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AN ALGEBRAIC DEFINITION OF (∞, N)-CATEGORIES

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ABSTRACT. In this paper we define a sequence of monads $\mathbb{T}^{(\infty,n)}(n \in \mathbb{N})$ on the category ∞-$\mathbb{G}$r of ∞-graphs. We conjecture that algebras for $\mathbb{T}^{(\infty,0)}$, which are defined in a purely algebraic setting, are models of ∞-groupoids. More generally, we conjecture that $\mathbb{T}^{(\infty,n)}$-algebras are models for (∞, n)-categories. We prove that our $(\infty, 0)$-categories are bigroupoids when truncated at level 2.

Introduction

The notion of weak (∞, n)-category can be made precise in many ways depending on our approach to higher categories. Intuitively this is a weak ∞-category such that all its cells of dimension greater than n are equivalences.

Models of weak $(\infty, 1)$-categories (case $n = 1$) are diverse: for example there are the quasicategories studied by Joyal and Tierney (see [24]), but also there are other models which have been studied like the Segal categories, the complete Segal spaces, the simplicial categories, the topological categories, the relative categories, and there are known to be equivalent (a survey of models of weak $(\infty, 1)$-categories can be found in [11]).

For any $n \in \mathbb{N}$, models of weak (∞, n)-categories have been studied especially by Segal, Simpson (based on Segal's idea; see [23, 30, 38]), Rezk (Θ_n*-categories*; see [36], but also see [35] for another approach), Bergner (see [10, 12]), and Barwick (who calls them *n-fold complete Segal spaces*; his approach is described in [5, 29]; see also [3]). It is known that some of these models are equivalent in an appropriate sense (see [35, 10, 12, 3]; see also the recent work of Dimitri Ara [2]).

However, these models of (∞, n)-categories are not of an algebraic nature. In this article we propose the first purely algebraic definition of weak (∞, n)-categories (or models of (∞, n)-categories to be more precise) in the globular setting, meaning that we describe these objects as algebras for some monad with good categorical properties. In particular these models are algebras for Batanin's ω-operad. We conjecture that the models of the (∞, n)-categories that we propose here, are equivalent to other existing models in a precise sense explained below. Grothendieck also proposed an algebraic definition of ∞-groupoids in Pursuing Stacks, and Maltsiniotis (see [32]) showed that the latter could be extended to a definition of weak ∞-categories resembling to Batanin's definition based on ω-operads. However, the algebraic nature of our definition of ∞-groupoid is stronger

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than in Grothendieck's approach, because for him, ∞-groupoids are models for a limit theory, while our concept gives algebras over a monad.

Our main motivation for introducing an algebraic model of (∞, n)-categories came from our wish to build a machinery which would lead to a proof of the "Grothendieck conjecture on homotopy types" and, possibly, it generalisation. This conjecture of Grothendieck (see [22, 7]), claims that weak ∞-groupoids encode all homotopical information about their associated topological spaces. In his seminal article (see [7]), Michael Batanin gave an accurate formulation of this conjecture by building a fundamental weak ∞-groupoid functor between the category of topological spaces $\mathbb{T}op$ to the category of the weak ∞-groupoids in his sense. This conjecture is not solved yet, and a good direction to solve it should be to build first a Quillen model structure on the category of weak ω-groupoids in the sense of Michael Batanin, and then show that his fundamental weak ∞-groupoid functor is the right part of a Quillen equivalence. One obstacle for building such a model structure is that the category of Batanin ∞-groupoids is defined in a nonalgebraic way. An important property of the category of weak ∞-groupoids $\mathbb{A}lg(\mathbb{T}^{(\infty,0)})$ (see Section 3.9) that we propose here is to be locally presentable (see Section 3). Therefore, we hope that this will allow us in the future to use Smith's theory on combinatorial model categories in our settings (see [9]).

More generally, we expect that it is be possible to build an adapted combinatorial model category structure for each category $\mathbb{A}lg(\mathbb{T}^{(\infty,n)})$ of these models of weak (∞, n)-categories (see Section 3.9) for arbitrary $n \in \mathbb{N}$, in order to be able to prove the existence of Quillen equivalences between our models of (∞, n)-categories and other models of (∞, n)-categories. This should be considered as a generalization of the Grothendieck conjecture for higher integers $n > 0$.

The aim of our present paper is to lay a categorical foundation for this multistage project. The model theoretical aspects of this project will be considered in future papers (but see Remark 3.7 about possible approaches).

Our algebraic description of weak (∞, n)-categories is an adaptation of the "philosophy" of categorical stretchings as developed by Jacques Penon in [34] to describe his weak ∞-categories (see also [16, 25]). Here we add the key concept of (∞, n)-graphs (see Section 1).

Weak ∞-categories in the sense of Penon can be seen as algebras for a specific ω-operad in the sense of Batanin, called by Batanin "the Penon operad" (see [8]). This result is deep and involves the complex machinery of higher computads. The goal of this article of Batanin was to prove that his weak ∞-categories were weaker than those of Penon. However, Batanin did not construct a specific ω-operad for each integer $n \in \mathbb{N}$ whose algebras were models of weak (∞, n)-categories. It is therefore impossible in the present paper to compare our weak (∞, n)-categories with anything in Batanin's work.

However it would be interesting to know whether our models of (∞, n)-categories do underlie a specific ω-operad for each integer $n \in \mathbb{N}$, of the kind Batanin produced for Penon's weak ∞-categories. That project is quite difficult and deserves further work which is indeed in progress.

Also it is important to notice that in [15], Dominique Bourn and Jacques Penon developed an inductive procedure to categorify structures defined by a cartesian monad. Basically they start with a cartesian monad, say for instance the monad of monoids, and then their procedure allows the categorification of this monad to produce the monad of monoidal categories, and so on. Their procedure applies to the monad of monoids, because it is a cartesian monad. Unfortunately we cannot use their inductive procedure in this article, because our monads are not cartesian. For example, the monad for groupoids is not cartesian. So we cannot use their inductive procedure to obtain a monad for weak ∞-groupoids (case $n = 0$) of the kind we are able to construct in the present article. In this present article we prefer to build these higher structures directly, by using adapted *stretchings* (see 3.8), thereby avoiding any inductive process, similar to what Penon did in [34].

The plan of this article is as follows.

In Section 1 we introduce *reversors*, which are the operations algebraically describing equivalences. These operations plus the brilliant idea of categorical stretching developed by Jacques Penon (see [34]) are in the heart of our approach to weak (∞, n)-categories.

Section 2 introduces the reader to strict (∞, n)-categories, where we point out the important fact that reversors are "canonical" in the "strict world". Reflexivity for strict (∞, n)-categories is seen as specific structure, using operations that we call *reflexors*, and we study in detail the relationships between *reversors* and *reflexors* (see 2.4). However most material of this section is well known.

Section 3 gives the steps in defining our algebraic approach to weak (∞, n)-categories. First we recall briefly the definition of weak ∞-categories in Penon's sense. Then we define (∞, n)-magmas (see 3.5), which are the "(∞, n)-analogue" of the ∞-magmas of Penon. Then we define (∞, n)-categorical stretching (see 3.8), which is the "(∞, n)-analogue" of the categorical stretching of Penon. In [34], Jacques Penon used categorical stretching to weakened strict ∞-categories. Roughly speaking, the philosophy of Penon follows the idea that the "weak" must be controlled by the "strict", and it is exactly what the (∞, n)-categorical stretchings do for the "(∞, n)-world". Thirdly we give the definition of weak (∞, n)-categories (see 3.9) as algebras for specific monads $\mathbb{T}^{(\infty,n)}$ on ∞-$\mathbb{G}$r. We show in 3.12 that each $\mathbb{T}^{(\infty,n)}$-algebra (G, v) puts on G a canonical (∞, n)-magma structure. Then, as we do for the strict case, we study the more subtle relationship between *reversors* and *reflexors* for weak (∞, n)-categories (see 3.13). Finally in 3.14, we make some computations for weak ∞-groupoids. We show that models of our weak ∞-groupoids in dimension 2 are bigroupoids.

The final section 4.2 explains how other choices of (∞, n)-*structure* could have been used to build other algebraic models of weak (∞, n)-categories, and among these choices, the *maximal* (∞, n)-*structure* is the one we use in this article, and the *minimal* (∞, n)-*structure* is another remarkable (∞, n)-structure.

The main ideas of this article were exposed for the first time in September 2011, in the Australian Category Seminar at Macquarie University [27].

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I dedicate this work to Ross Street.

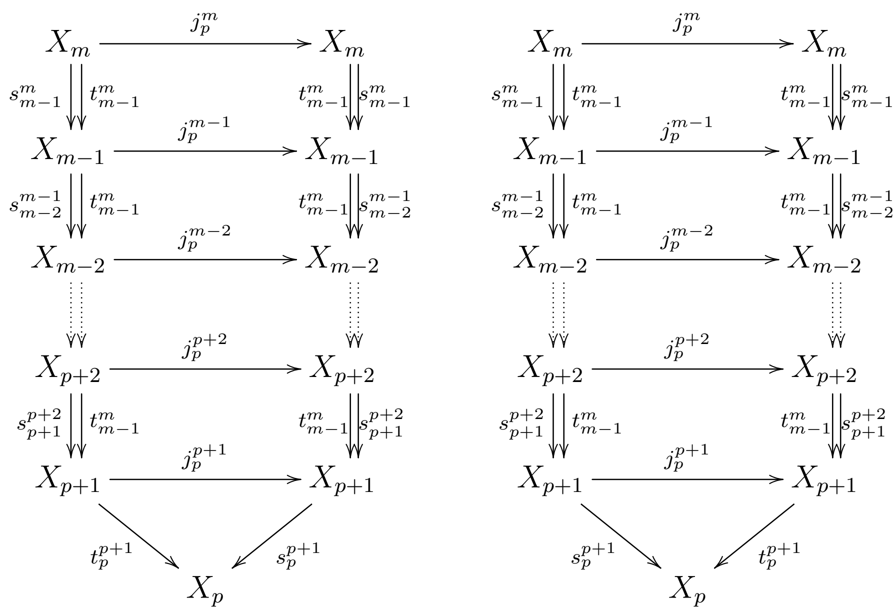
1. (∞, n)-graphs

Let $\mathbb{G}$ be the globe category defined as follows. For each $m \in \mathbb{N}$, objects of $\mathbb{G}$ are formal objects $\overline{m}$. Morphisms of $\mathbb{G}$ are generated by the formal cosource and cotarget $\overline{m} \underset{t_m^{m+1}}{\overset{s_m^{m+1}}{\rightrightarrows}} \overline{m+1}$ such that we have the following relations $s_m^{m+1}s_{m-1}^m = s_m^{m+1}t_{m-1}^m$ and $t_m^{m+1}t_{m-1}^m = t_m^{m+1}s_{m-1}^m$. An ∞-*graph* X is just a presheaf $\mathbb{G}^{op} \xrightarrow{X} \mathbb{S}et$. We denote by ∞-$\mathbb{G}$r $:= [\mathbb{G}^{op}, \mathbb{S}et]$ the category of ∞-graphs where morphisms are just natural transformations. If X is an ∞-graph, sources and targets are still denoted by s_m^{m+1} and t_m^{m+1}. If $0 \leq p < m$ we define $s_p^m := s_p^{p+1} \circ ... \circ s_{m-1}^m$ and $t_p^m := t_p^{p+1} \circ ... \circ t_{m-1}^m$.

An (∞, n)-*graph* is given by a couple $(X, (j_p^m)_{0 \leq n \leq p < m})$ where X is an ∞-graph (see [34]) or “globular set” (see [7]), and j_p^m are maps $(0 \leq n \leq p < m)$, called the *reversors*

$$X_m \xrightarrow{j_p^m} X_m \ ,$$

such that for all integers n, m, and p such that $0 \leq n \leq p < m$ we have the following two diagrams in $\mathbb{S}et$, each commuting serially.



We shall say also that an (∞, n)-*graph* is an ∞-graph X equipped with an (∞, n)-*structure.*

1.1. Remark. These two diagrams looks equal, but it is their bottoms which are different, and are one of the key of our approach of *reversibility.* Also, to describe it we have preferred to use diagrams than equations, which we believe make it easier to be understood for the reader.

1.2. Remark. In the last section 4.2 we will see other interesting (∞, n)-*structures* on ∞-graphs, where the (∞, n)-structure just above shall be called the *maximal* (∞, n)-*structure.*

A morphism of (∞, n)-graphs

$$(X, (j_p^m)_{0\le n\le p<m}) \xrightarrow{\varphi} (X', (j_p'^m)_{0\le n\le p<m})$$

is given by a morphism of ∞-graphs $X \xrightarrow{\varphi} X'$ which is compatible with the reversors: this means that, for integers $0 \le n \le p < m$, we have the following commutative square.

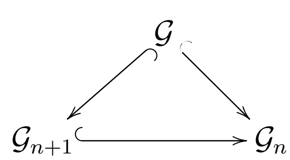
$$\begin{array}{ccc} X_m & \xrightarrow{\varphi_m} & X'_m \\ {\scriptstyle j_p^m}\downarrow & & \downarrow{\scriptstyle j_p'^m} \\ X_m & \xrightarrow[\varphi_m]{} & X'_m \end{array}$$

The category of (∞, n)-graphs is denoted (∞, n)-$\mathbb{G}$r.

1.3. Remark. In [25] we defined the category (∞, n)-$\mathbb{G}$r of "∞-graphes n-cellulaires" (n-cellular ∞-graphs) which is a completely different category from this category (∞, n)-$\mathbb{G}$r. The category (∞, n)-$\mathbb{G}$r was used to define an algebraic approach to "weak n-higher transformations", still in the same spirit of the weak ∞-categories of Penon (see [34], but a quick review of this approach is in 3.1 below).

1.4. Remark. Throughout this paper the reversors are denoted by the symbols "j^m_p" except with weak (∞, n)-categories (see comment in Section 3.12) where they are denoted by the symbols "i^m_p". Let us also make a little comment on reflexive ∞-graphs (see [34] for their definition). For us a reflexive ∞-graph $(X, (1^p_m)_{0\leq p<m})$ must be seen as a "structured ∞-graph" : that is, an ∞-graph X equipped with a structure $(1^p_m)_{0\leq p<m}$, where the maps $X(p) \xrightarrow{1^p_m} X(m)$ must be considered as specific operations that we call *reflexors*. Throughout this paper these operations are denoted by the symbols 1^p_m except for the underlying reflexive structure of weak (∞, n)-categories (see Section 3.12) where, instead, they are denoted by the symbols ι^p_m (with the Greek letter "iota"). Morphisms between reflexive ∞-graphs are morphisms of ∞-graphs which respect this structure. In [34] the category of reflexive ∞-graphs is denoted ∞-$\mathbb{G}$rr. The canonical forgetful functor $\infty\text{-}\mathbb{G}\mathrm{rr} \xrightarrow{\mathbb{U}} \infty\text{-}\mathbb{G}\mathrm{r}$ is a right adjoint, and gives rise to the very important monad $\mathbb{R}$ of reflexive ∞-graphs on ∞-graphs.

The reversors are built without using limits, and it is trivial to build the sketch[1] $\mathcal{G}_n$ of (∞, n)-graphs. It has no cones and no cocones, thus (∞, n)-$\mathbb{G}$r is just a category of presheaves $(\infty, n)\text{-}\mathbb{G}\mathrm{r} \simeq [\mathcal{G}_n, \mathbb{S}et]$. Denote by $\mathcal{G}$ the sketch of ∞-graphs (it is the category $\mathbb{G}^{op}$ at the beginning of this section). We have the inclusions



showing that the functor

$$(\infty, n)\text{-}\mathbb{G}\mathrm{r} \xrightarrow{M_n} (\infty, n+1)\text{-}\mathbb{G}\mathrm{r}$$

forgetting the reversors $(j^m_n)_{m\geq n+2}$ of each (∞, n)-graph has a left and a right adjoint: $L_n \dashv M_n \dashv R_n$. The functor L_n is the "free (∞, n)-graphisation functor" on $(\infty, n+1)$-graphs, and the functor R_n is the "internal (∞, n)-graphisation functor" on $(\infty, n+1)$-graphs. The forgetful functor

$$(\infty, n)\text{-}\mathbb{G}\mathrm{r} \xrightarrow{O_n} \infty\text{-}\mathbb{G}\mathrm{r}$$

which forgets all the reversors, has a left and a right adjoint: $G_n \dashv O_n \dashv D_n$. The functor G_n is the "free (∞, n)-graphisation functor" on ∞-graphs, and the functor D_n is

[1] see [18, 14] for good references on sketch theory.

the "internal (∞,n)-graphisation functor" on ∞-graphs. Both M_n and O_n are monadic because they are conservative and, as well as left adjoints, have rights adjoints (and so preserve all coequalizers).

2. Strict (∞, n)-categories ($n \in \mathbb{N}$)

2.1. DEFINITION. The definition of the category ∞-$\mathbb{C}at$ of ∞-categories, in the form needed here, can be found in [34]. A strict ∞-category C has operations $\circ^m_p$, for all $0 \leq p < m$, which are maps

$$\circ^m_p : C(m) \underset{C(p)}{\times} C(m) \longrightarrow C(m)$$

where $C(m) \underset{C(p)}{\times} C(m) = \{(y,x) \in C(m) \times C(m) : s^m_p(y) = t^m_p(x)\}$.

Recall that the ∞-graph domains and codomains of these operations must satisfy the following conditions: If $(y,x) \in C(m) \underset{C(p)}{\times} C(m)$, then

- for $0 \leq p < q < m$, $s^m_q(y \circ^m_p x) = s^m_q(y) \circ^q_p s^m_q(x)$ and $t^m_q(y \circ^m_p x) = t^m_q(y) \circ^q_p t^m_q(x)$
- for $0 \leq p = q < m$, $s^m_q(y \circ^m_p x) = s^m_q(x)$ and $t^m_q(y \circ^m_p x) = t^m_q(x)$.

These are the *positional axioms* in the terminology of [34].

If we denote by $(C, (1^p_m)_{0 \leq p < m})$ the underlying reflexivity structure on C, then the operations 1^p_m are just an abbreviation for $1^{m-1}_m \circ \ldots \circ 1^p_{p+1}$. These reflexivity maps 1^p_m are called *reflexors* to emphasise that we see the reflexivity as specific structure.

Now let $\alpha \in C(m)$ be an m-cell of C. We say that α has an $\circ^m_p$-inverse $(0 \leq p < m)$ if there is an m-cell $\beta \in C(m)$ such that $\alpha \circ^m_p \beta = 1^p_m(t^m_p(\alpha))$ and $\beta \circ^m_p \alpha = 1^p_m(s^m_p(\alpha))$.

A strict (∞, n)-category C is a strict ∞-category such that for all $0 \leq n \leq p < m$, every m-cell $\alpha \in C(m)$ has an $\circ^m_p$-inverse. If such an inverse exists then it is unique, because it is an inverse for a morphism in a category. Thus every strict (∞, n)-category C has an underlying canonical (∞, n)-graph $(C, (j^m_p)_{0 \leq n \leq p < m})$ such that the maps j^m_p give the unique $\circ^m_p$-inverse for each m-cell of C. In other words, for each m-cell α of C such that $0 \leq n \leq p < m$, we have $\alpha \circ^m_p j^m_p(\alpha) = 1^p_m(t^m_p(\alpha))$ and $j^m_p(\alpha) \circ^m_p \alpha = 1^p_m(s^m_p(\alpha))$. Strict ∞-functors respect the reversibility. As a matter of fact, consider two strict (∞, n)-categories C and C' and a strict ∞-functor $C \xrightarrow{F} C'$. If α is an m-cell of C, then for all $0 \leq n \leq p < m$, we have

$$\begin{aligned}
F(j^m_p(\alpha) \circ^m_p \alpha) &= F(j^m_p(\alpha)) \circ^m_p F(\alpha) \\
&= F(1^p_m(s^m_p(\alpha))) \\
&= 1^p_m(F(s^m_p(\alpha)))1^p_m(s^m_p(F(\alpha))) \\
&= j^m_p(F(\alpha)) \circ^m_p F(\alpha)
\end{aligned}$$

which shows, by the unicity of $j_p^m(F(\alpha))$, that $F(j_p^m(\alpha)) = j_p^m(F(\alpha))$. Thus morphisms between strict (∞, n)-categories are just strict ∞-functors. Thus the category of strict (∞, n)-categories, denoted by (∞, n)-$\mathbb{C}at$, is a full subcategory of ∞-$\mathbb{C}at$.

It is not difficult to see that there is a projective sketch $\mathcal{C}_n$ satisfying an equivalence of categories $\mathbb{M}od(\mathcal{C}_n) \simeq (\infty, n)$-$\mathbb{C}$at. Thus, for all $n \in \mathbb{N}$, the category (∞, n)-$\mathbb{C}at$ is locally presentable.

Furthermore, for each $n \in \mathbb{N}$, we have the following forgetful functor

$$(\infty, n)\text{-}\mathbb{C}\mathrm{at} \xrightarrow{U_n} \infty\text{-}\mathbb{G}\mathrm{r} .$$

There is an inclusion $\mathcal{G} \subset \mathcal{C}_n$, and this inclusion of sketches produces, on passing to models, a functor C_n between the categories of models

$$\mathbb{M}od(\mathcal{C}_n) \xrightarrow{C_n} \mathbb{M}od(\mathcal{G}) ,$$

and the associated sheaf theorem for sketches of Foltz (see [21]) yields that C_n has a left adjoint. Thus the following commutative square induced by the previous equivalence of categories

$$\begin{array}{ccc} \mathbb{M}od(\mathcal{C}_n) & \xrightarrow{C_n} & \mathbb{M}od(\mathcal{G}) \\ \wr\downarrow & & \downarrow\wr \\ (\infty, n)\text{-}\mathbb{C}\mathrm{at} & \xrightarrow{U_n} & \infty\text{-}\mathbb{G}\mathrm{r} \end{array}$$

produces the required left adjoint $F_n \dashv U_n : (\infty, n)\text{-}\mathbb{C}\mathrm{at} \longrightarrow \infty\text{-}\mathbb{G}\mathrm{r}$.

The unit and the counit of this adjunction are respectively denoted by $\lambda_s^{(\infty,n)}$ and $\varepsilon_s^{(\infty,n)}$. Using Beck's theorem of monadicity (see for instance [14]), we see that these functors U_n are monadic. This adjunction generates a monad $\mathbb{T}_s^{(\infty,n)} = (T_s^{(\infty,n)}, \mu_s^{(\infty,n)}, \lambda_s^{(\infty,n)})$ on ∞-$\mathbb{G}$r. It is the monad for strict (∞, n)-categories on ∞-graphs.

2.2. Remark. For each $n \in \mathbb{N}$, when no confusion appears, we will simplify the notation of these monads by omitting the symbol ∞; so $\mathbb{T}_s^n = (T_s^n, \mu_s^n, \lambda_s^n)$ is the same as $\mathbb{T}_s^{(\infty,n)} = (T_s^{(\infty,n)}, \mu_s^{(\infty,n)}, \lambda_s^{(\infty,n)})$.

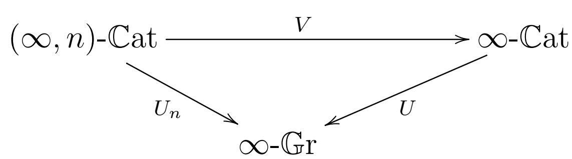
As we did for (∞, n)-graphs (see Section 1) by building functors of "(∞, n)-graphisation", we are going to build some functors of "strict (∞, n)-categorification" by using systematically the Dubuc adjoint triangle theorem (see theorem 1 page 72 in [20]).

For all $n \in \mathbb{N}$ we have the following triangle in $\mathbb{C}AT$

$$\begin{array}{ccc} (\infty, n)\text{-}\mathbb{C}\mathrm{at} & \xrightarrow{\quad V_n \quad} & (\infty, n+1)\text{-}\mathbb{C}\mathrm{at} \\ & {\scriptstyle U_n}\searrow \quad \swarrow{\scriptstyle U_{n+1}} & \\ & \infty\text{-}\mathbb{G}\mathrm{r} & \end{array}$$

where the functor V_n forgets the reversors $(j_n^m)_{m\geq n+2}$ for each strict (∞,n)-category, and we have the adjunctions $F_n \dashv U_n$ and $F_{n+1} \dashv U_{n+1}$, where in particular $U_{n+1}V_n = U_n$ and U_{n+1} is monadic. So we can apply the Dubuc adjoint triangle theorem which shows that the functor V_n has a left adjoint: $L_n \dashv V_n$. For each strict $(\infty, n+1)$-category C, the left adjoint L_n of V_n assigns the free strict (∞,n)-category $L_n(C)$ associated to C. The functor L_n is the "free strict (∞,n)-categorification functor" for strict $(\infty, n+1)$-categories. Notice that the functor V_n has an evident right adjoint R_n. For each strict $(\infty, n+1)$-category C, the right adjoint R_n of V_n assigns the maximal strict (∞,n)-category $R_n(C)$ associated to C. This is simply because, if D is an object of (∞,n)-$\mathbb{C}at$, then the unit map $D \xrightarrow{\eta_n} R_n(V_n(D))$ is just the identity 1_D, and its universality becomes straightforward.

We can apply the same argument to the following triangle in $\mathbb{C}AT$ (where here the functor V forgets all the reversors or can be seen as an inclusion)



to prove that the functor V has a left adjoint: $L \dashv V$. For each strict ∞-category C, the left adjoint L of V assigns the free strict (∞,n)-category $L_n(C)$ associated to C. The functor L is the "free strict (∞,n)-categorification functor" for strict ∞-categories. Notice also that the functor V has an evident right adjoint R, by the same argument as before, for the adjunction $V_n \dashv R_n$.

2.3. Remark. The previous functors V_n and V are, from our point of view, not only inclusions but also "trivial forgetful functors". Indeed for instance, they occur in the paper [1] where they do not see strict ∞-groupoids (which are in our terminology $(\infty,0)$-categories) as strict ∞-categories equipped with canonical reversible structures. So from their point of view V is just an inclusion. We do not claim their point of view is incorrect but we believe that our point of view, which is more algebraic (the reversors j_p^m must be seen as unary operations), shows clearly that this inclusion is also a forgetful functor which forgets the canonical and unique reversible structures of some specific strict ∞-categories. Basically in our point of view, a strict (∞,n)-category ($n \in \mathbb{N}$) is a strict ∞-category equipped with some canonical specific structure.

2.4. (∞,n)-Involutive structures and (∞,n)-reflexivity structures. Involutive properties and reflexive structures (see below) are an important part of each strict (∞,n)-categories ($n \in \mathbb{N}$). We could have spoken about strict (∞,n)-categories without referring to these two specific structures. Yet we believe it is informative to especially point out that, while these two structures are canonical in the world of strict (∞,n)-categories, they are not canonical in the world of weak (∞,n)-categories (see Section 3.13). In particular we will show that they cannot be weakened for weak (∞,n)-categories, but only for some specific equalities which are part of these two kinds of structures (see Section 3).

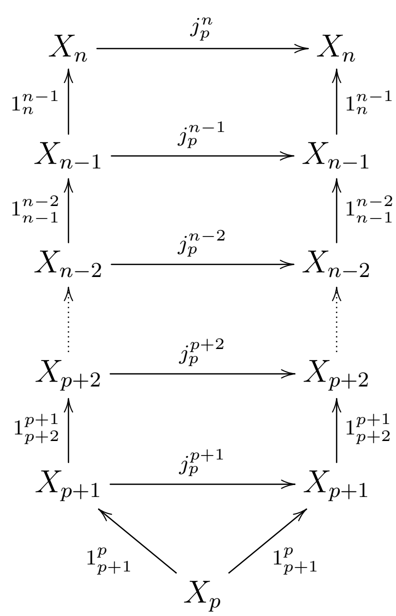
This observation indicates that some properties or structures, true in the world of strict (∞,n)-categories, might or not be weakened in the world of weak (∞,n)-categories.

An *involutive* (∞,n)-*graph* is an (∞,n)-graph $(X,(j^m_p)_{0\leq n\leq p<m})$ satisfying $j^m_p\circ j^m_p=1_{X_m}$. Involutive (∞,n)-graphs form a full reflexive subcategory $i(\infty,n)$-$\mathbb{G}$r of (∞,n)-$\mathbb{G}$r. For each $n\in\mathbb{N}$, each strict (∞,n)-category C has its underlying (∞,n)-graph $(C,(j^m_p)_{0\leq n\leq p<m})$ an involutive (∞,n)-graph. Indeed, for each $0\leq n\leq p<m$ and each m-cell $\alpha\in C(m)$, we have

$$j^m_p(j^m_p(\alpha))\circ^m_p j^m_p(\alpha)=1^p_m(s^m_p(j^m_p(\alpha)))=1^p_m(t^m_p(\alpha))\ ;$$

thus $j^m_p(j^m_p(\alpha))$ is an $\circ^m_p$-inverse of $j^m_p(\alpha)$. By uniqueness, $j^m_p(j^m_p(\alpha))=\alpha$.

A *reflexive* (∞,n)-*graph* is a triple $(X,(1^p_m)_{0\leq p<m},(j^m_p)_{0\leq n\leq p<m})$ where $(X,(1^p_m)_{0\leq p<m})$ is an ∞-graph equipped with a reflexivity structure $(1^p_m)_{0\leq p<m}$, where $(X,(j^m_p)_{0\leq n\leq p<m})$ is an (∞,n)-graph, and such that we have the commutative diagram



in $\mathbb{S}et$, expressing the relation between the truncation at level n of the reflexors 1^p_m and the reversors j^m_p $(0\leq n\leq p<m)$. Thus, for all $0\leq n\leq q<m$ and $q\geq p\geq 0$, we have $j^m_q(1^p_m(\alpha))=1^p_m(\alpha)$ and, for all $0\leq n\leq q<p<m$, we have $j^m_q(1^p_m(\alpha))=1^p_m(j^p_q(\alpha))$. Morphisms between reflexive (∞,n)-graphs are those which are morphisms of reflexive ∞-graphs and morphisms of (∞,n)-graphs. The category of reflexive (∞,n)-graphs is denoted by (∞,n)-$\mathbb{G}$rr.

For each $n\in\mathbb{N}$, each strict (∞,n)-category C has its underlying (∞,n)-graph $(C,(j^m_p)_{0\leq n\leq p<m})$ equipped with a reflexive (∞,n)-graph $(C,(1^p_m)_{0\leq p<m},(j^m_p)_{0\leq n\leq p<m})$ structure where $(1^p_m)_{0\leq p<m}$ is the reflexive structure of the underlying strict ∞-category of C. As a matter of fact for all $0\leq n\leq q<p<m$, we have $j^m_q(1^p_m(\alpha))\circ^m_q 1^p_m(\alpha)=1^q_m(s^m_q(1^p_m(\alpha)))=1^q_m(s^p_q(\alpha))$. Also we have the following axiom for strict ∞-categories: if $q<p<m$ and $s^p_q(y)=t^p_q(x)$ then $1^p_m(y\circ^p_q x)=1^p_m(y)\circ^m_q 1^p_m(x)$ (see [34]). But

here we have $s^p_q(j^p_q(\alpha)) = t^{q+1}_q s^{q+2}_{q+1}...s^p_{p-1}(\alpha) = t^p_q(\alpha)$, thus we can apply this axiom: $1^p_m(j^p_q(\alpha)) \circ^m_q 1^p_m(\alpha) = 1^p_m(j^p_q(\alpha) \circ^p_q \alpha) = 1^p_m(1^q_p(s^p_q(\alpha))) = 1^q_m(s^p_q(\alpha))$ which shows that $1^p_m(j^p_q(\alpha))$ is the unique $\circ^m_q$-inverse of $1^p_m(\alpha)$ and thus $1^p_m(j^p_q(\alpha)) = j^m_q(1^p_m(\alpha))$. Also for all $0 \leq n \leq q < m$ and $q \geq p \geq 0$, we have $j^m_q(1^p_m(\alpha)) \circ^m_q 1^p_m(\alpha) = 1^q_m(s^m_q(1^p_m(\alpha))) = 1^q_m(1^p_q(\alpha)) = 1^p_m(\alpha)$ and $1^p_m(\alpha) \circ^m_q 1^p_m(\alpha) = 1^p_m(\alpha)$ because $q \geq p$, thus $1^p_m(\alpha)$ is the unique $\circ^m_q$-inverse of $1^p_m(\alpha)$ and thus $1^p_m(\alpha) = j^m_q(1^p_m(\alpha))$.

As in Section 2, it is not difficult to show some similar results for the category $i(\infty,n)$-$\mathbb{G}$r and the category (∞,n)-$\mathbb{G}$rr ($n \in \mathbb{N}$):

- For each $n \in \mathbb{N}$, the categories $i(\infty,n)$-$\mathbb{G}$r and (∞,n)-$\mathbb{G}$rr are both locally presentable.
- For each $n \in \mathbb{N}$, there is a monad $\mathbb{I}^{(\infty,n)}_i = (I^{(\infty,n)}_i, \mu^{(\infty,n)}_i, \lambda^{(\infty,n)}_i)$ on ∞-$\mathbb{G}$r (i is for "involutive") with $\mathbb{A}lg(\mathbb{I}^{(\infty,n)}_i) \simeq i(\infty,n)$-$\mathbb{G}$r, and a monad $\mathbb{R}^{(\infty,n)}_r = (R^{(\infty,n)}_r, \mu^{(\infty,n)}_r, \lambda^{(\infty,n)}_r)$ on ∞-$\mathbb{G}$r (r is for "reflexive") with $\mathbb{A}lg(\mathbb{R}^{(\infty,n)}_r) \simeq (\infty,n)$-$\mathbb{G}$rr.
- We can also consider the category $i(\infty,n)$-$\mathbb{G}$rr of *involutive* (∞,n)-*graphs equipped with a specific reflexivity structure*, whose morphisms are those of (∞,n)-$\mathbb{G}$r which respect the reflexivity structure. This category $i(\infty,n)$-$\mathbb{G}$rr is also locally presentable. For each $n \in \mathbb{N}$, there is a monad $\mathbb{K}^{(\infty,n)}_{ir} = (K^{(\infty,n)}_{ir}, \mu^{(\infty,n)}_{ir}, \lambda^{(\infty,n)}_{ir})$ on ∞-$\mathbb{G}$r (ir is for "involutive-reflexive") with $\mathbb{A}lg(\mathbb{K}^{(\infty,n)}_{ir}) \simeq i(\infty,n)$-$\mathbb{G}$rr. There is a forgetful functor from the category $i(\infty,n)$-$\mathbb{G}$rr to the category (∞,n)-$\mathbb{G}$rr which has a left adjoint, the functor "(∞,n)-involution" of any reflexive (∞,n)-graph. There is a forgetful functor from the category $i(\infty,n)$-$\mathbb{G}$rr to the category $i(\infty,n)$-$\mathbb{G}$r which has a left adjoint, the functor "(∞,n)-reflexivisation" of any involutive (∞,n)-graph. These left adjoints are built using the Dubuc adjoint triangle theorem.

3. Weak (∞,n)-categories ($n \in \mathbb{N}$)

In this section we define our algebraic point of view of weak (∞,n)-categories for all $n \in \mathbb{N}$. As the reader will see, many kind of filtrations as in Section 2 could be studied here, because their filtered colimits do exist. But we have avoided that, because all the filtrations involved here are not built with "inclusion functors" but are all right adjunctions, and the author has not found a good description of their corresponding filtered colimits. We do hope to afford it in a future work because we believe that these filtered colimits have their own interest in abstract homotopy theory, and also in higher category theory. We start this section by recalling briefly[2] the definition of the weak ∞-categories in Penon's sense.

[2]For a self-contained text and the convenience of the reader.

3.1. Weak ∞-categories in Penon's sense. For all $n \in \mathbb{N}$, each model of weak (∞, n)-category that we are going to define (see 3.9) is a weak ∞-categories in Penon's sense.

3.2. Definition. *An ∞-magma*[3] *is a reflexive ∞-graph M equipped for all $0 \leq p < m$ with operations $\circ_p^m$*

$$\circ_p^m : M(m) \underset{M(p)}{\times} M(m) \longrightarrow M(m)$$

where $M(m) \underset{M(p)}{\times} M(m) = \{(y,x) \in M(m) \times M(m) : s_p^m(y) = t_p^m(x)\}$, and the operations $\circ_p^m$ satisfy only positional axioms as in 2.1. Morphisms between ∞-magmas are morphisms of reflexive ∞-graphs which preserve these operations. We write ∞-$\mathbb{M}ag$ for the category of ∞-magmas.

3.3. Definition. *A categorical stretching is given by a quadruple*

$$\mathbb{E} = (M, C, \pi, ([-,-]_m)_{m\in\mathbb{N}})$$

where M is an ∞-magma, C is a strict ∞-category, π is a morphism in ∞-$\mathbb{M}ag$, and $([-,-]_m)_{m\in\mathbb{N}})$ is an extra structure called the "bracketing structure", and which is the key structure of the Penon approach to weakening the axioms of strict ∞-categories; let be more precise about it: If $m \geq 1$, two m-cells c_1, c_0 of M are parallel if $t_{m-1}^m(c_1) = t_{m-1}^m(c_0)$ and if $s_{m-1}^m(c_1) = s_{m-1}^m(c_0)$, and in that case we denote it $c_1 \| c_0$.

$$([-,-]_m : \widetilde{M}_m \longrightarrow M_{m+1})_{m\in\mathbb{N}}$$

is a sequence of maps, where

$$\widetilde{M}_m = \{(c_1, c_0) \in M_m \times M_m : c_1 \| c_0 \text{ and } \pi_m(c_1) = \pi_m(c_0)\} ,$$

and such that

- $\forall (c_1, c_0) \in \widetilde{M}_m,\ t_{m-1}^m([c_1, c_0]_m) = c_1,\ s_{m-1}^m([c_1, c_0]_m) = c_0,$
- $\pi_{m+1}([c_1, c_0]_m) = 1_m^{m-1}(\pi_m(c_1)) = 1_m^{m-1}(\pi_m(c_0)),$
- $\forall c \in M_m, [c, c]_m = 1_{m+1}^m(c).$

A morphism of categorical stretchings,

$$\mathbb{E} \xrightarrow{(m,c)} \mathbb{E}'$$

[3] In [26] ∞-magmas are defined without the reflexive structure. In [26] we can see that this approach of ∞-magmas have its own interest.

is given by the following commutative square in ∞-$\mathbb{M}ag$*,*

$$\begin{array}{ccc} M & \xrightarrow{m} & M' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ C & \xrightarrow[c]{} & C' \end{array}$$

such that for all $m \in \mathbb{N}$*, and for all* $(c_1, c_0) \in \widetilde{M}_m$*,*

$$m_{m+1}([c_1, c_0]_m) = [m_m(c_1), m_m(c_0)]_m.$$

Let ∞-$\mathbb{E}t\mathbb{C}at$ *denote the category of categorical stretchings.*

Now consider the forgetful functor:

$$\infty\text{-}\mathbb{E}\mathrm{t}\mathbb{C}\mathrm{at} \xrightarrow{U} \infty\text{-}\mathbb{G}\mathrm{r}$$

given by $(M, C, \pi, ([,])_{m\in\mathbb{N}}) \longmapsto M$. This functor has a left adjoint which produces a monad $\mathbb{T}^P = (T^P, \mu^P, \lambda^P)$ on the category of ∞-graphs called the *Penon's monad*.

3.4. Definition. *Weak* ∞*-categories in the sense of Penon are algebras for the monad* $\mathbb{T}^P$ *above.*

The original approach of the Penon's monad is defined on the category ∞-$\mathbb{G}$rr of reflexive ∞-graphs, however Michael Batanin has proved in [8] that we obtain a better approach of this monad by considering ∞-graphs instead. See also the work in the article [16].

3.5. (∞, n)-Magmas. An (∞, n)-magma is an ∞-magma such that its underlying reflexive ∞-graph is equipped with a specific (∞, n)-structure in the sense of 1.

3.6. Remark. The reversibility part of an (∞, n)-magma has no relation with its reflexivity structure, neither with the involutive properties, contrary to strict (∞, n)-categories where their reversible structures, their involutive structures and their reflexivity structures are all related to one another (see 2.4). Instead we are going to see in this Section 3, that each underlying (∞, n)-categorical stretching of any weak (∞, n)-category ($n \in \mathbb{N}$) is especially going to be weakened, for the specific relation between the reversibility structure and the involutive structure, inside its underlying reflexive (∞, n)-magma the equalities $j^{n+1}_n \circ j^{n+1}_n = 1_{M_{n+1}}$. Also we are going to see in 3.8, that each (∞, n)-categorical stretching is especially going to be weakened, for the specific relation between the reversibility structure and the reflexibility structure, inside its underlying reflexive (∞, n)-magma the equalities $j^m_q \circ 1^{m-1}_m = 1^{m-1}_m \circ j^{m-1}_q$ and the equalities $j^m_{m-1} \circ 1^{m-1}_m = 1^{m-1}_m$.

The basic examples of (∞, n)-magmas are strict (∞, n)-categories. Let us denote by $\mathbb{M} = (M, (j^m_p)_{0\leq n\leq p<m})$ and $\mathbb{M}' = (M', (j'^{m'}_{p'})_{0\leq n\leq p'<m'})$ two (∞, n)-magmas where M and M' are respectively their underlying ∞-magmas, and $(j^m_p)_{0\leq n\leq p<m}$ and $(j'^{m'}_{p'})_{0\leq n\leq p'<m'}$ are respectively their underlying reversors. A morphism between these (∞, n)-magmas

$$\mathbb{M} \xrightarrow{\varphi} \mathbb{M}'$$

is given by its underlying morphism of ∞-magmas

$$M \xrightarrow{\varphi} M'$$

such that φ preserves the (∞, n)-structure; this means that for integers $0 \leq n \leq p < m$ we have the following commutative square.

$$\begin{array}{ccc} M_m & \xrightarrow{\varphi_m} & M'_m \\ {\scriptstyle j^m_p}\downarrow & & \downarrow{\scriptstyle j'^m_p} \\ M_m & \xrightarrow[\varphi_m]{} & M'_m \end{array}$$

The category of the (∞, n)-magmas is denoted by (∞, n)-$\mathbb{M}ag$; clearly it is not a full subcategory of ∞-$\mathbb{M}ag$.

As in Section 2, it is not difficult to show the following similar results for (∞, n)-magmas ($n \in \mathbb{N}$):

- For each $n \in \mathbb{N}$, the category (∞, n)-$\mathbb{M}ag$ is locally presentable.
- For each $n \in \mathbb{N}$, there is a monad $\mathbb{T}^{(\infty,n)}_m = (T^{(\infty,n)}_m, \mu^{(\infty,n)}_m, \lambda^{(\infty,n)}_m)$ on ∞-$\mathbb{G}$r (m is for "magmatic") with $\mathbb{A}lg(\mathbb{T}^{(\infty,n)}_m) \simeq (\infty, n)$-$\mathbb{M}ag$.
- By using Dubuc's adjoint triangle theorem we can build functors of "(∞, n)-magmafication" similar to those in Section 2.

3.7. Remark. Let us explain some informal intuition related to homotopy. The reader may notice that we can imagine many variations of "∞-magmas" similar to those of [34], or those that we propose in this paper (see above), but which still need to keep the presence of "higher equivalences", encoded by the reversors (see the Section 1), or, in a less obvious way, by the reflexors plus some compositions $\circ^m_p$ (see Section 2.4). For instance we can build kinds of "∞-magma", their adapted "stretchings" (similar to those of Section 3.8), and their corresponding "weak ∞-structures" (similar to those of Section 3.9). All that just by using reversors, reflexors plus compositions. Such variations of "higher structures" must be all the time projectively sketchable (see Section 2). If we restrict to taking models of such sketches in $\mathbb{S}et$, then these categories should be locally presentables and there are strong reasons to believe that there exists an interesting Quillen model structure on it. The Smith theorem could bring simplification to proving these intuitions. For instance, in [28], the authors have built a folk Quillen model structure on ω-$\mathbb{C}at$, by using the Smith theorem, and ω-$\mathbb{C}at$ is such a "higher structure" where weak equivalences were build only with reflexors and compositions. So, even though the goal of this paper is to give an algebraic approach of weak (∞, n)-categories, we believe that such structures and variations should provide us many categories with interesting Quillen model structure. Our slogan is: "enough reversors, and (or) reflexors plus some higher compositions" captures enough equivalences for doing abstract homotopy theory, based on higher category theory.

3.8. (∞, n)-Categorical stretchings. Now we are going to define (∞, n)-categorical stretchings ($n \in \mathbb{N}$), which are for the weak (∞, n)-categories what categorical stretchings are for weak ∞-categories (see 3.1), and we are going to use these important tools to weaken the axioms of strict (∞, n)-categories.

An (∞, n)-categorical stretching is given by a categorical stretching $\mathbb{E}^n = (M^n, C^n, \pi^n, ([-,-]_m)_{m\in\mathbb{N}})$ where M^n is an (∞, n)-magma, C^n is a strict (∞, n)-category, π^n is a morphism in (∞, n)-$\mathbb{M}ag$. A morphism of (∞, n)-categorical stretchings

$$\mathbb{E} \xrightarrow{(m,c)} \mathbb{E}'$$

is given by the following commutative square in (∞, n)-$\mathbb{M}ag$,

$$\begin{array}{ccc} M & \xrightarrow{m} & M' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ C & \xrightarrow[c]{} & C' \end{array}$$

such that for all $m \in \mathbb{N}$, and for all $(c_1, c_0) \in \widetilde{M}_m$,

$$m_{m+1}([c_1, c_0]_m) = [m_m(c_1), m_m(c_0)]_m \ .$$

Let (∞, n)-$\mathbb{E}t\mathbb{C}at$ denote the category of (∞, n)-categorical stretchings.

As in Section 2, it is not difficult to show the following similar results for (∞, n)-categorical stretchings ($n \in \mathbb{N}$):

- For each $n \in \mathbb{N}$, the category (∞, n)-$\mathbb{E}t\mathbb{C}at$ is locally presentable (see also 3.9).
- By using Dubuc's adjoint triangle theorem we can build functors of "(∞, n)-categorisation stretching" for any $(\infty, n+1)$-categorical stretching, and for any categorical stretching.

3.9. Definition. For each $n \in \mathbb{N}$ consider the forgetful functors

$$(\infty, n)\text{-}\mathbb{E}\mathrm{t}\mathbb{C}\mathrm{at} \xrightarrow{U_n} \infty\text{-}\mathbb{G}\mathrm{r}$$

given by $(M, C, \pi, ([,])_{m\in\mathbb{N}}) \longmapsto M$

Also, for each $n \in \mathbb{N}$ the categories (∞, n)-$\mathbb{E}\mathrm{t}\mathbb{C}\mathrm{at}$ and ∞-$\mathbb{G}\mathrm{r}$ are sketchable (in Section 2 we call $\mathcal{G}$ the sketch of ∞-graphs). Let us call $\mathcal{E}_n$ the sketch of (∞, n)-categorical stretchings. These sketches are both projective and there is an easy inclusion $\mathcal{G} \subset \mathcal{E}_n$. This inclusion of sketches produces, in passing to models, a functor W_n:

$$\mathbb{M}od(\mathcal{E}_n) \xrightarrow{W_n} \mathbb{M}od(\mathcal{G})$$

and the associated sheaf theorem for sketches of Foltz ([21]) proves that W_n has a left adjoint. Furthermore, there is an equivalence of categories $\mathbb{M}od(\mathcal{E}_n) \simeq (\infty, n)\text{-}\mathbb{E}t\mathbb{C}at$. Thus the following commutative square induced by these equivalences

$$\begin{array}{ccc} \mathbb{M}od(\mathcal{E}_n) & \xrightarrow{W_n} & \mathbb{M}od(\mathcal{G}) \\ \wr\downarrow & & \downarrow\wr \\ (\infty, n)\text{-}\mathbb{E}t\mathbb{C}at & \xrightarrow{U_n} & \infty\text{-}\mathbb{G}\text{r} \end{array}$$

produces the required left adjoint F_n of U_n.

$$(\infty, n)\text{-}\mathbb{E}\text{t}\mathbb{C}\text{at} \underset{F_n}{\overset{U_n}{\underset{\leftarrow}{\overset{\rightarrow}{\top}}}} \infty\text{-}\mathbb{G}\text{r}$$

The unit and the counit of this adjunction are respectively denoted $\lambda^{(\infty,n)}$ and $\varepsilon^{(\infty,n)}$.

This adjunction generates a monad $\mathbb{T}^{(\infty,n)} = (T^{(\infty,n)}, \mu^{(\infty,n)}, \lambda^{(\infty,n)})$ on ∞-$\mathbb{G}$r.

3.10. Definition. *For each $n \in \mathbb{N}$, a weak (∞, n)-category is an algebra for the monad $\mathbb{T}^{(\infty,n)} = (T^{(\infty,n)}, \mu^{(\infty,n)}, \lambda^{(\infty,n)})$ on ∞-$\mathbb{G}r$.*

3.11. Remark. For each $n \in \mathbb{N}$, when no confusion occurs, we will simplify the notation of these monads: $\mathbb{T}^n = (T^n, \mu^n, \lambda^n) = \mathbb{T}^{(\infty,n)} = (T^{(\infty,n)}, \mu^{(\infty,n)}, \lambda^{(\infty,n)})$, by omitting the symbol ∞.

For each $n \in \mathbb{N}$, the category $\mathbb{A}lg(\mathbb{T}^{(\infty,n)})$ is locally presentable. As a matter of fact, the adjunction $(\infty, n)\text{-}\mathbb{E}\text{t}\mathbb{C}\text{at} \underset{F_n}{\overset{U_n}{\underset{\leftarrow}{\overset{\rightarrow}{\top}}}} \infty\text{-}\mathbb{G}\text{r}$ involves the categories (∞, n)-$\mathbb{E}$t$\mathbb{C}$at and ∞-$\mathbb{G}$r which are both accessible (because they are both projectively sketchable thus locally presentable). But the forgetful functor U_n has a left adjoint, thus thanks to proposition 5.5.6 of [14], it preserves filtered colimits. Thus the monad $\mathbb{T}^n$ preserves filtered colimits in the locally presentable category ∞-$\mathbb{G}$r, and theorem 5.5.9 of [14] implies that the category $\mathbb{A}lg(\mathbb{T}^{(\infty,n)})$ is locally presentable as well.

Now we are going to build some functors of "weak (∞, n)-categorification" by using systematically Dubuc's adjoint triangle theorem (see [20]). For all $n \in \mathbb{N}$ we have the following triangle in $\mathbb{C}AT$

$$\begin{array}{ccc} \mathbb{A}lg(\mathbb{T}^n) & \xrightarrow{\quad V_n \quad} & \mathbb{A}lg(\mathbb{T}^{n+1}) \\ & {\scriptstyle U_n}\searrow \quad \swarrow{\scriptstyle U_{n+1}} & \\ & \infty\text{-}\mathbb{G}\text{r} & \end{array}$$

The functors V_n can be thought of as forgetful functors which forget the reversors $(i_n^m)_{m \geq n+2}$ for each weak (∞, n)-category (see 3.12 for the definition of the reversors produced by each weak (∞, n)-category). We have the adjunctions $F_n \dashv U_n$ and $F_{n+1} \dashv U_{n+1}$, where in particular $U_{n+1}V_n = U_n$ and U_{n+1} is monadic. So we can apply Dubuc's adjoint triangle theorem (see [20]) to show that the functor V_n has a left adjoint: $L_n \dashv V_n$. For each weak $(\infty, n+1)$-category C, the left adjoint L_n of V_n yields the free weak (∞, n)-category $L_n(C)$ associated to C. L_n can be seen as the "free weak (∞, n)-categorification functor" for weak $(\infty, n+1)$-categories.

We can apply the same argument to the following triangles in $\mathbb{C}AT$ (where here the functor V forgets all the reversors; see also 3.1)

$$\begin{array}{ccc} \mathbb{A}lg(\mathbb{T}^n) & \xrightarrow{\quad V \quad} & \mathbb{A}lg(\mathbb{T}^P) \\ & {\scriptstyle U_n}\searrow \quad \swarrow{\scriptstyle U} & \\ & \infty\text{-}\mathbb{G}\mathrm{r} & \end{array}$$

to prove that the functor V has a left adjoint: $L \dashv V$. For each weak ∞-category C, the left adjoint L of V builds the free weak (∞, n)-category $L_n(C)$ associated to C. L is the "free weak (∞, n)-categorification functor" for weak ∞-categories.

In [8] Batanin has proved that Penon's monad $\mathbb{T}^P$ is in fact a contractible ω-operad[4] equipped with a composition system, and thus algebras for $\mathbb{T}^P$ are weak ∞-categories in Batanin's sense. Thus for each $n \in \mathbb{N}$ and thanks to the forgetful functor V above, our models of (∞, n)-categories are weak ∞-categories in Batanin's sense.

3.12. Magmatic properties of weak (∞, n)-categories ($n \in \mathbb{N}$). If (G, v) is a $\mathbb{T}^n$-algebra then $G \xrightarrow{\lambda^n_G} \mathbb{T}^n(G)$ is the associated universal map and $M^n(G) \xrightarrow{\pi^n_G} C^n(G)$ is the free (∞, n)-categorical stretching associated to G, and we write $(\star^m_p)_{0 \leq p < m}$ for the composition laws of $M^n(G)$. Also let us define the following composition laws on G: If $a, b \in G(m)$ are such that $s^m_p(a) = t^m_p(b)$ then we put

$$a \circ^m_p b = v_m(\lambda^n_G(a) \star^m_p \lambda^n_G(b)) \ ,$$

if $a \in G(p)$ then we put

$$\iota^p_m(a) := v_m(1^p_m(\lambda^n_G(a))) \ ,$$

and if $a \in G(m)$ and $0 \leq n \leq p < m$ then we put

$$i^m_p(a) := v_m(j^m_p(\lambda^n_G(a))) \ .$$

It is easy to show that with these definitions, the $\mathbb{T}^n$-algebra (G, v) puts an (∞, n)-magma structure on G.

In [34] the author showed that if a, b are m-cells of $\mathbb{T}^n(G)$ such that $s^m_p(a) = t^m_p(b)$ then $v_m(a \star^m_p b) = v_m(a) \circ^m_p v_m(b)$. We are going to show that if a is a p-cell of $\mathbb{T}^n(G)$ such that $0 \leq n \leq p < m$ then $v_m(1^p_m(a)) = \iota^p_m(v_p(a))$ and if a is an m-cell of $\mathbb{T}^n(G)$

[4]We use here the notation ω instead of ∞, in order to make clear that we are dealing with higher operads in Batanin's sense and not with ∞-operads as defined in Jacob Lurie's book (see [31])

such that $0 \leq n \leq p < m$ then $v_m(j^m_p(a)) = i^m_p(v_m(a))$. In other words, each underlying morphism $\mathbb{T}^n(G) \xrightarrow{v} G$ in ∞-$\mathbb{G}$r of a weak (∞, n)-category (G, v) is also a morphism of (∞, n)-$\mathbb{M}ag$ when we consider them as equipped with the (∞, n)-magmatic structures defined above. Proofs of these magmatic properties becomes standard after the work in [25, 34], but for the comfort of the reader we shall give a complete proof.

All reversors for algebras are denoted "i^m_p" because there is no risk of confusion. The $\mathbb{T}^n$-algebra $(\mathbb{T}^n(G), \mu^n_G)$ on $\mathbb{T}^n(G)$ is an (∞, n)-structure $(i^m_p)_{0\leq n\leq p<m}$ such that for all t in $\mathbb{T}^n(G)(m)$ we have $j^m_p(t) = i^m_p(t)$. As a matter of fact

$$i^m_p(t) := \mu^n_G(j^m_p(\lambda^n_{T^n(G)}(t))) = j^m_p(\mu^n_G(\lambda^n_{T^n(G)}(t)))$$

because μ^n_G forgets that a morphism preserves the involutions, so $i^m_p(t) = j^m_p(t)$. Furthermore a morphism of $\mathbb{T}^n$-algebras $(G, v) \xrightarrow{f} (G', v')$ is such that for all $t \in G(m)$ with $0 \leq n \leq p < m$ we have $f(i^m_p(t)) = i^m_p(f(t))$. Indeed

$$\begin{aligned} f(i^m_p(t)) &= f(v_m(j^m_p(\lambda^n_G(t)))) \\ &= v'_m(T^n(f)(j^m_p(\lambda^n_G(t)))) \\ &= v'_m(j^m_p(T^n(f)(\lambda^n_G(t)))) \end{aligned}$$

because $\mathbb{T}^n(f)$ forgets that a morphism preserves the reversors. Thus

$$f(i^m_p(t)) = v'_m(j^m_p(T^n(f)(\lambda^n_G(t)))) = v'(j^m_p(\lambda^n_{G'}(f(t)) \ .$$

Because a $\mathbb{T}^n$-algebra (G, v) determines a morphism of $\mathbb{T}^n$-algebras

$$(\mu^n_G, \mathbb{T}^n(G)) \xrightarrow{v} (G, v) \ ,$$

we deduce the useful formula $v_m(j^m_p(t)) = i^m_p(v_m(t))$.

All reflexors for algebras are denoted "ι^p_m" because there is no risk of confusion, and we use the symbols "1^p_m" for reflexors coming from the free (∞, n)-categorical stretchings. The $\mathbb{T}^n$-algebra $(\mathbb{T}^n(G), \mu^n_G)$ puts a reflexive structure $(\iota^p_m)_{0\leq p<m}$ on $\mathbb{T}^n(G)$ such that for all t in $\mathbb{T}^n(G)(p)$ we have $1^p_m(t) = \iota^p_m(t)$. As a matter of fact

$$\iota^p_m(t) := \mu^n_G(1^p_m(\lambda^n_{\mathbb{T}^n(G)}(t))) = 1^p_m(\mu^n_G(\lambda^n_{\mathbb{T}^n(G)}(t)))$$

because μ^n_G forgets that a morphism preserves the reflexivities, so $\iota^p_m(t) = 1^p_m(t)$. Furthermore a morphism of $\mathbb{T}^n$-algebras $(G, v) \xrightarrow{f} (G', v')$ is such that for all $t \in G(p)$ with $0 \leq p < m$, we have $f(\iota^p_m(t)) = \iota^p_m(f(t))$. Indeed

$$\begin{aligned} f(\iota^p_m(t)) &= f(v_m(1^p_m(\lambda^n_G(t)))) \\ &= v'_m(\mathbb{T}^n(f)(1^p_m(\lambda^n_G(t)))) \\ &= v'_m(1^p_m(T^n(f)(\lambda^n_G(t)))) \end{aligned}$$

because $\mathbb{T}^n(f)$ forgets that a morphism preserves the reflexors, so

$$f(\iota^p_m(t)) = v'(1^p_m(\mathbb{T}^n(f)(\lambda^n_G(t)))) = v'(1^p_m(\lambda^n_{G'}(f(t)))) = \iota^p_m(f(t)) \ .$$

Thus, because a $\mathbb{T}^n$-algebra (G, v) is also a morphism

$$(\mu^n_G, \mathbb{T}^n(G)) \xrightarrow{v} (G, v)$$

of $\mathbb{T}^n$-algebras, we have the useful formula $v_m(1^p_m(t)) = \iota^p_m(v_m(t))$.

3.13. Interactions between reversibility structures, involutive structures, and reflexivity structures. The reversors for strict (∞, n)-categories and for (∞, n)-magmas are denoted by "j", whereas the reversors for weak (∞, n)-categories are denoted by "i". Let us fix an $n \in \mathbb{N}$ and a strict (∞, n)-category C. We know that in C (see Section 2) we have for each $0 \leq n \leq p < m$ the involutive properties $j^m_p \circ j^m_p = 1_{X_m}$. However for weak (∞, n)-categories this property does not hold even up to coherence cell; yet reversors of type i^{m+1}_m do permit a weakened version of the involutive property. As a matter of fact consider now a weak (∞, n)-category (G, v), the free (∞, n)-categorical stretching $M^n(G) \xrightarrow{\pi^n_G} C^n(G)$ associated to G and the universal map $G \xrightarrow{\lambda^n_G} \mathbb{T}^n(G)$. For each $\alpha \in G(m+1)$ we have

$$\begin{aligned} i^{m+1}_m(i^{m+1}_m(\alpha)) &= i^{m+1}_m(v(j^{m+1}_m(\lambda^m_G(\alpha)))) \\ &= v(j^{m+1}_m(j^{m+1}_m(\lambda^m_G(\alpha)))), \end{aligned}$$

because the algebra (G, v) preserves the reversibility structure (see 3.14). This implies $i^{m+1}_m(i^{m+1}_m(\alpha)) \| \alpha$ because

$$\begin{aligned} s^{m+1}_m(i^{m+1}_n(i^{m+1}_m(\alpha))) &= s^{m+1}_m(v_{m+1}(j^{m+1}_m(j^{m+1}_m(\lambda^m_G(\alpha))))) \\ &= v_m(s^{m+1}_m(j^{m+1}_m(j^{m+1}_m(\lambda^m_G(\alpha))))) \\ &= v_m(t^{m+1}_m(j^{m+1}_m(\lambda^m_G(\alpha)))) \\ &= v_m(s^{m+1}_m(\lambda^m_G(\alpha))) \\ &= v_m(\lambda^m_G(s^{m+1}_m(\alpha))) \\ &= s^{m+1}_m(\alpha), \end{aligned}$$

and similarly we show that $t^{m+1}_m(i^{m+1}_m(i^{m+1}_m(\alpha))) = t^{m+1}_m(\alpha)$.

But also in the free (∞, n)-categorical stretching associated to G, which controls the algebraic nature of (G, v), creates between the $(m+1)$-cells $j^{m+1}_m(j^{m+1}_m(\lambda^m_G(\alpha)))$ and $\lambda^m_G(\alpha)$, an $(m+2)$-cell of coherence:

$$[j^{m+1}_m(j^{m+1}_m(\lambda^m_G(\alpha))); \lambda^m_G(\alpha)]_{m+1}.$$

Thus at the level of algebras it shows that there is a coherence cell between $i^{m+1}_m(i^{m+1}_m(\alpha))$ and α.

The other equalities $j^m_p \circ j^m_p = 1_{X_m}$ $(0 \leq n < p < m)$ which are valid in any strict (∞, n)-category have no reason to be weakened in any weak (∞, n)-category for the simple reason that the axioms of (∞, n)-structure do not imply parallelism between the m-cells $i^m_p(i^m_p(\alpha))$ and α when $p > n$.

Now let us fix an $n \in \mathbb{N}$ and a strict (∞, n)-category C. We know that for each p-cell α of C and for each $0 \leq n \leq q < p < m$ we have the equalities $j^m_q(1^p_m(\alpha)) = 1^p_m(j^p_q(\alpha))$ but also for each $0 \leq n \leq q < m$ and $0 \leq p \leq q$ we have the equalities $j^m_q(1^p_m(\alpha)) = 1^p_m(\alpha)$ (see Section 2). However for a weak (∞, n)-category (G, v) and for any p-cell α in it, the (∞, n)-structure, for each $0 \leq n \leq q < p < m$, does not ensure the parallelism between the m-cells $i^m_q(\iota^p_m(\alpha))$ and $\iota^p_m(i^p_q(\alpha))$, and for each $0 \leq n \leq q < m$ and $0 \leq p \leq q$, does not ensure the parallelism between the m-cells $i^m_q(\iota^p_m(\alpha))$ and $\iota^p_m(\alpha)$.

Thus these equalities which are true in the strict case are not necessarily weakened in the weak case. However there are certain situations where in the weak case these equalities are replaced by some coherence cells. As a matter of fact if now (G, v) is a weak (∞, n)-category then it is easy to check, thanks to the axioms for the (∞, n)-structure (see Section 1) that if $p = m - 1$ and $0 \leq n \leq q < m - 1$, then for any $(m-1)$-cell α of (G, v) we have $i^m_q(\iota^{m-1}_m(\alpha)) \| \iota^{m-1}_m(i^{m-1}_q(\alpha))$. Indeed, we have

$$\begin{aligned} i^m_q(\iota^{m-1}_m(\alpha)) &= i^m_q(v_m(1^{m-1}_m(\lambda^n_G(\alpha)))) \\ &= v_m(j^m_q(1^{m-1}_m(\lambda^n_G(\alpha)))). \end{aligned}$$

Thus

$$\begin{aligned} s^m_{m-1}(i^m_q(\iota^{m-1}_m(\alpha))) &= s^m_{m-1}(v_m(j^m_q(1^{m-1}_m(\lambda^n_G(\alpha))))) \\ &= v_{m-1}(s^m_{m-1}(j^m_q(1^{m-1}_m(\lambda^n_G(\alpha))))) \\ &= v_{m-1}(j^{m-1}_q(s^m_{m-1}(1^{m-1}_m(\lambda^n_G(\alpha))))) \\ &= v_{m-1}(j^{m-1}_q(\lambda^n_G(\alpha))) \\ &= i^{m-1}_q(\alpha) \\ &= v_{m-1}(\lambda^n_G(i^{m-1}_q(\alpha))) \\ &= v_{m-1}(s^m_{m-1}(1^{m-1}_m(\lambda^n_G(i^{m-1}_q(\alpha))))) \\ &= s^m_{m-1}(v_m(1^{m-1}_m(\lambda^n_G(i^{m-1}_q(\alpha))))) \\ &= s^m_{m-1}(\iota^{m-1}_m(i^{m-1}_q(\alpha))), \end{aligned}$$

and similarly we see that $t^m_{m-1}(i^m_q(\iota^{m-1}_m(\alpha))) = t^m_{m-1}(\iota^{m-1}_m(i^{m-1}_q(\alpha)))$. But also in the free (∞, n)-categorical stretching associated to G, which controls the algebraic nature of (G, v), between the m-cells $j^m_q(1^{m-1}_m(\lambda^n_G(\alpha)))$ and $1^{m-1}_m(j^{m-1}_q(\lambda^n_G(\alpha)))$, an $(m+1)$-cell of coherence is created:

$$[j^m_q(1^{m-1}_m(\lambda^n_G(\alpha))); 1^{m-1}_m(j^{m-1}_q(\lambda^n_G(\alpha)))]_m.$$

Thus at the level of algebras it shows that there is a coherence cell between $i^m_q(\iota^{m-1}_m(\alpha))$ and $\iota^{m-1}_m(i^{m-1}_q(\alpha))$.

Furthermore, thanks to the (∞,n)-structure (see Section 1) we easily prove that for $p=q=m-1$ we have, for any $(m-1)$-cells α of any weak (∞,n)-category (G,v), the parallelism $i^m_{m-1}(\iota^{m-1}_m(\alpha))\|\iota^{m-1}_m(\alpha)$. As a matter of fact

$$\begin{aligned} i^m_{m-1}(\iota^{m-1}_m(\alpha)) &= i^m_{m-1}(v_m(1^{m-1}_m(\lambda^n_G(\alpha)))) \\ &= v_m(j^m_{m-1}(1^{m-1}_m(\lambda^n_G(\alpha)))), \end{aligned}$$

Thus

$$\begin{aligned} s^m_{m-1}(i^m_{m-1}(\iota^{m-1}_m(\alpha))) &= s^m_{m-1}(v_m(j^m_{m-1}(1^{m-1}_m(\lambda^n_G(\alpha))))) \\ &= v_{m-1}(s^m_{m-1}(j^m_{m-1}(1^{m-1}_m(\lambda^n_G(\alpha))))) \\ &= v_{m-1}(t^m_{m-1}(1^{m-1}_m(\lambda^n_G(\alpha)))) \\ &= v_{m-1}(\lambda^n_G(\alpha)) \\ &= \alpha \\ &= s^m_{m-1}(\iota^{m-1}_m(\alpha)), \end{aligned}$$

and similarly we show that $t^m_{m-1}(i^m_{m-1}(1^m_{m-1}(\alpha))) = t^m_{m-1}(1^m_{m-1}(\alpha))$. But also in the free (∞,n)-categorical stretching associated to G, which controls the algebricity of (G,v), between the m-cells $j^m_{m-1}(1^{m-1}_m(\lambda^n_G(\alpha)))$ and $1^{m-1}_m(\lambda^n_G(\alpha))$ creates an $(m+1)$-cell of coherence

$$[j^m_{m-1}(1^{m-1}_m(\lambda^n_G(\alpha))); 1^{m-1}_m(\lambda^n_G(\alpha))]_m.$$

Thus at the level of algebras it shows that there is a coherence cell between $i^m_{m-1}(\iota^{m-1}_m(\alpha))$ and $\iota^{m-1}_m(\alpha)$.

For the other equalities $j^m_q(1^p_m(\alpha)) = 1^p_m(j^p_q(\alpha))$ (for $0\leq n\leq q<p<m$ and $p\neq m-1$) and $j^m_q(1^p_m(\alpha)) = 1^p_m(\alpha)$ (for $0\leq n\leq q<m$, $0\leq p\leq q$ and $p,q\neq m-1$), which are valid in any strict (∞,n)-category, they have no reason to be weakened in any weak (∞,n)-category for the simple reason that the axioms of the (∞,n)-structure does not imply the parallelism between these m-cells $i^m_q(\iota^p_m(\alpha))$ and $\iota^p_m(i^p_q(\alpha))$, or between the m-cells $i^m_q(\iota^p_m(\alpha))$ and $\iota^p_m(\alpha)$.

3.14. Computations in dimensions 2. We are going to see that in dimension 2, algebras for the monad $\mathbb{T}^0$ (see Remark 3.11 for this notation) for weak $(\infty,0)$-categories (see Section 3), commonly called in the literature weak ∞-groupoids, are bigroupoids in the usual sense.

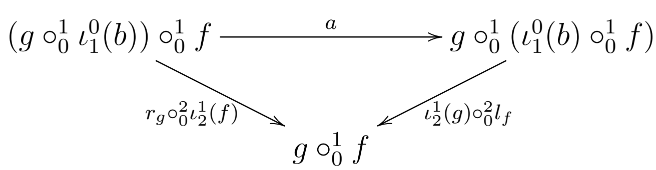
A bigroupoid is just a bicategory such that its 1-cells are equivalences and its 2-cells are $\circ^2_1$-isomorphism (see below). In [34] Penon has proved that weak ∞-categories in his sense are bicategories in dimension 2. However as we said in 3.1, this approach is slightly different from our approach because he used reflexive ∞-graphs whereas we use here ∞-graphs instead. It is important to recall that our models of weak $(\infty,0)$-categories are weak ∞-categories in Penon sense, where these weak ∞-categories are obtained with the monad $\mathbb{T}^P=(T^P,\mu^P,\lambda^P)$ described in 3.1.

First let us recall some basic definitions that we can find in [25]. A reflexive ∞-graph has dimension $p\in\mathbb{N}$ if all its q-cells for which $q>p$ are identity cells and if there is at least one p-cell which is not an identity cell. Thus reflexive ∞-graphs of dimension 0 are just sets. An $(\infty, 0)$-categorical stretching $\mathbb{E}^0 = (M^0, C^0, \pi^0, ([,])_{m\in\mathbb{N}})$ (that we should also call "groupoidal stretching" by analogy with the "categorical stretchings" of Penon) has dimension $p \in \mathbb{N}$ if the underlying reflexive ∞-graph of M^0 has dimension p. A $\mathbb{T}^0$-algebra $(G; v)$ has dimension $p \in \mathbb{N}$ if G has dimension p when G is considered with its canonical reflexivity structure (see 3.12).

Because Penon has proved in [34] that algebras for the monad $\mathbb{T}^P$ on reflexive ∞-graphs are bicategories in dimension 2, we just need to prove the identities axiom for the bicategorical part : Basically, it says that given a $\mathbb{T}^0$-algebra (G, v) of dimension 2, and given a diagram in G of the type

$$a \xrightarrow{f} b \xrightarrow{\iota_1^0(b)} b \xrightarrow{g} c$$

then the following commutative diagram in G holds

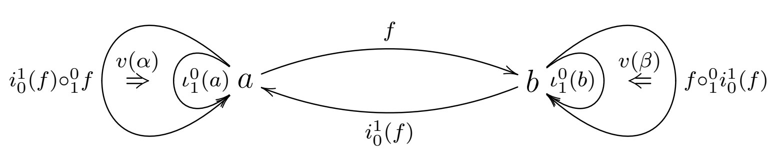


where a is the coherence cell of associativity isomorphism described in [34], and $g \circ_0^1 1_1^0(b) \xrightarrow{r_g} g$, $1_1^0(b) \circ_0^1 f \xrightarrow{l_f} f$ are respectively the right unit isomorphism cell and the left unit isomorphism cell.

Also we need to prove that 1-cells are equivalences and 2-cells are $\circ_1^2$-isomorphisms. We first start to prove these facts and then finish by proving the bicategorical nature of such algebras.

3.15. Proposition. *Let $(G; v)$ be a $\mathbb{T}^0$-algebra of dimension 2 and let $a \xrightarrow{f} b$ be a 1-cell of G. Then f is an equivalence.*

Proof. Actually we are going to exhibit a diagram in G of the following form



and show that the 2-cells $v(\alpha)$ and $v(\beta)$ are $\circ_1^2$-isomorphism. Let us denote by $G \xrightarrow{\lambda_G^0} \mathbb{T}^0(G)$ the universal map associated to G, and by $M^0(G) \xrightarrow{\pi_G^0} C^0(G)$ the free groupoidal stretching associated to G.

Consider the 2-cell

$$\beta = \big[\lambda_G^0(f) \star_0^1 j_0^1\big(\lambda_G^0(f)\big) \,;\, 1_1^0\big(\lambda_G^0(b)\big)\big]_1$$

in $M^0(G)$. We are going to show that a diagram of the type

$$\begin{array}{c}\big[\lambda^0_G(f)\star^1_0 j^1_0\big(\lambda^0_G(f)\big)\,;\,1^0_1\big(\lambda^0_G(b)\big)\big]_1\star^2_1 j^2_1\big(\big[\lambda^0_G(f)\star^1_0 j^1_0\big(\lambda^0_G(f)\big)\,;\,1^0_1\big(\lambda^0_G(b)\big)\big]_1\big)\\ \Big\Downarrow \lambda^0_f\\ 1^0_2\big(\lambda^0_G(b)\big)\end{array}$$

lives in $M^0(G)$. Because π^0_G is a morphism of $(\infty,0)$-magmas, we have

$$\begin{aligned}&\pi^0_G\big(\big[\lambda^0_G(f)\star^1_0 j^1_0\big(\lambda^0_G(f)\big)\,;\,1^0_1\big(\lambda^0_G(b)\big)\big)]_1\\ &\qquad\star^2_1 j^2_1\big(\big[\lambda^0_G(f)\star^1_0 j^1_0\big(\lambda^0_G(f)\big)\,;\,1^0_1\big(\lambda^0_G(b)\big)\big]_1\big)\big)\\ &=\pi^0_G([\lambda^0_G(f)\star^1_0 j^1_0(\lambda^0_G(f));1^0_1(\lambda^0_G(b))]_1)\\ &\qquad\circ^2_1\pi^0_G(j^2_1([\lambda^0_G(f)\star^1_0 j^1_0(\lambda^0_G(f));1^0_1(\lambda^0_G(b))]_1))\\ &=\pi^0_G([\lambda^0_G(f)\star^1_0 j^1_0(\lambda^0_G(f));1^0_1(\lambda^0_G(b))]_1)\\ &\qquad\circ^2_1 j^2_1(\pi^0_G([\lambda^0_G(f)\star^1_0 j^1_0(\lambda^0_G(f));1^0_1(\lambda^0_G(b))]_1))\\ &=1^1_2(\pi^0_G(\lambda^0_G(f)\star^1_0 j^1_0(\lambda^0_G(f))))\\ &\qquad\circ^2_1 j^2_1(1^1_2(\pi^0_G(\lambda^0_G(f)\star^1_0 j^1_0(\lambda^0_G(f)))))\\ &=1^1_2(\pi^0_G(\lambda^0_G(f))\circ^1_0 j^1_0\pi_G((\lambda^0_G(f))))\\ &\qquad\circ^2_1 j^2_1(1^1_2(\pi^0_G(\lambda^0_G(f))\circ^1_0 j^1_0\pi^0_G((\lambda^0_G(f)))))\\ &=1^1_2(1^0_1(\pi^0_G(\lambda^0_G(b))))\circ^2_1 j^2_1(1^1_2(1^0_1(\pi^0_G(\lambda^0_G(b)))))\\ &=1^0_2(\pi^0_G(\lambda^0_G(b)))\circ^2_1 j^2_1(1^0_2(\pi^0_G(\lambda^0_G(b))))\\ &=1^0_2(\pi^0_G(\lambda^0_G(b)))\circ^2_1 1^0_2(\pi^0_G(\lambda^0_G(b)))\\ &=1^0_2(\pi^0_G(\lambda^0_G(b)))=\pi^0_G(1^0_2(\lambda^0_G(b)))\,.\end{aligned}$$

The second equality holds because π^0_G respects the $(\infty,0)$-reversible structure and the third equality holds because of the definition of an $(\infty,0)$-stretching (see 3.8), the fourth equality is because π^0_G is a morphism of ∞-magmas and π^0_G preserves the reversible structure, whereas the fifth equality is because $\pi^0_G(\lambda^0_G(f))$ and $j^1_0\pi^0_G((\lambda^0_G(f))$ are $\circ^1_0$-inverse in the strict ∞-groupoid $C^0(G)$. The seventh equality is because $j^2_1\circ 1^0_2=1^0_2$; see 2.4. Thus by the contractibility structure in $M^0(G)$, we get the coherence 3-cell

$$\begin{array}{c}[\lambda^0_G(f)\star^1_0 j^1_0(\lambda^0_G(f));1^0_1(\lambda^0_G(b))]_1\star^2_1 j^2_1([\lambda^0_G(f)\star^1_0 j^1_0(\lambda^0_G(f));1^0_1(\lambda^0_G(b))]_1)\\ \Big\Downarrow \lambda^0_f\\ 1^0_2(\lambda^0_G(b))\end{array}$$

in $M^0(G)$, where λ^0_f is the 3-cell

$$\begin{aligned}&[[\lambda^0_G(f)\star^1_0 j^1_0(\lambda^0_G(f))\,;1^0_1(\lambda^0_G(b))]_1\star^2_1 j^2_1([\lambda^0_G(f)\star^1_0 j^1_0(\lambda^0_G(f))\,;1^0_1(\lambda^0_G(b))]_1)\\ &\qquad;1^0_2(\lambda^0_G(b))]_2\ .\end{aligned}$$

By applying to it the $\mathbb{T}^0$-algebra $(G;v)$ we obtain the 3-cell in G

$$\begin{array}{c} v([\lambda^0_G(f)\star^1_0 j^1_0(\lambda^0_G(f));1^0_1(\lambda^0_G(b))]_1\star^2_1 j^2_1([\lambda^0_G(f)\star^1_0 j^1_0(\lambda^0_G(f));1^0_1(\lambda^0_G(b))]_1)) \\ \Big\Downarrow{\scriptstyle v(\lambda^0_f)} \\ v(1^0_2(\lambda^0_G(b))) \end{array}$$

with

$$\begin{aligned} & v([\lambda^0_G(f)\star^1_0 j^1_0(\lambda^0_G(f));1^0_1(\lambda^0_G(b))]_1 \\ &\qquad \star^2_1 j^2_1([\lambda^0_G(f)\star^1_0 j^1_0(\lambda^0_G(f));1^0_1(\lambda^0_G(b))]_1) \\ &= v([\lambda^0_G(f)\star^1_0 j^1_0(\lambda^0_G(f));1^0_1(\lambda^0_G(b))]_1) \\ &\qquad \circ^2_1 v(j^2_1([\lambda^0_G(f)\star^1_0 j^1_0(\lambda^0_G(f));1^0_1(\lambda^0_G(b))]_1)) \end{aligned}$$

because v is a morphism of $(\infty,0)$-magmas

$$\begin{aligned} &= v([\lambda^0_G(f)\star^1_0 j^1_0(\lambda^0_G(f));1^0_1(\lambda^0_G(b))]_1) \\ &\qquad \circ^2_1 i^2_1(v([\lambda^0_G(f)\star^1_0 i^1_0(\lambda^0_G(f));1^0_1(\lambda^0_G(b))]_1)) \\ &= v(\beta)\circ^2_1 i^2_1(v(\beta))\ . \end{aligned}$$

Thus we obtain the 3-cell

$$v(\beta)\circ^2_1 i^2_1(v(\beta)) \xRightarrow{v(\lambda^0_f)} \iota^0_2(b)$$

in G. But the $\mathbb{T}^0$-algebra $(G;v)$ has dimension 2 forcing $v(\lambda^0_f)$ to be an identity, proving $v(\beta)\circ^2_1 i^2_1(v(\beta)) = \iota^0_2(b)$. By the same method we can prove that $i^2_1(v(\beta))\circ^2_1 v(\beta) = \iota^1_2(f\circ^1_0 i^1_0(f))$ which shows that $v(\beta)$ is an $\circ^2_1$-isomorphism.

Furthermore with the 2-cell

$$\alpha = [j^1_0(\lambda^0_G(f))\star^1_0 \lambda^0_G(f);1^0_1(\lambda^0_G(a))]_1$$

in $M^0(G)$, we can build a 3-cell in $M^0(G)$ of the type

$$\begin{array}{c} [j^1_0(\lambda^0_G(f))\star^1_0 \lambda^0_G(f);1^0_1(\lambda^0_G(a))]_1]\star^2_1 j^2_1([j^1_0(\lambda^0_G(f))\star^1_0 \lambda^0_G(f);1^0_1(\lambda^0_G(a))]_1]) \\ \Big\Downarrow{\scriptstyle \rho^0_f} \\ 1^0_2(\lambda^0_G(a)) \end{array}$$

and with the same kind of arguments as above we can prove that in G we have the following 2-cell

$$i^1_0(f)\circ^1_0 f \xRightarrow{v(\alpha)} \iota^1_0(a)$$

which is an $\circ^2_1$-isomorphism, that is we have $v(\alpha)\circ^2_1 i^2_1(v(\alpha)) = \iota^0_2(a)$ and $i^2_1(v(\alpha))\circ^2_1 v(\alpha) = \iota^1_2(i^1_0(f)\circ^1_0 f)$, which finally shows that f is an equivalence. ∎

3.16. PROPOSITION. *Let $(G;v)$ be a $\mathbb{T}^0$-algebra of dimension 2 and let $f \xrightarrow{\alpha} g$ be a 2-cell of G. Then α is an $\circ_1^2$-isomorphism.*

PROOF. Let us denote by $G \xrightarrow{\lambda_G^0} \mathbb{T}^0(G)$ the universal map associated to G, and by $M^0(G) \xrightarrow{\pi_G^0} C^0(G)$ the free groupoidal stretching associated to G.

Consider the following diagram in $M^0(G)$

$$\lambda_G^0(g) \xrightarrow{j_1^2(\lambda_G^0(\alpha))} \lambda_G^0(f) \xrightarrow{\lambda_G^0(\alpha)} \lambda_G^0(g) \xrightarrow{j_1^2(\lambda_G^0(\alpha))} \lambda_G^0(f)$$

The 2-cells $\lambda_G^0(\alpha)\star_1^2 j_1^2(\lambda_G^0(\alpha))$ and $1_2^1(\lambda_G^0(g))$ are parallels and are connected by a coherence 3-cell A because

$$\begin{aligned}
\pi_G^0\big(\lambda_G^0(\alpha) \star_1^2 j_1^2(\lambda_G^0(\alpha))\big) &= \pi_G^0\big(\lambda_G^0(\alpha)\big) \circ_1^2 \pi_G^0\big(j_1^2(\lambda_G^0(\alpha))\big) \\
&= \pi_G^0\big(\lambda_G^0(\alpha)\big) \circ_1^2 j_1^2\big(\pi_G^0(\lambda_G^0(\alpha))\big) \\
&= 1_2^1(\pi_G^0\big(\lambda_G^0(g)\big)) \\
&= \pi_G^0(1_2^1(\lambda_G^0(g)))) \,.
\end{aligned}$$

Also, the 2-cells $j_1^2(\lambda_G^0(\alpha)) \star_1^2 \lambda_G^0(\alpha)$ and $1_2^1(\lambda_G^0(f))$ are parallels and are connected by a coherence 3-cell B because

$$\begin{aligned}
\pi_G^0\big(j_1^2(\lambda_G^0(\alpha)) \star_1^2 \lambda_G^0(\alpha)\big) &= \pi_G^0\big(j_1^2(\lambda_G^0(\alpha))\big) \circ_1^2 \pi_G^0\big(\lambda_G^0(\alpha)\big) \\
&= j_1^2\big(\pi_G^0(\lambda_G^0(\alpha))\big) \circ_1^2 \pi_G^0\big(\lambda_G^0(\alpha)\big) \\
&= 1_2^1(\pi_G^0(\lambda_G^0(f))) \\
&= \pi_G^0(1_2^1(\lambda_G^0(f))) \,.
\end{aligned}$$

These two equalities hold because π_G^0 is a morphism of $(\infty,0)$-magmas.

But v is also a morphism of $(\infty,0)$-magmas, thus we have $v(\lambda_G^0(\alpha) \star_1^2 j_1^2(\lambda_G^0(\alpha))) = \alpha \circ_1^2 i_1^2(\alpha)$ and $v(j_1^2(\lambda_G^0(\alpha)) \star_1^2 \lambda_G^0(\alpha)) = i_1^2(\alpha) \circ_1^2 \alpha$. Also $(G;v)$ has dimension 2 thus it shows that $v(A)$ and $v(B)$ are identity cells, which brings equalities $\alpha \circ_1^2 i_1^2(\alpha) = \iota_2^1(g)$ and $i_1^2(\alpha) \circ_1^2 \alpha = \iota_2^1(f)$, which proves that α is indeed an $\circ_1^2$-isomorphism.

■

3.17. PROPOSITION. *If $(G;v)$ is a $\mathbb{T}^0$-algebra of dimension 2, then it puts on G a bicategorical structure.*

PROOF. The reflexive structure that $(G;v)$ put on G has been defined in 3.12, and the associativity axiom for bicategories has been already proved in [34].

Consider the following diagram in G

$$a \xrightarrow{f} b \xrightarrow{\iota_1^0(b)} b \xrightarrow{g} c$$

We have to show that we have the following commutative diagram in G

$$\begin{array}{ccc} (g \circ^1_0 \iota^0_1(b)) \circ^1_0 f & \xrightarrow{a} & g \circ^1_0 (\iota^0_1(b) \circ^1_0 f) \\ & {\scriptstyle r_g \circ^2_0 \iota^1_2(f)} \searrow \quad \swarrow {\scriptstyle \iota^1_2(g) \circ^2_0 l_f} & \\ & g \circ^1_0 f & \end{array}$$

where a is the coherence cell of associativity isomorphism described in [34], and $g \circ^1_0 \iota^0_1(b) \xrightarrow{r_g} g$, $\iota^0_1(b) \circ^1_0 f \xrightarrow{l_f} f$ are respectively the right unit isomorphism cell and the left unit isomorphism cell. First we define r_g and l_f. Let us denote by $G \xrightarrow{\lambda^0_G} \mathbb{T}^0(G)$ the universal map associated to G, and by $M^0(G) \xrightarrow{\pi^0_G} C^0(G)$ the free groupoidal stretching associated to G. Consider the coherence 2-cells

$$R_g = \big[\lambda^0_G(g) \star^1_0 1^1_0\big(\lambda^0_G(b)\big)\,;\, \lambda^0_G(g)\big]_1$$

and

$$L_f = \big[1^1_0\big(\lambda^0_G(b)\big) \star^1_0 \lambda^0_G(f)\,;\, \lambda^0_G(f)\big]_1$$

in $M^0(G)$. By definition we put $r_g = v(R_g)$ and $l_f = v(L_f)$

Now we have the following diagram in $M^0(G)$

$$\begin{array}{ccc} \big(\lambda^0_G(g) \star^1_0 1^1_0\big(\lambda^0_G(b)\big)\big) \star^1_0 \lambda^0_G(f) & \xrightarrow{A} & \lambda^0_G(g) \star^1_0 \big(1^1_0\big(\lambda^0_G(b)\big) \star^1_0 \lambda^0_G(f)\big) \\ & {\scriptstyle R_g \star^2_0 1^1_2(\lambda^0_G(f))} \searrow \quad \swarrow {\scriptstyle 1^1_2(\lambda^0_G(g)) \star^2_0 L_f} & \\ & \lambda^0_G(g) \star^1_0 \lambda^0_G(f) & \end{array}$$

such that the $v(A) = a$ and which is commutative up to a 3-coherence cell

$$\begin{array}{c} \big(1^1_2(\lambda^0_G(g)) \star^2_0 L_f\big) \star^2_1 A \\ \Big\Downarrow u \\ R_g \star^2_0 1^1_2(\lambda^0_G(f)) \end{array}$$

This coherence 3-cell u exist because $1^1_2(\lambda^0_G(g)) \star^2_0 L_f\big) \star^2_1 A$ and $R_g \star^2_0 1^1_2(\lambda^0_G(f))$ are parallels; because π^0_G is a morphism of $(\infty,0)$-magmas, we have

$$\begin{aligned} \pi^0_G\big(\big(1^1_2(\lambda^0_G(g)) \star^2_0 L_f\big) \star^2_1 A\big) &= \pi^0_G\big(1^1_2(\lambda^0_G(g)) \star^2_0 L_f\big) \circ^2_1 \pi^0_G(A) \\ &= \big(\pi^0_G(1^1_2(\lambda^0_G(g))) \circ^2_0 \pi^0_G(L_f)\big) \circ^2_1 1^1_2(f \circ^1_0 g) \\ &= 1^1_2(\pi^0_G(\lambda^0_G(g))) \circ^2_0 \pi^0_G(L_f) \\ &= 1^1_2(\pi^0_G(\lambda^0_G(g))) \circ^2_0 1^1_2(\pi^0_G(\lambda^0_G(f))) \\ &= \pi^0_G(R_g) \circ^2_0 1^1_2(\pi^0_G(\lambda^0_G(f))) \\ &= \pi^0_G(R_g) \circ^2_0 \pi^0_G(1^1_2(\lambda^0_G(f))) \\ &= \pi^0_G(R_g \star^2_0 1^1_2(\lambda^0_G(f)))\,. \end{aligned}$$

which shows existence of this coherence 3-cell u between the 2-cells $\big(1^1_2(\lambda^0_G(g)) \star^2_0 L_f\big) \star^2_1 A$ and $R_g \star^2_0 1^1_2(\lambda^0_G(f))$.

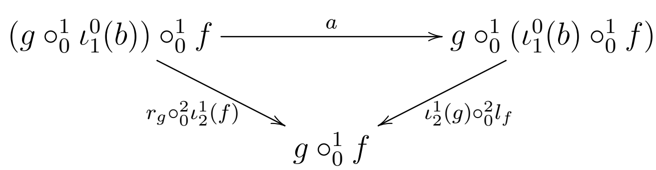
Finally v is also a morphism of $(\infty, 0)$-magmas (see 3.12), thus

- $v(\lambda^0_G(g) \star^1_0 1^1_0\big(\lambda^0_G(b)\big)\big) \star^1_0 \lambda^0_G(f) = (g \circ^1_0 \iota^0_1(b)) \circ^1_0 f$,
- $v(\lambda^0_G(g) \star^1_0 \big(1^1_0\big(\lambda^0_G(b)\big) \star^1_0 \lambda^0_G(f)\big)\big) = g \circ^1_0 (\iota^0_1(b) \circ^1_0 f)$,
- $v(\lambda^0_G(g) \star^1_0 \lambda^0_G(f)) = g \circ^1_0 f$,

and

- $v(\big(1^1_2(\lambda^0_G(g)) \star^2_0 L_f\big) \star^2_1 A) = (\iota^1_2(g) \circ^2_0 l_f) \circ^2_1 a$,
- $v(R_g \star^2_0 1^1_2(\lambda^0_G(f))) = r_g \circ^2_0 \iota^1_2(f)$.

If we apply v to this 3-cell u we obtain the required commutative diagram



because $v(u)$ is an identity cell which is a consequence of the dimension 2 of the algebra $(G; v)$. ∎

3.18. Proposition. *If $(G; v)$ is a $\mathbb{T}^0$-algebra of dimension 2, then it puts on G a bigroupoidal structure.*

Proof. It is a consequence of the propositions above ∎

4. (∞, n)-structures

This last section was motivated by the suggestion of the anonymous referee who told us to include the proof that in dimension 2 our models of weak ∞-groupoids are bigroupoids.

Actually, we could have built all our article with many other combinatorics similar to those of the first section 1 and for each $n \in \mathbb{N}$ it should give different flavours of what we suspect to be other models of weak (∞, n)-categories, where in dimension 2 corresponding models of weak ∞-groupoids are bigroupoids (see below).

These combinatorics are called *regular* (∞, n)*-structure*, and in our article we have used the *maximal* (∞, n)*-structure* (see below).

Surprisingly, the proposition 4.1 below related to strict ∞-groupoids, has helped us to understand better the key ingredients (see definitions 4.2 and 4.4) of all variations possible of models of weak ∞-groupoids which in dimension 2 produce bigroupoids.

Strict ∞-groupoids have been defined in 2.1 under the name of a strict $(\infty, 0)$-categories. Also we have the following proposition

4.1. PROPOSITION. *A strict ∞-category is a strict ∞-groupoid if the following equivalent conditions hold*

1. *For each $m \geq 1$, each m-cell has an $\circ^m_{m-1}$-inverse,*

2. *For each $m \geq 1$, each m-cell has an $\circ^m_p$-inverse for one p such that $0 \leq p < m$,*

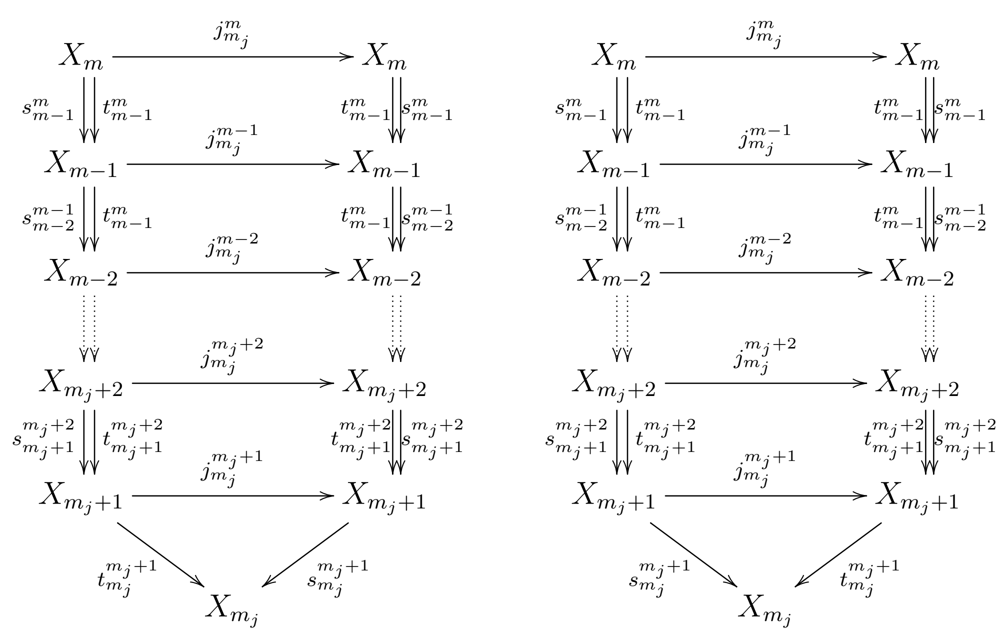
3. *For each $m \geq 1$, each m-cell has an $\circ^m_p$-inverse for all p such that $0 \leq p < m$.*

Proof of this proposition can be found in [1]. The underlying technology of stretchings that we used in this article is to use the "strict world" to build the "weak world". This proposition informs us that other candidates of (∞, n)-structure (see below) could have been chosen for other approach of algebraic model of weak (∞, n)-categories. This proposition motivates the following definition

4.2. DEFINITION. *An $(\infty, 0)$-structure on an ∞-graph X is the data of a family of diagrams constructed as follows : $\forall m > 0$, $\forall s \in \mathbb{N}$ such that $1 \leq s \leq m+1$, we consider a finite sequence in $\mathbb{N}$*

$$0 \leq m_1 < m_2 < ... < m_j < ... < m_s \leq m-1$$

where for all $j \in \mathbb{N}$ such that $1 \leq j \leq s$, it corresponds two diagrams in $\mathbb{S}et$, each commuting serially



Now let us fix an integer $n \in \mathbb{N}$. An (∞, n)-structure on an ∞-graph X is the data of a family of diagrams that are obtained by n-truncation of an $(\infty, 0)$-structure on an ∞-graph X'[5]*, that is we consider only diagrams of such $(\infty, 0)$-structure for which $m > n$, and where X' is replaced by X. It is equivalent to say that an (∞, n)-structure on an ∞-graph X is the realisation of the sketch obtained by n-truncation of the sketch of an $(\infty, 0)$-structure on an ∞-graph X'. An (∞, n)-structure on an ∞-graph X is denoted by*

$$R = (X, (j^m_{m_j})_{1 \leq j \leq s; 1 \leq s \leq m-n+1; m>n})$$

4.3. REMARK. Morphisms between two ∞-graphs equipped with the same kind of (∞, n)-structure, are build as in 1 : Such morphisms need just to preserve underlying reversors of such (∞, n)-structure.

4.4. DEFINITION. *Consider an (∞, n)-structure on an ∞-graph X as above. If for each $m > n$ and for each $1 \leq s \leq m - n + 1$ we have $m_s = m - 1$, then we say that this (∞, n)-structure is regular.*

For each regular (∞, n)-structure $R = (X, (j^m_{m_j})_{1 \leq j \leq s; 1 \leq s \leq m-n+1; m>n})$, the corresponding monad on ∞-$\mathbb{G}$r is denoted by

$$\mathbb{T}^{(\infty,n,R)} = (T^{(\infty,n,R)}, \mu^{(\infty,n,R)}, \lambda^{(\infty,n,R)})$$

and its algebras are models of weak (∞, n)-categories build with this specific regular (∞, n)-structure. We believe that these regular (∞, n)-structures R produce categories $\mathbb{A}lg(\mathbb{T}^{(\infty,n,R)})$ which are Quillen equivalents.

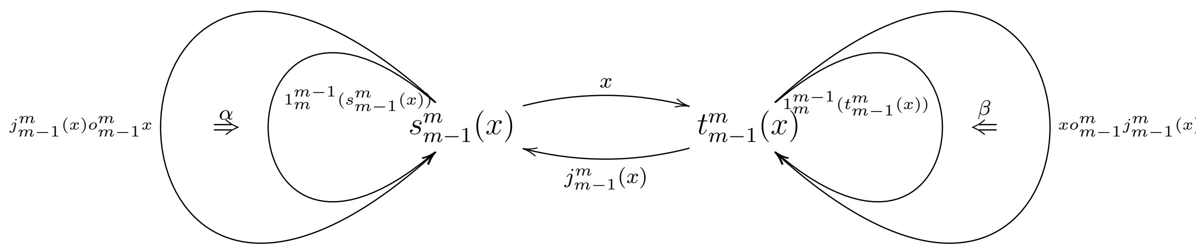
The *maximal* (∞, n)*-structure* $R_{\max}$ that we used in our article, is to consider all kind of reversors j^m_p with $0 \leq n \leq p < m$. It is a regular (∞, n)-structure as well, and we have shown in 3.18 that the corresponding models of weak ∞-groupoids in dimension 2 are bigroupoids. These are bigroupoids because the maximal (∞, n)-structure is regular, which forces the existence of the reversor j^2_1, and thus the proof of the bigroupoidal nature in dimension 2.

More generally, it is not difficult to check that with any regular (∞, n)-structure, the corresponding models of weak ∞-groupoids in dimension 2 are also bigroupoids, because regularity forces the existence of the reversor j^2_1. However it is not true for a non-regular (∞, n)-structure which doesn't contain the reversor j^2_1, that the corresponding models of weak ∞-groupoids in dimension 2 are bigroupoids, just for the simple reason that each 2-cell in it have no reasons to be $\circ^2_1$-isomorphisms.

Also each regular (∞, n)-structure captures many kinds of *higher equivalences*, depending on the kinds of reversors we used. More precisely, consider for example the following definition build by decreasing induction :

4.5. DEFINITION. *Let us consider a strict ∞-category C and $x \in C(m)$. We say that x is an $(\circ^m_{m-1}, q)$-equivalence ($q \in \mathbb{N}$) if there exist a diagram*

[5] X' could be equal to X but not necessarily.



such that α and β are $(\circ_m^{m+1}, q-1)$-equivalences. By convention, for each $0 \leq s < r$, $(\circ_s^r, 0)$-equivalences are $\circ_s^r$-isomorphisms.

This definition gives also an intuition of what could be an $(\circ_{m-1}^m, q)$-equivalence when q is taken to the infinite.

Also for any regular (∞, n)-structure R, for each $n \in \mathbb{N}$, and for each integer m such that $0 \leq n < m$, the m-cells of a $\mathbb{T}^{(\infty,n,R)}$-algebras are $(\circ_{m-1}^m, q)$-equivalences where q could be taken to the infinite.

Let us finish by a fact that we believe is important : A remarkable regular (∞, n)-structure called the *minimal* (∞, n)*-structure* $R_{\min}$, is to consider for each $m > n$, $s = 1$ and :

$$n \leq m_1 = m - 1.$$

and this (∞, n)-structure and the maximal (∞, n)-structure $R_{\max}$ are both extrema in the sense of the evident following proposition :

4.6. Proposition. *For any regular (∞, n)-structure R we have the following functorial inclusions*

$$\mathbb{A}lg(\mathbb{T}^{(\infty,n,R_{max})}) \hookrightarrow \mathbb{A}lg(\mathbb{T}^{(\infty,n,R)} \hookrightarrow \mathbb{A}lg(\mathbb{T}^{(\infty,n,R_{min})})$$

This proposition raises the question that the collection of regular (∞, n)-structures might form a lattice. Other members of the collection, apart from the minimal and maximal, may provide useful models of weak (∞, n)-categories.

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