1(\alpha_2\#_0f)) \\ \#_2((f'\#_0(\gamma_1\#_1\beta_1))\#_1\alpha_3); \\ \gamma_1\#_1\beta_1\#_1\alpha_1, \gamma_2\#_1\beta_2\#_1\alpha_2, g_2, m_2; g_0, g_1, m_0, m_1, f, f' \end{pmatrix} \\
&= \begin{pmatrix} (((\gamma_3\#_1(\beta_2\#_0f))\#_2((f'\#_0\gamma_1)\#_1\beta_3)) \\ \#_1(\alpha_2\#_0f))\#_2((f'\#_0(\gamma_1\#_1\beta_1))\#_1\alpha_3); \\ \gamma_1\#_1\beta_1\#_1\alpha_1, \gamma_2\#_1\beta_2\#_1\alpha_2, \\ g_2, m_2; g_0, g_1, m_0, m_1, f, f' \end{pmatrix} = \begin{pmatrix} (\gamma_3\#_1(\beta_2\#_0f)\#_1(\alpha_2\#_0f)) \\ \#_2(((f'\#_0\gamma_1)\#_1\beta_3)\#_1(\alpha_2\#_0f)) \\ \#_2((f'\#_0(\gamma_1\#_1\beta_1))\#_1\alpha_3); \\ \gamma_1\#_1\beta_1\#_1\alpha_1, \gamma_2\#_1\beta_2\#_1\alpha_2, g_2, m_2; \\ g_0, g_1, m_0, m_1, f, f' \end{pmatrix} \\
&= \begin{pmatrix} (\gamma_3\#_1((\beta_2\#_1\alpha_2)\#_0f))\#_2((f'\#_0\gamma_1)\#_1\beta_3\#_1(\alpha_2\#_0f)) \\ \#_2((f'\#_0\gamma_1)\#_1(f'\#_0\beta_1)\#_1\alpha_3); \\ \gamma_1\#_1\beta_1\#_1\alpha_1, \gamma_2\#_1\beta_2\#_1\alpha_2, g_2, m_2; g_0, g_1, m_0, m_1, f, f' \end{pmatrix} \\
&= \begin{pmatrix} (\gamma_3\#_1((\beta_2\#_1\alpha_2)\#_0f))\#_2((f'\#_0\gamma_1) \\ \#_1\underbrace{((\beta_3\#_1(\alpha_2\#_0f))\#_2((f'\#_0\beta_1)\#_1\alpha_3))}_{\zeta_3}); \\ \gamma_1\#_1\beta_1\#_1\alpha_1, \gamma_2\#_1\beta_2\#_1\alpha_2, g_2, m_2; g_0, g_1, m_0, m_1, f, f' \end{pmatrix} \\
&= \begin{pmatrix} (\gamma_3\#_1((\beta_2\#_1\alpha_2)\#_0f)) \\ \#_2((f'\#_0\gamma_1)\#_1\zeta_3); \\ \gamma_1\#_1\beta_1\#_1\alpha_1, \gamma_2\#_1\beta_2\#_1\alpha_2, g_2, m_2; g_0, g_1, m_0, m_1, f, f' \end{pmatrix} \\
&= \gamma\square_1 \begin{pmatrix} \zeta_3; \beta_1\#_1\alpha_1, \beta_2\#_1\alpha_2, \\ g_2, k_2; g_0, g_1, k_0, k_1, f, f' \end{pmatrix} = \gamma\square_1(\beta\square_1\alpha)\,.
\end{aligned}$$

We check that 2-1-whiskering in $\overrightarrow{\mathbb{H}}$ is functorial, that is, $m\Box_0(\beta\Box_1\alpha) = (m\Box_0\beta)\Box_1(m\Box_0\alpha)$. In diagram (35) the diagonal is $m\Box_0(\beta\Box_1\alpha)$ and left and down is $(m\Box_0\beta)\Box_1(m\Box_0\alpha)$. 1-2-whiskering in $\overrightarrow{\mathbb{H}}$ is functorial by duality.

It is obvious that 3-1-whiskering is 2-functorial, that is,

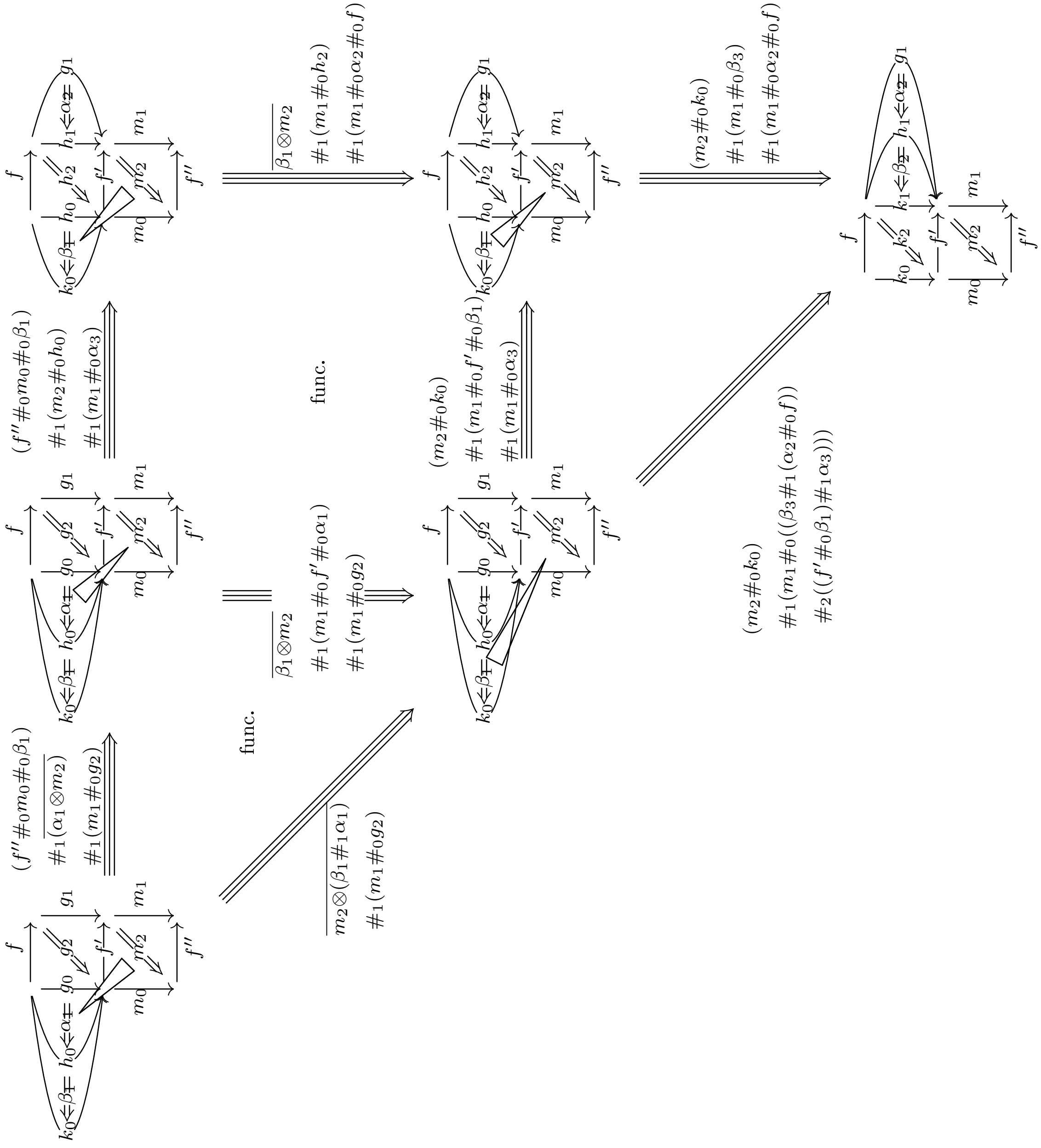
$$\begin{aligned}
&(m_0, m_1, m_2)\Box_0((\Delta_1, \Delta_2)\Box_2(\Gamma_1, \Gamma_2))\\
&\qquad = (m_0, m_1, m_2)\Box_0(\Delta_1\#_2\Gamma_1, \Delta_2\#_2\Gamma_2)\\
&\qquad = (m_0\#_0(\Delta_1\#_2\Gamma_1), m_1\#_0(\Delta_2\#_2\Gamma_2))\\
&\qquad = (((m_0\#_0\Delta_1)\#_2(m_0\#_0\Gamma_1)), ((m_1\#_0\Delta_2)\#_2(m_1\#_0\Gamma_2)))\\
&\qquad = ((m_0\#_0\Delta_1), (m_1\#_0\Delta_2))\Box_2((m_0\#_0\Gamma_1), (m_1\#_0\Gamma_2))\\
&\qquad = ((m_0, m_1, m_2)\Box_0(\Delta_1, \Delta_2))\Box_2((m_0, m_1, m_2)\Box_0(\Gamma_1, \Gamma_2))\,.
\end{aligned}$$

By duality, 1-2-whiskering in $\overrightarrow{\mathbb{H}}$ is functorial as well. And the 3-2-whiskering thus defined is functorial with respect to vertical composition of 3-cells, that is, $\gamma\Box_1(\Gamma\Box_2\Delta) = (\gamma\Box_1\Gamma)\Box_2(\gamma\Box_1\Delta)$, as can seen by inspecting the following diagram:



Theory and Applications of Categories, Vol. 29, No. 5, 2014, pp. 129–187.

(35)



We see that 2-3-whiskering is functorial:

$$\begin{aligned}
&(\Delta\Box_1\beta)\Box_2(\gamma\Box_1\Gamma)\\
&\qquad= (\Delta_1\#_1\beta_1,\Delta_2\#_1\beta_2)\Box_2(\gamma_1\#_1\Gamma_1,\gamma_2\#_1\Gamma_2)\\
&\qquad= ((\Delta_1\#_1\beta_1)\#_2(\gamma_1\#_1\Gamma_1),((\Delta_2\#_1\beta_2)\#_2(\gamma_2\#_1\Gamma_2))\\
&\qquad= ((\delta_1\#_1\Gamma_1)\#_2(\Delta_1\#_1\alpha_1),(\delta_2\#_1\Gamma_2)\#_2(\Delta_2\#_2\alpha_2))\\
&\qquad= (\delta_1\#_1\Gamma_1,\delta_2\#_1\Gamma_2)\Box_2(\Delta_1\#_1\alpha_1,\Delta_2\#_1\alpha_2)\\
&\qquad\qquad= (\delta\Box_1\Gamma)\Box_2(\Delta\Box_1\alpha)\,.
\end{aligned}$$

So we can conclude that $\overrightarrow{\mathbb{H}}$ is locally a 2-category.

That interchange $\boxtimes$ is natural and functorial in both arguments follows immediately from the respective properties of $\otimes$ in $\mathbb{H}$. Thus we have:

3.15. Lemma. *The path space* $\overrightarrow{\mathbb{H}}$ *for a* Gray*-category* $\mathbb{H}$ *is again a* Gray*-category.* ☐

3.16. Lemma. *Given a* Gray*-functor* $F\colon \mathbb{G}\longrightarrow\mathbb{H}$ *there is a canonical* Gray*-functor* $\overrightarrow{F}\colon\overrightarrow{\mathbb{G}}\longrightarrow\overrightarrow{\mathbb{H}}$.

Proof The Gray-functor $\overrightarrow{F}$ acts by applying F to all components of the cells of $\overrightarrow{\mathbb{G}}$:

$$\left(\begin{tikzcd} x \arrow[r,"f"] & y\end{tikzcd}\right)\mapsto\left(\begin{tikzcd} Fx \arrow[r,"Ff"] & Fy\end{tikzcd}\right)$$

$$\left(\begin{tikzcd} {} \arrow[r,"f"] \arrow[d,"g_0"'] \arrow[dr, phantom, "\overset{}{\swarrow}\,g_2"] & {} \arrow[d,"g_1"] \\ {} \arrow[r,"f'"'] & {} \end{tikzcd}\right)\mapsto\left(\begin{tikzcd} {} \arrow[r,"Ff"] \arrow[d,"Fg_0"'] \arrow[dr, phantom, "\swarrow\,Fg_2"] & {} \arrow[d,"Fg_1"] \\ {} \arrow[r,"Ff'"'] & {} \end{tikzcd}\right)$$

$$\left(\begin{tikzcd} {} \arrow[r,"f"] \arrow[d,"g_0"] \arrow[d, bend right=80, "h_0"'] \arrow[d, phantom, bend right=40, "\underset{\alpha_1}{\Leftarrow}"] \arrow[dr, phantom, "\swarrow\,g_2"] & {} \arrow[d,"g_1"] \arrow[r, phantom, "\overset{\alpha_3}{\Rrightarrow}"] & {} \arrow[r,"f"] \arrow[d,"h_0"'] \arrow[dr, phantom, "\swarrow\,h_2"] & {} \arrow[d,"h_1"] \arrow[d, bend left=80, "g_1"] \arrow[d, phantom, bend left=40, "\underset{\alpha_2}{\Leftarrow}"] \\ {} \arrow[r,"f'"'] & {} & {} \arrow[r,"f'"'] & {} \end{tikzcd}\right)\mapsto\left(\begin{tikzcd} {} \arrow[r,"Ff"] \arrow[d,"Fg_0"] \arrow[d, bend right=80, "Fh_0"'] \arrow[d, phantom, bend right=40, "\underset{F\alpha_1}{\Leftarrow}"] \arrow[dr, phantom, "\swarrow\,Fg_2"] & {} \arrow[d,"Fg_1"] \arrow[r, phantom, "\overset{F\alpha_3}{\Rrightarrow}"] & {} \arrow[r,"Ff"] \arrow[d,"Fh_0"'] \arrow[dr, phantom, "\swarrow\,Fh_2"] & {} \arrow[d,"Fh_1"] \arrow[d, bend left=80, "Fg_1"] \arrow[d, phantom, bend left=40, "\underset{F\alpha_2}{\Leftarrow}"] \\ {} \arrow[r,"Ff'"'] & {} & {} \arrow[r,"Ff'"'] & {} \end{tikzcd}\right)$$

$$\left(\begin{tikzcd} {} \arrow[r,"f"] \arrow[d,"g_0"] \arrow[d, bend right=80, "h_0"'] \arrow[d, phantom, bend right=40, "\overset{\alpha_1}{\Leftarrow}"] \arrow[dr, phantom, "\swarrow\,g_2"] & {} \arrow[d,"g_1"] \arrow[r, phantom, "\overset{\alpha_3}{\Rrightarrow}"] & {} \arrow[r,"f"] \arrow[d,"h_0"'] \arrow[dr, phantom, "\swarrow\,h_2"] & {} \arrow[d,"h_1"] \arrow[d, bend left=80, "g_1"] \arrow[d, phantom, bend left=40, "\overset{\alpha_2}{\Leftarrow}"] \\ {} \arrow[r,"f'"'] \arrow[d, phantom, "\Bigg\Downarrow{\scriptstyle (f'\#_0\underline{\Gamma_1})\#_1g_2}"] & {} & {} \arrow[r,"f'"'] & {} \arrow[d, phantom, "\Bigg\Downarrow{\scriptstyle h_2\#_1(\underline{\Gamma_2}\#_0f)}"] \\ {} \arrow[r,"f"] \arrow[d,"g_0"] \arrow[d, bend right=80, "h_0"'] \arrow[d, phantom, bend right=40, "\overset{\beta_1}{\Leftarrow}"] \arrow[dr, phantom, "\swarrow\,g_2"] & {} \arrow[d,"g_1"] \arrow[r, phantom, "\underset{\underline{\beta_3}}{\Rrightarrow}"] & {} \arrow[r,"f"] \arrow[d,"h_0"'] \arrow[dr, phantom, "\swarrow\,h_2"] & {} \arrow[d,"h_1"] \arrow[d, bend left=80, "g_1"] \arrow[d, phantom, bend left=40, "\overset{\beta_2}{\Leftarrow}"] \\ {} \arrow[r,"f'"'] & {} & {} \arrow[r,"f'"'] & {} \end{tikzcd}\right)\mapsto\left(\begin{tikzcd} {} \arrow[r,"Ff"] \arrow[d,"Fg_0"] \arrow[d, bend right=80, "Fh_0"'] \arrow[d, phantom, bend right=40, "\overset{F\alpha_1}{\Leftarrow}"] \arrow[dr, phantom, "\swarrow\,Fg_2"] & {} \arrow[d,"Fg_1"] \arrow[r, phantom, "\overset{F\alpha_3}{\Rrightarrow}"] & {} \arrow[r,"Ff"] \arrow[d,"Fh_0"'] \arrow[dr, phantom, "\swarrow\,Fh_2"] & {} \arrow[d,"Fh_1"] \arrow[d, bend left=80, "Fg_1"] \arrow[d, phantom, bend left=40, "\overset{F\alpha_2}{\Leftarrow}"] \\ {} \arrow[r,"Ff'"'] \arrow[d, phantom, "\Bigg\Downarrow{\scriptstyle (Ff'\#_0\underline{F\Gamma_1})\#_1Fg_2}"] & {} & {} \arrow[r,"Ff'"'] & {} \arrow[d, phantom, "\Bigg\Downarrow{\scriptstyle Fh_2\#_1(\underline{F\Gamma_2}\#_0Ff)}"] \\ {} \arrow[r,"Ff"] \arrow[d,"Fg_0"] \arrow[d, bend right=80, "Fh_0"'] \arrow[d, phantom, bend right=40, "\overset{F\beta_1}{\Leftarrow}"] \arrow[dr, phantom, "\swarrow\,Fg_2"] & {} \arrow[d,"Fg_1"] \arrow[r, phantom, "\underset{\underline{F\beta_3}}{\Rrightarrow}"] & {} \arrow[r,"Ff"] \arrow[d,"Fh_0"'] \arrow[dr, phantom, "\swarrow\,Fh_2"] & {} \arrow[d,"Fh_1"] \arrow[d, bend left=80, "Fg_1"] \arrow[d, phantom, bend left=40, "\overset{F\beta_2}{\Leftarrow}"] \\ {} \arrow[r,"Ff'"'] & {} & {} \arrow[r,"Ff'"'] & {} \end{tikzcd}\right)$$

This preserves the structure of $\overrightarrow{\mathbb{G}}$ since F preserves all commuting diagrams on the nose. ☐

3.17. THEOREM. *Furthermore,* $\overrightarrow{(-)}$ *is canonically an endofunctor of* $\mathsf{GrayCat}$.

PROOF Obviously $\overrightarrow{GF} = \overrightarrow{G}\,\overrightarrow{F}$. ☐

We finally note the following:

3.18. LEMMA. *The functor* $\overrightarrow{(-)}\colon \mathsf{GrayCat} \longrightarrow \mathsf{GrayCat}$ *preserves limits.*

PROOF This is obviously true for products.

For the equalizer $\mathbb{E}$ of two strict maps F, G we remember that the action of $\overrightarrow{F}$ and $\overrightarrow{G}$ is defined by the component wise action of F and G, that is, a cell of $\overrightarrow{\mathbb{E}}$ is equal under $\overrightarrow{F}$ and $\overrightarrow{G}$ iff its components are so under F and G. ☐

A straightforward calculation shows how this forms part of an adjunction

$$\begin{tikzcd} \mathsf{GrayCat} \arrow[r, bend left=20, "\overrightarrow{(\_)}"] \arrow[r, phantom, "\top"] & \mathsf{GrayCat} \arrow[l, bend left=20, "\_\otimes\mathbb{I}"] \end{tikzcd}$$

where $\mathbb{I}$ is the free Gray-category on a single 1-cell $(01)\colon 0 \longrightarrow 1$ and $\otimes$ is Crans' tensor of Gray-categories.

4. Composition of Paths

We want to turn the path space that we constructed in the previous section into the arrow part of an internal category, which requires us to define a composition map as follows:

4.1. DEFINITION. *We define the* **composite of paths** *as a pseudo* Q^1 *graph map* $m\colon \overrightarrow{\mathbb{H}} \times_{\mathbb{H}} \overrightarrow{\mathbb{H}} \nrightarrow \overrightarrow{\mathbb{H}}$ *by horizontal pasting in the following fashion:*

1. *0-cells*

$$\left(y \xrightarrow{\widehat{f}} z\ ,\ x \xrightarrow{f} y \right) \mapsto \left(x \xrightarrow{\widehat{f}\#_0 f} z \right)$$

2. *1-cells*

$$\left(\begin{tikzcd} {} \arrow[r, "\widehat{f}"] \arrow[d, "\widehat{g}_0=g_1"'] & {} \arrow[d, "\widehat{g}_1"] \arrow[dl, Rightarrow, "\widehat{g}_2"] \\ {} \arrow[r, "\widehat{f}'"'] & {} \end{tikzcd} ,\ \begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "g_0"'] & {} \arrow[d, "g_1"] \arrow[dl, Rightarrow, "g_2"] \\ {} \arrow[r, "f'"'] & {} \end{tikzcd} \right) \mapsto \left(\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "g_0"'] & {} \arrow[r, "\widehat{f}"] \arrow[d, "g_1"] \arrow[dl, Rightarrow, "g_2"] & {} \arrow[d, "\widehat{g}_1"] \arrow[dl, Rightarrow, "\widehat{g}_2"] \\ {} \arrow[r, "f'"'] & {} \arrow[r, "\widehat{f}'"'] & {} \end{tikzcd} \right)$$

$$= \left(\begin{tikzcd} {} \arrow[rr, "\widehat{f}\#_0 f"] \arrow[d, "g_0"'] & & {} \arrow[d, "\widehat{g}_1"] \arrow[dll, Rightarrow, "(\widehat{f}'\#_0 g_2) \#_1 (\widehat{g}_2\#_0 f)"] \\ {} \arrow[rr, "\widehat{f}'\#_0 f'"'] & & {} \end{tikzcd} \right)$$

3. 2-cells

$$
\left(
\begin{tikzcd} {} \arrow[r, "\widehat{f}"] \arrow[d, "\widehat{g_0}"] \arrow[d, bend right=80, "\widehat{h_0}"'] \arrow[d, phantom, bend right=45, "\overset{\Leftarrow}{\widehat{\alpha_1}=\alpha_2}"] & {} \arrow[d, "\widehat{g_1}"] \arrow[dl, Rightarrow, "\widehat{g_2}"] \\ {} \arrow[r, "\widehat{f}'"'] & {} \end{tikzcd}
\overset{\widehat{\alpha_3}}{\Rrightarrow}
\begin{tikzcd} {} \arrow[r, "\widehat{f}"] \arrow[d, "\widehat{h_0}"'] & {} \arrow[d, "\widehat{h_1}"'] \arrow[dl, Rightarrow, "\widehat{h_2}"'] \arrow[d, bend left=80, "\widehat{g_1}"] \arrow[d, phantom, bend left=45, "\overset{\Leftarrow}{\widehat{\alpha_2}}"] \\ {} \arrow[r, "\widehat{f}'"'] & {} \end{tikzcd}
,\;
\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "g_0"] \arrow[d, bend right=80, "h_0"'] \arrow[d, phantom, bend right=45, "\overset{\Leftarrow}{\alpha_1}"] & {} \arrow[d, "g_1"] \arrow[dl, Rightarrow, "g_2"] \\ {} \arrow[r, "f'"'] & {} \end{tikzcd}
\overset{\alpha_3}{\Rrightarrow}
\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "h_0"'] & {} \arrow[d, "h_1"'] \arrow[dl, Rightarrow, "h_2"'] \arrow[d, bend left=80, "g_1"] \arrow[d, phantom, bend left=45, "\overset{\Leftarrow}{\alpha_2}"] \\ {} \arrow[r, "f'"'] & {} \end{tikzcd}
\right)
$$

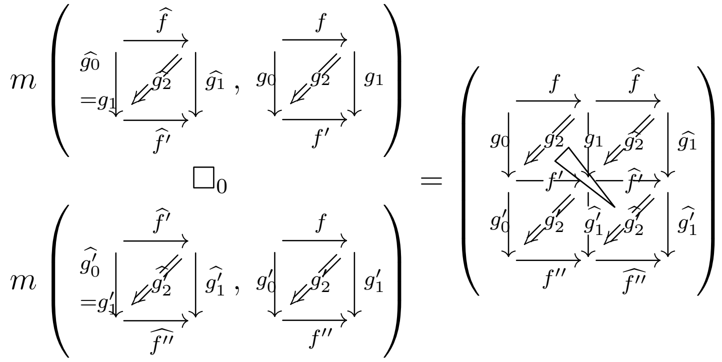
$$
\mapsto
\left(
\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "g_0"] \arrow[d, bend right=80, "h_0"'] \arrow[d, phantom, bend right=45, "\overset{\Leftarrow}{\alpha_1}"] & {} \arrow[r, "\widehat{f}"] \arrow[d, "g_1"] \arrow[dl, Rightarrow, "g_2"] & {} \arrow[d, "\widehat{g_1}"] \arrow[dl, Rightarrow, "\widehat{g_2}"] \\ {} \arrow[r, "f'"'] & {} \arrow[r, "\widehat{f}'"'] & {} \end{tikzcd}
\overset{\substack{(\widehat{f}'\#_0\alpha_3)\\ \#_1(\widehat{g_2}\#_0f)}}{\Rrightarrow}
\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "h_0"'] & {} \arrow[r, "\widehat{f}"] \arrow[d, bend right=50, "h_1"'] \arrow[d, bend left=50, "g_1"] \arrow[d, phantom, "\overset{\Leftarrow}{\alpha_2}"] \arrow[dl, Rightarrow, "h_2"] & {} \arrow[d, "\widehat{g_1}"] \arrow[dl, Rightarrow, "\widehat{g_2}"] \\ {} \arrow[r, "f'"'] & {} \arrow[r, "\widehat{f}'"'] & {} \end{tikzcd}
\overset{\substack{(\widehat{f}'\#_0h_2)\\ \#_1(\widehat{\alpha_3}\#_0f)}}{\Rrightarrow}
\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "h_0"'] & {} \arrow[r, "\widehat{f}"] \arrow[d, "h_1"] \arrow[dl, Rightarrow, "h_2"] & {} \arrow[d, "\widehat{h_1}"'] \arrow[dl, Rightarrow, "\widehat{h_2}"] \arrow[d, bend left=80, "h_0"] \arrow[d, phantom, bend left=45, "\overset{\Leftarrow}{\widehat{\alpha_2}}"] \\ {} \arrow[r, "f'"'] & {} \arrow[r, "\widehat{f}'"'] & {} \end{tikzcd}
\right)
$$

4. 3-cells

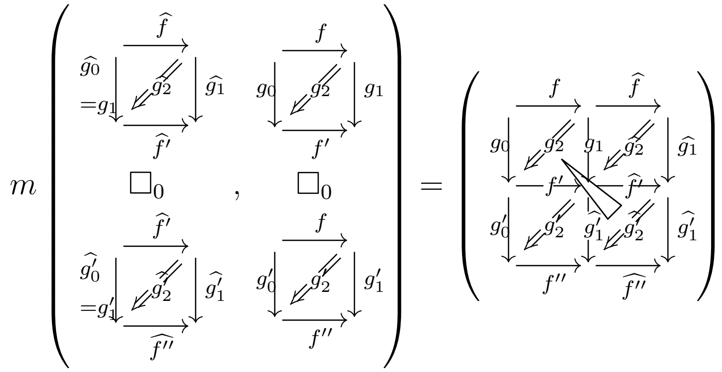
$$
\left(
\begin{array}{ccccc}
\begin{tikzcd} {} \arrow[r, "\widehat{f}"] \arrow[d, "\widehat{g_0}"] \arrow[d, bend right=80, "\widehat{h_0}"'] \arrow[d, phantom, bend right=45, "\overset{\Leftarrow}{\widehat{\alpha_1}=\alpha_2}"] & {} \arrow[d, "\widehat{g_1}"] \arrow[dl, Rightarrow, "\widehat{g_2}"] \\ {} \arrow[r, "\widehat{f}'"'] & {} \end{tikzcd}
\overset{\widehat{\alpha_3}}{\Rrightarrow}
\begin{tikzcd} {} \arrow[r, "\widehat{f}"] \arrow[d, "\widehat{h_0}"'] & {} \arrow[d, "\widehat{h_1}"'] \arrow[dl, Rightarrow, "\widehat{h_2}"'] \arrow[d, bend left=80, "\widehat{g_1}"] \arrow[d, phantom, bend left=45, "\overset{\Leftarrow}{\widehat{\alpha_2}}"] \\ {} \arrow[r, "\widehat{f}'"'] & {} \end{tikzcd}
& &
\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "g_0"] \arrow[d, bend right=80, "h_0"'] \arrow[d, phantom, bend right=45, "\overset{\Leftarrow}{\alpha_1=\alpha_2}"] & {} \arrow[d, "g_1"] \arrow[dl, Rightarrow, "g_2"] \\ {} \arrow[r, "f'"'] & {} \end{tikzcd}
\overset{\alpha_3}{\Rrightarrow}
\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "h_0"'] & {} \arrow[d, "h_1"'] \arrow[dl, Rightarrow, "h_2"'] \arrow[d, bend left=80, "g_1"] \arrow[d, phantom, bend left=45, "\overset{\Leftarrow}{\alpha_2}"] \\ {} \arrow[r, "f'"'] & {} \end{tikzcd}
\\
\substack{(\widehat{f}'\#_0\widehat{\Gamma_1})\#_1\widehat{g_2}\\ =(\widehat{f}'\#_0\Gamma_2)\#_1\widehat{g_2}}\Bigg\Downarrow \qquad\qquad \Bigg\Downarrow\widehat{h_2}\#_1(\widehat{\Gamma_2}\#_0\widehat{f})
& , &
(f'\#_0\Gamma_1)\#_1g_2\Bigg\Downarrow \qquad\qquad \Bigg\Downarrow h_2\#_1(\Gamma_2\#_0f)
\\
\begin{tikzcd} {} \arrow[r, "\widehat{f}"] \arrow[d, "\widehat{g_0}"] \arrow[d, bend right=80, "\widehat{h_0}"'] \arrow[d, phantom, bend right=45, "\overset{\Leftarrow}{\widehat{\beta_1}=\beta_2}"] & {} \arrow[d, "\widehat{g_1}"] \arrow[dl, Rightarrow, "\widehat{g_2}"] \\ {} \arrow[r, "\widehat{f}'"'] & {} \end{tikzcd}
\underset{\widehat{\beta_3}}{\Rrightarrow}
\begin{tikzcd} {} \arrow[r, "\widehat{f}"] \arrow[d, "\widehat{h_0}"'] & {} \arrow[d, "\widehat{h_1}"'] \arrow[dl, Rightarrow, "\widehat{h_2}"'] \arrow[d, bend left=80, "\widehat{g_1}"] \arrow[d, phantom, bend left=45, "\overset{\Leftarrow}{\widehat{\beta_2}}"] \\ {} \arrow[r, "\widehat{f}'"'] & {} \end{tikzcd}
& &
\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "g_0"] \arrow[d, bend right=80, "h_0"'] \arrow[d, phantom, bend right=45, "\overset{\Leftarrow}{\beta_1=\beta_2}"] & {} \arrow[d, "g_1"] \arrow[dl, Rightarrow, "g_2"] \\ {} \arrow[r, "f'"'] & {} \end{tikzcd}
\underset{\beta_3}{\Rrightarrow}
\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "h_0"'] & {} \arrow[d, "h_1"'] \arrow[dl, Rightarrow, "h_2"'] \arrow[d, bend left=80, "g_1"] \arrow[d, phantom, bend left=45, "\overset{\Leftarrow}{\beta_2}"] \\ {} \arrow[r, "f'"'] & {} \end{tikzcd}
\end{array}
\right)
$$

$$
\mapsto
\left(
\begin{array}{ccccc}
\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "g_0"] \arrow[d, bend right=80, "h_0"'] \arrow[d, phantom, bend right=45, "\overset{\Leftarrow}{\alpha_1}"] & {} \arrow[r, "\widehat{f}"] \arrow[d, "g_1"] \arrow[dl, Rightarrow, "g_2"] & {} \arrow[d, "\widehat{g_1}"] \arrow[dl, Rightarrow, "\widehat{g_2}"] \\ {} \arrow[r, "f'"'] & {} \arrow[r, "\widehat{f}'"'] & {} \end{tikzcd}
& \overset{\substack{(\widehat{f}'\#_0\alpha_3)\\ \#_1(\widehat{g_2}\#_0f)}}{\Rrightarrow} &
\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "h_0"'] & {} \arrow[r, "\widehat{f}"] \arrow[d, bend right=50, "h_1"'] \arrow[d, bend left=50, "g_1"] \arrow[d, phantom, "\overset{\Leftarrow}{\alpha_2}"] \arrow[dl, Rightarrow, "h_2"] & {} \arrow[d, "\widehat{g_1}"] \arrow[dl, Rightarrow, "\widehat{g_2}"] \\ {} \arrow[r, "f'"'] & {} \arrow[r, "\widehat{f}'"'] & {} \end{tikzcd}
& \overset{\substack{(\widehat{f}'\#_0h_2)\\ \#_1(\widehat{\alpha_3}\#_0f)}}{\Rrightarrow} &
\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "h_0"'] & {} \arrow[r, "\widehat{f}"] \arrow[d, "h_1"] \arrow[dl, Rightarrow, "h_2"] & {} \arrow[d, "\widehat{h_1}"'] \arrow[dl, Rightarrow, "\widehat{h_2}"] \arrow[d, bend left=80, "h_0"] \arrow[d, phantom, bend left=45, "\overset{\Leftarrow}{\widehat{\alpha_2}}"] \\ {} \arrow[r, "f'"'] & {} \arrow[r, "\widehat{f}'"'] & {} \end{tikzcd}
\\
\substack{(\widehat{f}'\#_0f'\#_0\Gamma_1)\\ \#_1(\widehat{f}'\#_0g_2)\\ \#_1(\widehat{g_2}\#_0f)}\Bigg\Downarrow
& &
\begin{matrix} \Vert \\ (\widehat{f}'\#_0h_2)\#_1(\widehat{f}'\#_0\Gamma_2\#_0f)\#_1(\widehat{g_2}\#_0f) \\ \Downarrow \end{matrix}
& &
\Bigg\Downarrow\substack{(\widehat{f}'\#_0h_2)\\ \#_1(\widehat{h_2}\#_0f)\\ \#_1(\widehat{\Gamma_2}\#_0\widehat{f}\#_0f)}
\\
\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "g_0"] \arrow[d, bend right=80, "h_0"'] \arrow[d, phantom, bend right=45, "\overset{\Leftarrow}{\beta_1}"] & {} \arrow[r, "\widehat{f}"] \arrow[d, "g_1"] \arrow[dl, Rightarrow, "g_2"] & {} \arrow[d, "\widehat{g_1}"] \arrow[dl, Rightarrow, "\widehat{g_2}"] \\ {} \arrow[r, "f'"'] & {} \arrow[r, "\widehat{f}'"'] & {} \end{tikzcd}
& \underset{\substack{(\widehat{f}'\#_0\beta_3)\\ \#_1(\widehat{g_2}\#_0f)}}{\Rrightarrow} &
\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "h_0"'] & {} \arrow[r, "\widehat{f}"] \arrow[d, bend right=50, "h_1"'] \arrow[d, bend left=50, "g_1"] \arrow[d, phantom, "\overset{\Leftarrow}{\beta_2}"] \arrow[dl, Rightarrow, "h_2"] & {} \arrow[d, "\widehat{g_1}"] \arrow[dl, Rightarrow, "\widehat{g_2}"] \\ {} \arrow[r, "f'"'] & {} \arrow[r, "\widehat{f}'"'] & {} \end{tikzcd}
& \underset{\substack{(\widehat{f}'\#_0h_2)\\ \#_1(\widehat{\beta_3}\#_0f)}}{\Rrightarrow} &
\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "h_0"'] & {} \arrow[r, "\widehat{f}"] \arrow[d, "h_1"] \arrow[dl, Rightarrow, "h_2"] & {} \arrow[d, "\widehat{h_1}"'] \arrow[dl, Rightarrow, "\widehat{h_2}"] \arrow[d, bend left=80, "h_0"] \arrow[d, phantom, bend left=45, "\overset{\Leftarrow}{\widehat{\beta_2}}"] \\ {} \arrow[r, "f'"'] & {} \arrow[r, "\widehat{f}'"'] & {} \end{tikzcd}
\end{array}
\right)
$$

5. the 2-cocycle: for a (vertically) composable pair in $\overrightarrow{\mathbb{H}} \times_{\mathbb{H}} \overrightarrow{\mathbb{H}}$ *we have the composite* *of the images and the image of the composites under m:*

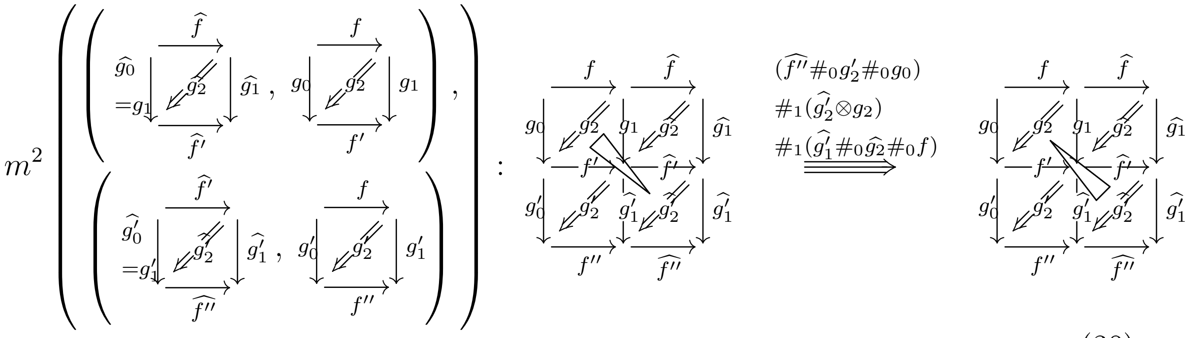


(36)



(37)

And the 2-cocycle going between them is:



(38)

For completeness' sake we give it in the algebraic notation:

$$\begin{pmatrix} (\widehat{f''}\#_0 g_2'\#_0 g_0)\#_1(\widehat{g_2'}\otimes g_2)\#_1(\widehat{g_1'}\#_0\widehat{g_2}\#_0 f);\\ \mathrm{id}_{g_0'\#_0 g_0}, \mathrm{id}_{\widehat{g_1'}\#_0\widehat{g_1}},\\ (\widehat{f''}\#_0 g_2'\#_0 g_0)\#_1(\widehat{g_2'} \lhd g_2)\#_1(\widehat{g_1'}\#_0\widehat{g_2}\#_0 f),\\ (\widehat{f''}\#_0 g_2'\#_0 g_0)\#_1(\widehat{g_2'} \rhd g_2)\#_1(\widehat{g_1'}\#_0\widehat{g_2}\#_0 f);\\ g_0'\#_0 g_0, \widehat{g_1'}\#_0\widehat{g_1}, g_0'\#_0 g_0, \widehat{g_1'}\#_0\widehat{g_1}, \widehat{f}\#_0 f, \widehat{f''}\#_0 f'' \end{pmatrix}$$

4.2. Lemma. *The map* $m\colon \overrightarrow{\mathbb{H}} \times_{\mathbb{H}} \overrightarrow{\mathbb{H}} \nrightarrow \overrightarrow{\mathbb{H}}$ *is a pseudo* Q^1 *graph map and hence by lemma 2.25 uniquely defines a pseudo* Gray*-functor.*

Proof As defined above, m is obviously a 3-globular map. We verify that it is locally a sesquifunctor: Let (β^1, β^2) and (α^1, α^2) be two pairs of 2-cells in $\overrightarrow{\mathbb{H}} \times_{\mathbb{H}} \overrightarrow{\mathbb{H}}$ composable along a pair of 1-cells. Then

$$m((\beta^1, \beta^2)\Box_1(\alpha^1, \alpha^2)) = m((\beta^1\Box_1\alpha^1), (\beta^2\Box_1\alpha^2)) = m(\beta^1, \beta^2)\Box_1 m(\alpha^1, \alpha^2)$$

follows obviously from the fact that in $\mathbb{H}$ 3-cells compose along a 2-cells interchangeably. Let (Δ^1, Δ^2) and (Γ^1, Γ^2) be two pairs of 3-cells in $\overrightarrow{\mathbb{H}} \times_{\mathbb{H}} \overrightarrow{\mathbb{H}}$ composable along a pair of 2-cells. Then

$$\begin{aligned}
m((\Delta^1, \Delta^2)\Box_2(\Gamma^1, \Gamma^2)) &= m((\Delta^1\Box_2\Gamma^1), (\Delta^2\Box_2\Gamma^2)) \\
&= m((\Delta^1_1\#_2\Gamma^1_1, \Delta^1_2\#_2\Gamma^1_2), (\Delta^2_1\#_2\Gamma^2_1, \Delta^2_2\#_2\Gamma^2_2)) = (\Delta^1_1\#_2\Gamma^1_1, \Delta^2_2\#_2\Gamma^2_2) \\
&= (\Delta^1_1, \Delta^2_2)\Box_2(\Gamma^1_1, \Gamma^2_2) = m((\Delta^1_1, \Delta^1_2), (\Delta^2_1, \Delta^2_2))\Box_2 m((\Gamma^1_1, \Gamma^1_2), (\Gamma^2_1, \Gamma^2_2)) \\
&= m(\Delta^1, \Delta^2)\Box_2 m(\Gamma^1, \Gamma^2)\,.
\end{aligned}$$

For the vertical composition of 3-cells see (31), their images under m are pastings of commuting diagrams, so preservation is immediate. Preservation of whiskers of 3-cells by 2-cells given for each component of $\overrightarrow{\mathbb{H}} \times_{\mathbb{H}} \overrightarrow{\mathbb{H}}$ in (34), again according to definition 4.1.4 m pastes two such commuting diagrams horizontally. Preservation of units is trivially satisfied. This concludes verification of 2.24.1.

We verify that m^2 is a 2-cocycle in (39). Note that in the last column of (39)

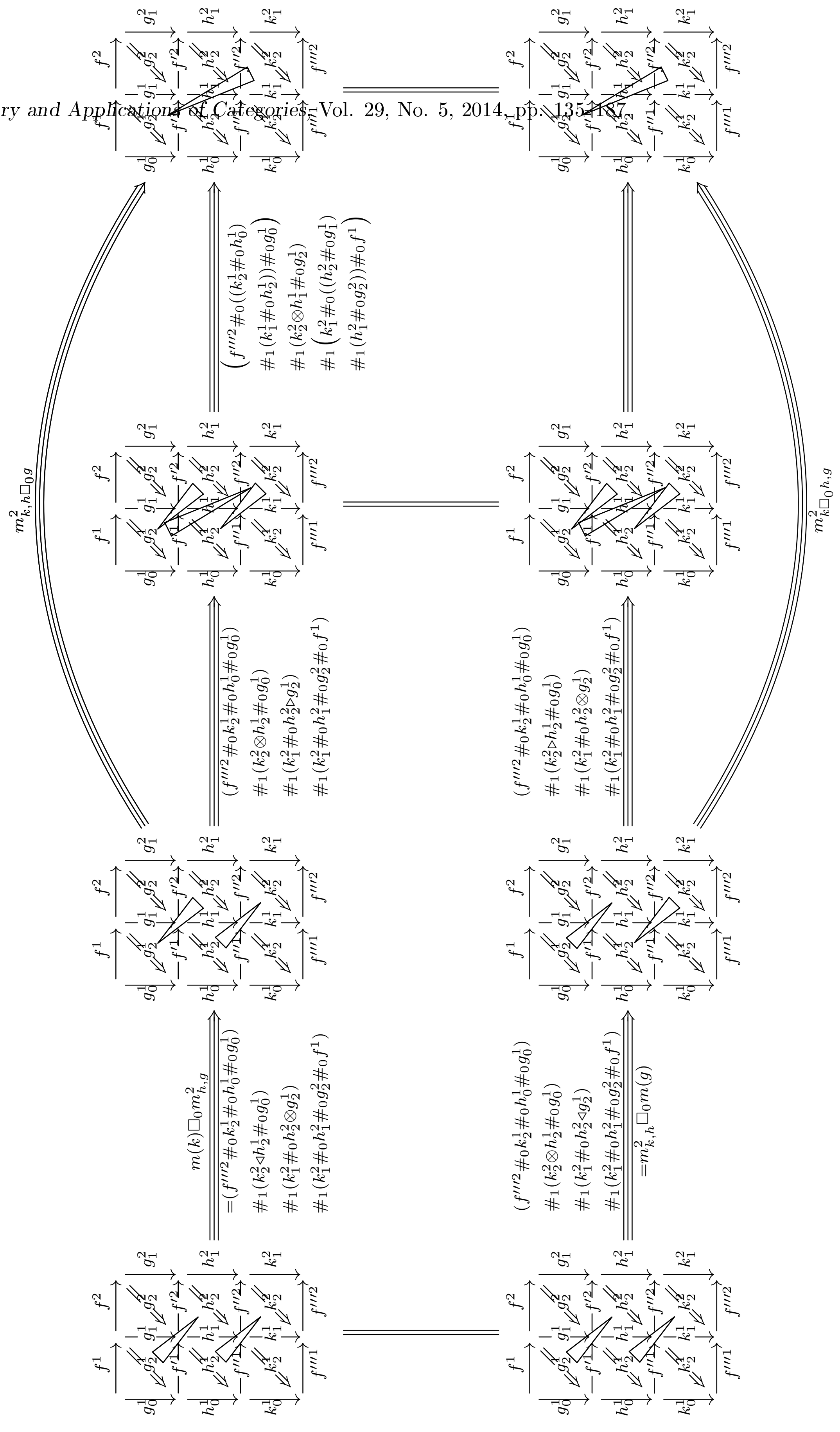
$$\begin{pmatrix}
(f'''^2\#_0 k^1_{\underline{2}}\#_0 h^1_0\#_0 g^1_0) \\
\\
\#_1(k^2_{\underline{2}} \triangleright \underline{h^1_2}\#_0 g^1_0) \\
\#_1(\overline{k^2_1}\#_0 \underline{h^2_2} \triangleright \underline{g^1_2}) \\
\\
\#_1(k^2_1\#_0 h^2_1\#_0 \underline{g^2_2}\#_0 f^1)
\end{pmatrix}
=
\begin{pmatrix}
(f'''^2\#_0 \underline{k^1_2}\#_0 h^1_0\#_0 g^1_0) \\
\#_1(f'''^2\overline{\#_0} k^1_1\#_0 h^1_{\underline{2}}\#_0 g^1_0) \\
\#_1(k^2_{\underline{2}}\#_0 h^1_1\#_0 \overline{f'^1}\#_0 g^1_0) \\
\#_1(\overline{k^2_1}\#_0 f''^2\#_0 h^1_1\#_0 \underline{g^1_2}) \\
\#_1(k^2_1\#_0 \underline{h^2_2}\#_0 g^1_1\#_0 f^1) \\
\#_1(k^2_1\#_0 \overline{h^2_1}\#_0 \underline{g^2_2}\#_0 f^1)
\end{pmatrix}
=
\begin{pmatrix}
(f'''^2\#_0((\underline{k^1_2}\#_0 h^1_0) \\
\#_1(k^1_1\#_0 \overline{h^1_2}))\#_0 g^1_0) \\
\#_1(\underline{k^2_2} \triangleleft \overline{h^1_1\#_0} \underline{g^1_2}) \\
\#_1\big(\overline{k^2_1}\#_0((\underline{h^2_2}\overline{\#_0 g^1_1}) \\
\#_1(h^2_1\#_0 \underline{g^2_2}))\#_0 f^1)
\end{pmatrix},$$

showing how the multiple horizontal composites of squares can be simplified. And the left hand rectangle in (39) commutes by local interchange. Also, m^2 is normalized by the unitality of the tensor in $\mathbb{H}$.

We check the coherent preservation of whiskers of 2-cells by 1-cells on the left, that is,

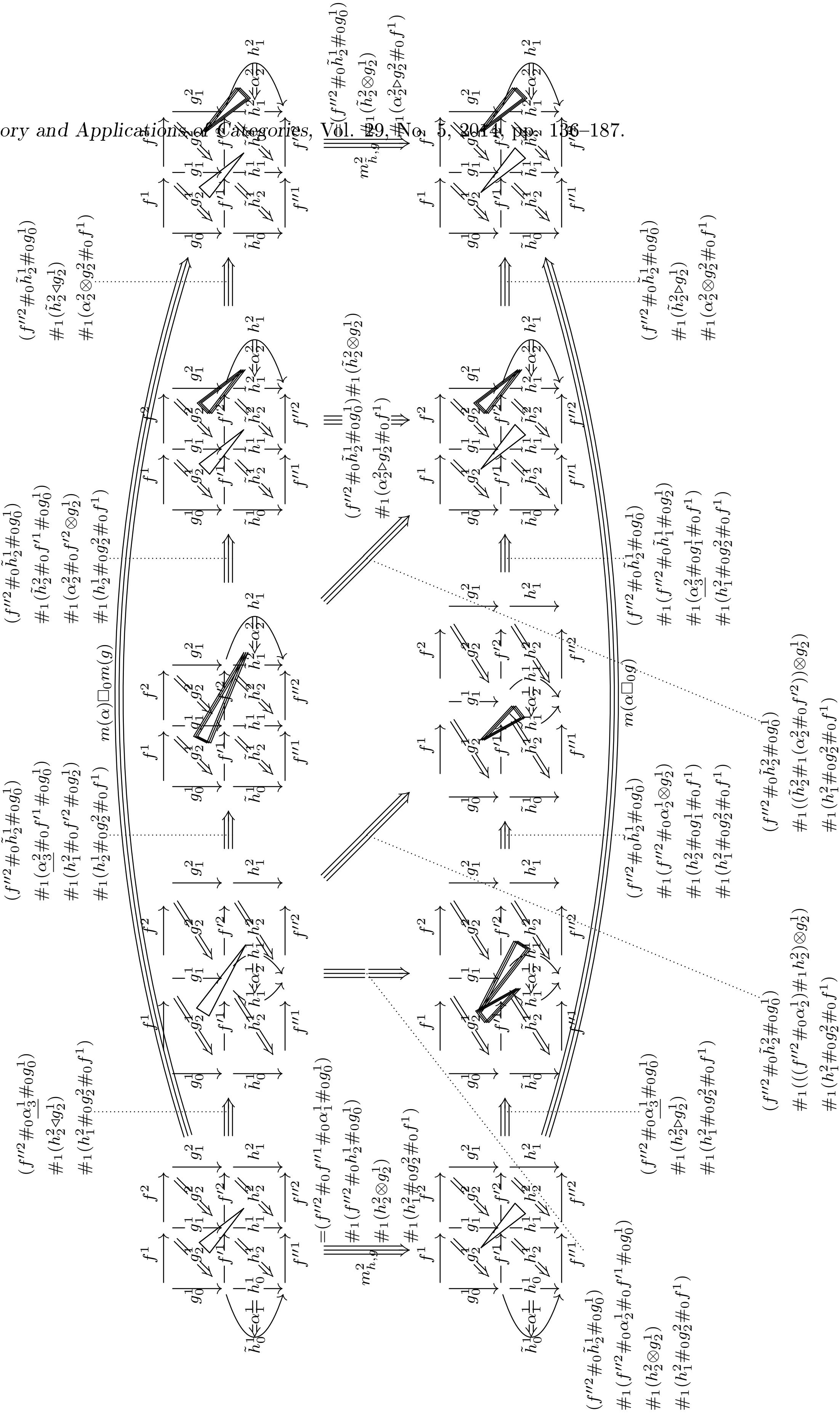
$$m^2_{h,g}\Box_1(m(\alpha)\Box_0 m(g)) = m(\alpha\Box_0 g)\Box_1 m^2_{h,g}$$

in (40), where the parts commute by the naturality of the tensor and the local interchange. The corresponding condition for right whiskers is verified similarly. Coherent preservation of whiskers of 3-cells by 1-cells is checked in the same way using in addition the naturality of the horizontal composition of a 3-cell by a 2-cell along a 0-cell. This proves conditions (12) and (13).


$m(k)\Box_0 m^2_{h,g}$
$=(f'''^2\#_0k^1_2\#_0h^1_0\#_0g^1_0)$ $\#_1(k^2_2\lhd h^1_2\#_0g^1_0)$ $\#_1(k^2_1\#_0h^2_2\otimes g^1_2)$ $\#_1(k^2_1\#_0h^2_1\#_0g^2_2\#_0f^1)$
$(f'''^2\#_0k^1_2\#_0h^1_0\#_0g^1_0)$ $\#_1(k^2_2\otimes h^1_2\#_0g^1_0)$ $\#_1(k^2_1\#_0h^2_2\rhd g^1_2)$ $\#_1(k^2_1\#_0h^2_1\#_0g^2_2\#_0f^1)$
$\Big(f'''^2\#_0((k^1_2\#_0h^1_0)$ $\#_1(k^1_1\#_0h^1_2))\#_0g^1_0\Big)$ $\#_1(k^2_2\otimes h^1_1\#_0g^1_2)$ $\#_1\Big(k^2_1\#_0((h^2_2\#_0g^1_1)$ $\#_1(h^2_1\#_0g^2_2))\#_0f^1\Big)$
$(f'''^2\#_0k^1_2\#_0h^1_0\#_0g^1_0)$ $\#_1(k^2_2\otimes h^1_2\#_0g^1_0)$ $\#_1(k^2_1\#_0h^2_2\lhd g^1_2)$ $\#_1(k^2_1\#_0h^2_1\#_0g^2_2\#_0f^1)$
$=m^2_{k,h}\Box_0 m(g)$
$(f'''^2\#_0k^1_2\#_0h^1_0\#_0g^1_0)$ $\#_1(k^2_2\rhd h^1_2\#_0g^1_0)$ $\#_1(k^2_1\#_0h^2_2\otimes g^1_2)$ $\#_1(k^2_1\#_0h^2_1\#_0g^2_2\#_0f^1)$
$m^2_{k,h\Box_0 g}$
$m^2_{k\Box_0 h,g}$

(39)

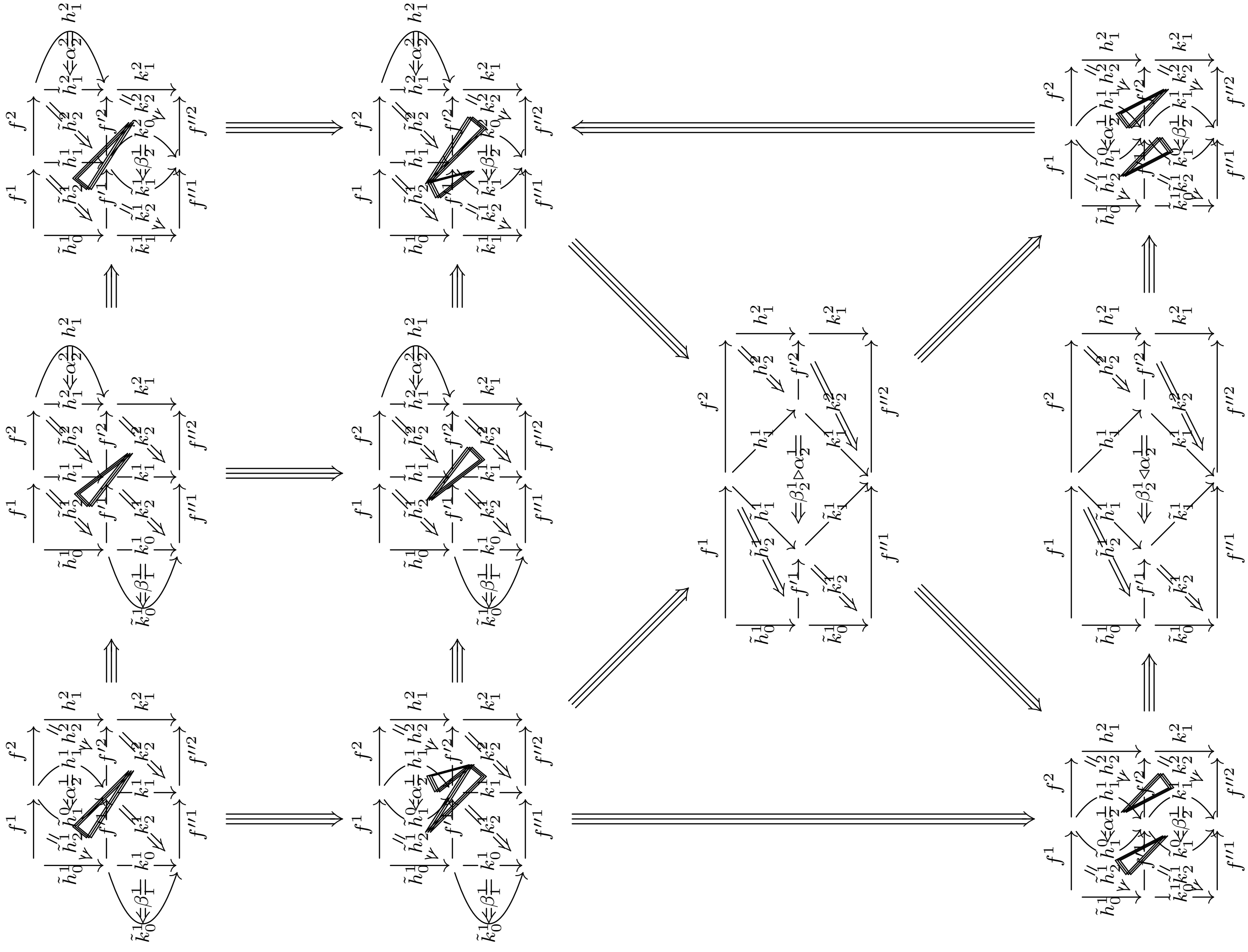
Theory and Applications of Categories, Vol. 29, No. 5, 2014, pp. 136–187.



(40)

Theory and Applications of Categories, Vol. 29, No. 5, 2014, pp. 137–187.

(41)



We verify the coherent preservation of tensors, i. e. that

$$m(\beta \boxtimes \alpha)\Box_1 m^2_{k,h} = m^2_{\tilde{k},\tilde{h}}\Box_1(m(\beta) \boxtimes m(\alpha))\,, \tag{42}$$

where $\alpha, \beta, k, h, \tilde{k}, \tilde{h}$ are 2- and 1-cells respectively in $\overrightarrow{\mathbb{H}} \times_{\mathbb{H}} \overrightarrow{\mathbb{H}}$. In terms of constituent cells (42) can be drawn as (43), where the pasting of the center and right squares corresponds to the right hand side of the equation (42), and the pasting of the left and outer squares corresponds to the left hand side. Equality in (42) is equivalent to the top and bottom squares commuting, since the aforementioned ones do so by assumption.

We thus spell out the details of the top and bottom squares in (43): The diagram (44) shows the details of the top square of (43). The central octagon of (44) is broken down in (41). The parts of these two diagrams commute essentially by the Gray-category axioms and the definitions of 2- and 3-cells in the path space. The bottom square on (43) is analogous.

This proves (14).

Furthermore, we check that tensors of cocycle elements are trivial: We calculate according to section 3.12:

$$m^2_{f_1,f_2} \boxtimes m^2_{f_3,f_4} = ((m^2_{f_1,f_2})_1 \otimes (m^2_{f_3,f_4})_1, (m^2_{f_1,f_2})_2 \otimes (m^2_{f_3,f_4})_2)\,,$$

where according to (38) all the arguments on the right are trivial, hence their tensors are trivial, that is, (15) holds.

Lastly, images of 2-cells tensor trivially with co-cycle components by the unitality of the tensor in $\mathbb{H}$ and the fact that the 2-cell faces of m^2 are trivial, hence verifying (16) and (17). ☐

4.3. THEOREM. *There is a pseudo* Gray*-functor m such that*

$$\begin{tikzcd} \overrightarrow{\mathbb{H}} \times_{\mathbb{H}} \overrightarrow{\mathbb{H}} \arrow[r, "m", "/" marking] & \overrightarrow{\mathbb{H}} \arrow[r, bend left, "d_1"] \arrow[r, bend right, "d_0"'] & \mathbb{H} \arrow[l, "i" description] \end{tikzcd} \tag{45}$$

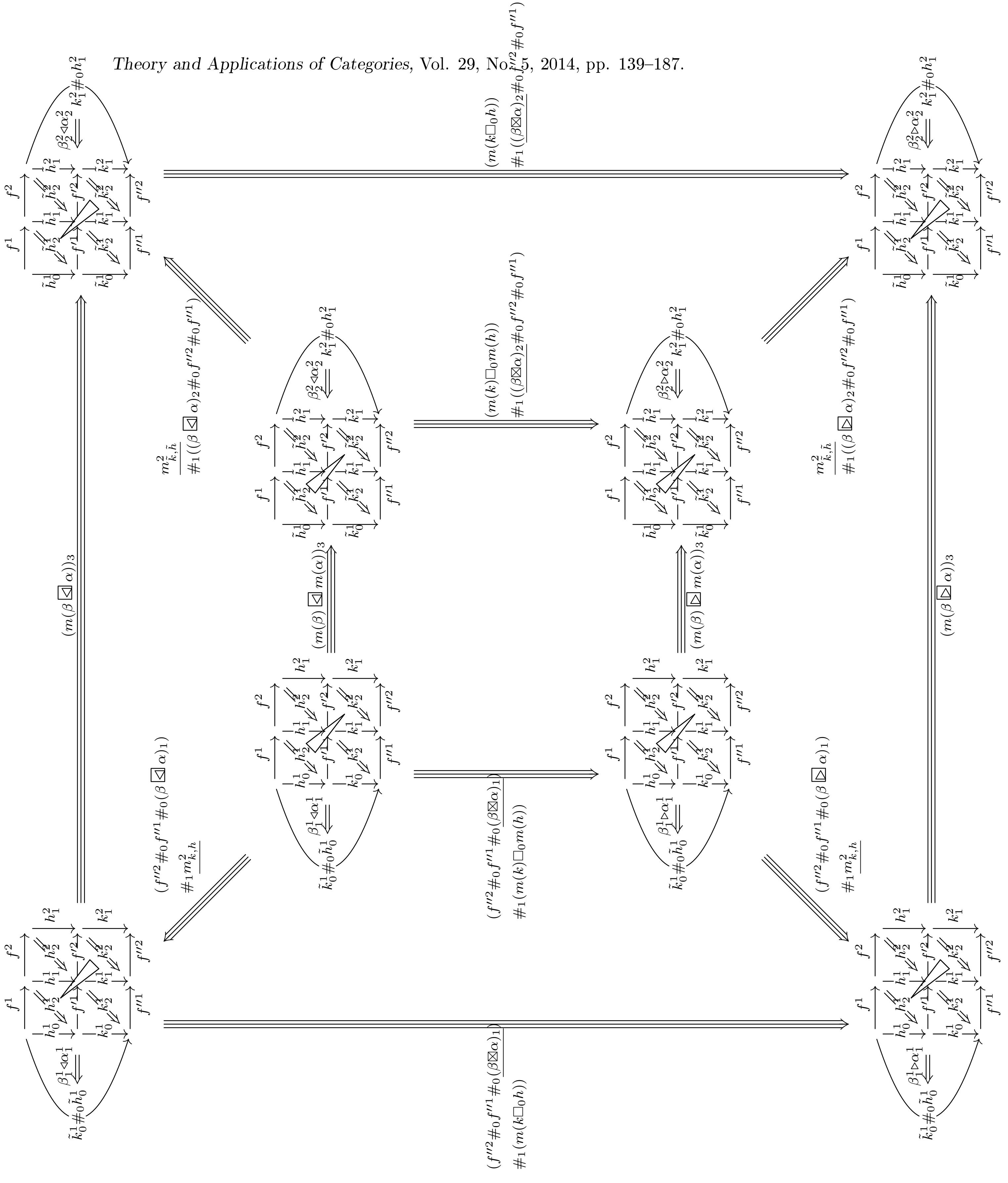
is an internal category object in $\mathsf{GrayCat}_{\mathrm{Q}^1}$.

PROOF We need to verify that m is an associative and unital operation. We need to check first that

$$\begin{tikzcd} \overrightarrow{\mathbb{H}} \times_{d_0,d_1} \overrightarrow{\mathbb{H}} \times_{d_0,d_1} \overrightarrow{\mathbb{H}} \arrow[r, "\overrightarrow{\mathbb{H}} \times m", "/" marking] \arrow[d, "m \times \overrightarrow{\mathbb{H}}"', "/" marking] & \overrightarrow{\mathbb{H}} \times_{d_0,d_1} \overrightarrow{\mathbb{H}} \arrow[d, "m", "/" marking] \\ \overrightarrow{\mathbb{H}} \times_{d_0,d_1} \overrightarrow{\mathbb{H}} \arrow[r, "m"', "/" marking] & \overrightarrow{\mathbb{H}} \end{tikzcd} \quad ,$$

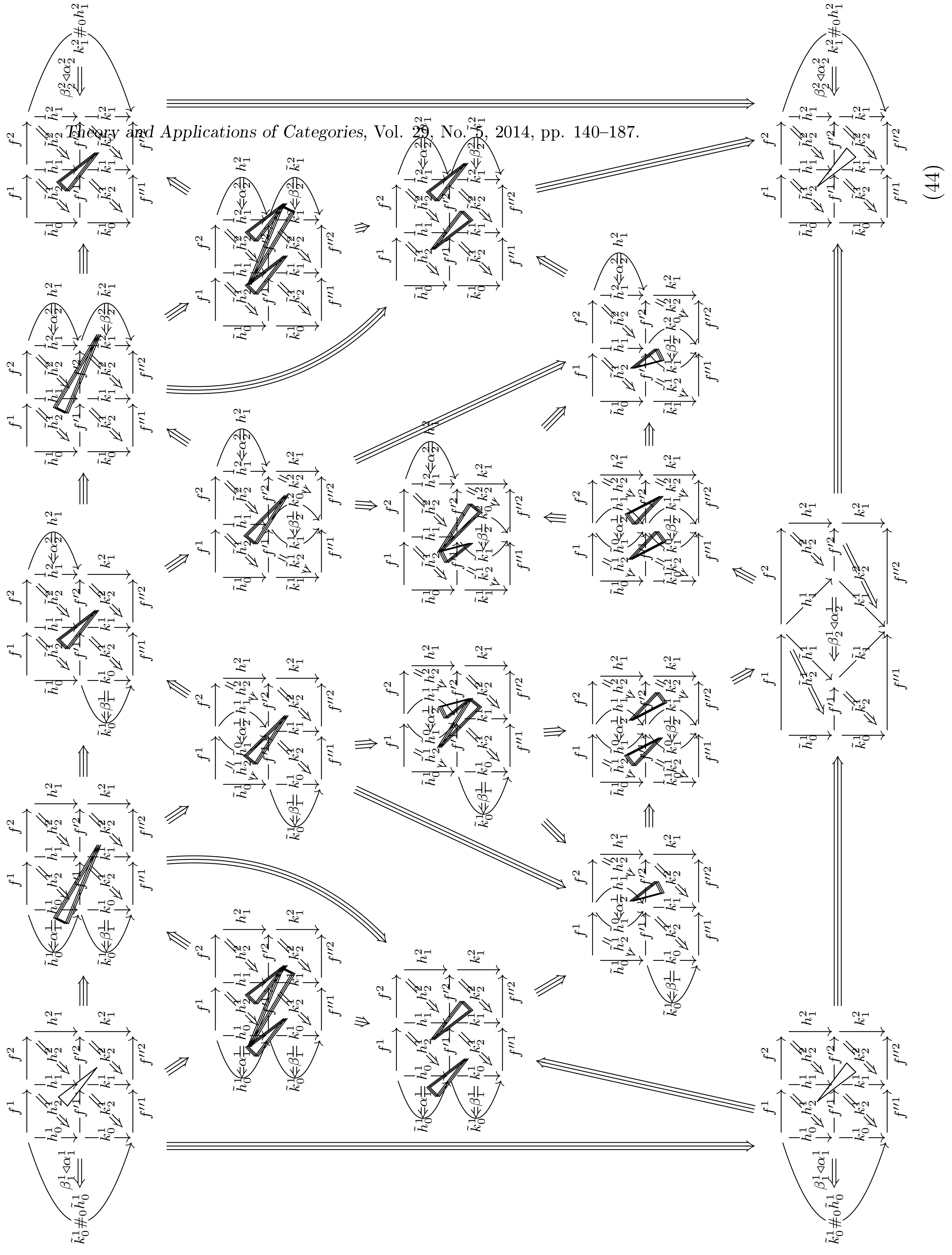
where $m \times \overrightarrow{\mathbb{H}}$ and $\overrightarrow{\mathbb{H}} \times m$ exist by the observation in remark 2.20. On the level of globular maps this is obvious, since it is just pasting according to definition 4.1. Proving that the cocycles both ways around are the same, means drawing a diagram that looks like (39) with each array transposed.

Theory and Applications of Categories, Vol. 29, No. 5, 2014, pp. 139–187.



(43)

Theory and Applications of Categories, Vol. 29, No. 5, 2014, pp. 140–187.



(44)

Unitality is obvious, source and target conditions

$$\begin{tikzcd}
& & \overrightarrow{\mathbb{H}} \times_{d_0,d_1} \overrightarrow{\mathbb{H}} \arrow[dl] \arrow[d, "m", "\shortmid" marking] \arrow[dr] & & \\
& \overrightarrow{\mathbb{H}} \arrow[dl, "d_1"'] \arrow[dr, "d_0"] & \overrightarrow{\mathbb{H}} \arrow[dll, "d_1"'] \arrow[drr, "d_0"] & \overrightarrow{\mathbb{H}} \arrow[dl, "d_1"'] \arrow[dr, "d_0"] & \\
\mathbb{H} & & \mathbb{H} & & \mathbb{H}
\end{tikzcd}$$

hold by definition 4.1. In particular, the 2-cell components of m^2 are trivial, thus d_0m and d_1m are strict Gray-functors, even though m is pseudo. ☐

4.4. Lemma. *For a strict* Gray*-functor F the multiplication map m is natural, that is*

$$\begin{tikzcd}
\overrightarrow{\mathbb{H}} \times_{d_0,d_1} \overrightarrow{\mathbb{H}} \arrow[r, "m", "\shortmid" marking] \arrow[d, "\overrightarrow{F}\times\overrightarrow{F}"'] & \overrightarrow{\mathbb{H}} \arrow[d, "F"] \\
\overrightarrow{\mathbb{K}} \times_{d_0,d_1} \overrightarrow{\mathbb{K}} \arrow[r, "m"', "\shortmid" marking] & \overrightarrow{\mathbb{K}}
\end{tikzcd} \quad . \tag{46}$$

Note that by (9) we have $(\overrightarrow{F}e)\dot{\times}(\overrightarrow{F}e) = (\overrightarrow{F} \times \overrightarrow{F})e$.

Proof Verifying (46) elementwise is straightforward. ☐

We can define the 1-cell inverse to

$$\begin{tikzcd}
{} \arrow[r, "f"] \arrow[d, "g_0"'] & {} \arrow[d, "g_1"] \arrow[dl, Rightarrow, shorten=8pt, "g_2"] \\
{} \arrow[r, "f'"'] & {}
\end{tikzcd} \tag{47}$$

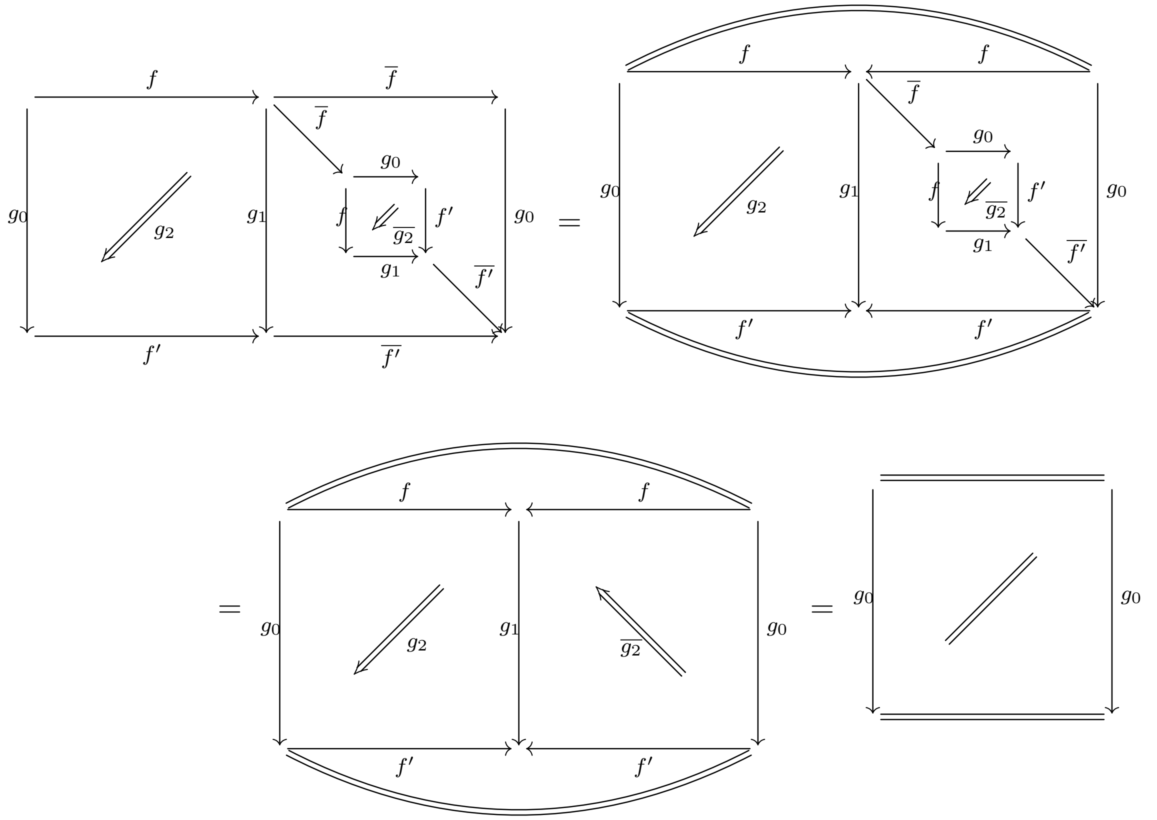
with respect to m as

$$\begin{tikzcd}
{} \arrow[rrr, "\overline{f}"] \arrow[ddd, "g_1"'] \arrow[dr, "\overline{f}"] & & & {} \arrow[ddd, "g_0"] \\
& {} \arrow[r, "g_0"] \arrow[d, "f"'] & {} \arrow[d, "f'"] \arrow[dl, Rightarrow, shorten=5pt, "\overline{g_2}"] & \\
& {} \arrow[r, "g_1"'] & {} \arrow[dr, "\overline{f'}"] & \\
{} \arrow[rrr, "\overline{f'}"'] & & & {}
\end{tikzcd} \tag{48}$$

where $\overline{(\_)}$ is the respective vertical inverse in $\mathbb{H}$.

4.5. Lemma. *The path space 1-cell in* (48) *is a left and right inverse to* (47) *with respect to* m.

Proof



And similarly for the right inverse. □

Furthermore, these inverses behave well with respect to the internal category structure:

4.6. Theorem. *Given the situation in* (45), *assume* $\mathbb{H}$ *is a* Gray*-groupoid, then there is a* Q^1*-map* $o\colon \overrightarrow{\mathbb{H}} \nrightarrow \overrightarrow{\mathbb{H}}$ *("opposite") such that* (45) *becomes an internal groupoid in* $\mathsf{GrayCat}_{\mathrm{Q}^1}$.

Proof The action of o on 0- and 1-cells is already given in (48), the effect on 2- and 3-cells of $\overrightarrow{\mathbb{H}}$ is analogous.

Furthermore, we need to give a 2-cocycle $o^2_{h,g}\colon o(h)\square_0 o(g) \longrightarrow o(h\square_0 g)$ the non-trivial part of which is the following 3-cell:

$$o\left(\begin{tikzcd} {} \arrow[r, "f'"] \arrow[d, "h_0"'] & {} \arrow[d, "h_1"] \arrow[dl, Rightarrow, shorten=2mm, "h_2"] \\ {} \arrow[r, "f''"'] & {} \end{tikzcd}\right) \square_0 o\left(\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "g_0"'] & {} \arrow[d, "g_1"] \arrow[dl, Rightarrow, shorten=2mm, "g_2"] \\ {} \arrow[r, "f'"'] & {} \end{tikzcd}\right) = \begin{tikzcd}
{} \arrow[rrr, "\overline{f}"] \arrow[dr, "\overline{f}"] \arrow[ddd, "g_1"'] & & & {} \arrow[ddd, "g_0"] \\
& {} \arrow[r, "g_0"] \arrow[d, "f"'] & {} \arrow[d, "f'"] \arrow[dl, Rightarrow, shorten=2mm, "\overline{g_2}"] & \\
& {} \arrow[r, "g_1"'] & {} \arrow[dr, "\overline{f'}"] & \\
{} \arrow[rrr, "\overline{f'}"] \arrow[dr, "\overline{f'}"] \arrow[ddd, "h_1"'] & & & {} \arrow[ddd, "h_0"] \\
& {} \arrow[r, "h_0"] \arrow[d, "f'"'] & {} \arrow[d, "f''"] \arrow[dl, Rightarrow, shorten=2mm, "\overline{h_2}"] & \\
& {} \arrow[r, "h_1"'] & {} \arrow[dr, "\overline{f''}"] & \\
{} \arrow[rrr, "\overline{f''}"'] & & & {}
\end{tikzcd}$$

$$= \begin{tikzcd}
{} \arrow[rrr, "\overline{f}"] \arrow[dr, "\overline{f}"] \arrow[ddd, "g_1"'] & & & {} \arrow[ddd, "g_0"] \\
& {} \arrow[r, "g_0"] \arrow[d, "f"'] & {} \arrow[d, "f'"] \arrow[dl, Rightarrow, shorten=2mm, "\overline{g_2}"] & \\
& {} \arrow[r, "g_1"'] & {} \arrow[ddl, "\overline{f'}"] & \\
{} \arrow[ddd, "h_1"'] & & & {} \arrow[ddd, "h_0"] \\
& {} \arrow[r, "h_0"] \arrow[d, "f'"'] & {} \arrow[d, "f''"] \arrow[dl, Rightarrow, shorten=2mm, "\overline{h_2}"] & \\
& {} \arrow[r, "h_1"'] & {} \arrow[dr, "\overline{f''}"] & \\
{} \arrow[rrr, "\overline{f''}"'] & & & {}
\end{tikzcd} \overset{\substack{\overline{f''}\#_0 \\ \big((\overline{h_2}\#_0\overline{f'})\otimes\overline{g_2}\big) \\ \#_0\overline{f}}}{\Rrightarrow} \begin{tikzcd}
{} \arrow[rrr, "\overline{f}"] \arrow[dr, "\overline{f}"] \arrow[ddd, "g_1"'] & & & {} \arrow[ddd, "g_0"] \\
& {} \arrow[r, "g_0"] \arrow[d, "f"'] & {} \arrow[d, "f'"] \arrow[dl, Rightarrow, shorten=2mm, "\overline{g_2}"] & \\
& {} \arrow[r, "g_1"'] & {} \arrow[ddl, "\overline{f'}"] & \\
{} \arrow[ddd, "h_1"'] & & & {} \arrow[ddd, "h_0"] \\
& {} \arrow[r, "h_0"] \arrow[d, "f'"'] & {} \arrow[d, "f''"] \arrow[dl, Rightarrow, shorten=2mm, "\overline{h_2}"] & \\
& {} \arrow[r, "h_1"'] & {} \arrow[dr, "\overline{f''}"] & \\
{} \arrow[rrr, "\overline{f''}"'] & & & {}
\end{tikzcd}$$

$$= \begin{tikzcd}
{} \arrow[rrr, "\overline{f}"] \arrow[dr, "\overline{f}"] \arrow[ddd, "h_1\#_0 g_1"'] & & & {} \arrow[ddd, "h_0\#_0 g_0"] \\
& {} \arrow[r, "h_0\#_0 g_0"] \arrow[d, "f"'] & {} \arrow[d, "f''"] \arrow[dl, Rightarrow, shorten=2mm] & \\
& {} \arrow[r, "h_1\#_0 g_1"'] & {} \arrow[dr, "\overline{f''}"] & \\
{} \arrow[rrr, "\overline{f''}"'] & & & {}
\end{tikzcd} \quad \begin{array}{l} (\overline{h_2}\#_0\overline{f'})\triangleright\overline{g_2} \\ =\overline{(h_2\#_0\overline{f'})\triangleleft g_2} \\ =\overline{h_2\triangleleft(\overline{f'}\#_0 g_2)} \\ =\overline{(h_2\#_0 g_0)\#_1(h_1\#_0 g_2)} \end{array} \quad = o\left(\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "g_0"'] & {} \arrow[d, "g_1"] \arrow[dl, Rightarrow, shorten=2mm, "g_2"] \\ {} \arrow[r, "f'"] \arrow[d, "h_0"'] & {} \arrow[d, "h_1"] \arrow[dl, Rightarrow, shorten=2mm, "h_2"] \\ {} \arrow[r, "f''"'] & {} \end{tikzcd}\right)$$

For the relationship between horizontal composition and pasting of squares see remark 3.9.

We check that o^2 is indeed a 2-cocycle. Given suitably incident 1-cells of $\mathbb{H}$ we need to verify that the analog of (11) hold, that is,

$$o^2_{k,h\Box_0 g}\Box_1(o(k)\Box_0 o^2_{h,g}) = o^2_{k\Box_0 h,g}\Box_1(o^2_{k,h}\Box_0 o(g)),$$

hence (49) commutes. $\Box$

5. Higher Cells

In order to describe higher transformations between maps of Gray-categories we construct an internal Gray-category in $\mathsf{GrayCat}_{\mathrm{Q}^1}$ as a substructure of the iterated path space.

5.1. Combining Path Spaces and Resolutions. We begin by describing explicitly the action of $\overrightarrow{e}\colon \overrightarrow{\mathrm{Q}^1\mathbb{G}} \longrightarrow \overrightarrow{\mathbb{G}}$ as follows:

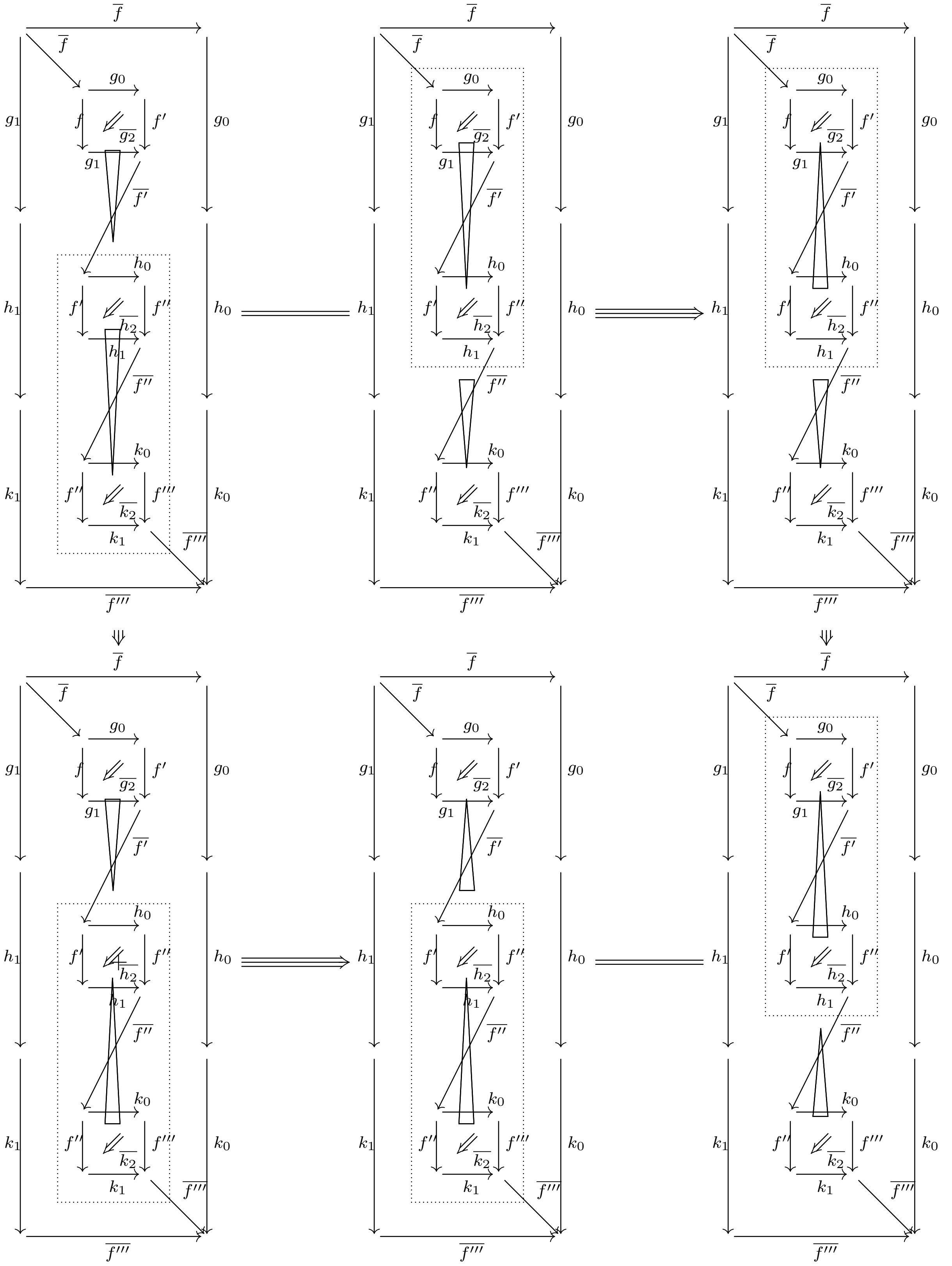
$$\overrightarrow{e}\left(\xrightarrow{[f_1,\dots,f_{n_f}]}\right) = \left(\xrightarrow{f_1\#_0\cdots\#_0 f_{n_f}}\right)$$

$$\overrightarrow{e}\left(\begin{tikzcd}[column sep=huge, row sep=huge] {} \arrow[r, "{[f_1,\dots,f_{n_f}]}"] \arrow[d, "{[g_{0,0},\dots,g_{0,n_{g_0}}]}"'] & {} \arrow[d, "{[g_{1,0},\dots,g_{1,n_{g_1}}]}"] \arrow[dl, Rightarrow, "{(g_2;[g_{1,0},\dots,g_{1,n_{g_1}},f_1,\dots,f_{n_f}],[f'_1,\dots,f'_{n_{f'}},g_{0,0},\dots,g_{0,n_{g_0}}])}"] \\ {} \arrow[r, "{[f'_1,\dots,f'_{n_{f'}}]}"'] & {} \end{tikzcd}\right) = \left(\begin{tikzcd}[column sep=huge, row sep=huge] {} \arrow[r, "f_1\#_0\cdots\#_0 f_{n_f}"] \arrow[d, "g_{0,0}\#_0\cdots\#_0 g_{0,n_{g_0}}"'] & {} \arrow[d, "g_{1,0}\#_0\cdots\#_0 g_{1,n_{g_1}}"] \arrow[dl, Rightarrow, "g_2"] \\ {} \arrow[r, "f'_1\#_0\cdots\#_0 f'_{n_{f'}}"'] & {} \end{tikzcd}\right)$$

$$\overrightarrow{e}\begin{pmatrix} \begin{pmatrix} \alpha_3;[g_{1,1},\dots,g_{1,n_{g_1}},f_{1,1},\dots,f_{1,n_f}], \\ [f'_{1,1},\dots,f'_{1,n_{f'}},h_{0,1},\dots,h_{0,n_{h_0}}] \end{pmatrix}; \\ (\alpha_1;[g_{0,1},\dots,g_{0,n_{g_0}}],[h_{0,1},\dots,h_{0,n_{h_0}}]), \\ (\alpha_2;[g_{1,1},\dots,g_{1,n_{g_1}}],[h_{1,1},\dots,h_{1,n_{h_1}}]), \\ \begin{pmatrix} g_2;[g_{1,1},\dots,g_{1,n_{g_1}},f_{1,1},\dots,f_{1,n_f}], \\ [f'_{1,1},\dots,f'_{1,n_{f'}},g_{0,1},\dots,g_{0,n_{h_0}}] \end{pmatrix}, \\ \begin{pmatrix} h_2;[h_{1,1},\dots,h_{1,n_{h_1}},f_{1,1},\dots,f_{1,n_f}], \\ [f'_{1,1},\dots,f'_{1,n_{f'}},h_{0,1},\dots,h_{0,n_{h_0}}] \end{pmatrix}; \\ [g_{0,1},\dots,g_{0,n_{g_0}}],[g_{1,1},\dots,g_{1,n_{g_1}}], \\ [h_{0,1},\dots,h_{0,n_{h_0}}],[h_{1,1},\dots,h_{1,n_{h_1}}], \\ [f_{1,1},\dots,f_{1,n_f}],[f'_{1,1},\dots,f'_{1,n_{f'}}] \end{pmatrix} = \begin{pmatrix} \alpha_3;\alpha_1,\alpha_2,g_2,h_2; \\ g_{0,1}\#_0\cdots\#_0 g_{0,n_{g_0}},g_{1,1}\#_0\cdots\#_0 g_{1,n_{g_1}}, \\ h_{0,1}\#_0\cdots\#_0 h_{0,n_{h_0}},h_{1,1}\#_0\cdots\#_0 h_{1,n_{h_1}}, \\ f_{1,1}\#_0\cdots\#_0 f_{1,n_f},f'_{1,1}\#_0\cdots\#_0 f'_{1,n_{f'}} \end{pmatrix}$$

$$\overrightarrow{e}\begin{pmatrix} (\Gamma_1;\alpha_1,\beta_1,[g_{0,1},\dots,g_{0,n_{g_0}}],[h_{0,1},\dots,h_{0,n_{h_0}}]), \\ (\Gamma_2;\alpha_2,\beta_2,[g_{1,1},\dots,g_{1,n_{g_1}}],[h_{1,1},\dots,h_{1,n_{h_1}}]) \end{pmatrix} = (\Gamma_1,\Gamma_2)$$

where for the 3-cells we used the abbreviated notation of (25).


$\overline{f}$
$\overline{f}$
g_0
g_1
f
f'
$\overline{g_2}$
g_1
g_0
$\overline{f'}$
h_0
h_1
f'
f''
$\overline{h_2}$
h_1
h_0
$\overline{f''}$
k_0
k_1
f''
f'''
$\overline{k_2}$
k_1
k_0
$\overline{f'''}$
$\overline{f'''}$

(49)

5.2. LEMMA. *The map* $\overrightarrow{e}\colon \overrightarrow{\mathrm{Q}^1\mathbb{G}} \longrightarrow \overrightarrow{\mathbb{G}}$ *is Cartesian with respect* $(\_)_1$*.*

PROOF $\overrightarrow{e}$ is obviously surjective on 0- and 1-cells and 2-locally an isomorphism. □

Let $F \dashv U\colon \mathsf{Cat} \longrightarrow \mathsf{RGrph}$ be the usual adjunction, then $(\overrightarrow{e})_1\colon \overrightarrow{\mathrm{Q}^1\mathbb{G}_1} \longrightarrow \overrightarrow{\mathbb{G}}_1$ has a splitting $s\colon U(\overrightarrow{\mathbb{G}}_1) \longrightarrow U(\overrightarrow{\mathrm{Q}^1\mathbb{G}_1})$ under U as follows:

$$s\left(\xrightarrow{f}\right) = \left(\xrightarrow{[f]}\right)$$

$$s\left(\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "g_0"'] \arrow[rd, phantom, "\Swarrow g_2"] & {} \arrow[d, "g_1"] \\ {} \arrow[r, "f'"'] & {} \end{tikzcd}\right) = \left(\begin{tikzcd}[column sep=huge, row sep=huge] {} \arrow[r, "{[f]}"] \arrow[d, "{[g_0]}"'] \arrow[rd, phantom, "\Swarrow (g_2;[g_1,f],[f',g_0])"] & {} \arrow[d, "{[g_1]}"] \\ {} \arrow[r, "{[f']}"'] & {} \end{tikzcd}\right)$$

Obviously in RGrph we have $U(\overrightarrow{e}_1)s = \mathrm{id}_{U(\overrightarrow{\mathbb{G}}_1)}$, taking the transpose $\overline{s}$ we get

$$\begin{tikzcd} FU(\overrightarrow{\mathbb{G}}_1) = \mathrm{Q}^1\overrightarrow{\mathbb{G}}_1 \arrow[r, "\overline{s}"] \arrow[rd, "\varepsilon = e_1"'] & \overrightarrow{\mathrm{Q}^1\mathbb{G}_1} \arrow[d, "\overrightarrow{e}_1"] \\ & \overrightarrow{\mathbb{G}}_1 \end{tikzcd} , \tag{50}$$

since $\overrightarrow{e}$ is Cartesian we can lift $\overline{s}$ through $(\_)_1$ to obtain $\psi\colon \mathrm{Q}^1\overrightarrow{\mathbb{G}} \longrightarrow \overrightarrow{\mathrm{Q}^1\mathbb{G}}$ satisfying

$$\begin{tikzcd} \mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[r, "\psi_{\mathbb{G}}"] \arrow[rd, "e_{\overrightarrow{\mathbb{G}}}"'] & \overrightarrow{\mathrm{Q}^1\mathbb{G}} \arrow[d, "\overrightarrow{e}_{\mathbb{G}}"] \\ & \overrightarrow{\mathbb{G}} \end{tikzcd} . \tag{51}$$

Let us consider the action of $\overline{s}\colon \mathrm{Q}^1\overrightarrow{\mathbb{G}}_1 \longrightarrow \overrightarrow{\mathrm{Q}^1\mathbb{G}_1}$. On 0-cells it acts just like s, on

1-cells we have the assignment:

$$\overline{s}\left(\begin{array}{c}\begin{tikzcd} {} \arrow[r, "f^n"] \arrow[d, "g_0^n"'] & {} \arrow[d, "g_1^n"] \arrow[dl, Rightarrow, "g_2^n"] \\ {} \arrow[r, "f^{n-1}"'] & {} \end{tikzcd} \\ \vdots \\ \begin{tikzcd} {} \arrow[r, "f^1"] \arrow[d, "g_0^1"'] & {} \arrow[d, "g_1^1"] \arrow[dl, Rightarrow, "g_2^1"] \\ {} \arrow[r, "f^0"'] & {} \end{tikzcd}\end{array}\right) = \left(\begin{tikzcd}[column sep=huge, row sep=huge] {} \arrow[r, "{[f^n]}"] \arrow[d, "{[g_0^1,\ldots,g_0^n]}"'] & {} \arrow[d, "{[g_1^1,\ldots,g_1^n]}"] \arrow[dl, Rightarrow, "{\begin{array}{l}\big((g_2^1\#_0 g_0^2\#_0\cdots\#_0 g_0^n)\\ \#_1\cdots\\ \#_1(g_1^1\#_0\cdots\#_0 g_2^i\#_0\cdots\#_0 g_0^n)\#_1\cdots\\ \#_1(g_1^1\#_0\cdots\#_0 g_1^{n-1}\#_0 g_2^n);\\ {[g_1^1,\ldots,g_1^n,f^n]},{[f^0,g_0^1,\ldots,g_0^n]}\big)\end{array}}"] \\ {} \arrow[r, "{[f^0]}"'] & {} \end{tikzcd}\right)$$

5.3. LEMMA. *The family ψ is natural with respect to maps $F\colon \mathbb{G} \longrightarrow \mathbb{H}$.*

PROOF Consider the diagram

$$\begin{tikzcd} \mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[r, "\psi_{\mathbb{G}}"] \arrow[d, "\mathrm{Q}^1\overrightarrow{F}"'] \arrow[rr, bend left, "e_{\overrightarrow{\mathbb{G}}}"] & \overrightarrow{\mathrm{Q}^1\mathbb{G}} \arrow[r, "\overrightarrow{e_{\mathbb{G}}}"] \arrow[d, "\overrightarrow{\mathrm{Q}^1F}"'] & \overrightarrow{\mathbb{G}} \arrow[d, "\overrightarrow{F}"] \\ \mathrm{Q}^1\overrightarrow{\mathbb{H}} \arrow[r, "\psi_{\mathbb{H}}"'] \arrow[rr, bend right, "e_{\overrightarrow{\mathbb{H}}}"'] & \overrightarrow{\mathrm{Q}^1\mathbb{H}} \arrow[r, "\overrightarrow{e_{\mathbb{H}}}"'] & \overrightarrow{\mathbb{H}} \end{tikzcd},$$

since the top and bottom triangles as well as the right hand square commute we obtain $\overrightarrow{e_{\mathbb{H}}}\psi_{\mathbb{H}}\mathrm{Q}^1\overrightarrow{F} = \overrightarrow{e_{\mathbb{H}}}\overrightarrow{\mathrm{Q}^1F}\psi_{\mathbb{G}}$. Since $\psi_1 = \overline{s}$ we need to only verify that $\overline{s}_{\mathbb{H}}(\mathrm{Q}^1\overrightarrow{F})_1 = (\overrightarrow{\mathrm{Q}^1F})_1\overline{s}_{\mathbb{G}}$, but this is immediate from the action of $\overrightarrow{(\_)}$ and Q^1. Naturality then follows by remark 2.10. □

It remains to be verified that ψ is compatible with the co-multiplication $d\colon \mathrm{Q}^1 \longrightarrow \mathrm{Q}^1\mathrm{Q}^1$, that is,

$$\begin{tikzcd} \mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[r, "d_{\overrightarrow{\mathbb{G}}}"] \arrow[d, "\psi_{\mathbb{G}}"'] & \mathrm{Q}^1\mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[r, "\mathrm{Q}^1\psi_{\mathbb{G}}"] & \mathrm{Q}^1\overrightarrow{\mathrm{Q}^1\mathbb{G}} \arrow[d, "\psi_{\mathrm{Q}^1\mathbb{G}}"] \\ \overrightarrow{\mathrm{Q}^1\mathbb{G}} \arrow[rr, "\overrightarrow{d_{\mathbb{G}}}"'] & & \overrightarrow{\mathrm{Q}^1\mathrm{Q}^1\mathbb{G}} \end{tikzcd} \tag{52}$$

commutes. We will prove this using, again, remark 2.10 with $\overrightarrow{e}$ and the commutativity of the underlying diagram of categories

$$\begin{tikzcd}
FU(\overrightarrow{\mathbb{G}}_1) \arrow[r, "F\eta U"] \arrow[d, "\overline{s}"'] & FUFU(\overrightarrow{\mathbb{G}}_1) \arrow[r, "FU\overline{s}"] & FU(\overrightarrow{\mathrm{Q}^1\mathbb{G}_1}) \arrow[d, "\overline{s}"] \\
\overrightarrow{\mathrm{Q}^1\mathbb{G}_1} \arrow[rr, "\overrightarrow{d_{\mathbb{G}1}}"'] & & \overrightarrow{\mathrm{Q}^1\mathrm{Q}^1\mathbb{G}_1} .
\end{tikzcd}$$

But because the upper left object is free over the reflexive graph $U(\overrightarrow{\mathbb{G}}_1)$ it is sufficient to check for generating 0- and 1-cells.

For 0-cells we compute:

$$\begin{aligned}\overrightarrow{d_{\mathbb{G}1}}\overline{s}\left(\xrightarrow{f}\right) &= \overrightarrow{d_{\mathbb{G}1}}\left(\xrightarrow{[f]}\right) = \left(\xrightarrow{[[f]]}\right)\\ &= \overline{s}\left(\xrightarrow{[f]}\right) = \overline{s}(FU\overline{s})\left(\xrightarrow{f}\right) = \overline{s}(FU\overline{s})(F\eta U)\left(\xrightarrow{f}\right)\end{aligned}$$

And likewise for 1-cells:

$$\overrightarrow{d_{\mathbb{G}1}}\overline{s}\left(\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "g_0"'] \arrow[rd, phantom, "\swarrow g_2"] & {} \arrow[d, "g_1"] \\ {} \arrow[r, "f'"'] & {} \end{tikzcd}\right) = \overrightarrow{d_{\mathbb{G}1}}\left(\begin{tikzcd} {} \arrow[r, "{[f]}"] \arrow[d, "{[g_0]}"'] \arrow[rd, phantom, "\swarrow (g_2;[g_1,f],[f',g_0])"] & {} \arrow[d, "{[g_1]}"] \\ {} \arrow[r, "{[f']}"'] & {} \end{tikzcd}\right) = \left(\begin{tikzcd} {} \arrow[r, "{[[f]]}"] \arrow[d, "{[[g_0]]}"'] \arrow[rd, phantom, "\swarrow (g_2;[[g_1],[f]],[[f]',[g_0]])"] & {} \arrow[d, "{[[g_1]]}"] \\ {} \arrow[r, "{[[f']]}"'] & {} \end{tikzcd}\right)$$

$$= \overline{s}\left(\begin{tikzcd} {} \arrow[r, "{[f]}"] \arrow[d, "{[g_0]}"'] \arrow[rd, phantom, "\swarrow (g_2;[g_1,f],[f',g_0])"] & {} \arrow[d, "{[g_1]}"] \\ {} \arrow[r, "{[f']}"'] & {} \end{tikzcd}\right) = \overline{s}(FU\overline{s})\left(\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "g_0"'] \arrow[rd, phantom, "\swarrow g_2"] & {} \arrow[d, "g_1"] \\ {} \arrow[r, "f'"'] & {} \end{tikzcd}\right) = \overline{s}(FU\overline{s})(F\eta U)\left(\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "g_0"'] \arrow[rd, phantom, "\swarrow g_2"] & {} \arrow[d, "g_1"] \\ {} \arrow[r, "f'"'] & {} \end{tikzcd}\right)$$

Furthermore, we can check that post-composing (52) with $\overrightarrow{e}$ gives a commuting diagram:

$$\begin{tikzcd}
\mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[r, "d_{\overrightarrow{\mathbb{G}}}"] \arrow[d, "\psi_{\mathbb{G}}"'] \arrow[rd, "\mathrm{Q}^1\overrightarrow{\mathbb{G}}"'] & \mathrm{Q}^1\mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[r, "\mathrm{Q}^1\psi_{\mathbb{G}}"] \arrow[d, "e_{\mathrm{Q}^1\overrightarrow{\mathbb{G}}}"'] & \mathrm{Q}^1\overrightarrow{\mathrm{Q}^1\mathbb{G}} \arrow[r, "\psi_{\mathrm{Q}^1\mathbb{G}}"] \arrow[rdd, "e_{\overrightarrow{\mathrm{Q}^1\mathbb{G}}}"] & \overrightarrow{\mathrm{Q}^1\mathrm{Q}^1\mathbb{G}} \arrow[dd, "\overrightarrow{e_{\mathrm{Q}^1\mathbb{G}}}"] \\
\overrightarrow{\mathrm{Q}^1\mathbb{G}} \arrow[d, "\overrightarrow{d_{\mathbb{G}}}"'] \arrow[rrrd, "\overrightarrow{\mathrm{Q}^1\mathbb{G}}"'] & \mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[rrd, "\psi_{\mathbb{G}}"] & & \\
\overrightarrow{\mathrm{Q}^1\mathrm{Q}^1\mathbb{G}} \arrow[rrr, "\overrightarrow{e_{\mathrm{Q}^1\mathbb{G}}}"'] & & & \overrightarrow{\mathrm{Q}^1\mathbb{G}}
\end{tikzcd}$$

where we use (51), naturality of ψ in lemma 5.3, and the fact that Q^1 is a comonad. Hence we can cancel $\overrightarrow{e}$ and obtain (52).

So, we have proved the following

5.4. LEMMA. *There is a natural transformation* $\psi\colon \mathrm{Q}^1\overrightarrow{(\_)} \longrightarrow \overrightarrow{\mathrm{Q}^1(\_)}$ *satisfying properties* (51) *and* (52). *We call* ψ *a* **semi-distributive law**. ☐

5.5. REMARK. *In terms of formal category theory the pair* $(\overrightarrow{(\_)}, \psi)$ *is an endomorphism of the comonad* (Q^1, d, e)*, that is,*

$$\begin{tikzcd}
\mathsf{GrayCat} \arrow[r, "\overrightarrow{(\_)}"] \arrow[d, "\mathrm{Q}^1", ""'{name=Q}] \arrow[d, bend right=70, "\mathrm{id}"', ""{name=I}] & \mathsf{GrayCat} \arrow[d, "\mathrm{Q}^1"] \\
\mathsf{GrayCat} \arrow[r, "\overrightarrow{(\_)}"'] & \mathsf{GrayCat}
\arrow[Rightarrow, from=Q, to=I, "e"']
\arrow[Rightarrow, from=1-2, to=2-1, shorten <=14pt, shorten >=14pt, "\psi"']
\end{tikzcd}
=
\begin{tikzcd}
\mathsf{GrayCat} \arrow[r, "\overrightarrow{(\_)}"] \arrow[d, "\mathrm{id}"'] & \mathsf{GrayCat} \arrow[d, "\mathrm{id}"', ""{name=J}] \arrow[d, bend left=70, "\mathrm{Q}^1", ""'{name=R}] \\
\mathsf{GrayCat} \arrow[r, "\overrightarrow{(\_)}"'] & \mathsf{GrayCat}
\arrow[Rightarrow, from=R, to=J, "e"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathsf{GrayCat} \arrow[r, "\overrightarrow{(\_)}"] \arrow[d, "\mathrm{Q}^1", ""'{name=Q}] \arrow[d, bend right=70, "\mathrm{Q}^1\mathrm{Q}^1"', ""{name=QQ}] & \mathsf{GrayCat} \arrow[d, "\mathrm{Q}^1"] \\
\mathsf{GrayCat} \arrow[r, "\overrightarrow{(\_)}"'] & \mathsf{GrayCat}
\arrow[Rightarrow, from=Q, to=QQ, "d"']
\arrow[Rightarrow, from=1-2, to=2-1, shorten <=14pt, shorten >=14pt, "\psi"']
\end{tikzcd}
=
\begin{tikzcd}
\mathsf{GrayCat} \arrow[r, "\overrightarrow{(\_)}"] \arrow[d, "\mathrm{Q}^1"'] & \mathsf{GrayCat} \arrow[d, "\mathrm{Q}^1"] \arrow[dd, bend left=70, "\mathrm{Q}^1", ""'{name=R}] \\
\mathsf{GrayCat} \arrow[r, "\overrightarrow{(\_)}"] \arrow[d, "\mathrm{Q}^1"'] & \mathsf{GrayCat} \arrow[d, "\mathrm{Q}^1"] \\
\mathsf{GrayCat} \arrow[r, "\overrightarrow{(\_)}"'] & \mathsf{GrayCat}
\arrow[Rightarrow, from=1-2, to=2-1, shorten <=14pt, shorten >=14pt, "\psi"']
\arrow[Rightarrow, from=2-2, to=3-1, shorten <=14pt, shorten >=14pt, "\psi"']
\arrow[Rightarrow, from=R, to=2-2, shorten >=10pt, "d"']
\end{tikzcd}
.$$

5.6. LEMMA. *The functor* $\overrightarrow{(\_)}$ *extends canonically to an endofunctor* $\mathcal{P}$ *of* $\mathsf{GrayCat}_{\mathrm{Q}^1}$ *by*

$$\mathcal{P}\left(\mathbb{G} \overset{f}{\nrightarrow} \mathbb{H}\right) = \left(\mathrm{Q}^1\overrightarrow{\mathbb{G}} \xrightarrow{\psi} \overrightarrow{\mathrm{Q}^1\mathbb{G}} \xrightarrow{\overrightarrow{f}} \overrightarrow{\mathbb{H}}\right) = \left(\overrightarrow{\mathbb{G}} \overset{\mathcal{P}(f)}{\nrightarrow} \overrightarrow{\mathbb{H}}\right).$$

Furthermore, it preserves strictness of maps.

PROOF We use the properties of ψ to check that this assignment is functorial. Given two maps $f\colon \mathbb{G} \nrightarrow \mathbb{H}$ and $g\colon \mathbb{H} \nrightarrow \mathbb{K}$ we compare $\mathcal{P}(g)\mathcal{P}(f)$ at the top and $\mathcal{P}(gf)$ at the bottom:

$$\begin{tikzcd}
\mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[r, "d"] \arrow[dr, "\psi"'] & \mathrm{Q}^1\mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[r, "\mathrm{Q}^1\psi"] & \mathrm{Q}^1\overrightarrow{\mathrm{Q}^1\mathbb{G}} \arrow[r, "\mathrm{Q}^1\overrightarrow{f}"] \arrow[d, "\psi"'] & \mathrm{Q}^1\overrightarrow{\mathbb{H}} \arrow[r, "\psi"] & \overrightarrow{\mathrm{Q}^1\mathbb{H}} \arrow[r, "\overrightarrow{g}"] & \overrightarrow{\mathbb{K}} \\
& \overrightarrow{\mathrm{Q}^1\mathbb{G}} \arrow[r, "\overrightarrow{d}"'] & \overrightarrow{\mathrm{Q}^1\mathrm{Q}^1\mathbb{G}} \arrow[urr, "\overrightarrow{\mathrm{Q}^1 f}"'] & & &
\end{tikzcd}
.$$

The naturality of ψ and (52) make sure they are equal. Preservation of units is exactly (51).

We remember that a strict map in $\mathsf{GrayCat}_{\mathrm{Q}^1}$ is given by $fe_{\mathbb{G}}$ where $f\colon \mathbb{G} \longrightarrow \mathbb{H}$ is from $\mathsf{GrayCat}$ and e is the co-unit of Q^1. Then by (51) we get

$$\mathcal{P}(fe_{\mathbb{G}}) = \overrightarrow{f}\overrightarrow{e_{\mathbb{G}}}\psi_{\mathbb{G}} = \overrightarrow{f}e_{\overrightarrow{\mathbb{G}}}\,,$$

Meaning that $\mathcal{P}$ acts on strict maps like $\overrightarrow{(\_)}$, in particular, it takes identities to identities. ☐

5.7. LEMMA. *The functor* $\mathcal{P}\colon \mathsf{GrayCat}_{\mathrm{Q}^1} \longrightarrow \mathsf{GrayCat}_{\mathrm{Q}^1}$ *preserves limits of diagrams of strict maps.*

PROOF Finally, by lemma 3.18 the restriction $\overrightarrow{(\_)}$ of $\mathcal{P}$ to $\mathsf{GrayCat}$ preserves limits: Let $p_i\colon \lim\{\mathbb{H}_i, b_k\} \longrightarrow \mathbb{H}_i$ be a limit cone in $\mathsf{GrayCat}$, let $f_i\colon \mathbb{G} \nrightarrow \overrightarrow{\mathbb{H}_i}$ be a cone in $\mathsf{GrayCat}_{\mathrm{Q}^1}$.

$$\begin{tikzcd}
\mathrm{Q}^1\mathbb{G} \arrow[r, dashed, "\langle f_i\rangle"] \arrow[dr, "f_i"'] & \overrightarrow{\lim\{\mathbb{H}_i, b_k\}} \arrow[d, "\overrightarrow{p_i}"] \\
& \overrightarrow{\mathbb{H}_i}
\end{tikzcd}$$

$\overrightarrow{p_i}$ is a limit cone, hence there is the unique weak map $\langle f_i\rangle\colon \mathbb{G} \nrightarrow \overrightarrow{\lim\{\mathbb{H}_i, b_k\}}$. ☐

5.8. LEMMA. *The functor* $\mathcal{P}\colon \mathsf{GrayCat}_{\mathrm{Q}^1} \longrightarrow \mathsf{GrayCat}_{\mathrm{Q}^1}$ *preserves induced maps of limits of strict diagrams, that is,* $\mathcal{P}(\dot{\lim} f_i) = \dot{\lim}(\mathcal{P} f_i)$.

PROOF Consider

$$\begin{tikzcd}
\mathrm{Q}^1\overrightarrow{\lim\{\mathbb{G}_i, a_k\}} \arrow[rr, "\psi"] \arrow[dr, "\mathrm{Q}^1\langle\overrightarrow{p_i}\rangle"] \arrow[dd, "\mathrm{Q}^1\overrightarrow{p_i}"'] & & \overrightarrow{\mathrm{Q}^1\lim\{\mathbb{G}_i, a_k\}} \arrow[r, "\overrightarrow{\dot{\lim} f_i}"] \arrow[dd, "\overrightarrow{\mathrm{Q}^1 p_i}"] & \overrightarrow{\lim\{\mathbb{H}_i, b_k\}} \arrow[dd, "\overrightarrow{p'_i}"] \\
& \mathrm{Q}^1\lim\{\overrightarrow{\mathbb{G}_i}, \overrightarrow{a_k}\} \arrow[urr, "\dot{\lim}\mathcal{P} f_i"'] \arrow[dl, "\mathrm{Q}^1 p''_i"] & & \\
\mathrm{Q}^1\overrightarrow{\mathbb{G}_i} \arrow[rr, "\psi"'] & & \overrightarrow{\mathrm{Q}^1\mathbb{G}_i} \arrow[r, "\overrightarrow{f_i}"'] & \overrightarrow{\mathbb{H}_i}
\end{tikzcd}$$

using the conventions of remark 2.20. Also, note that $\overrightarrow{\dot{\lim} f_i}\psi = \mathcal{P}(\dot{\lim} f_i)$ by definition. $\dot{\lim} f_i$ is the induced arrow for the source $f_i(\mathrm{Q}^1 p_i)$, $\dot{\lim}\mathcal{P} f_i$ is the induced arrow for $\mathcal{P}(f_i)\mathrm{Q}^1(p''_i)$. Since

$$\overrightarrow{p'_i}(\dot{\lim}\mathcal{P} f_i)\mathrm{Q}^1\langle\overrightarrow{p_i}\rangle = \overrightarrow{p'_i}\,\overrightarrow{\dot{\lim} f_i}\psi$$

and $\overrightarrow{p'_i}$ is a limit cone we obtain

$$(\dot{\lim}\mathcal{P} f_i)\mathrm{Q}^1\langle\overrightarrow{p_i}\rangle = \overrightarrow{\dot{\lim} f_i}\psi\,.$$

☐

If the limit is, for example, a product we may now say that

$$\mathcal{P}(f \dot{\times} g) = \mathcal{P} f \dot{\times} \mathcal{P} g\,. \tag{53}$$

From now on however we shall use × for the product of arrows in $\mathsf{GrayCat}_{\mathrm{Q}^1}$.

5.9. Lemma. *The face maps are natural with respect to weak maps, that is*

$$\begin{tikzcd}
\overrightarrow{\mathbb{G}} \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, "\mathcal{P}f"'] & \mathbb{G} \arrow[d, "f"] \\
\overrightarrow{\mathbb{H}} \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & \mathbb{H}
\end{tikzcd} \tag{54}$$

commutes.

Proof We write (54) in terms of its underlying maps:

$$\begin{tikzcd}
\mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[r, "d"] \arrow[d, "d"] & \mathrm{Q}^1\mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[r, "\mathrm{Q}^1e"] & \mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[r, shift left, "\mathrm{Q}^1d_0"] \arrow[r, shift right, "\mathrm{Q}^1d_1"'] \arrow[d, "\psi"] & \mathrm{Q}^1\mathbb{G} \arrow[d, equal] \\
\mathrm{Q}^1\mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[r, "\mathrm{Q}^1\psi"'] \arrow[rru, "e"] & \mathrm{Q}^1\overrightarrow{\mathrm{Q}^1\mathbb{G}} \arrow[r, "e"] \arrow[d, "\mathrm{Q}^1\overrightarrow{f}"'] & \overrightarrow{\mathrm{Q}^1\mathbb{G}} \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, "\overrightarrow{f}"'] & \mathrm{Q}^1\mathbb{G} \arrow[d, "f"] \\
 & \mathrm{Q}^1\overrightarrow{\mathbb{H}} \arrow[r, "e"'] & \overrightarrow{\mathbb{H}} \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & \mathbb{H}
\end{tikzcd} , \tag{55}$$

that is, (54) commuting is equivalent to the outer frame in (55) commuting. All parts are given by naturality and the co-unit laws of Q^1, except the upper right square.

We use remark 2.10 to conclude $d_0\psi = \mathrm{Q}^1d_0$ and $d_1\psi = \mathrm{Q}^1d_1$: By naturality and semi-distributivity we get $ed_0\psi = d_0\overrightarrow{e}\psi = d_0e = e\mathrm{Q}^1d_0$, furthermore, $(d_0\psi)_1 = (\mathrm{Q}^1d_0)_1$ is immediate from the definition of ψ. The map d_1 is obviously treated in the same way. □

5.10. Lemma. *The degeneracy maps of the path space are natural with respect to weak maps:*

$$\begin{tikzcd}
\mathbb{G} \arrow[r, "i"] \arrow[d, "f"'] & \overrightarrow{\mathbb{G}} \arrow[d, "\mathcal{P}f"] \\
\mathbb{H} \arrow[r, "i"'] & \overrightarrow{\mathbb{H}}
\end{tikzcd} .$$

Proof Consider

$$\begin{tikzcd}
\mathrm{Q}^1\mathbb{G} \arrow[r, "d"] \arrow[d, "d"] & \mathrm{Q}^1\mathrm{Q}^1\mathbb{G} \arrow[r, "\mathrm{Q}^1e"] & \mathrm{Q}^1\mathbb{G} \arrow[r, "\mathrm{Q}^1i"] \arrow[d, equal] & \mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[d, "\psi"] \\
\mathrm{Q}^1\mathrm{Q}^1\mathbb{G} \arrow[rr, "e"] \arrow[d, "\mathrm{Q}^1f"'] & & \mathrm{Q}^1\mathbb{G} \arrow[r, "i"] \arrow[d, "f"] & \overrightarrow{\mathrm{Q}^1\mathbb{G}} \arrow[d, "\overrightarrow{f}"] \\
\mathrm{Q}^1\mathbb{H} \arrow[rr, "e"'] & & \mathbb{H} \arrow[r, "i"'] & \overrightarrow{\mathbb{H}}
\end{tikzcd} .$$

We conclude that then top right square commutes by computing $\overrightarrow{e}i = ie = e\mathrm{Q}^1 i = \overrightarrow{e}\psi\mathrm{Q}^1 i$ and checking that $(\psi\mathrm{Q}^1 i)_1 = i_1$ and again applying remark 2.10 together with lemma 5.2. □

The functor $\mathcal{P}$ can also be applied to Q^1-graph maps by setting $\mathcal{P}' = (\mathcal{P}\tilde{G})^\vee$; see lemma 2.25 for the notation. For the sake of completeness we describe briefly the effect of $\mathcal{P}'$ at the level of 1-cells as well as its 2-co-cycle. Let $G\colon \mathbb{G} \longrightarrow \mathbb{H}$ be a Q^1-graph map. We take a 1-cell $g\colon f \longrightarrow f'$ from $\overrightarrow{\mathbb{G}}$ and calculate:

$$(\mathcal{P}'G)(g) = \left(\overrightarrow{\tilde{G}}\psi\right)^\vee(g) = \overrightarrow{\tilde{G}}\psi\left[\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "g_0"'] & {} \arrow[d, "g_1"] \arrow[dl, Rightarrow, "g_2" description] \\ {} \arrow[r, "f'"'] & {} \end{tikzcd}\right]$$

$$= \left(\begin{tikzcd}[column sep=huge, row sep=huge] {} \arrow[r, "{\tilde{G}[f]}"] \arrow[d, "{\tilde{G}[g_0]}"'] & {} \arrow[d, "{\tilde{G}[g_1]}"] \arrow[dl, Rightarrow, "{\tilde{G}(g_2;[g_1,f],[f,g_0])}" description] \\ {} \arrow[r, "{\tilde{G}[f']}"'] & {} \end{tikzcd}\right) = \left(\begin{tikzcd}[column sep=huge, row sep=huge] {} \arrow[r, "Gf"] \arrow[d, "Gg_0"'] & {} \arrow[d, "Gg_1"] \arrow[dl, Rightarrow, "{\overline{G^2_{f',g_0}}\#_1 Gg_2 \#_1 G^2_{g_1,f}}" description] \\ {} \arrow[r, "Gf'"'] & {} \end{tikzcd}\right) \tag{56}$$

Taking two composable 1-cells $g\colon f \longrightarrow f'$ and $h\colon f' \longrightarrow f''$ of $\overrightarrow{\mathbb{G}}$ we get a 2-cocycle with components as shown in (57), where in the end the $\tilde{G}\kappa_{\dots}$ are iterated 2-cocycles of G.

5.11. ITERATING THE PATH SPACE CONSTRUCTION.

5.12. REMARK. *As a consequence of lemma 5.9, lemma 5.10, and lemma 4.4 the maps* i, d_0, d_1 *and* m *for all* Gray*-categories* $\mathbb{H}$ *constitute natural transformations with respect to strict maps.*

For reference, this means that for all $f\colon \mathbb{H} \longrightarrow \mathbb{K}$ the following diagram commutes sequentially:

$$\begin{tikzcd}[column sep=large] \overrightarrow{\mathbb{H}} \times_{\mathbb{H}} \overrightarrow{\mathbb{H}} \arrow[r, "m", "/" marking] \arrow[d, "\overrightarrow{f} \times \overrightarrow{f}"'] & \overrightarrow{\mathbb{H}} \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] \arrow[d, "\overrightarrow{f}"'] & \mathbb{H} \arrow[l, "i" description] \arrow[d, "f"] \\ \overrightarrow{\mathbb{K}} \times_{\mathbb{K}} \overrightarrow{\mathbb{K}} \arrow[r, "m"', "/" marking] & \overrightarrow{\mathbb{K}} \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & \mathbb{K} \arrow[l, "i" description] \end{tikzcd}$$

Iterating the arrow construction yields an internal cubical set, so it allows us to talk about higher cells in the internal language of GrayCat. But since we want to construct an internal Gray-category we need to restrict to cubical cells with certain degeneracies. The general recipe beyond the construction in section 3 is to apply $\overrightarrow{(\_)}$ and squash the excess faces given by $\overrightarrow{d_{0,1}}$ so that the only non-trivial faces of each cubical element are the ones given by $d_{0,1}$.

$$((\mathcal{P}'G)^\vee)^2_{h,g} = ((\overrightarrow{\tilde{G}}\psi)^\vee)^2_{h,g} = \overrightarrow{\tilde{G}}\psi(\kappa_{h,g})$$

$$= \overrightarrow{\tilde{G}}\psi\left(\begin{array}{c}
\begin{tikzcd}[column sep=large, row sep=large]
{} \arrow[r, "f"] \arrow[d, bend right=70, "h_0\#_0g_0"'] \arrow[d, "h_0\#_0g_0"] \arrow[d, phantom, shift right=5ex, "\Leftarrow_{\mathrm{id}}"] \arrow[dr, phantom, "(h_2\#_0g_0)\#_1(h_1\#_0g_2)"] & {} \arrow[d, "h_1\#_0g_1"] \\
{} \arrow[r, "f''"'] & {}
\end{tikzcd} \\
\Big\Downarrow \mathrm{id}_{(h_2\#_0g_0)\#_1(h_1\#_0g_2)}; \\
\begin{tikzcd}[column sep=large, row sep=large]
{} \arrow[r, "f"] \arrow[d, "h_0\#_0g_0"'] \arrow[dr, phantom, "(h_2\#_0g_0)\#_1(h_1\#_0g_2)"] & {} \arrow[d, "h_1\#_0g_1"'] \arrow[d, bend left=70, "h_1\#_0g_1"] \arrow[d, phantom, shift left=5ex, "\Leftarrow_{\mathrm{id}}"] \\
{} \arrow[r, "f''"'] & {}
\end{tikzcd}
\end{array}
\left[\begin{tikzcd}
{} \arrow[r, "f'"] \arrow[d, "h_0"'] \arrow[dr, phantom, "h_2"] & {} \arrow[d, "h_1"] \\
{} \arrow[r, "f''"'] & {}
\end{tikzcd}, \begin{tikzcd}
{} \arrow[r, "f"] \arrow[d, "g_0"'] \arrow[dr, phantom, "g_2"] & {} \arrow[d, "g_1"] \\
{} \arrow[r, "f'"'] & {}
\end{tikzcd}\right], \left[\begin{tikzcd}[column sep=large, row sep=large]
{} \arrow[r, "f"] \arrow[d, "h_0\#_0g_0"'] \arrow[dr, phantom, "(h_2\#_0g_0)\#_1(h_1\#_0g_2)"] & {} \arrow[d, "h_1\#_0g_1"] \\
{} \arrow[r, "f''"'] & {}
\end{tikzcd}\right]\right)$$

$$= \overrightarrow{\tilde{G}}\left(
\begin{tikzcd}[column sep=huge, row sep=huge]
{} \arrow[r, "{[f]}"] \arrow[d, bend right=70, "{[h_0\#_0g_0]}"'] \arrow[d, "{[h_0,g_0]}"] \arrow[d, phantom, shift right=5ex, "\Leftarrow_{\kappa_{h_0,g_0}}"] \arrow[dr, phantom, "{((h_2\#_0g_0)\#_1(h_1\#_0g_2);[h_1,g_1,f],[f'',h_0,g_0])}"] & {} \arrow[d, "{[h_1,g_1]}"] \\
{} \arrow[r, "{[f'']}"'] & {}
\end{tikzcd}
\overset{\begin{array}{l}(\mathrm{id}_{(h_2\#_0g_0)\#_1(h_1\#_0g_2)};\\ (h_2\#_0g_0)\#_1(h_1\#_0g_2),\\ (h_2\#_0g_0)\#_1(h_1\#_0g_2);\\ {[h_1,g_1,f]},[f'',h_0\#_0g_0])\end{array}}{\Rrightarrow}
\begin{tikzcd}[column sep=huge, row sep=huge]
{} \arrow[r, "{[f]}"] \arrow[d, "{[h_0\#_0g_0]}"'] \arrow[dr, phantom, "{((h_2\#_0g_0)\#_1(h_1\#_0g_2);[h_1\#_0g_1,f],[f'',h_0\#_0g_0])}"] & {} \arrow[d, "{[h_1\#_0g_1]}"'] \arrow[d, bend left=70, "{[h_1,g_1]}"] \arrow[d, phantom, shift left=5ex, "\Leftarrow_{\kappa_{h_1,g_1}}"] \\
{} \arrow[r, "{[f'']}"'] & {}
\end{tikzcd}
\right)$$

$$= \left(
\begin{tikzcd}[column sep=huge, row sep=huge]
{} \arrow[r, "{\tilde{G}[f]}"] \arrow[d, bend right=70, "{\tilde{G}[h_0\#_0g_0]}"'] \arrow[d, "{\tilde{G}[h_0,g_0]}"] \arrow[d, phantom, shift right=5ex, "\Leftarrow_{\tilde{G}\kappa_{h_0,g_0}}"] \arrow[dr, phantom, "{\tilde{G}((h_2\#_0g_0)\#_1(h_1\#_0g_2);[h_1,g_1,f],[f'',h_0,g_0])}"] & {} \arrow[d, "{\tilde{G}[h_1,g_1]}"] \\
{} \arrow[r, "{\tilde{G}[f'']}"'] & {}
\end{tikzcd}
\overset{\begin{array}{l}\tilde{G}(\mathrm{id}_{(h_2\#_0g_0)\#_1(h_1\#_0g_2)};\\ (h_2\#_0g_0)\#_1(h_1\#_0g_2),\\ (h_2\#_0g_0)\#_1(h_1\#_0g_2);\\ {[h_1,g_1,f]},[f'',h_0\#_0g_0])\end{array}}{\Rrightarrow}
\begin{tikzcd}[column sep=huge, row sep=huge]
{} \arrow[r, "{\tilde{G}[f]}"] \arrow[d, "{\tilde{G}[h_0\#_0g_0]}"'] \arrow[dr, phantom, "{\tilde{G}((h_2\#_0g_0)\#_1(h_1\#_0g_2);[h_1\#_0g_1,f],[f'',h_0\#_0g_0])}"] & {} \arrow[d, "{\tilde{G}[h_1\#_0g_1]}"'] \arrow[d, bend left=70, "{\tilde{G}[h_1,g_1]}"] \arrow[d, phantom, shift left=5ex, "\Leftarrow_{\tilde{G}\kappa_{h_1,g_1}}"] \\
{} \arrow[r, "{\tilde{G}[f'']}"'] & {}
\end{tikzcd}
\right)$$

$$= \left(
\begin{tikzcd}[column sep=huge, row sep=huge]
{} \arrow[r, "Gf"] \arrow[d, bend right=70, "G(h_0\#_0g_0)"'] \arrow[d, "Gh_0\#_0Gg_0"] \arrow[d, phantom, shift right=5ex, "\Leftarrow_{G^2_{h_0,g_0}}"] \arrow[dr, phantom, "\overline{\tilde{G}\kappa_{f'',h_0,g_0}}\#_1G((h_2\#_0g_0)\#_1(h_1\#_0g_2))\#_1\tilde{G}\kappa_{h_1,g_1,f}"] & {} \arrow[d, "Gh_1\#_0Gg_1"] \\
{} \arrow[r, "Gf''"'] & {}
\end{tikzcd}
\overset{\mathrm{id}}{\Rrightarrow}
\begin{tikzcd}[column sep=huge, row sep=huge]
{} \arrow[r, "Gf"] \arrow[d, "G(h_0\#_0g_0)"'] \arrow[dr, phantom, "{G((h_2\#_0g_0)\#_1(h_1\#_0g_2);[h_1\#_0g_1,f],[f'',h_0\#_0g_0])}"] & {} \arrow[d, "G(h_1\#_0g_1)"'] \arrow[d, bend left=70, "Gh_1\#_0Gg_1"] \arrow[d, phantom, shift left=5ex, "\Leftarrow_{\tilde{G}\kappa_{h_1,g_1}}"] \\
{} \arrow[r, "{\tilde{G}[f'']}"'] & {}
\end{tikzcd}
\right) \tag{57}$$

This general procedure will canonically yield an internal reflexive n-graph, we will furthermore have to provide the operations in each degree to actually obtain a Gray-category. We carry out this construction for the degrees 2 and 3 in sections 5.12.1 and 5.22.1.

5.12.1. 2-PATHS. We construct the space of 2-paths $\overline{\overline{\overline{\mathbb{H}}}}$ over $\overrightarrow{\mathbb{H}}$ and give the vertical composition of 2-paths and their whiskers by 1-paths.

The 0-cells in $\overrightarrow{\overrightarrow{\mathbb{H}}}$ are squares, and we want to filter out those square that are actually bigons, that is, have identity arrows as left and right sides. That is exactly what we get by forming the double pullback on the left:

$$\begin{tikzcd}
\overline{\overline{\overline{\mathbb{H}}}} \arrow[r, "j"] \arrow[d, shift right, "\overline{d_0}"'] \arrow[d, shift left, "\overline{d_1}"] & \overrightarrow{\overrightarrow{\mathbb{H}}} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[d, shift right, "\overrightarrow{d_0}"'] \arrow[d, shift left, "\overrightarrow{d_1}"] & \overrightarrow{\mathbb{H}} \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
\mathbb{H} \arrow[r, "i"'] & \overrightarrow{\mathbb{H}} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & \mathbb{H}
\end{tikzcd} \tag{58}$$

where $\overline{\overline{\overline{\mathbb{H}}}}$ is the intersection of the pullbacks of d_0 and d_1 along i. Let $d_0^j = d_0 j$ and $d_1^j = d_1 j$.

5.13. LEMMA. *The diagram*

$$\begin{tikzcd}
\overline{\overline{\overline{\mathbb{H}}}} \arrow[r, shift left, "d_1^j"] \arrow[r, shift right, "d_0^j"'] & \overrightarrow{\mathbb{H}} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & \mathbb{H}
\end{tikzcd} \tag{59}$$

is a globular object, i. e. $d_0 d_0^j = d_0 d_1^j$ *and* $d_1 d_0^j = d_1 d_1^j$.

PROOF Using the naturality of d_0 and d_1 we calculate:

$$d_0 d_0^j = d_0 d_0 j = d_0 \overrightarrow{d_0} j = d_0 i \overline{d_0} = d_1 i \overline{d_0} = d_1 \overrightarrow{d_0} j = d_0 d_1 j = d_1 d_0^j \,,$$

and similarly for d_1. ☐

To get a unit for $\overline{\overline{\overline{\mathbb{H}}}}$, that is, an identity 2-paths for 1-paths, we consider the following diagram:

$$\begin{tikzcd}
\overrightarrow{\mathbb{H}} \arrow[dr, dotted, "\bar{i}"] \arrow[drr, "i"] \arrow[ddr, shift right, "d_0"'] \arrow[ddr, shift left, "d_1"] & & & \\
& \overline{\overline{\overline{\mathbb{H}}}} \arrow[r, "j"] \arrow[d, shift right, "\overline{d_0}"'] \arrow[d, shift left, "\overline{d_1}"] & \overrightarrow{\overrightarrow{\mathbb{H}}} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[d, shift right, "\overrightarrow{d_0}"'] \arrow[d, shift left, "\overrightarrow{d_1}"] & \overrightarrow{\mathbb{H}} \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
& \mathbb{H} \arrow[r, "i"'] & \overrightarrow{\mathbb{H}} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & \mathbb{H}
\end{tikzcd}$$

The upper left span is a compatible source by the naturality of i. The induced arrow $\bar{i}$ is a joint section of d_0^j and d_1^j. Hence we get:

5.14. Lemma. *The diagram*

$$\begin{tikzcd}
\overline{\overline{\mathbb{H}}} \arrow[r, shift left=2, "d_1^j"] \arrow[r, shift right=2, "d_0^j"'] & \overrightarrow{\mathbb{H}} \arrow[l, "\bar{i}" description]
\end{tikzcd} \tag{60}$$

is a reflexive graph. ☐

5.15. Lemma. *The mapping* $\overline{\overline{(-)}}$ *extends to a sub-functor of* $\overset{\rightrightarrows}{(-)}$: $\mathsf{GrayCat} \longrightarrow \mathsf{GrayCat}$ *with natural embedding* j.

Proof For each $\mathbb{H}$ the map j is a monomorphism by construction and $\overline{\overline{(-)}}$ extends to morphisms by the universal property. ☐

5.16. Lemma. *There is a multiplication*

$$\overline{\overline{\mathbb{H}}} \times_{d_0^j,d_1^j} \overline{\overline{\mathbb{H}}} \xrightarrow{\overline{m}} \overline{\overline{\mathbb{H}}}$$

with

$$\begin{aligned} d_0^j\overline{m} &= d_0^j p_1 \\ d_1^j\overline{m} &= d_1^j p_0 \end{aligned} \tag{61}$$

uniquely induced by $m_{\overrightarrow{\mathbb{H}}}$.

Proof All we need to show is that $m(j \times j)$ factors through j, that is, show that the two outer rectangles commute:

$$\begin{tikzcd}
\overline{\overline{\mathbb{H}}} \times_{d_0^j,d_1^j} \overline{\overline{\mathbb{H}}} \arrow[r, "j\times j"] \arrow[d, shift right=1, "p_0"'] \arrow[d, shift left=1, "p_1"] \arrow[d, dotted, bend left=60, "\overline{m}", "|" marking] \arrow[dd, bend right=40, shift right=1, "d_0'"'] \arrow[dd, bend right=40, shift left=1, "d_1'"] & \overset{\rightrightarrows}{\mathbb{H}} \times_{d_0,d_1} \overset{\rightrightarrows}{\mathbb{H}} \arrow[d, "m", "|" marking] \\
\overline{\overline{\mathbb{H}}} \arrow[r, "j"] \arrow[d, shift right=1, "\overline{d}_0"'] \arrow[d, shift left=1, "\overline{d}_1"] & \overset{\rightrightarrows}{\mathbb{H}} \arrow[d, shift right=1, "\overrightarrow{d}_0"'] \arrow[d, shift left=1, "\overrightarrow{d}_1"] \\
\mathbb{H} \arrow[r, "i"'] & \overrightarrow{\mathbb{H}}
\end{tikzcd} \tag{62}$$

that is, we shall verify that

$$\begin{aligned} \overrightarrow{d_0}m(j\times j) &= id_0' \\ \overrightarrow{d_1}m(j\times j) &= id_1' \end{aligned}$$

in order to obtain $\overline{m}$ as a universally induced arrow.

First we prove that $\overline{d}_0 p_0 = \overline{d}_0 p_1$:

$$\overline{d}_0 p_0 = d_0 i \overline{d}_0 p_0 = d_0 \overrightarrow{d}_0 j p_0 = d_0 d_0 j p_0 = d_0 d_0^j p_0 = d_0 d_1^j p_1 = d_0 d_0^j p_1 = \overline{d}_0 p_1 \tag{63}$$

which holds by (60), (59) and (58). Similarly $\overline{d}_1p_0 = \overline{d}_1p_1$. Thus we may define $d_0' = \overline{d}_0p_0$ and $d_1' = \overline{d}_1p_0$. Note that $j \times j$ is universally induced by $d_0jp_0 = d_1jp_1$.

Furthermore, we need that $(i\overline{d}_0 \times i\overline{d}_0) = (i,i)d_0'$ and $(i\overline{d}_1 \times i\overline{d}_1) = (i,i)d_1'$. Consider

$$\begin{tikzcd}
\overline{\overline{\mathbb{H}}} \times_{d_0^j,d_1^j} \overline{\overline{\mathbb{H}}} \arrow[rrr, "p_1"] \arrow[dr, "d_0'"] \arrow[ddr, bend right, "(i\overline{d}_0\times i\overline{d}_0)"'] \arrow[ddd, "p_0"'] & & & \overline{\overline{\mathbb{H}}} \arrow[d, "\overline{d}_0"] \\
 & \mathbb{H} \arrow[rr] \arrow[dr, "(i{,}i)"] \arrow[dd] & & \mathbb{H} \arrow[d, "i"] \\
 & & \overrightarrow{\mathbb{H}} \times_{d_0,d_1} \overrightarrow{\mathbb{H}} \arrow[r] \arrow[d] & \overrightarrow{\mathbb{H}} \arrow[d, "d_1"] \\
\overline{\overline{\mathbb{H}}} \arrow[r, "\overline{d}_0"'] & \mathbb{H} \arrow[r, "i"'] & \overrightarrow{\mathbb{H}} \arrow[r, "d_0"'] & \mathbb{H}
\end{tikzcd}$$

The top and left squares commute by (63), and (59) makes the pair $(i\overline{d}_0p_0, i\overline{d}_0p_1)$ a compatible source for lower right pullback square. The universality thus proves our equation.

Finally, we verify that

$$\overrightarrow{d}_0m(j \times j) = m(\overrightarrow{d}_0 \times \overrightarrow{d}_0)(j \times j) = m(\overrightarrow{d}_0j \times \overrightarrow{d}_0j) = m(i\overline{d}_0j \times i\overline{d}_0j) = m(i,i)d_0' = id_0' \,.$$

By the same token $d_1m(j \times j) = id_1'$ hence we get the desired $\overline{m}$.

To check (61) we calculate:

$$d_0^j\overline{m} = d_0j\overline{m} = d_0m(j \times j) = d_0p_1(j \times j) = d_0jp_1 = d_0^jp_1 \,.$$

□

5.17. Lemma. *The composition $\overline{m}$ is unital and associative, that is, it makes* (60) *a category.*

Proof Obviously since $m_{\overset{\rightrightarrows}{\mathbb{H}}}$ is so: Using the notation of (62) we can formulate the associativity condition as the two composites in the left hand column being equal:

$$\begin{tikzcd}
(\overline{\overline{\mathbb{H}}})^3 \arrow[r, "j\times j\times j"] \arrow[d, shift right, "\overline{\overline{\mathbb{H}}}\times\overline{m}"', "\shortmid"{marking}] \arrow[d, shift left, "\overline{m}\times\overline{\overline{\mathbb{H}}}", "\shortmid"{marking}] & (\overset{\rightrightarrows}{\mathbb{H}})^3 \arrow[d, shift right, "\overset{\rightrightarrows}{\mathbb{H}}\times m"', "\shortmid"{marking}] \arrow[d, shift left, "m\times\overset{\rightrightarrows}{\mathbb{H}}", "\shortmid"{marking}] \\
\overline{\overline{\mathbb{H}}} \times_{d_0^j,d_1^j} \overline{\overline{\mathbb{H}}} \arrow[r, "j\times j"] \arrow[d, "\overline{m}"', "\shortmid"{marking}] & \overset{\rightrightarrows}{\mathbb{H}} \times_{d_0,d_1} \overset{\rightrightarrows}{\mathbb{H}} \arrow[d, "m", "\shortmid"{marking}] \\
\overline{\overline{\mathbb{H}}} \arrow[r, "j"'] & \overset{\rightrightarrows}{\mathbb{H}}
\end{tikzcd}$$

whence we conclude that $j\overline{m}(\overline{\overline{\overline{\mathbb{H}}}} \times \overline{m}) = j\overline{m}(\overline{m} \times \overline{\overline{\overline{\mathbb{H}}}})$, and by j mono we get the desired $\overline{m}(\overline{\overline{\overline{\mathbb{H}}}} \times \overline{m}) = \overline{m}(\overline{m} \times \overline{\overline{\overline{\mathbb{H}}}})$.

For the unit we can argue in the same manner:

$$\begin{tikzcd}
\overline{\overline{\mathbb{H}}} \arrow[rr, "j"] \arrow[dr, "\left\langle \overline{\overline{\mathbb{H}}}, \bar{i} \right\rangle"] \arrow[dd, "\overline{\overline{\mathbb{H}}}"'] & & \overrightarrow{\overrightarrow{\mathbb{H}}} \arrow[dr, "\left\langle \overrightarrow{\overrightarrow{\mathbb{H}}}, i \right\rangle"] \arrow[dd, "\overline{\overline{\mathbb{H}}}"' near start] & \\
& \overline{\overline{\mathbb{H}}} \times_{d_0^j, d_1^j} \overline{\overline{\mathbb{H}}} \arrow[rr, "j \times j" near start] \arrow[dl, "\overline{m}", "\shortmid" marking] & & \overrightarrow{\overrightarrow{\mathbb{H}}} \times_{d_0, d_1} \overrightarrow{\overrightarrow{\mathbb{H}}} \arrow[dl, "m", "\shortmid" marking] \\
\overline{\overline{\mathbb{H}}} \arrow[rr, "j"'] & & \overrightarrow{\overrightarrow{\mathbb{H}}} &
\end{tikzcd} .$$

$\square$

5.18. Lemma. *Applying $\mathcal{P}$ to an internal category*

$$\begin{tikzcd}
\mathbb{K} \times_{d_0, d_1} \mathbb{K} \arrow[r, "m", "\shortmid" marking] & \mathbb{K} \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & \mathbb{H} \arrow[l, "i" description]
\end{tikzcd} \tag{64}$$

yields an internal category

$$\begin{tikzcd}
\overrightarrow{\mathbb{K}} \times_{\overrightarrow{d_0}, \overrightarrow{d_1}} \overrightarrow{\mathbb{K}} \arrow[r, equal] & \overrightarrow{\mathbb{K} \times_{d_0, d_1} \mathbb{K}} \arrow[r, "\mathcal{P}m", "\shortmid" marking] & \overrightarrow{\mathbb{K}} \arrow[r, shift left=2, "\overrightarrow{d_1}"] \arrow[r, shift right=2, "\overrightarrow{d_0}"'] & \overrightarrow{\mathbb{H}} \arrow[l, "\overrightarrow{i}" description]
\end{tikzcd} .$$

Proof This is true since $\mathcal{P}$ is an endofunctor of $\mathsf{GrayCat}_{\mathrm{Q}^1}$ that by lemma 3.18 preserves pullbacks of strict diagrams. In particular

$$\begin{tikzcd}
\overrightarrow{\mathbb{K}} \times_{\overrightarrow{d}_0, \overrightarrow{d}_1} \overrightarrow{\mathbb{K}} \times_{\overrightarrow{d}_0, \overrightarrow{d}_1} \overrightarrow{\mathbb{K}} \arrow[r, "\overrightarrow{\mathbb{K}} \dot{\times} \mathcal{P}m", "\shortmid" marking] \arrow[d, "\mathcal{P}m \dot{\times} \overrightarrow{\mathbb{K}}"', "\shortmid" marking] & \overrightarrow{\mathbb{K}} \times_{\overrightarrow{d}_0, \overrightarrow{d}_1} \overrightarrow{\mathbb{K}} \arrow[d, "\mathcal{P}m", "\shortmid" marking] \\
\overrightarrow{\mathbb{K}} \times_{\overrightarrow{d}_0, \overrightarrow{d}_1} \overrightarrow{\mathbb{K}} \arrow[r, "\mathcal{P}m"', "\shortmid" marking] & \overrightarrow{\mathbb{K}}
\end{tikzcd}$$

commutes since by (53) $\mathcal{P}(\mathbb{K} \dot{\times} m) = \overrightarrow{\mathbb{K}} \dot{\times} \mathcal{P}m$. $\square$

5.19. Lemma. *There are left and right whiskering maps*

$$\begin{tikzcd}
\overline{\overline{\mathbb{H}}} \times_{\overline{d_0}, d_1} \overrightarrow{\mathbb{H}} \arrow[r, "w_\ell", "\shortmid" marking] & \overline{\overline{\mathbb{H}}}
\end{tikzcd}$$

$$\begin{tikzcd}
\overrightarrow{\mathbb{H}} \times_{d_0, \overline{d_1}} \overline{\overline{\mathbb{H}}} \arrow[r, "w_r", "\shortmid" marking] & \overline{\overline{\mathbb{H}}}
\end{tikzcd}$$

induced uniquely by $\mathcal{P}(m)$.

Proof We construct a restricted horizontal composition $m'_r\colon \overrightarrow{\mathbb{H}} \times_{d_0,\overline{d_1}} \overline{\overline{\mathbb{H}}} \nrightarrow \overrightarrow{\overrightarrow{\mathbb{H}}}$ in the following diagram:

$$\begin{tikzcd}
\overrightarrow{\mathbb{H}} \times_{d_0,\bar{d}_1} \overline{\overline{\mathbb{H}}} \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[rrr, dashed, "i\times j"] \arrow[rrrrr, rounded corners, to path={[pos=0.5] -| (\tikztotarget)}, "\shortmid" marking, "m'_r"'] & \overrightarrow{\mathbb{H}} \arrow[d, "d_0"] \arrow[rrrr, "i"] & & & & \\
\overline{\overline{\mathbb{H}}} \arrow[r, "\bar{d}_1"] \arrow[rrr, "j"'] & \mathbb{H} \arrow[rrrr, "i"] & & \overrightarrow{\mathbb{H}} \times_{\overrightarrow{d}_0,\overrightarrow{d}_1} \overrightarrow{\overrightarrow{\mathbb{H}}} \arrow[r] \arrow[d] \arrow[rrdd, bend right, "\shortmid" marking, "\mathcal{P}m"'] & \overrightarrow{\overrightarrow{\mathbb{H}}} \arrow[r, "\overrightarrow{d}_1"] \arrow[d, "\overrightarrow{d}_0"] & \overrightarrow{\mathbb{H}} \\
& & & \overrightarrow{\overrightarrow{\mathbb{H}}} \arrow[r, "\overrightarrow{d}_1"'] \arrow[d, "\overrightarrow{d}_0"'] & \overrightarrow{\mathbb{H}} & \\
& & & \overrightarrow{\mathbb{H}} & & \overrightarrow{\overrightarrow{\mathbb{H}}} \arrow[lll, "\overrightarrow{d}_0"] \arrow[uu, "\overrightarrow{d}_1"']
\end{tikzcd}$$

where $i \times j$ is universally induced and m'_r is defined as the composite $\mathcal{P}(m)(i \times j)$. We need to show that m'_r factors through $\overline{\overline{\overline{\mathbb{H}}}}$.

Consider the defining pullback for $\overline{\overline{\overline{\mathbb{H}}}}$:

$$\begin{tikzcd}
\overrightarrow{\mathbb{H}} \times_{d_0,\overline{d_1}} \overline{\overline{\mathbb{H}}} \arrow[dr, dotted, "\shortmid" marking, "w_r"] \arrow[drr, "\shortmid" marking, "m'_r"] \arrow[ddr, shift right, "\overline{d_0}p_0"'] \arrow[ddr, shift left, "d_1p_1"] & & & \\
& \overline{\overline{\overline{\mathbb{H}}}} \arrow[r, "j"] \arrow[d, shift right, "\overline{d_0}"'] \arrow[d, shift left, "\overline{d_1}"] & \overrightarrow{\overrightarrow{\mathbb{H}}} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[d, shift right, "\overrightarrow{d_0}"'] \arrow[d, shift left, "\overrightarrow{d_1}"] & \overrightarrow{\mathbb{H}} \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
& \mathbb{H} \arrow[r, "i"'] & \overrightarrow{\mathbb{H}} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & \mathbb{H}\,.
\end{tikzcd} \tag{65}$$

We need to show that $\overrightarrow{d}_0 m'_r = i\bar{d}_0 p_0$ and $\overrightarrow{d}_1 m'_r = i d_1 p_1$ to obtain a universal w_r, hence we calculate:

$$\overrightarrow{d}_0 m'_r = \overrightarrow{d}_0 \mathcal{P}(m)(i \times j) = \overrightarrow{d}_0 j p_0 = \bar{d}_0 p_0$$
$$\overrightarrow{d}_1 m'_r = \overrightarrow{d}_1 \mathcal{P}(m)(i \times j) = \overrightarrow{d}_1 i p_1 = \bar{d}_1 p_1$$

using the definitions of $i \times j$ and j as well as the naturality of i.

For w_ℓ there is a corresponding argument. □

5.20. LEMMA. *Left and right whiskering are compatible and associative, that is, the diagrams*

$$\begin{tikzcd}
\overrightarrow{\mathbb{H}} \times_{d_0,d_1} \overrightarrow{\mathbb{H}} \times_{d_0,\overline{d}_1} \overline{\overline{\mathbb{H}}} \arrow[r, "\overrightarrow{\mathbb{H}}\times w_r", "/" marking] \arrow[d, "m\times\overrightarrow{\mathbb{H}}"', "/" marking] & \overrightarrow{\mathbb{H}} \times_{d_0,\overline{d}_1} \overline{\overline{\mathbb{H}}} \arrow[d, "w_r", "/" marking] \\
\overrightarrow{\mathbb{H}} \times_{d_0,\overline{d}_1} \overline{\overline{\mathbb{H}}} \arrow[r, "w_r"', "/" marking] & \overline{\overline{\mathbb{H}}}
\end{tikzcd}$$

$$\begin{tikzcd}
\overline{\overline{\mathbb{H}}} \times_{\overline{d}_0,d_1} \overrightarrow{\mathbb{H}} \times_{d_0,d_1} \overrightarrow{\mathbb{H}} \arrow[r, "w_\ell\times\overrightarrow{\mathbb{H}}", "/" marking] \arrow[d, "\overrightarrow{\mathbb{H}}\times m"', "/" marking] & \overline{\overline{\mathbb{H}}} \times_{\overline{d}_0,d_1} \overrightarrow{\mathbb{H}} \arrow[d, "w_\ell", "/" marking] \\
\overline{\overline{\mathbb{H}}} \times_{\overline{d}_0,d_1} \overrightarrow{\mathbb{H}} \arrow[r, "w_\ell"', "/" marking] & \overline{\overline{\mathbb{H}}}
\end{tikzcd}$$

$$\begin{tikzcd}
\overrightarrow{\mathbb{H}} \times_{d_0,\overline{d}_1} \overline{\overline{\mathbb{H}}} \times_{\overline{d}_0,d_1} \overrightarrow{\mathbb{H}} \arrow[r, "w_r\times\overrightarrow{\mathbb{H}}", "/" marking] \arrow[d, "\overrightarrow{\mathbb{H}}\times w_\ell"', "/" marking] & \overline{\overline{\mathbb{H}}} \times_{\overline{d}_0,d_1} \overrightarrow{\mathbb{H}} \arrow[d, "w_\ell", "/" marking] \\
\overrightarrow{\mathbb{H}} \times_{d_0,\overline{d}_1} \overline{\overline{\mathbb{H}}} \arrow[r, "w_r"', "/" marking] & \overline{\overline{\mathbb{H}}}
\end{tikzcd}$$

commute.

PROOF The objects in the above diagram embed into pullbacks of $\overrightarrow{\overrightarrow{\mathbb{H}}}$ by j and these pullbacks being preserved by $\mathcal{P}$ and the monicity of j yield the desired result. □

5.21. LEMMA. *w_ℓ and w_r extend m. That is*

$$\begin{tikzcd}
\overrightarrow{\mathbb{H}} \times_{d_0,\overline{d_1}} \overline{\overline{\mathbb{H}}} \arrow[r, "w_r", "/" marking] \arrow[d, shift right=3, "\overrightarrow{\mathbb{H}}\times d_0^j"'] \arrow[d, shift left=3, "\overrightarrow{\mathbb{H}}\times d_1^j"] & \overline{\overline{\mathbb{H}}} \arrow[d, shift right=3, "d_0^j"'] \arrow[d, shift left=3, "d_1^j"] \\
\overrightarrow{\mathbb{H}} \times_{d_0,d_1} \overrightarrow{\mathbb{H}} \arrow[r, "m"', "/" marking] \arrow[u, "\overrightarrow{\mathbb{H}}\times\bar{i}" description] & \overrightarrow{\mathbb{H}} \arrow[u, "\bar{i}" description]
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{\overline{\mathbb{H}}} \times_{\overline{d_0},d_1} \overrightarrow{\mathbb{H}} \arrow[r, "w_\ell", "/" marking] \arrow[d, shift right=3, "d_0\times\overrightarrow{\mathbb{H}}"'] \arrow[d, shift left=3, "d_1\times\overrightarrow{\mathbb{H}}"] & \overline{\overline{\mathbb{H}}} \arrow[d, shift right=3, "d_0^j"'] \arrow[d, shift left=3, "d_1^j"] \\
\overrightarrow{\mathbb{H}} \times_{d_0,d_1} \overrightarrow{\mathbb{H}} \arrow[r, "m"', "/" marking] \arrow[u, "\overrightarrow{\mathbb{H}}\times\bar{i}" description] & \overrightarrow{\mathbb{H}} \arrow[u, "\bar{i}" description]
\end{tikzcd}$$

commute serially, and the outside 0-faces are preserved:

$$\begin{aligned}
\overline{d_0}w_r &= \overline{d_0}p_1 & \qquad \overline{d_0}w_\ell &= d_0p_1 \\
\overline{d_1}w_r &= d_1p_0 & \qquad \overline{d_1}w_\ell &= \overline{d_1}p_0
\end{aligned} \tag{66}$$

PROOF Considering the proof of lemma 5.19 we calculate:

$$d_0^j w_r = d_0 j w_r = d_0 m_r' = d_0 \mathcal{P} m(i \times j) = m(d_0 \times d_0)(i \times j) = m(\overrightarrow{\mathbb{H}} \times d_0^j)\,.$$

Similarly for d_1^j and w_ℓ.

The equations (66) hold by the construction as given in (65). □

Lemma 5.21 allows us to define left and right horizontal composites. Call the composite along the middle in the following diagram $h_\ell \colon \overline{\overline{\overline{\mathbb{H}}}} \times_{\overline{d_0},\overline{d_1}} \overline{\overline{\overline{\mathbb{H}}}} \nrightarrow \overline{\overline{\overline{\mathbb{H}}}}$:

$$\begin{tikzcd}
\overrightarrow{\mathbb{H}} \times_{d_0,\overline{d_1}} \overline{\overline{\overline{\mathbb{H}}}} \arrow[rr, "w_r", "/" marking] & & \overline{\overline{\overline{\mathbb{H}}}} & \\
\overline{\overline{\overline{\mathbb{H}}}} \times_{\overline{d_0},\overline{d_1}} \overline{\overline{\overline{\mathbb{H}}}} \arrow[u, "d_0^j \times \overline{\overline{\overline{\mathbb{H}}}}"] \arrow[d, "\overline{\overline{\overline{\mathbb{H}}}} \times d_1^j"'] \arrow[rr, dotted, "/" marking] & & \overline{\overline{\overline{\mathbb{H}}}} \times_{d_0^j,d_1^j} \overline{\overline{\overline{\mathbb{H}}}} \arrow[u] \arrow[d] \arrow[r, "\overline{m}", "/" marking] & \overline{\overline{\overline{\mathbb{H}}}} \\
\overline{\overline{\overline{\mathbb{H}}}} \times_{\overline{d_0},d_1} \overrightarrow{\mathbb{H}} \arrow[rr, "w_\ell"', "/" marking] & & \overline{\overline{\overline{\mathbb{H}}}} &
\end{tikzcd} \quad , \tag{67}$$

and correspondingly $h_r \colon \overline{\overline{\overline{\mathbb{H}}}} \times_{\overline{d_0},\overline{d_1}} \overline{\overline{\overline{\mathbb{H}}}} \nrightarrow \overline{\overline{\overline{\mathbb{H}}}}$:

$$\begin{tikzcd}
\overline{\overline{\overline{\mathbb{H}}}} \times_{\overline{d_0},d_1} \overrightarrow{\mathbb{H}} \arrow[rr, "w_\ell", "/" marking] & & \overline{\overline{\overline{\mathbb{H}}}} & \\
\overline{\overline{\overline{\mathbb{H}}}} \times_{\overline{d_0},\overline{d_1}} \overline{\overline{\overline{\mathbb{H}}}} \arrow[u, "\overline{\overline{\overline{\mathbb{H}}}} \times d_1^j"] \arrow[d, "d_0^j \times \overline{\overline{\overline{\mathbb{H}}}}"'] \arrow[rr, dotted, "/" marking] & & \overline{\overline{\overline{\mathbb{H}}}} \times_{d_0^j,d_1^j} \overline{\overline{\overline{\mathbb{H}}}} \arrow[u] \arrow[d] \arrow[r, "\overline{m}", "/" marking] & \overline{\overline{\overline{\mathbb{H}}}} \\
\overrightarrow{\mathbb{H}} \times_{d_0,\overline{d_1}} \overline{\overline{\overline{\mathbb{H}}}} \arrow[rr, "w_r"', "/" marking] & & \overline{\overline{\overline{\mathbb{H}}}} &
\end{tikzcd} \quad . \tag{68}$$

5.22. Lemma. *Left and right horizontal composites give a globular object*

$$\begin{tikzcd}
\overline{\overline{\overline{\mathbb{H}}}} \times_{\overline{d_0},\overline{d_1}} \overline{\overline{\overline{\mathbb{H}}}} \arrow[r, shift left, "h_\ell", "/" marking] \arrow[r, shift right, "h_r"', "/" marking] & \overline{\overline{\overline{\mathbb{H}}}} \arrow[r, shift left, "d_1^j"] \arrow[r, shift right, "d_0^j"'] & \overrightarrow{\mathbb{H}}
\end{tikzcd} \; . \tag{69}$$

Proof We calculate:

$$\begin{aligned}
d_0^j h_\ell &\overset{(67)}{=} d_0 j \overline{m} \left\langle w_r(d_0^j \times \overline{\overline{\overline{\mathbb{H}}}}), w_\ell(\overline{\overline{\overline{\mathbb{H}}}} \times d_1^j) \right\rangle \\
&\overset{(62)}{=} d_0 m(j \times j) \left\langle w_r(d_0^j \times \overline{\overline{\overline{\mathbb{H}}}}), w_\ell(\overline{\overline{\overline{\mathbb{H}}}} \times d_1^j) \right\rangle \\
&\overset{(65)}{=} d_0 p_0 \left\langle m'_r(d_0^j \times \overline{\overline{\overline{\mathbb{H}}}}), m'_\ell(\overline{\overline{\overline{\mathbb{H}}}} \times d_1^j) \right\rangle \\
&= d_0 m'_r(d_0^j \times \overline{\overline{\overline{\mathbb{H}}}}) \\
&= d_0 \mathcal{P} m(i \times j)(d_0^j \times \overline{\overline{\overline{\mathbb{H}}}}) \\
&\overset{(54)}{=} m(d_0 \times d_0)(i \times j)(d_0^j \times \overline{\overline{\overline{\mathbb{H}}}}) \\
&= m(d_0^j \times d_0^j)
\end{aligned}$$

and by the same token

$$d_0^j h_r = m(d_0^j \times d_0^j) \, . \tag{70}$$

Analogously for d_1^j. □

5.22.1. 3-PATHS. We proceed to construct the internal 3-path object and the operations involving 3-cells. Note that the $\overline{(\_)}$ and $\widetilde{(\_)}$ used in this section are not at all functors.

We apply the construction in (58) to (60) as follows:

$$\begin{tikzcd}
\overline{\overline{\overline{\mathbb{H}}}} \arrow[r, "j"] \arrow[d, shift right, "\overline{d_0^j}"'] \arrow[d, shift left, "\overline{d_1^j}"] & \overrightarrow{\overline{\overline{\mathbb{H}}}} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[d, shift right, "\overrightarrow{d_0^j}"'] \arrow[d, shift left, "\overrightarrow{d_1^j}"] & \overline{\overline{\mathbb{H}}} \arrow[d, shift right, "d_0^j"'] \arrow[d, shift left, "d_1^j"] \\
\overrightarrow{\mathbb{H}} \arrow[r, "i"'] & \overrightarrow{\overrightarrow{\mathbb{H}}} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & \overrightarrow{\mathbb{H}}
\end{tikzcd}$$

By (60) we get a reflexive graph

$$\begin{tikzcd}
\overline{\overline{\overline{\mathbb{H}}}} \arrow[r, shift left=2, "d_1^j"] \arrow[r, shift right=2, "d_0^j"'] & \overline{\overline{\mathbb{H}}} \arrow[l, "\overline{i}" description]
\end{tikzcd}$$

where by (59)

$$\begin{tikzcd}
\overline{\overline{\overline{\mathbb{H}}}} \arrow[r, shift left, "d_1^j"] \arrow[r, shift right, "d_0^j"'] & \overline{\overline{\mathbb{H}}} \arrow[r, shift left, "d_1^j"] \arrow[r, shift right, "d_0^j"'] & \overrightarrow{\mathbb{H}} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & \mathbb{H}
\end{tikzcd}$$

is a 3-globular object. Furthermore, by applying the reasoning of lemma 5.16 we get a vertical multiplication map

$$\begin{tikzcd}
\overline{\overline{\overline{\mathbb{H}}}} \times_{d_0^j, d_1^j} \overline{\overline{\overline{\mathbb{H}}}} \arrow[r, "\overline{\overline{m}}", "|" marking] & \overline{\overline{\overline{\mathbb{H}}}}
\end{tikzcd}$$

arising as a restriction of $m_{\overline{\overline{\mathbb{H}}}}$:

$$\begin{tikzcd}
\overline{\overline{\overline{\mathbb{H}}}} \times_{d_0^j, d_1^j} \overline{\overline{\overline{\mathbb{H}}}} \arrow[rr, "j \times j"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] \arrow[d, dotted, bend left=60, "\overline{\overline{m}}", "|" marking] \arrow[dd, bend right=50, shift right, "d'_0"'] \arrow[dd, bend right=50, shift left, "d'_1"] & & \overrightarrow{\overline{\overline{\mathbb{H}}}} \times_{d_0, d_1} \overrightarrow{\overline{\overline{\mathbb{H}}}} \arrow[d, "m", "|" marking] \\
\overline{\overline{\overline{\mathbb{H}}}} \arrow[rr, "j"] \arrow[d, shift right, "\overline{d}_0"'] \arrow[d, shift left, "\overline{d}_1"] & & \overrightarrow{\overline{\overline{\mathbb{H}}}} \arrow[d, shift right, "\overrightarrow{d}_0"'] \arrow[d, shift left, "\overrightarrow{d}_1"] \\
\overrightarrow{\mathbb{H}} \arrow[rr, "i"'] & & \overline{\overline{\mathbb{H}}}
\end{tikzcd}$$

where $d'_0 = \overline{d_0} p_0$ and $d'_1 = \overline{d_1} p_1$.

5.23. LEMMA. *There are left and right whiskering maps*

$$\begin{tikzcd}
\overline{\overline{\overline{\mathbb{H}}}} \times_{d_0 \overline{d_0^j}, d_1} \overrightarrow{\mathbb{H}} \arrow[r, "\overline{w_\ell}", "|" marking] & \overline{\overline{\overline{\mathbb{H}}}}
\end{tikzcd}$$

$$\begin{tikzcd}
\overrightarrow{\mathbb{H}} \times_{d_0, d_1 \overline{d_1^j}} \overline{\overline{\overline{\mathbb{H}}}} \arrow[r, "\overline{w_r}", "|" marking] & \overline{\overline{\overline{\mathbb{H}}}}
\end{tikzcd}$$

induced uniquely by $\mathcal{P}w_\ell$ and $\mathcal{P}w_r$.

Proof We define $\overline{w_\ell}$ as the universally induced arrow in the following diagram:

$$\begin{tikzcd}
\overline{\overline{\overline{\mathbb{H}}}}\times_{d_0\overline{d^j_0},d_1}\overrightarrow{\mathbb{H}} \arrow[r, "j\times i"] \arrow[d, dashed, "\overline{w_\ell}", "\shortmid" marking] \arrow[dd, bend right=50, "r_0"', "\shortmid" marking] \arrow[dd, bend right=35, "r_1", "\shortmid" marking] & \overrightarrow{\overline{\overline{\mathbb{H}}}}\times_{\overrightarrow{d_0},\overrightarrow{d_1}}\overrightarrow{\overline{\mathbb{H}}} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[d, "\mathcal{P}w_\ell", "\shortmid" marking] & \overline{\overline{\mathbb{H}}}\times_{\overline{d_0},d_1}\overrightarrow{\mathbb{H}} \arrow[d, "w_\ell"', "\shortmid" marking] \arrow[dr, shift left, "d^j_1\times\overrightarrow{\mathbb{H}}"] \arrow[dr, shift right, "d^j_0\times\overrightarrow{\mathbb{H}}"'] & \\
\overline{\overline{\overline{\mathbb{H}}}} \arrow[r, "j"] \arrow[d, shift right, "\overline{d^j_0}"'] \arrow[d, shift left, "\overline{d^j_1}"] & \overrightarrow{\overline{\overline{\mathbb{H}}}} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[d, shift right, "\overrightarrow{d^j_0}"'] \arrow[d, shift left, "\overrightarrow{d^j_1}"] & \overline{\overline{\mathbb{H}}} \arrow[d, shift right, "d^j_0"'] \arrow[d, shift left, "d^j_1"] & \overrightarrow{\mathbb{H}}\times_{d_0,d_1}\overrightarrow{\mathbb{H}} \arrow[dl, "m", "\shortmid" marking] \\
\overrightarrow{\mathbb{H}} \arrow[r, "i"'] & \overrightarrow{\overline{\mathbb{H}}} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & \overrightarrow{\overline{\mathbb{H}}} &
\end{tikzcd}\tag{71}$$

where $r_0 = m(\overline{d^j_0}\times\overrightarrow{\mathbb{H}})$ and $r_1 = m(\overline{d^j_1}\times\overrightarrow{\mathbb{H}})$. We calculate

$$\begin{aligned}
&ir_0\\
&= im(\overline{d^j_0}\times\overrightarrow{\mathbb{H}}) = \mathcal{P}m(i\times i)(\overline{d^j_0}\times\overrightarrow{\mathbb{H}}) = \mathcal{P}m(i\overline{d^j_0}\times i) = \mathcal{P}m(\overrightarrow{d^j_0}j\times i) = \mathcal{P}(d^j_0 w_\ell)(j\times i)\\
&\qquad = \overrightarrow{d^j_0}\mathcal{P}w_\ell(j\times i)\,,
\end{aligned}$$

and likewise for r_1 and $\overrightarrow{d^j_1}$. And hence we obtain $\overline{w_\ell}$, and $\overline{w_r}$ by analogy. $\square$

5.24. Lemma. *$\overline{w_\ell}$ and $\overline{w_r}$ extend w_ℓ and w_r respectively. That is*

$$\begin{tikzcd}
\overrightarrow{\mathbb{H}}\times_{d_0,d_1\overline{d^j_1}}\overline{\overline{\overline{\mathbb{H}}}} \arrow[r, "\overline{w_r}", "\shortmid" marking] \arrow[d, shift right, "\overrightarrow{\mathbb{H}}\times d^j_0"'] \arrow[d, shift left, "\overrightarrow{\mathbb{H}}\times d^j_1"] & \overline{\overline{\overline{\mathbb{H}}}} \arrow[d, shift right, "d^j_0"'] \arrow[d, shift left, "d^j_1"] & \overline{\overline{\overline{\mathbb{H}}}}\times_{d_0\overline{d^j_0},d_1}\overrightarrow{\mathbb{H}} \arrow[r, "\overline{w_\ell}", "\shortmid" marking] \arrow[d, shift right, "d^j_0\times\overrightarrow{\mathbb{H}}"'] \arrow[d, shift left, "d^j_1\times\overrightarrow{\mathbb{H}}"] & \overline{\overline{\overline{\mathbb{H}}}} \arrow[d, shift right, "d^j_0"'] \arrow[d, shift left, "d^j_1"] \\
\overrightarrow{\mathbb{H}}\times_{d_0,d^j_1}\overline{\overline{\mathbb{H}}} \arrow[r, "w_r"', "\shortmid" marking] & \overline{\overline{\mathbb{H}}} & \overline{\overline{\mathbb{H}}}\times_{d^j_0,d_1}\overrightarrow{\mathbb{H}} \arrow[r, "w_\ell"', "\shortmid" marking] & \overline{\overline{\mathbb{H}}}
\end{tikzcd}\tag{72}$$

commute serially.

Proof Inspecting (71) we can calculate

$$\begin{aligned}
&d^j_0\overline{w_\ell}\\
&\qquad = d_0 j\overline{w_\ell} = d_0\mathcal{P}(w_\ell)(j\times i) = w_\ell d_0(j\times i) = w_\ell(d_0\times d_0)(j\times i)\\
&\qquad = w_\ell(d^j_0\times\overrightarrow{\mathbb{H}})\,.
\end{aligned}$$

And likewise for the other squares in (72). $\square$

Lastly, we need the whiskering of a 3-path by a 2-path along a 1-path. We can reapply the basic scheme of lemma 5.19.

5.25. Lemma. *There are left and right whiskering maps*

$$\begin{tikzcd} \overline{\overline{\overline{\mathbb{H}}}} \times_{\overline{d_0^j},d_1^j} \overline{\overline{\mathbb{H}}} \arrow[r, "\tilde{w}_\ell", "\shortmid" marking] & \overline{\overline{\overline{\mathbb{H}}}} \end{tikzcd}$$

$$\begin{tikzcd} \overline{\overline{\mathbb{H}}} \times_{d_0^j,\overline{d_1^j}} \overline{\overline{\overline{\mathbb{H}}}} \arrow[r, "\tilde{w}_r", "\shortmid" marking] & \overline{\overline{\overline{\mathbb{H}}}} \end{tikzcd}$$

induced uniquely by $\mathcal{P}(\overline{m})$.

And these extend $\overline{m}$, that is

$$d_0^j\tilde{w}_r = \overline{m}(\overline{\overline{\mathbb{H}}} \times d_0^j) \qquad d_1^j\tilde{w}_r = \overline{m}(\overline{\overline{\mathbb{H}}} \times d_1^j) \tag{73}$$

$$d_0^j\tilde{w}_\ell = \overline{m}(d_0^j \times \overline{\overline{\mathbb{H}}}) \qquad d_1^j\tilde{w}_\ell = \overline{m}(d_1^j \times \overline{\overline{\mathbb{H}}})\,. \tag{74}$$

Proof The desired map arises as a universal arrow in the following diagram:

$$\begin{tikzcd}
\overline{\overline{\mathbb{H}}} \times_{d_0^j,\overline{d_1^j}} \overline{\overline{\overline{\mathbb{H}}}} \arrow[rr, "i\times j"] \arrow[dr, dotted, "\tilde{w}_r", "\shortmid" marking] \arrow[ddr, shift right, "\overline{d_0^1}p_0"'] \arrow[ddr, shift left, "d_1^jp_1"] & & \overrightarrow{\overline{\overline{\mathbb{H}}}} \times_{\overrightarrow{d_0^j},\overrightarrow{d_1^j}} \overrightarrow{\overline{\overline{\mathbb{H}}}} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[d, "\mathcal{P}\overline{m}", "\shortmid" marking] & \overline{\overline{\mathbb{H}}} \times_{d_0^j,d_1^j} \overline{\overline{\mathbb{H}}} \arrow[d, "\overline{m}", "\shortmid" marking] \\
& \overline{\overline{\overline{\mathbb{H}}}} \arrow[r, "j"] \arrow[d, shift right, "\overline{d_0^j}"'] \arrow[d, shift left, "\overline{d_1^j}"] & \overrightarrow{\overline{\overline{\mathbb{H}}}} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[d, shift right, "\overrightarrow{d_0^j}"'] \arrow[d, shift left, "\overrightarrow{d_1^j}"] & \overline{\overline{\mathbb{H}}} \arrow[d, shift right, "d_0^j"'] \arrow[d, shift left, "d_1^j"] \\
& \overrightarrow{\mathbb{H}} \arrow[r, "i"'] & \overrightarrow{\overrightarrow{\mathbb{H}}} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & \mathbb{H}\,.
\end{tikzcd} \quad . \tag{75}$$

Now, we can verify $i\overline{d_0^j}p_0 = \overrightarrow{d_0^j}jp_0 = \overrightarrow{d_0^j}p_0(i \times j) = \overrightarrow{d_0^j}\mathcal{P}\overline{m}(i \times j)$ and $id_1^jp_1 = \overrightarrow{d_1^j}jp_1 = \overrightarrow{d_1^j}p_1(i \times j) = \overrightarrow{d_1^j}\mathcal{P}\overline{m}(i \times j)$.

The equations (73) are now immediate. □

5.26. The Space of Parallel Cells. For a Gray-category $\mathbb{H}$ we define the space of parallel 1-cells $P^1(\mathbb{H})$ as the following limit:

$$\begin{tikzcd}
& P^1(\mathbb{H}) \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\
\overrightarrow{\mathbb{H}} \arrow[d, "d_0"'] \arrow[drr, "d_1" near start] & & \overrightarrow{\mathbb{H}} \arrow[d, "d_1"] \arrow[dll, "d_0"' near start] \\
\mathbb{H} & & \mathbb{H}
\end{tikzcd}$$

$$\begin{tikzcd}
 & P^2(\mathbb{H}) \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\
\overline{\overline{\overline{\mathbb{H}}}} \arrow[d, "d_0^j"'] \arrow[drr, "d_1^j" near start] & & \overline{\overline{\overline{\mathbb{H}}}} \arrow[d, "d_1^j"] \arrow[dll, "d_0^j"' near start] \\
\overrightarrow{\mathbb{H}} & & \overrightarrow{\mathbb{H}}
\end{tikzcd} \tag{76}$$

5.27. Lemma. *The canonical map* $\left\langle d_0^j, d_1^j \right\rangle \colon \overline{\overline{\overline{\overline{\mathbb{H}}}}} \longrightarrow P^2(\mathbb{H})$ *is 1-Cartesian.*

Proof Consider the following cells in $\overline{\overline{\overline{\overline{\mathbb{H}}}}}$

$$\begin{aligned}
f &= (f_4; f_2, f_3; f_0, f_1) \\
g &= (g_4; g_2, g_3; g_0, g_1) \\
h &= (h_4, h_5; h_2, h_3; h_0, h_1) \colon f \longrightarrow g \\
k &= (k_4, k_5; k_2, k_3; k_0, k_1) \colon f \longrightarrow g \\
\alpha &= (\alpha_3; \alpha_1, \alpha_2) \colon h \Longrightarrow k
\end{aligned}$$

By construction the map $\left\langle d_0^j, d_1^j \right\rangle$ acts on this data as follows:

$$\begin{aligned}
f &\mapsto ((f_2; f_0, f_1), (f_3; f_0, f_1)) \\
g &\mapsto ((g_2; g_0, g_1), (g_3; g_0, g_1)) \\
h &\mapsto ((h_4; h_2, h_3; h_0, h_1), (h_5; h_2, h_3; h_0, h_1)) \\
k &\mapsto ((k_4; k_2, k_3; k_0, k_1), (k_5; k_2, k_3; k_0, k_1)) \\
\alpha &\mapsto ((\alpha_3; \alpha_1, \alpha_2), (\alpha_3; \alpha_1, \alpha_2))
\end{aligned}$$

where on the right we find parallel pairs of cells from $\overline{\overline{\overline{\mathbb{H}}}}$, that is, in (77) the central square, the outer square, and the left and right hand trapezoids commute by assumption.

The requisite compatibility conditions for f, g, h, k, α to be cells of $\overline{\overline{\overline{\mathbb{H}}}}$ are displayed in (77). We obverse that the remaining trapezoids at the top and the bottom commute by naturality of $\#_1$ and $\otimes$ in $\mathbb{H}$. Hence we conclude that given 1-cells h, k in $\overline{\overline{\overline{\overline{\mathbb{H}}}}}$ all higher cells, including 3-cells, between them are determined by their image under $\left\langle d_0^j, d_1^j \right\rangle$. ☐

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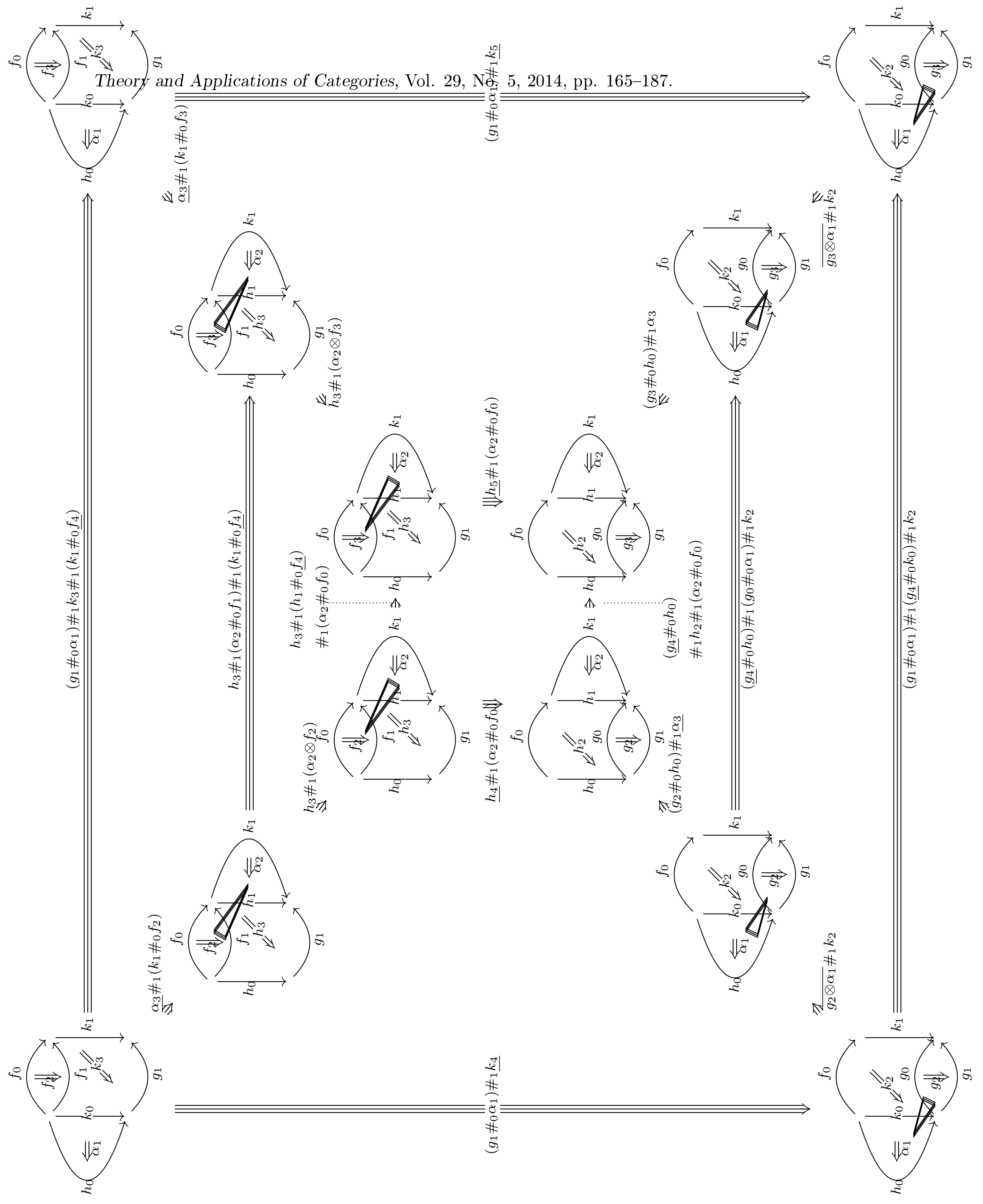


(77)

5.28. LEMMA. *The 3-paths compose horizontally along 2-paths, that is,*

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\overline{\overline{\overline{\mathbb{H}}}} \times_{\overline{d_0^j},\overline{d_1^j}} \overline{\overline{\overline{\mathbb{H}}}} \arrow[r, "\left\langle \tilde{w}_\ell(\overline{\overline{\overline{\mathbb{H}}}}\times d_1^j),\tilde{w}_r(d_1^j0\times\overline{\overline{\overline{\mathbb{H}}}})\right\rangle"] \arrow[d, "\left\langle \tilde{w}_r(d_1^j\times\overline{\overline{\overline{\mathbb{H}}}}),\tilde{w}_\ell(\overline{\overline{\overline{\mathbb{H}}}}\times d_0^j)\right\rangle"'] & \overline{\overline{\overline{\mathbb{H}}}} \times_{d_0^j,d_1^j} \overline{\overline{\overline{\mathbb{H}}}} \arrow[d, "\overline{\overline{m}}"] \\
\overline{\overline{\overline{\mathbb{H}}}} \times_{d_0^j,d_1^j} \overline{\overline{\overline{\mathbb{H}}}} \arrow[r, "\overline{\overline{m}}"'] & \overline{\overline{\overline{\mathbb{H}}}}
\end{tikzcd}$$

commutes. □

5.29. THE TENSOR MAP. Given that by lemma 5.27 we have a 1-Cartesian map $\left\langle d_0^j, d_1^j\right\rangle\overline{\overline{\overline{\mathbb{H}}}} \longrightarrow P^2(\mathbb{H})$ we consider the following diagram in $\mathsf{GrayCat}_{\mathrm{Q}^1}$

$$\begin{tikzcd}
\overline{\overline{\overline{\mathbb{H}}}} \times_{\overline{d_0},\overline{d_1}} \overline{\overline{\overline{\mathbb{H}}}} \arrow[dr, dotted, "t"'] \arrow[drr, "\langle h_\ell,h_r\rangle", "/"{marking}] & & \\
& \overline{\overline{\overline{\mathbb{H}}}} \arrow[r, "\left\langle d_0^j,d_1^j\right\rangle"'] & P^2(\mathbb{H})
\end{tikzcd} \tag{78}$$

where h_ℓ and h_r are given by (67) and (68) respectively. By (69) we know that (h_ℓ, h_r) is a source for (76) hence we obtain $\langle h_\ell, h_r\rangle$.

There is a map $t_1\colon (\overline{\overline{\overline{\mathbb{H}}}} \times_{\overline{d_0},\overline{d_1}} \overline{\overline{\overline{\mathbb{H}}}})_1 \longrightarrow (\overline{\overline{\overline{\mathbb{H}}}})_1$ in $\mathsf{Cat}_{\mathrm{Q}^1}$ given by:

$$(g,f) = ((g_2; g_0, g_1), (f_2; f_0, f_1)) = \left(\begin{tikzcd}
{} \arrow[r, bend left=60, "f_0"{name=A}] \arrow[r, bend right=60, "f_1"'{name=B}] \arrow[Rightarrow, from=A, to=B, "f_2"'] & {} \arrow[r, bend left=60, "g_0"{name=C}] \arrow[r, bend right=60, "g_1"'{name=D}] \arrow[Rightarrow, from=C, to=D, "g_2"'] & {}
\end{tikzcd} \right)$$

$$\mapsto (g_2\otimes f_2; g_2 \triangleleft f_2, g_2 \triangleright f_2; g_0\#_0 f_0, g_1\#_0 f_1) = \left(\begin{tikzcd}
{} \arrow[r, bend left=60, "g_0\#_0 f_0"{name=A}] \arrow[r, bend right=60, "g_1\#_0 f_1"'{name=B}] \arrow[Rightarrow, from=A, to=B, "g_2\triangleleft f_2"'] & {}
\end{tikzcd} \overset{g_2\otimes f_2}{\Rrightarrow} \begin{tikzcd}
{} \arrow[r, bend left=60, "g_0\#_0 f_0"{name=C}] \arrow[r, bend right=60, "g_1\#_0 f_1"'{name=D}] \arrow[Rightarrow, from=C, to=D, "g_2\triangleright f_2"'] & {}
\end{tikzcd} \right)$$

and

$$\big((k,h)\colon (g,f) \longrightarrow (g',f')\big) = \begin{pmatrix} (k_4; k_2, k_3; h_1, k_1), \\ (h_4; h_2, h_3; h_0, h_1) \end{pmatrix} = \left(\begin{array}{c} \begin{tikzcd}[ampersand replacement=\&] {} \arrow[r, bend left=50, "f_0"{name=F0}] \arrow[r, bend right=50, "f_1"'{name=F1}] \arrow[d, "h_0"'] \& {} \arrow[r, bend left=50, "g_0"{name=G0}] \arrow[r, bend right=50, "g_1"'{name=G1}] \arrow[d, "h_1"] \& {} \arrow[d, "k_1"] \\ {} \arrow[r, bend right=50, "f_1'"'] \& {} \arrow[r, bend right=50, "g_1'"'] \& {} \arrow[from=F0, to=F1, Rightarrow, "f_2"'] \arrow[from=G0, to=G1, Rightarrow, "g_2"'] \arrow[from=1-2, to=2-1, Rightarrow, shorten=6mm, "h_3"] \arrow[from=1-3, to=2-2, Rightarrow, shorten=6mm, "k_3"] \end{tikzcd} \\ \begin{tikzcd}[ampersand replacement=\&] {} \arrow[d, Rrightarrow, "{(k_4,h_4)}"] \\ {} \end{tikzcd} \\ \begin{tikzcd}[ampersand replacement=\&] {} \arrow[r, bend left=50, "f_0"] \arrow[d, "h_0"'] \& {} \arrow[r, bend left=50, "g_0"] \arrow[d, "h_1"'] \& {} \arrow[d, "k_1"] \\ {} \arrow[r, bend left=50, "f_0'"{name=F0p}] \arrow[r, bend right=50, "f_1'"'{name=F1p}] \& {} \arrow[r, bend left=50, "g_0'"{name=G0p}] \arrow[r, bend right=50, "g_1'"'{name=G1p}] \& {} \arrow[from=F0p, to=F1p, Rightarrow, "f_2'"'] \arrow[from=G0p, to=G1p, Rightarrow, "g_2'"'] \arrow[from=1-2, to=2-1, Rightarrow, shorten=6mm, "h_2"] \arrow[from=1-3, to=2-2, Rightarrow, shorten=6mm, "k_2"] \end{tikzcd} \end{array} \right)$$

$$\mapsto \begin{pmatrix} \omega_1, \omega_2; (g_0' \#_0 h_2) \#_1 (k_2 \#_0 f_0), \\ (g_1' \#_0 h_3) \#_1 (k_3 \#_0 f_1); h_0, k_1 \end{pmatrix},$$

where ω_1 and ω_2 are defined as the vertical composites in (79), by definition these constitute the components of a 1-cell in $\overline{\overline{\overline{\mathbb{H}}}}$.

such that

5.30. LEMMA. *$\langle h_\ell, h_r \rangle_1 = \langle d_0^j, d_1^j \rangle_1 t_1$ in* RGrph.

PROOF One checks that $(h_\ell)_1 = (d_0^j t)_1$ and $(h_r)_1 = (d_1^j t)_1$ as graph maps using definitions (67) and (68). □

5.31. LEMMA. *The 3-globular set*

$$\begin{tikzcd} P^2(\mathbb{H}) \arrow[r, shift left=2, "p^1"] \arrow[r, shift right=2, "p_0"'] & \overline{\overline{\mathbb{H}}} \arrow[l, "\Delta" description] \arrow[r, shift left=2, "d^1"] \arrow[r, shift right=2, "d_0"'] & \overrightarrow{\mathbb{H}} \arrow[l, "i" description] \arrow[r, shift left=2, "d^1"] \arrow[r, shift right=2, "d_0"'] & \mathbb{H} \arrow[l, "i" description] \end{tikzcd}$$

is an internal Gray*-category.*

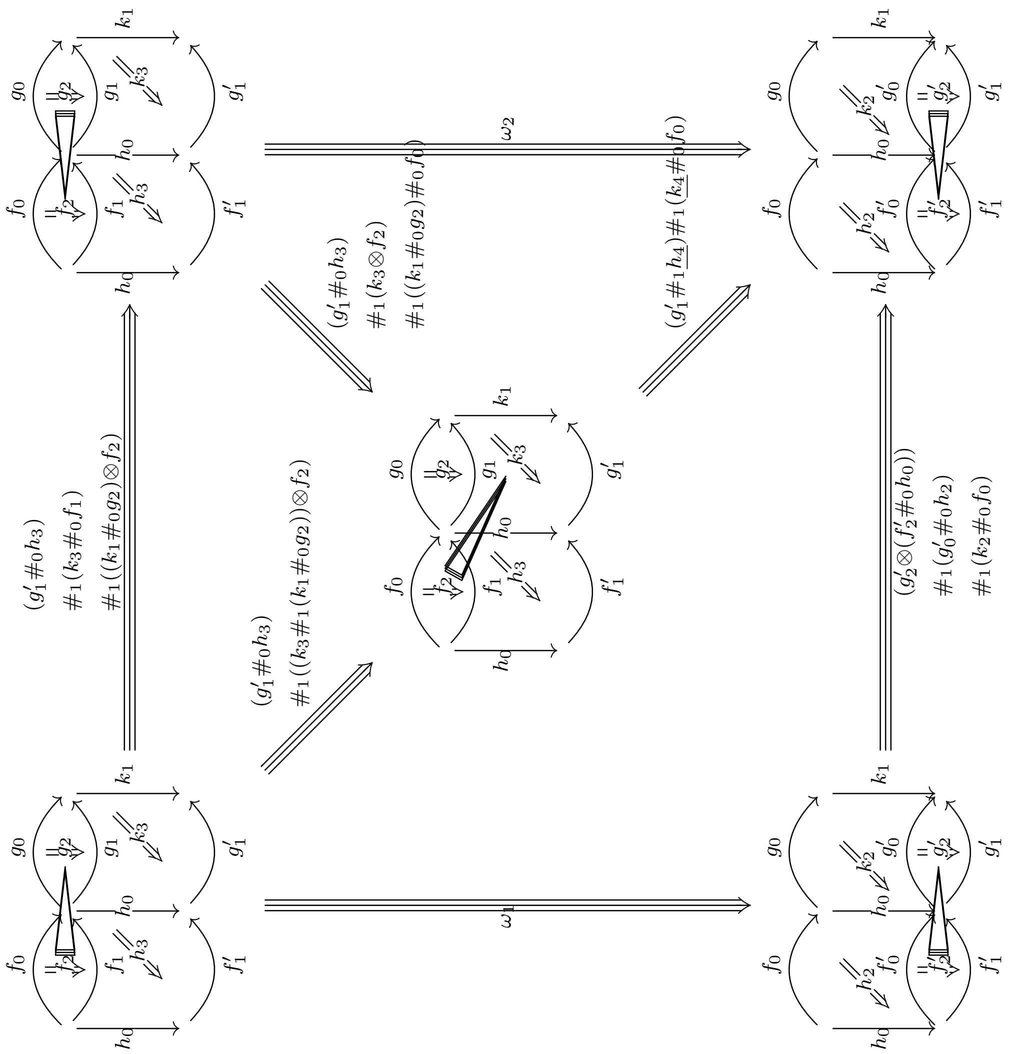
PROOF We already know that its three lower stages constitute a sesqui-category. The three top parts are trivially a 2-category. The tensor map is given by

$$\overline{\overline{\mathbb{H}}} \times_{\overline{d_0}, \overline{d_1}} \overline{\overline{\mathbb{H}}} \xrightarrow[]{\langle h_\ell, h_r \rangle} \!\!\!\!\!\!\!\!\!\!\!\! \not\;\;\;\;\;\; P^2(\mathbb{H})$$

which satisfies the tensor axioms by construction. □

We can finally prove our desired theorem:

(79)


ω_2
ω_1
$(g'_1\#_0h_3)\#_1(k_3\#_0f_1)\#_1((k_1\#_0g_2)\otimes f_2)$
$(g'_1\#_0h_3)\#_1((k_3\#_1(k_1\#_0g_2))\otimes f_2)$
$(g'_1\#_0h_3)\#_1(k_3\otimes f_2)\#_1((k_1\#_0g_2)\#_0f_0)$
$(g'_1\#_1\underline{h_4})\#_1(\underline{k_4}\#_0f_0)$
$(g'_2\otimes(f'_2\#_0h_0))\#_1(g'_0\#_0h_2)\#_1(k_2\#_0f_0)$
f_0 g_0 f_1 g_1 f'_0 g'_0 f'_1 g'_1 h_0 k_1 f_2 g_2 f'_2 g'_2 h_2 k_2 h_3 k_3

5.32. THEOREM. *Given a* Gray*-category* $\mathbb{H}$ *there is an internal* Gray*-category in* $\mathsf{GrayCat}_{\mathrm{Q}^1}$

$$\begin{tikzcd}
\overline{\overline{\overline{\mathbb{H}}}} \arrow[r, shift left=2, "d^1"] \arrow[r, shift right=2, "d_0"'] & \overline{\overline{\mathbb{H}}} \arrow[l, "i" description] \arrow[r, shift left=2, "d^1"] \arrow[r, shift right=2, "d_0"'] & \overrightarrow{\mathbb{H}} \arrow[l, "i" description] \arrow[r, shift left=2, "d^1"] \arrow[r, shift right=2, "d_0"'] & \mathbb{H} \arrow[l, "i" description]
\end{tikzcd} \tag{80}$$

with composition operations $m, \overline{m}, \overline{\overline{m}}, w_\ell, w_r, \overline{w_\ell}, \overline{w_r}, \tilde{w}_\ell, \tilde{w}_r,$ *and tensor* t.

PROOF We have a globular map

$$\begin{tikzcd}
\overline{\overline{\overline{\mathbb{H}}}} \arrow[r, shift left=2, "d^1"] \arrow[r, shift right=2, "d_0"'] \arrow[d, "\langle d_0^j, d_1^j \rangle"'] & \overline{\overline{\mathbb{H}}} \arrow[l, "i" description] \arrow[r, shift left=2, "d^1"] \arrow[r, shift right=2, "d_0"'] \arrow[d] & \overrightarrow{\mathbb{H}} \arrow[l, "i" description] \arrow[r, shift left=2, "d^1"] \arrow[r, shift right=2, "d_0"'] \arrow[d] & \mathbb{H} \arrow[l, "i" description] \arrow[d] \\
P^2(\mathbb{H}) \arrow[r, shift left=2, "p^1"] \arrow[r, shift right=2, "p_0"'] & \overline{\overline{\mathbb{H}}} \arrow[l, "\Delta" description] \arrow[r, shift left=2, "d^1"] \arrow[r, shift right=2, "d_0"'] & \overrightarrow{\mathbb{H}} \arrow[l, "i" description] \arrow[r, shift left=2, "d^1"] \arrow[r, shift right=2, "d_0"'] & \mathbb{H} \arrow[l, "i" description]
\end{tikzcd}$$

This globular map is an internal sesqui-functor in the lower and at the upper degrees, and by (78) it preserves the tensor:

$$\begin{tikzcd}
\overline{\overline{\mathbb{H}}} \times_{\overline{d_0}, \overline{d_1}} \overline{\overline{\mathbb{H}}} \arrow[r, "t", "/" marking] \arrow[d] & \overline{\overline{\overline{\mathbb{H}}}} \arrow[d, "\langle d_0^j, d_1^j \rangle"] \\
\overline{\overline{\mathbb{H}}} \times_{\overline{d_0}, \overline{d_1}} \overline{\overline{\mathbb{H}}} \arrow[r, "\langle h_\ell, h_r \rangle", "/" marking] & P^2(\mathbb{H})
\end{tikzcd}$$

Using the results of sections 4 and 5 this proves that (80) is an internal Gray-category. ☐

5.33. LEMMA. *The operations* $\overline{\overline{m}}$, w_ℓ, w_r, $\tilde{w}_\ell$, $\tilde{w}_r$,$\overline{w_\ell}$, $\overline{w_r}$ *and* t *are natural with respect to strict* Gray*-functors.*

PROOF This can be shown using the universality of the respective constructions and the fact that m is natural with respect to strict Gray-functors, i. e. lemma 4.4. ☐

6. The Internal Hom Functor

We can finally define the internal hom of $\mathsf{GrayCat}_{\mathrm{Q}^1}$

$$[\mathbb{G}, \mathbb{H}]$$

$$= \left(\begin{tikzcd}[column sep=small]
\mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, \overline{\overline{\overline{\mathbb{H}}}}) \arrow[r, shift left=2, "d_{1*}"] \arrow[r, shift right=2, "d_{0*}"'] & \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, \overline{\overline{\mathbb{H}}}) \arrow[l, "i_*" description] \arrow[r, shift left=2, "d_{1*}"] \arrow[r, shift right=2, "d_{0*}"'] & \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, \overrightarrow{\mathbb{H}}) \arrow[l, "i_*" description] \arrow[r, shift left=2, "d_{1*}"] \arrow[r, shift right=2, "d_{0*}"'] & \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, \mathbb{H}) \arrow[l, "i_*" description]
\end{tikzcd} \right) \tag{81}$$

by applying $\mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, -)$ to the diagram (80), where the lower star means action by post-composition in the co-Kleisli sense. This includes the various induced composition operations m_*, $\overline{m}_*$, $\overline{\overline{m}}_*$, $w_{\ell *}$, w_{r*}, $\tilde{w}_{\ell *}$, $\tilde{w}_{r*}$,$\overline{w_{\ell *}}$, $\overline{w_{r*}}$ and t_*. Because $\mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, -)$ by definition preserves limits in the second variable, it takes internal Gray-categories in $\mathsf{GrayCat}_{\mathrm{Q}^1}$ to such in Set, that is, to ordinary Gray-categories. In analogy with our earlier notation we write the compositions on $[\mathbb{G}, \mathbb{H}]$ as $*_n$ where n is the dimension of the incident cell, we use $*$ for the tensor of transformations incident on a functor.

Explicitly, for example, given

$$\begin{tikzcd}[row sep=large]
\mathbb{G} \arrow[r, bend left=70, "G", "\shortmid" marking, ""{name=U, below}] \arrow[r, "H" description, "\shortmid" marking, ""{name=M}] \arrow[r, bend right=70, "K"', "\shortmid" marking, ""{name=D, above}] & \mathbb{H}
\arrow[Rightarrow, from=U, to=M, "\alpha"]
\arrow[Rightarrow, from=M, to=D, "\beta"]
\end{tikzcd}$$

the composite $\beta *_0 \alpha$ is defined as

$$\begin{tikzcd}
\mathbb{G} \arrow[r, "\langle \beta, \alpha \rangle", "\shortmid" marking] & \overrightarrow{\mathbb{H}} \times_{d_0, d_1} \overrightarrow{\mathbb{H}} \arrow[r, "m", "\shortmid" marking] & \overrightarrow{\mathbb{H}}
\end{tikzcd} \quad .$$

that is, $\beta *_0 \alpha = m\mathrm{Q}^1\langle \beta, \alpha \rangle d$.

To be slightly more explicit, at the level if 0-, and 1-cells of $[\mathbb{G}, \mathbb{H}]$, that is, pseudo-functors and transformations the composition works as follows:

$$\begin{tikzcd}[column sep=tiny]
 & & \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, \overrightarrow{\mathbb{H}}) \times_{d_{0*}, d_{1*}} \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, \overrightarrow{\mathbb{H}}) \arrow[d, "\cong"] & & \\
 & & \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, \overrightarrow{\mathbb{H}} \times_{d_0, d_1} \overrightarrow{\mathbb{H}}) \arrow[dl] \arrow[d, "m_*"] \arrow[dr] & & \\
 & \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, \overrightarrow{\mathbb{H}}) \arrow[dl, "d_{0*}"'] \arrow[dr, "d_{1*}"] & \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, \overrightarrow{\mathbb{H}}) \arrow[dll, "d_{0*}"'] \arrow[drr, "d_{1*}"'] & \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, \overrightarrow{\mathbb{H}}) \arrow[dl, "d_{0*}"] \arrow[dr, "d_{1*}"] & \\
\mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, \mathbb{H}) & & \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, \mathbb{H}) & & \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, \mathbb{H})
\end{tikzcd}$$

6.1. REMARK. *The* Gray-*category* $[\mathbb{G}, \mathbb{H}]$ *is a* Gray-*groupoid if* $\mathbb{H}$ *is one.*

6.2. THEOREM. *Given a morphism* $F \colon \mathbb{G}' \nrightarrow \mathbb{G}$ *in* $\mathsf{GrayCat}_{\mathrm{Q}^1}$*, the map*

$$F^* = [F, \mathbb{H}] \colon [\mathbb{G}, \mathbb{H}] \longrightarrow [\mathbb{G}', \mathbb{H}]$$

acting by pre-composition in the co-Kleisli sense is a Gray-*functor, that is, a strict morphism.*

Proof Assume a situation $\begin{tikzcd} \mathbb{G}' \arrow[r, "F", "\shortmid" marking] & \mathbb{G} \arrow[r, bend left=60, "G", "\shortmid" marking, ""{name=U, below}] \arrow[r, "H" description, ""{name=M}] \arrow[r, bend right=60, "K"', "\shortmid" marking, ""{name=D, above}] & \mathbb{H} \arrow[Rightarrow, from=U, to=M, "\alpha"] \arrow[Rightarrow, from=M, to=D, "\beta"] \end{tikzcd}$ then we have

$$F^*(\beta *_0 \alpha) = (\beta *_0 \alpha)F = m\langle \beta, \alpha\rangle F$$
$$= m\langle \beta F, \alpha F\rangle = (\beta F) *_0 (\alpha F) = (F^*\beta) *_0 (F^*\alpha)\,.$$

Also, for identity transformations we have:

$$F^*\mathrm{id}_G = iGF = \mathrm{id}_{GF}\,,$$

hence F^* is a functor. By the same reasoning the higher operations including the tensor, are preserved as well. □

6.3. Remark. *This way* $[-,\mathbb{H}]\colon \mathsf{GrayCat}^{\mathrm{op}}_{\mathrm{Q}^1} \longrightarrow \mathsf{GrayCat}_{\mathrm{Q}^1}$ *is a functor for each* $\mathbb{H}$.

6.4. Theorem. *Given a strict morphism* $F\colon \mathbb{H} \longrightarrow \mathbb{H}'$ *in* $\mathsf{GrayCat}$*, the map*

$$F_* = [\mathbb{G}, F]\colon [\mathbb{G},\mathbb{H}] \longrightarrow [\mathbb{G},\mathbb{H}']$$

acting by post-composition is a Gray*-functor, that is, a strict morphism.*

Proof Assume a situation $\begin{tikzcd} \mathbb{G} \arrow[r, bend left=60, "G", "\shortmid" marking, ""{name=U, below}] \arrow[r, "H" description, ""{name=M}] \arrow[r, bend right=60, "K"', "\shortmid" marking, ""{name=D, above}] & \mathbb{H} \arrow[r, "F", "\shortmid" marking] & \mathbb{H}' \arrow[Rightarrow, from=U, to=M, "\alpha"] \arrow[Rightarrow, from=M, to=D, "\beta"] \end{tikzcd}$ then we have

$$F * (\beta *_0 \alpha) = \overrightarrow{F} m\mathrm{Q}^1\langle \beta, \alpha\rangle d = m\mathrm{Q}^1(\overrightarrow{F} \times \overrightarrow{F})\mathrm{Q}^1\langle \beta, \alpha\rangle d$$
$$= m\mathrm{Q}^1(\Big\langle \overrightarrow{F}\beta, \overrightarrow{F}\alpha\Big\rangle)d = (F * \beta) *_0 (F * \alpha)\,,$$

where we use lemma 5.33. Also, for identity transformations we have:

$$\overrightarrow{F} * \mathrm{id}_G = \overrightarrow{F} iG = iFG = \mathrm{id}_{F*G}$$

hence F^* is a functor.

The other operations are preserved similarly by applying lemma 5.33. □

We now proceed to constructing the restricted mapping space $\{\mathbb{G},\mathbb{H}\}$. We pull back all the parts of (81) along e^* given in (4) to obtain

$$\begin{tikzcd}[column sep=large]
\{\mathbb{G},\mathbb{H}\}_3 \arrow[r, shift left=2, "d_{1*}"] \arrow[r, shift right=2, "d_{0*}"'] \arrow[d, "\overline{e^*}"'] & \{\mathbb{G},\mathbb{H}\}_2 \arrow[l, "i_*" description] \arrow[r, shift left=2, "d_{1*}"] \arrow[r, shift right=2, "d_{0*}"'] \arrow[d, "\overline{e^*}"'] & \{\mathbb{G},\mathbb{H}\}_1 \arrow[l, "i_*" description] \arrow[r, shift left=2, "d_{1*}"] \arrow[r, shift right=2, "d_{0*}"'] \arrow[d, "\overline{e^*}"'] & \mathsf{GrayCat}(\mathbb{G},\mathbb{H}) \arrow[l, "i_*" description] \arrow[d, "e^*"] \\
\mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G},\overline{\overline{\overline{\mathbb{H}}}}) \arrow[r, shift left=2, "d_{1*}"] \arrow[r, shift right=2, "d_{0*}"'] & \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G},\overline{\overline{\mathbb{H}}}) \arrow[l, "i_*" description] \arrow[r, shift left=2, "d_{1*}"] \arrow[r, shift right=2, "d_{0*}"'] & \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G},\overrightarrow{\mathbb{H}}) \arrow[l, "i_*" description] \arrow[r, shift left=2, "d_{1*}"] \arrow[r, shift right=2, "d_{0*}"'] & \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G},\mathbb{H}) \arrow[l, "i_*" description]
\end{tikzcd}\,, \tag{82}$$

and we set $\{\mathbb{G},\mathbb{H}\}_0 = \mathsf{GrayCat}(\mathbb{G},\mathbb{H})$. We call $\{\mathbb{G},\mathbb{H}\}_1$ the set of malleable transformations, c. f. definition 7.2. Obviously the left and right actions of strict functors described in theorems 6.4 and 6.2 restrict to the restricted mapping space.

Hence for strict morphisms $F\colon \mathbb{G}' \longrightarrow \mathbb{G}$ and $G\colon \mathbb{H} \longrightarrow \mathbb{H}'$ we get a commuting square of Gray-functors

$$\begin{tikzcd}
\{\mathbb{G},\mathbb{H}\} \arrow[r, "F^*"] \arrow[d, "G_*"'] & \{\mathbb{G}',\mathbb{H}\} \arrow[d, "G_*"] \\
\{\mathbb{G},\mathbb{H}'\} \arrow[r, "F^*"'] & \{\mathbb{G}',\mathbb{H}'\}
\end{tikzcd} .$$

In conclusion, we get the following interesting structure on $\mathsf{GrayCat}$, and leave the question as to further, higher structure open:

6.5. Theorem. *The category* $\mathsf{GrayCat}$ *of* Gray*-categories, strict* Gray*-functors and malleable transformations is a sesquicategory.* ☐

6.6. Remark. *By section 2.1* $\{\mathbb{G},\mathbb{H}\}$ *is a* Gray*-category and* $\overline{e^*}\colon \{\mathbb{G},\mathbb{H}\} \longrightarrow [\mathbb{G},\mathbb{H}]$ *is a strict* Gray*-functor.*

For $\mathbb{G}$ free up to order 1 the maps e and k discussed in (2) give natural transformations

$$\begin{tikzcd}
\mathsf{GrayCat}(\mathbb{G},\_) \arrow[r, "e^*"] \arrow[rr, bend right, "\mathsf{GrayCat}(\mathbb{G},\_)"'] & \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G},\_) \arrow[r, "k^*"] & \mathsf{GrayCat}(\mathbb{G},\_)
\end{tikzcd} ,$$

where the maps act by precomposition in $\mathsf{GrayCat}$.

6.7. Lemma. *Given a* Gray*-category* $\mathbb{G}$ *free up to order 1 there are canonical transformations*

$$\begin{tikzcd}
\mathbb{G} \arrow[r, bend left=50, "F", ""{name=U, below}] \arrow[r, bend right=50, "Fk"', ""{name=D, above}] & \mathbb{H} \arrow[Rightarrow, from=U, to=D, "\rho"]
\end{tikzcd}$$

that is the identity on objects.[1]

Proof We need to give a Q^1 graph map $\rho\colon \mathbb{G} \twoheadrightarrow \overrightarrow{\mathbb{H}}$ with $d_1\rho = Fke$ and $d_0\rho = F$:

1. 0-cells

$$x \mapsto x \xrightarrow{\mathrm{id}_x} x$$

---

[1] I. e. basically icons in the sense of Lack [2007], except our constraint 2-cell points the other way.

2. 1-cells

$$f = [f_1, \dots, f_n] \mapsto \begin{tikzcd}[column sep=large, row sep=large] {} \arrow[r, "\mathrm{id}_x"] \arrow[d, "Ff"'] & {} \arrow[d, "F[f_1]\#_0\cdots\#_0F[f_n]"] \arrow[dl, Rightarrow, "F^2_{[f_1],\dots,[f_n]}"] \\ {} \arrow[r, "\mathrm{id}_y"'] & {} \end{tikzcd}$$

3. 2-cells

$$(\alpha\colon f \Longrightarrow f') \mapsto \begin{tikzcd}[column sep=large, row sep=large] {} \arrow[r, "\mathrm{id}_x"] \arrow[d, "Ff"] \arrow[d, bend right=80, "Ff'"'] \arrow[d, phantom, shift right=5ex, "\overset{F\alpha}{\Longleftarrow}"] & {} \arrow[d, "F[f_1]\#_0\cdots\#_0F[f_n]"] \arrow[dl, Rightarrow, "F^2_{[f_1],\dots,[f_n]}"] \\ {} \arrow[r, "\mathrm{id}_y"'] & {} \end{tikzcd} = \begin{tikzcd}[column sep=large, row sep=large] {} \arrow[r, "\mathrm{id}_x"] \arrow[d, "Ff'"'] & {} \arrow[d, "F[f'_1]\#_0\cdots\#_0F[f'_{n'}]"'] \arrow[d, bend left=80, "F[f_1]\#_0\cdots\#_0F[f_n]"] \arrow[d, phantom, shift left=5ex, "\overset{\omega}{\Longleftarrow}"] \arrow[dl, Rightarrow, "F^2_{[f_1],\dots,[f_n]}"'] \\ {} \arrow[r, "\mathrm{id}_y"'] & {} \end{tikzcd}$$

where ω is $\overline{F^2_{[f'_1],\dots,[f'_{n'}]}}\#_1F\alpha\#_1F^2_{[f_1],\dots,[f_n]}$.

4. 3-cells

$$(\Gamma\colon \alpha \Rrightarrow \alpha') \mapsto \begin{array}{ccc} \begin{tikzcd}[column sep=large, row sep=large] {} \arrow[r, "\mathrm{id}_x"] \arrow[d, "Ff"] \arrow[d, bend right=80, "Ff'"'] \arrow[d, phantom, shift right=5ex, "\overset{F\alpha}{\Longleftarrow}"] & {} \arrow[d, "F[f_1]\#_0\cdots\#_0F[f_n]"] \arrow[dl, Rightarrow, "F^2_{[f_1],\dots,[f_n]}"] \\ {} \arrow[r, "\mathrm{id}_y"'] & {} \end{tikzcd} & = & \begin{tikzcd}[column sep=large, row sep=large] {} \arrow[r, "\mathrm{id}_x"] \arrow[d, "Ff'"'] & {} \arrow[d, "F[f'_1]\#_0\cdots\#_0F[f'_{n'}]"'] \arrow[d, bend left=80, "F[f_1]\#_0\cdots\#_0F[f_n]"] \arrow[d, phantom, shift left=5ex, "\overset{\omega}{\Longleftarrow}"] \arrow[dl, Rightarrow, "F^2_{[f_1],\dots,[f_n]}"'] \\ {} \arrow[r, "\mathrm{id}_y"'] & {} \end{tikzcd} \\ {\scriptstyle F\Gamma\#_1F^2_{[f_1],\dots,[f'_{n'}]}}\ \Big\Downarrow\!\!\Big\Downarrow & & \Big\Downarrow\!\!\Big\Downarrow\ {\scriptstyle F\Gamma\#_1F^2_{[f_1],\dots,[f_n]}} \\ \begin{tikzcd}[column sep=large, row sep=large] {} \arrow[r, "\mathrm{id}_x"] \arrow[d, "Ff"] \arrow[d, bend right=80, "Ff'"'] \arrow[d, phantom, shift right=5ex, "\overset{F\alpha'}{\Longleftarrow}"] & {} \arrow[d, "F[f_1]\#_0\cdots\#_0F[f_n]"] \arrow[dl, Rightarrow, "F^2_{[f_1],\dots,[f_n]}"] \\ {} \arrow[r, "\mathrm{id}_y"'] & {} \end{tikzcd} & = & \begin{tikzcd}[column sep=large, row sep=large] {} \arrow[r, "\mathrm{id}_x"] \arrow[d, "Ff'"'] & {} \arrow[d, "F[f'_1]\#_0\cdots\#_0F[f'_{n'}]"'] \arrow[d, bend left=80, "F[f_1]\#_0\cdots\#_0F[f_n]"] \arrow[d, phantom, shift left=5ex, "\overset{\omega'}{\Longleftarrow}"] \arrow[dl, Rightarrow, "F^2_{[f_1],\dots,[f_n]}"'] \\ {} \arrow[r, "\mathrm{id}_y"'] & {} \end{tikzcd} \end{array}$$

where $\omega = \overline{F^2_{[f'_1],\dots,[f'_{n'}]}}\#_1 F\alpha \#_1 F^2_{[f_1],\dots,[f_n]}$ and $\omega' = \overline{F^2_{[f'_1],\dots,[f'_{n'}]}}\#_1 F\alpha' \#_1 F^2_{[f_1],\dots,[f_n]}$.

5. For a composable pair of 1-cells f', f a 2-cocycle element

$$\begin{tikzcd}
{} & {} \arrow[r, "\mathrm{id}_x"] \arrow[d, "Ff"] \arrow[dd, bend right=70, "F(f'\#_0 f)"'] & {} \arrow[d, "F{[f_1]}\#_0\cdots\#_0 F{[f_n]}"] \arrow[dl, Rightarrow, shorten=4mm, "F^2_{[f_1],\dots,[f_n]}"'] & & {} \arrow[r, "\alpha_x"] \arrow[dd, "F(f'\#_0 f)"'] & {} \arrow[dd, "F{[f'_1]}\#_0\cdots\#_0 F{[f'_{n'}]}\#_0 F{[f_1]}\#_0\cdots\#_0 F{[f_n]}"] \arrow[ddl, Rightarrow, shorten=6mm, "F^2_{[f'_1],\dots,[f'_{n'}],[f_1],\dots,[f_n]}"'] \arrow[drr, "F{[f_1]}\#_0\cdots\#_0 F{[f_n]}"] & & \\
{} & {} \arrow[r, "\alpha_y"'] \arrow[d, "Ff'"] \arrow[l, phantom, "\Longleftarrow F^2_{f',f}"'] & {} \arrow[d, "F{[f'_1]}\#_0\cdots\#_0 F{[f'_{n'}]}"] \arrow[dl, Rightarrow, shorten=4mm, "F^2_{[f'_1],\dots,[f'_{n'}]}"'] & = & & & = & {} \arrow[dll, "F{[f'_1]}\#_0\cdots\#_0 F{[f'_{n'}]}"] \\
{} & {} \arrow[r, "\alpha_z"'] & {} & & {} \arrow[r, "\alpha_z"'] & {} & &
\end{tikzcd}$$

.

The equation holds by 15 and 11.

The verification that this is a Q^1-graph map is straightforward. □

7. Putting it all together

7.1. Definition. *A* **lax transformation** $\alpha\colon F \longrightarrow G$ *between pseudo-functors* $F, G\colon \mathbb{G} \nrightarrow \mathbb{H}$ *of* Gray*-categories is a pseudo-functor* $\alpha\colon \mathbb{G} \nrightarrow \overrightarrow{\mathbb{H}}$ *such that* $d_0\alpha = F$ *and* $d_1\alpha = G$.

7.2. Definition. *A* **malleable transformation** $\alpha\colon F \longrightarrow G$ *between strict functors* $F, G\colon \mathbb{G} \longrightarrow \mathbb{H}$ *of* Gray*-categories is a pseudo-functor* $\alpha\colon \mathbb{G} \nrightarrow \overrightarrow{\mathbb{H}}$ *such that* $d_0\alpha = F$ *and* $d_1\alpha = G$.

This was introduced in (82).

7.3. Remark. *Using the definition of path spaces in definition 3.1 and the characterization of pseudo-maps in definition 2.24 we note for reference that a lax transformation* α *is given by the following underlying data:*

1. *for each 0-cell* x *of* $\mathbb{G}$ *a 1-cell* $\alpha_x\colon Fx \longrightarrow Gx$,
2. *for each 1-cell* $f\colon x \longrightarrow y$ *of* $\mathbb{G}$ *a 2-cell*

$$\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"'] & Gx \arrow[d, "Gf"] \arrow[dl, Rightarrow, shorten=2mm, "\alpha_f"] \\
Fy \arrow[r, "\alpha_y"'] & Gy
\end{tikzcd}$$

3. *for each 2-cell $g\colon f \longrightarrow f'$ of $\mathbb{G}$ a 3-cell of $\mathbb{H}$*

$$\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"] \arrow[d, bend right=70, "Ff'"'] \arrow[d, phantom, shift right=4ex, "\overset{\Leftarrow}{Fg}"] & Gx \arrow[d, "Gf"] \arrow[dl, Rightarrow, "\alpha_f"] \arrow[rr, phantom, shift right=4ex, "\overset{\alpha_g}{\Rrightarrow}"] & & Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff'"'] & Gx \arrow[d, "Gf'"] \arrow[d, bend left=70, "Gf"] \arrow[dl, Rightarrow, "\alpha_{f'}"] \arrow[d, phantom, shift left=4ex, "\overset{\Leftarrow}{Gg}"] \\
Fy \arrow[r, "\alpha_y"'] & Gy & & Fy \arrow[r, "\alpha_y"'] & Gy
\end{tikzcd}$$

4. *for each pair of composable 1-cells $f\colon x \longrightarrow y$, $f'\colon y \longrightarrow z$ an invertible 3-cell*

$$\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"] \arrow[dd, bend right=80, "F(f'\#_0 f)"'] & Gx \arrow[d, "Gf"] \arrow[dl, Rightarrow, "\alpha_f"] & & Fx \arrow[r, "\alpha_x"] \arrow[dd, "F(f'\#_0 f)"'] & Gx \arrow[dd, "G(f'\#_0 f)"] \arrow[dr, "Gf"] \arrow[ddl, Rightarrow, "\alpha_{f'\#_0 f}"'] & \\
Fy \arrow[r, "\alpha_y"'] \arrow[d, "Ff'"] \arrow[l, phantom, "\underset{F^2_{f',f}}{\Leftarrow}"] & Gy \arrow[d, "Gf'"] \arrow[dl, Rightarrow, "\alpha_{f'}"] \arrow[rr, phantom, "\overset{\alpha^2_{f',f}}{\Rrightarrow}"] & & & \overset{G^2_{f',f}}{\Leftarrow} & Gy \arrow[dl, "Gf'"] \\
Fz \arrow[r, "\alpha_z"'] & Gz & & Fz \arrow[r, "\alpha_z"'] & Gz &
\end{tikzcd}\quad .$$

Furthermore, these data have to satisfy the following equations:

1. *On identities of 0-cells:*

$$\alpha_{\mathrm{id}_x} = \mathrm{id}_{\alpha_x}$$

2. *for each 3-cell $\Gamma\colon g \longrightarrow g'$ the square of 3-cells in $\mathbb{H}$*

$$\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"] \arrow[d, bend right=70, "Ff'"'] \arrow[d, phantom, shift right=4ex, "\Leftarrow Fg"] & Gx \arrow[d, "Gf"] \arrow[dl, Rightarrow, "\alpha_f"] \arrow[rr, phantom, shift right=4ex, "\overset{\alpha_g}{\Rrightarrow}"] & & Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff'"'] & Gx \arrow[d, "Gf'"] \arrow[d, bend left=70, "Gf"] \arrow[dl, Rightarrow, "\alpha_{f'}"] \arrow[d, phantom, shift left=4ex, "\Leftarrow Gg"] \\
Fy \arrow[r, "\alpha_y"'] \arrow[d, Rrightarrow, "(\alpha_y \#_0 F\Gamma)\#_1 \alpha_f"'] & Gy & & Fy \arrow[r, "\alpha_y"'] & Gy \arrow[d, Rrightarrow, "\alpha_{f'}\#_1(G\Gamma \#_0 \alpha_x)"] \\
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"] \arrow[d, bend right=70, "Ff'"'] \arrow[d, phantom, shift right=4ex, "\Leftarrow Fg'"] & Gx \arrow[d, "Gf"] \arrow[dl, Rightarrow, "\alpha_f"] \arrow[rr, phantom, shift right=4ex, "\underset{\alpha_{g'}}{\Rrightarrow}"] & & Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff'"'] & Gy \arrow[d, "Gf'"] \arrow[d, bend left=70, "Gf"] \arrow[dl, Rightarrow, "\alpha_{f'}"] \arrow[d, phantom, shift left=4ex, "\Leftarrow Gg'"] \\
Fy \arrow[r, "\alpha_y"'] & Gy & & Fy \arrow[r, "\alpha_y"'] & Gy
\end{tikzcd}$$

commutes. This condition obviously comes from the definition of 3-cells in the path space.

3. *For every pair $g\colon f \Longrightarrow f', g'\colon f' \Longrightarrow f''$:*

$$\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[dd, "Ff"] \arrow[dd, bend right=40, "Ff'"'] \arrow[dd, bend right=70, "Ff''"'] \arrow[dd, phantom, bend right=25, "Fg"'] \arrow[dd, phantom, bend right=55, "Fg'"'] & Gx \arrow[dd, "Gf"] \arrow[ddl, Rightarrow, "\alpha_f"] & & Fx \arrow[r, "\alpha_x"] \arrow[dd, "Ff'"] \arrow[dd, bend right=40, "Ff"'] \arrow[dd, phantom, bend right=20, "Fg"'] & Gx \arrow[dd, "Gf"] \arrow[dd, bend left=40, "Gf"] \arrow[dd, phantom, bend left=20, "Gg"] \arrow[ddl, Rightarrow, "\alpha_{f'}"] & & Fx \arrow[r, "\alpha_x"] \arrow[dd, "Ff''"'] & Gx \arrow[dd, "Gf"] \arrow[dd, bend left=40, "Gf'"] \arrow[dd, bend left=70, "Gf''"] \arrow[dd, phantom, bend left=25, "Gg"] \arrow[dd, phantom, bend left=55, "Gg'"] \arrow[ddl, Rightarrow, "\alpha_{f''}"] \\
& \arrow[rr, Rrightarrow, "(\alpha_y\#_0Fg')\#_1\alpha_g"] & & & \arrow[rr, Rrightarrow, "\alpha_{g'}\#_1(Gg\#_0\alpha_x)"] & & & \\
Fy \arrow[r, "\alpha_y"'] & Gy \arrow[rrrrr, Rrightarrow, bend right, "\alpha_{g'\#_1g}"'] & & Fx \arrow[r, "\alpha_y"'] & Gy & & Fy \arrow[r, "\alpha_y"'] & Gy
\end{tikzcd},$$

and for identity 2-cells $\mathrm{id}_f\colon f \Longrightarrow f$ *we have an identity 3-cell*

$$\alpha_{\mathrm{id}_f} = \mathrm{id}_{\alpha_f}\,.$$

4. *The family of 3-cells has to satisfy a kind of cocycle condition: For a composable triple f, f', f'' of 1-cells α^2 has to satisfy equation (83). furthermore, α^2 has to satisfy the normalization condition:*

$$\alpha^2_{f',f} = \begin{cases} \mathrm{id}_{\alpha_{f'}} & \textit{if } f' = \mathrm{id}_y \\ \mathrm{id}_{\alpha_f} & \textit{if } f = \mathrm{id}_x \end{cases}$$

5. *The family of 3-cells α^2 has to be compatible with left and right whiskering according to (84) and (85).*

*These conditions are derived from the ones in the definition of pseudo-*Gray*-functors 2.24. Note how conditions 4, 5, 6 of definition 2.24 are trivially satisfied for transformations.*

7.4. Definition. *A transformation $\alpha\colon F \longrightarrow G$ where the cocycle α^2 has only trivial components we call a* **stiff transformation***.*

7.5. Lemma. *A stiff transformation $\alpha\colon F \longrightarrow G$ with F and G strict* Gray*-functors is a 1-transfor in the sense of [Crans 1999].* ☐

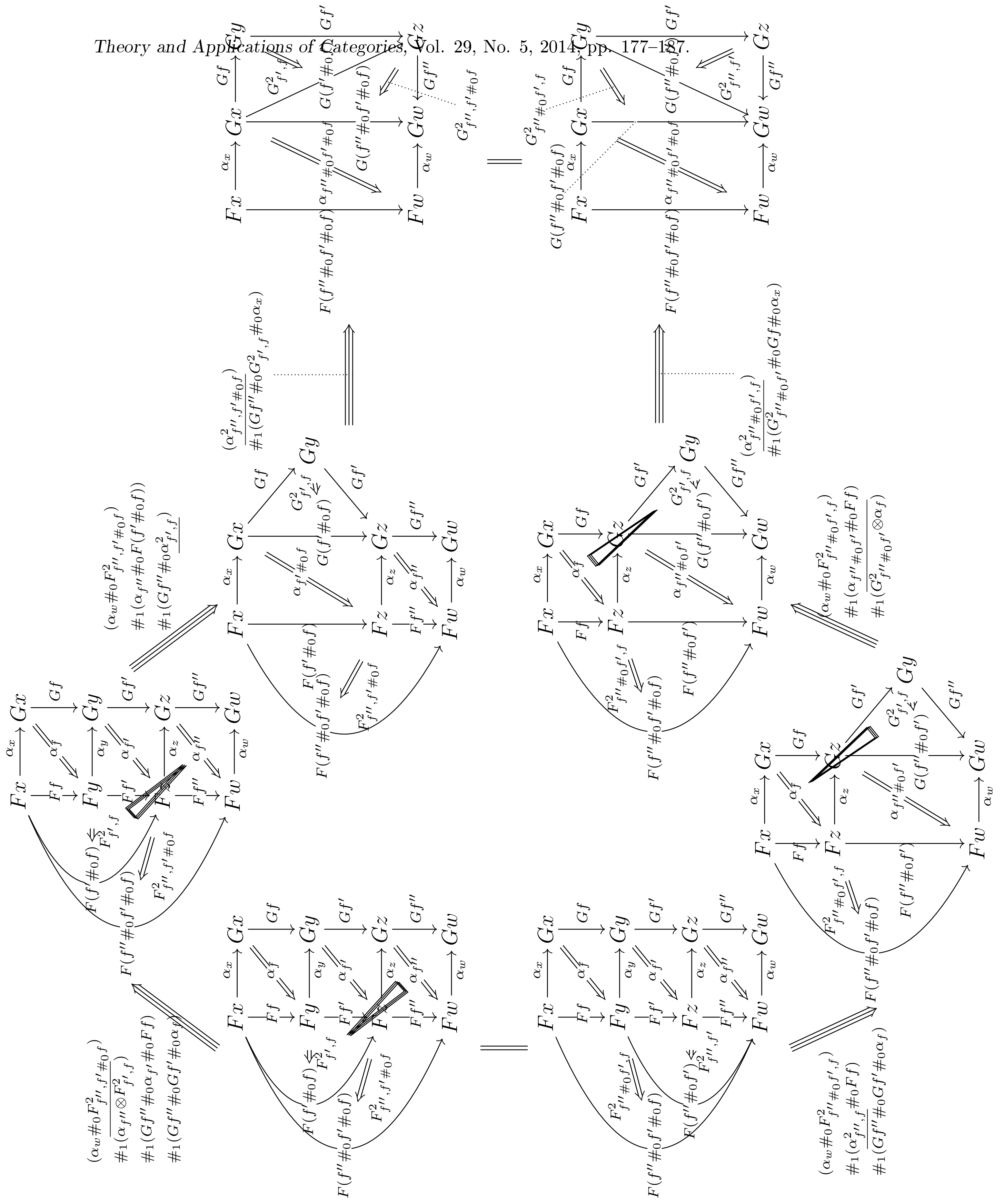
7.6. Remark. *Given two lax-transformations $F \xrightarrow{\alpha} G \xrightarrow{\beta} H$ their composite $\beta * \alpha$ given by $m\langle\beta,\alpha\rangle$ and has the following components:*

1. *for each 0-cell x of $\mathbb{G}$ the 1-cell*

$$Fx \xrightarrow{(\beta*\alpha)_x} Hx \;=\; Fx \xrightarrow{\alpha_x} Gx \xrightarrow{\beta_x} Hx\,,$$

2. *for each 1-cell $f\colon x \longrightarrow y$ of $\mathbb{G}$ the 2-cell*

$$\begin{tikzcd}
Fx \arrow[r, "(\beta*\alpha)_x"] \arrow[d, "Ff"'] & Hx \arrow[d, "Hf"] \arrow[dl, Rightarrow, "(\beta*\alpha)_f"'] \\
Fy \arrow[r, "(\beta*\alpha)_y"'] & Hy
\end{tikzcd}
=
\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"'] & Gx \arrow[r, "\beta_x"] \arrow[d, "Gf"] \arrow[dl, Rightarrow, "\alpha_f"] & Hx \arrow[d, "Hf"] \arrow[dl, Rightarrow, "\beta_f"] \\
Fy \arrow[r, "\alpha_y"'] & Gy \arrow[r, "\beta_y"'] & Hy
\end{tikzcd}$$


$(\alpha_w \#_0 F^2_{f'',f'\#_0 f})$
$\#_1(\overline{\alpha_{f''} \otimes F^2_{f',f}})$
$\#_1(Gf'' \#_0 \alpha_{f'} \#_0 Ff)$
$\#_1(Gf'' \#_0 Gf' \#_0 \alpha_f)$
$(\alpha_w \#_0 F^2_{f'',f'\#_0 f})$
$\#_1(\alpha_{f''} \#_0 F(f'\#_0 f))$
$\#_1(Gf'' \#_0 \underline{\alpha^2_{f',f}})$
$(\alpha^2_{f'',f'\#_0 f})$
$\#_1(Gf'' \#_0 G^2_{f',f} \#_0 \alpha_x)$
$(\alpha^2_{f''\#_0 f',f})$
$\#_1(G^2_{f'',f'} \#_0 Gf \#_0 \alpha_x)$
$(\alpha_w \#_0 F^2_{f''\#_0 f',f})$
$\#_1(\alpha_{f''\#_0 f'} \#_0 Ff)$
$\#_1(\overline{G^2_{f'',f'}} \otimes \alpha_f)$
$(\alpha_w \#_0 F^2_{f''\#_0 f',f})$
$\#_1(\alpha^2_{f'',f'} \#_0 Ff)$
$\#_1(\overline{Gf'' \#_0 Gf' \#_0 \alpha_f})$

(83)

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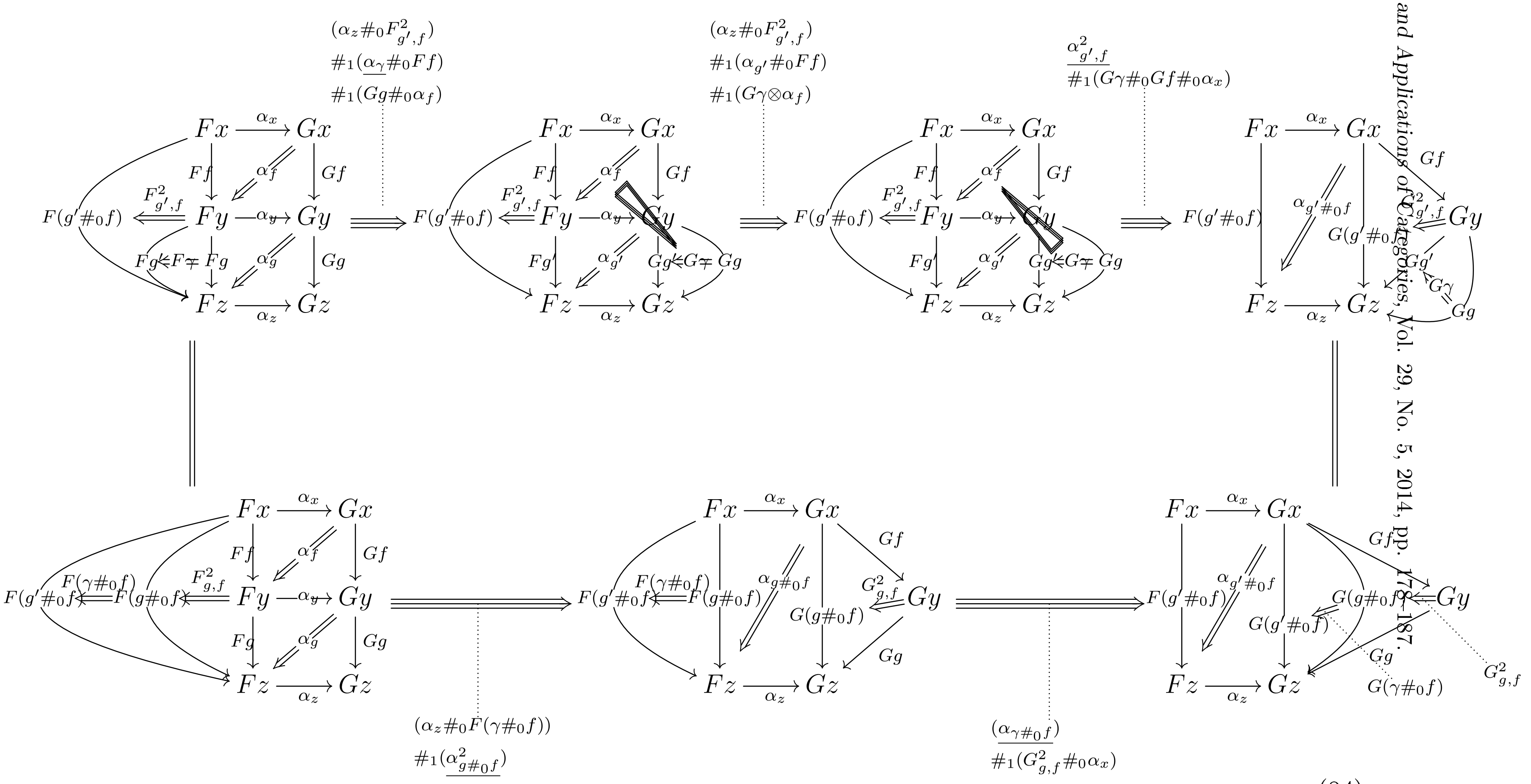
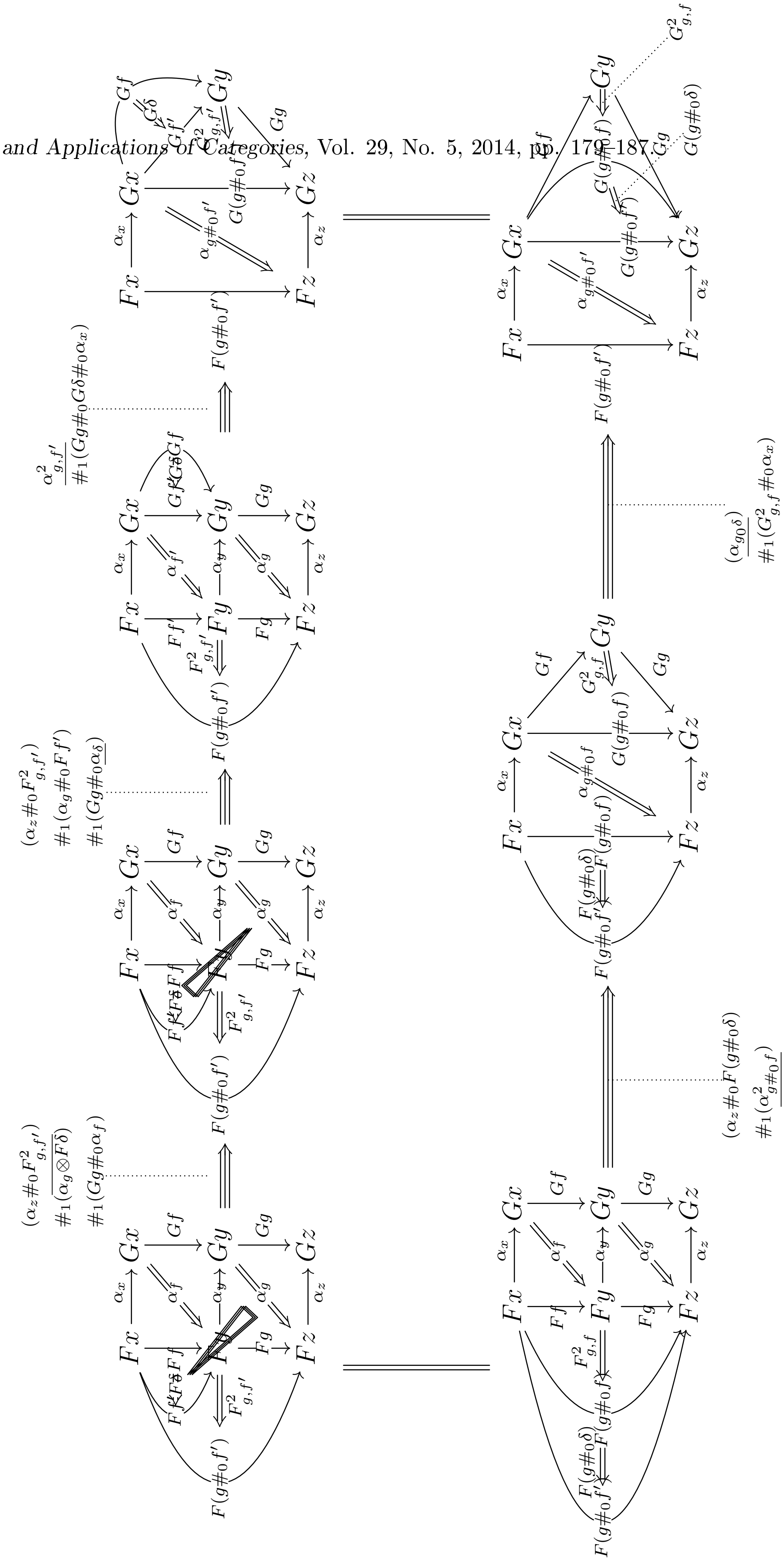


(84)

Compatibility of the cocycle α^2 with left whiskers $\gamma\#_0f$.



(85)

Compatibility of the cocycle α^2 with right whiskers $g\#_0\delta$.

Theory and Applications of Categories, Vol. 29, No. 5, 2014, pp. 180–187.

$$\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"] \arrow[d, bend right=70, "Ff'"'] \arrow[d, phantom, bend right=35, "\overset{\Leftarrow}{Fg}"] & Hx \arrow[d, "Hf"] \arrow[dl, Rightarrow, "(\beta*\alpha)_f"] & {} \arrow[r, phantom, "\overset{(\beta*\alpha)_g}{\Rrightarrow}"] & {} & Fx \arrow[r, "(\beta*\alpha)_x"] \arrow[d, "Ff'"'] & Hx \arrow[d, "Hf'"] \arrow[d, bend left=70, "Hf"] \arrow[d, phantom, bend left=35, "\overset{\Leftarrow}{Hg}"] \arrow[dl, Rightarrow, "(\beta*\alpha)_{f'}"] \\
Fy \arrow[r, "(\beta*\alpha)_y"'] & Hy & & & Fy \arrow[r, "(\beta*\alpha)_y"'] & Hy
\end{tikzcd}$$

$$= \begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"] \arrow[d, bend right=70, "Ff'"'] \arrow[d, phantom, bend right=35, "\overset{\Leftarrow}{Fg}"] & Gx \arrow[r, "\beta_x"] \arrow[d, "Gf"] \arrow[dl, Rightarrow, "\alpha_f"] & Hx \arrow[d, "Hf"] \arrow[dl, Rightarrow, "\beta_f"] & {} \arrow[r, phantom, "\overset{(\beta_y\#_0\alpha_g)}{\underset{}{\overset{\#_1(\beta_f\#_0\alpha_x)}{\Rrightarrow}}}"] & {} & Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff'"'] & Gx \arrow[r, "\beta_x"] \arrow[d, bend right, "Gf'"'] \arrow[d, bend left, "Gf"] \arrow[d, phantom, "\overset{\Leftarrow}{Gg}"] \arrow[dl, phantom, "\alpha_{f'}\swarrow"] & Hx \arrow[d, "Hf"] \arrow[dl, phantom, "\beta_f\swarrow"] & {} \arrow[r, phantom, "\overset{(\beta_y\#_0\alpha_{f'})}{\overset{\#_1(\beta_g\#_0\alpha_x)}{\Rrightarrow}}"] & {} & Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff'"'] & Gx \arrow[r, "\beta_x"] \arrow[d, "Gf'"] \arrow[dl, Rightarrow, "\alpha_{f'}"] & Hx \arrow[d, "Hf'"] \arrow[d, bend left=70, "Hf"] \arrow[d, phantom, bend left=35, "\overset{\Leftarrow}{Hg}"] \arrow[dl, Rightarrow, "\beta_{f'}"] \\
Fy \arrow[r, "\alpha_y"'] & Gy \arrow[r, "\beta_y"'] & Hy & & & Fy \arrow[r, "\alpha_y"'] & Gy \arrow[r, "\beta_y"'] & Hy & & & Fy \arrow[r, "\alpha_y"'] & Gy \arrow[r, "\beta_y"'] & Hy
\end{tikzcd} \tag{86}$$

3. *for each 2-cell* $g\colon f \longrightarrow f'$ *of* $\mathbb{G}$ *the 3-cell of* $\mathbb{H}$ *shown in* (86)

4. *for each pair of composable 1-cells* $f\colon x \longrightarrow y$, $f'\colon y \longrightarrow z$ *a 3-cell shown in* (87)

7.7. Definition. *Assuming* α *and* β *are as in definition 7.1 and* F *and* G *are pseudo-functors* $\mathbb{G} \nrightarrow \mathbb{H}$, *a* **modification** $A\colon \alpha \longrightarrow \beta\colon F \longrightarrow G$ *is a pseudo-functor* $A\colon \mathbb{G} \nrightarrow \overline{\overline{\overline{\mathbb{H}}}}$, *such that* $d_0A = \alpha$ *and* $d_1A = \beta$.

7.8. Remark. *A modification* $A\colon \alpha \longrightarrow \beta$ *according to definitions 7.7 and 2.24 is given by the following data:*

1. *For every 0-cell* x *in* $\mathbb{G}$ *a 2-cell*

$$\begin{tikzcd}
Fx \arrow[r, bend left, "\alpha_x", ""{name=U, below}] \arrow[r, bend right, "\beta_x"', ""{name=D}] & Gx \arrow[Rightarrow, from=U, to=D, "A_x"]
\end{tikzcd}$$

2. *For every 1-cell* $f\colon x \longrightarrow y$ *a 3-cell in* $\mathbb{H}$

$$\begin{tikzcd}
Fx \arrow[r, bend left, "\alpha_x", ""{name=U, below}] \arrow[r, bend right, "\beta_x"', ""{name=D}] \arrow[d, "Ff"'] & Gx \arrow[d, "Gf"] \arrow[dl, phantom, "\overset{\beta_f}{\Swarrow}"] \\
Fy \arrow[r, bend right, "\beta_y"'] & Gy
\arrow[Rightarrow, from=U, to=D, "A_x"]
\end{tikzcd}
\overset{A_f}{\Rrightarrow}
\begin{tikzcd}
Fx \arrow[r, bend left, "\alpha_x"] \arrow[d, "Ff"'] & Gx \arrow[d, "Gf"] \arrow[dl, phantom, "\overset{\alpha_f}{\Swarrow}"] \\
Fy \arrow[r, bend left, "\alpha_y", ""{name=U2, below}] \arrow[r, bend right, "\beta_y"', ""{name=D2}] & Gy
\arrow[Rightarrow, from=U2, to=D2, "A_y"]
\end{tikzcd}$$

This data has to satisfy the following conditions:

1. *Units are preserved:*

$$A_{\mathrm{id}_x} = \mathrm{id}_{A_x}$$

2. *Compatibility with the cocycles of* F, G, α, β *according to (88)*

3. *For 2-cells* $g\colon f \Longrightarrow f'$ *in* $\mathbb{G}$ *the images under* F *and* G *as well the data of* A, α *and* β *are compatible as shown in (89)*

7.9. Lemma. *A transformation* $A\colon \alpha \longrightarrow \beta$ *where* $\alpha, \beta\colon F \longrightarrow G$ *are stiff and* F, G *are strict is a 2-transfor in the sense of [Crans 1999].* □

7.10. Definition. *Given modifications* $A, B\colon \alpha \longrightarrow \beta$ *a* **perturbation** *is a pseudo-*Gray-*functor* $\sigma\colon \mathbb{G} \nrightarrow \overline{\overline{\overline{\overline{\mathbb{H}}}}}$ *such that* $d_0\sigma = A$ *and* $d_1\sigma = B$.

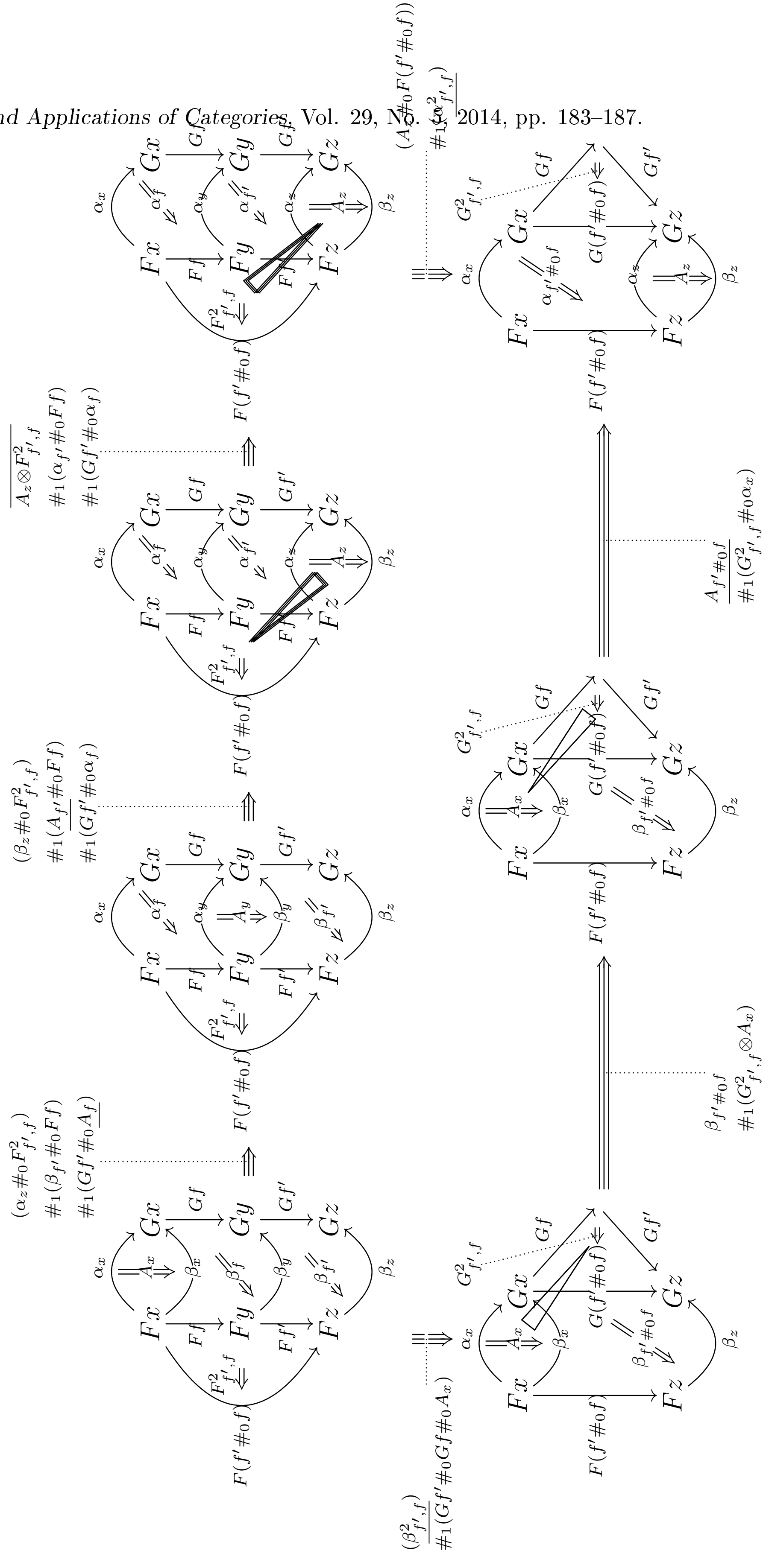
$$\begin{tikzcd}
Fx \arrow[r, "(\beta*\alpha)_x"] \arrow[d, "Ff"] \arrow[dd, bend right=60, "F(f'\#_0f)"'] \arrow[dd, phantom, bend right=30, "\Leftarrow F^2_{f',f}"] & Hx \arrow[d, "Hf"] \arrow[dl, phantom, "\Swarrow (\beta*\alpha)_f"] \\
Fy \arrow[r, "(\beta*\alpha)_y"] \arrow[d, "Ff'"] & Hy \arrow[d, "Hf'"] \arrow[dl, phantom, "\Swarrow (\beta*\alpha)_{f'}"] \\
Fz \arrow[r, "(\beta*\alpha)_z"'] & = Hz
\end{tikzcd}
\overset{(\beta*\alpha)^2_{f',f}}{\Rrightarrow}
\begin{tikzcd}
Fx \arrow[r, "(\beta_\alpha)_x"] \arrow[d, "F(f'\#_0f)"'] & Hx \arrow[d, "H(f'\#_0f)"] \arrow[d, bend left=80, "Hf'\#_0Hf"] \arrow[d, phantom, bend left=40, "\Leftarrow H^2_{f',f}"] \arrow[dl, phantom, "\Swarrow (\beta*\alpha)_{f'\#_0f}"] \\
Fy \arrow[r, "(\beta*\alpha)_z"'] & Hy
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"] \arrow[dd, bend right=60, "F(f'\#_0f)"'] \arrow[dd, phantom, bend right=30, "\Leftarrow F^2_{f',f}"] & Gx \arrow[r, "\beta_x"] \arrow[d, "Gf"] \arrow[dl, phantom, "\Swarrow \alpha_f"] & Hx \arrow[d, "Hf"] \arrow[dl, phantom, "\Swarrow \beta_f"] \\
Fy \arrow[r, "\alpha_y"] \arrow[d, "Ff'"] & {} \arrow[r, "\beta_y"] \arrow[d, "Gf'"] \arrow[dl, phantom, "\Swarrow \alpha_{f'}"] & Hy \arrow[d, "Hf'"] \arrow[dl, phantom, "\Swarrow \beta_{f'}"] \\
Fz \arrow[r, "\alpha_z"'] & Gz \arrow[r, "\beta_z"'] & Hz
\end{tikzcd}
\overset{\substack{(\beta_z\#_0\alpha_z\#_0F^2_{f',f}) \\ \#_1(\beta_z\#_0\alpha_{f'}\#_0Ff) \\ \#_1(\beta_{f'}\otimes\alpha_f) \\ \#_1(Hf'\#_0\beta_f\#_0\alpha_x)}}{\Rrightarrow}
\begin{tikzcd}[column sep=huge, row sep=huge]
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff'\#_0Ff"] \arrow[d, bend right=80, "F(f'\#_0f)"'] \arrow[d, phantom, bend right=40, "\Leftarrow F^2_{f',f}"] & Gx \arrow[r, "\beta_x"] \arrow[d, "Gf'\#_0Gf"] \arrow[dl, phantom, "\substack{(\alpha_{f'}\#_0Ff) \\ \#_1(Gf'\#_0\alpha_f)}"] & Hx \arrow[d, "Hf'\#_0Hf"] \arrow[dl, phantom, "\substack{(\beta_{f'}\#_0Gf) \\ \#_1(Hf'\#_0\beta_f)}"] \\
Fz \arrow[r, "\alpha_z"'] & Gz \arrow[r, "\beta_z"'] & Hz
\end{tikzcd}$$

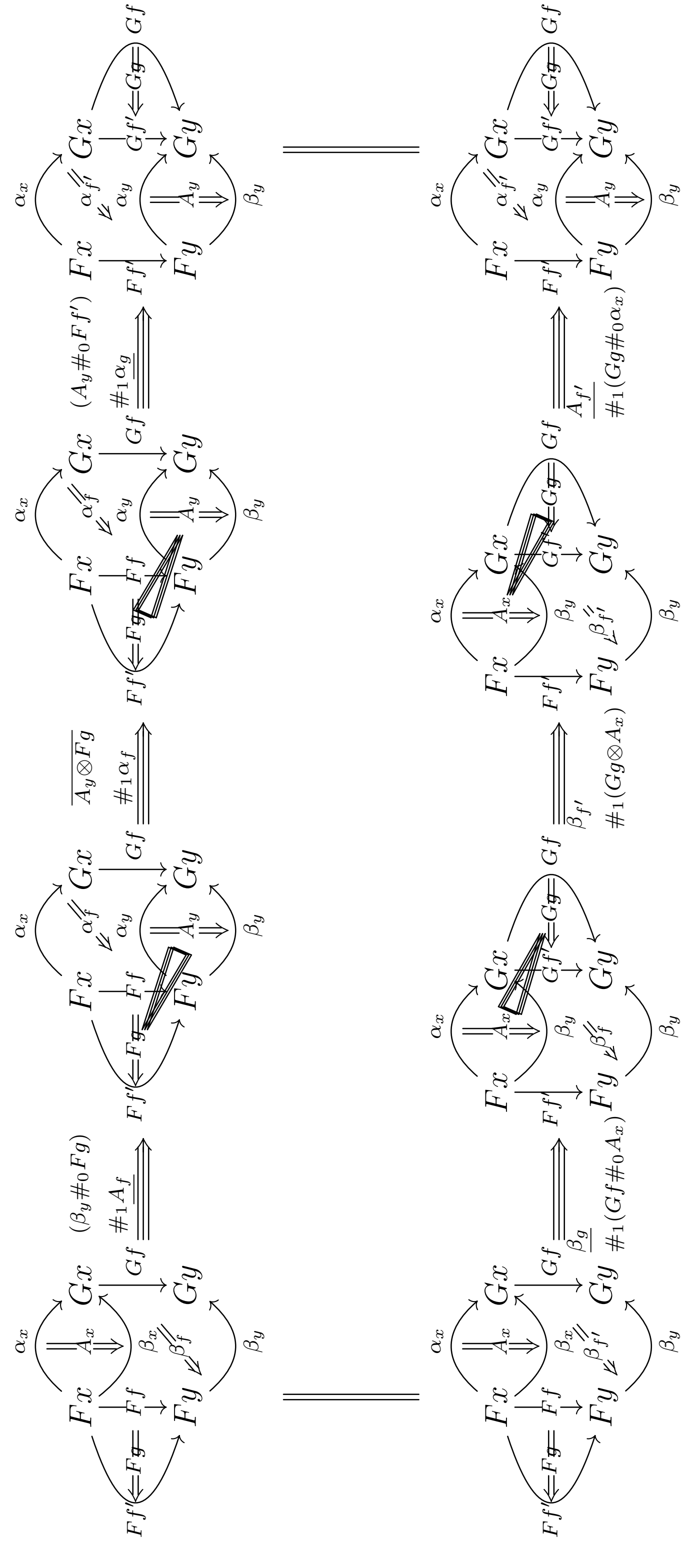
$$\overset{\substack{(\beta_z\#_0\alpha^2_{f',f}) \\ \#_1(((\beta_{f'}\#_0Gf) \\ \#_1(Hf'\#_0\beta_f))\#_0\alpha_x)}}{\Rrightarrow}$$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
Fx \arrow[r, "\alpha_x"] \arrow[d, "F(f'\#_0f)"'] & Gx \arrow[r, "\beta_x"] \arrow[d, bend right=30, "G(f'\#_0f)"'] \arrow[d, bend left=30, "G(f'\#_0f)"] \arrow[d, phantom, "\Swarrow G^2_{f',f}"] \arrow[dl, phantom, "\Swarrow \alpha_{f'\#_0f}"] & Hx \arrow[d, "Hf'\#_0Hf"] \arrow[dl, phantom, "\substack{(\beta_{f'}\#_0Gf) \\ \#_1(Hf'\#_0\beta_f)}"] \\
Fz \arrow[r, "\alpha_z"'] & Gz \arrow[r, "\beta_z"'] & Hz
\end{tikzcd}
\overset{}{\underset{\substack{(\beta_z\#_0\alpha_{f'\#_0f}) \\ \#_1(\beta^2_{f',f}\#_0\alpha_x)}}{\Rrightarrow}}
\begin{tikzcd}[column sep=huge, row sep=huge]
Fx \arrow[r, "\alpha_x"] \arrow[d, "F(f'\#_0f)"'] & Gx \arrow[r, "\beta_x"] \arrow[d, "G(f'\#_0f)"] \arrow[dl, phantom, "\Swarrow \alpha_{f'\#_0f}"] & Hx \arrow[d, "H(f'\#_0f)"] \arrow[d, bend left=80, "Hf'\#_0Hf"] \arrow[d, phantom, bend left=40, "\Leftarrow H^2_{f',f}"] \arrow[dl, phantom, "\Swarrow \beta_{f'\#_0f}"] \\
Fz \arrow[r, "\alpha_z"'] & Gz \arrow[r, "\beta_z"'] & Hz
\end{tikzcd}
\tag{87}$$



(88)

Compatibility of the modification A with the cocycles of F, G, α, β



(89)

Compatibility of 2-cells with A, α and β

7.11. Remark. *According to definition 7.10 a perturbation is given by a 3-cell in* $\mathbb{H}$

$$\begin{tikzcd}
Fx \arrow[r, bend left=50, "\alpha_x"] \arrow[r, bend right=50, "\beta_x"'] \arrow[r, phantom, "\Downarrow A_x"] & Gx \arrow[r, Rrightarrow, "\sigma_x"] & Fx \arrow[r, bend left=50, "\alpha_x"] \arrow[r, bend right=50, "\beta_x"'] \arrow[r, phantom, "\Downarrow B_x"] & Gx
\end{tikzcd}$$

for each 0-cell x in $\mathbb{G}$ such that

$$\begin{tikzcd}
Fx \arrow[r, bend left=50, "\alpha_x"] \arrow[r, bend right=50, "\beta_x"'] \arrow[r, phantom, "\Downarrow A_x"] \arrow[d, "Ff"'] & Gx \arrow[d, "Gf"] \arrow[dl, Rightarrow, shorten=4mm, "\beta_f"] \arrow[rr, Rrightarrow, "\beta_f \#_1(Gf\#_0\sigma_x)"] & & Fx \arrow[r, bend left=50, "\alpha_x"] \arrow[r, bend right=50, "\beta_x"'] \arrow[r, phantom, "\Downarrow B_x"] \arrow[d, "Ff"'] & Gx \arrow[d, "Gf"] \arrow[dl, Rightarrow, shorten=4mm, "\beta_f"] \\
Fy \arrow[r, bend right=50, "\beta_y"'] \arrow[d, Rrightarrow, "A_f"'] & Gy & & Fy \arrow[r, bend right=50, "\beta_y"'] \arrow[d, Rrightarrow, "B_f"] & Gy \\
Fx \arrow[r, bend left=50, "\alpha_x"] \arrow[d, "Ff"'] & Gx \arrow[d, "Gf"] \arrow[dl, Rightarrow, shorten=4mm, "\alpha_f"] \arrow[rr, Rrightarrow, "(\sigma_y\#_0Ff)\#_1\alpha_f"'] & & Fx \arrow[r, bend left=50, "\alpha_x"] \arrow[d, "Ff"'] & Gx \arrow[d, "Gf"] \arrow[dl, Rightarrow, shorten=4mm, "\alpha_f"] \\
Fy \arrow[r, bend left=50, "\alpha_y"] \arrow[r, bend right=50, "\beta_y"'] \arrow[r, phantom, "\Downarrow A_y"] & Gy & & Fy \arrow[r, bend left=50, "\alpha_y"] \arrow[r, bend right=50, "\beta_y"'] \arrow[r, phantom, "\Downarrow B_y"] & Gy
\end{tikzcd}$$

commutes.

7.12. Lemma. *A perturbation $\sigma\colon A \longrightarrow B$ fulfilling the conditions of lemma 7.9 is a 3-transfor in the sense of [Crans 1999].* ☐

A. Adjunctions

We can embed the ideas developed in section 2 in a more global picture. The functor $\mathrm{Q}^1\colon \mathsf{GrayCat} \longrightarrow \mathsf{GrayCat}$ is part of the following adjunction of fibered categories:

$$\begin{tikzcd}
F^*(\mathsf{GrayCat}) \arrow[r, bend left, "(\_)_1{}^*(F)"] \arrow[r, phantom, "\perp"] \arrow[d, "F^*((\_)_1)"'] & \mathsf{GrayCat} \arrow[l, bend left, "\underline{U}"] \arrow[d, "(\_)_1"] \\
\mathsf{RGrph} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{Cat} \arrow[l, bend left, "U"]
\end{tikzcd}$$

where F means “free category over a reflexive graph” and U means “underlying reflexive graph of a category”, $(\_)_1$ means “underlying category of a Gray-category. According to [Hermida 1999, 4.1] the adjunction $F \dashv U$ lifts canonically to an adjunction

$((\_)_1{}^*(F), F) \dashv (\underline{U}, U)$ of fibered categories. Which means in particular that $(\_)_1{}^*(F) \dashv \underline{U}$ is an adjunction and our Q^1 can be defined as $(\_)_1{}^*(F)\underline{U}$.

The objects of $\mathsf{Graph} \times \mathsf{GrayCat}$ might be called 1-free Gray-categories.

A.1. Remark. *Let $P\colon \mathcal{E} \longrightarrow \mathcal{B}$ be a 2-fibration in the sense of Hermida [1999]. Given $u\colon I \longrightarrow PX$ and $u'\colon I' \longrightarrow PX$ for X an object in $\mathcal{E}$; and an equivalence $h\colon I \longrightarrow I'$ such that $u'h = u$. Then the unique filler $\widehat{h}$ over h is an equivalence as well.*

In particular, given the comparison functor $K\colon \mathsf{X}_{FU} \longrightarrow \mathsf{A}$ for the comonad induced by $F \dashv U\colon \mathsf{A} \longrightarrow \mathsf{X}$ lifts to a comparison functor $\widehat{K}$.

A.2. Lemma. *If F is comonadic, then so is $((\_)_1{}^*(F), F)$.*

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