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OPERADIC DEFINITIONS OF WEAK N-CATEGORY: COHERENCE AND COMPARISONS

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ABSTRACT. This paper concerns the relationships between notions of weak n-category defined as algebras for n-globular operads, as well as their coherence properties. We focus primarily on the definitions due to Batanin and Leinster.

A correspondence between the contractions and systems of compositions used in Batanin's definition, and the unbiased contractions used in Leinster's definition, has long been suspected, and we prove a conjecture of Leinster that shows that the two notions are in some sense equivalent. We then prove several coherence theorems which apply to algebras for any operad with a contraction and system of compositions or with an unbiased contraction; these coherence theorems thus apply to weak n-categories in the senses of Batanin, Leinster, Penon and Trimble.

We then take some steps towards a comparison between Batanin weak n-categories and Leinster weak n-categories. We describe a canonical adjunction between the categories of these, giving a construction of the left adjoint, which is applicable in more generality to a class of functors induced by monad morphisms. We conclude with some preliminary statements about a possible weak equivalence of some sort between these categories.

1. Introduction

Of the various definitions of weak n-category that have been proposed, many can be expressed in the form "a weak n-category is an algebra for a certain n-globular operad". This includes Batanin's definition [Batanin, 1998] and its variants [Leinster 2004b, Berger 2002, Leinster 2002, Cisinski 2007, Garner 2010, van den Berg–Garner 2011, Cheng 2011, Batanin–Cisinski–Weber 2013], Penon's definition [Penon 1999, Batanin 2002], and Trimble's definition [Trimble 1999, Cheng 2011]. The established method of ensuring that an n-globular operad gives rise to a suitably coherent definition of weak n-category is to use some sort of contraction on the operad. There are two approaches to this: a binary-biased approach due to Batanin (which also uses a system of compositions), and an unbiased approach due to Leinster; both of these can be applied algebraically (equipping an operad with a specified contraction) or non-algebraically (requiring a suitable contraction to exist). The aims of this paper are to formalise the relationship between the binary-biased and unbiased approaches, to compare the resulting definitions of weak n-category (specifically those of Batanin and Leinster), and to establish what coherence

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properties these contractions give rise to.

We recall the necessary preliminary definitions in Sections 2 and 3: in Section 2 we recall the definitions of generalised operads and their algebras, and in Section 3 we recall the operadic definitions of weak n-category due to Batanin and Leinster. In Section 4 we make precise the correspondence between operads with contractions and systems of compositions, and operads with unbiased contractions; specifically, we prove a conjecture of Leinster [Leinster 2004a, Section 10.1] stating that any operad with a contraction and system of compositions can be equipped with an unbiased contraction (the converse is already known [Leinster 2004a, Examples 10.1.2 and 10.1.4]).

In Section 5 we prove three coherence theorems for algebras for n-globular operads; these results are not surprising, but have not previously been proved. These theorems hold for the algebras for any n-globular operad equipped either with a contraction and system of compositions, or with an unbiased contraction. By the result from Section 4, for each theorem we can pick whichever notion is most convenient for the purposes of the proof.

In Sections 6, 7 and 8 we take several steps towards a comparison between Batanin weak n-categories and Leinster weak n-categories. It has been widely believed that these definitions are in some sense equivalent (see [Leinster 2004b, end of Section 4.5]), but no attempt to formalise this statement has been made. In Section 6 we derive comparison functors between the categories of Batanin weak n-categories and Leinster weak n-categories, and discuss how close these functors are to being equivalences of categories. One of these comparison functors is canonical, and in Section 7 we construct its left adjoint, thus giving a canonical adjunction between the categories of Batanin weak n-categories and Leinster weak n-categories; this construction is valid in much greater generality, as noted in the section. In Section 8 we investigate what happens when we take a Leinster weak n-category and apply first the comparison functor to the category of Batanin weak n-categories, then the comparison functor back to the category of Leinster weak n-categories. We believe that the Leinster weak n-category we obtain is in some sense equivalent to the one with which we started, and take a preliminary step towards formalising this statement.

Notation and terminology. Throughout this paper, the letter n always denotes a fixed natural number, which is assumed to be the highest dimension of cell in the definition(s) of weak n-category being discussed. All definitions in this paper are of the n-dimensional case, but it is straightforward and well-established how to modify the definitions to the ω-dimensional case [Batanin, 1998, Leinster 1998]. The results in Section 5 are mostly not applicable in the ω-dimensional case, since most of the coherence theorems concern behaviour of cells at dimension n (for example, stating that certain diagrams of n-cells commute).

All of the definitions of weak n-category in this paper use n-globular sets as their underlying data. An n-globular set is a presheaf on the *n-globe category* $\mathbb{G}$, which is defined as the category with

- objects: natural numbers $0, 1, \ldots, n-1, n$;

- morphisms generated by, for each $1 \leq m \leq n$, morphisms

$$\sigma_m, \tau_m \colon (m-1) \to m$$

such that $\sigma_{m+1}\sigma_m = \tau_{m+1}\sigma_m$ and $\sigma_{m+1}\tau_m = \tau_{m+1}\tau_m$ for $m \geq 2$ (called the "globularity conditions").

For an n-globular set $X \colon \mathbb{G}^{\mathrm{op}} \to \mathbf{Set}$, we write s for $X(\sigma_m)$, and t for $X(\tau_m)$, regardless of the value of m, and refer to them as the *source* and *target* maps respectively. We denote the set $X(m)$ by X_m. We say that two m-cells x, $y \in X_m$ are *parallel* if $s(x) = s(y)$ and $t(x) = t(y)$; note that all 0-cells are considered to be parallel. We write n-$\mathbf{GSet}$ for the category of n-globular sets $[\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$.

Finally, for any monad K, we denote its unit by $\eta^K \colon 1 \Rightarrow K$ and its multiplication by $\mu^K \colon K^2 \Rightarrow K$.

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2. Generalised operads

In this section we recall the definitions of generalised operads and their algebras. The material in this section originates in [Leinster 2004a], with the special case of n-globular operads originating in [Batanin, 1998].

A classical operad has a set of operations, each equipped with an arity: a natural number which is to be thought of as the number of inputs that the operation has. In the definition of generalised operad, we replace $\mathbf{Set}$ with any category $\mathcal{C}$ that has all pullbacks and a terminal object, denoted 1. The arities of the generalised operad are then generated by applying a suitably well-behaved monad T to the terminal object, giving an "object of arities" $T1$ in $\mathcal{C}$; hence such a generalised operad is called a "T-operad". Before giving the definition of T-operad, we must first state formally what it means for T to be "suitably well-behaved".

2.1. Definition. *A category is said to be* cartesian *if it has all pullbacks. A functor is said to be* cartesian *if it preserves pullbacks. A natural transformation is said to be* cartesian *if all of its naturality squares are pullback squares. A map of monads is said to be* cartesian *if its underlying natural transformation is cartesian. A monad is said to be* cartesian *if its functor part is a cartesian functor and its unit and counit are cartesian natural transformations.*

We now recall the definition of T-collections, the underlying data for T-operads.

2.2. Definition. *Let $\mathcal{C}$ be a cartesian category with a terminal object* 1*, and let T be a cartesian monad on $\mathcal{C}$. The* category of T-collections *is the slice category $\mathcal{C}/T1$.*

*We obtain from $\mathcal{C}/T1$ the monoidal category of collections T-**Coll** by equipping it with the following tensor product: let $k\colon K \to T1$, $k'\colon K' \to T1$ be collections; then their tensor product is defined to be the composite along the top of the diagram*

$$\begin{tikzcd} K \otimes K' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TK' \arrow[r, "Tk'"] \arrow[d, "T!"] & T^2 1 \arrow[r, "\mu_1^T"] & T1 \\ K \arrow[r, "k"'] & T1 & & \end{tikzcd}$$

where ! is the unique map $K' \to 1$ in $\mathcal{C}$ (since 1 is terminal). The unit for this tensor product is the collection

$$\begin{tikzcd} 1 \arrow[d, "\eta_1^T"] \\ T1. \end{tikzcd}$$

*We will be particularly interested in the case in which $\mathcal{C} = n$-**GSet**, and T is the free strict n-category monad, as this is the case that gives n-globular operads. In this case $T1$ is the n-globular set whose elements are globular pasting diagrams, and a T-collection is called an* n-globular collection*. We write n-**Coll** for the monoidal category of n-globular collections.*

We now give the definition of a T-operad.

2.3. Definition. *Let $\mathcal{C}$ be a cartesian category with a terminal object* 1*, and let T be a cartesian monad on $\mathcal{C}$. A* T-operad *is a monoid in the monoidal category T-**Coll**.*

*In the case in which $\mathcal{C} = n$-**GSet**, and T is the free strict n-category monad, a T-operad is called an* n-globular operad*.*

For brevity, we will often refer to "a T-operad K" when we really mean a T-operad with underlying T-collection $K \xrightarrow{k} T1$, unit map η^K and multiplication map μ^K.

In an n-globular operad, each operation has a pasting diagram as its arity, and should be thought of as a way of composing a diagram of cells of that shape. Since n-globular operads are the only kind of operads used in this paper, we will often refer to them simply as "operads".

The algebras for a T-operad are the algebras for a particular induced monad, which we now define.

2.4. Definition. *Let $\mathcal{C}$ be a cartesian category with a terminal object* 1*, let T be a cartesian monad on $\mathcal{C}$ and let K be a T-operad. Then there is an induced monad on $\mathcal{C}$, which by abuse of notation we denote (K, η^K, μ^K) (so the endofunctor part of the monad is denoted by the same letter as the underlying n-globular set of the operad, and we use* *the same notation for the unit and multiplication of the monad as we do for those of the operad). The endofunctor*

$$K\colon \mathcal{C} \to \mathcal{C}$$

is defined as follows: on objects, given an object X in $\mathcal{C}$, KX is defined by the pullback:

$$\begin{tikzcd} KX \arrow[r, "K!"] \arrow[d, "k_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & K \arrow[d, "k"] \\ TX \arrow[r, "T!"'] & T1, \end{tikzcd}$$

where $!$ is the unique morphism $X \to 1$ in $\mathcal{C}$; on morphisms, given a morphism $u\colon X \to Y$ in $\mathcal{C}$, Ku is defined to be the unique map induced by the universal property of the pullback defining KY such that the diagram

$$\begin{tikzcd} KX \arrow[r, dashed, "Ku"] \arrow[d, "k_X"'] \arrow[rr, bend left, "K!"] \arrow[dr, phantom, "\lrcorner", very near start] & KY \arrow[r, "K!"] \arrow[d, "k_Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & K \arrow[d, "k"] \\ TX \arrow[r, "Tu"'] \arrow[rr, bend right, "T!"'] & TY \arrow[r, "T!"'] & T1 \end{tikzcd}$$

commutes. Observe that commutativity of the left-hand square in the diagram above shows that k is a natural transformation $K \Rightarrow T$; the fact that this square is a pullback square shows that this natural transformation is cartesian.

The unit map $\eta^K\colon 1 \Rightarrow K$ for the monad K has, for each $X \in \mathcal{C}$, a component $\eta^K_X\colon X \to KX$ which is the unique map such that the diagram

$$\begin{tikzcd} X \arrow[rr, "!"] \arrow[dr, dashed, "\eta^K_X"] \arrow[ddr, bend right, "\eta^T_X"'] & & 1 \arrow[d, "\epsilon"] \arrow[dd, bend left, "\eta^T_1"] \\ & KX \arrow[r, "K!"] \arrow[d, "k_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & K \arrow[d, "k"] \\ & TX \arrow[r, "T!"'] & T1, \end{tikzcd}$$

commutes.

The multiplication map $\mu^K\colon K^2 \Rightarrow K$ for the monad K has, for each object X in $\mathcal{C}$, a *component* $\mu^K_A \colon K^2X \to KX$ *which is the defined to be unique map such that the diagram*

$$
\begin{tikzcd}
 & & & K^2X \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45] \arrow[dddd, dashed, "\mu^K_X"] & & & \\
 & & K \otimes K \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45] \arrow[dddd, bend right=60, "\mu^K"'] & & TKX \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45] & & \\
 & K \arrow[dr, "k"'] & & TK \arrow[dl, "T!"] \arrow[dr, "Tk"'] & & T^2X \arrow[dl, "T^2!"] \arrow[dddl, bend left=40, "\mu^T_X"] & \\
 & & T1 & & T^21 & & \\
 & & & KX \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45] & & & \\
 & & K \arrow[dr, "k"'] & & TX \arrow[dl, "T!"] & & \\
 & & & T1 & & &
\end{tikzcd}
$$

commutes.

2.5. Definition. *Let $\mathcal{C}$ be a cartesian category with a terminal object* 1*, let T be a cartesian monad on $\mathcal{C}$ and let K be a T-operad. An* algebra *for the operad K, referred to as a K-*algebra*, is defined to be an algebra for the induced monad (K, η^K, μ^K). Similarly, a* map of algebras for the T-operad K *is a map of algebras for the induced monad, and the category of algebras for the T-operad K is K-$\mathbf{Alg}$, the category of algebras for the induced monad.*

3. Weak n-categories

Throughout the rest of this paper we are concerned only with the case of n-globular operads, so, from here onwards, we will let $\mathcal{C} = n$-$\mathbf{GSet}$, T will denote the free strict n-category monad, and 1 denotes the terminal n-globular set, which has precisely one m-cell for each $0 \leq m \leq n$. This is the monad induced by the adjunction

$$
\begin{tikzcd}
n\text{-}\mathbf{GSet} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & n\text{-}\mathbf{Cat}, \arrow[l, shift left]
\end{tikzcd}
$$

where n-$\mathbf{Cat}$ is the category of strict n-categories, and the right adjoint is the forgetful functor sending a strict n-category to its underlying n-globular set.

In this section we recall two operadic definitions of weak n-category: Batanin weak n-categories, originally defined in [Batanin, 1998], and Leinster weak n-categories, a variant of Batanin's definition originating in [Leinster 1998].

We recall Batanin's definition first. In order to identify an appropriate operad to use, Batanin's approach is to define two pieces of extra structure on an operad:

- a system of compositions: this picks out binary composition operations at each dimension;
- a contraction on the underlying collection: this ensures that we have contraction operations which give the constraint cells in algebras for the operad; it also ensures that composition is strict at dimension n.

Operads equipped with contractions and systems of compositions form a category, and this category has an initial object; a Batanin weak n-category is defined to be an algebra for this initial operad.

In fact, the approach described here is slightly different from that of [Batanin, 1998], in which Batanin uses contractible operads rather than operads equipped with a specified contraction. Since contractibility is non-algebraic, there is no initial object in the category of contractible operads with systems of compositions, so Batanin explicitly constructs an operad that is weakly initial in this category. He then states that, if we use specified contractions, this operad is initial [Batanin, 1998, Section 8, Remark 2], so the operad we describe is the same as Batanin's, even though the approach is slightly different. Note that this alternative approach is completely standard (see, for example, [Leinster 2002]).

We begin by defining what it means for an operad to be equipped with a system of compositions. To do this, we define a collection that contains precisely one binary composition operation for each dimension of cell and boundary; in order for the sources and targets of these operations to be well-defined, it also contains a unary operation (i.e. one whose arity is a single globular cell) at each dimension. A map from this collection into the underlying collection of an operad then picks out the desired binary composition operations in that operad.

3.1. Definition. *Let $0 \le m \le n$, and write $\eta_m := \eta^T_m(1)$, the single m-cell in the image of the unit map $\eta^T \colon 1 \to T1$. Define, for $0 \le p \le m \le n$,*

$$\beta^m_p = \begin{cases} \eta_m & \text{if } p = m, \\ \eta_m \circ^m_p \eta_m & \text{if } p < m, \end{cases}$$

where $\circ^m_p$ denotes composition of m-cells along boundary p-cells. Define an n-globular collection $S \xrightarrow{s} T1$, in which

$$S_m := \{\beta^m_p \mid 0 \le p \le m\} \subseteq T1_m,$$

and s is the inclusion function; define a unit map $\eta^S \colon 1 \to S$ by $\eta^S_m(1) = \beta^m_m$.

Let $K \xrightarrow{k} T1$ be an n-globular operad. A system of compositions *on K consists of a map of collections*

$$\begin{tikzcd} S \arrow[rr, "\sigma"] \arrow[dr, "s"'] & & K \arrow[dl, "k"] \\ & T1 & \end{tikzcd}$$

such that the diagram

$$\begin{tikzcd} 1 \arrow[r, "\eta^S"] \arrow[rr, bend right, "\eta^K"'] & S \arrow[r, "\sigma"] & K \end{tikzcd}$$

commutes.

We now define what it means for an operad to be equipped with a contraction. Informally, this means that any two parallel operations of the same arity should be related – either by a mediating cell at the dimension above, or by an equality if they are at the highest dimension. Since this does not require the operad structure, the notion of contraction is defined on n-globular collections.

Let $K \xrightarrow{k} T1$ be an n-globular collection. We will define, for each globular pasting diagram π, a set $C_K(\pi)$ whose elements are parallel pairs of cells in K, the first of which maps to the source of π under k, and the second of which maps to the target of π under k. When $\pi = \mathrm{id}_\alpha$ for some $\alpha \in T1$, we can think of $C_K(\pi)$ as a set of contraction cells living over π, since every such pair requires a contraction cell for there to be a contraction on the map k. (We will use all pasting diagrams π in $T1$, not just those of the form $\pi = \mathrm{id}_\alpha$ for some $\alpha \in T1$, later, in Definition 3.5.)

To define $C_K(\pi)$, we first define, for all $0 \leq m \leq n$, $x \in T1_m$, a set

$$K(x) = \{a \in K_m \mid k(a) = x\};$$

that is, the preimage of x under k. Then, for all $1 \leq m \leq n$, $\pi \in T1_m$, we define

$$C_K(\pi) = \begin{cases} K(s(\pi)) \times K(t(\pi)) & \text{if } m = 1, \\ \{(a,b) \in K(s(\pi)) \times K(t(\pi)) \mid s(a) = s(b), t(a) = t(b)\} & \text{if } m > 1. \end{cases}$$

3.2. Definition. *A* contraction γ *on an n-globular collection* $K \xrightarrow{k} T1$ *consists of, for all $1 \leq m \leq n$, and for each $\alpha \in (T1)_{m-1}$, a function*

$$\gamma_{\mathrm{id}_\alpha} \colon C_K(\mathrm{id}_\alpha) \to K(\mathrm{id}_\alpha)$$

such that, for all $(a,b) \in C_K(\mathrm{id}_\alpha)$,

$$s\gamma_{\mathrm{id}_\alpha}(a,b) = a,\ t\gamma_{\mathrm{id}_\alpha}(a,b) = b$$

We also require the following "tameness" condition (terminology due to Leinster [Leinster 2004a, Definition 9.3.1]): for α, $\beta \in K_n$, if

$$s(\alpha) = s(\beta),\ t(\alpha) = t(\beta),\ k(\alpha) = k(\beta),$$

then $\alpha = \beta$.

Operads with contractions and systems of compositions form a category, which we now define.

3.3. Definition. *Define* **OCS** *to be the category with*

- *objects: operads* $K \xrightarrow{k} T1$ *equipped with a contraction* γ *and a system of compositions* $\sigma\colon S \to K$*;*
- *morphisms: for operads* $K \xrightarrow{k} T1$ *,* $K' \xrightarrow{k'} T1$ *, respectively equipped with contraction* γ*,* γ'*, and systems of compositions* σ*,* σ'*, a morphism* $u\colon K \to K'$ *consists of a map* u *of the underlying operads such that*
  - *the diagram*

$$\begin{tikzcd} & S \arrow[dl, "\sigma"'] \arrow[dr, "\sigma'"] & \\ K \arrow[rr, "u"'] & & K' \end{tikzcd}$$

    commutes;
  - *for all* $1 \le m \le n$*,* $\alpha \in T1_{m-1}$*,* $(a,b) \in C_K(\mathrm{id}_\alpha)$*,*

$$u_m(\gamma_{\mathrm{id}_\alpha}(a,b)) = \gamma'_{\mathrm{id}_\alpha}(u_{m-1}(a), u_{m-1}(b)).$$

We often refer to an operad with a contraction and system of compositions simply as a *Batanin operad.* The category **OCS** has an initial object, denoted

$$\begin{tikzcd} B \arrow[d, "b"] \\ T1. \end{tikzcd}$$

This initial object is in some sense the "simplest" operad in **OCS**. It has precisely the operations required to have a system of compositions, a contraction, and an operad structure, and no more; furthermore, it has no spurious relations between these operations.

3.4. Definition. *A Batanin weak* n*-category is an algebra for the* n*-globular operad* $B \xrightarrow{b} T1$ *. The category of Batanin weak* n*-categories is* B*-***Alg***.*

Note that the presence of a system of compositions and a contraction on an operad does not affect the category of algebras for that operad. The algebras depend only on the operad itself; systems of compositions and contractions are used purely as a tool for making an appropriate choice of operad.

We now recall Leinster's variant of Batanin's definition of weak n-category [Leinster 1998]. The key distinction between Leinster's variant and Batanin's original definition is that, rather than using a contraction and system of compositions, Leinster ensures the existence of both composition operations and contraction operations using a single piece of extra structure, called an "unbiased contraction" (note that Leinster simply uses the term "contraction" for this concept, and uses the term "coherence" for Batanin's contractions).

An unbiased contraction on an operad lifts all cells from $T1$, not just identity cells as in a contraction. As well as giving the usual constraint cells, an unbiased contraction gives a composition operation for each non-identity cell in $T1$. Thus for any globular pasting diagram there is an operation, specified by the unbiased contraction, which we think of as telling us how to compose a pasting diagram of that shape "all at once". Consequently, when using unbiased contractions we have no need for a system of compositions. Operads equipped with unbiased contractions form a category, and this category has an initial object; a Leinster weak n-category is defined to be an algebra for this initial operad.

3.5. Definition. *An* unbiased contraction γ *on an n-globular collection*

$$K \xrightarrow{k} T1$$

consists of, for all $1 \leq m \leq n$, and for each $\pi \in T1_m$, a function

$$\gamma_\pi \colon C_K(\pi) \to K(\pi)$$

such that, for all $(a, b) \in C_K(\pi)$,

$$s\gamma_\pi(a, b) = a, \; t\gamma_\pi(a, b) = b.$$

We also require that, for α, $\beta \in K_n$, if

$$s(\alpha) = s(\beta), \; t(\alpha) = t(\beta), \; k(\alpha) = k(\beta),$$

then $\alpha = \beta$.

3.6. Definition. *Define* **OUC** *to be the category with*

- *objects: operads $K \xrightarrow{k} T1$ equipped with an unbiased contraction γ;*
- *morphisms: for operads $K \xrightarrow{k} T1$, $K' \xrightarrow{k'} T1$, respectively equipped with unbiased contractions γ, γ', a morphisms $u \colon K \to K'$ consists of a map of the underlying operads such that, for all $1 \leq m \leq n$, $\pi \in (T1)_m$, $(a, b) \in C_K(\pi)$,*

$$u_m(\gamma_\pi(a, b)) = \gamma'_\pi(u_{m-1}(a), u_{m-1}(b)).$$

We often refer to an operad with an unbiased contraction simply as a *Leinster operad*.

3.7. Lemma. *The category* **OUC** *has an initial object, denoted $L \xrightarrow{l} T1$.*

This lemma was originally proved by Leinster in his thesis [Leinster 2004b]; an explicit construction of $L \xrightarrow{l} T1$ is given by Cheng in [Cheng 2010].

3.8. Definition. *A Leinster weak n-category is an algebra for the n-globular operad $L \xrightarrow{l} T1$. The category of Leinster weak n-categories is L-**Alg**.*

4. The relationship between Batanin operads and Leinster operads

We now discuss the relationship between the contractions and systems of compositions used by Batanin, and the unbiased contractions used by Leinster. We recall the following theorem of Leinster [Leinster 2004a, Examples 10.1.2 and 10.1.4]:

4.1. THEOREM. *Let K be an n-globular operad with unbiased contraction γ. Then K can be equipped with a contraction and a system of compositions in a canonical way.*

This tells us that every Leinster operad can be given the structure of a Batanin operad in a canonical way. In this section we prove the converse of this, a conjecture of Leinster [Leinster 2004a, Section 10.1]:

4.2. THEOREM. *Let K be an n-globular operad with contraction γ and system of compositions σ. Then K can be equipped with an unbiased contraction.*

The proof consists of picking a binary bracketing for each pasting diagram in $T1$, then composing these bracketings with contraction cells to obtain unbiased contraction cells with the correct sources and targets. We have to make arbitrary choices of bracketings during this process, so there is no canonical way doing this.

Since the algebras for an operad are not affected by the choice of system of compositions, contraction, or unbiased contraction, one consequence of these theorems is that any result that holds for algebras for a Batanin operad also holds for algebras for a Leinster operad (and vice versa). We use this fact in Section 5 to prove several coherence theorems that are valid for the algebras for any Batanin operad or Leinster operad, whilst working with whichever notion is more technically convenient in the case of each proof.

Our approach to prove Theorem 4.2 is as follows: first, we define a map $\hat{k}\colon T1 \to K$, which uses the contraction on k to lift identity cells in $T1$, and picks a binary bracketing for each non-identity cell. This bracketing is constructed using the system of compositions on K; the choice of bracketing is arbitrary. To extend this to an unbiased contraction on k we need to specify, for all $1 \leq m \leq n$, and for each $\pi \in T1_m$ and $(a, b) \in C_K(\pi)$, an unbiased contraction cell

$$\gamma_\pi(a, b)\colon a \longrightarrow b.$$

To obtain this unbiased contraction cell we start with the cell $\hat{k}(\pi)$; since $\hat{k}$ is a section to k this cell maps to π under k, but in general it does not have the desired source and target. In order to obtain a cell with source a and target b we compose $\hat{k}(\pi)$ with contraction cells, first composing $\hat{k}(\pi)$ with contraction 1-cells to obtain a cell with the desired source and target 0-cells, then composing the resulting cell with contraction 2-cells to obtain a cell with the desired source and target 1-cells, and so on; this composition is performed using the system of compositions on K. The resulting cell has the desired source and target and, since contraction cells map to identities under k, and $\hat{k}$ is a section to k, this cell maps to π under k.

4.3. Lemma. *Let K be an n-globular operad with contraction γ and system of compositions σ. Then k has a section $\hat{k}\colon T1 \to K$ in n-**GSet**, so for all $0 \leq m \leq n$, $k_m\colon K_m \to T1_m$ is surjective.*

Proof. Our approach is first to define $\hat{k}$, then show it is a section to k and therefore each k_m is surjective. To define $\hat{k}\colon T1 \to K$, we use a description of $T1$ due to Leinster [Leinster 2004a, Section 8.1]. For a set X, write X^* for the underlying set of the free monoid on X (so X^* is the set of all finite strings of elements of X, including the empty string, which we write as $\emptyset$). Define $T1$ inductively as follows:

- $T1_0 = 1$;
- for $1 \leq m \leq n$, $T1_m = T1^*_{m-1}$.

The source and target maps are defined as follows:

- for $m = 1$, $s = t = !\colon T1_m \to T1_0$;
- for $m > 1$, $s = t\colon T1_m \to T1_{m-1}$ is defined by, for $(\pi_1, \pi_2, \ldots, \pi_i) \in T1_m$,

$$s(\pi_1, \pi_2, \ldots, \pi_i) = (s(\pi_1), s(\pi_2), \ldots, s(\pi_i)).$$

This description of $T1$ is technically convenient, but it hides what is going on conceptually. The element $(\pi_1, \pi_2, \ldots, \pi_i)$ of $T1_m$ should not be visualised as a string of $(m-1)$-cells; instead, we increase the dimension of each cell in each π_i by 1, then compose $\pi_1, \pi_2, \ldots, \pi_i$ along their boundary 0-cells. So the element

$$\underbrace{(\bullet, \bullet, \ldots, \bullet)}_{i}$$

of $T1_1$ should be thought of as

$$\underbrace{\bullet \longrightarrow \bullet \longrightarrow \bullet \cdots \bullet \longrightarrow \bullet,}_{i \text{ 1-cells}}$$

the element

$$(\emptyset, \bullet \longrightarrow \bullet \longrightarrow \bullet, \bullet \longrightarrow \bullet)$$

of $T1_2$ should be thought of as

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\Downarrow"] \arrow[r, phantom, shift right=3, "\Downarrow"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet, \end{tikzcd}$$

and so on.

We now define $\hat{k}\colon T1 \to K$ by defining its components $\hat{k}_m$, for $0 \leq m \leq n$, inductively over m. We use the following notational abbreviations:

- for each m we write η_m for the m-cell $\sigma_m(\beta^m_m) = \eta^K_m(1)$ of K (recall from Definition 3.1 that $\beta^m_m = \eta^T_m(1)$ for all m);
- for $m \geq 1$ we write id_m for the identity m-cell on η_0. Recall that identity cells in K are defined via the contraction on k, so id_m is defined inductively over m as follows:
  - when $m = 1$, $\mathrm{id}_m := \gamma_\emptyset(1, 1)$;
  - when $m > 1$, $\mathrm{id}_m := \gamma_\emptyset(\mathrm{id}_{m-1}, \mathrm{id}_{m-1})$.

We also denote binary composition of m-cells along p-cells, defined using the system of compositions on K, by $\circ^m_p$.

When $m = 0$, define

$$\hat{k}_0(\bullet) = \eta_0.$$

When $1 \leq m \leq n$ the construction becomes notationally complicated, so we first describe it by example in the cases $m = 1$, 2.

When $m = 1$, by the construction of $T1$ above, an element of $T1_m$ is a string

$$\underbrace{(\bullet, \bullet, \ldots, \bullet)}_{i}$$

for some natural number i. When $i = 0$, define

$$\hat{k}_1(\emptyset) = \mathrm{id}_1.$$

When $i \geq 1$, there are three steps to the construction of $\hat{k}_1$. First, we apply $\hat{k}_0$ to all elements in the string, which gives

$$\underbrace{(\eta_0, \eta_0, \ldots, \eta_0)}_{i},$$

a string of 0-cells in K. Now, we add 1 to the dimension of each cell in the string by replacing each instance of η_0 with η_1, which gives

$$\underbrace{(\eta_1, \eta_1, \ldots, \eta_1)}_{i},$$

a string of 1-cells in K. Finally, we compose these 1-cells along boundary 0-cells, using the system of compositions on K, with the bracketing on the left. Thus, for example, in the case of $i = 4$, we obtain

$$\hat{k}_1(\bullet, \bullet, \bullet, \bullet) := \big(\big(\eta_1 \circ^1_0 \eta_1\big) \circ^1_0 \eta_1\big) \circ^1_0 \eta_1.$$

When $m = 2$, an element of $T1_m$ is a string of elements of $T1_1$

$$\pi = (\pi_1, \pi_2, \ldots, \pi_i),$$

for some natural number i. When $i = 0$, define

$$\hat{k}_2(\emptyset) = \mathrm{id}_2.$$

For the case $i \geq 1$, we explain with reference to the example

$$\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[r, bend left=60, ""{name=A, below}] \arrow[r, ""{name=B, above}, ""{name=C, below}] \arrow[r, bend right=60, ""{name=D, above}] \arrow[from=A, to=B, phantom, "\Downarrow"] \arrow[from=C, to=D, phantom, "\Downarrow"] & \bullet \arrow[r, bend left=45] \arrow[r, bend right=45] \arrow[r, phantom, "\Downarrow"] & \bullet .
\end{tikzcd}$$

Recall that, as a string of elements of $T1_1$, this is written as

$$(\pi_1, \pi_2, \pi_3) = (\emptyset, \bullet \longrightarrow \bullet \longrightarrow \bullet, \bullet \longrightarrow \bullet).$$

As in the case $m = 1$, there are three steps to the construction of $\hat{k}_2(\pi_1, \pi_2, \pi_3)$. First, we apply $\hat{k}_1$ to all elements in the string, which gives

$$\left(\hat{k}_1(\pi_1), \hat{k}_1(\pi_2), \hat{k}_1(\pi_3)\right) = \left(\mathrm{id}_1, \eta_1 \circ_0^1 \eta_1, \eta_1\right).$$

In general each $\hat{k}_1(\pi_j)$ is either id_1 or a composite of η_1's. The next step is to add 1 to the dimension of each $\hat{k}_1(\pi_j)$ by replacing

- every instance of id_1 with id_2;
- every instance of η_1 with η_2;
- every instance of $\circ_0^1$ with $\circ_1^2$.

The cell we obtain from $\hat{k}_1(\pi_j)$ is denoted $\hat{k}_1^+(\pi_j)$. Thus our example becomes

$$\left(\hat{k}_1^+(\pi_1), \hat{k}_1^+(\pi_2), \hat{k}_1^+(\pi_3)\right) = \left(\mathrm{id}_2, \eta_2 \circ_1^2 \eta_2, \eta_2\right).$$

Finally, we compose these cells along boundary 0-cells, using the system of compositions on K, with the bracketing on the left. In our example, this gives

$$\hat{k}_2(\pi) := \left(\eta_2 \circ_1^2 \left(\eta_2 \circ_1^2 \eta_2\right)\right) \circ_0^2 \mathrm{id}_2.$$

We now describe the construction in general for $1 \leq m \leq n$. Suppose that we have defined $\hat{k}_{m-1}$ in such a way that, for all $\pi \in T1_{m-1}$, $\hat{k}_{m-1}(\pi)$ consists of a composite of copies of η_{m-1} and id_{m-1}, composed via operations of the form $\circ_p^{m-1}$ for some $0 \leq p < m-1$.

Let $(\pi_1, \pi_2, \ldots, \pi_i)$ be an element of $T1_m$. When $i = 0$, we define

$$\hat{k}_m(\pi_1, \pi_2, \ldots, \pi_i) = \hat{k}_m(\emptyset) = \mathrm{id}_m.$$

When $i \geq 1$ we define $\hat{k}_m(\pi_1, \pi_2, \ldots, \pi_i)$ in three steps, as described above. First, we apply $\hat{k}_{m-1}$ to each π_j to obtain

$$\left(\hat{k}_{m-1}(\pi_1), \hat{k}_{m-1}(\pi_2), \ldots, \hat{k}_{m-1}(\pi_i)\right).$$

Next, we obtain from each $\hat{k}_{m-1}(\pi_j)$ a cell $\hat{k}_{m-1}^+(\pi_j) \in K_m$ by replacing

- every instance of id_{m-1} with id_m;
- every instance of η_{m-1} with η_m;
- every instance of $\circ_p^{m-1}$, for all $0 \leq p < m-1$, with $\circ_{p+1}^m$.

This gives
$$\left(\hat{k}_{m-1}^+(\pi_1), \hat{k}_{m-1}^+(\pi_2), \ldots, \hat{k}_{m-1}^+(\pi_i)\right).$$
Finally, we compose these cells along boundary 0-cells, using the system of compositions on K, with the bracketing on the left. This gives

$$\begin{aligned}&\hat{k}_m(\pi_1, \pi_2, \ldots, \pi_i) := \\ &\left(\ldots\left(\hat{k}_{m-1}^+(\pi_i) \circ_0^m \hat{k}_{m-1}^+(\pi_{i-1})\right) \circ_0^m \cdots \circ_0^m \hat{k}_{m-1}^+(\pi_2)\right) \circ_0^m \hat{k}_{m-1}^+(\pi_1).\end{aligned}$$

This defines a map of n-globular sets $\hat{k}\colon T1 \longrightarrow K$.

We now show that $\hat{k}$ is a section to k. At dimension 0, $k_0\hat{k}_0 = \mathrm{id}_{T1_0}$ since $T1_0$ is terminal, so $\hat{k}_0$ is a section to k_0. Suppose we have shown that, for $1 \leq m \leq n$, $k_{m-1}\hat{k}_{m-1} = \mathrm{id}_{T1_{m-1}}$. For $\pi \in T1_{m-1}$,

$$k_m\hat{k}_{m-1}^+(\pi) = (\pi),$$

so for $(\pi_1, \pi_2, \ldots, \pi_i) \in T1_m$, we have

$$k_m\hat{k}_m(\pi_1, \pi_2, \ldots, \pi_i) = (\pi_1, \pi_2, \ldots, \pi_i),$$

as required. When $i = 0$,
$$k_m\hat{k}_m(\emptyset) = \emptyset.$$
Hence $\hat{k}$ is a section to k. ■

We now use the map $\hat{k}$ to define an unbiased contraction on $k\colon K \to T1$.

Proof of Theorem 4.2. We define an unbiased contraction δ on the operad K; that is, for all $1 \leq m \leq n$, and for each $\pi \in T1_m$, a function

$$\delta_\pi\colon C_K(\pi) \to K(\pi)$$

such that, for all $(a, b) \in C_K(\pi)$,

$$s\delta_\pi(a, b) = a,\ t\delta_\pi(a, b) = b.$$

To make the construction easier to follow, we first present the cases $m = 1$ and $m = 2$ separately, before giving the construction for general m. Throughout the construction, we use the map $\hat{k}\colon T1 \to K$ defined in the proof of Lemma 4.3, which we showed to be a section to $k\colon K \to T1$.

Let $m = 1$, let $\pi \in T1_m = T1_1$, and let $(a, b) \in C_K(\pi)$. If $\pi = \mathrm{id}_\alpha$ for some $\alpha \in T1_0$ we already have a corresponding contraction cell from the contraction γ on k, so we define

$$\delta_\pi(a, b) := \gamma_{\mathrm{id}_\alpha}(a, b).$$

Now suppose that $\pi \neq \mathrm{id}_\alpha$ for any $\alpha \in T1_0$. We seek a 1-cell

$$\delta_\pi(a, b) \colon a \longrightarrow b$$

in K such that $k_1\delta_\pi(a, b) = \pi$. We have a 1-cell $\hat{k}_1(\pi)$ in K, and since $\hat{k}$ is a section to k, we have

$$k_1\hat{k}_1(\pi) = \pi.$$

However, $\hat{k}_1(\pi)$ does not necessarily have the required source and target. In order to obtain a cell with the desired source and target, we first observe that

$$k_1 s\hat{k}_1(\pi) = sk_1\hat{k}_1(\pi) = s(\pi)$$

and

$$k_1 t\hat{k}_1(\pi) = tk_1\hat{k}_1(\pi) = t(\pi).$$

Thus, from the contraction γ, we have contraction 1-cells

$$\gamma_{\mathrm{id}_{k_0(a)}}(a, s\hat{k}_1(\pi)) \colon a \longrightarrow s\hat{k}_1(\pi)$$

and

$$\gamma_{\mathrm{id}_{k_0(b)}}(t\hat{k}_1(\pi), b) \colon t\hat{k}_1(\pi) \longrightarrow b$$

in K. Thus in K we have composable 1-cells

$$a \dashrightarrow \bullet \xrightarrow{\hat{k}_1(\pi)} \bullet \dashrightarrow b,$$

where the dashed arrows denote the contraction cells. We define the contraction cell $\delta_\pi(a, b)$ to be given by a composite of these cells; as in the definition of $\hat{k}$, we bracket this composite on the left, so

$$\delta_\pi(a, b) := \Big(\gamma_{\mathrm{id}_{k(b)}}(t\hat{k}(\pi), b) \circ^1_0 \hat{k}(\pi)\Big) \circ^1_0 \gamma_{\mathrm{id}_{k(a)}}(a, s\hat{k}(\pi)).$$

Since k maps the contraction cells to identities and $\hat{k}_1(\pi)$ to π, and since in K the arity of a composite is the composite of the arities, we have

$$k\delta_\pi(a, b) = \pi,$$

as required. This defines the unbiased contraction δ on $k \colon K \to T1$ at dimension 1.

Before defining δ for $m = 2$ or for general m, we establish some notation. For repeated application of source and target maps in K, we write

$$s^p := \underbrace{s \circ s \circ \cdots \circ s}_{p \text{ times}}, \; t^p := \underbrace{t \circ t \circ \cdots \circ t}_{p \text{ times}},$$

so for $1 \leq p < m \leq n$, and for an m-cell α of K, $s^p(\alpha)$ is the source $(m-p)$-cell of α, and $t^p(\alpha)$ is the target $(m-p)$-cell of α. For all $m < l \leq n$, we write $\mathrm{id}^l\alpha$ for the identity l-cell on α; so, for example,

$$\mathrm{id}^{m+1}\alpha = \mathrm{id}_\alpha,\ \mathrm{id}^{m+2}\alpha = \mathrm{id}_{\mathrm{id}_\alpha},$$

and so on.

Now let $m = 2$, let $\pi \in T1_m = T1_2$, and let $(a, b) \in C_K(\pi)$. As in the case $m = 1$, if $\pi = \mathrm{id}_\alpha$ for some $\alpha \in T1_1$, define

$$\delta_\pi(a, b) := \gamma_{\mathrm{id}_\alpha}(a, b)$$

for all $(a, b) \in C_K(\pi)$.

Now suppose that $\pi \neq \mathrm{id}_\alpha$ for any $\alpha \in T1_1$. We seek a 2-cell

$$\delta_\pi(a, b) \colon a \Longrightarrow b$$

in K such that $k_2\delta_\pi(a, b) = \pi$. We have a 2-cell $\hat{k}_2(\pi)$ in K, and since $\hat{k}$ is a section to k, we have

$$k_2\hat{k}_2(\pi) = \pi.$$

However, $\hat{k}_2(\pi)$ does not necessarily have the required source and target cells at any dimension. We construct $\delta_\pi(a, b)$ from $\hat{k}_2(\pi)$ in two stages: first we compose with contraction 1-cells to obtain a 2-cell with the required source and target 0-cells, then we compose this with contraction 2-cells to obtain a 2-cell with the required source and target 1-cells. To obtain a cell with the required source and target 0-cells, observe that, since $T1_0$ is terminal,

$$ks(a) = ks^2\hat{k}(\pi)$$

and

$$kt(b) = kt^2\hat{k}(\pi).$$

Thus, from the contraction γ, we have contraction 1-cells

$$\gamma_{\mathrm{id}_1}(s(a), s^2\hat{k}(\pi)) \colon s(a) \longrightarrow s^2\hat{k}(\pi)$$

and

$$\gamma_{\mathrm{id}_1}(t^2\hat{k}(\pi), t(b)) \colon t^2\hat{k}(\pi) \longrightarrow t(b)$$

in K. Thus we have the following composable diagram of cells in K:

$$\begin{tikzcd} s(a) \arrow[r, dashed] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \hat{k}_2(\pi)"] & \bullet \arrow[r, dashed] & t(b), \end{tikzcd}$$

where the dashed arrows denote identity 2-cells on the contraction cells mentioned above. We compose this diagram to obtain a 2-cell in K with the required source and target 0-cells, which we denote $\delta^0_\pi(a, b)$. Formally, this is defined by

$$\delta^0_\pi(a, b) := \left(\mathrm{id}^2\gamma_{\mathrm{id}_1}(t^2\hat{k}(\pi), t(b)) \circ^2_0 \hat{k}(\pi)\right) \circ^2_0 \mathrm{id}^2\gamma_{\mathrm{id}_1}(s(a), s^2\hat{k}(\pi)).$$

As before, we bracket this composite on the left, though this choice is arbitrary.

We now repeat this process at dimension 2 to obtain a cell with the required source and target 1-cells. We have

$$s(a) = s(b) = s^2\delta^0_\pi(a,b)$$

and

$$t(a) = t(b) = t^2\delta^0_\pi(a,b),$$

so we have contraction 2-cells

$$\gamma_{\mathrm{id}_{k(a)}}(a, s\delta^0_\pi(a,b))\colon a \Longrightarrow s\delta^0_\pi(a,b)$$

and

$$\gamma_{\mathrm{id}_{k(b)}}(t\delta^0_\pi(a,b), b)\colon t\delta^0_\pi(a,b) \Longrightarrow b$$

in K. Thus we have the following composable diagram of cells in K:

$$\begin{tikzcd}
s(a) \arrow[r, dashed] \arrow[rrr, bend left=60, "a"] \arrow[rrr, bend right=60, "b"'] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \hat{k}_2(\pi)"] & \bullet \arrow[r, dashed] & t(b),
\end{tikzcd}$$

where the dashed arrows denote contraction cells. We compose this diagram to obtain the unbiased contraction cell $\delta_\pi(a,b)$ in K. Formally, this is defined by

$$\delta_\pi(a,b) := \Big(\gamma_{\mathrm{id}_{k(b)}}(t\delta^0_\pi(a,b), b) \circ^2_1 \delta^0_\pi(a,b)\Big) \circ^2_1 \gamma_{\mathrm{id}_{k(a)}}(a, s\delta^0_\pi(a,b)).$$

By construction, we see that $s\delta_\pi(a,b) = a$, $t\delta_\pi(a,b) = b$. As before, since k maps the contraction cells to identities and $\hat{k}_2(\pi)$ to π, and since in K the arity of a composite is the composite of the arities, we have

$$k\delta_\pi(a,b) = \pi,$$

as required. This defines the unbiased contraction δ on $k\colon K \to T1$ at dimension 2.

We now give the definition of δ for higher dimensions. Our approach is the same as that for dimensions 1 and 2; we build our contraction cells in stages, first constructing a cell with the desired source and target 0-cells, then constructing from that a cell with the desired source and target 1-cells, and so on.

Let $3 \leq m \leq n$, let $\pi \in T1_m$, and let $(a,b) \in C_K(\pi)$. If $\pi = \mathrm{id}_\alpha$ for some $\alpha \in T1_{m-1}$, we define

$$\delta_\pi(a,b) := \gamma_{\mathrm{id}_\alpha}(a,b).$$

Now suppose that $\pi \neq \mathrm{id}_\alpha$ for any $\alpha \in T1_{m-1}$. We seek an m-cell

$$\delta_\pi(a,b)\colon a \longrightarrow b$$

in K such that $k_m\delta_\pi(a,b) = \pi$. As before, we have an m-cell $\hat{k}_m(\pi)$ in K, and since $\hat{k}$ is a section to k, we have

$$k_m\hat{k}_m(\pi) = \pi.$$

However, $\hat{k}_m(\pi)$ does not necessarily have the required source and target cells at any dimension. We obtain a cell with the required source and target by defining, for each $0 \leq j \leq m-1$, an m-cell $\delta^j_\pi(a,b)$ which has the required source and target j-cells, and maps to π under k. We define this by induction over j. Note that, since this construction is very notation heavy, we henceforth omit subscripts indicating the dimensions of components of maps of n-globular sets, so we write k for k_m, $\hat{k}$ for $\hat{k}_m$, etc.

Let $j = 0$. Since $T1_0$ is the terminal set, we have

$$ks^{m-1}(a) = ks^m\hat{k}(\pi)$$

and

$$kt^{m-1}(b) = kt_m\hat{k}(\pi)$$

in K, so we have contraction 1-cells

$$\gamma_{\mathrm{id}_1}(s^{m-1}(a), s^m\hat{k}(\pi))$$

and

$$\gamma_{\mathrm{id}_1}(t^m\hat{k}(\pi), t_{m-1}(b))$$

in K. We obtain $\delta^0_\pi(a,b)$ by composing $\hat{k}(\pi)$ with the m-cell identities on these contraction cells, so we define

$$\delta^0_\pi(a,b) := \left(\mathrm{id}^m\gamma_{\mathrm{id}_1}(t^m\hat{k}(\pi), t_{m-1}(b)) \circ^m_0 \hat{k}(\pi)\right) \circ^m_0 \mathrm{id}^m\gamma_{\mathrm{id}_1}(s^{m-1}(a), s^m\hat{k}(\pi)).$$

By construction, we have

$$s^{m-1}(a) = s^{m-1}(b) = s^m\delta^0_\pi(a,b)$$

and

$$t^{m-1}(a) = t^{m-1}(b) = t^m\delta^0_\pi(a,b),$$

so this has the required source and target 0-cells. Since k sends contraction cells to identities, and since $\hat{k}$ is a section to k, we have

$$k\delta^0_\pi(a,b) = \pi.$$

Now let $0 \leq j < m-1$, and suppose we have defined $\delta^j_\pi(a,b)$ such that

$$s^{m-j-1}(a) = s^{m-j-1}(b) = s^{m-j}\delta^j_\pi(a,b),$$

$$t^{m-j-1}(a) = t^{m-j-1}(b) = t^{m-j}\delta_\pi^j(a,b),$$

so $\delta_\pi^j(a,b)$ has the required source and target j-cells, and

$$k\delta_\pi^j(a,b) = \pi.$$

Applying k to the source and target conditions above, we have

$$ks^{m-j-2}(a) = ks^{m-j-1}\delta_\pi^j(a,b)$$

and

$$kt^{m-j-2}(b) = kt^{m-j-1}\delta_\pi^j(a,b).$$

Thus we have contraction cells

$$\gamma_{\mathrm{id}_{ks^{m-j-2}(a)}}(s^{m-j-2}(a), s^{m-j-1}\delta_\pi^j(a,b)),$$

and

$$\gamma_{\mathrm{id}_{ks^{m-j-2}(b)}}(t^{m-j-1}\delta_\pi^j(a,b), t^{m-j-2}(b)).$$

in K. We obtain $\delta_\pi^{j+1}(a,b)$ by composing $\delta_\pi^j(a,b)$ with the m-cell identities on these contraction cells (or with the contraction cells themselves in the case $j+1=m$), so we define

$$\begin{aligned}\delta_\pi^{j+1}(a,b) := \Big(&\mathrm{id}^m\gamma_{\mathrm{id}_{ks^{m-j-2}(b)}}(t^{m-j-1}\delta_\pi^j(a,b), t^{m-j-2}(b)) \circ_{j+1}^m \delta_\pi^j(a,b)\Big) \\ &\circ_{j+1}^m \mathrm{id}^m\gamma_{\mathrm{id}_{ks^{m-j-2}(a)}}(s^{m-j-2}(a), s^{m-j-1}\delta_\pi^j(a,b)).\end{aligned}$$

By construction, we see that

$$s^{m-j-1}\delta_\pi^{j+1}(a,b) = s^{m-j-2}(a)$$

and

$$t^{m-j-1}\delta_\pi^{j+1}(a,b) = t^{m-j-2}(b),$$

so $\delta_\pi^{j+1}(a,b)$ has the required source and target $(j+1)$-cells. Since

$$k\delta_\pi^j(a,b) = \pi,$$

and k maps contraction cells to identities, we have

$$k\delta_\pi^{j+1}(a,b) = \pi.$$

This defines an m-cell $\delta_\pi^j(a,b)$ in K, for each $0 \le j \le m-1$, with the required source and target j-cells, and such that

$$k\delta_\pi^j(a,b) = \pi.$$

In particular, we have

$$\delta_\pi^{m-1}(a,b)\colon a \longrightarrow b.$$

Thus we define

$$\delta_\pi(a,b) := \delta_\pi^{m-1}(a,b).$$

This defines an unbiased contraction δ on the operad K, as required. ∎

Thus any operad with a contraction and system of compositions can be equipped with an unbiased contraction. In the proof above we had to make several arbitrary choices. Most of these involved picking a binary bracketing for a composite; we also chose to define the unbiased contraction to be the same as the original contraction on all cells for which this makes sense, which we did not have to do. There is no canonical choice in any of these cases, and thus no canonical way of equipping an operad in **OCS** with an unbiased contraction.

Note that various authors use variants of Batanin's definition in which a choice of n-globular operad is not specified, and instead a weak n-category is defined either to be an algebra for any operad that *can be* equipped with a contraction and system of compositions, or an algebra for any operad that *can be* equipped with an unbiased contraction ([Leinster 2002, Definitions B2 and L2], [Berger 2002, Garner 2010, van den Berg–Garner 2011, Cheng 2011]). By Theorems 4.1 and 4.2, these two "less algebraic" variants of Batanin's definition are equivalent, since any operad that can be equipped with a contraction and system of compositions can also be equipped with an unbiased contraction, and vice versa.

5. Coherence for algebras for n-globular operads

In this section we prove three new coherence theorems for algebras for any Batanin operad or Leinster operad K. Roughly speaking, our coherence theorems say the following:

- every free K-algebra is equivalent to a free strict n-category;
- every diagram of constraint n-cells commutes in a free K-algebra;
- in any K-algebra there is a certain class of diagrams of constraint n-cells that always commute; these should be thought of as the diagrams of shapes that can arise in a free algebra.

In the first two of these theorems freeness is crucial; these theorems do not hold in general for non-free K-algebras, so this does not mean that every weak n-category is equivalent to a strict one, which we know should not be true for $n \geq 3$ in a fully weak theory. All of these theorems have analogues in the case of tricategories, which appear in Gurski's thesis [Gurski 2006] and book [Gurski 2013] on coherence for tricategories; these are noted throughout the section. Note that there is no theorem corresponding to the coherence theorem for tricategories that states "every tricategory is triequivalent to a **Gray**-category" [Gordon–Power–Street 1995, Theorem 8.1], since we have no analogue of **Gray**-categories in this case. There are also no coherence theorems for maps of K-algebras, since there is no well-established notion of weak map of K-algebras.

These coherence theorems hold for the algebras for any Batanin operad or Leinster operad; hence, they hold for Batanin weak n-categories and Leinster weak n-categories. They also hold for the weak n-categories of Penon, since these are algebras for a Batanin operad [Batanin 2002, Theorem 3.1], and those of Trimble, since these are algebras for a Leinster operad [Cheng 2011, Theorem 4.8].

Note that, by Theorems 4.1 and 4.2, we need only prove each coherence theorem either in the case of algebras for a Batanin operad or algebras for a Leinster operad; thus in each case we use whichever of these is more technically convenient for the purposes of the proof. Throughout this section we write K to denote either a Batanin operad or Leinster operad (with the exception of Definition 5.1 and Proposition 5.2, in which a little more generality is possible).

Our first coherence theorem corresponds to the coherence theorem for tricategories stating that the free tricategory on a **Cat**-enriched 2-graph X is triequivalent to the free strict 3-category on X [Gurski 2013, Theorem 10.4]. Since the theorem involves comparing K-algebras with strict n-categories, before stating the theorem we first define, for any n-globular operad K, a functor $T\text{-}\mathbf{Alg} \to K\text{-}\mathbf{Alg}$; in fact, we do this for a T-operad K for any suitable choice of monad T. Recall from Definition 2.4 that every T-operad K has a natural transformation $k\colon K \Rightarrow T$. The functor $T\text{-}\mathbf{Alg} \to K\text{-}\mathbf{Alg}$ is induced by this natural transformation. We then prove that, under certain circumstances (and in particular, when K is an n-globular operad with unbiased contraction), this functor is full, faithful, and injective on objects, so we can consider $T\text{-}\mathbf{Alg}$ to be a full subcategory of $K\text{-}\mathbf{Alg}$. This tells us that, for any definition of weak n-categories as algebras for an n-globular operad, every strict n-category is a weak n-category. The fact that the inclusion functor is full comes from the fact that, since $K\text{-}\mathbf{Alg}$ is the category of algebras for a monad, we only have strict maps of K-algebras.

5.1. Definition. *Let T be a cartesian monad on a cartesian category $\mathcal{C}$, which has an initial object 1, and let K be a T-operad. Then there is a functor $-\circ k\colon T\text{-}\mathbf{Alg} \to K\text{-}\mathbf{Alg}$ defined by*

$$-\circ k\colon T\text{-}\mathbf{Alg} \longrightarrow K\text{-}\mathbf{Alg}$$

$$\begin{tikzcd} TX \arrow[r, "\phi"] \arrow[d, "Tu"'] & X \arrow[d, "u"] \\ TY \arrow[r, "\psi"'] & Y \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} KX \arrow[r, "k_X"] \arrow[d, "Ku"'] & TX \arrow[r, "\phi"] & X \arrow[d, "u"] \\ KY \arrow[r, "k_Y"'] & TY \arrow[r, "\psi"'] & Y \end{tikzcd}$$

5.2. Proposition. *Let T be a cartesian monad on a cartesian category $\mathcal{C}$, which has an initial object 1, and let K be a T-operad such that, for any object X in $\mathcal{C}$, the component $k_X\colon KX \to TX$ of the natural transformation $k\colon K \Rightarrow T$ is an epimorphism. Then the functor $-\circ k\colon T\text{-}\mathbf{Alg} \to K\text{-}\mathbf{Alg}$ is full, faithful, and injective on objects; hence we can consider $T\text{-}\mathbf{Alg}$ to be a full subcategory of $K\text{-}\mathbf{Alg}$.*

Proof. First, faithfulness is immediate since when we apply $-\circ k$ to a map of T-algebras it retains the same underlying map of n-globular sets.

For fullness, suppose we have T-algebras $TX \xrightarrow{\phi} X$, $TY \xrightarrow{\psi} Y$, and a map u

between their images in K-**Alg**. By naturality of k,

$$\begin{tikzcd} KX \arrow[r, "Ku"] \arrow[d, "k_X"'] & KY \arrow[d, "k_Y"] \\ TX \arrow[r, "Tu"'] & TY \end{tikzcd}$$

commutes, so

$$\begin{tikzcd} KX \arrow[r, "k_X"] \arrow[d, "k_X"'] & TX \arrow[d, "Tu"] \\ TX \arrow[d, "\phi"'] & TY \arrow[d, "\psi"] \\ X \arrow[r, "u"'] & Y \end{tikzcd}$$

commutes. Since k_X is an epimorphism, the diagram above gives us that

$$\begin{tikzcd} TX \arrow[r, "Tu"] \arrow[d, "\phi"'] & TY \arrow[d, "\psi"] \\ X \arrow[r, "u"'] & Y \end{tikzcd}$$

commutes, so u is a map of T-algebras. Hence $- \circ k$ is full.

Finally, suppose we have T-algebras $TX \xrightarrow{\phi} X$, $TX \xrightarrow{\psi} X$, with

$$- \circ k\big(TX \xrightarrow{\phi} X \big) = - \circ k\big(TX \xrightarrow{\psi} X \big).$$

Then

$$\begin{tikzcd} KX \arrow[r, "k_X"] \arrow[d, "k_X"'] & TX \arrow[d, "\psi"] \\ TX \arrow[r, "\phi"'] & X \end{tikzcd}$$

commutes. Since k_X is an epimorphism, this gives us that $\phi = \psi$, so $- \circ k$ is injective on objects. ∎

In the case in which K is a Batanin operad or Leinster operad, each component k_X is surjective on all dimensions of cell (a consequence of Lemma 4.3), so we have the following corollary.

5.3. Corollary. *Let K be a Batanin operad or Leinster operad. Then the functor $-\circ k\colon T\text{-}\mathbf{Alg} \to K\text{-}\mathbf{Alg}$ is full, faithful, and injective on objects.*

For the remainder of this section, when we say "strict n-category", we mean it in the sense of a K-algebra in the image of the functor $-\circ k\colon T\text{-}\mathbf{Alg} \to K\text{-}\mathbf{Alg}$.

Before we state our first coherence theorem, we must also define what it means for two K-algebras to be equivalent.

5.4. Definition. *Let K be an n-globular operad, and let*

$$KX \xrightarrow{\theta} X\ ,\ KY \xrightarrow{\phi} Y$$

be K-algebras. We say that the algebras $KX \xrightarrow{\theta} X$ and $KY \xrightarrow{\phi} Y$ are equivalent *if there exists a map of K-algebras $u\colon X \to Y$ or $u\colon Y \to X$ such that u is surjective on 0-cells, full on m-cells for all $1 \leq m \leq n$, and faithful on n-cells. The map u is referred to as an* equivalence of K-algebras.

Observe that, since maps of K-algebras preserve the K-algebra structure strictly, this definition of equivalence is much more strict (and thus much less general) than it "ought" to be. This is also why we require that the map u can go in either direction; having a map $X \to Y$ satisfying the conditions does not imply the existence of a map $Y \to X$ satisfying the conditions. We will use this definition of equivalence only in the next theorem, and, in spite of its lack of generality, it is sufficient for our purposes. If we required a more general definition of equivalence of K-algebras, there are various approaches we could take. One option would be to replace the map u with a weak map of K-algebras; a definition of weak maps of K-algebras is given by Garner in [Garner 2010], and is valid for any n-globular operad K. Another option is to replace the map u with a span of maps of K-algebras, similar to the approach used by Smyth and Woolf to define an equivalence of Whitney n-categories [Smyth–Woolf 2011]. However, pursuing definitions of equivalence given by either of these approaches is beyond the scope of this paper. We give a weaker definition of equivalence later, in Definition 8.2.

In this definition of equivalence we asked for surjectivity on 0-cells, rather than essential surjectivity. This is another way in which our definition of equivalence is less general than it "ought" to be, but once again, asking for surjectivity is enough for our purposes. This approach of using surjectivity instead of essential surjectivity to simplify the definition of equivalence has previously been taken by Simpson [Simpson 1997].

5.5. Theorem. *Let K be an n-globular operad with unbiased contraction γ, and let X be an n-globular set. Then the free K-algebra on X is equivalent to the free strict n-category on X.*

Proof. As a K-algebra, the free strict n-category on X is

$$KTX \xrightarrow{k_{TX}} T^2X \xrightarrow{\mu^T_X} TX.$$

We first show that k_X is a map of K-algebras, and then show that it is an equivalence of K-algebras.

The diagram

$$\begin{tikzcd}
K^2X \arrow[rr, "Kk_X"] \arrow[dr, "k_{KX}"] \arrow[ddd, "\mu^K_X"'] & & KTX \arrow[d, "k_{TX}"] \\
& TKX \arrow[r, "Tk_X"] & T^2X \arrow[dd, "\mu^T_X"] \\
& & \\
KX \arrow[rr, "k_X"'] & & TX
\end{tikzcd}$$

commutes; the top square is a naturality square for k, and the bottom pentagon comes from the definition of μ^K, in Definition 2.4. Thus k_X is a map of K-algebras, as required.

We now show that k_X is surjective on 0-cells. By definition of the unit $\eta^K\colon 1 \Rightarrow K$, the diagram

$$\begin{tikzcd}
X \arrow[r, "\eta^K_X"] \arrow[dr, "\eta^T_X"'] & KX \arrow[d, "k_X"] \\
& TX
\end{tikzcd}$$

commutes. We have $TX_0 = X_0$, and $(\eta^T_X)_0 = \mathrm{id}_X$, so at dimension 0 the diagram above becomes

$$\begin{tikzcd}
X_0 \arrow[r, "(\eta^K_X)_0"] \arrow[dr, "\mathrm{id}_{X_0}"'] & KX_0 \arrow[d, "(k_X)_0"] \\
& X_0.
\end{tikzcd}$$

Hence $(k_X)_0$ is surjective, i.e. k_X is surjective on 0-cells.

We now show that k_X is full on m-cells for all $1 \leq m \leq n$. Let (α, p), $(\beta, q) \in KX_{m-1}$ be parallel $(m-1)$-cells, and let $\pi\colon k_X(f) \to k_X(g)$ be an m-cell in TX. Then we have an m-cell

$$(\pi, \gamma_{T!(\pi)}(p, q))\colon (\alpha, p) \longrightarrow (\beta, q)$$

in KX with $k_X(\pi, \gamma_{T!(\pi)}(p, q)) = \pi$. Hence k_X is full at dimension m.

Finally, we show that k_X is faithful at dimension n. Let (α, p), (β, q) be n-cells in KA, such that

$$s(\alpha, p) = s(\beta, q),\ t(\alpha, p) = t(\beta, q),\ k_X(\alpha, p) = k_X(\beta, q).$$

The first two equations above give us that $s(p) = s(q)$ and $t(p) = t(q)$, and the third equation gives

$$\alpha = k_X(\alpha, p) = k_X(\beta, q) = \beta.$$

Now, since (α, p), $(\beta, q) \in KX_n$, and since $\alpha = \beta$, we have

$$k(p) = T!(\alpha) = T!(\beta) = k(q),$$

and since k has an unbiased contraction γ, it is faithful at dimension n, and we get that $p = q$. Hence $(\alpha, p) = (\beta, q)$, so k_X is faithful at dimension n.

Hence k_X is an equivalence of K-algebras, so the free K-algebra on X is equivalent to the free strict n-category on X. ∎

The remaining coherence theorems require only a contraction on the operad K, not a system of compositions or an unbiased contraction. These theorems concern which diagrams of constraint cells commute in a K-algebra, so in order to state them, we must first define what we mean by a "diagram" in a K-algebra, and what it means for a diagram to commute.

5.6. Definition. *Let K be an n-globular operad, let $KX \xrightarrow{\theta} X$ be a K-algebra, and let $1 \leq m \leq n$. A* diagram of m-cells *in $KX \xrightarrow{\theta_X} X$ consists of an unordered pair of m-cells (α, p), (β, q) in KX_m such that $\theta(\alpha, p)$ and $\theta(\beta, q)$ are parallel, i.e.*

$$s\theta(\alpha, p) = s\theta(\beta, q),\ t\theta(\alpha, p) = t\theta(\beta, q).$$

We write such a diagram as $((\alpha, p), (\beta, q))$.

We say that the diagram $((\alpha, p), (\beta, q))$ commutes *if*

$$\theta(\alpha, p) = \theta(\beta, q).$$

Our second coherence theorem states that in a free K-algebra every diagram of constraint n-cells commutes. This corresponds to the coherence theorem for tricategories due to Gurski which states that, in the free tricategory on a **Cat**-enriched 2-graph whose set of 3-cells is empty, every diagram of 3-cells commutes ([Gurski 2013, Corollary 10.6], originally [Gurski 2006, Theorem 10.2.2]). Since the constraint 3-cells in a free tricategory do not depend on the generating 3-cells, this implies that in a free tricategory all diagrams of constraint 3-cells commute. Our theorem is analogous to this last result, and our approach is the same as that of Gurski: first, we prove a lemma which states that, in the free K-algebra on an n-globular set whose set of n-cells is empty, all diagrams of n-cells commute; note that in a free K-algebra of this type, all n-cells are constraint cells. We then use this lemma, combined with the fact that the constraint n-cells in a free K-algebra depend only on dimension $n-1$, to prove the theorem.

5.7. Lemma. *Let K be an n-globular operad with contraction γ, and let X be an n-globular set with $X_n = \emptyset$. Then in the free K-algebra on X, every diagram of n-cells commutes.*

Proof. Let $((\alpha, p), (\beta, q))$ be a diagram of n-cells in $K^2X \xrightarrow{\mu_A^X} KX$. Since $X_n = \emptyset$, the only n-cells in TX are identities, so we have $TX_n \cong TX_{n-1}$, and the source and target maps s, $t\colon TX_n \to TX_{n-1}$ are isomorphisms with $s(\pi) = t(\pi)$ for all $\pi \in TX_n$. Since (α, p), (β, q) are parallel and k_X is a map of n-globular sets, so preserves sources and targets, $k_X(\alpha, p)$ and $k_X(\beta, q)$ are parallel n-cells in TX so must be equal. As shown in the proof of Theorem 5.5, k_X is faithful at dimension n, hence $(\alpha, p) = (\beta, q)$. ∎

Before we use Lemma 5.7 to prove our second coherence theorem, we must first give a formal definition of constraint cells in a K-algebra. Constraint cells are cells that arise from the contraction on $k\colon K \to T1$; these include identities, and mediating cells between different composites of the same pasting diagram. Note that constraint m-cells for $m < n$ depend on the choice of contraction on k, even though the algebras for K do not; constraint n-cells do not depend on the choice of contraction on k, since faithfulness of k at dimension n ensures that there is only ever one valid choice at this dimension.

To define the constraint cells in a K-algebra $KX \xrightarrow{\theta} X$, we first lift the contraction cells in K to contraction cells in KX. Recall from Definition 2.4 that KX is defined by the following pullback square:

$$\begin{tikzcd} KX \arrow[r, "K!"] \arrow[d, "k_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & K \arrow[d, "k"] \\ TX \arrow[r, "T!"'] & T1. \end{tikzcd}$$

The contraction on $k\colon K \to T1$ lifts along this pullback square to a contraction on $k_X\colon KX \to TX$ as follows: let (π, p), (π', q) be two $(m-1)$-cells in KX, where $\pi, \pi' \in TX_{m-1}$ and $p, q \in K_{m-1}$. Suppose that these cells require a contraction m-cell, so the cells are parallel, and $k_X(\pi, q) = k_X(\pi', q)$. Since (π, p), (π', q) are parallel, p and q are parallel, and the equation $k_X(\pi, q) = k_X(\pi', q)$ gives $\pi = \pi'$. Hence, by commutativity of the pullback square defining KX, we have $k(p) = k(q)$. Thus, since p and q are parallel cells in K with the same image under k, there is a contraction cell in K from p to q; this lifts to a contraction cell in KX from (π, p) to (π, q).

5.8. Definition. *Let K be an n-globular operad with unbiased contraction γ, and let $KX \xrightarrow{\theta} X$ be a K-algebra. There is a contraction δ on $k_X\colon KX \to TX$ given by, for each $1 \leq m \leq n$, $\pi \in TX_{m-1}$, the function*

$$\begin{aligned} \delta_{\mathrm{id}_\pi}\colon C_{KX}(\mathrm{id}_\pi) &\longrightarrow KX(\mathrm{id}_\pi) \\ ((\pi, p), (\pi, q)) &\longmapsto (\mathrm{id}_\pi, \gamma_{T!(\mathrm{id}_\pi)}(p, q)). \end{aligned}$$

A constraint m-cell *in $KX \xrightarrow{\theta} X$ is an m-cell of X in the image of the map*

$$C_{KX}(\mathrm{id}_\pi) \xrightarrow{\delta_{\mathrm{id}_\pi}} KX(\mathrm{id}_\pi) \hookrightarrow (KX)_m \xrightarrow{\theta_m} X_m,$$

for some $\pi \in TX_{m-1}$.

5.9. Corollary. *Let K be an n-globular operad with contraction γ, and let X be an n-globular set. In the free K-algebra on X, $K^2X \xrightarrow{\mu^K_X} KX$, every diagram of constraint n-cells commutes.*

PROOF. Write X' for the n-globular set defined by

$$X'_m = \begin{cases} X_m & \text{if } m < n, \\ \emptyset & \text{if } m = n, \end{cases}$$

with source and target maps the same as those in X for dimensions $m < n$, and write $u\colon X' \to X$ for the map which is the identity on all dimensions $m < n$. For all $\pi \in TKX_{n-1} = TKX'_{n-1}$ we have $C_{K^2X'}(\mathrm{id}_\pi) = C_{K^2X}(\mathrm{id}_\pi)$, and the diagram

$$\begin{tikzcd} C_{K^2X'}(\mathrm{id}_\pi) \arrow[r, "\delta'_{\mathrm{id}_\pi}"] \arrow[d, equal] & K^2X'(\mathrm{id}_\pi) \arrow[r, hook] \arrow[d, "K^2u_n"'] & (K^2X')_n \arrow[r, "\mu^K_{X'}"] \arrow[d, "K^2u_n"'] & KX'_m, \arrow[d, "Ku_n"] \\ C_{K^2X}(\mathrm{id}_\pi) \arrow[r, "\delta_{\mathrm{id}_\pi}"] & K^2X(\mathrm{id}_\pi) \arrow[r, hook] & K^2X_n \arrow[r, "\mu^K_X"] & KX_m, \end{tikzcd}$$

commutes.

Let $((\alpha, p), (\beta, q))$ be a diagram of n-cells in K^2X such that α, $\beta \in TKX_n$ are composites of constraint n-cells of KX. Since constraint n-cells are determined by $(n-1)$-cells, and $TKX_{n-1} = TKX'_{n-1}$, we have (α, p), $(\beta, q) \in K^2X'_n$, with $\mu^K_{X'}(\alpha, p)$ and $\mu^K_{X'}(\beta, q)$ parallel. Thus, by Lemma 5.7, $(\alpha, p) = (\beta, q)$. ∎

The final coherence theorem describes a class of diagrams of constraint n-cells which commute in any K-algebra. These diagrams should be thought of as those that are "free-shaped", i.e. they are diagrams of constraint cells that could arise in a free K-algebra. This rules out diagrams in which the sources and targets of the constraint cells involve non-constraint cells with constraint cells in their boundaries, and non-composite cells with composites in their boundaries. This is the analogue of a coherence theorem for tricategories due to Gurski, which describes a similar class of diagrams of constraint 3-cells in the context of tricategories [Gurski 2006, Corollary 10.2.5]. We call such a diagram F_K-admissible, where F_K is the left adjoint to the forgetful functor

$$U_K\colon K\text{-}\mathbf{Alg} \longrightarrow n\text{-}\mathbf{GSet},$$

which sends a K-algebra to its underlying n-globular set; this terminology is taken from the theorem of Gurski mentioned above.

5.10. DEFINITION. *Let K be an n-globular operad with contraction γ, and let $KX \xrightarrow{\theta} X$ be a K-algebra. A diagram $((\alpha, p), (\beta, q))$ of constraint n-cells in X is said to be F_K-*admissible *if there exists a sub-n-globular set E of X, with $E_n = \emptyset$ and inclusion map $i\colon E \hookrightarrow X$, and a diagram $((\alpha', p'), (\beta', q'))$ of constraint n-cells in F_KE such that the diagram $((\alpha, p), (\beta, q))$ is the image of $((\alpha', p'), (\beta', q'))$ under the map*

$$\begin{tikzcd} K^2E \arrow[r, "K\bar{i}"] \arrow[d, "\mu^K_E"'] & KX \arrow[d, "\theta"] \\ KE \arrow[r, "\bar{i}"'] & X, \end{tikzcd}$$

where $\bar{i}$ is the transpose under the adjunction $F_K \dashv U_K$ of i.

The following is now an immediate corollary of Lemma 5.7, and the fact that $\bar{i}$ is a map of K-algebras.

5.11. Corollary. *Let K be an n-globular operad with contraction γ and let $KX \xrightarrow{\theta} X$ be a K-algebra. Then every F_K-admissible diagram of constraint n-cells in $KX \xrightarrow{\theta} X$ commutes.*

6. Comparison functors between B-**Alg** and L-**Alg**

The fact that both Batanin weak n-categories and Leinster weak n-categories are defined as algebras for n-globular operads means we can make some statements about the relationship between the two definitions by comparing the operads B and L. In this section we use the correspondence between Batanin operads and Leinster operads (Theorems 4.1 and 4.2), along with the universal properties of the operads B and L, to derive comparison functors

$$u_* \colon L\text{-}\mathbf{Alg} \longrightarrow B\text{-}\mathbf{Alg} \text{ and } v_* \colon B\text{-}\mathbf{Alg} \longrightarrow L\text{-}\mathbf{Alg}.$$

In the following Sections 7 and 8 we will use these functors to give some steps towards a comparison between Batanin weak n-categories and Leinster weak n-categories. Some of these statements are preliminary, but we hope that they will pave the way for a more comprehensive comparison in the future.

Note that, in the case $n = 2$, the relationship is already understood: Leinster has shown that Batanin weak 2-categories are the same as bicategories, and Leinster weak 2-categories are the same as unbiased bicategories [Leinster 2002], and that bicategories and unbiased bicategories are equivalent [Leinster 2004a, Section 3.4].

Recall from Definition 3.3 that we write **OCS** for the category of Batanin operads, and from Definition 3.6 that we write **OUC** for the category of Leinster operads. By Theorem 4.1 we have a canonical functor

$$\mathbf{OUC} \longrightarrow \mathbf{OCS}$$

which is the identity on the underlying operads. Applying this functor to L equips it with a contraction and a system of compositions. Thus, since B is initial in **OCS**, there is a unique map

$$\begin{tikzcd} B \arrow[rr, "u"] \arrow[dr, "b"'] & & L \arrow[dl, "l"] \\ & T1 & \end{tikzcd}$$

in **OCS**.

By Theorem 4.2 we can equip the operad B with an unbiased contraction to obtain an object of **OUC**. However, unlike the process of equipping L with a contraction and system of compositions, there is no canonical way of doing this; the unbiased contraction on B depends on a choice of section to b and various choices of bracketings, as described in Lemma 4.3. Suppose we have chosen a section to b and thus equipped B with an unbiased contraction. Since L is initial in **OUC**, there is a unique map

$$\begin{tikzcd} L \arrow[rr, "v"] \arrow[dr, "l"'] & & B \arrow[dl, "b"] \\ & T1 & \end{tikzcd}$$

in **OUC**.

Every map of operads gives rise to a corresponding map of the induced monads [Leinster 2004a, Corollary 6.2.4]. Thus the maps u and v induce functors between the categories of algebras B-**Alg** and L-**Alg**; we write

$$u_*\colon L\text{-}\mathbf{Alg} \longrightarrow B\text{-}\mathbf{Alg}$$

for the functor induced by u, and

$$v_*\colon B\text{-}\mathbf{Alg} \longrightarrow L\text{-}\mathbf{Alg}$$

for the functor induced by v.

6.1. Proposition. *The functor v_* is a section of the functor u_*, i.e.*

$$u_* v_* = \mathrm{id}_{B\text{-}\mathbf{Alg}}.$$

Proof. Theorem 4.1 gives a canonical functor $\mathbf{OUC} \to \mathbf{OCS}$. Applying this functor to v gives that v is a map in **OCS**, so the composite

$$\begin{tikzcd} B \arrow[r, "u"] \arrow[dr, "b"'] & L \arrow[r, "v"] \arrow[d, "l"] & B \arrow[dl, "b"] \\ & T1 & \end{tikzcd}$$

is the identity id_B, since B is initial in **OCS**. Thus $u_* v_*$ is the functor induced by $vu = \mathrm{id}_B$, so $u_* v_* = \mathrm{id}_{B\text{-}\mathbf{Alg}}$, as required. ∎

We now consider the composite

$$L\text{-}\mathbf{Alg} \xrightarrow{u_*} B\text{-}\mathbf{Alg} \xrightarrow{v_*} L\text{-}\mathbf{Alg}.$$

Note that $v_* u_*$ does not change the underlying n-globular set of an L-algebra, it only changes the algebra structure. We describe a small example which illustrates the way in which the new algebra structure differs from the original one. In Section 8 we investigate the relationship between an L-algebra and its image under the functor $v_* u_*$ more fully, using this example to motivate a definition of weak map of L-algebras that we use in the general case. Let $n \geq 2$ and let A denote the n-globular set consisting of three composable 1-cells:

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet$$

We consider the free L-algebra on A, i.e.

$$\begin{tikzcd} L^2A \arrow[d, "\mu^L_A"] \\ LA. \end{tikzcd}$$

This has:

- 0-cells: the same as those of A;
- 1-cells:
  - generating cells f, g, h,
  - binary composites $g \circ f$, $h \circ g$, $h \circ (g \circ f)$, $(h \circ g) \circ f$,
  - a ternary composite $h \circ g \circ f$,
  - identities and composites involving identities;
- 2-cells: for every pair of parallel 1-cells a, $b \in LA_1$, a constraint cell which we write as
$$[a, b] \colon a \Longrightarrow b.$$
In particular, this includes constraint cells mediating between different composites of the same cells, e.g.
$$[h \circ g \circ f, (h \circ g) \circ f],$$
$$[(h \circ g) \circ f, h \circ (g \circ f)],$$
etc. We also have freely generated composites of these;
- m-cells for $m \geq 3$: constraint cells, and composites of constraint cells.

Applying $v_* u_*$ to this gives the L-algebra

$$\begin{tikzcd} L^2A \arrow[d, "v_{LA}"] \\ BLA \arrow[d, "u_{LA}"] \\ L^2A \arrow[d, "\mu^L_A"] \\ LA, \end{tikzcd}$$

which has the same underlying n-globular set as the free L-algebra on A, but has a different composition structure. Write $\odot$ for the new composition operation on 1-cells, which is defined as follows:

- binary composition remains the same, so we have

$$g \odot f = g \circ f,\ h \odot g = h \circ g,\ h \odot (g \odot f) = h \circ (g \circ f),$$

  etc.;

- ternary composition is given by bracketing on the left, i.e.

$$h \odot g \odot f = (h \circ g) \circ f.$$

Consider the diagram

$$\begin{tikzcd}
L^2A \arrow[r, "L\mathrm{id}_{LA}"] \arrow[d, "v_{LA}"'] & L^2A \arrow[ddd, "\mu^L_A"] \\
BLA \arrow[d, "u_{LA}"'] & \\
L^2A \arrow[d, "\mu^L_A"'] & \\
LA \arrow[r, "\mathrm{id}_{LA}"'] & LA
\end{tikzcd}$$

in n-**GSet**. If this diagram commuted it would be a map of L-algebras, and since its underlying map of n-globular sets is an identity, this would show that the free L-algebra on A is isomorphic to its image under v_*u_*. In fact this diagram does not commute; although it does commute on the underlying B-algebra structure, i.e. it commutes on generating cells, binary composites, and constraint cells mediating between these, but it does not commute on cells that only exist in the L-algebra structure, such as ternary composites. Consider the freely generated ternary composite

$$h \circ g \circ f \in L^2A.$$

We have

- $\theta \circ L\mathrm{id}_{LA}(h \circ g \circ f) = \theta(h \circ g \circ f) = h \circ g \circ f$;
- $\mathrm{id}_{LA} \circ \theta \circ u_{LA} \circ v_{LA}(h \circ g \circ f) = h \odot g \odot f = (h \circ g) \circ f$,

and

$$h \circ g \circ f \neq (h \circ g) \circ f,$$

so the diagram does not commute. However, there is a constraint cell mediating between these two 1-cells:

$$[h \circ g \circ f, (h \circ g) \circ f] \colon h \circ g \circ f \Rightarrow (h \circ g) \circ f.$$

Similarly, for any other cell in L^2A that is not part of the underlying B-algebra structure (such as non-binary composites of 1-cells involving identities, and non-binary composites at higher dimensions) we also have a constraint cell mediating between its images under the maps $\mu^L_A \circ L\mathrm{id}_{LA}$ and $\mathrm{id}_{LA} \circ \mu^L_A \circ u_{LA} \circ v_{LA}$. Thus, we can think of the diagram as “commuting up to a constraint cell”. By the definition of constraint cells as those induced by the contraction L, combined with the fact that all diagrams of constraint n-cells commute in a free L-algebra (Corollary 5.9), these constraint cells are equivalences in the L-algebra, and any diagram of them commutes up to a constraint cell at the dimension above, with strict commutativity for diagrams of constraint n-cells. Thus these constraint cells are “well-behaved enough” to act as the mediating cells in a weak map; any commutativity conditions we would need to check are automatically satisfied by coherence for L-algebras.

7. Left adjoint to u_*

In this section we construct a functor

$$F\colon B\text{-}\mathbf{Alg} \longrightarrow L\text{-}\mathbf{Alg},$$

and prove that this is left adjoint to the functor u_*. Recall that u_* is the functor induced by the unique map of Batanin operads

$$u\colon B \longrightarrow L$$

induced by the universal property of B, the initial object in $\mathbf{OCS}$. We can think of u_* as a forgetful functor that sends an L-algebra to its underlying B-algebra by forgetting its unbiased composition structure, and remembering only the binary composition structure and the necessary constraint cells. The left adjoint F takes a B-algebra and freely adds an unbiased composition structure, along with all the required constraint cells to make an L-algebra, but retains the original binary composition structure (note that new binary composites are not added freely).

It is a result of Blackwell–Kelly–Power [Blackwell–Kelly–Power 1989, Theorem 5.12] that any functor induced by a map of monads in this way has a left adjoint (their result is for 2-monads, but can be applied to monads by considering them as a special case of 2-monads). Consequently, one may ask why the adjunction

$$\begin{tikzcd} B\text{-}\mathbf{Alg} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & L\text{-}\mathbf{Alg} \arrow[l, shift left, "u_*"] \end{tikzcd}$$

should be considered significant, and, in particular, why it is more significant than the adjunction in which v_* is the right adjoint. There are two reasons for this. First, $u : B \to L$ is canonical in the sense that it is the only such map of monads that preserves the contraction and system of compositions on B. In contrast, $v\colon L \to B$ is not canonical; there is no canonical way of equipping B with an unbiased contraction (Theorem 4.2), so v depends on the choices we made when doing so. Second, this adjunction formalises the idea that the key difference between B-algebras and L-algebras is that B-algebras have binary-biased composition whereas L-algebras have unbiased composition, and describes how to obtain an L-algebra from a B-algebra by adding unbiased composites, as well as the necessary constraint cells, freely.

The construction of the left adjoint described in this section is valid in greater generality than just this case; we can replace n-**GSet** with any cocomplete category, L with any finitary monad, B with any other monad on the same category, and $u\colon B \to L$ with any map of monads whose functor part is the identity. We first explain the construction with reference to the specific case of a left adjoint to u_*, then state the construction in more generality.

Note that the left adjoint we construct is not induced by a map of monads; a functor B-**Alg** $\to$ L-**Alg** induced by a map of monads $L \to B$ would leave the underlying n-globular set of a B-algebra unchanged, but the left adjoint to u_* freely adds unbiased composites (and various contraction cells) to obtain an L-algebra structure, rather than using cells already present in the original B-algebra.

Let

$$\begin{tikzcd} BX \arrow[d, "\theta"] \\ X, \end{tikzcd}$$

be a B-algebra; we will now construct an n-globular set $\bar{X}$, which will be the underlying n-globular set of the L-algebra obtained by applying F to the B-algebra above. First, we apply L to X, which freely adds an L-algebra structure, while ignoring the existing B-algebra structure. This free L-algebra structure has a free B-algebra structure inside it, which is picked out by the map

$$u_X\colon BX \longrightarrow LX.$$

We identify this free B-algebra structure with the original B-algebra structure on X by taking the following pushout:

$$\begin{tikzcd} BX \arrow[r, "u_A"] \arrow[d, "\theta"'] & LX \arrow[d, "\phi^{(1)}"] \\ X \arrow[r, "x^{(1)}"'] & X^{(1)}. \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

Taking this pushout identifies any cell in the free B-algebra structure inside LX with the corresponding cell in the original B-algebra on X. So, for example, any free binary composite in LX is identified with the binary composite of the same cells, as evaluated by θ, in X.

However, this is not the end of the construction for two reasons: first, in principle the act of identifying cells causes more cells to share common boundaries, thus making more cells composable; second, taking this pushout does nothing to cells in LX that involve both the B-algebra and non-B-algebra structure. Such cells include non-binary composites of binary composites; for example, suppose we have a string of four composable 1-cells

$$\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \xrightarrow{c} \bullet \xrightarrow{d} \bullet$$

in X. In $X^{(1)}$, we have distinct cells

$$d \circ c \circ (b \circ a) \neq d \circ c \circ \theta(b \circ a),$$

but in the L-algebra we are constructing we want these cells to be equal.

To rectify these problems we apply L to $X^{(1)}$, thus freely adding composites of the newly composable cells, then identify the free L-algebra structure on $LX^{(1)}$ with the partial L-algebra structure on $X^{(1)}$ given by $\phi^{(1)}\colon LX \to X^{(1)}$ by taking the pushout

$$\begin{tikzcd}
L^2X \arrow[r, "L\phi^{(1)}"] \arrow[d, "\mu^L_X"'] & LX^{(1)} \arrow[dd, "\phi^{(2)}"] \\
LX \arrow[d, "\phi^{(1)}"'] & \\
X^{(1)} \arrow[r, "x^{(2)}"'] \arrow[ur, phantom, "\ulcorner", very near end] & X^{(2)}.
\end{tikzcd}$$

Once again, the act of identifying cells causes more cells to become composable. Also, although in $X^{(1)}$ we now have the desired equalities between non-binary composites involving binary composites such as

$$d \circ c \circ (b \circ a) = d \circ c \circ \theta(b \circ a),$$

this is not true for composites whose binary parts appear at greater "depths", such as non-binary composites of non-binary composites of binary composites. We thus must repeat the procedure above indefinitely to obtain the following sequence of pushouts in n-**GSet**:

$$\begin{tikzcd}
 & & L^2X^{(1)} \arrow[r, "L\phi^{(2)}"] \arrow[d, "\mu^L_{X^{(1)}}"'] & LX^{(2)} \arrow[ddd, "\phi^{(3)}"] & \cdots \\
 & L^2X \arrow[r, "L\phi^{(1)}"] \arrow[d, "\mu^L_X"'] & LX^{(1)} \arrow[dd, "\phi^{(2)}"] & & \\
BX \arrow[r, "u_A"] \arrow[d, "\phi^{(0)}=\theta"'] & LX \arrow[d, "\phi^{(1)}"] & & & \\
X = X^{(0)} \arrow[r, "x^{(1)}"'] & X^{(1)} \arrow[r, "x^{(2)}"'] & X^{(2)} \arrow[r, "x^{(3)}"'] & X^{(3)} & \cdots
\end{tikzcd}$$

The bottom row of this diagram is a sequence of n-globular sets

$$\{X^{(i)}\}_{i \geq 0}.$$

We define $\bar{X}$ to be given by

$$\bar{X} := \operatorname*{colim}_{i \geq 0} X^{(i)},$$

We now describe the construction in general. Throughout the rest of this section, let $\mathcal{C}$ denote a cocomplete category, let R and S be monads on $\mathcal{C}$ with S finitary (i.e. the functor part of S preserves filtered colimits), and let $p\colon R \to S$ be a map of monads whose functor part is the identity. The map p induces a functor

$$p_*\colon S\text{-}\mathbf{Alg} \longrightarrow R\text{-}\mathbf{Alg},$$

and we will construct a left adjoint F to p_*. Let

$$\begin{tikzcd} RX \arrow[d, "\theta"] \\ X, \end{tikzcd}$$

be an R-algebra. We define a sequence

$$\{X^{(i)}\}_{i\geq 0}.$$

of objects in $\mathcal{C}$ by the following sequence of pushouts in $\mathcal{C}$:

$$\begin{tikzcd}
 & & S^2X^{(1)} \arrow[r, "S\phi^{(2)}"] \arrow[d, "\mu^S_{X^{(1)}}"'] & SX^{(2)} \arrow[ddd, "\phi^{(3)}"] & \cdots \\
 & S^2X \arrow[r, "S\phi^{(1)}"] \arrow[d, "\mu^S_X"'] & SX^{(1)} \arrow[dd, "\phi^{(2)}"] & & \\
RX \arrow[r, "u_A"] \arrow[d, "\phi^{(0)}=\theta"'] & SX \arrow[d, "\phi^{(1)}"] & & & \\
X = X^{(0)} \arrow[r, "x^{(1)}"'] & X^{(1)} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, "x^{(2)}"'] & X^{(2)} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, "x^{(3)}"'] & X^{(3)} \arrow[ul, phantom, "\ulcorner", very near start] & \cdots
\end{tikzcd}$$

We then define an object $\bar{X}$ of $\mathcal{C}$ by

$$\bar{X} := \operatorname*{colim}_{i\geq 0} X^{(i)}.$$

This will be the underlying object of $\mathcal{C}$ of the S-algebra obtained by applying the functor F to the R-algebra $\theta\colon RX \to X$.

We now equip $\bar{X}$ with an S-algebra action

$$\phi\colon S\bar{X} \longrightarrow \bar{X}.$$

Since S is finitary, we can write $S\bar{X}$ as

$$S\bar{X} = \operatorname*{colim}_{i\geq 0} SX^{(i)}.$$

We wish to use the universal property of this colimit to define the S-algebra action ϕ. To do so, we now describe the cocone that induces ϕ, and prove that it commutes.

7.1. Lemma. *There is a cocone under the diagram*

$$\{SX^{(i)}\}_{i\geq 0}$$

with vertex $\bar{X}$*, given by*

$$\begin{tikzcd}
SX^{(0)} \arrow[r, "Sx^{(1)}"] \arrow[d, "\phi^{(1)}"'] & SX^{(1)} \arrow[r, "Sx^{(2)}"] \arrow[d, "\phi^{(2)}"'] & SX^{(2)} \arrow[r, "Sx^{(3)}"] \arrow[d, "\phi^{(3)}"'] & \cdots \\
X^{(1)} \arrow[dr, bend right, "c^{(1)}"'] & X^{(2)} \arrow[d, "c^{(2)}"'] & X^{(3)} \arrow[dl, "c^{(3)}"] & \\
 & \bar{X} & &
\end{tikzcd}$$

Proof. We must show that, for each $i \geq 0$, the diagram

$$\begin{tikzcd}
SX^{(i)} \arrow[rr, "Sx^{(i+1)}"] \arrow[d, "\phi^{(i+1)}"'] & & SX^{(i+1)} \arrow[d, "\phi^{(i+2)}"] \\
X^{(i+1)} \arrow[dr, "c^{(i+1)}"'] & & X^{(i+2)} \arrow[dl, "c^{(i+2)}"] \\
 & \bar{X} &
\end{tikzcd}$$

commutes. We can write this diagram as

$$\begin{tikzcd}
SX^{(i)} \arrow[d, "S\eta^S_{X^{(i)}}"'] \arrow[dr, "Sx^{(i+1)}"] \arrow[dd, bend right=60, "\mathrm{id}_{SX^{(i)}}"'] & \\
S^2X^{(i)} \arrow[r, "S\phi^{(i+1)}"'] \arrow[d, "\mu^S_{X^{(i)}}"'] & SX^{(i+1)} \arrow[dd, "\phi^{(i+2)}"] \\
SX^{(i)} \arrow[d, "\phi^{(i+1)}"'] & \\
X^{(i+1)} \arrow[r, "x^{(i+2)}"'] \arrow[dr, "c^{(i+1)}"'] & X^{(i+2)} \arrow[d, "c^{(i+2)}"] \\
 & \bar{X}.
\end{tikzcd}$$

The rectangle commutes since it is the pushout square defining $X^{(i+2)}$, the top-left triangle commutes by the unit axiom for the monad S, and the bottom triangle commutes by definition of $\bar{X}$; thus we need only check that the top-right triangle commutes. We do so by showing that, for all $i \geq 0$, the diagram

$$\begin{tikzcd} X^{(i)} \arrow[d, "\eta^S_{X^{(i)}}"'] \arrow[dr, "x^{(i+1)}"] & \\ SX^{(i)} \arrow[r, "\phi^{(i+1)}"'] & X^{(i+1)} \end{tikzcd}$$

commutes, then applying S to this diagram.

When $i = 0$, $X^{(i)} = X^{(0)} = X$, and the diagram above becomes

$$\begin{tikzcd} X \arrow[d, "\eta^S_X"'] \arrow[dr, "x^{(1)}"] & \\ SX \arrow[r, "\phi^{(1)}"'] & X^{(1)}. \end{tikzcd}$$

This diagram can be written as

$$\begin{tikzcd} X \arrow[d, "\eta^R_X"'] \arrow[dr, "\eta^S_X"] \arrow[dd, bend right=60, "\mathrm{id}_X"'] & \\ RX \arrow[r, "p_X"] \arrow[d, "\theta"'] & SX \arrow[d, "\phi^{(1)}"] \\ X \arrow[r, "x^{(1)}"'] & X^{(1)}. \end{tikzcd}$$

The square commutes since it is the pushout square defining $X^{(1)}$, the left-hand triangle commutes by the unit axiom for θ, and the top-right triangle commutes by the unit axiom for the monad map p. Thus this diagram commutes.

Now let $i \geq 1$. The diagram

$$\begin{tikzcd} SX^{(i-1)} \arrow[r, "\phi^{(i)}"] \arrow[d, "\eta^S_{SX^{(i-1)}}"'] \arrow[dd, bend right=60, "\mathrm{id}_{SX^{(i-1)}}"'] & X^{(i)} \arrow[d, "\eta^S_{X^{(i)}}"] \\ S^2X^{(i-1)} \arrow[r, "S\phi^{(i)}"'] \arrow[d, "\mu^S_{X^{(i-1)}}"'] & SX^{(i)} \arrow[dd, "\phi^{(i+1)}"] \\ SX^{(i-1)} \arrow[d, "\phi^{(i)}"'] & \\ X^{(i)} \arrow[r, "x^{(i+1)}"'] & X^{(i+1)} \end{tikzcd}$$

commutes; the bottom rectangle commutes since it is the pushout square defining $X^{(i+1)}$, the top square commutes since it is a naturality square for η^S, and the top-left triangle commutes by the unit axiom for the monad S. We wish to cancel the $\phi^{(i)}$'s in the diagram above, in order to obtain the desired triangle. We can do this if $\phi^{(i)}$ is an epimorphism; we now show that this is true by induction over i.

To show that this is true when $i = 1$, observe that η_X^R is a section to θ, so θ is epic; since the pushout of an epimorphism is also an epimorphism [Borceux 1994, Proposition 2.5.3], we have that $\phi^{(1)}$ is epic.

Now suppose $i > 1$ and that we have shown that $\phi^{(i-1)}$ is epic. By the unit axiom for the monad S, $\eta^S_{SX^{(i-2)}}$ is a section to $\mu^S_{X^{(i-2)}}$, so $\mu^S_{X^{(i-2)}}$ is epic. Hence the composite

$$\phi^{(i-1)} \circ \mu^S_{X^{(i-2)}}$$

is epic; since the pushout of an epimorphism is also an epimorphism, we have that $\phi^{(i)}$ is epic.

Hence, for each $i \geq 0$, the diagram

$$\begin{tikzcd}
X^{(i)} \arrow[d, "\eta^S_{X^{(i)}}"'] \arrow[dr, "x^{(i+1)}"] & \\
SX^{(i)} \arrow[r, "\phi^{(i+1)}"'] & X^{(i+1)}
\end{tikzcd}$$

commutes, and thus the diagram

$$\begin{tikzcd}
SX^{(i)} \arrow[rr, "Sx^{(i+1)}"] \arrow[d, "\phi^{(i+1)}"'] & & SX^{(i+1)} \arrow[d, "\phi^{(i+2)}"] \\
X^{(i+1)} \arrow[dr, "c^{(i+1)}"'] & & X^{(i+2)} \arrow[dl, "c^{(i+2)}"] \\
& \bar{X} &
\end{tikzcd}$$

commutes, as required. ■

We now define $\phi\colon S\bar{X} \to \bar{X}$ to be the unique map induced by the universal property of $S\bar{X}$ such that, for all $i \geq 0$, the diagram

$$\begin{tikzcd}
SX^{(i)} \arrow[r, "Sc^{(i)}"] \arrow[d, "\phi^{(i+1)}"'] & S\bar{X} \arrow[d, dashed, "\phi"] \\
X^{(i+1)} \arrow[r, "c^{(i+1)}"'] & \bar{X}
\end{tikzcd}$$

commutes. To check that

$$\begin{tikzcd} S\bar{X} \arrow[d, "\phi"] \\ \bar{X} \end{tikzcd}$$

is an S-algebra we must show that it satisfies the S-algebra axioms.

7.2. Lemma. *The map $\phi\colon S\bar{X} \to \bar{X}$ satisfies the S-algebra axioms.*

Proof. For the unit axiom, we must check that the diagram

$$\begin{tikzcd} \bar{X} \arrow[r, "\eta^S_{\bar{X}}"] \arrow[dr, "\mathrm{id}_{\bar{X}}"'] & S\bar{X} \arrow[d, "\phi"] \\ & \bar{X} \end{tikzcd}$$

commutes. Since $\bar{X}$ is defined as a colimit, we check this by comparing the cocones corresponding to the maps on either side of the diagram. The cocone corresponding to $\phi \circ \eta^S_{\bar{X}}$ is given by, for each $i \geq 0$, the composite

$$X^{(i)} \xrightarrow{\eta^S_{X^{(i)}}} SX^{(i)} \xrightarrow{\phi^{(i+1)}} X^{(i+1)} \xrightarrow{c^{(i+1)}} \bar{X}.$$

The cocone corresponding to $\mathrm{id}_{\bar{X}}$ is the universal cocone given by the coprojections $c^{(i)}$. The diagram

$$\begin{tikzcd} X^{(i)} \arrow[r, "\eta^S_{X^{(i)}}"] \arrow[dr, "x^{(i)}"] \arrow[ddr, "c^{(i)}"'] & SX^{(i)} \arrow[d, "\phi^{(i+1)}"] \\ & X^{(i+1)} \arrow[d, "c^{(i+1)}"] \\ & \bar{X}. \end{tikzcd}$$

commutes, so these cocones are equal. Thus the unit axiom is satisfied.

For the multiplication axiom, we must check that the diagram

$$\begin{tikzcd} S^2\bar{X} \arrow[r, "S\phi"] \arrow[d, "\mu^S_{\bar{X}}"'] & S\bar{X} \arrow[d, "\phi"] \\ S\bar{X} \arrow[r, "\phi"'] & \bar{X} \end{tikzcd}$$

commutes. Since S is finitary, we have

$$S^2\bar{X} = \operatorname*{colim}_{i \geq 0} S^2 X^{(i)}.$$

Thus we can check that the diagram commutes by comparing the cocones corresponding to the maps on either side of the diagram. The cocone corresponding to $\phi \circ S\phi$ is given by, for each $i \geq 0$, the composite

$$S^2X^{(i)} \xrightarrow{S\phi^{(i+1)}} SX^{(i+1)} \xrightarrow{\phi^{(i+2)}} X^{(i+2)} \xrightarrow{c^{(i+2)}} \bar{X}$$

The cocone corresponding to $\phi \circ \mu^S_{\bar{X}}$ is given by, for each $i \geq 0$, the composite

$$S^2X^{(i)} \xrightarrow{\mu^S_{X^{(i)}}} SX^{(i)} \xrightarrow{\phi^{(i+1)}} X^{(i+1)} \xrightarrow{x^{(i+2)}} X^{(i+2)} \xrightarrow{c^{(i+2)}} \bar{X}.$$

From the definition of $\bar{X}$, for all $i \geq 0$, the diagram

$$\begin{tikzcd}
S^2X^{(i)} \arrow[r, "S\phi^{(i+1)}"] \arrow[d, "\mu^S_{X^{(i)}}"'] & SX^{(i+1)} \arrow[dd, "\phi^{(i+2)}"] \\
SX^{(i)} \arrow[d, "\phi^{(i+1)}"'] & \\
X^{(i+1)} \arrow[r, "x^{(i+2)}"'] \arrow[dr, "c^{(i+1)}"'] & X^{(i+2)} \arrow[d, "c^{(i+2)}"] \\
& \bar{X}
\end{tikzcd}$$

commutes, so these cocones are equal. Thus the associativity axiom is satisfied.

Hence

$$\begin{tikzcd}
S\bar{X} \arrow[d, "\phi"] \\
\bar{X}
\end{tikzcd}$$

is an S-algebra. ∎

This gives us the action of the left adjoint to p_* on objects. To prove that this gives a left adjoint, we use the following result of Mac Lane [Mac Lane 1998, Theorem IV.1.2], which allows us to avoid describing the action of the left adjoint on morphisms.

7.3. Lemma. [Mac Lane] *Given a functor $U\colon \mathcal{D} \longrightarrow \mathcal{C}$, an adjunction*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "U"]
\end{tikzcd}$$

is completely determined by, for all objects x in $\mathcal{C}$, an object $F_0(x)$ in $\mathcal{D}$ and a universal arrow $\eta_x\colon x \to UF_0(x)$ from x to U.

As is suggested by the notation, here the assignment $F_0(x)$ gives the action of the left adjoint F on objects, and the maps η_x are the components of the unit of the adjunction.

7.4. Proposition. *There is an adjunction $F \dashv p_*$.*

Proof. As discussed above, we prove this using Lemma 7.3, thus allowing us to avoid constructing the action of F on morphisms. Let

$$\begin{tikzcd} RX \arrow[d, "\theta"] \\ X \end{tikzcd}$$

be an R-algebra. By the construction described earlier we have a corresponding S-algebra

$$\begin{tikzcd} S\bar{X} \arrow[d, "\phi"] \\ \bar{X}. \end{tikzcd}$$

To show that this gives the action on objects of the left adjoint $F\colon R\text{-}\mathbf{Alg} \to S\text{-}\mathbf{Alg}$ to p_*, we require a map of R-algebras

$$\eta_X\colon X \longrightarrow \bar{X}$$

which is a universal arrow from X to p_*. For this we take $\eta_X = c^{(0)}\colon X \to \bar{X}$. This is indeed a map of R-algebras, since the diagram

$$\begin{tikzcd} RX \arrow[rr, "Rc^{(0)}"] \arrow[dr, "p_X"] \arrow[dd, "\theta"'] & & R\bar{X} \arrow[d, "p_{\bar{X}}"] \\ & SX \arrow[r, "Sc^{(0)}"] \arrow[d, "\phi^{(1)}"'] & S\bar{X} \arrow[d, "\phi"] \\ X \arrow[r, "x^{(1)}"'] \arrow[rr, bend right, "c^{(0)}"'] & X^{(1)} \arrow[r, "c^{(1)}"'] & \bar{X} \end{tikzcd}$$

commutes. We now show universality. Suppose we have an S-algebra

$$\begin{tikzcd} SY \arrow[d, "\psi"] \\ Y \end{tikzcd}$$

and a map of R-algebras

$$\begin{tikzcd} RX \arrow[r, "Rf"] \arrow[dd, "\theta"'] & RY \arrow[d, "p_Y"] \\ & SY \arrow[d, "\psi"] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

We seek a unique map of S-algebras

$$\begin{tikzcd}
S\bar{X} \arrow[r, "S\bar{f}"] \arrow[d, "\phi"'] & SY \arrow[d, "\psi"] \\
\bar{X} \arrow[r, "\bar{f}"'] & Y
\end{tikzcd}$$

such that the diagram

$$\begin{tikzcd}
X \arrow[r, "c^{(0)}"] \arrow[dr, "f"'] & \bar{X} \arrow[d, "\bar{f}"] \\
& Y
\end{tikzcd}$$

commutes. We define $\bar{f}$ by defining a cocone

$$\bar{f}^{(i)} \colon X^{(i)} \longrightarrow Y$$

by induction over i.

When $i = 0$, $X^{(i)} = X$, and we define $\bar{f}^{(i)} = \bar{f}^{(0)}$ to be given by $f \colon X \to Y$.

When $i = 1$, $\bar{f}^{(i)} = \bar{f}^{(1)}$ is the unique map such that the diagram

$$\begin{tikzcd}
RX \arrow[r, "p_X"] \arrow[dd, "\theta"'] & SX \arrow[dr, "Sf"] \arrow[dd, "\phi^{(1)}"] & \\
& & SY \arrow[dd, "\psi"] \\
X \arrow[r, "x^{(1)}"'] \arrow[drr, bend right, "f"'] & X^{(1)} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\bar{f}^{(1)}"] & \\
& & Y
\end{tikzcd}$$

commutes. To check that this is well-defined we must check that the outside of this diagram commutes; this is true, since

$$\begin{tikzcd}
RX \arrow[r, "p_X"] \arrow[dr, "Rf"] \arrow[dd, "\theta"'] & SX \arrow[dr, "Sf"] & \\
& RY \arrow[r, "p_Y"'] & SY \arrow[dd, "\psi"] \\
X \arrow[drr, bend right, "f"'] & & \\
& & Y
\end{tikzcd}$$

commutes.

Now let $i > 1$ and suppose that we have defined $\bar{f}^{(i-1)} \colon Y^{(i-1)} \to \bar{Y}$. We define $\bar{f}^{(i)}$ to be the unique map induced by the universal property of the pushout $X^{(i)}$ such that the diagram

$$\begin{tikzcd}
S^2X^{(i-2)} \arrow[r, "S\phi^{(i-1)}"] \arrow[d, "\mu^S_{X^{(i-2)}}"'] & SX^{(i-1)} \arrow[dd, "\phi^{(i)}"] \arrow[dr, "S\bar{f}^{(i-1)}"] & \\
SX^{(i-2)} \arrow[d, "\phi^{(i-1)}"'] & & SY \arrow[dd, "\psi"] \\
X^{(i-1)} \arrow[r, "x^{(i)}"'] \arrow[drr, bend right, "\bar{f}^{(i-1)}"'] & X^{(i)} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\bar{f}^{(i)}"] & \\
& & Y
\end{tikzcd}$$

commutes. Note that the fact that the bottom triangle in this diagram commutes gives us commutativity of the cocone. To check that this is well-defined we must check that the outside of this diagram commutes; this is true, since

$$\begin{tikzcd}
S^2X^{(i-2)} \arrow[r, "S\phi^{(i-1)}"] \arrow[dr, "S^2\bar{f}^{(i-2)}"] \arrow[dd, "\mu^S_{X^{(i-2)}}"'] & SX^{(i-1)} \arrow[r, "S\bar{f}^{(i-1)}"] & SY \arrow[dddd, "\psi"] \\
& S^2Y \arrow[ur, "S\psi"'] \arrow[dd, "\mu^S_X"] & \\
SX^{(i-2)} \arrow[dr, "S\bar{f}^{(i-2)}"'] \arrow[dd, "\phi^{(i-1)}"'] & & \\
& SY \arrow[dr, "\psi"'] & \\
X^{(i-1)} \arrow[rr, "\bar{f}^{(i-1)}"'] & & Y
\end{tikzcd}$$

commutes.

We then define $\bar{f}$ to be the unique map such that, for each $i \geq 0$, the diagram

$$\begin{tikzcd}
X^{(i)} \arrow[r, "c^{(i)}"] \arrow[dr, "\bar{f}^{(i)}"'] & \bar{X} \arrow[d, "\bar{f}"] \\
& Y
\end{tikzcd}$$

commutes. When $i = 0$ this gives us the required commutativity condition for $c^{(0)}$ to be a universal arrow, and uniqueness of $\bar{f}$ comes from the universal property of $\bar{X}$. All that remains is to check that $\bar{f}$ is a map of S-algebras, i.e. that the diagram

$$\begin{tikzcd}
S\bar{X} \arrow[r, "S\bar{f}"] \arrow[d, "\phi"'] & SY \arrow[d, "\psi"] \\
\bar{X} \arrow[r, "\bar{f}"'] & Y
\end{tikzcd}$$

commutes. Since S is finitary, we can write $S\bar{X}$ as

$$S\bar{X} = \operatorname*{colim}_{i \geq 0} SX^{(i)}.$$

Thus we can check that the square above commutes by comparing the cocones corresponding to the maps $\psi \circ S\bar{f}$ and $\bar{f} \circ \phi$. The cocone corresponding to $\psi \circ S\bar{f}$ has components given, for each $i \geq 1$, by the composite

$$SX^{(i)} \xrightarrow{\bar{f}^{(i)}} SY \xrightarrow{\psi} Y.$$

The cocone corresponding to $\bar{f} \circ \phi$ has components given, for each $i \geq 1$, by the composite

$$SX^{(i)} \xrightarrow{\phi^{(i+1)}} X^{(i+1)} \xrightarrow{\bar{f}^{(i+1)}} Y.$$

From the definition of $\bar{f}^{(i+1)}$ we see that the diagram

$$\begin{tikzcd}
SX^{(i)} \arrow[r, "\bar{f}^{(i)}"] \arrow[d, "\phi^{(i+1)}"'] & SY \arrow[d, "\psi"] \\
X^{(i+1)} \arrow[r, "\bar{f}^{(i+1)}"'] & Y
\end{tikzcd}$$

commutes for all $i \geq 0$; hence the cocones described above are equal, so $\bar{f}$ is a map of S-algebras.

Hence we have an adjunction $F \dashv p_*$, as required. ∎

Finally, to show that this construction does indeed give an adjunction

$$\begin{tikzcd}
B\text{-}\mathbf{Alg} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & L\text{-}\mathbf{Alg} \arrow[l, shift left, "u_*"]
\end{tikzcd}$$

in the case $\mathcal{C} = n\text{-}\mathbf{GSet}$, $R = B$, $S = L$, $p = u$, we must show that L is finitary. In fact, this is true of any monad induced by an n-globular operad.

7.5. Lemma. *Let K be an n-globular operad. Then the monad induced by K is finitary, i.e. its underlying endofunctor preserves filtered colimits.*

Proof. It is a result of Leinster that the free strict n-category monad T is finitary [Leinster 2004a, Theorem F.2.2]; the proof that the monad K is finitary is an application of this and of the fact that filtered colimits commute with pullbacks in **Set** [Mac Lane 1998, Theorem IX.2.1].

Let $\mathbb{I}$ be a small, filtered category and let

$$D \colon \mathbb{I} \longrightarrow n\text{-}\mathbf{GSet}$$

be a diagram in n-**GSet**. Then for each $i \in \mathbb{I}$, $KD(i)$ is given by the pullback

$$\begin{tikzcd} KD(i) \arrow[r, "K!"] \arrow[d, "k_{D(i)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & K \arrow[d, "k"] \\ TD(i) \arrow[r, "T!"'] & T1, \end{tikzcd}$$

in n-**GSet**. Write

$$X := \operatorname*{colim}_{i \in \mathbb{I}} D(i).$$

Then KX is given by the pullback

$$\begin{tikzcd} KX \arrow[r, "K!"] \arrow[d, "k_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & K \arrow[d, "k"] \\ TX \arrow[r, "T!"'] & T1, \end{tikzcd}$$

in n-**GSet**. Since T is finitary, we have

$$TX \cong \operatorname*{colim}_{i \in \mathbb{I}} TD(i).$$

Since filtered colimits commute with pullbacks in **Set**, and since limits and colimits are computed pointwise in n-**GSet**, we have that filtered colimits commutes with pullbacks in n-**GSet**, so

$$KX = \operatorname*{colim}_{i \in \mathbb{I}} KD(i).$$

Hence K preserves filtered colimits, i.e. K is finitary. ∎

Hence there is an adjunction $F \dashv u_*$.

8. The relationship between u_* and v_*

The functors u_* and v_* are not equivalences of categories; they should be higher-dimensional equivalences of some kind, but we do not have a formal way of saying this, so instead we approximate this statement. To do so, we consider what happens when we start with an L-algebra, apply u_* to obtain a B-algebra, then apply v_* to that to obtain an L-algebra; in particular, we take some steps towards investigating the relationship between the resulting L-algebra and the original L-algebra. We expect these L-algebras to be in some sense equivalent, but it is not clear how to make this precise, due to the lack of a well-established notion of weak map of L-algebras. The underlying n-globular sets of these L-algebras are the same; they differ only on their algebra actions. We argue that these algebra actions differ only "up to a constraint cell"; we make this statement precise, defining a new notion of weak map of L-algebras in the process.

Recall that, in Proposition 6.1, we showed that $u_* v_* = \mathrm{id}_{B\text{-}\mathbf{Alg}}$. We then gave a small example of an L-algebra

$$\begin{tikzcd} L^2A \arrow[d, "\mu^L_A"] \\ LA. \end{tikzcd}$$

and described its image under the composite

$$L\text{-}\mathbf{Alg} \xrightarrow{u_*} B\text{-}\mathbf{Alg} \xrightarrow{v_*} L\text{-}\mathbf{Alg}.$$

Specifically, we argued that the diagram

$$\begin{tikzcd} L^2A \arrow[r, "L\mathrm{id}_{LA}"] \arrow[d, "v_{LA}"'] & L^2A \arrow[ddd, "\mu^L_A"] \\ BLA \arrow[d, "u_{LA}"'] & \\ L^2A \arrow[d, "\mu^L_A"'] & \\ LA \arrow[r, "\mathrm{id}_{LA}"'] & LA \end{tikzcd}$$

in n-**GSet** "commutes up to a contraction cell". We now extend these ideas to a definition of weak map of L-algebras that uses constraint cells for mediating cells, in order to formalise this idea and thus investigate the relationship between a general L-algebra and its image under the functor $v_* u_*$ more fully. (In fact, our definition is given for algebras for any Batanin operad or Leinster operad.) The idea is that, by using constraint cells, any axioms we would require will automatically be satisfied, so we do not have to state any axioms in the definition. This approach is beneficial, since it is straightforward to specify the data required for a weak map of L-algebras (i.e. to specify where we require mediating cells), but difficult to state the axioms that this data must satisfy.

Note that the definition of weak map that this approach gives is not optimal, for several reasons. First, the fact that the mediating cells must be constraint cells means that this definition lacks generality, since in a fully general definition of weak map we would be able to use any choice of cells that interacted with one another in a suitably coherent way. Second, the composite of two weak maps is not necessarily a weak map, since the mediating cells in the composite are composites of constraint cells, and these are not necessarily constraint cells. Finally, in a non-free L-algebra not all diagrams of constraint n-cells commute, and not all diagrams of constraint cells commute up to a higher constraint cell.

8.1. Definition. *Let K be a Batanin operad or Leinster operad and let*

$$\begin{tikzcd} KX \arrow[d, "\theta"'] & KY \arrow[d, "\phi"] \\ X, & Y, \end{tikzcd}$$

be K-algebras. A weak map of K-algebras *consists of a (not necessarily commuting) square*

$$\begin{tikzcd} KX \arrow[r, "Kf"] \arrow[d, "\theta"'] & KY \arrow[d, "\phi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

*in n-**GSet**, equipped with the following constraint cells:*

- *for all 0-cells x in KX, a constraint 1-cell*

$$f_x \colon \phi \circ Kf(x) \longrightarrow f \circ \theta(x)$$

  in Y;

- *for all 1-cells $a\colon x \to y$ in KX, a constraint 2-cell*

$$\begin{tikzcd} \phi \circ Kf(x) \arrow[r, "\phi\circ Kf(a)"] \arrow[d, "f_x"'] \arrow[dr, phantom, "\swarrow f_a"] & \phi \circ Kf(y) \arrow[d, "f_y"] \\ f \circ \theta(x) \arrow[r, "f\circ\theta(a)"'] & f \circ \theta(y) \end{tikzcd}$$

  in Y;

- *for all 2-cells*

$$\begin{tikzcd} x \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\alpha"] & y \end{tikzcd}$$

  in KX, a constraint 3-cell

$$\begin{tikzcd} \bullet \arrow[r, bend left=50] \arrow[r, bend right=30] \arrow[r, phantom, "\Downarrow\phi\circ Kf(\alpha)"] \arrow[d, "f_x"'] \arrow[dr, phantom, "\swarrow f_b"] & \bullet \arrow[d, "f_y"] & & \bullet \arrow[r, bend left=50] \arrow[d, "f_x"'] \arrow[dr, phantom, "\swarrow f_a"] & \bullet \arrow[d, "f_y"] \\ \bullet \arrow[r, bend right=50] & \bullet & \Rrightarrow_{f_\alpha} & \bullet \arrow[r, bend left=30] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow f\circ\theta(\alpha)"] & \bullet \end{tikzcd}$$

  in Y. We abuse notation slightly and write this as

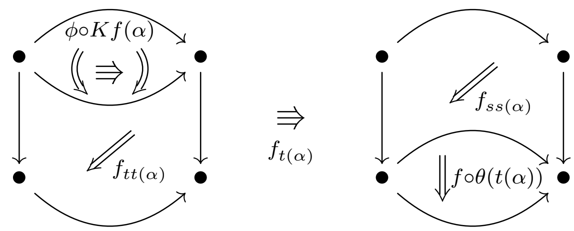
$$f_\alpha \colon f_b \circ (\phi \circ Kf(\alpha)) \Rrightarrow (f \circ \theta(\alpha)) \circ f_a,$$

  omitting the 1-cells f_x and f_y; this makes little difference here, but at higher dimensions it allows us to avoid unwieldy notation;

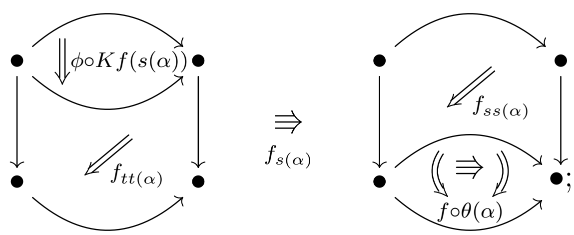
- *for $3 \leq m \leq n-1$, and for all m-cells α in KX, a constraint cell*

$$f_\alpha \colon f_{t(\alpha)} \circ (\phi \circ Kf(\alpha)) \to (f \circ \theta(\alpha)) \circ f_{s(\alpha)}.$$

As described above, we omit lower-dimensional constraint cells from the source and target to avoid unwieldy notation. When $m = 3$, f_α is a constraint 4-cell with source



and target



- *for all n-cells α in KX, an equality (which we can think of as a "constraint $(n+1)$-cell")*

$$f_{t(\alpha)} \circ (\phi \circ Kf(\alpha)) = (f \circ \theta(\alpha)) \circ f_{s(\alpha)}.$$

Note that, given two weak maps of K-algebras

$$\begin{tikzcd}
KX \arrow[r, "Kf"] \arrow[d, "\theta"'] & KY \arrow[r, "Kg"] \arrow[d, "\phi"] & KZ \arrow[d, "\psi"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z,
\end{tikzcd}$$

although their underlying maps of n-globular sets are composable, this composite is not necessarily a weak map of K-algebras, since in a weak map we require the mediating cells to be constraint cells, whereas in the composite gf we only have composites of constraint cells. Thus there is no category of K-algebras with morphisms given by weak maps. There are two ways we could get around this: we could either modify our definition of weak map to allow us to use composites of constraint cells as mediating cells, or we could take the closure under composition of the class of weak maps; either approach is beyond the scope of this paper.

Recall that in Section 5 we gave a definition of equivalence of K-algebras which used strict maps (Definition 5.4). We now modify this definition by using weak maps instead of strict maps, to obtain a notion of weak equivalence of K-algebras.

8.2. Definition. *Let K be an n-globular operad with a contraction and system of compositions. We say that two K-algebras*

$$\begin{array}{ccc} KX & & KY \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\phi} \\ X, & & Y \end{array}$$

are weakly equivalent *if there exists a weak map*

$$\begin{tikzcd} KX \arrow[r, "Kf"] \arrow[d, "\theta"'] & KY \arrow[d, "\phi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} KY \arrow[r, "Kf"] \arrow[d, "\phi"'] & KX \arrow[d, "\theta"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

such that f is surjective on 0-cells, full on m-cells for all $1 \leq m \leq n$, and faithful on n-cells. The map f is referred to as a weak equivalence of K-algebras*.*

As in Definition 5.4, in this definition we require that the weak equivalence can go in either direction, since having a weak equivalence in one direction does not guarantee the existence of a weak equivalence in the opposite direction. This is caused by the fact that our definition of weak map only allows for mediating cells that are constraint cells, rather than allowing any suitably coherent choice of cells.

We now consider the composite

$$L\text{-}\mathbf{Alg} \xrightarrow{u_*} B\text{-}\mathbf{Alg} \xrightarrow{v_*} L\text{-}\mathbf{Alg},$$

and show that any L-algebra is weakly equivalent to its image under this functor.

8.3. Proposition. *Let*

$$LX \xrightarrow{\theta} X$$

be an L-algebra. Then the diagram

$$\begin{tikzcd} LX \arrow[r, "L\mathrm{id}_X"] \arrow[d, "v_X"'] & LX \arrow[ddd, "\theta"] \\ BX \arrow[d, "u_X"'] & \\ LX \arrow[d, "\theta"'] & \\ X \arrow[r, "\mathrm{id}_X"'] & X \end{tikzcd}$$

can be equipped with the structure of a weak map of L-algebras, and this weak map is a weak equivalence.

Proof. We need only show that this diagram can be equipped with the structure of a weak map of L-algebras; if so, it will automatically be a weak equivalence since its underlying map of n-globular sets is the identity. In this proof we write $f := \mathrm{id}_X$ to avoid double subscripts and misleading notation for the mediating constraint cells. Note that although L is defined using an unbiased contraction, in this proof we only use its contraction and system of compositions, as described in Theorem 4.1.

Recall from Definition 2.4 that LX is defined by the pullback

$$\begin{tikzcd} LX \arrow[r, "L!"] \arrow[d, "l_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & L \arrow[d, "l"] \\ TX \arrow[r, "T!"'] & T1. \end{tikzcd}$$

For $1 \leq m \leq n$ we write the elements of LX_m in the form (α, χ), where $\alpha \in TX$, $\chi \in L$, and $T!(\alpha) = l(\chi)$ (note that we do not do this when $m = 0$ since L_0 has only one element). Our approach is to find the required constraint cells using the fact that

$$\begin{tikzcd} L \arrow[r, "v"] \arrow[dr, "l"'] & B \arrow[r, "u"] \arrow[d, "b"] & L \arrow[dl, "l"] \\ & T1 & \end{tikzcd}$$

commutes; thus when we apply $u_X v_X$ to a cell in LX we end up "a contraction cell away" from where we started.

Since the lower-dimensional mediating cells appear in the sources and targets of the mediating cells at the dimensions above, we must define our choice of mediating cells by induction over dimension. As in Definition 8.1, for dimensions greater than 1 we abuse notation slightly by omitting lower-dimensional constraint cells from sources and targets.

At dimension 0 the diagram commutes, so for all $x \in LX_0$ we define

$$f_x := \mathrm{id}_{\theta(x)} \colon \theta(x) \longrightarrow \theta u_X v_X(x).$$

At dimension 1, let

$$(a, p) \colon x \longrightarrow y$$

be a 1-cell in LX, where $a \in TX_1$ and $p \in L_1$. We seek a constraint 2-cell

$$\begin{tikzcd} \phi \circ Kf(x) \arrow[r, "\phi \circ Kf(a,p)"] \arrow[d, "f_x = \mathrm{id}_{\theta(x)}"'] \arrow[dr, phantom, "\Swarrow f_{(a,p)}"] & \phi \circ Kf(y) \arrow[d, "f_y = \mathrm{id}_{\theta(y)}"] \\ f \circ \theta(x) \arrow[r, "f \circ \theta(a,p)"'] & f \circ \theta(y) \end{tikzcd}$$

in LX. We have

$$u_X v_X(a, p) = (a, uv(p)),$$

and we can write the source and target of the required 2-cell as

$$\mathrm{id}_{\theta(y)} \circ \theta(a,p) = \theta(a, \mathrm{id} \circ p)$$

and

$$\theta(a, uv(p)) \circ \mathrm{id}_{\theta(x)} = \theta(a, uv(p) \circ \mathrm{id})$$

respectively, where id denotes the identity on the unique 0-cell of L. Now, since $(uv)_0 = \mathrm{id}_{LX}$, we have

$$s(\mathrm{id} \circ p) = s(uv(p) \circ \mathrm{id}),$$
$$t(\mathrm{id} \circ p) = t(uv(p) \circ \mathrm{id}),$$
$$l(\mathrm{id} \circ p) = l(uv(p) \circ \mathrm{id}).$$

Hence there is a contraction 2-cell $\gamma(\mathrm{id} \circ p, uv(p) \circ \mathrm{id})$ in L. We denote this by κ_p, and define $f_{(a,p)}$ to be the constraint cell

$$f_{(a,p)} := \theta\big(\mathrm{id}_a, \kappa_p\big) \colon \theta(a, \mathrm{id} \circ p) \Rightarrow \theta(a, uv(p) \circ \mathrm{id})$$

in X.

Now let $2 \leq m \leq n$, and suppose that for all $j < m$ and for all j-cells (a,p) in LX we have defined a constraint cell

$$f_{(a,p)} = \theta(\mathrm{id}_a, \kappa_p) \colon \theta(a, \kappa_{t(p)} \circ p) \Rightarrow \theta(a, uv(p) \circ \kappa_{s(p)})$$

in X. Let (α, χ) be an m-cell in LX. We have $u_X v_X(\alpha,\chi) = (\alpha, uv(\chi))$, so we seek a constraint $(m+1)$-cell

$$f_{(\alpha,\chi)} \colon f_{t(\alpha,\chi)} \circ \theta(\alpha, \chi) \to \theta(\alpha, uv(\chi)) \circ f_{s(\alpha,\chi)}.$$

We can write the source of this as

$$f_{t(\alpha,\chi)} \circ \theta(\alpha,\chi) = \theta(\alpha, \kappa_{t(\chi)} \circ \chi),$$

and the target as

$$\theta(\alpha, uv(\chi)) \circ f_{s(\alpha,\chi)} = \theta(\alpha, uv(\chi) \circ \kappa_{s(\chi)}).$$

The cells $\kappa_{t(\chi)} \circ \chi$ and $uv(\chi) \circ \kappa_{s(\chi)}$ are parallel; since the diagram

$$\begin{tikzcd} L \arrow[r, "v"] \arrow[dr, "l"'] & B \arrow[r, "u"] \arrow[d, "b"] & L \arrow[dl, "l"] \\ & T1 & \end{tikzcd}$$

commutes, and since l maps contraction cells to identities in $T1$, we have

$$l(\kappa_{t(\chi)} \circ \chi) = l(\chi) = luv(\chi) = l(uv(\chi) \circ \kappa_{s(\chi)}).$$

Hence there is a contraction $(m+1)$-cell

$$\gamma(\kappa_{t(\chi)} \circ \chi, uv(\chi) \circ \kappa_{s(\chi)})$$

in L (an equality if $m = n$). We denote this by κ_χ, and define $f_{(\alpha,\chi)}$ to be the constraint cell

$$f_{(\alpha,\chi)} = \theta(\mathrm{id}_\alpha, \kappa_\chi) \colon \theta(\alpha, \kappa_{t(\chi)} \circ \chi) \to \theta(\alpha, uv(\chi) \circ \kappa_{s(\chi)}).$$

This equips the diagram

$$\begin{tikzcd}
LX \arrow[r, "L\mathrm{id}_X"] \arrow[d, "v_X"'] & LX \arrow[ddd, "\theta"] \\
BX \arrow[d, "u_X"'] & \\
LX \arrow[d, "\theta"'] & \\
X \arrow[r, "\mathrm{id}_X"'] & X
\end{tikzcd}$$

with the structure of a weak map of L-algebras; thus, since its underlying map of n-globular sets is id_X, it is a weak equivalence of L-algebras. ■

We now justify that, in the example above, the use of constraint cells as mediating cells allows us to avoid having to check any axioms. All of the constraint cells we used in this example were first formed in LX, with the correct source and target; we then applied θ to obtain a constraint cell in X. Thus any diagram we would want to commute, as one of the axioms for a weak map, is the image under θ of a diagram of constraint cells in the free L-algebra

$$L^2X \xrightarrow{\mu^L_X} LX;$$

thus any such diagram commutes (up to a constraint cell of the dimension above in the case of diagrams of cells of dimension less than n), by coherence for L-algebras (see Corollary 5.9). Note that this will not be true for a general weak map of L-algebras, since in a non-free L-algebra not all diagrams of constraint cells commute.

All of this highlights many of the difficulties involved in defining and working with weak maps when using an algebraic definition of weak n-category. The use of weak maps is necessary since, if the definitions of Batanin weak n-category and Leinster weak n-category are equivalent, they are equivalent in some weak sense that cannot be described by strict maps alone. However, the naturally arising notion of map in the algebraic setting is that of strict map; to define a notion of weak map we need to specify a large amount of extra structure and axioms. The idea behind the approach we took in Definition 8.1 was to avoid having to specify all of this extra structure and any axioms by using constraint cells to give maps that are "automatically coherent". This is comparable to the approach taken in non-algebraic definitions, such as those of Street [Street 1987], Tamsamani–Simpson [Tamsamani 1999, Simpson 1997], Joyal [Joyal 1997], and the opetopic definitions [Baez–Dolan 98, Hermida–Makkai–Power 2000, Hermida–Makkai–Power 2001, Hermida–Makkai–Power 2002, Leinster 1998]. In the non-algebraic setting it is meaningless to say that a map is strict, since we have no specified composites for maps to preserve. The natural notion of map is more like a weak map (or a normalised map if the definition has a notion of degeneracies), and consists simply of a map of the underlying data. This is sufficient since the roles of the cells are encoded in their shapes, which are recorded in the underlying data; this is in contrast to the algebraic case in which, once we have applied an algebra action, all cells are globular so we are unable to tell what role they play in the algebra.

References

[Batanin, 1998] M. A. Batanin. Monoidal globular categories as a natural environment for the theory of weak n-categories. *Adv. Math.*, 136(1):39–103, 1998.

[Batanin 2002] M. A. Batanin. On the Penon method of weakening algebraic structures. *J. Pure Appl. Algebra*, 172(1):1–23, 2002.

[Batanin–Cisinski–Weber 2013] Michael Batanin, Denis-Charles Cisinski, and Mark Weber. Multitensor lifting and strictly unital higher category theory. *Theory Appl. Categ.*, 28:No. 25, 804–856, 2013.

[Baez–Dolan 98] John C. Baez and James Dolan. Higher-dimensional algebra. III. n-categories and the algebra of opetopes. *Adv. Math.*, 135(2):145–206, 1998.

[Berger 2002] Clemens Berger. A cellular nerve for higher categories. *Adv. Math.*, 169(1):118–175, 2002.

[van den Berg–Garner 2011] Benno van den Berg and Richard Garner. Types are weak ω-groupoids. *Proc. Lond. Math. Soc. (3)*, 102(2):370–394, 2011.

[Blackwell–Kelly–Power 1989] R. Blackwell, G. M. Kelly, and A. J. Power. Two-dimensional monad theory. *J. Pure Appl. Algebra*, 59(1):1–41, 1989.

[Borceux 1994] Francis Borceux. *Handbook of categorical algebra. 1*, volume 50 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994. Basic category theory.

[Cheng 2010] Eugenia Cheng. Monad interleaving: a construction of the operad for Leinster's weak ω-categories. *J. Pure Appl. Algebra*, 214(6):809–821, 2010.

[Cheng 2011] Eugenia Cheng. Comparing operadic theories of n-category. *Homology Homotopy Appl.*, 13(2):217–248, 2011.

[Cisinski 2007] Denis-Charles Cisinski. Batanin higher groupoids and homotopy types. In *Categories in algebra, geometry and mathematical physics*, volume 431 of *Contemp. Math.*, pages 171–186. Amer. Math. Soc., Providence, RI, 2007.

[Garner 2010] Richard Garner. Homomorphisms of higher categories. *Adv. Math.*, 224(6):2269–2311, 2010.

[Gordon–Power–Street 1995] R. Gordon, A. J. Power, and Ross Street. Coherence for tricategories. *Mem. Amer. Math. Soc.*, 117(558):vi+81, 1995.

[Gurski 2006] Nick Gurski. *An algebraic theory of tricategories*. ProQuest LLC, Ann Arbor, MI, 2006. Thesis (Ph.D.)–The University of Chicago.

[Gurski 2013] Nick Gurski. *Coherence in three-dimensional category theory*, volume 201 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2013.

[Hermida–Makkai–Power 2000] Claudio Hermida, Michael Makkai, and John Power. On weak higher dimensional categories. I. 1. *J. Pure Appl. Algebra*, 154(1-3):221–246, 2000. Category theory and its applications (Montreal, QC, 1997).

[Hermida–Makkai–Power 2001] Claudio Hermida, Michael Makkai, and John Power. On weak higher-dimensional categories. I.2. *J. Pure Appl. Algebra*, 157(2-3):247–277, 2001.

[Hermida–Makkai–Power 2002] Claudio Hermida, Michael Makkai, and John Power. On weak higher-dimensional categories. I. 3. *J. Pure Appl. Algebra*, 166(1-2):83–104, 2002.

[Joyal 1997] André Joyal. Disks, duality and θ-categories, preprint. c. 1997.

[Leinster 1998] Tom Leinster. Structures in higher-dimensional category theory. 1998, eprint `arXiv:math/0109021v1`.

[Leinster 2002] Tom Leinster. A survey of definitions of n-category. *Theory Appl. Categ.*, 10:1–70 (electronic), 2002.

[Leinster 2004a] Tom Leinster. *Higher operads, higher categories*, volume 298 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2004.

[Leinster 2004b] Tom Leinster. Operads in higher-dimensional category theory. *Theory Appl. Categ.*, 12:No. 3, 73–194, 2004.

[Mac Lane 1998] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[Penon 1999] Jacques Penon. Approche polygraphique des ∞-categories non strictes. *Cahiers Topologie Géom. Différentielle Catég.*, 40(1):31–80, 1999.

[Simpson 1997] Carlos Simpson. A closed model structure for n-categories. 1997, eprint `arXiv:alg-geom/9704006v2`.

[Street 1987] Ross Street. The algebra of oriented simplexes. *J. Pure Appl. Algebra*, 49(3):283–335, 1987.

[Smyth–Woolf 2011] Conor Smyth and Jon Woolf. Whitney categories and the Tangle Hypothesis. 2011, eprint `arXiv:1108.3724v1`.

[Tamsamani 1999] Zouhair Tamsamani. Sur des notions de n-catégorie et n-groupoïde non strictes via des ensembles multi-simpliciaux. *K-Theory*, 16(1):51–99, 1999.

[Trimble 1999] Todd Trimble. What are 'fundamental n-groupoids'? 1999. Seminar at DPMMS, Cambridge, 24 August 1999.

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