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GAUGE INVARIANT SURFACE HOLONOMY AND MONOPOLES

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Abstract. There are few known computable examples of non-abelian surface holonomy. In this paper, we give several examples whose structure 2-groups are covering 2-groups and show that the surface holonomies can be computed via a simple formula in terms of paths of 1-dimensional holonomies inspired by earlier work of Chan Hong-Mo and Tsou Sheung Tsun on magnetic monopoles. As a consequence of our work and that of Schreiber and Waldorf, this formula gives a rigorous meaning to non-abelian magnetic flux for magnetic monopoles. In the process, we discuss gauge covariance of surface holonomies for spheres for any 2-group, therefore generalizing the notion of the reduced group introduced by Schreiber and Waldorf. Using these ideas, we also prove that magnetic monopoles form an abelian group.

Contents

1. Introduction

1.1. Background, motivation, and overview. Ordinary holonomy along paths for principal group bundles has been studied for over 40 years in the context of gauge theories in physics and in the context of fiber bundles in mathematics. Recently, with ideas from higher category theory, it has been possible to extend these ideas to holonomy along surfaces. Although higher holonomy, and more generally higher gauge theory, has been studied in the context of *abelian* gauge theory for higher-dimensional manifolds, it was thought for some time that non-abelian generalizations were not possible [Te86]. Today, we understand this as being due to the fact that a group object in the category of groups is an abelian group. By "categorifying" well-known concepts, and considering group objects

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in the category of categories, one can avoid this restriction. The language of higher categories allows us to give a resolution to this problem.

The data needed for defining surface holonomy for abelian structure groups has been known for quite some time under the name *abelian gerbes with connection* with a formal presentation offered by Gawedzki [Ga88] in 1988 in the context of the WZW model, with further work in 2002 with Reis [GaRe02]. Further development under the name of *non-abelian gerbes*, *higher bundles*, and so on were carried out in the following years starting with the foundational work of Breen and Messing [BrMe05] in 2001, where the data for connections on non-abelian gerbes first appeared. In [BaSc04], Baez and Schreiber gave a definition of non-abelian gerbes with connection in terms of parallel transport using the notion of a 2-group. The most up-to-date theoretical framework in terms of category theory, which provides a language easily adaptable for non-abelian generalizations, was established by Schreiber and Waldorf in [ScWa13]. In this categorical setting, higher principal bundles with connections are described by transport functors.

The motivation for transport functors comes from observations originally made by Barrett in [Ba91] and expanded on by Caetano and Picken in [CaPi94] by describing a bundle with connection in terms of its holonomies. In [ScWa09], Schreiber and Waldorf use a categorical perspective to prove that a principal group bundle with connection over a smooth manifold determines, and is determined by, a transport functor defined on the thin path groupoid of that manifold with values in a fattened version of the structure group viewed as a one-object category. The upshot of this equivalence is that it is conceptually simple to go from categories and functors to 2-categories and 2-functors. In [ScWa11], [ScWa], and [ScWa13], Schreiber and Waldorf take advantage of this equivalence and abstract the definition so that it can be used to *define* principal 2-group 2-bundles with connection allowing a conceptually simple formulation of surface holonomy.

In the present article, we review the theory of transport functors formalized by Schreiber and Waldorf in [ScWa09], [ScWa11], [ScWa], and [ScWa13] with an emphasis on examples and explicit computations. Besides this, we accomplish several new results. First, we provide a definition of holonomy along spheres modulo thin homotopy *without* representing a sphere as a bigon (Definition 3.50). The target of this holonomy is an analogue of conjugacy classes, which is used for ordinary holonomy along loops, called α-conjugacy classes. To prove this, we introduce a procedure that turns an arbitrary transport functor into a group-valued transport functor. In [ScWa13], the authors forced their surface holonomy to land in a rather restrictive quotient of the structure 2-group to prove gauge invariance of holonomy. Our perspective is to take the smallest quotient possible, and we show our quotient surjects onto the one of [ScWa13].

We then focus on transport functors with a particular class of 2-groups, termed *covering 2-groups*, given by a Lie group G and a covering space of G. We provide a *simple* formula, motivated by constructions in [HoTs93], for holonomy along surfaces in a local trivialization and show that this formula agrees with the surface-ordered integral in [ScWa11]. This gives an interesting relationship between (i) well-known formulas in the physics literature for computing the magnetic flux in terms of a loop of holonomies and (ii) non-abelian surface-ordered integrals in terms of 1- and 2-forms of [ScWa11]. Physically, we argue that the latter is the correct analogue to computing the magnetic flux as a surface integral and our formula tells us that this agrees with the usual definition given in the physics literature. This is all done without the introduction of a Higgs field, completing the ideas in [GoNuOl77].

Then we consider an entire collection of examples of transport 2-functors constructed from an ordinary principal G-bundle with connection along with a choice of a subgroup N of $\pi_1(G)$, the fundamental group of G (such a choice of subgroup determines a covering 2-group). We show that when the subgroup N is chosen to be $\pi_1(G)$ itself, our example reduces to the curvature 2-functor defined by Schreiber and Waldorf in [ScWa13]. We instead focus on the other extreme, namely when the subgroup N is chosen to be the trivial group $\{1\}$, to calculate four examples of surface holonomies associated to both abelian and non-abelian magnetic monopoles. But just as ordinary holonomy is not exactly group-valued on the space of all loops (due to conjugation issues), surface holonomy isn't in general either. Using our results on gauge invariance of sphere holonomy for arbitrary 2-groups, we prove that the surface holonomies for magnetic monopoles are not only gauge invariant but also form an abelian group.

1.2. Outline of paper along with main results. In Section 2, we review the main definitions of transport functors along with an equivalence between local descent data and global transport functors. We follow the recent work of Schreiber and Waldorf [ScWa09] who describe it precisely and categorically in a framework that is suitable for generalizations to surfaces. We briefly discuss the relationship to principal G-bundles with connection, where G is a Lie group, in their usual formulation by introducing the category of G-torsors (manifolds with free and transitive right G-actions). The equivalence between the two descriptions was proved in [ScWa09]. We also review the relationship between local descent data and differential cocycle data for principal group bundles, recalling the well-known formula for parallel transport in terms of a path-ordered integral. To obtain group-valued holonomies, we introduce a procedure (60) described as a functor that takes an arbitrary transport functor and produces a group-valued transport functor in Section 2.31. The presentation differs a bit from that of [ScWa09] so we describe it in some detail.

In Section 3, we review how to 'categorify' the definitions and statements of Section 2 in order to define transport 2-functors. The main references for this section include [ScWa11], [ScWa], and [ScWa13]. We only briefly review the technical points but spend more time on a computational understanding of surface holonomy and also supply an iterated integral expression for surface holonomy including a picture (Figure 15) that we think will be useful for lattice gauge theory. As in the case of holonomy along loops, we introduce a procedure (169) to obtain group-valued surface holonomy. This lets us discuss gauge covariance and gauge invariance simply and in full detail without referring to the *reduced group* of [ScWa13]. However, we restrict ourselves to holonomy along spheres as opposed to surfaces of arbitrary genus. We show, in Theorem 3.49, that our holonomy along spheres lands in a set that surjects onto the reduced group and give a simple example, in Lemma 3.54, where this surjection has nontrivial kernel.

In Section 4, we consider transport 2-functors with structure 2-group given by a covering 2-group. We give a new and simple formula valid for *all* such transport 2-functors in Corollary 4.21 for surface holonomy in a local trivialization in terms of homotopy classes of paths of holonomies along loops. This construction was inspired by work of physicists for computing magnetic charge as a topological number [HoTs93]. In Definition 4.15, we give our main construction of a transport 2-functor, called the *path-curvature 2-functor*, associated to every principal G-bundle with connection and to any subgroup of $\pi_1(G)$. We prove that this assignment is functorial. Furthermore, the path-curvature 2-functor is shown to reduce to the example of Schreiber and Waldorf known as the *curvature 2-functor* in [ScWa13] when the subgroup of $\pi_1(G)$ is chosen to be $\pi_1(G)$ itself. We describe this construction on four levels: (i) global transport functors (ii) functors with smooth trivialization data chosen (iii) descent data (iv) differential cocycle data. This allows one to work with either construction at whatever level he or she pleases. We then summarize our result as a list of commutative diagrams of functors in (251), (255), and (257).

In Section 5, we consider special cases of covering 2-groups and give several examples all of which are known as magnetic monopoles [HoTs93]. The first example is obtained from any principal $U(1)$-bundle with connection over the two-sphere S^2. It is shown that the surface holonomy along this sphere coming from the path-curvature 2-functor defined in Section 4 is precisely the integral of the curvature form of the principal $U(1)$-bundle along this sphere, which in this case is the integral of the first Chern class over the sphere. This example is precisely the Dirac monopole [Di31] and the surface holonomy gives the magnetic charge as the integral of a magnetic flux. We then discuss non-abelian examples starting with a principal $SO(3)$-bundle with connection over the sphere and compute the surface holonomy explicitly using both our simple formula and the formula in terms of path-ordered integrals using differential forms. In the case of a non-trivial bundle, the surface holonomy along the sphere is given by the element $\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)$ in $SU(2)$, the universal cover of $SO(3)$, which is the nontrivial element in the kernel of the covering map $\tau : SU(2) \longrightarrow SO(3)$. We do this same computation in other examples including $SU(n) \longrightarrow SU(n)/Z(n)$, where $Z(n)$ is the center of $SU(n)$, and also for the case $SU(n) \times \mathbb{R} \longrightarrow U(n)$. This gives a rigorous meaning to the notion of *non-abelian magnetic flux* as a surface holonomy along a sphere (see Definition 5.6). Furthermore, it is shown that magnetic flux is a gauge-invariant quantity in Corollary 5.7.

Finally, the Appendix includes an overview of diffeological spaces which are used to describe several of the constructions involving infinite-dimensional manifolds and smooth maps between them.

In short, this article contains the following results.

- Theorem 2.47 allows one to define gauge-invariant holonomy along loops in the language of transport functors via Definition 2.48. The image lands in conjugacy classes instead of the abelianization.

- Theorem 3.49 accomplishes the analogous result for surface holonomy *along spheres* in Definition 3.50. The image lands in α-conjugacy classes (Definition 3.48) instead of the reduced group of [ScWa13]. The set of α-conjugacy classes surjects to the reduced group but is not in general injective as shown in Lemma 3.54. We also prove that the fixed points of this α action form a central subgroup of the group of surface holonomies in Lemma 3.56.

- The rest of the paper focuses on transport 2-functors whose structure 2-groups are covering 2-groups (Definition 4.8). They are called *path-curvature 2-functors* (Definition 4.15). These transport 2-functors are defined *without* using surface integrals, and we show, in Theorem 4.20 and Corollary 4.21, that locally, *any* transport 2-functor (defined as in [ScWa11] using surface integrals) with structure 2-group a covering 2-group, coincides with ours, thus enabling a simple formula for calculating surface holonomy.

- Section 5 includes several examples and explicit computations of surface holonomy. Due to the previously mentioned theorem, these examples can rightfully be called magnetic fluxes of magnetic monopoles from physics. We include several examples of non-abelian surface holonomy. We conclude with Corollary 5.7 that shows that the magnetic flux is a fixed point under the α action and therefore lands in the central subgroup mentioned earlier. In particular, this implies that the magnetic charge is an abelian group-valued quantity known as a *topological number*.

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1.4. Notations and conventions. We assume the reader is familiar with some basic concepts of 2-categories (the Appendix of [ScWa] explains most details needed for this paper) but our notation differs from the norm so we set it now.

Compositions of 1-morphisms is usually written from right to left as in

$$z \xleftarrow{\alpha} y \xleftarrow{\beta} x \qquad \mapsto \qquad z \xleftarrow{\alpha\circ\beta} x\,. \tag{1}$$

Vertical composition is written from top to bottom as

$$\begin{tikzcd}[column sep=large] y & x \arrow[l, bend right=60, "\beta"', ""{name=B, below}] \arrow[l, "\delta" description, ""{name=D1, above}, ""{name=D2, below}] \arrow[l, bend left=60, "\zeta", ""{name=Z, above}] \arrow[Rightarrow, from=B, to=D1, "\Sigma"] \arrow[Rightarrow, from=D2, to=Z, "\Omega"] \end{tikzcd} \quad \mapsto \quad \begin{tikzcd}[column sep=large] y & x \arrow[l, bend right=60, "\beta"', ""{name=B, below}] \arrow[l, bend left=60, "\zeta", ""{name=Z, above}] \arrow[Rightarrow, from=B, to=Z, "\Sigma \circ \Omega"] \end{tikzcd} . \tag{2}$$

Horizontal composition is written as

$$\begin{tikzcd} z & y \arrow[l, bend right=50, "\alpha"', ""{name=A, below}] \arrow[l, bend left=50, "\gamma", ""{name=G, above}] & x \arrow[l, bend right=50, "\beta"', ""{name=B, below}] \arrow[l, bend left=50, "\delta", ""{name=D, above}] \arrow[Rightarrow, from=A, to=G, "\Sigma"] \arrow[Rightarrow, from=B, to=D, "\Omega"] \end{tikzcd} \quad \mapsto \quad \begin{tikzcd}[column sep=large] z & x \arrow[l, bend right=50, "\alpha\circ\beta"', ""{name=A, below}] \arrow[l, bend left=50, "\gamma\circ\delta", ""{name=G, above}] \arrow[Rightarrow, from=A, to=G, "\Sigma\circ\Omega"] \end{tikzcd} . \tag{3}$$

Sources, targets, and identity-assigning functions are denoted by s, t, and i, respectively. We will always write the identity $i(x)$ at an object x as id_x, id_α for the vertical identity at a 1-morphisms α, and $\mathrm{id}_{\mathrm{id}_x}$ for the horizontal identity at an object x. Given a 2-category $\mathcal{C}$, the set of objects is typically denoted by $\mathcal{C}_0$, 1-morphisms by $\mathcal{C}_1$ and 2-morphisms by $\mathcal{C}_2$. In general, an overline such as $\overline{f}$ will denote weak inverses, vertical inverses, and reversing paths/bigons. It will be clear from context which is which. The first form of 2-categories appeared under the name *bi-categories* and were introduced by Bénabou [Bé67].

2. Principal bundles with connection are transport functors

In this section, we review the notion of transport functors mainly following [ScWa09]. We split up the discussion into several parts. We first discuss a Čech description of principal G-bundles (without connection), where G is a Lie group, in terms of smooth functors. Then we attempt a guess for describing principal G-bundles with connections in terms of smooth functors. This attempt fails as it only gives trivialized bundles, motivating the need to use transport functors. We then proceed to describing local trivialization data, descent data, and finally transport functors. The key feature of descent data is that it enables us to encode smoothness while still allowing the 'bundle' to have nontrivial topology. We then discuss a reconstruction functor that takes us from the category of descent data to the category of transport functors with chosen trivializations. It is here that we discuss a version of the Čech groupoid incorporating paths and 'jumps' that are necessary for transition functions. Then we move in the other direction and go from smooth descent data to locally defined differential forms, or more generally differential cocycle data. We also describe how to go from differential cocycle data back to smooth descent data. We then summarize the four different levels describing transport functors and their relationship to one another. Finally, we use these results to formulate a procedure that sends an arbitrary transport functor to a transport functor with group-valued parallel transport and discuss its gauge covariance and invariance stressing the use of conjugacy classes.

2.1. A Čech description of principal G-bundles. Let G be a Lie group. Principal G-bundles over a smooth manifold M can be described simply in terms of functors. Furthermore, an isomorphism of such bundles corresponds to a natural transformation of the corresponding functors. This is done as follows (this is an expansion of Remark II.13. in [Wo11]).

2.2. Definition. *Given an open cover $\{U_i\}_{i\in I}$ of M, the* Čech groupoid $\mathfrak{U}$ *is the category whose set of objects is given by*

$$\mathfrak{U}_0 := \coprod_{i\in I} U_i \tag{4}$$

and whose morphisms, called 'jumps,' are given by

$$\mathfrak{U}_1 := \coprod_{i,j\in I} U_{ij}, \tag{5}$$

where $U_{ij} := U_i \cap U_j$ and the order of the index is kept track of in the disjoint union. Explicitly, elements of $\mathfrak{U}_0$ are written as (x,i) and elements of $\mathfrak{U}_1$ are written as (x,i,j). The source and target maps are given by $s((x,i,j)) := (x,i)$ and $t((x,i,j)) := (x,j)$ for $(x,i,j) \in \mathfrak{U}_1$. The identity-assigning map is given by[1] *$i((x,i)) := (x,i,i)$. Let (x,i,j) and (x',i',j') be two morphisms with $t((x,i,j)) = s((x',i',j'))$, i.e. $(x,j) = (x',i')$. Renaming the index j' to k, the composition is defined to be*

$$(x,j,k) \circ (x,i,j) := (x,i,k). \tag{6}$$

2.3. Definition. *For every Lie group G, there is a one-object groupoid $\mathcal{B}G$ defined as follows. Denote the one object by $\bullet$. Let the set of morphisms from $\bullet$ to itself be given by the set G. Composition is given by group multiplication.*

The previous two groupoids have a smooth structure, formalized in the following definition.

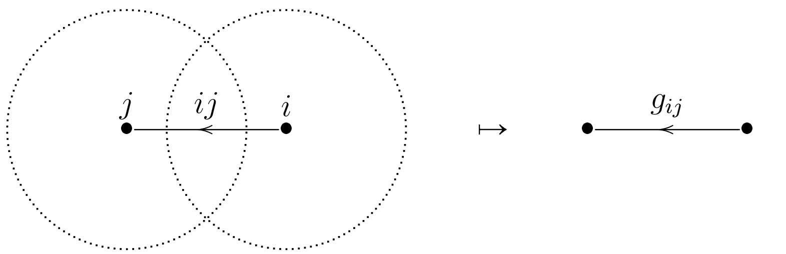
2.4. Definition. *A* Lie groupoid *is a (small) category, typically denoted by* Gr, *whose objects, morphisms, and sets of composable morphisms all form smooth manifolds. Furthermore, the source, target, identity-assigning, and composition maps are all smooth. In addition, every morphism has an inverse and the map that sends a morphism to its inverse is smooth.*

2.5. Example. *The Čech groupoid of Definition 2.2 and $\mathcal{B}G$ of Definition 2.3 are Lie groupoids with the appropriate (obvious) smooth structures.*

[1]Our apologies for this double usage of the letter i to mean both the identity-inclusion map and the index letter. We hope that it is not too confusing. Later, we will also use the letter i for several other purposes.

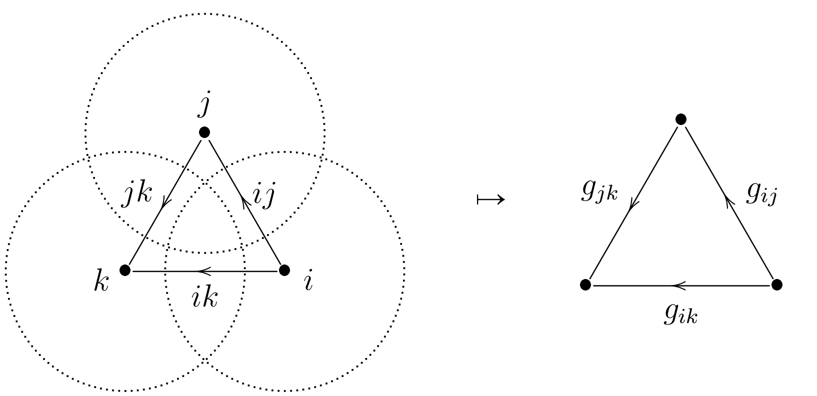
2.6. Definition. *A* smooth functor *from one Lie groupoid to another is an ordinary functor that is smooth on objects and morphisms. Likewise, a* smooth natural transformation *is a natural transformation whose function from objects to morphisms is smooth.*

Any smooth functor $\mathfrak{U} \longrightarrow \mathcal{B}G$ gives the Čech cocycle data of a principal G-bundle over M subordinate to the cover $\{U_i\}_{i \in I}$. To see this, simply recall what a functor does. To each object (x, i) in $\mathfrak{U}$, it assigns the single object $\bullet$ in $\mathcal{B}G$. To each jump (x, i, j), it assigns an element denoted by $g_{ij}(x) \in G$ in such a way that we get a smooth 1-cochain $g_{ij} : U_{ij} \longrightarrow G$



(7)

This picture should be interpreted as follows. To each $x \in U_{ij}$, we draw the jump (x, i, j) as the figure on the left. Its image under $\mathfrak{U} \longrightarrow \mathcal{B}G$ is $g_{ij}(x)$ drawn on the right (without explicitly writing x). To each triple intersection U_{ijk}, which corresponds to the composition of (x, i, j) in U_{ij} with (x, j, k) in U_{jk} as in (6), functoriality gives a cocycle condition



(8)

which says

$$g_{jk} g_{ij} = g_{ik}. \tag{9}$$

This convention was chosen to match that of [ScWa09] and [ScWa13] so that the reader who is interested in further details can consult without too much trouble.

We now discuss refinements and morphisms between two such functors. Let $\{U_{i'}\}_{i' \in I'}$ be another cover of M with associated Čech groupoid $\mathfrak{U}'$. Let $P : \mathfrak{U} \longrightarrow \mathcal{B}G$ and $P' : \mathfrak{U}' \longrightarrow \mathcal{B}G$ be two smooth functors. A morphism from P to P' consists of a common refinement $\{V_\alpha\}_{\alpha \in A}$, with associated Čech groupoid $\mathfrak{V}$, of both $\{U_i\}_{i \in I}$ and $\{U_{i'}\}_{i' \in I'}$ along with a smooth natural transformation

$$
\begin{tikzcd}
 & \mathfrak{U} \arrow[dr, "P"] \arrow[dd, Rightarrow, "h"] & \\
\mathfrak{V} \arrow[ur, "\alpha"] \arrow[dr, "\alpha'"'] & & \mathcal{B}G . \\
 & \mathfrak{U}' \arrow[ur, "P"'] &
\end{tikzcd}
\tag{10}
$$

The refinement condition means that there are associated functions $\alpha : A \longrightarrow I$ and $\alpha' : A \longrightarrow I'$ so that $V_a \subset U_{\alpha(a)}$ and $V_a \subset U'_{\alpha(a)}$ for all $a \in A$. These functions determine the functors $\alpha : \mathfrak{V} \longrightarrow \mathfrak{U}$ and $\alpha' : \mathfrak{V} \longrightarrow \mathfrak{U}'$ drawn above. We denote the restrictions of $g_{\alpha(a)\alpha(b)}$ and $g'_{\alpha'(a)\alpha'(b)}$ to V_{ab} by g_{ab} and g'_{ab}, respectively. Any such smooth natural transformation gives an equivalence of Čech cocycle data of principle G-bundles. To see this, simply recall what a natural transformation does. To each object (x, a) in $\mathfrak{V}$ it assigns a group element $h_a(x) \in G$ in a smooth way. In other words, it gives a smooth function $h_a : V_\alpha \longrightarrow G$. To each jump (x, a, b) in $\mathfrak{V}$, the naturality condition

$$
\begin{tikzcd}
\bullet \arrow[d, "h_b"'] & \bullet \arrow[l, "g_{ab}"'] \arrow[d, "h_a"] \arrow[dl, phantom, "="] \\
\bullet & \bullet \arrow[l, "g'_{ab}"]
\end{tikzcd}
\tag{11}
$$

says that

$$h_b g_{ab} = g'_{ab} h_a \tag{12}$$

on V_{ab}. This is precisely the condition that says the principal G-bundles P and P' are isomorphic [St99].

2.7. A NAIVE GUESS FOR TRANSPORT FUNCTORS. Our goal in this section is to guess what a connection on a principal G-bundle over M should be in terms of functors. We will fail at this attempt, but will learn an important lesson which will motivate the modern definition in terms of transport functors. First, recall that in a principal G-bundle $P \longrightarrow M$, every fiber is a right G-torsor.

2.8. DEFINITION. *Let G be a Lie group. Let G-*Tor *be the category whose objects are right* <u>G-torsors</u>, *i.e. smooth manifolds equipped with a free and transitive right G-action, and whose morphisms are right G-equivariant maps.*

Furthermore, a connection on a principal G-bundle over M gives an assignment from paths in M to isomorphisms of fibers between the endpoints. This assignment is independent of the parametrization of the path, but it is even independent of the thin homotopy class of a path as discussed in [CaPi94]. To define this, we use the theory of smooth spaces, reviewed in Appendix A, which give natural definitions for smooth structures on subsets, mapping spaces, and quotient spaces.

2.9. Definition. *Let X be a smooth manifold. A* path with sitting instants *is a smooth map $\gamma : [0,1] \longrightarrow X$ such that there exists an ϵ with $\frac{1}{2} > \epsilon > 0$ and $\gamma(t)$ is constant for all $t \in [0,\epsilon] \cup [1-\epsilon, 1]$. For such paths γ with $\gamma(0) = x$ and $\gamma(1) = y$, we write*

$$y \xleftarrow{\gamma} x \, . \tag{13}$$

The set of paths with sitting instants in X will be denoted by PX.

Paths with sitting instants were first described in [CaPi94]. We reserve the notation $X^{[0,1]}$ for the set of (ordinary) smooth paths in X. Thus, $PX \subset X^{[0,1]}$.

2.10. Definition. *Two paths in X with sitting instants γ and γ' with the same endpoints, i.e. $\gamma(0) = \gamma'(0) = x$ and $\gamma(1) = \gamma'(1) = y$, are said to be* thinly homotopic *if there exists a smooth map $\Gamma : [0,1] \times [0,1] \longrightarrow X$ with the following two properties.*

(a) First, there exists an ϵ with $\frac{1}{2} > \epsilon > 0$ such that

$$\Gamma(t,s) = \begin{cases} x & \text{for all } (t,s) \in [0,\epsilon] \times [0,1] \\ y & \text{for all } (t,s) \in [1-\epsilon,1] \times [0,1] \\ \gamma(t) & \text{for all } (t,s) \in [0,1] \times [0,\epsilon] \\ \gamma'(t) & \text{for all } (t,s) \in [0,1] \times [1-\epsilon,1] \end{cases} \tag{14}$$

A map $\Gamma : [0,1] \times [0,1] \longrightarrow X$ satisfying just (14) is called a bigon *in X and is typically denoted by*

$$\begin{tikzcd} y & & x \arrow[ll, bend right, "\gamma"'] \arrow[ll, bend left, "\gamma'"] \arrow[ll, phantom, "\Downarrow\Gamma"] \end{tikzcd} \tag{15}$$

The set of bigons in X is denoted by BX.

(b) Second, the rank of Γ is strictly less than 2, i.e. the differential $D_{(t,s)}\Gamma : T_{(t,s)}([0,1] \times [0,1]) \longrightarrow T_{\Gamma(t,s)}X$, where T_yY denotes the tangent space to Y at the point $y \in Y$, has kernel of dimension at least one for all $(t,s) \in [0,1] \times [0,1]$.

Thin homotopy is an equivalence relation and the equivalence classes are called thin paths. *Denote the set of thin paths in X by P^1X.*

P^1X is naturally a smooth space since it is a quotient of PX, which is itself a subset of $X^{[0,1]}$, which has a natural smooth space structure as a mapping space. With these preliminaries, the definition of the thin path-groupoid of a smooth manifold X can be given (we refer the reader to [CaPi94] and [ScWa09] for more details).

2.11. Definition. *Let X be a smooth manifold. Let $\mathcal{P}_1(X)$ be the category whose objects are the points of the smooth manifold X and whose morphisms are the thin paths of X. The source and target of a thin path are defined by choosing a representative and taking the source and target, respectively. The identity at each point $x \in X$ is the thin path associated to the constant path at x. The composition of two thin paths is defined by choosing representatives and concatenating with double-speed parametrization. Namely, given two thin paths*

$$z \xleftarrow{\gamma'} y \xleftarrow{\gamma} x \,, \tag{16}$$

the composition is given by the thin homotopy class associated to

$$(\gamma' \circ \gamma)(t) := \begin{cases} \gamma(2t) & \text{for } 0 \leqslant t \leqslant \frac{1}{2} \\ \gamma'(2t-1) & \text{for } \frac{1}{2} \leqslant t \leqslant 1 \end{cases}. \tag{17}$$

Under the sitting instants assumption and the thin homotopy equivalence relation, the composition is well-defined, smooth, associative, has left and right units given by constant paths, and right and left inverses by reversing paths. By replacing the word "smooth manifold" with "smooth space" in Definition 2.4, $\mathcal{P}_1(X)$ is therefore a Lie groupoid.

With this definition of the thin path-groupoid of M, one might guess that a transport functor should be a smooth functor $\mathcal{P}_1(M) \longrightarrow G$-Tor. However, since G-Tor is not a Lie groupoid, there is no obvious way of demanding such a functor to be smooth. One might therefore try to use $\mathcal{B}G$ instead of G-Tor. Indeed, notice that there is a natural functor $i : \mathcal{B}G \longrightarrow G$-Tor defined by

$$\begin{aligned} \bullet &\mapsto G \\ g &\mapsto L_g, \end{aligned} \tag{18}$$

where G is viewed as a right G-torsor and L_g is left multiplication on G by g. One can think of G-Tor as a 'thickening' of $\mathcal{B}G$ because i is an equivalence of categories. We can then try to use $\mathcal{B}G$ for our target instead of G-Tor so that we can ask for smoothness. Then one might guess that a transport functor should be a smooth functor $\mathcal{P}_1(M) \longrightarrow \mathcal{B}G$. Unfortunately, now that we have smoothness, we've lost non-triviality because such smooth functors describe parallel transport on *trivialized* principal G-bundles (this fact follows from Section 2.27 particularly around equation (47)).

In order to encode local instead of global triviality, we have to combine these ideas with those of the previous section in terms of the Čech groupoid (we will also return to a more suitable combination of the path groupoid and the Čech groupoid in Section 2.24). To avoid a huge collection of indices again, we denote our open cover $\{U_i\}_{i \in I}$ of M simply by $Y := \coprod_{i \in I} U_i$ and we let $\pi : Y \longrightarrow M$ be the inclusion of these open sets into M. Note that π is a surjective submersion. Then, the next guess might be that we need to have a smooth functor $\mathcal{P}_1(Y) \longrightarrow \mathcal{B}G$, but we still need an assignment of fibers $\mathcal{P}_1(M) \longrightarrow G$-Tor. These assignments should be compatible in terms of the functor $i : \mathcal{B}G \longrightarrow G$-Tor and the submersion π. This is exactly what is done in [ScWa09] and we therefore now proceed to discussing local triviality of functors.

2.12. Local triviality of functors. Our first goal is to discuss local triviality of functors without making any assumptions on smoothness, which is left to the next section. The fibers of principal G-bundles were right G-torsors, which led us to consider the category G-Tor of G-torsors. One of the great features of Schreiber's and Waldorf's work [ScWa09] is their generality on the different flavors of bundles. If one wants to work with vector bundles one simply replaces G-Tor with Vect, the category of vector spaces (over some appropriate field such as $\mathbb{R}$ or $\mathbb{C}$), and if this vector bundle is an associated bundle for some representation of G, then this representation is precisely encoded by a functor $i : \mathcal{B}G \longrightarrow \mathrm{Vect}$. Fiber bundles can be defined similarly. Therefore, we've made two important observations. The first is that fibers of a bundle are objects of some category T. The second is that the structure group of the bundle is encoded by a functor $i : \mathcal{B}G \longrightarrow T$. Schreiber and Waldorf generalize this even further by considering any Lie groupoid Gr instead of the special one $\mathcal{B}G$. They define a π-local trivialization as follows (Definition 2.5. of [ScWa09]).

2.13. Definition. *Let* Gr *be a Lie groupoid,* T *a category,* $i : \mathrm{Gr} \longrightarrow T$ *a functor, and* M *a smooth manifold. Fix a surjective submersion* $\pi : Y \longrightarrow M$*. A* π-local i-trivialization *of a functor* $F : \mathcal{P}_1(M) \longrightarrow T$ *is a pair* (triv, t) *of a functor* $\mathrm{triv} : \mathcal{P}_1(Y) \longrightarrow \mathrm{Gr}$ *and a natural isomorphism* $t : \pi^*F \Rightarrow \mathrm{triv}_i$ *as in the diagram*

$$\begin{tikzcd} \mathcal{P}_1(M) \arrow[d, "F"'] & \mathcal{P}_1(Y) \arrow[l, "\pi_*"'] \arrow[d, "\mathrm{triv}"] \\ T & \mathrm{Gr} \arrow[l, "i"] \end{tikzcd} \quad \text{with } t: \mathcal{P}_1(M) \Rightarrow \mathrm{Gr} \text{ diagonal}. \tag{19}$$

The groupoid Gr *is called the* structure groupoid *for* F.

In this definition π_* is the pushforward defined sending points $y \in Y$ to $\pi(y)$ and sending thin paths $\gamma \in P^1Y$ to the thin homotopy class of $\pi \circ \gamma$ (by choosing a representative). $\pi^*F := F \circ \pi_*$ is the pullback of F along π and $\mathrm{triv}_i := i \circ \mathrm{triv}$. Functors $F : \mathcal{P}_1(M) \longrightarrow T$ equipped with π-local i-trivializations (triv, t) form the objects, written as triples (F, triv, t), of a category denoted by $\mathrm{Triv}^1_\pi(i)$.

2.14. Definition. *A* morphism $\alpha : (F, \mathrm{triv}, t) \longrightarrow (F', \mathrm{triv}', t')$ *in* $\mathrm{Triv}^1_\pi(i)$ of π-local i-trivializations *is a natural transformation* $\alpha : F \Rightarrow F'$*. Composition is given by vertical composition of natural transformations.*

2.15. Remark. *One might expect a morphism* $(F, \mathrm{triv}, t) \longrightarrow (F', \mathrm{triv}', t')$ *to consist of* $\alpha : F \Rightarrow F'$ *as well as a natural transformation* $h : \mathrm{triv} \Rightarrow \mathrm{triv}'$ *satisfying some compatibility condition with* α*,* t*, and* t'*. This natural compatibility condition completely determines* h *which is why it is excluded in the definition.*

In this description, it's not immediately obvious what transition functions are. This is part of the motivation for introducing descent objects (Definition 2.8. of [ScWa09]).

We use the notation $Y^{[n]}$ associated to a surjective submersion $\pi : Y \longrightarrow M$ to mean the n-fold fiber product defined by

$$Y^{[n]} := \{(y_1, \ldots, y_n) \in Y \times \cdots \times Y \mid \pi(y_1) = \cdots = \pi(y_n)\}\,. \tag{20}$$

There are several projection maps $\pi_{i_1 \cdots i_k} : Y^{[n]} \longrightarrow Y^{[n-k]}$ for all $n \geqslant 2$ and $0 < k < n$ with $1 < i_1 < \cdots < i_k < n$ that are defined by

$$Y^{[n]} \ni (y_1, \ldots, y_n) \mapsto (y_{i_1}, \ldots, y_{i_k}). \tag{21}$$

$Y^{[n]}$ is a smooth manifold for all n and all $\pi_{i_1 \cdots i_k}$ are smooth since π is a surjective submersion.

2.16. Definition. *Let* Gr *be a Lie groupoid, T a category, and $i : \mathrm{Gr} \longrightarrow T$ a functor. Fix a surjective submersion $\pi : Y \longrightarrow M$. A* <u>descent object</u> *is a pair* (triv, g) *consisting of a functor* $\mathrm{triv} : \mathcal{P}_1(Y) \longrightarrow \mathrm{Gr}$, *a natural isomorphism*

$$\begin{tikzcd} \mathcal{P}_1(Y) \arrow[d, "\mathrm{triv}_i"'] \arrow[dr, Rightarrow, "g"] & \mathcal{P}_1(Y^{[2]}) \arrow[l, "\pi_{1*}"'] \arrow[d, "\pi_{2*}"] \\ T & \mathcal{P}_1(Y) \arrow[l, "\mathrm{triv}_i"] \end{tikzcd} . \tag{22}$$

The pair (triv, g) *must satisfy*

$$\begin{matrix} \pi_{12}^* g \\ \circ \\ \pi_{23}^* g \end{matrix} = \pi_{13}^* g, \tag{23}$$

where the left-hand-side is vertical composition of natural transformations (read from top to bottom), and

$$\mathrm{id}_{\mathrm{triv}_i} = \Delta^* g, \tag{24}$$

where Δ is the diagonal $\Delta : Y \longrightarrow Y^{[2]}$ sending y to (y, y).

Descent objects form the objects of a category denoted by $\mathfrak{Des}^1_\pi(i)$.

2.17. Definition. *A* <u>descent morphism</u> $h : (\mathrm{triv}, g) \longrightarrow (\mathrm{triv}', g')$ *is a natural transformation* $h : \mathrm{triv}_i \Rightarrow \mathrm{triv}'_i$ *satisfying*

$$\begin{matrix} \pi_1^* h \\ \circ \\ g' \end{matrix} = \begin{matrix} g \\ \circ \\ \pi_2^* h \end{matrix}. \tag{25}$$

There is a functor $\mathrm{Ex}^1_\pi : \mathrm{Triv}^1_\pi(i) \longrightarrow \mathfrak{Des}^1_\pi(i)$ that extracts descent data from trivialization data. At the level of objects, this functor is defined as follows. Let (F, triv, t) be an object in $\mathrm{Triv}^1_\pi(i)$. For the pair (triv, g), take triv to be exactly the same. For g take the composition $g := \begin{smallmatrix} \pi_1^*\bar{t} \\ \circ \\ \pi_2^* t \end{smallmatrix}$ coming from the composition in the diagram

$$\begin{tikzcd}
 & \mathcal{P}_1(Y) \arrow[dl, "\mathrm{triv}"'] \arrow[d, "\pi_*"] \arrow[dr, equal, "\mathrm{id}"] & \mathcal{P}_1(Y^{[2]}) \arrow[l, "\pi_{1*}"'] \arrow[d, "\pi_{2*}"] \\
\mathrm{Gr} \arrow[r, Rightarrow, "\bar{t}"] \arrow[d, "i"'] & \mathcal{P}_1(M) \arrow[dl, "F"] \arrow[d, Rightarrow, "t"] & \mathcal{P}_1(Y) \arrow[l, "\pi_*"] \arrow[dl, "\mathrm{triv}"] \\
T & \mathrm{Gr} \arrow[l, "i"] &
\end{tikzcd}, \tag{26}$$

where $\bar{t}$ is the (vertical) inverse of t. This defines a descent object (Section 2.2 of [ScWa09]). On a morphism $\alpha : (F, \mathrm{triv}, t) \longrightarrow (F', \mathrm{triv}', t')$, the functor Ex^1_π is defined by setting

$$h := \begin{smallmatrix} \bar{t} \\ \circ \\ \pi^*\alpha \\ \circ \\ t' \end{smallmatrix} \tag{27}$$

coming from the composition in the diagram

$$\begin{tikzcd}
T & \mathcal{P}_1(M) \arrow[l, bend right, "F"'] \arrow[l, bend left, "F'"] \arrow[l, phantom, "\Downarrow\alpha"] & \mathcal{P}_1(Y) \arrow[l, "\pi_*"'] \arrow[ll, bend right=60, "\mathrm{triv}_i"'] \arrow[ll, bend left=60, "\mathrm{triv}'_i"] \arrow[l, phantom, bend right=50, "\Downarrow\bar{t}"] \arrow[l, phantom, bend left=50, "\Downarrow t'"]
\end{tikzcd}. \tag{28}$$

The functor Ex^1_π is part of an equivalence of categories between $\mathrm{Triv}^1_\pi(i)$ and $\mathfrak{Des}^1_\pi(i)$. This is done by constructing a weak inverse functor $\mathrm{Rec}^1_\pi : \mathfrak{Des}^1_\pi(i) \longrightarrow \mathrm{Triv}^1_\pi(i)$, which we will describe in Section 2.24.

2.18. DEFINITION. *Let (F, triv, t) be a π-local i-trivialization of a functor $F : \mathcal{P}_1(M) \longrightarrow T$, i.e. an object of $\mathrm{Triv}^1_\pi(i)$. The* <u>descent object associated to the π-local i-trivialization of F</u> *is $\mathrm{Ex}^1_\pi(F, \mathrm{triv}, t)$. Let $\alpha : (F, \mathrm{triv}, t) \longrightarrow (F', \mathrm{triv}', t')$ be a morphism in $\mathrm{Triv}^1_\pi(i)$. The* <u>descent morphism associated to the π-local i-trivialization of α</u> *is $\mathrm{Ex}^1_\pi(\alpha)$.*

2.19. TRANSPORT FUNCTORS. We now discuss smoothness of descent data and finally give a definition of transport functors.

2.20. DEFINITION. *A descent object (triv, g) as above is said to be* <u>smooth</u> *if $\mathrm{triv} : \mathcal{P}_1(Y) \longrightarrow \mathrm{Gr}$ is a smooth functor and if there exists a smooth natural isomorphism $\tilde{g} : \pi_1^*\mathrm{triv} \Rightarrow \pi_2^*\mathrm{triv}$ with $g = \mathrm{id}_i \circ \tilde{g}$, the horizontal composition of natural transformations id_i and $\tilde{g}$. A descent morphism $h : (\mathrm{triv}, g) \longrightarrow (\mathrm{triv}', g')$ as above is said to be* <u>smooth</u> *if there exists a smooth natural isomorphism $\tilde{h} : \mathrm{triv} \Rightarrow \mathrm{triv}'$ with $h = \mathrm{id}_i \circ \tilde{h}$.*

Smooth descent objects and morphisms form the objects and morphisms of a category denoted by $\mathfrak{Des}^1_\pi(i)^\infty$ and form a sub-category of $\mathfrak{Des}^1_\pi(i)$.

2.21. Definition. *A π-local i-trivialization (F, triv, t) is said to be* <u>smooth</u> *if the associated descent object $\mathrm{Ex}^1_\pi(F, \mathrm{triv}, t)$ is smooth. A morphism $\alpha : (F, \mathrm{triv}, t) \longrightarrow (F', \mathrm{triv}', t')$ is said to be* <u>smooth</u> *if the associated descent morphism $\mathrm{Ex}^1_\pi(\alpha)$ is smooth.*

Smooth local trivializations and their morphisms form the objects and morphisms of a category denoted by $\mathrm{Triv}^1_\pi(i)^\infty$ and form a sub-category of $\mathrm{Triv}^1_\pi(i)$. Ex^1_π restricts to an equivalence of categories $\mathrm{Triv}^1_\pi(i)^\infty \xrightarrow{\simeq} \mathfrak{Des}^1_\pi(i)^\infty$ of smooth data. Again, we will discuss an inverse functor in Section 2.24 since it will be necessary in discussing gauge invariant holonomy in Section 2.31. We now come to the definition of a transport functor (Definition 3.6 of [ScWa09]).

2.22. Definition. *Let* Gr *be a Lie groupoid, T a category, $i : \mathrm{Gr} \longrightarrow T$ a functor, and M a smooth manifold. A* <u>transport functor on M with values in a category T and with Gr-structure</u> *is a functor* $\mathrm{tra} : \mathcal{P}_1(M) \longrightarrow T$ *such that there exists a surjective submersion $\pi : Y \longrightarrow M$ and a smooth π-local i-trivialization* (triv, t) *of* tra.

Transport functors with values in T with Gr-structure form the objects of a category $\mathrm{Trans}^1_{\mathrm{Gr}}(M, T)$. We also define the morphisms of transport functors.

2.23. Definition. *A* <u>morphism η of transport functors on M from tra to tra$'$</u> *is a natural transformation $\eta : \mathrm{tra} \Rightarrow \mathrm{tra}'$ such that there exists a surjective submersion $\pi : Y \longrightarrow M$ and smooth π-local i-trivializations* (triv, t), (triv', t'), *and* $h : (\mathrm{triv}, t) \longrightarrow (\mathrm{triv}', t')$ *of* tra, tra$'$, *and η respectively.*

By using pullbacks, one can define the composition of such morphisms. We will not explicitly describe this now since we will come back to it later when discussing limit categories over surjective submersions in Section 2.30.

2.24. The reconstruction functor: local to global. In many situations, one works locally and pieces together data to construct globally defined quantities. In the case of parallel transport, one obtains group elements. An explicit construction of a (weak) inverse $\mathrm{Rec}^1_\pi : \mathfrak{Des}^1_\pi(i) \longrightarrow \mathrm{Triv}^1_\pi(i)$ to Ex^1_π will assist in this direction. Following Section 2.3 of [ScWa09], we introduce a category that combines the Čech groupoid with the path groupoid utilizing the surjective submersion $\pi : Y \longrightarrow M$.

2.25. Definition. *Let $\mathcal{P}^\pi_1(M)$ be the category, called the* <u>Čech path groupoid</u>, *whose set of objects are the elements of Y. The set of morphisms are freely generated by two types of morphisms (the generators) which are given as follows*

i) thin paths (see Definition 2.10) γ in Y with sitting instants and

ii) points α in $Y^{[2]}$ (thought of as morphisms $\pi_1(\alpha) \xrightarrow{\alpha} \pi_2(\alpha)$ and called <u>jumps</u>*).*

There are several relations imposed on the set of morphisms.

(a) For any thin path $\Theta : \alpha \longrightarrow \beta$ in $Y^{[2]}$ the diagram

$$\begin{tikzcd} \pi_1(\beta) \arrow[d, "\beta"'] & \pi_1(\alpha) \arrow[l, "\pi_1(\Theta)"'] \arrow[d, "\alpha"] \\ \pi_2(\beta) & \pi_2(\alpha) \arrow[l, "\pi_2(\Theta)"] \end{tikzcd} \tag{29}$$

commutes (see Figure 1 for a visualization of this).

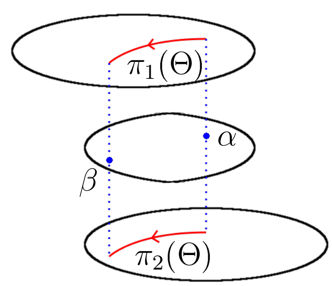


Figure 1: Thinking in terms of an open cover as a submersion, condition i) above says that if a path $\Theta : \alpha \to \beta$ is in a double intersection, it doesn't matter whether or not the jump is performed first and then the thin path is traversed or vice versa.

(b) For any point $\Xi \in Y^{[3]}$ the diagram

$$\begin{tikzcd} & \pi_2(\Xi) \arrow[dl, "\pi_{23}(\Xi)"'] & \\ \pi_3(\Xi) & & \pi_1(\Xi) \arrow[ul, "\pi_{12}(\Xi)"'] \arrow[ll, "\pi_{13}(\Xi)"] \end{tikzcd} \tag{30}$$

commutes.

(c) The free composition of two thin free paths is the usual composition of thin paths and for every point $y \in Y$, the thin homotopy class representing the constant path at y is equal to $\Delta(y) \in Y^{[2]}$ which is the formal identity for the composition.

The notation for the free composition will be $*$.

Item (b) together with item (c) demands that the jumps $\alpha \in Y^{[2]}$ are isomorphisms. A typical morphism in $\mathcal{P}_1^\pi(M)$ is depicted in Figure 2.

Associated to every descent object (triv, g) in $\mathfrak{Des}_\pi^1(i)$ is a functor $R_{(\mathrm{triv},g)} : \mathcal{P}_1^\pi(M) \longrightarrow T$ defined (on objects and generators) by

$$\begin{aligned} Y \ni y &\mapsto \mathrm{triv}_i(y), \\ P^1Y \ni \gamma &\mapsto \mathrm{triv}_i(\gamma), \quad \text{and} \\ Y^{[2]} \ni \alpha &\mapsto \Big(g(\alpha) : \mathrm{triv}_i(\pi_1(\alpha)) \longrightarrow \mathrm{triv}_i(\pi_2(\alpha)) \Big). \end{aligned} \tag{31}$$

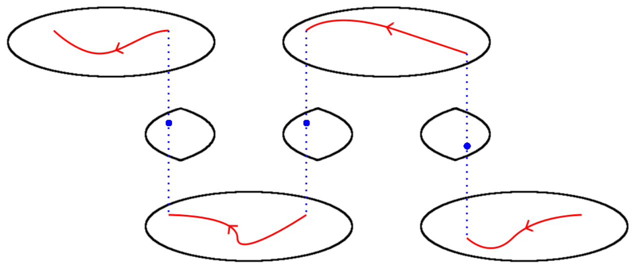


Figure 2: A generic representative of a morphism in $\mathcal{P}_1^\pi(M)$ is shown above for $Y = \coprod_{i\in I} U_i$, the disjoint union over an open cover. The larger ellipses indicate open sets and the smaller ones in the middle indicate intersections. The curves in the open sets indicate the paths and the dotted vertical lines indicate the jumps.

This assignment extends to a functor $R : \mathfrak{Des}_\pi^1(i) \longrightarrow \mathrm{Funct}(\mathcal{P}_1^\pi(M), T)$ (Lemma 2.14. of [ScWa09]). To a descent morphism $h : (\mathrm{triv}, g) \longrightarrow (\mathrm{triv}', g')$ it gives a natural transformation $R_h : R_{(\mathrm{triv},g)} \Rightarrow R_{(\mathrm{triv}',g')}$ defined by sending $y \in Y$ to $h(y)$ for all $y \in Y$.

The functor $\mathrm{Rec}_\pi^1 : \mathfrak{Des}_\pi^1(i) \longrightarrow \mathrm{Triv}_\pi^1(i)$ will be defined so that it factors through R. What will then remain is to define a functor $\mathrm{Funct}(\mathcal{P}_1^\pi(M), T) \longrightarrow \mathrm{Funct}(\mathcal{P}_1(M), T)$. In order to do this, we need to "lift" paths. First, notice that there is a canonical projection functor $p^\pi : \mathcal{P}_1^\pi(M) \longrightarrow \mathcal{P}_1(M)$ which sends objects $y \in Y$ to $\pi(y)$, thin paths γ to $\pi(\gamma)$, and points $\alpha \in Y^{[2]}$ to the identity. We will construct a weak inverse $s^\pi : \mathcal{P}_1(M) \longrightarrow \mathcal{P}_1^\pi(M)$.

Since $\pi : Y \longrightarrow M$ is surjective, for every $x \in M$, there exists a $y \in Y$ such that $\pi(y) = x$. Therefore, define $s^\pi : \mathcal{P}_1(M) \longrightarrow \mathcal{P}_1^\pi(M)$ on objects to be this assignment. Because $\pi : Y \longrightarrow M$ is a surjective submersion, there exists an open cover $\{U_i\}_{i\in I}$ of M with local sections $s_i : U_i \longrightarrow Y$ of π. Using these local sections, we can define $s^\pi : \mathcal{P}_1(M) \longrightarrow \mathcal{P}_1^\pi(M)$ on morphisms as follows. For every thin path $\gamma : x \longrightarrow x'$ in M there exists a collection of thin paths $\gamma_1, \cdots, \gamma_n$ with (representatives of) γ_j inside U_{i_j} for all $j = 1, \ldots, n$ and

$$x' \xleftarrow{\gamma} x \quad = \quad x' \xleftarrow{\gamma_n} x_{n-1} \xleftarrow{\gamma_{n-1}} \cdots \xleftarrow{\gamma_2} x_1 \xleftarrow{\gamma_1} x. \tag{32}$$

For such a choice define (we write s_j instead of s_{i_j} to avoid too many indices)

$$s^\pi(\gamma) := \alpha_{x'} * s_n(\gamma_n) * \alpha_{n-1} * s_{n-1}(\gamma_{n-1}) * \cdots * s_2(\gamma_2) * \alpha_1 * s_1(\gamma_1) * \alpha_x, \tag{33}$$

where α_x is the unique isomorphism from $s^\pi(x)$ to $s_1(x)$, α_j is the unique isomorphism from $s_{j-1}(x_j)$ to $s_j(x_j)$, and $\alpha_{x'}$ is the unique isomorphism from $s_n(x)$ to $s^\pi(x')$. This definition comes from Figure 3.

The functor s^π is a weak inverse to p^π (Lemma 2.15. of [ScWa09]). For reference, by definition this means there exists a natural isomorphism

$$\zeta : s^\pi \circ p^\pi \Rightarrow \mathrm{id}_{\mathcal{P}_1^\pi(M)}. \tag{34}$$

that is part of an adjoint equivalence given by the quadruple $(s^\pi, p^\pi, \zeta, \mathrm{id}_{p^\pi \circ s^\pi})$ since $p^\pi \circ s^\pi = \mathrm{id}_{\mathcal{P}_1(M)}$. This natural isomorphism ζ is the one that sends $y \in Y$ to the unique jump, an isomorphism, from y to $s^\pi(\pi(y))$. It is natural by relation i) in Definition 2.25.

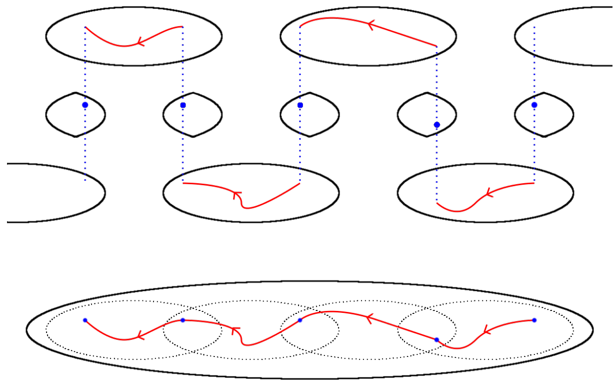


Figure 3: By choosing a decomposition of every path to land in open sets one can lift using the locally defined sections. At the beginning and end of the path, one must apply a jump since the map s defined on objects might not coincide with the lift of the endpoint of the path.

2.26. REMARK. *Note that we have not put a smooth structure on $\mathcal{P}_1^\pi(M)$ nor will we (although it is done in [ScWa09]). Indeed, the choice of lifts for the points could be sporadic. All the smoothness for transport functors is contained in the descent data.*

The functor $s^\pi : \mathcal{P}_1(M) \longrightarrow \mathcal{P}_1^\pi(M)$ induces a pullback functor $s^{\pi*} : \mathrm{Funct}(\mathcal{P}_1^\pi(M), T) \to \mathrm{Funct}(\mathcal{P}_1(M), T)$ defined by $s^{\pi*}(F) := F \circ s^\pi$ on functors $F : \mathcal{P}_1^\pi(M) \longrightarrow T$ and by $s^{\pi*}(\rho) := \rho \circ \mathrm{id}_{s^\pi}$ on natural transformations $\rho : F \Rightarrow G$. Finally, Rec_π^1 is defined as the composition in the diagram

$$\begin{tikzcd} \mathrm{Funct}(\mathcal{P}_1(M), T) & & \mathfrak{Des}_\pi^1(i) \arrow[ll, "\mathrm{Rec}_\pi^1"'] \arrow[dl, "R"] \\ & \mathrm{Funct}(\mathcal{P}_1^\pi(M), T) \arrow[ul, "s^{\pi*}"] & \end{tikzcd} . \tag{35}$$

The image of $\mathfrak{Des}_\pi^1(i)$ under Rec_π^1 is actually in $\mathrm{Triv}_\pi^1(i)$. This means at the level of objects that associated to $R_{(\mathrm{triv},g)} \circ s^\pi$ there exists a π-local i- trivialization. We take triv itself for the first part of this datum. To define $t : \pi^*\big(s^{\pi*}(R_{(\mathrm{triv},g)})\big) \Rightarrow \mathrm{triv}_i$ we take the composition defined by the diagram

$$\begin{tikzcd}
\mathcal{P}_1(M) \arrow[dd, "s^\pi"'] \arrow[ddr, phantom, "\Downarrow\zeta"] & & \mathcal{P}_1(Y) \arrow[ll, "\pi_*"'] \arrow[dl, hook] \arrow[ddd, "\mathrm{triv}"] \\
& \mathcal{P}_1^\pi(M) \arrow[ul, "p^\pi"'] \arrow[dl, "\mathrm{id}"] \arrow[u, phantom, "\| \mathrm{id}"] \arrow[ddr, equal, "\mathrm{id}"'] & \\
\mathcal{P}_1^\pi(M) \arrow[d, "R_{(\mathrm{triv},g)}"'] & & \\
T & & \mathrm{Gr} \arrow[ll, "i"']
\end{tikzcd} , \tag{36}$$

where the functor $\mathcal{P}_1(Y) \hookrightarrow \mathcal{P}_1^\pi(M)$ is the inclusion. The rest of the proof, namely the fact that the image of a morphism lands in $\mathrm{Triv}_\pi^1(i)$ under Rec_π^1, is explained in Appendix B.1. of [ScWa09].

2.27. Differential cocycle data. We now switch gears a bit and go in the other (infinitesimal) direction. We describe this in several parts. We focus on a local description first in terms of 'trivialized' transport functors. We extract the differential cocycle data from functors and then we construct functors from differential cocycle data. This is a brief and simplified account of the material covered in Section 4 of [ScWa09] since we do not prove any results.

2.27.1. From functors to 1-forms. Throughout this article, let $\underline{G}$ denote the Lie algebra of G. Given a smooth functor $F : \mathcal{P}_1(X) \longrightarrow \mathcal{B}G$, we will define a $\underline{G}$-valued 1-form A pointwise for every $x \in X$ and $v \in T_xX$ as follows. Let $\gamma : \mathbb{R} \longrightarrow X$ be a curve in X with $\gamma(0) = x$ and $\frac{d\gamma}{dt}(0) = v$. $\gamma : \mathbb{R} \longrightarrow X$ induces a smooth pushforward functor $\gamma_* : \mathcal{P}_1(\mathbb{R}) \longrightarrow \mathcal{P}_1(X)$. At the level of morphisms, the space $P^1\mathbb{R}$ of thin homotopy classes of paths in $\mathbb{R}$ is actually smoothly equivalent to $\mathbb{R}\times\mathbb{R}$. The diffeomorphism $\gamma_{\mathbb{R}} : \mathbb{R}\times\mathbb{R} \longrightarrow P^1\mathbb{R}$ is defined by sending (s,t) to the thin homotopy class of a path in $\mathbb{R}$ determined by its source point s and target t as shown schematically in Figure 4.

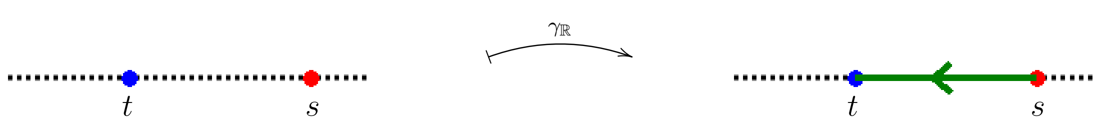


Figure 4: A point (s,t) in $\mathbb{R}^2$ is drawn as two points on $\mathbb{R}$ and gets mapped to the thin path in $\mathbb{R}$ from the point s to the point t with a representative shown on the right under the map $\gamma_{\mathbb{R}}$.

Therefore, we obtain a function $F_1 \circ \gamma_* \circ \gamma_{\mathbb{R}}$ from the composition

$$G \xleftarrow{F_1} P^1X \xleftarrow{\gamma_*} P^1\mathbb{R} \xleftarrow{\gamma_{\mathbb{R}}} \mathbb{R}\times\mathbb{R}. \tag{37}$$

Here F_1 is F restricted to the set of morphisms P^1X. Using this, we define

$$A_x(v) := -\frac{d}{dt}\bigg|_{t=0} F_1\Big(\gamma_*\big(\gamma_{\mathbb{R}}(0,t)\big)\Big). \tag{38}$$

$A_x(v)$ is independent of γ and only depends on x and v. Furthermore, it defines a 1-form $A \in \Omega^1(X;\underline{G})$.

2.27.2. From 1-forms to functors. Starting with a $\underline{G}$-valued 1-form $A \in \Omega^1(X;\underline{G})$ on X we want to define a smooth functor $\mathcal{P}_1(X) \longrightarrow \mathcal{B}G$. To do this, we first define a function, referred to as the *path transport*, $k_A : PX \longrightarrow G$ on paths in X with sitting instants (we do *not* mod out by thin homotopy). Given $\gamma \in PX$, we can pull back the 1-form A to $\mathbb{R}$, namely $\gamma^*(A) \in \Omega^1([0,1];\underline{G})$. We then define $k_A(\gamma)$ using the path-ordered-exponential

$$k_A(\gamma) := \mathcal{P}\exp\left\{\int_0^1 A_t\left(\frac{\partial}{\partial t}\right)dt\right\}. \tag{39}$$

Recall that this path-ordered exponential is defined by[2]

$$\mathcal{P}\exp\left\{\int_0^1 A_t\left(\frac{\partial}{\partial t}\right)dt\right\} := \sum_{n=0}^{\infty}\frac{1}{n!}\int_0^1 dt_n \cdots \int_0^1 dt_1\, \mathcal{T}\left[A_{t_n}\left(\frac{\partial}{\partial t}\right)\cdots A_{t_1}\left(\frac{\partial}{\partial t}\right)\right], \tag{40}$$

where the *time-ordering operator* $\mathcal{T}$ is defined by

$$\mathcal{T}[A_tA_s] := \begin{cases} A_tA_s & \text{if } t \geqslant s \\ A_sA_t & \text{if } s \geqslant t \end{cases}. \tag{41}$$

The $n = 0$ term on the right-hand side of equation (40) is the identity. We can picture the path-ordered exponential schematically as a power series of graphs with marked points as in Figure 5.

k_A only depends on the thin homotopy class of γ and therefore factors through a smooth map $F_A : P^1X \longrightarrow G$ on thin paths (see Definition 2.10). This map defines a smooth functor $F_A : \mathcal{P}_1(X) \longrightarrow \mathcal{B}G$ (see Proposition 4.3. and Lemma 4.5. of [ScWa09]).

2.27.3. Local differential cocycles for transport functors. The above constructions can be extended to smooth natural transformations between smooth functors. Given a smooth natural transformation $h : F \Rightarrow F'$ of smooth functors $F, F' : \mathcal{P}_1(X) \longrightarrow \mathcal{B}G$ we obtain a function, written somewhat abusively also as $h : X \longrightarrow G$ satisfying

$$h(y)F(\gamma) = F'(\gamma)h(x) \tag{42}$$

for all thin paths $\gamma : x \longrightarrow y$ in X. If we differentiate this condition, we obtain

$$A' = \mathrm{Ad}_h(A) - h^*\overline{\theta}, \tag{43}$$

[2] In this expression, we are assuming that G is a matrix Lie group.

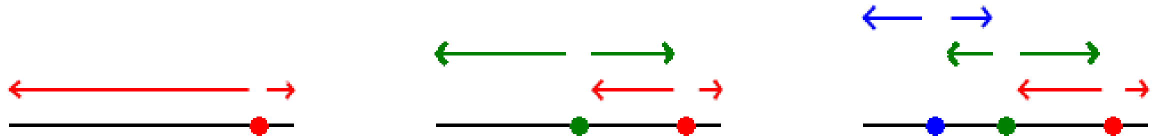


Figure 5: The path-ordered integral is depicted as a power series over integrals. The first term (not drawn) is the identity. The second term is the integral of A_t (depicted as a bullet on the interval) over all t from the right to the left (the orientation goes from right to left). The third term is the integral of $A_t A_s$ over the interval but keeping earlier operators to the right. This is drawn by showing the bullet on the right being able to move along the interval provided it stays behind the bullet to its left. The fourth term involves three operators. Higher terms are not drawn. All terms are summed with appropriate factors.

where $\overline{\theta}$ is right Maurer-Cartan form, sometimes written as dgg^{-1} for matrix groups, A is the 1-form corresponding to F, A' is the 1-form corresponding to F', and Ad is the adjoint action on the Lie algebra $\underline{G}$ defined by

$$\mathrm{Ad}_h(T) := \frac{d}{dt}\bigg|_{t=0}\Big(h\exp\{tT\}h^{-1}\Big) \tag{44}$$

for all $T \in \underline{G}$. This motivates the following definition.

2.28. Definition. *Let $Z^1_X(G)^\infty$ be the category whose objects are 1-forms $A \in \Omega^1(X;\underline{G})$ and a morphism from A to A' is a function $h : X \longrightarrow G$ satisfying*

$$A' = \mathrm{Ad}_h(A) - h^*\overline{\theta}. \tag{45}$$

The composition is defined by

$$\Big(A'' \xleftarrow{h'} A' \xleftarrow{h} A\Big) \quad \mapsto \quad \Big(A'' \xleftarrow{h'h} A\Big), \tag{46}$$

where $h'h$ is (pointwise) multiplication of G-valued functions.

This (and the previous section) defines two functors

$$Z^1_X(G)^\infty \underset{\mathcal{D}_X}{\overset{\mathcal{P}_X}{\rightleftarrows}} \mathrm{Funct}^\infty(\mathcal{P}_1(X), \mathcal{B}G)\ , \tag{47}$$

where $\mathrm{Funct}^\infty(\mathcal{P}_1(X), \mathcal{B}G)$ is the category of smooth functors and smooth natural transformations from the thin path groupoid of X to $\mathcal{B}G$. These functors are defined on objects by $\mathcal{D}_X(F) := A$ from (38) and $\mathcal{P}_X(A) := F_A$ from (39). These two functors are *inverses* of each other, and not just an *equivalence* pair (Proposition 4.7. of [ScWa09]).

All of this was for globally defined smooth functors. Transport functors on M are not necessarily smooth globally. However, there must exist a surjective submersion $\pi : Y \longrightarrow M$ with a smooth π-local i-trivialization. The smooth functor $\mathrm{triv} : \mathcal{P}_1(Y) \longrightarrow \mathcal{B}G$ corresponds to a 1-form $A \in \Omega^1(Y; \underline{G})$, which is an object in $Z^1_Y(G)^\infty$. The natural transformation $g : \pi_1^*\mathrm{triv}_i \Rightarrow \pi_2^*\mathrm{triv}_i$ factors through a smooth natural transformation $\tilde{g} : \pi_1^*\mathrm{triv} \Rightarrow \pi_2^*\mathrm{triv}$, which is a morphism in the category $Z^1_{Y^{[2]}}(G)^\infty$ from π_1^*A to π_2^*A. This means

$$\pi_2^*A = \mathrm{Ad}_{\tilde{g}}(\pi_1^*A) - \tilde{g}^*\overline{\theta}. \tag{48}$$

The condition

$$\begin{matrix} \pi_{12}^*g \\ \circ \\ \pi_{23}^*g \end{matrix} = \pi_{13}^*g \tag{49}$$

translates to

$$\pi_{23}^*\tilde{g}\ \pi_{12}^*\tilde{g} = \pi_{13}^*\tilde{g}, \tag{50}$$

where the concatenation indicates multiplication of G-valued functions. A morphism of transport functors subordinate to the same surjective submersion is a natural transformation $h : \mathrm{triv}_i \Rightarrow \mathrm{triv}'_i$ that factors through a smooth natural transformation $\tilde{h} : \mathrm{triv} \Rightarrow \mathrm{triv}'$ and therefore defines a morphism from A to A' in $Z^1_Y(G)^\infty$. This motivates the following definition of local differential cocycles.

2.29. Definition. *Let $\pi : Y \longrightarrow M$ be a surjective submersion. Define the category $Z^1_\pi(G)^\infty$ of* <u>differential cocycles subordinate to π</u> *as follows. An object of $Z^1_\pi(G)^\infty$ is a pair (A, g), where A is an object in $Z^1_Y(G)^\infty$, g is a morphism from π_1^*A to π_2^*A in $Z^1_{Y^{[2]}}(G)^\infty$. A morphism from (A, g) to (A', g') is a morphism h from A to A' in $Z^1_Y(G)^\infty$. The composition of morphisms in $Z^1_\pi(G)^\infty$ is defined by*

$$\Big((A'', g'') \xleftarrow{h'} (A', g') \xleftarrow{h} (A, g)\Big) \quad \mapsto \quad \Big((A'', g'') \xleftarrow{h'h} (A, g)\Big). \tag{51}$$

The above generalizations produce functors

$$Z^1_\pi(G)^\infty \underset{\mathcal{D}_\pi}{\overset{\mathcal{P}_\pi}{\rightleftarrows}} \mathfrak{Des}^1_\pi(i)^\infty \tag{52}$$

exhibiting an equivalence of categories whenever $i : \mathcal{B}G \longrightarrow T$ is an equivalence (Corollary 4.9. in [ScWa09]).

2.30. Limit over surjective submersions. Here we give a brief summary of the four levels of construction introduced and the notation of the functors relating these categories. To do this, we get rid of the dependence on the surjective submersion in the categories introduced in the prequel. Several of our categories depended on the choice of a surjective submersion. These categories were $\mathrm{Triv}^1_\pi(i)^\infty, \mathfrak{Des}^1_\pi(i)^\infty$, and $Z^1_\pi(G)^\infty$. On the contrast, the category of transport functors $\mathrm{Trans}^1_{\mathcal{B}G}(M, T)$ does not depend on π. To relate these categories better, we will take limits over π. Changing the surjective submersion gives a collection of categories depending on such surjective submersions. One can take a limit over the collection of surjective submersions in this case.

The general construction is done as follows. Let S_π be a family of categories parametrized by surjective submersions $\pi : Y \longrightarrow M$ and let $F(\zeta) : S_\pi \longrightarrow S_{\pi\circ\zeta}$ be a family of functors for every refinement $\zeta : Y' \longrightarrow Y$ of π satisfying the condition that for any iterated refinement $\zeta' : Y'' \longrightarrow Y'$ and $\zeta : Y' \longrightarrow Y$ of $\pi : Y \longrightarrow M$ then $F(\zeta' \circ \zeta) = F(\zeta') \circ F(\zeta)$. In this case, an object of $\varinjlim_\pi S_\pi$ is given by a pair (π, X) of a surjective submersion $\pi : Y \longrightarrow M$ and an object X of S_π. A morphism from (π_1, X_1) to (π_2, X_2) consists of an equivalence class of a common refinement

$$\begin{tikzcd} & Z \arrow[dl, "y_1"'] \arrow[dd, "\zeta"] \arrow[dr, "y_2"] & \\ Y_1 \arrow[dr, "\pi_1"'] & & Y_2 \arrow[dl, "\pi_2"] \\ & M & \end{tikzcd} \tag{53}$$

together with a morphism $f : (F(y_1))(X_1) \longrightarrow (F(y_2))(X_2)$ in S_ζ. It is written as a pair (ζ, f). Two such (ζ, f) and (ζ', f') are equivalent if they agree (on the nose) on their common pullback. The composition

$$(\pi_3, X_3) \xleftarrow{(\zeta_{23}, g)} (\pi_2, X_2) \xleftarrow{(\zeta_{12}, f)} (\pi_1, X_1) \tag{54}$$

is defined by choosing representatives and taking the pullback refinement

$$\begin{tikzcd} & & Z_{13} \arrow[dl, "\mathrm{pr}_{12}"'] \arrow[dr, "\mathrm{pr}_{23}"] & & \\ & Z_{12} \arrow[dl] \arrow[dr] \arrow[ddr, "\zeta_{12}"'] & & Z_{23} \arrow[dl] \arrow[dr] \arrow[ddl, "\zeta_{23}"] & \\ Y_1 \arrow[drr, "\pi_1"'] & & Y_2 \arrow[d, "\pi_2"] & & Y_3 \arrow[dll, "\pi_3"] \\ & & M & & \end{tikzcd} \tag{55}$$

along with the composition $(F(\mathrm{pr}_{23}))(g) \circ (F(\mathrm{pr}_{12}))(f)$. One can check this definition does not depend on the equivalence class chosen.

After getting rid of the specific choices of the surjective submersions, we can take the limits of all the categories we have introduced. We set the following notation, slightly differing from that of [ScWa13]:

$$\mathrm{Triv}^1_M(i)^\infty := \varinjlim_\pi \mathrm{Triv}^1_\pi(i)^\infty \tag{56}$$

$$\mathfrak{Des}^1_M(i)^\infty := \varinjlim_\pi \mathfrak{Des}^1_\pi(i)^\infty \tag{57}$$

$$Z^1(M; G)^\infty := \varinjlim_\pi Z^1_\pi(G)^\infty. \tag{58}$$

Because a limit of such equivalences is still an equivalence, the following facts, summarizing the several previous sections, hold. The categories $Z^1(M;G)^\infty$ and $\mathfrak{Des}^1_M(i)^\infty$ are equivalent under the condition that $i : \mathcal{B}G \longrightarrow T$ is an equivalence of categories. $\mathfrak{Des}^1_M(i)^\infty$ and $\mathrm{Triv}^1_M(i)^\infty$ are equivalent for *any* i. Let $v : \mathrm{Triv}^1_M(i)^\infty \longrightarrow \mathrm{Trans}^1_{\mathcal{B}G}(M,T)$ be the forgetful functor which forgets the specific local trivialization. If i is full and faithful, then $v : \mathrm{Triv}^1_M(i)^\infty \longrightarrow \mathrm{Trans}^1_{\mathcal{B}G}(M,T)$ is part of an equivalence of categories. All these statements are proved in [ScWa09] (except the last one, but it follows from Lemma 3.3 in [ScWa09]).

For the reader's convenience, we collect the categories and equivalences (assuming i is an equivalence) introduced in the past few sections

$$Z^1(M;G)^\infty \underset{\mathcal{D}}{\overset{\mathcal{P}}{\rightleftarrows}} \mathfrak{Des}^1_M(i)^\infty \underset{\mathrm{Ex}^1}{\overset{\mathrm{Rec}^1}{\rightleftarrows}} \mathrm{Triv}^1_M(i)^\infty \underset{c}{\overset{v}{\rightleftarrows}} \mathrm{Trans}^1_{\mathcal{B}G}(M,T)\ , \tag{59}$$

where we've introduced the notation $\mathcal{P} := \varinjlim_\pi \mathcal{P}_\pi$ and similarly for the other functors. c is a weak inverse to v and chooses a π-local i-trivialization for transport functors.

2.31. Parallel transport, holonomy, and gauge invariance. Holonomy for principal G-bundles with connection is defined in several different ways. In all cases, it is a special case of parallel transport where one restricts attention to paths whose target match their source, i.e. *marked loops*.[3] Holonomy along a marked loop is an isomorphism of the fiber over the endpoint. However, for computational purposes, it is convenient to express such isomorphisms as group elements. One common way of doing this is to choose an open cover over which the bundle trivializes, choose a trivialization, and for each path, choose a decomposition of that path subordinate to the cover and parallel transport along each piece while patching the terms together using the transition functions. This is the procedure we discussed in Section 2.24. The problem with this procedure is that it depends on several choices. One purpose of this section is to analyze the dependence on these choices. The second purpose is to discuss (and make precise) the dependence of such group elements on the marking chosen for loops. The punchline is that to obtain a well-defined holonomy independent of such choices, one needs to pass to conjugacy classes in G.

The first goal is accomplished by starting with a transport functor $F : \mathcal{P}_1(M) \longrightarrow T$, choosing a local trivialization, extracting the descent data, and using the descent data to reconstruct a transport functor. This procedure can be described as a functor, which we call ℓ, from $\mathrm{Trans}^1_{\mathcal{B}G}(M,T)$ to itself (see Definition 2.33). Although all the ingredients for the functor ℓ were described in [ScWa09], this procedure was not discussed. Here, we formulate this procedure and use it to analyze holonomy along loops. Thus, starting with a transport functor F we obtain a new transport functor $\ell(F)$ that produces group-valued holonomies along loops under suitable assumptions. The first choice we made in this procedure is the transport functor F itself. One could have chosen a different, but

[3]The terminology "marked" is chosen over "based" to avoid confusion with the based loop space, which is the space of loops with a single base point. We allow our basepoints to vary.

naturally isomorphic, transport functor F' to obtain $\mathscr{t}(F')$. The other choices made were those defining $\mathscr{t}$. Abstract nonsense tells us there is a morphism $F \longrightarrow \mathscr{t}(F)$ of transport functors. Different choices of local trivializations and reconstructions are thus described in terms of natural isomorphisms. Formulated this way, it becomes a tautology that holonomy along loops is independent of these choices once one passes to conjugacy classes in G.

2.32. Remark. *One might argue that such a complicated formalism to obtain the well-known fact that holonomy is defined only with respect to conjugacy classes of G is overkill. While this is true for holonomy along loops, this formalism extends naturally to holonomy along surfaces, which is our main objective, and the proofs are similar since they are expressed in terms of category theory. In the case of surfaces, we will use these ideas to generalize the results of Section 5.2 of [ScWa13]. It is therefore important to study the simpler case of holonomy along loops first.*

The second goal, namely the dependence on markings, is accomplished by showing that for any two loops that are thinly homotopic, but not necessarily thinly homotopic preserving their marking, the group-valued holonomy using $\mathscr{t}(F)$ is well-defined up to conjugation. Using all these observations, we define, for every isomorphism class of transport functors, a holonomy map $L^1M \longrightarrow G/\mathrm{Inn}(G)$ from the space of thin homotopy classes of free loops (see Definition 2.36) to the conjugacy classes of G.

We now define precisely what we mean by (functorially) extracting group-valued parallel transport from arbitrary transport functors. In order to accomplish this, we restrict our discussion to transport functors with $\mathcal{B}G$-structure and with values in T and assume once and for all that $i : \mathcal{B}G \longrightarrow T$ is full and faithful.

2.33. Definition. *A* <u>group-valued transport extraction</u> *is a composition of functors (starting at the left and moving clockwise)*

$$\begin{tikzcd} & \mathrm{Triv}^1(i)^\infty \arrow[dr, "\mathrm{Ex}^1"] & \\ \mathrm{Trans}^1_{\mathcal{B}G}(M,T) \arrow[ur, "c"] & & \mathfrak{Des}^1(i)^\infty \arrow[dl, "\mathrm{Rec}^1"] \\ & \mathrm{Triv}^1(i)^\infty \arrow[ul, "v"] & \end{tikzcd} \tag{60}$$

and consists of a choice of a weak inverse c of the forgetful functor v and a reconstruction functor Rec^1 (which itself depends on the choice of a lifting of paths as in (33)). Such a functor is written as $\mathscr{t} := v \circ \mathrm{Rec}^1 \circ \mathrm{Ex}^1 \circ c$. The notation $\mathscr{t}$ stands for (local) trivialization.

2.34. Remark. *Although the functor $\mathscr{t}$ depends on both c and s^π (which defines Rec^1) we suppress the notation. The reason is because if we change c and/or s^π, the functor $\mathscr{t}$ will change to a naturally isomorphic one and only this fact will matter in any computation.*

The purpose of $\mathscr{t}$ is that it assigns group elements to thin paths for *every* transport functor F and *also* assigns group-valued gauge transformations for *every* morphism

$\eta : F \longrightarrow F'$ of transport functors (this will be reviewed in the following paragraphs). Furthermore, we know that a natural isomorphism $\imath : \mathrm{id} \Rightarrow \ell$ exists because all the functors in (60) are (part of) equivalences of categories. *Choosing* such a natural isomorphism specifies isomorphisms from the original fibers to the fiber G viewed as a G-torsor and relates our original parallel transports to the group elements defined from ℓ.

To see this, first recall what ℓ does. For a transport functor F, c chooses a local trivialization $c(F) := (\pi, F, \mathrm{triv}, t)$. Then we extract the smooth local descent object $\mathrm{Ex}^1(\pi, F, \mathrm{triv}, t) := (\pi, \mathrm{triv}, g)$. Then, we reconstruct a π-local i-trivialization $\mathrm{Rec}^1(\pi, \mathrm{triv}, g)$ and then forget the trivialization data keeping just the transport functor $v(\mathrm{Rec}^1(\pi, \mathrm{triv}, g))$. The resulting transport functor, written as ℓ_F (as opposed to $\ell(F)$), is defined by (see the paragraph after Definition 2.25)

$$
\begin{aligned}
\mathcal{P}_1(M) &\xrightarrow{\ell_F} T \\
M \ni x &\mapsto i(\bullet) =: \mathrm{triv}_i(s^\pi(x)) \\
P^1 M \ni \gamma &\mapsto R_{\mathrm{Ex}^1(c(F))}(s^\pi(\gamma)).
\end{aligned}
\tag{61}
$$

Here $\mathrm{triv} : \mathcal{P}_1(Y) \longrightarrow \mathcal{B}G$ is the "local" transport, $s^\pi : \mathcal{P}_1(M) \longrightarrow \mathcal{P}_1^\pi(M)$ is a choice of lifting points and paths, and $R_{\mathrm{Ex}^1(c(F))}(s^\pi(\gamma)) : i(\bullet) \longrightarrow i(\bullet)$ is an element of G because i is full and faithful. This element of G is defined by choosing a lift of the path γ (see Figure 3) and applying trivialized transport on the pieces and transition functions on the jumps (see Section 2.24). Note that in the special case that $T = G$-Tor, $i(\bullet)$ can be taken to be G itself and then $\ell_F(\gamma)$ for a thin path γ is left multiplication by some uniquely specified group element.

To a morphism $\eta : F \longrightarrow F'$ of transport functors, the resulting morphism of transport functors, written as ℓ_η, is defined as follows. First, c chooses surjective submersions $\pi : Y \longrightarrow M$ and $\pi' : Y' \longrightarrow M$ for F and F', respectively, along with local trivializations (triv, t) and (triv', t'). This means that under c the morphism $c(\eta)$ is defined on a common refinement $\zeta : Z \longrightarrow M$ of both π and π'. The same thing applies to the extracted descent morphism $\mathrm{Ex}^1(c(\eta)) = (\zeta, h)$. Since our domain is changed under the refinement, h is defined by the composition

$$
\begin{tikzcd}
 & {} & \mathcal{P}_1(Y) \arrow[dl, "\pi_*"'] \arrow[dll, bend right=40, "\mathrm{triv}_i"'] \arrow[dd, phantom, "\Downarrow \mathrm{id}"] & \\
T & \mathcal{P}_1(M) \arrow[l, bend right, "F"'] \arrow[l, bend left, "F'"] \arrow[l, phantom, "\Downarrow \eta"] \arrow[u, phantom, "\Downarrow \bar{t}"] \arrow[d, phantom, "\Downarrow t'"] & & \mathcal{P}_1(Z) \arrow[ul, "y_*"'] \arrow[dl, "y'_*"] \\
 & {} & \mathcal{P}_1(Y') \arrow[ul, "\pi'_*"] \arrow[ull, bend left=40, "\mathrm{triv}'_i"] &
\end{tikzcd}
\tag{62}
$$

This composition satisfies the condition

$$
\begin{matrix} y^{[2]*} g \\ \circ \\ \zeta_2^* h \end{matrix} \quad = \quad \begin{matrix} \zeta_1^* h \\ \circ \\ y'^{[2]*} g' \end{matrix} .
\tag{63}
$$

The notation means the following. A map $y : Z \longrightarrow Y$ (and similarly for $y' : Z \longrightarrow Y'$) determines a unique map $y^{[2]} : Z^{[2]} \longrightarrow Y^{[2]}$ defined by $y^{[2]}(z, z') := (y(z), y(z'))$. The maps $\zeta_1, \zeta_2 : Z^{[2]} \longrightarrow Z$ are the two projections.

The reconstruction functor $\mathrm{Rec}^1 : \mathfrak{Des}^1_M(i) \longrightarrow \mathrm{Triv}^1_M(i)$ sends the morphism h in (62) to $\mathrm{Rec}^1(\zeta, h) := s^{\zeta *} R_{(\zeta,h)}$ which is a morphism of transport functors from $\mathrm{Rec}^1(y^*(\pi, \mathrm{triv}, g))$ to $\mathrm{Rec}^1(y'^*(\pi', \mathrm{triv}', g'))$ with respect to this common refinement and where

$$s^\zeta : \mathcal{P}_1(M) \longrightarrow \mathcal{P}_1^\zeta(M).$$

$\mathrm{Rec}^1(\zeta, h)$ is defined by sending $x \in M$ to $h(s^\zeta(x))$ which is a morphism from $\mathrm{triv}_i(y(s^\zeta(x)))$ to $\mathrm{triv}'_i(y'(s^\zeta(x)))$.

Now, the natural isomorphism $\mathscr{r} : \mathrm{id} \Rightarrow \mathscr{t}$ assigns to every transport functor F a morphism of transport functors $\mathscr{r}_F : F \longrightarrow \mathscr{t}_F$. This means (see Definition 2.23) that associated to every $x \in M$ is an isomorphism $\mathscr{r}_F(x) : F(x) \longrightarrow i(\bullet)$ satisfying naturality, which means that to every thin path $\gamma \in P^1M$ from x to y, the diagram

$$\begin{tikzcd} i(\bullet) \arrow[d, "\mathscr{t}_F(\gamma)"'] & F(x) \arrow[l, "\mathscr{r}_F(x)"'] \arrow[d, "F(\gamma)"] \\ i(\bullet) & F(y) \arrow[l, "\mathscr{r}_F(y)"] \end{tikzcd} \tag{64}$$

commutes.

2.35. Remark. *In Section 3.2, [ScWa09]* define *the Wilson line, what we're calling $\mathscr{t}_F(\gamma)$, in terms of (64) as the composition $\mathscr{r}_F(y) \circ F(\gamma) \circ \mathscr{r}_F(x)^{-1} : i(\bullet) \longrightarrow i(\bullet)$ using that i is full and faithful so that this composition defines a unique group element. Our viewpoint is to define the Wilson line functorially and globally by using the group-valued transport extraction procedure $\mathscr{t}$.*

Since $\mathscr{r}$ itself is a natural transformation, to every morphism $\eta : F \longrightarrow F'$ of transport functors, the diagram

$$\begin{tikzcd} \mathscr{t}_F \arrow[d, "\mathscr{t}_\eta"'] & F \arrow[l, "\mathscr{r}_F"'] \arrow[d, "\eta"] \\ \mathscr{t}_{F'} & F' \arrow[l, "\mathscr{r}_{F'}"] \end{tikzcd} \tag{65}$$

commutes.

To analyze holonomy, we need to restrict parallel transport to thin paths whose source and target are the same, i.e. thin marked loops, and eventually thin free loops.

2.36. Definition. *The* <u>marked loop space of M</u> *is the set*

$$\mathfrak{L}M := \{\gamma \in PM \mid s(\gamma) = t(\gamma)\} \tag{66}$$

equipped with the subspace smooth structure (see Example A.4). Elements of $\mathfrak{L}M$ *are called* marked loops. *The* thin marked loop space of M *is the set*

$$\mathfrak{L}^1 M := \{\gamma \in P^1 M \mid s(\gamma) = t(\gamma)\} \tag{67}$$

equipped with the subspace smooth structure. Elements of $\mathfrak{L}^1 M$ *are called* thin marked loops.

2.37. DEFINITION. *The* ℓ-holonomy of F, *written as* hol^F_{ℓ}, *is defined as the restriction of parallel transport of a transport functor* F *to the thin marked loop space* $\mathfrak{L}^1 M$ *of* M*:*

$$\mathrm{hol}^F_{\ell} := \ell_F\Big|_{\mathfrak{L}^1 M} : \mathfrak{L}^1 M \longrightarrow G. \tag{68}$$

We now pose three questions that will motivate the rest of our discussion on holonomy along thin marked loops.

i) How does hol^F_{ℓ} depend on the choice of basepoint? Namely, suppose that two thin marked loops $\gamma : x \longrightarrow x$ and $\gamma' : x' \longrightarrow x'$ are thinly homotopic *without preserving the marking*[4] (see Definition 2.38). Then, how is $\mathrm{hol}^F_{\ell}(\gamma)$ related to $\mathrm{hol}^F_{\ell}(\gamma')$?

ii) How does hol^F_{ℓ} depend on F? Namely, suppose that $\eta : F \longrightarrow F'$ is a morphism of transport functors. How is hol^F_{ℓ} related to $\mathrm{hol}^{F'}_{\ell}$ in terms of η?

iii) How does hol^F_{ℓ} depend on ℓ, i.e. the choices of c and s^π? Namely, suppose that ℓ' is another trivialization. Then how is hol^F_{ℓ} related to $\mathrm{hol}^F_{\ell'}$?

We first define what we mean by the thin free loop space and then we proceed to answer the above questions. Denote the smooth space of loops in M by $LM := \{\gamma : S^1 \longrightarrow M \mid \gamma \text{ smooth}\}$.

2.38. DEFINITION. *Two smooth loops* $\gamma, \gamma' \in LM$ *are* thinly homotopic *if there exists a smooth map* $h : S^1 \times [0,1] \longrightarrow M$ *such that*

i) there exists an $\epsilon > 0$ *with* $h(t,s) = \gamma(t)$ *for* $s \leqslant \epsilon$ *and* $h(t,s) = \gamma'(t)$ *for* $s \geqslant 1 - \epsilon$ *and for all* $t \in S^1$ *and*

ii) the smooth map h *has rank* $\leqslant 1$.

Such a smooth map h *is called an* unmarked thin homotopy. *The smooth space of such thin homotopy classes of loops is denoted by* $L^1 M$ *and is called the* thin free loop space of M. *Elements of* $L^1 M$ *are called* thin free loops *or just* thin loops.

The first condition guarantees that unmarked thin homotopy defines an equivalence relation and $L^1 M$ is well-defined. The second condition is where the thin structure is buried. We need to discuss a few definitions and facts before relating thin loops to thin marked loops. For the purposes of being absolutely clear, from Lemma 2.40 through Lemma 2.44 we will distinguish between representatives of loops and thin homotopy equivalence classes by using brackets [ ]. However, afterwards, we will abuse notation and will rarely make the distinction.

[4]The notion of thin homotopy introduced in Definition 2.10 does not make sense when $x \neq x'$.

2.39. Definition. *The function* $f : \mathfrak{L}M \longrightarrow LM$ *defined by sending a marked loop* $\gamma : [0,1] \longrightarrow M$ *to the associated map* $f(\gamma) : S^1 \longrightarrow M$ *obtained from identifying the endpoints of* $[0,1]$ *is called the* forgetful map.

2.40. Lemma. *There exists a unique map* $f^1 : \mathfrak{L}^1 M \longrightarrow L^1 M$ *such that the diagram*

$$\begin{tikzcd} \mathfrak{L}M \arrow[r] \arrow[d, "f"'] & \mathfrak{L}^1 M \arrow[d, "f^1"] \\ LM \arrow[r] & L^1 M \end{tikzcd} \tag{69}$$

commutes (the horizontal arrows are the projections onto thin homotopy classes).

Proof. The map is constructed by choosing a representative, applying f, and then projecting to $L^1 M$. Let $[\gamma] : x \longrightarrow x$ be an element of $\mathfrak{L}^1 M$ and let $\gamma : x \longrightarrow x$ and $\gamma' : x \longrightarrow x$ be two representatives in $\mathfrak{L}M$. Then there exists a thin homotopy $h : [0,1] \times [0,1] \longrightarrow M$ from γ to γ'. Because $h(t,s) = x$ for all $s \in [0,1]$ and all $t \in [0,\epsilon] \cup [1-\epsilon, 0]$ for some $\epsilon > 0$, the two ends of the first $[0,1]$ factor can be identified resulting in a smooth map $\tilde{h} : S^1 \times [0,1]$. This gives the desired homotopy from $f(\gamma)$ to $f(\gamma')$. ■

Note that there is also a function $\mathrm{ev}_0 : \mathfrak{L}^1 M \longrightarrow M$ given by evaluating a thin loop at its endpoint. This function forgets the loop and remembers only the basepoint.

2.41. Definition. *A* marking of thin loops *is a section (*not *necessarily smooth)* $\mathfrak{m} : L^1 M \longrightarrow \mathfrak{L}^1 M$ *of* $f^1 : \mathfrak{L}^1 M \longrightarrow L^1 M$, *i.e.* $f^1 \circ \mathfrak{m} = \mathrm{id}$.

2.42. Remark. *A marking of ordinary loops cannot be defined in this way as a section of* $f : \mathfrak{L}M \longrightarrow LM$ *because an arbitrary smooth map* $S^1 \longrightarrow M$ *need not have a sitting instant at any point.*

2.43. Proposition. *A marking of thin loops exists.*

Actually, much more is true. Because the fact is somewhat surprising and interesting (and only holds due to the thin homotopy equivalence relation), we include it here. Let $\pi_0 M$ denote the set of components of M and $p : M \longrightarrow \pi_0 M$ the canonical function sending a point to its component. Let $c_0 : L^1 M \longrightarrow \pi_0 M$ denote the canonical function sending a thin loop to the component in which it (every representative) lies. A marking of thin loops $\mathfrak{m}$ determines a function $\beta : L^1 M \longrightarrow M$ given by $\beta := \mathrm{ev}_0 \circ \mathfrak{m}$ that satisfies the condition that

$$\begin{tikzcd} & M \arrow[d, "p"] \\ L^1 M \arrow[ur, "\beta"] \arrow[r, "c_0"'] & \pi_0 M \end{tikzcd} \tag{70}$$

commutes.

2.44. Lemma. *Let $\beta : L^1M \longrightarrow M$ be* any *function such that the diagram in (70) commutes. Then there exists a marking of thin loops $\mathfrak{m} : L^1M \longrightarrow \mathfrak{L}^1M$ such that the diagram*

$$\begin{tikzcd} \mathfrak{L}^1M \arrow[r, "\mathrm{ev}_0"] & M \\ L^1M \arrow[u, "\mathfrak{m}"] \arrow[ur, "\beta"'] & \end{tikzcd} \tag{71}$$

commutes.

Proof. A function $\mathfrak{m}$ can be defined as follows. For any thin loop $[\gamma] \in L^1M$, let $\gamma : S^1 \longrightarrow M$ be a representative. Then there exists an unmarked thin homotopy h from γ to a loop γ_β with sitting instants at $\beta([\gamma])$ because (70) commutes. To see this, one can simply pick a point on the loop and extend the loop out to the basepoint and come back without sweeping out any area (see Figure 6). Then project γ_β to $\mathfrak{L}^1M$. Thus, set

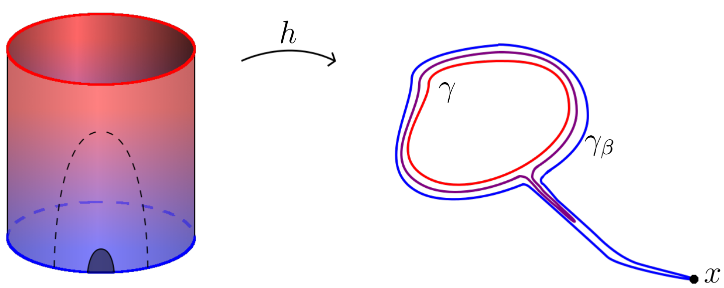


Figure 6: Let $[\gamma]$ be a thin free loop, $x := \beta([\gamma])$ a point in the same connected component as $[\gamma]$, and γ a representative loop (in red). Then there exists a path $\gamma' : x \to x$ with sitting instants (in blue) and an unmarked thin homotopy $h : \gamma \Rightarrow \gamma'$. The cylinder depicts such a homotopy with the middle loop (in purple) indicating an intermediate loop. The dashed line on the cylinder indicates that the loops begin to extend outwardly towards the marking without sweeping any area. The "mouse-hole" on the cylinder indicates that the loops from the homotopy eventually sit at x.

$\mathfrak{m}([\gamma]) := [\gamma_\beta]$. To see that this is well-defined, let γ' be another representative of $[\gamma]$ and let $\tilde{h}$ be an unmarked thin homotopy from γ' to γ. Then composing the two unmarked thin homotopies $h \circ \tilde{h}$ gives an unmarked thin homotopy from γ' to γ_β. Of course, there are many possible choices for γ_β for a given β that will give different markings $\mathfrak{m}$. ∎

2.45. Remark. *If β is chosen so that the diagram in (70) does not commute, a marking $\mathfrak{m}$ satisfying (71) does not exist.*

We now proceed to answering the above questions in order.

i) Let $\mathfrak{m}, \mathfrak{m}' : L^1M \longrightarrow \mathfrak{L}^1M$ be two markings of thin loops in M. Let $[\gamma] \in L^1M$ and denote $x := \mathrm{ev}_0(\mathfrak{m}([\gamma]))$ and $x' := \mathrm{ev}_0(\mathfrak{m}'([\gamma]))$. A choice of representatives $\gamma : x \longrightarrow x$

and $\gamma' : x' \longrightarrow x'$ as paths with sitting instants of $\mathfrak{m}([\gamma])$ and $\mathfrak{m}'([\gamma])$, respectively, need not have the same image. In particular, x and x' might not lie on each others images. Figure 7 gives an example. This makes it impossible to compare their holonomies using thin bigons in the usual way (because no such bigon exists).

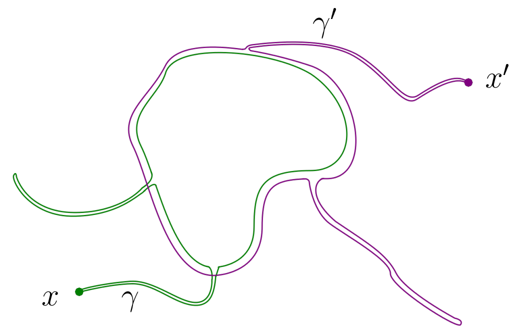


Figure 7: Two representatives γ and γ' of two markings of a single thin loop are shown. Their respective basepoints x and x' do not lie on each others images.

However, there is an *unmarked* thin homotopy $h : S^1 \times [0,1] \longrightarrow M$ with $h(t,s) = \gamma(t)$ for $s \leqslant \epsilon$ and $h(t,s) = \gamma'(t)$ for $s \geqslant 1-\epsilon$ for some $\epsilon > 0$. Therefore, one can choose a loop $\tilde{\gamma}$ and two *paths* with sitting instants $\gamma_{x'x} : x \longrightarrow x'$ and $\gamma_{xx'} : x' \longrightarrow x$ with the following three properties. First, as a loop, $\tilde{\gamma}$ can be written as the composition $\gamma_{x'x}$ and $\gamma_{xx'}$ in some order, i.e. using the map f of Definition 2.39, $\tilde{\gamma} = f(\gamma_{x'x} \circ \gamma_{xx'})$ or $f(\gamma_{xx'} \circ \gamma_{x'x})$. Second, the composition $\gamma_{xx'} \circ \gamma_{x'x}$ is thinly homotopic to γ preserving the basepoint x. Third, the composition $\gamma_{x'x} \circ \gamma_{xx'}$ is thinly homotopic to γ' preserving the basepoint x'. This is depicted in Figure 8.

This says that given two marked loops, with possibly different markings, that are thinly homotopic *without* preserving the marking, one can always choose a representative of such a thin loop in M with *two* marked points so that the associated two *marked* loops (coming from starting at either marking) are thinly homotopic to the original two with a thin homotopy that preserves the marking. More precisely, we proved the following fact.

2.46. LEMMA. *Let $\mathfrak{m}, \mathfrak{m}' : L^1M \longrightarrow \mathfrak{L}^1M$ be two markings. Let $[\gamma] \in L^1M$ be a thin loop in M and write $x := \mathrm{ev}_0(\mathfrak{m}([\gamma]))$ and $x' := \mathrm{ev}_0(\mathfrak{m}'([\gamma]))$. Then, there exist two paths $\gamma_{x'x} : x \longrightarrow x'$ and $\gamma_{xx'} : x' \longrightarrow x$ with sitting instants such that the following three properties hold (see Figure 9).*

i) The composition of $\gamma_{xx'}$ and $\gamma_{x'x}$ (in either order) and forgetting the marking is a representative of $[\gamma]$.

ii) $\gamma_{xx'} \circ \gamma_{x'x}$ is a representative of $\mathfrak{m}([\gamma])$ as a path with sitting instants.

iii) $\gamma_{x'x} \circ \gamma_{xx'}$ is a representative of $\mathfrak{m}'([\gamma])$ as a path with sitting instants.

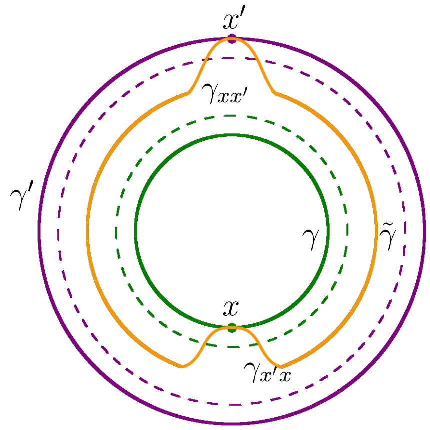


Figure 8: The domain of the unmarked thin homotopy $h : S^1 \times [0,1] \to M$ is drawn as an annulus depicting the domain of γ as the inner circle and that of γ' as the outer circle. The homotopy allows us to choose a loop $\tilde{\gamma}$, drawn somewhat in the middle (in orange), that contains both x and x' and is thinly homotopic to both γ and γ'. This loop $\tilde{\gamma}$ is decomposed into two paths $\gamma_{x'x} : x \to x'$ and $\gamma_{xx'} : x' \to x$. The dashed lines indicate the regions of sitting instants. All paths are oriented counter-clockwise. Note that, by a reparametrization if necessary, the homotopy h may be chosen to separate the two basepoints into the northern and southern hemispheres as drawn.

Therefore, without loss of generality, we can choose a *single* representative $\tilde{\gamma}$ of a thin free loop $[\gamma]$ with a decomposition as in the Lemma. We denote $\gamma' := \gamma_{x'x} \circ \gamma_{xx'}$ and $\gamma := \gamma_{xx'} \circ \gamma_{x'x}$. Thus $\tilde{\gamma}$ is one of $f(\gamma)$ or $f(\gamma')$. Note that γ' and $\overline{\gamma_{xx'}} \circ \gamma \circ \gamma_{xx'}$ are thinly homotopic. For convenience, from now on we abuse notation often and do not distinguish between the actual paths versus the thin homotopy classes as elements of P^1M.

By functoriality of the transport functor $\mathcal{t}_F$, we have

$$\begin{aligned} \mathrm{hol}^F_{\ell}(\gamma') &= \mathcal{t}_F(\gamma') \\ &= \mathcal{t}_F(\overline{\gamma_{xx'}} \circ \gamma \circ \gamma_{xx'}) \\ &= \mathcal{t}_F(\overline{\gamma_{xx'}})\mathcal{t}_F(\gamma)\mathcal{t}_F(\gamma_{xx'}) \\ &= (\mathcal{t}_F(\gamma_{xx'}))^{-1}\mathrm{hol}^F_{\ell}(\gamma)\mathcal{t}_F(\gamma_{xx'}) \end{aligned} \tag{72}$$

so that hol^F_{ℓ} changes by conjugation in G when the marking is changed.

ii) Suppose that $\eta : F \longrightarrow F'$ is a morphism of transport functors. Then, for every thin

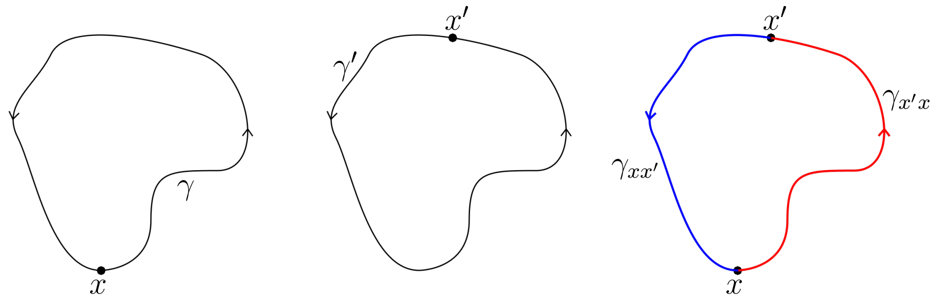


Figure 9: For two markings with associated basepoints x and x' of a thin loop $[\gamma]$, there exist representatives paths with sitting instants (shown on the right) $\gamma_{x'x} : x \to x'$ (in red) and $\gamma_{xx'} : x' \to x$ (in blue) such that $\gamma := \gamma_{xx'} \circ \gamma_{x'x}$ (shown on the left) represents one marking and $\gamma' := \gamma_{x'x} \circ \gamma_{xx'}$ (shown in the middle) represents the other. Note that γ' and $\overline{\gamma_{xx'}} \circ \gamma \circ \gamma_{xx'}$ are thinly homotopic.

path $\gamma : x \longrightarrow y$ we have a commutative diagram

$$\begin{tikzcd} \mathscr{t}_{F'}(x) \arrow[d, "\mathscr{t}_{F'}(\gamma)"'] & \mathscr{t}_F(x) \arrow[l, "\mathscr{t}_\eta(x)"'] \arrow[d, "\mathscr{t}_F(\gamma)"] \\ \mathscr{t}_{F'}(y) & \mathscr{t}_F(y) \arrow[l, "\mathscr{t}_\eta(y)"] \end{tikzcd} , \tag{73}$$

which says

$$\mathscr{t}_\eta(y)\mathscr{t}_F(\gamma) = \mathscr{t}_{F'}(\gamma)\mathscr{t}_\eta(x). \tag{74}$$

If we restrict this to a thin marked loop γ with $y = x$, then

$$\mathrm{hol}^{F'}_{\mathscr{t}}(\gamma) = (\mathscr{t}_\eta(x))^{-1}\mathrm{hol}^F_{\mathscr{t}}(\gamma)\mathscr{t}_\eta(x) \tag{75}$$

so that again, $\mathrm{hol}^F_{\mathscr{t}}$ changes under conjugation when the functor F is changed to an isomorphic one.

iii) Suppose that another trivialization $\mathscr{t}\,'$ was chosen. Then following the comments after Remark 2.34, we can choose natural isomorphisms $\mathscr{r} : \mathrm{id} \Rightarrow \mathscr{t}$ and $\mathscr{r}' : \mathrm{id} \Rightarrow \mathscr{t}\,'$ resulting in a natural isomorphism $\mathscr{s} := \begin{smallmatrix}\overline{\mathscr{r}'}\\ \circ \\ \mathscr{r}\end{smallmatrix} : \mathscr{t}\,' \Rightarrow \mathscr{t}$. This means every transport functor F gets assigned a morphism of transport functors $\mathscr{s}_F : \mathscr{t}_F' \longrightarrow \mathscr{t}_F$ satisfying naturality. This means to every $x \in M$ we have a morphism $\mathscr{s}_F(x) : \mathscr{t}_F'(x) \longrightarrow \mathscr{t}_F(x)$ satisfying naturality, i.e. to every path $\gamma : x \longrightarrow y$ the diagram

$$\begin{tikzcd} \mathscr{t}_F(x) \arrow[d, "\mathscr{t}_F(\gamma)"'] & \mathscr{t}_F'(x) \arrow[l, "\mathscr{s}_F(x)"'] \arrow[d, "\mathscr{t}_F'(\gamma)"] \\ \mathscr{t}_F(y) & \mathscr{t}_F'(y) \arrow[l, "\mathscr{s}_F(y)"] \end{tikzcd} \tag{76}$$

commutes. In case γ is a thin loop at x, this gives

$$\mathrm{hol}^F_{\ell'}(\gamma) = (\mathscr{I}_F(x))^{-1}\mathrm{hol}^F_{\ell}(\gamma)\mathscr{I}_F(x). \tag{77}$$

In conclusion, the answer to every one of the three questions is conjugation. This is what is called *gauge covariance*. To get something gauge invariant, we first denote the quotient map from G to its conjugacy classes by $q : G \longrightarrow G/\mathrm{Inn}(G)$, where $\mathrm{Inn}(G)$ stands for the inner automorphisms of G and the quotient $G/\mathrm{Inn}(G)$ is given by the conjugation action of G on itself. All of the above considerations show that the following theorem holds.

2.47. Theorem. *Let M be a smooth manifold, G be a Lie group, T a category, and suppose that $i : \mathcal{B}G \longrightarrow T$ is full and faithful. Let $F \in \mathrm{Trans}^1_{\mathcal{B}G}(M,T)$ be a transport functor and ℓ a group-valued transport extraction. Let $L^1M, \mathfrak{L}^1M, \mathfrak{m}, \mathrm{hol}^F_{\ell}$ and q be defined as above. Then the composition*

$$G/\mathrm{Inn}(G) \xleftarrow{q} G \xleftarrow{\mathrm{hol}^F_{\ell}} \mathfrak{L}^1M \xleftarrow{\mathfrak{m}} L^1M \tag{78}$$

is

i) independent of $\mathfrak{m}$,

ii) independent of the isomorphism class of F,

iii) and independent of the isomorphism class of ℓ.

Notice that this theorem lets us make the following definition.

2.48. Definition. *Let $[F]$ be an isomorphism class of transport functors. The* gauge invariant holonomy *of $[F]$ is defined to be the map in the previous theorem, namely*

$$\mathrm{hol}^{[F]} := q \circ \mathrm{hol}^F_{\ell} \circ \mathfrak{m} : L^1M \longrightarrow G/\mathrm{Inn}(G) \tag{79}$$

where F is a representative of $[F]$, ℓ is a group-valued transport extraction, and $\mathfrak{m} : L^1M \longrightarrow \mathfrak{L}^1M$ is a marking of thin loops in M. Let $\gamma \in L^1M$. If $\mathrm{hol}^{[F]}(\gamma)$ is such that $q^{-1}(\mathrm{hol}^{[F]}(\gamma))$ is a single element, we will say that $\mathrm{hol}^{[F]}(\gamma)$ is gauge invariant *and abusively write $\mathrm{hol}^{[F]}(\gamma)$ instead of $q^{-1}(\mathrm{hol}^{[F]}(\gamma))$.*

3. Transport 2-functors and gauge invariant surface holonomy

In the present section, we review the basics of transport 2-functors and also provide some new and interesting results. As a preliminary, we briefly set our notation and review some facts about (strict) 2-groups and crossed modules. Then we split up the discussion into several parts and follow a similar pattern to the transport functor case. However, since we are now aware of what local triviality should mean, we skip the guess-work and head straight to the correct theory. We start with a Čech description of ordinary principal (strict) 2-group 2-bundles (without connection) in terms of smooth 2-functors. We then discuss how to add connection data by introducing transport functors, local triviality, and descent data. The discussion of the reconstruction functor is more involved, and because it is important for the calculation, we spend some time on it. Nevertheless, we skip some technical details (such as compositors and unifiers). Then we consider the differential cocycle data and discuss a formula for higher holonomy in terms of an iterated surface integral. We summarize the results as before. Sections 3.1 through 3.35 are a summary of [ScWa11], [ScWa], and [ScWa13].

Finally, in Section 3.36, we discuss some results on surface holonomy and its gauge covariance. We introduce a notion of *α-conjugacy classes* for a 2-group in Definition 3.48 and prove in Theorem 3.49 that surface holonomy along *spheres* is well-defined in α-conjugacy classes generalizing the reduced group of [ScWa13] (it is not yet known whether this generalization will work for more general surfaces). In the process, the procedure of group-valued transport extraction is categorified for the purposes of (i) proving this theorem and (ii) providing a functorial description for computing transport locally, which we utilize in Section 5.

We assume the reader is familiar with the basics of 2-categories. A review sufficient for most of our purposes can be found in Appendix A of [ScWa].

3.1. 2-group conventions. The theory of 2-groups is discussed in great detail in the article [BaLa04]. However, to simplify the discussion, we will define a (strict) *2-group* as a strict one-object 2-groupoid, i.e. a strict 2-category with inverses for all 1- and 2-morphisms. Normally, one defines a 2-group as a groupal groupoid as in [BaLa04], but we find this unnecessary. However, to be consistent with notation in the literature, we will write our 2-groups as $\mathcal{B}\mathfrak{G}$ and use the notation $\mathfrak{G}$ where appropriate.

There is a 2-category of strict 2-groups denoted by 2-Grp whose 1-morphisms and 2-morphisms are functors and natural transformations, respectively. It is useful to relate this higher-categorical definition to one involving ordinary groups. Although this is standard, we set the notation, which may differ from some authors.

3.2. Definition. *A* crossed module *is a quadruple* (H, G, τ, α) *of two groups,* G *and* H*, and group homomorphisms* $\tau : H \longrightarrow G$ *and* $\alpha : G \longrightarrow \mathrm{Aut}(H)$ *satisfying the two conditions*

$$\alpha_{\tau(h)}(h') = hh'h^{-1} \tag{80}$$

and

$$\tau(\alpha_g(h)) = g\tau(h)g^{-1}. \tag{81}$$

In this definition, $\mathrm{Aut}(H)$ is the automorphism group of H. The collection of crossed modules form the objects of a 2-category CrsMod.

3.3. Theorem. *The 2-categories* CrsMod *and* 2-Grp *are equivalent.*

This theorem has been known for quite some time in several different forms. A simple place to start for this is in the article [BaHu11] with more information in [BaLa04].

Proof. We only prove the equivalence at the level of objects and in only one direction. This will set up our conventions throughout the paper. Given a crossed module (H, G, τ, α) the associated 2-group $\mathcal{B}\mathfrak{G}$ is defined to have a single object $\bullet$, G as its set of 1-morphisms, and $H \rtimes G$ as its set of 2-morphisms. Composition of 1-morphisms is given by multiplication in G. The source and target maps of 2-morphisms are defined pictorially by

$$\begin{tikzcd} \bullet & \bullet \arrow[l, bend right=50, "g"', ""{name=U, below}] \arrow[l, bend left=50, "\tau(h)g", ""{name=D, above}] \arrow[Rightarrow, from=U, to=D, "(h,g)"] \end{tikzcd} . \tag{82}$$

Vertical and horizontal compositions are defined by

$$\begin{tikzcd} \bullet & \bullet \arrow[l, bend right=70, "g"', ""{name=U, below}] \arrow[l, "\tau(h)g" description, ""{name=M}] \arrow[l, bend left=70, "\tau(\tilde{h})\tau(h)g", ""{name=D, above}] \arrow[Rightarrow, from=U, to=M, "(h,g)"] \arrow[Rightarrow, from=M, to=D, "(\tilde{h},\tau(h)g)"] \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \bullet & \bullet \arrow[l, bend right=50, "g"', ""{name=U, below}] \arrow[l, bend left=50, "\tau(\tilde{h}h)g", ""{name=D, above}] \arrow[Rightarrow, from=U, to=D, "(\tilde{h}h,g)"] \end{tikzcd} \tag{83}$$

and

$$\begin{tikzcd} \bullet & \bullet \arrow[l, bend right=50, "g'"', ""{name=U1, below}] \arrow[l, bend left=50, "\tau(h')g'", ""{name=D1, above}] \arrow[Rightarrow, from=U1, to=D1, "(h',g')"] & \bullet \arrow[l, bend right=50, "g"', ""{name=U2, below}] \arrow[l, bend left=50, "\tau(h)g", ""{name=D2, above}] \arrow[Rightarrow, from=U2, to=D2, "(h,g)"] \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \bullet & \bullet \arrow[l, bend right=50, "g'g"', ""{name=U, below}] \arrow[l, bend left=50, "\tau(h')g'\tau(h)g", ""{name=D, above}] \arrow[Rightarrow, from=U, to=D, "(h'\alpha_{g'}(h),g'g)"] \end{tikzcd} , \tag{84}$$

respectively. When writing 2-group multiplication, we will always drop the composition symbol $\circ$, which is a common practice for ordinary group multiplication. ∎

The above proof sets up our convention for 2-group multiplication. Equation (82) shows that what's needed to specify a 2-morphism is an element of G, the source of the 2-morphism, and an element of H. Thus, if the source is already known, the element in H specifies the 2-morphism. Equation (83) defines vertical composition and equation (84) defines horizontal composition. Please be aware that different authors have different conventions (since the 2-categories CrsMod and 2-Grp are equivalent in many ways).

The following is a simple but important fact (which we use in studying gauge invariance, mainly Corollary 5.7).

3.4. Lemma. *Let (H, G, τ, α) be a crossed module. Then $\ker\tau := \{h \in H \mid \tau(h) = e\}$ is a central subgroup of H.*

Proof. Let $k \in \ker\tau$ and $h \in H$. Then

$$kh = khk^{-1}k = \alpha_{\tau(k)}(h)k = \alpha_e(h)k = hk. \tag{85}$$

∎

3.5. Definition. *A* Lie crossed module *is a crossed module* (H, G, τ, α) *with* G *and* H *Lie groups and where* τ *and* α *are smooth maps, where* α *being smooth technically means that the adjoint map* $G \times H \longrightarrow H$ *is smooth.*

3.6. Definition. *A* Lie 2-groupoid *is a strict 2-category* Gr *whose objects, 1-morphisms, and 2-morphisms are all smooth spaces and all structure maps are smooth. Furthermore, all 1- and 2-morphisms are invertible and the inversion maps are all smooth.*

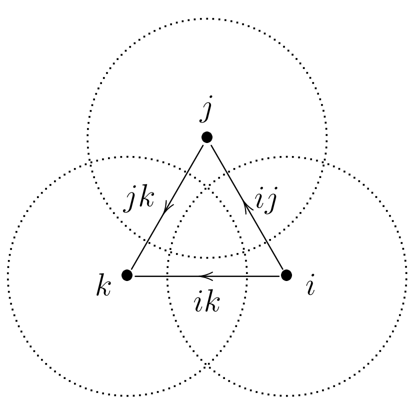
3.7. Definition. *A* Lie 2-group *is a Lie 2-groupoid with a single object.*

3.8. Remark. *Lie crossed modules form the objects of a 2-category and Lie 2-groups form the objects of a 2-category. A similar proof shows that these 2-categories are also equivalent.*

3.9. A Čech description of principal $\mathfrak{G}$-2-bundles. Let $\mathcal{B}\mathfrak{G}$ be a Lie 2-group and denote the associated crossed module by (H, G, τ, α). Principal $\mathfrak{G}$-2-bundles over a manifold M can be described in terms of 2-functors using the Čech groupoid as well (this also comes from Remark II.13. of [Wo11]). However, since we are dealing with 2-categories we need to slightly modify the Čech groupoid of Definition 2.2. The way we do this is just by throwing on identity 2-morphisms. In other words, given an open cover $\{U_i\}_{i \in I}$ of M, a 2-morphism from (x, i, j) to (x', i', j') exists only if $x' = x, i' = i$, and $j' = j$ and in this case there is only the identity 2-morphism. Composition is uniquely defined by this. This defines the *Čech 2-groupoid*, also written as $\mathfrak{U}$. This is a Lie 2-groupoid.

3.10. Definition. *2-functors between Lie 2-groupoids are* smooth *if they assign data smoothly. Similarly, pseudonatural transformations and modifications are smooth when the assignments defining them are smooth.*

Any smooth 2-functor $\mathfrak{U} \longrightarrow \mathcal{B}\mathfrak{G}$ gives the Čech cocycle data of a principal $\mathfrak{G}$-2-bundle over M subordinate to the cover $\{U_i\}_{i \in I}$. To see this, simply recall what a 2-functor does (see Definition A.5. of [ScWa]). To each object (x, i) in $\mathfrak{U}$ it assigns the single object $\bullet$ in $\mathcal{B}\mathfrak{G}$. To each jump (x, i, j), it assigns an element denoted by $g_{ij}(x) \in G$ in such a way that we get a smooth 1-cochain $g_{ij} : U_{ij} \longrightarrow G$ as in Section 2.1. However, to each triple intersection U_{ijk}, which corresponds to the composition of U_{ij} with U_{jk}, it assigns an element $f_{ijk}(x) \in H$ in such a way that we get a smooth 2-cochain $f_{ijk} : U_{ijk} \longrightarrow H$

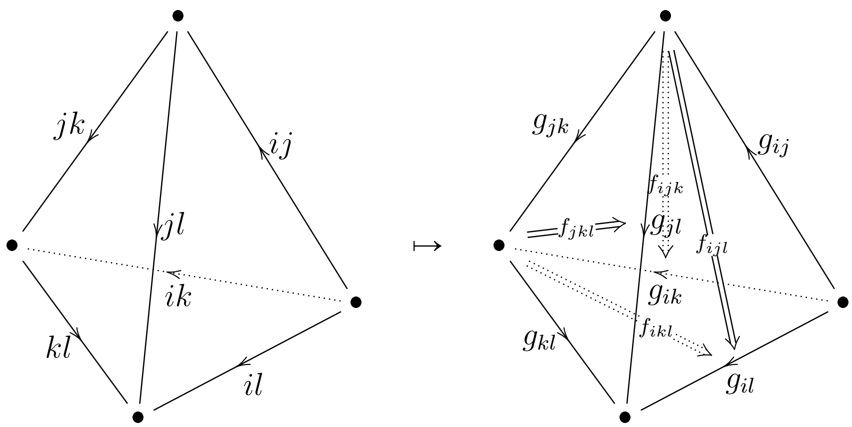


$$\mapsto \begin{tikzcd} & \bullet \arrow[dl, "g_{jk}"'] \arrow[d, phantom, "\Downarrow (f_{ijk}, g_{jk}g_{ij})"] & \\ \bullet & & \bullet \arrow[ul, "g_{ij}"'] \arrow[ll, "g_{ik}"] \end{tikzcd}, \qquad (86)$$

which says

$$\tau(f_{ijk})g_{jk}g_{ij} = g_{ik}. \tag{87}$$

The 2-functor satisfies an associativity condition which is translated into a condition on quadruple intersections giving a "cocycle condition"



(88)

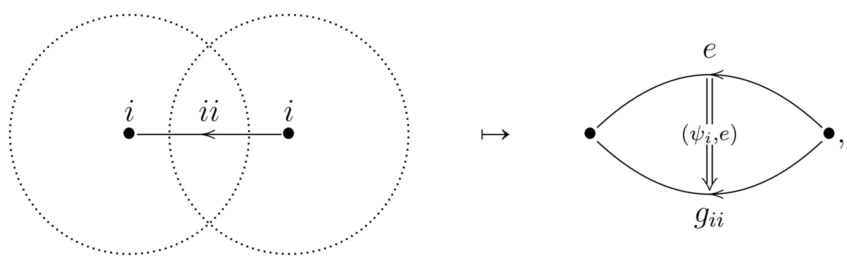
where f_{ijk} is short for $(f_{ijk}, g_{jk}g_{ij})$, etc. This condition says

$$\begin{matrix}(f_{jkl}, g_{kl}g_{jk})(e, g_{ij}) \\ (f_{ijl}, g_{jl}g_{ij})\end{matrix} = \begin{matrix}(e, g_{kl})(f_{ijk}, g_{jk}g_{ij}) \\ (f_{ikl}, g_{kl}g_{ik})\end{matrix}, \tag{89}$$

which after multiplying out (using the rules of Section 3.1) and projecting both sides to H gives

$$f_{ijl}f_{jkl} = f_{ikl}\alpha_{g_{kl}}(f_{ijk}). \tag{90}$$

The 2-functor also assigns 0-cochains $\psi_i : U_i \longrightarrow H$



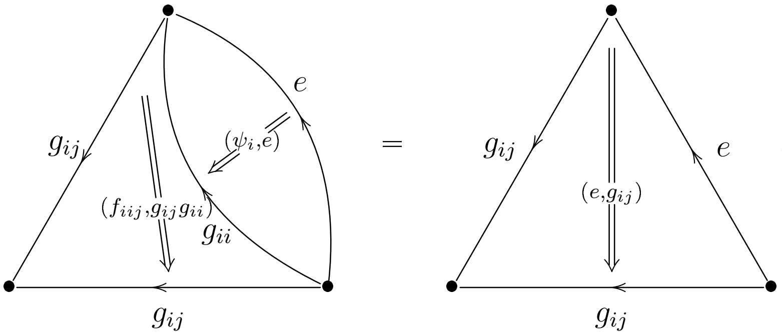
(91)

which says

$$\tau(\psi_i) = g_{ii}. \tag{92}$$

These satisfy two "degenerate" cocycle conditions on each double intersection U_{ij} of M

for the two ways one edge can be collapsed on the triangle. One is

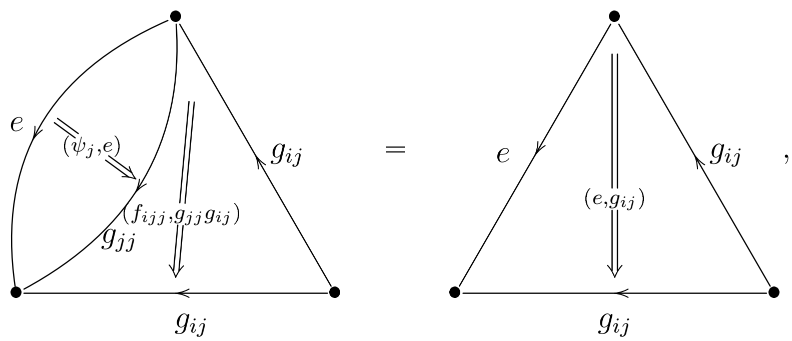


(93)

which after multiplying out and projecting to H says

$$f_{iij}\alpha_{g_{ij}}(\psi_i) = e. \tag{94}$$

The other cocycle condition is



(95)

which after multiplying out and projecting to H says

$$f_{ijj}\psi_j = e. \tag{96}$$

Refinements and 1-morphisms between two such 2-functors is similar to the ordinary functor case from Section 2.1 but a bit more subtle due to modifications (which we won't discuss now anyway). Let $\{U_{i'}\}_{i'\in I'}$ be another cover of M with associated Čech 2-groupoid $\mathfrak{U}'$. Let $P : \mathfrak{U} \longrightarrow \mathcal{B}\mathfrak{G}$ and $P' : \mathfrak{U}' \longrightarrow \mathcal{B}\mathfrak{G}$ be two smooth 2-functors. A 1-morphism from P to P' consists of a common refinement $\{V_a\}_{a\in A}$ of both $\{U_i\}_{i\in I}$ and $\{U_{i'}\}_{i'\in I'}$ along with a smooth pseudo-natural transformation

$$\begin{tikzcd} & \mathfrak{U} \arrow[dr, "P"] \arrow[dd, phantom, "\Downarrow h"] & \\ \mathfrak{V} \arrow[ur, "\alpha"] \arrow[dr, "\alpha'"'] & & \mathcal{B}\mathfrak{G} \\ & \mathfrak{U}' \arrow[ur, "P"'] & \end{tikzcd} . \tag{97}$$

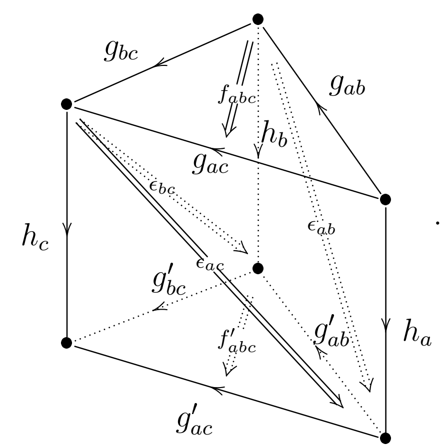
By definition (see Definition A.6. in [ScWa]), to each object (x, a) in $\mathfrak{V}$ such a pseudo-natural transformation gives a smooth function $h_a : V_a \longrightarrow G$ as before, but also to each jump (x,a,b) in $\mathfrak{V}$, it gives another smooth function $\epsilon_{ab} : V_{ab} \longrightarrow H$ fitting in the diagram

$$\begin{tikzcd} \bullet \arrow[d, "h_b"'] & \bullet \arrow[l, "g_{ab}"'] \arrow[d, "h_a"] \\ \bullet & \bullet \arrow[l, "g'_{ab}"] \arrow[from=1-1, to=2-2, Rightarrow, "(\epsilon_{ab}, h_b g_{ab})" description] \end{tikzcd}, \tag{98}$$

which says that

$$\tau(\epsilon_{ab})h_b g_{ab} = g'_{ab}h_a. \tag{99}$$

The higher naturality conditions of a pseudo-natural transformation are given as follows. In general, to every 2-morphism, there is an associated naturality condition, but because the 2-morphisms in $\mathfrak{U}$ are all identities, this condition is vacuously true. To every pair of composable 1-morphisms (x,i,j) and (x,j,k) we get



. (100)

Commutativity of this diagram says

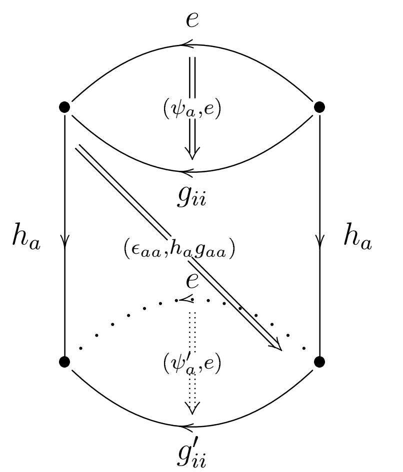
$$\begin{matrix} (e,h_c)(f_{abc}, g_{bc}g_{ab}) \\ (\epsilon_{ac}, h_c g_{ac}) \end{matrix} = \begin{matrix} (\epsilon_{bc}, h_c g_{bc})(e, g_{ab}) \\ (e, g'_{bc})(\epsilon_{ab}, h_b g_{ab}), \\ (f'_{abc}, g'_{bc}g'_{ab}) \end{matrix} \tag{101}$$

which after equating both projections to H gives

$$\epsilon_{ac}\alpha_{h_c}(f_{abc}) = f'_{abc}\alpha_{g'_{bc}}(\epsilon_{ab})\epsilon_{bc} \tag{102}$$

for all $a,b,c \in A$.

Finally, to every object (x, i) we get on each open set U_i



, (103)

where the back face of the cylinder is the identity 2-morphism (e, h_a). This reads

$$\begin{array}{c} (e,h_a)(\psi_a,e) \\ (\epsilon_{aa}, h_a g_{aa}) \end{array} = \begin{array}{c} (e,h_a) \\ (\psi'_a, e)(e, h_a) \end{array} , \tag{104}$$

which after projecting to H says

$$\epsilon_{aa}\alpha_{h_a}(\psi_a) = \psi'_a. \tag{105}$$

Therefore, a 1-morphism of such principal 2-bundles as described above defines an equivalence of principal 2-bundles as described in [Wo11].

We won't discuss 2-morphisms now because we will see that the above construction is a special case of the concept of limits of 2-categories in Section 3.35.

3.11. Local triviality of 2-functors. Just as transport functors describe parallel transport along paths, transport 2-functors describe parallel transport along paths and surfaces. They exhibit a formulation of a generalization of bundles with connection that describe such transport. We start by generalizing the thin path groupoid $\mathcal{P}_1(X)$ to the thin path 2-groupoid $\mathcal{P}_2(X)$. At this point, one should recall Definition 2.38 where bigons are introduced.

3.12. Definition. *Let X be a smooth manifold. Two bigons Γ and Γ' are said to be* thinly homotopic *if there exists a smooth map $A : [0,1] \times [0,1] \times [0,1] \longrightarrow X$ with the following two properties.*

i) First, there exists an ϵ with $\frac{1}{2} > \epsilon > 0$ such that

$$A(t,s,r) = \begin{cases} x & \textit{for all } (t,s,r) \in [0,\epsilon] \times [0,1] \times [0,1] \\ y & \textit{for all } (t,s,r) \in [1-\epsilon,1] \times [0,1] \times [0,1] \\ \gamma(t) & \textit{for all } (t,s,r) \in [0,1] \times [0,\epsilon] \times [0,1] \\ \gamma'(t) & \textit{for all } (t,s,r) \in [0,1] \times [1-\epsilon,1] \times [0,1] \\ \Gamma(t,s) & \textit{for all } (t,s,r) \in [0,1] \times [0,1] \times [0,\epsilon] \\ \Gamma'(t,s) & \textit{for all } (t,s,r) \in [0,1] \times [0,1] \times [1-\epsilon,1] \end{cases} \tag{106}$$

ii) Second, the rank of A is strictly less than 3 for all $(t,s,r) \in [0,1] \times [0,1] \times [0,1]$ and is strictly less than 2 for all $(t,s,r) \in [0,1] \times ([0,\epsilon] \cup [1-\epsilon,1]) \times [0,1]$.

In this case, A is said to be a thin homotopy *from Γ to Γ'. The set of equivalence classes of bigons under thin homotopy is denoted by P^2X. Elements of P^2X are called* thin bigons.

3.13. Definition. *Let X be a smooth manifold. The* thin path-2-groupoid *is a 2-category $\mathcal{P}_2(X)$ defined as follows. The set of objects and 1-morphisms of $\mathcal{P}_2(X)$ coincide with that of $\mathcal{P}_1(X)$. The set of 2-morphisms of $\mathcal{P}_2(X)$ is P^2X. Let $[\Gamma]$ be a thin bigon. The source and targets are defined by choosing a representative bigon Γ and taking the thin homotopy equivalence classes of the paths $t \mapsto \Gamma(t,0)$ and $t \mapsto \Gamma(t,1)$, respectively. For a thin path $[\gamma]$, the identity at $[\gamma]$ is the thin homotopy class of the bigon $(t,s) \mapsto \gamma(t)$.*

The various compositions in $\mathcal{P}_2(X)$ are the usual ones of composing paths and homotopies by either stacking squares vertically or horizontally and parametrizing via double speed vertically or horizontally, respectively. More concretely, given two vertically composable thin bigons[5]

$$\begin{tikzcd} y & & x \arrow[ll, bend right=60, "{[\gamma]}"', ""{name=U}] \arrow[ll, "{[\delta]}" description, ""{name=M}] \arrow[ll, bend left=60, "{[\epsilon]}", ""{name=D}] \arrow[from=U, to=M, Rightarrow, "{[\Gamma]}"] \arrow[from=M, to=D, Rightarrow, "{[\Delta]}"] \end{tikzcd} \tag{107}$$

the vertical composition is given by first choosing representatives δ for the target of Γ and δ' for the source of Δ. Then, there exists a thin (rank strictly less than 2) homotopy $\Sigma : \delta \Rightarrow \delta'$. Using this thin homotopy, the vertical composition is the thin homotopy class associated to the bigon

$$\underset{\Delta}{\overset{\Gamma}{\circ}}(t,s) := \begin{cases} \Gamma(t,3s) & \text{for } 0 \leqslant s \leqslant \frac{1}{3} \\ \Sigma(t,3s-1) & \text{for } \frac{1}{3} \leqslant s \leqslant \frac{2}{3} \\ \Delta(t,3s-2) & \text{for } \frac{2}{3} \leqslant s \leqslant 1 \end{cases}, \quad t \in [0,1]. \tag{108}$$

Given two horizontally composable thin bigons

$$\begin{tikzcd} z & & y \arrow[ll, bend right=50, "{[\gamma']}"', ""{name=A}] \arrow[ll, bend left=50, "{[\delta']}", ""{name=B}] \arrow[from=A, to=B, Rightarrow, "{[\Gamma']}"] & & x \arrow[ll, bend right=50, "{[\gamma]}"', ""{name=C}] \arrow[ll, bend left=50, "{[\delta]}", ""{name=D}] \arrow[from=C, to=D, Rightarrow, "{[\Gamma]}"] \end{tikzcd} \tag{109}$$

the horizontal composition is given by the thin homotopy class associated to

$$(\Gamma' \circ \Gamma)(t,s) := \begin{cases} \Gamma(2t,s) & \text{for } 0 \leqslant t \leqslant \frac{1}{2} \\ \Gamma'(2t-1,s) & \text{for } \frac{1}{2} \leqslant t \leqslant 1 \end{cases}, \quad s \in [0,1]. \tag{110}$$

[5]To be absolutely clear, we write square brackets to denote the thin homotopy equivalence classes. After this definition, we will generally *not* do this, unless otherwise specified.

All such compositions are well-defined, smooth, associative, have left and right units given by constant bigons for horizontal composition and paths viewed as bigons for vertical composition respectively, and satisfy the interchange law. $\mathcal{P}_2(X)$ is a Lie 2-groupoid since thin homotopy classes of bigons are invertible in both ways and the functions that assign every class to its vertical and horizontal inverses are both smooth.

3.14. Remark. *In the definition of vertical composition (108), we can always choose representatives of* Γ *and* Δ *so that* $\delta = \delta'$ *and we can ignore* Σ *for all practical purposes of this paper. Therefore, we will always write the vertical composition as*

$$\begin{smallmatrix}\Gamma\\ \circ\\ \Delta\end{smallmatrix}(t,s) := \begin{cases} \Gamma(t,2s) & \text{for } 0 \leqslant s \leqslant \frac{1}{2} \\ \Delta(t,2s-1) & \text{for } \frac{1}{2} \leqslant s1 \end{cases}, \ t \in [0,1]. \tag{111}$$

3.15. Definition. *Let* Gr *be a Lie 2-groupoid,* T *be a 2-category,* $i : \mathrm{Gr} \longrightarrow T$ *a 2-functor, and* M *a smooth manifold. Fix a surjective submersion* $\pi : Y \longrightarrow M$*. A* <u>$\pi$-local i-trivialization</u> *of a 2-functor* $F : \mathcal{P}_2(M) \longrightarrow T$ *is a pair* (triv, t) *of a strict 2-functor* $\mathrm{triv} : \mathcal{P}_2(Y) \longrightarrow \mathrm{Gr}$ *and a pseudonatural equivalence*

$$\begin{tikzcd} \mathcal{P}_2(M) \arrow[d, "F"'] \arrow[dr, Rightarrow, "t"] & \mathcal{P}_2(Y) \arrow[l, "\pi^*"'] \arrow[d, "\mathrm{triv}"] \\ T & \mathrm{Gr} \arrow[l, "i"] \end{tikzcd} \tag{112}$$

meaning that there exist a weak inverse $\bar{t}$ *along with modifications (see Definition A.8. in [ScWa])* $i_t : \begin{smallmatrix}t\\ \circ\\ \bar{t}\end{smallmatrix} \Rrightarrow \mathrm{id}_{\pi^* F}$ *and* $j_t : \mathrm{id}_{\mathrm{triv}_i} \Rrightarrow \begin{smallmatrix}\bar{t}\\ \circ\\ t\end{smallmatrix}$ *satisfying the zig-zag identities (see Definition 7. of [BaLa04] and particularly their discussion on string diagrams). The 2-groupoid* Gr *is called the* <u>structure 2-groupoid</u> *for* F*.*

2-functors $F : \mathcal{P}_2(M) \longrightarrow T$ equipped with π-local i-trivializations (triv, t) form the objects, written as triples (F, triv, t), of a 2-category denoted by $\mathrm{Triv}^2_\pi(i)$.

3.16. Definition. *A* <u>1-morphism of π-local i-trivializations</u> $\alpha : (F, \mathrm{triv}, t) \longrightarrow (F', \mathrm{triv}', t')$ *in* $\mathrm{Triv}^2_\pi(i)$ *is a pseudo-natural transformation* $\alpha : F \Rightarrow F'$*. A* <u>2-morphism</u> $\alpha \Rrightarrow \alpha'$ *is a modification.*

3.17. Definition. *Let* Gr *be a Lie 2-groupoid,* T *a 2-category,* $i : \mathrm{Gr} \longrightarrow T$ *a 2-functor and* $\pi : Y \longrightarrow M$ *a surjective submersion. A* <u>descent object</u> *is a quadruple* $(\mathrm{triv}, g, \psi, f)$ *consisting of a strict 2-functor* $\mathrm{triv} : \mathcal{P}_2(Y) \longrightarrow \mathrm{Gr}$, *a pseudonatural equivalence*

$$\begin{tikzcd} \mathcal{P}_2(Y) \arrow[d, "\mathrm{triv}_i"'] \arrow[dr, Rightarrow, "g"] & \mathcal{P}_2(Y^{[2]}) \arrow[l, "\pi_{1*}"'] \arrow[d, "\pi_{2*}"] \\ T & \mathcal{P}_2(Y) \arrow[l, "\mathrm{triv}_i"] \end{tikzcd} \tag{113}$$

and invertible modifications

$$f : \begin{matrix} \pi_{12}^* g \\ \circ \\ \pi_{23}^* g \end{matrix} \Rrightarrow \pi_{13}^* g \tag{114}$$

and

$$\psi : \mathrm{id}_{\mathrm{triv}_i} \Rrightarrow \Delta^* g. \tag{115}$$

These modifications must satisfy the coherence conditions which are explicitly given in Definition 2.2.1. of [ScWa] (in the examples of this current paper, the above modifications will actually be trivial and the coherence conditions will automatically be satisfied, which is why we leave them out).

Descent objects form the objects of a 2-category denoted by $\mathfrak{Des}_\pi^2(i)$. Morphisms and 2-morphisms are defined as follows.

3.18. Definition. *A* descent 1-morphism *from* $(\mathrm{triv}, g, \psi, f)$ *to* $(\mathrm{triv}', g', \psi', f')$ *is a pair* (h, ϵ) *with* h *a pseudo-natural transformation* $h : \mathrm{triv}_i \Rightarrow \mathrm{triv}'_i$ *and* ϵ *an invertible modification*

$$\epsilon : \begin{matrix} g \\ \circ \\ \pi_2^* h \end{matrix} \Rrightarrow \begin{matrix} \pi_1^* h \\ \circ \\ g' \end{matrix}. \tag{116}$$

These must satisfy certain identities explained in Definition 2.2.2. of [ScWa].

3.19. Definition. *Let* (h, ϵ) *and* (h', ϵ') *be two descent 1-morphisms from* $(\mathrm{triv}, g, \psi, f)$ *to* $(\mathrm{triv}', g', \psi', f')$. *A* descent 2-morphism *from* (h, ϵ) *to* (h', ϵ') *is a modification* $E : h \Rrightarrow h'$ *satisfying a certain identity explained in Definition 2.2.3. of [ScWa].*

There is a 2-functor $\mathrm{Ex}_\pi^2 : \mathrm{Triv}_\pi^2(i) \longrightarrow \mathfrak{Des}_\pi^2(i)$ that extracts descent data from trivialization data. At the level of objects, this functor is defined as follows. Let (F, triv, t) be an object in $\mathrm{Triv}_\pi^2(i)$. For the quadruple $(\mathrm{triv}, g, \psi, f)$, take triv to be exactly the same. For g take the composition $g := \begin{smallmatrix} \pi_1^* \bar{t} \\ \circ \\ \pi_2^* t \end{smallmatrix}$ coming from the composition in the diagram

$$\begin{tikzcd}
& \mathcal{P}_2(Y) \arrow[dl, "\mathrm{triv}"'] \arrow[dd, "\pi_*"] \arrow[ddr, equal, "\mathrm{id}"] & \mathcal{P}_2(Y^{[2]}) \arrow[l, "\pi_{1*}"'] \arrow[dd, "\pi_{2*}"] \\
\mathrm{Gr} \arrow[dr, Rightarrow, "\bar{t}"] \arrow[dd, "i"'] & & \\
& \mathcal{P}_2(M) \arrow[dl, "F"'] \arrow[d, Rightarrow, "t"] & \mathcal{P}_2(Y) \arrow[l, "\pi_*"] \arrow[dl, "\mathrm{triv}"] \\
T & \mathrm{Gr} \arrow[l, "i"] &
\end{tikzcd} \tag{117}$$

just as before but this time $\bar{t}$ is a *weak* (vertical) inverse to t. By definition f should be a modification $f : \begin{smallmatrix} \pi_{12}^* g \\ \circ \\ \pi_{23}^* g \end{smallmatrix} \Rrightarrow \pi_{13}^* g$. Using our definition of g, this means that we can break it down as follows

$$f : \begin{matrix} \pi_{12}^* g \\ \circ \\ \pi_{23}^* g \end{matrix} = \begin{matrix} \pi_{12}^* \begin{pmatrix} \pi_1^* \bar{t} \\ \circ \\ \pi_2^* t \end{pmatrix} \\ \circ \\ \pi_{23}^* \begin{pmatrix} \pi_1^* \bar{t} \\ \circ \\ \pi_2^* t \end{pmatrix} \end{matrix} = \begin{matrix} \begin{pmatrix} \pi_1^* \bar{t} \\ \circ \\ \pi_2^* t \end{pmatrix} \\ \circ \\ \begin{pmatrix} \pi_2^* \bar{t} \\ \circ \\ \pi_3^* t \end{pmatrix} \end{matrix} \Rrightarrow \begin{matrix} \pi_1^* \bar{t} \\ \circ \\ \pi_3^* t \end{matrix} = \pi_{13}^* g, \tag{118}$$

where all equalities hold by commutativity of certain diagrams and the leftover $\Rrightarrow$ is specified by the following sequence of modifications

$$\begin{pmatrix}\pi_1^*\bar{t}\\ \circ\\ \pi_2^*t\end{pmatrix} \atop {\circ \atop \begin{pmatrix}\pi_2^*\bar{t}\\ \circ\\ \pi_3^*t\end{pmatrix}} \xRrightarrow{\text{associators}} \begin{pmatrix}\pi_1^*\bar{t}\\ \circ\\ \begin{pmatrix}\pi_2^*t\\ \circ\\ \pi_2^*\bar{t}\end{pmatrix}\\ \circ\\ \pi_3^*t\end{pmatrix} \xRrightarrow{\begin{pmatrix}\mathrm{id}_{\pi_1^*\bar{t}}\\ \circ\\ \pi_2^*i_t\end{pmatrix} \circ\, \mathrm{id}_{\pi_3^*t}} \begin{pmatrix}\pi_1^*\bar{t}\\ \circ\\ \pi_2^*\mathrm{id}_{\pi^*F}\end{pmatrix} \atop {\circ \atop \pi_3^*t} \xRrightarrow{\pi_1^*l \,\circ\, \mathrm{id}_{\pi_3^*t}} \begin{matrix}\pi_1^*\bar{t}\\ \circ\\ \pi_3^*t\end{matrix}\,, \tag{119}$$

where i_t is part of the pseudo-natural equivalence from t and $\bar{t}$, and l is a left unifier.

Finally, by definition ψ should be a modification $\psi : \mathrm{id}_{\mathrm{triv}_i} \Rrightarrow \Delta^*g$. Using our definition of g, we can break it down as follows

$$\psi : \mathrm{id}_{\mathrm{triv}_i} = \Delta^*\pi_1^*\mathrm{id}_{\mathrm{triv}_i} \Rrightarrow \Delta^*\begin{pmatrix}\pi_1^*\bar{t}\\ \circ\\ \pi_2^*t\end{pmatrix} = \Delta^*g \tag{120}$$

and such a modification can be achieved by

$$\Delta^*\pi_1^*\mathrm{id}_{\mathrm{triv}_i} \xRrightarrow{\Delta^*\pi_1^*j_t} \Delta^*\pi_1^*\begin{pmatrix}\bar{t}\\ \circ\\ t\end{pmatrix} = \Delta^*\begin{pmatrix}\pi_1^*\bar{t}\\ \circ\\ \pi_2^*t\end{pmatrix}, \tag{121}$$

where j_t is the other part of the pseudo-natural equivalence from t and $\bar{t}$. This indeed defines a descent object and that this assignment of descent data to trivialization data extends to a 2-functor $\mathrm{Ex}^2_\pi : \mathrm{Triv}^2_\pi(i) \longrightarrow \mathfrak{Des}^2_\pi(i)$ to include 1-morphisms and 2-morphisms (see Lemma 2.3.1., Lemma 2.3.2., and Lemma 2.3.3. of [ScWa]).

3.20. Definition. *Let (F, triv, t) be a π-local i-trivialization of a 2-functor $F : \mathcal{P}_2(M) \longrightarrow T$, i.e. an object of* $\mathrm{Triv}^2_\pi(i)$. *The* <u>descent object associated to the π-local i-trivialization</u> *is* $\mathrm{Ex}^2_\pi(F, \mathrm{triv}, t)$. *A similar definition is made for 1- and 2-morphisms.*

3.21. Transport 2-functors. We now wish to discuss smoothness for descent data. However, to do this is not so simple as it was for ordinary functors. We will have to make a detour to describe how to think of natural transformations as functors and modifications as natural transformations by altering the source and target categories. For the purposes of this document, we will make stricter assumptions than is done in [ScWa13] that are sufficient for our purposes and simplify several of the arguments and constructions.

Let $\mathcal{C}$ and $\mathcal{D}$ be two strict 2-categories. Let $\mathcal{C}_{0,1}$ denote the category whose objects and morphisms are the objects and 1-morphisms of $\mathcal{C}$ respectively. Because $\mathcal{C}$ is strict, this defines a category. Let $\Lambda\mathcal{D}$ be the category whose objects are morphisms $X_f \xrightarrow{f} Y_f$ of $\mathcal{D}$. The set of morphisms in $\Lambda\mathcal{D}$ from $X_f \xrightarrow{f} Y_f$ to $X_g \xrightarrow{g} Y_g$ are pairs of morphisms

$(x : X_f \longrightarrow X_g, y : Y_f \longrightarrow Y_g)$ along with a 2-morphism $\varphi : g \circ x \Rightarrow y \circ f$ as in the diagram

$$\begin{tikzcd} X_g \arrow[d, "g"'] \arrow[dr, Rightarrow, "\varphi"] & X_f \arrow[l, "x"'] \arrow[d, "f"] \\ Y_g & Y_f \arrow[l, "y"] \end{tikzcd} . \tag{122}$$

The composition is given by stacking

$$\begin{tikzcd} X_h \arrow[d, "h"'] \arrow[dr, Rightarrow, "\psi"] & X_g \arrow[l, "x'"'] \arrow[d, "g"] \arrow[dr, Rightarrow, "\varphi"] & X_f \arrow[l, "x"'] \arrow[d, "f"] \\ Y_h & Y_g \arrow[l, "y'"] & Y_f \arrow[l, "y"] \end{tikzcd} = \begin{tikzcd} X_h \arrow[d, "h"'] \arrow[dr, Rightarrow, "\substack{\psi\circ\mathrm{id}_x \\ \circ \\ \mathrm{id}_{y'}\circ\varphi}"] & X_f \arrow[l, "x'\circ x"'] \arrow[d, "f"] \\ Y_h & Y_f \arrow[l, "y'\circ y"] \end{tikzcd} . \tag{123}$$

One can check that under our assumptions, this forms a category.

Notice that $\Lambda\mathcal{D}$ has a bit more structure. It also has a partially defined operation on objects and 1-morphisms given by "stacking vertically." Suppose that $X_f \xrightarrow{f} Y_f$ and $Y_f \xrightarrow{f'} Z_f$ are two 1-morphisms in $\mathcal{D}$ then one can compose them and this gives a partially defined associative and unital operation on objects of $\Lambda\mathcal{D}$. Similarly, given morphisms in $\Lambda\mathcal{D}$ which can be vertically stacked as in the diagram

$$\begin{tikzcd} X_g \arrow[d, "g"'] \arrow[dr, Rightarrow, "\varphi"] & X_f \arrow[l, "x"'] \arrow[d, "f"] \\ Y_g \arrow[d, "g'"'] \arrow[dr, Rightarrow, "\varphi'"] & Y_f \arrow[l, "y" description] \arrow[d, "f'"] \\ Z_g & Z_f \arrow[l, "z"] \end{tikzcd} = \begin{tikzcd} X_g \arrow[d, "g'\circ g"'] \arrow[dr, Rightarrow, "\substack{\mathrm{id}_{g'}\circ\varphi \\ \circ \\ \varphi'\circ\mathrm{id}_f}"] & X_f \arrow[l, "x"'] \arrow[d, "f'\circ f"] \\ Y_g & Y_f \arrow[l, "z"] \end{tikzcd} . \tag{124}$$

This additional partially defined composition is written as $\otimes$ in [ScWa13] so we stick with this notation.

Associated to a pseudo-natural transformation ρ as in

$$\begin{tikzcd} \mathcal{D} & \mathcal{C} \arrow[l, bend right=50, "F"', ""{name=U, below}] \arrow[l, bend left=50, "G", ""{name=D, above}] \arrow[Rightarrow, from=U, to=D, "\rho"] \end{tikzcd} \tag{125}$$

is a functor $\mathcal{F}(\rho) : \mathcal{C}_{0,1} \longrightarrow \Lambda\mathcal{D}$ defined by

$$X \quad \overset{\mathcal{F}(\rho)}{\longmapsto} \quad \begin{tikzcd} FX \arrow[d, "\rho(X)"] \\ GX \end{tikzcd} \tag{126}$$

on objects X in $\mathcal{C}_{0,1}$, i.e. objects in $\mathcal{C}$, and

$$
Y \xleftarrow{\;f\;} X \quad \stackrel{\mathcal{F}(\rho)}{\longmapsto} \quad
\begin{tikzcd}
FY \arrow[d, "\rho(Y)"'] \arrow[dr, Rightarrow, "\rho(f)"] & FX \arrow[l, "Ff"'] \arrow[d, "\rho(X)"] \\
GY & GX \arrow[l, "Gf"]
\end{tikzcd}
\tag{127}
$$

on morphisms in $\mathcal{C}_{0,1}$, i.e. 1-morphisms in $\mathcal{C}$. One can check this defines a functor.

Associated to a modification A as in

$$
\begin{tikzcd}
\mathcal{D} & & \mathcal{C} \arrow[ll, bend right=50, "F"'{name=U}] \arrow[ll, bend left=50, "G"{name=D}]
\arrow[from=U, to=D, Rightarrow, bend right=40, "\sigma"'{name=S}]
\arrow[from=U, to=D, Rightarrow, bend left=40, "\rho"{name=R}]
\arrow[from=R, to=S, phantom, "\overset{A}{\Leftarrow}"]
\end{tikzcd}
\tag{128}
$$

is a natural transformation $\mathcal{F}(A) : \mathcal{F}(\rho) \Rightarrow \mathcal{F}(\sigma)$ defined by

$$
X \quad \stackrel{\mathcal{F}(A)}{\longmapsto} \quad
\begin{tikzcd}
FX \arrow[d, "\sigma(X)"'] \arrow[dr, Rightarrow, "A(X)"] & FX \arrow[l, "\mathrm{id}_{FX}"'] \arrow[d, "\rho(X)"] \\
GX & GX \arrow[l, "\mathrm{id}_{GX}"]
\end{tikzcd} .
\tag{129}
$$

This defines a functor $\mathcal{F} : \mathrm{Hom}(F,G) \longrightarrow \mathrm{Funct}(\mathcal{C}_{0,1}, \Lambda\mathcal{D})$, where $\mathrm{Hom}(F,G)$ is the category whose objects are pseudonatural transformations and morphisms are modifications while $\mathrm{Funct}(\mathcal{E},\mathcal{E}')$ (between two ordinary categories $\mathcal{E}$ and $\mathcal{E}'$) is the category whose objects are functors from $\mathcal{E}$ to $\mathcal{E}'$ and whose morphisms are natural transformations.

Separately, notice also that if $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a 2-functor then there is a functor $\Lambda F : \Lambda\mathcal{C} \longrightarrow \Lambda\mathcal{D}$ defined by

$$
\begin{tikzcd}
X_f \arrow[d, "f"] \\
Y_f
\end{tikzcd}
\quad \stackrel{\Lambda F}{\longmapsto} \quad
\begin{tikzcd}
FX_f \arrow[d, "Ff"] \\
FY_f
\end{tikzcd}
\tag{130}
$$

on objects and

$$
\begin{tikzcd}
X_g \arrow[d, "g"'] \arrow[dr, Rightarrow, "\sigma"] & X_f \arrow[l, "x"'] \arrow[d, "f"] \\
Y_g & Y_f \arrow[l, "y"]
\end{tikzcd}
\quad \stackrel{\Lambda F}{\longmapsto} \quad
\begin{tikzcd}
FX_g \arrow[d, "Fg"'] \arrow[dr, Rightarrow, "F\sigma"] & FX_f \arrow[l, "Fx"'] \arrow[d, "Ff"] \\
FY_g & FY_f \arrow[l, "Fy"]
\end{tikzcd}
\tag{131}
$$

on morphisms.

3.22. Definition. *A descent object* $(\mathrm{triv}, g, \psi, f)$ *as in Definition 3.17 is said to be* smooth *if*

i) the 2-functor $\mathrm{triv} : \mathcal{P}_2(Y) \longrightarrow \mathrm{Gr}$ *is smooth,* *ii) the functor* $\mathcal{F}(g) : \mathcal{P}_1(Y^{[2]}) \longrightarrow \Lambda T$ *is a transport functor with* $\Lambda\mathrm{Gr}$*-structure, and*

iii) the natural transformations $\mathcal{F}(\psi) : \mathcal{F}(\mathrm{id}_{\mathrm{triv}_i}) \Rightarrow \Delta^*\mathcal{F}(g)$ *and* $\mathcal{F}(f) : \pi_{23}^*\mathcal{F}(g) \otimes \pi_{12}^*\mathcal{F}(g) \Rightarrow \pi_{13}^*\mathcal{F}(g)$ *are morphisms between transport functors.*

Smooth descent objects form the objects of a 2-category denoted by $\mathfrak{Des}_\pi^2(i)^\infty$ and form a sub-2-category of $\mathfrak{Des}_\pi^2(i)$. Smoothness of descent 1-morphisms and descent 2-morphisms is discussed in [ScWa13] following Definition 3.1.2.

3.23. Definition. *A* π*-local* i*-trivialization* (F, triv, t) *is said to be* smooth *if the associated descent object* $\mathrm{Ex}_\pi^2(F, \mathrm{triv}, t)$ *is smooth. The same can be said of 1-morphisms and 2-morphisms.*

Smooth local trivializations, 1-morphisms, and 2-morphisms form a sub-2-category denoted by $\mathrm{Triv}_\pi^2(i)^\infty$ of $\mathrm{Triv}_\pi^2(i)$. Furthermore, Ex_π^2 restricts to an equivalence of 2-categories of smooth data (Lemma 3.2.3. of [ScWa13]).

After all this formalism, it should be more or less clear now what the definition of a transport 2-functor is by just abstracting what we did for the one-dimensional case (Definition 3.2.1. of [ScWa13]).

3.24. Definition. *Let* Gr *be a Lie 2-groupoid,* T *a 2-category,* $i : \mathrm{Gr} \longrightarrow T$ *a 2-functor, and* M *a smooth manifold. A* transport 2-functor on M with values in a 2-category T and with Gr-structure *is a 2-functor* $\mathrm{tra} : \mathcal{P}_2(M) \longrightarrow T$ *such that there exists a surjective submersion* $\pi : Y \longrightarrow M$ *and a smooth* π*-local* i*-trivialization* (triv, t).

Transport 2-functors over M with values in T with Gr-structure form the objects of a 2-category $\mathrm{Trans}_{\mathrm{Gr}}^2(M, T)$. A 1-morphism of transport functors is a pseudo-natural transformation of 2-functors for which there exists a common surjective submersion π and smooth π-local i-trivializations of both 2-functors so that the associated descent 1-morphism is smooth. A similar definition exists for 2-morphisms.

As a short summary, in the past two sections we introduced three categories for describing transport 2-functors. These were $\mathfrak{Des}_\pi^2(i)$, $\mathrm{Triv}_\pi^2(i)$, and $\mathrm{Trans}_{\mathrm{Gr}}^2(M, T)$. The category $\mathrm{Triv}_\pi^2(i)$ was used to describe local triviality of transport 2-functors and their morphisms in $\mathrm{Trans}_{\mathrm{Gr}}^2(M, T)$. We then constructed a 2-functor $\mathrm{Ex}_\pi^2 : \mathrm{Triv}_\pi^2(i) \longrightarrow \mathfrak{Des}_\pi^2(i)$ that allowed us to describe smoothness via the subcategory $\mathfrak{Des}_\pi^2(i)^\infty \subset \mathfrak{Des}_\pi^2(i)$ from which we defined $\mathrm{Triv}_\pi^2(i)^\infty \subset \mathrm{Triv}_\pi^2(i)$.

3.25. The reconstruction 2-functor: from local to global. The 2-functor $\mathrm{Ex}_\pi^2 : \mathrm{Triv}_\pi^2(i) \longrightarrow \mathfrak{Des}_\pi^2(i)$ is an equivalence of 2-categories (Proposition 4.1.1. of [ScWa]). To construct a (weak) inverse $\mathrm{Rec}_\pi^2 : \mathfrak{Des}_\pi^2(i) \longrightarrow \mathrm{Triv}_\pi^2(i)$, we need to enhance the Čech path groupoid so that it includes more data.

We do not require the full general definition of $\mathcal{P}_2^\pi(M)$ in Section 3.1 of [ScWa] for our purposes, but briefly the general definition is obtained by keeping the same objects and morphisms but replacing the relations that we imposed by 2-morphisms and setting relations on those. There are also additional 2-morphisms given by thin bigons, thin paths on intersections, and other formal 2-morphisms such as associators, unitors, and

2-morphisms relating the formal product to the usual composition of paths. We therefore warn the reader that although the following definition is not the same as that in [ScWa], we use their general results and theorems which in fact rely on their more general definition.

3.26. Definition. *Let M be a smooth manifold and let $\pi : Y \longrightarrow M$ be a surjective submersion. The* <u>Čech path 2-groupoid of M</u> *is the 2-category $\mathcal{P}_2^{\pi}(M)$ whose set of objects and 1-morphisms are the objects and morphisms of $\mathcal{P}_1^{\pi}(M)$, respectively. The set of 2-morphisms are freely generated by*

i) thin bigons Γ in Y,

ii) thin paths $\Theta : \alpha \longrightarrow \beta$ in $Y^{[2]}$ with sitting instants thought of as 2-isomorphisms

$$\begin{tikzcd} \pi_1(\beta) \arrow[d, "\beta"'] & \pi_1(\alpha) \arrow[l, "\pi_1(\Theta)"'] \arrow[d, "\alpha"] \\ \pi_2(\beta) & \pi_2(\alpha) \arrow[l, "\pi_2(\Theta)"] \arrow[ul, Rightarrow, "\Theta"'] \end{tikzcd} \tag{132}$$

(one should think of this as weakening the first relation in Definition 2.25 of $\mathcal{P}_1^{\pi}(M)$—see Figure 10 for a visualization of this),

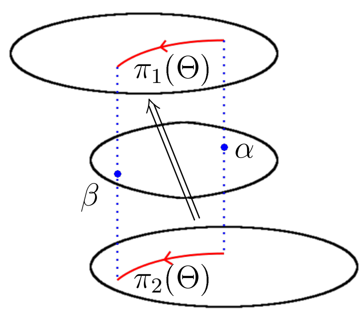


Figure 10: Thinking in terms of an open cover as a submersion, condition ii) above says that if a thin path $\Theta : \alpha \to \beta$ (with chosen representative) is in a double intersection, there is a relationship between going along the path first and then jumping versus jumping first and then going along the path. The two need not be equal.

iii) points Ξ in $Y^{[3]}$ thought of as 2-isomorphisms

$$\begin{tikzcd} & \pi_2(\Xi) \arrow[dl, "\pi_{23}(\Xi)"'] \arrow[d, Rightarrow, "\Xi"] & \\ \pi_3(\Xi) & \, & \pi_1(\Xi) \arrow[ul, "\pi_{12}(\Xi)"'] \arrow[ll, "\pi_{13}(\Xi)"] \end{tikzcd} \tag{133}$$

(one should think of this as weakening the second relation in Definition 2.25 of $\mathcal{P}_1^{\pi}(M)$),

iv) points a in Y thought of as 2-isomorphisms (id_a^* is the formal identity)

$$
\begin{tikzcd}
a & a \arrow[l, bend right, "\mathrm{id}_a^*"'] \arrow[l, bend left, "\Delta(a)"] \arrow[l, phantom, "\Downarrow \Delta_a"]
\end{tikzcd}
\tag{134}
$$

(one should think of this as weakening part of the third relation in Definition 2.25 of $\mathcal{P}_1^{\pi}(M)$),

v) and several other more technical generators that will not be discussed here.

There are several relations imposed on the set of 1-morphisms and 2-morphisms. We will not discuss any of them, and the reader is referred to Section 3.1 of [ScWa] for the details. As before, the compositions will be written with $*$ and will be drawn vertically or horizontally when dealing with 2-morphisms.

As before, we associate to every object $(\mathrm{triv}, g, \psi, f)$ in $\mathfrak{Des}^2_{\pi}(i)$ a functor $R_{(\mathrm{triv},g,\psi,f)} : \mathcal{P}_2^{\pi}(M) \longrightarrow T$ defined as follows. It sends $y \in Y$ to $\mathrm{triv}_i(y)$, thin paths γ in Y to $\mathrm{triv}_i(\gamma)$, and jumps $\alpha \in Y^{[2]}$ to $g(\alpha) : \mathrm{triv}_i(\pi_1(\alpha)) \longrightarrow \mathrm{triv}_i(\pi_2(\alpha))$. For the basic 2-morphisms, it makes the following assignments

$$
\begin{tikzcd}
y & x \arrow[l, bend right, "\gamma"'] \arrow[l, bend left, "\delta"] \arrow[l, phantom, "\Downarrow \Gamma"]
\end{tikzcd}
\xmapsto{R_{(\mathrm{triv},g,\psi,f)}}
\begin{tikzcd}
\mathrm{triv}_i(y) & \mathrm{triv}_i(x) \arrow[l, bend right, "\mathrm{triv}_i(\gamma)"'] \arrow[l, bend left, "\mathrm{triv}_i(\delta)"] \arrow[l, phantom, "\Downarrow \mathrm{triv}_i(\Gamma)"]
\end{tikzcd}
\tag{135}
$$

for thin bigons $\Gamma : \gamma \Rightarrow \delta$ in Y,

$$
\begin{tikzcd}
\pi_2(\alpha) \arrow[d, "\pi_2(\Theta)"'] \arrow[dr, phantom, "\Theta"] & \pi_1(\alpha) \arrow[l, "\alpha"'] \arrow[d, "\pi_1(\Theta)"] \\
\pi_2(\beta) & \pi_1(\beta) \arrow[l, "\beta"]
\end{tikzcd}
\xmapsto{R_{(\mathrm{triv},g,\psi,f)}}
\begin{tikzcd}
\mathrm{triv}_i(\pi_2(\alpha)) \arrow[d, "\mathrm{triv}_i(\pi_2(\Theta))"'] \arrow[dr, phantom, "g(\Theta)"] & \mathrm{triv}_i(\pi_1(\alpha)) \arrow[l, "g(\alpha)"'] \arrow[d, "\mathrm{triv}_i(\pi_1(\Theta))"] \\
\mathrm{triv}_i(\pi_2(\beta)) & \mathrm{triv}_i(\pi_1(\beta)) \arrow[l, "g(\beta)"]
\end{tikzcd}
\tag{136}
$$

for thin paths $\Theta : \alpha \longrightarrow \beta$ in $Y^{[2]}$,

$$
\begin{tikzcd}
 & \pi_2(\Xi) \arrow[dl, "\pi_{23}(\Xi)"'] \arrow[d, phantom, "\Downarrow \Xi"] & \\
\pi_3(\Xi) & & \pi_1(\Xi) \arrow[ll, "\pi_{13}(\Xi)"] \arrow[ul, "\pi_{12}(\Xi)"']
\end{tikzcd}
\xmapsto{R_{(\mathrm{triv},g,\psi,f)}}
\begin{tikzcd}
 & \mathrm{triv}_i(\pi_2(\Xi)) \arrow[dl, "g(\pi_{23}(\Xi))"'] \arrow[d, phantom, "\Downarrow f(\Xi)"] & \\
\mathrm{triv}_i(\pi_3(\Xi)) & & \mathrm{triv}_i(\pi_1(\Xi)) \arrow[ll, "g(\pi_{13}(\Xi))"] \arrow[ul, "g(\pi_{12}(\Xi))"']
\end{tikzcd}
\tag{137}
$$

for points Ξ in $Y^{[3]}$, and

$$\begin{tikzcd} a & a \arrow[l, bend right, "\mathrm{id}_a^*"'] \arrow[l, bend left, "\Delta(a)"] \arrow[l, phantom, "\Downarrow \Delta_a"] \end{tikzcd} \xmapsto{R_{(\mathrm{triv},g,\psi,f)}} \begin{tikzcd} \mathrm{triv}_i(a) & \mathrm{triv}_i(a) \arrow[l, bend right, "1_{\mathrm{triv}_i(a)}"'] \arrow[l, bend left, "g(\Delta(a))"] \arrow[l, phantom, "\Downarrow \psi(a)"] \end{tikzcd} \tag{138}$$

for points a in Y. This defines a 2-functor $R : \mathfrak{Des}^2_\pi(i) \longrightarrow \mathrm{Funct}(\mathcal{P}^\pi_2(M), T)$ at the level of objects. The rest of this 2-functor is defined in Proposition 3.3.2. of [ScWa].

There is a canonical projection functor $p^\pi : \mathcal{P}^\pi_2(M) \longrightarrow \mathcal{P}_2(M)$ defined in the same way as $p^\pi : \mathcal{P}^\pi_1(M) \longrightarrow \mathcal{P}_1(M)$ on the level of objects and morphisms. On the level of 2-morphisms, p^π sends a thin bigon Γ in Y to a the thin bigon $\pi(\Gamma)$ in M. It sends a thin path Θ in $Y^{[2]}$ to the identity thin bigon $\mathrm{id}_{\pi(\Theta)}$ (the vertical identity) in M and it sends a point Ξ in $Y^{[3]}$ to the constant thin bigon at the point $\pi(\Xi)$ in M. Finally, it sends a point a in Y to the constant thin bigon at the point $\pi(a)$ in M. We now move on to defining, as before, a weak inverse $s^\pi : \mathcal{P}_2(M) \longrightarrow \mathcal{P}^\pi_2(M)$ of the canonical projection functor. To define s^π, we will constantly use the following important fact (Lemma 3.2.2. of [ScWa]).

3.27. LEMMA. *Let $\gamma : x \longrightarrow x'$ be a thin path in M and let $\tilde{\gamma}$ and $\tilde{\gamma}'$ be two lifts of γ as 1-morphisms in $\mathcal{P}^\pi_2(M)$ (the existence follows from our choices above when we defined $s^\pi : \mathcal{P}_1(M) \longrightarrow \mathcal{P}^\pi_1(M)$). Then there exists a unique 2-isomorphism $A : \tilde{\gamma} \Rightarrow \tilde{\gamma}'$ in $\mathcal{P}^\pi_2(M)$ such that $p^\pi(A) = \mathrm{id}_\gamma$.*

We will use this to define $s^\pi : \mathcal{P}_2(M) \longrightarrow \mathcal{P}^\pi_2(M)$ on thin bigons (we have already defined s^π near (33) on objects and 1-morphisms). Let $\Gamma : \gamma \Rightarrow \delta$ be any thin bigon in M as in Figure 11.

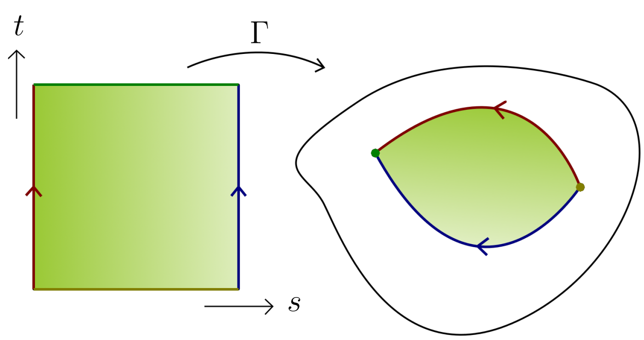


Figure 11: A representative of a thin bigon Γ in M drawn as a map of a square into M. The $s = 0$ line is drawn on top in the figure on the right while the $s = 1$ line is drawn on the bottom. The entire $t = 0$ line gets mapped to the source point and the $t = 1$ line gets mapped to the target point.

As in the case of a path, because the domain is compact, there exists a decomposition of the bigon Γ (we abuse notation and write Γ to mean a bigon and its thin homotopy class relying on context to distinguish them) into smaller bigons $\{\Gamma_j\}_j$, as in Figure 12, each of which fits into an open set U_j. We use the same notation $s_j : U_j \longrightarrow Y$ as before for our local sections.

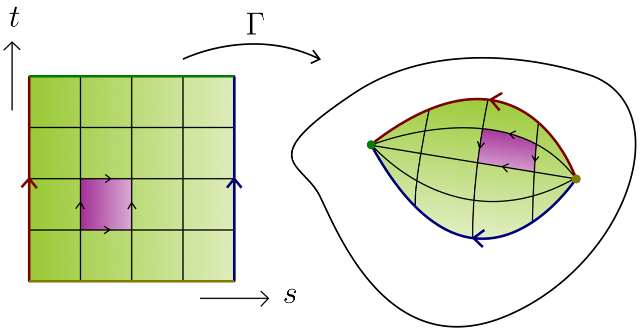


Figure 12: A decomposition of a representative of a thin bigon Γ in M with a single subbigon Γ_j highlighted. $s^\pi(\Gamma)$ will be defined as a composition of several $s^\pi(\Gamma_j)$. Of course, a general decomposition would not necessarily look like this, but such a decomposition always exists by a thin homotopy so that the decomposed pieces are bigons.

Therefore, it suffices to define $s^\pi(\Gamma_j)$ for a single one of the associated thin bigons provided that we match up all sources and targets for the individual ones. Denote the thin bigon by

$$\begin{tikzcd} x'_j & x_j \arrow[l, bend right, "\gamma_j"'] \arrow[l, bend left, "\delta_j"] \arrow[l, phantom, "\Downarrow \Gamma_j"] \end{tikzcd} . \tag{139}$$

Then the image of this under s^π is defined as the composition

$$\begin{tikzcd} s^\pi(x'_j) \arrow[r] & s_j(x'_j) & s_j(x_j) \arrow[l, bend right, "s_j(\gamma_j)"'] \arrow[l, bend left, "s_j(\delta_j)"] \arrow[l, phantom, "\Downarrow s_j(\Gamma_j)"] & s^\pi(x_j) \arrow[l] \arrow[lll, bend right=40, "s^\pi(\gamma_j)"'] \arrow[lll, bend left=40, "s^\pi(\delta_j)"] \end{tikzcd} . \tag{140}$$

In other words, we have lifted Γ_j using the section $s_j : U_j \longrightarrow Y$, but to make sure that this image matches up with how s^π was already defined on objects and 1-morphisms, we use the obvious jumps and the unique 2-isomorphisms from Lemma 3.27 to match everything (these are the unlabeled 1-morphisms and 2-morphisms). The image of the entire thin bigon Γ is then defined by vertical and horizontal compositions of all the $s^\pi(\Gamma_j)$ so that s^π respects compositions.

The 2-functor s^π is a weak inverse to p^π as in the case for the path groupoid (Proposition 3.2.1. of [ScWa]). However, a weak inverse in 2-category theory in this case means that there exists a pseudo-natural equivalence $\zeta : s^{\pi} \circ p^{\pi} \Rightarrow \mathrm{id}_{\mathcal{P}_2^{\pi}(M)}$ since $p^{\pi} \circ s^{\pi} = \mathrm{id}_{\mathcal{P}_2(M)}$. This means there exists a weak inverse to ζ which is written as $\xi : \mathrm{id}_{\mathcal{P}_2^{\pi}(M)} \Rightarrow s^{\pi} \circ p^{\pi}$. "Weak" means that there are invertible modifications $i_{\zeta} : \xi \circ \zeta \Rrightarrow \mathrm{id}_{s^{\pi} \circ p^{\pi}}$ and $j_{\zeta} : \mathrm{id}_{\mathrm{id}_{\mathcal{P}_2^{\pi}(M)}} \Rrightarrow \zeta \circ \xi$ that satisfy the zig-zag identities. The details are irrelevant for our purposes but can be found in Section 3.2 of [ScWa]. An important consequence of s^{π} being a weak inverse to p^{π} is the following (general categorical) fact reproduced here for convenience (Corollary 3.2.5. of [ScWa]).

3.28. Corollary. *Any two weak inverses $s^{\pi}, s'^{\pi} : \mathcal{P}_2(M) \longrightarrow \mathcal{P}_2^{\pi}(M)$ of p^{π} are pseudo-naturally equivalent.*

We can define such a pseudo-natural equivalence $\eta : s^{\pi} \Rightarrow s'^{\pi}$ by the following assignment $M \ni x \mapsto$ the jump from $s^{\pi}(x)$ to $s'^{\pi}(x)$ and $P^1M \ni \gamma \mapsto$ the unique 2-isomorphism $s^{\pi}(\gamma) \Rightarrow s'^{\pi}(\gamma)$ specified by Lemma 3.27. We will exploit this fact when discussing examples of higher holonomy in Section 5.

As before, the 2-functor

$$s^{\pi} : \mathcal{P}_2(M) \longrightarrow \mathcal{P}_2^{\pi}(M)$$

induces a 2-functor $s^{\pi *} : \mathrm{Funct}(\mathcal{P}_2^{\pi}(M), T) \to \mathrm{Funct}(\mathcal{P}_2(M), T)$, the pullback along s^{π}. Similarly, Rec_{π}^2 is defined as the composition in the diagram

$$\begin{tikzcd} \mathrm{Funct}(\mathcal{P}_2(M), T) & & \mathfrak{Des}_{\pi}^2(i) \arrow[ll, "\mathrm{Rec}_{\pi}^2"'] \arrow[dl, "R"] \\ & \mathrm{Funct}(\mathcal{P}_2^{\pi}(M), T) \arrow[ul, "s^{\pi *}"] & \end{tikzcd} . \tag{141}$$

As before, the image of $\mathfrak{Des}_{\pi}^2(i)$ under Rec_{π}^2 lands in $\mathrm{Triv}_{\pi}^2(i)$ and the definition is the same as it was before, only this time ζ is a pseudo-natural equivalence between 2-functors between 2-categories.

As a short summary, in this section we introduced a weak inverse functor $\mathrm{Rec}_{\pi}^2 : \mathfrak{Des}_{\pi}^2(i) \longrightarrow \mathrm{Triv}_{\pi}^2(i)$ for $\mathrm{Ex}_{\pi}^2 : \mathrm{Triv}_{\pi}^2(i) \longrightarrow \mathfrak{Des}_{\pi}^2(i)$ by using the 2-groupoid $\mathcal{P}_2^{\pi}(M)$ associated to the surjective submersion $\pi : Y \longrightarrow M$ to lift points, thin paths, and thin bigons in M to points, thin paths and/or jumps, and thin bigons and/or jumps in $\mathcal{P}_2^{\pi}(M)$, respectively.

3.29. Differential cocycle data. In this section, we will give a brief review of an equivalence between differential forms and smooth 2-functors following Section 2 of [ScWa11]. This will allow us to describe parallel transport locally in terms of differential cocycle data. We will leave out several proofs but will provide pictures that we find illustrate the necessary ideas behind the statements. We first remind the reader of the "Lie algebra" of a Lie crossed module.

Given a Lie crossed module (H, G, τ, α) (recall Definition 3.2) there is an associated <u>*differential Lie crossed module*</u> $(\underline{H}, \underline{G}, \underline{\tau}, \underline{\alpha})$, where $\underline{\tau} : \underline{H} \longrightarrow \underline{G}$ is the differential of $\tau : H \longrightarrow G$, $\underline{\alpha} : \underline{G} \longrightarrow \mathrm{Der}(\underline{H})$ is the differential of the associated action (given the same name) $\alpha : G \times H \longrightarrow H$ ("Der" stands for derivations). The differential Lie crossed module data satisfy

$$\underline{\alpha}_{\underline{\tau}(B')}(B) = [B', B] \tag{142}$$

and

$$\underline{\tau}(\underline{\alpha}_A(B)) = [A, \underline{\tau}(B)] \tag{143}$$

for all $A \in \underline{G}$ and $B, B' \in \underline{H}$.

Note that by restricting the action α to $\{g\} \times H$ for any $g \in G$ and differentiating with respect to the second coordinate, we obtain a Lie algebra homomorphism $\underline{\alpha_g} : \underline{H} \longrightarrow \underline{H}$. Both $\underline{\alpha}$ and $\underline{\alpha_g}$ are important for understanding the differential cocycle data of Section 3.29. A more thorough review can be found in [BaHu11].

3.29.1. From 2-functors to 2-forms. Let $\mathcal{B}\mathfrak{G}$ be a Lie 2-group and (H, G, τ, α) its corresponding crossed module. Given a strict smooth 2-functor $F : \mathcal{P}_2(X) \longrightarrow \mathcal{B}\mathfrak{G}$, we will obtain differential forms $A \in \Omega^1(X; \underline{G})$ and $B \in \Omega^2(X; \underline{H})$. These will form the objects of a 2-category $Z^2_X(\mathfrak{G})$. By our previous discussion and since our 2-categories $\mathcal{P}_2(X)$ and $\mathcal{B}\mathfrak{G}$ are strict and the 2-functor F is strict, the restriction of F to $\mathcal{P}_1(X)$ is smooth. Therefore, we obtain a differential form $A \in \Omega^1(X; \underline{G})$ by the results of Section 2.27. To obtain the differential form $B \in \Omega^2(X; \underline{H})$ we will "differentiate" the composition

$$H \xleftarrow{p_H} H \rtimes G \xleftarrow{F_2} P^2X, \tag{144}$$

where p_H is the projection onto the H factor and F_2 is F restricted to 2-morphisms.

Infinitesimally, a bigon is determined by a point and the two tangent vectors that begin to span it. Therefore, let $x \in X$ and $v_1, v_2 \in T_xX$ and let $\Gamma : \mathbb{R}^2 \longrightarrow X$ be a smooth map such that

$$\Gamma((0,0)) = x, \quad \left.\frac{\partial}{\partial s}\right|_{s=0} \Gamma(s, t=0) = v_1, \quad \& \quad \left.\frac{\partial}{\partial t}\right|_{t=0} \Gamma(s=0, t) = v_2. \tag{145}$$

Let $\Sigma_{\mathbb{R}} : \mathbb{R}^2 \longrightarrow P^2\mathbb{R}^2$ be the (smooth) map that sends (s,t) to the thin homotopy class of the bigon in Figure 13. This is unambiguously defined after modding out by thin homotopy because a thin bigon in $\mathbb{R}^2$ is determined by its source and target thin paths in $\mathbb{R}^2$.

Then we use this to define a smooth map F_Γ by the composition of smooth maps

$$H \xleftarrow{p_H} H \rtimes G \xleftarrow{F_2} P^2X \xleftarrow{\Gamma_*} P^2\mathbb{R}^2 \xleftarrow{\Sigma_{\mathbb{R}}} \mathbb{R}^2. \tag{146}$$

This gives an element of the Lie algebra $\underline{H}$ by taking derivatives

$$B_x(v_1, v_2) := -\left.\frac{\partial^2 F_\Gamma}{\partial s \partial t}\right|_{(0,0)} \in \underline{H}. \tag{147}$$

Furthermore, this element is independent of the choice of Γ provided that equation (145) still holds. In fact, we get a smooth differential form $B \in \Omega^2(X; \underline{H})$.

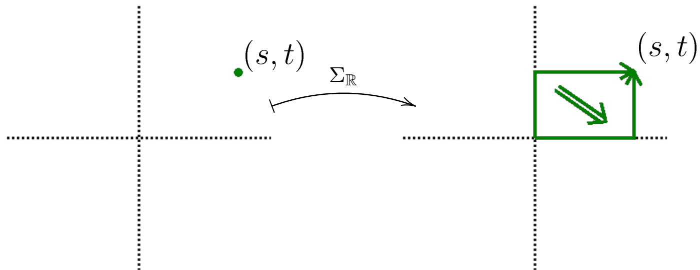


Figure 13: A point (s,t) in $\mathbb{R}^2$ gets mapped to the bigon in $\mathbb{R}^2$ shown on the right under the map $\Sigma_{\mathbb{R}}$.

Now let $\Gamma : \gamma \Rightarrow \delta$ be a thin bigon between two thin paths. The *source-target matching condition*, which says $\tau(p_H(F(\Gamma)))F(\gamma) = F(\delta)$, implies

$$dA + \frac{1}{2}[A, A] = \underline{\tau}(B). \tag{148}$$

All of these claims are proved in Section 2.2.1 of [ScWa11].

3.29.2. From 2-forms to 2-functors. Starting with a $\underline{G}$-valued 1-form $A \in \Omega^1(X; \underline{G})$ on X and a $\underline{H}$-valued 2-form $B \in \Omega^2(X; \underline{H})$ on X we want to define a smooth functor $\mathcal{P}_2(X) \longrightarrow \mathcal{B}\mathfrak{G}$. From Section 2.27.2, we have already defined the functor at the level of objects and thin paths. What remains is to define $F_2 : P^2X \longrightarrow H \rtimes G$. To do this, we will define a function $k_{A,B} : BX \longrightarrow H$ on bigons in X (we do not mod out by thin homotopy). Given a bigon $\Sigma : [0,1] \times [0,1] \longrightarrow X$, we can pull back the 1-form A and the 2-form B to $[0,1] \times [0,1]$, obtaining $\Sigma^*(A) \in \Omega^1([0,1] \times [0,1]; \underline{G})$ and $\Sigma^*(B) \in \Omega^2([0,1] \times [0,1]; \underline{H})$.

To define $k_{A,B}$, we first introduce an $\underline{H}$-valued 1-form $\mathcal{A}_\Sigma \in \Omega^1([0,1]; \underline{H})$ obtained by integrating over one of the directions for the bigon. It is defined by

$$(\mathcal{A}_\Sigma)_s \left(\frac{d}{ds} \right) := - \int_0^1 dt \; \underline{\alpha_{F_1(\Sigma_* \gamma_{s,t})^{-1}}} \left((\Sigma^* B)_{(s,t)} \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right) \right), \tag{149}$$

where $\gamma_{s,t}$ is defined to be the straight vertical path from $(s,0)$ to (s,t) in $[0,1] \times [0,1]$ as in Figure 14. Note that in expression (149), it is assumed that $\Sigma_* \gamma_{s,t}$ refers to the thin homotopy class of the path (otherwise, applying the function F_1 would not make sense). Therefore, the parametrization of $\gamma_{s,t}$ is irrelevant.

Besides the path-ordered integral expression from the term $F_1(\Sigma_*(\gamma_{s,t}))$, the expression for $\mathcal{A}_\Sigma$ is an ordinary integral. Also note that $\mathcal{A}_\Sigma$ depends on Σ. In particular, it is *not* invariant under thin homotopy.

3.30. Remark. *Incidentally, although Schreiber and Waldorf in [ScWa11] made their own arguments for how to obtain such a formula for $\mathcal{A}_\Sigma$, this formula appears in a special case as early as 1977 in the work of Goddard, Nuyts, and Olive on magnetic*

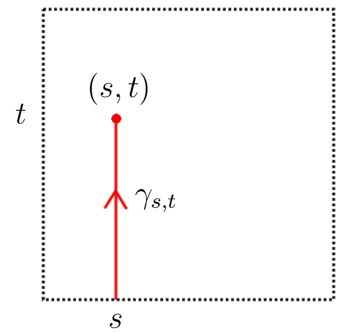


Figure 14: The path $\gamma_{s,t}$ is the straight vertical path from the point $(s, 0)$ to (s, t) in $[0, 1] \times [0, 1]$.

monopoles [GoNuOl77] on the right-hand side of equation (2.9) and it may have been known earlier [Ch75]. The special case [GoNuOl77] considered is the case of the crossed module $(G, G, \mathrm{id}, \alpha)$ *with* α *being the ordinary conjugation action.*

Finally, to every bigon $\Sigma : \gamma \Rightarrow \delta$, we define

$$k_{A,B}(\Sigma) := \alpha_{F_1(\gamma)} \left(\mathcal{P} \exp \left\{ - \int_0^1 \mathcal{A}_\Sigma \right\} \right). \tag{150}$$

In Figure 15, this integral is schematically drawn as a power series of graphs with marked points and paths analogous to Figure 5. Each of the paths drawn has a path-ordered integral expression attached to it, and therefore each expression has an additional power series of the form we discussed for the ordinary path-ordered integral.

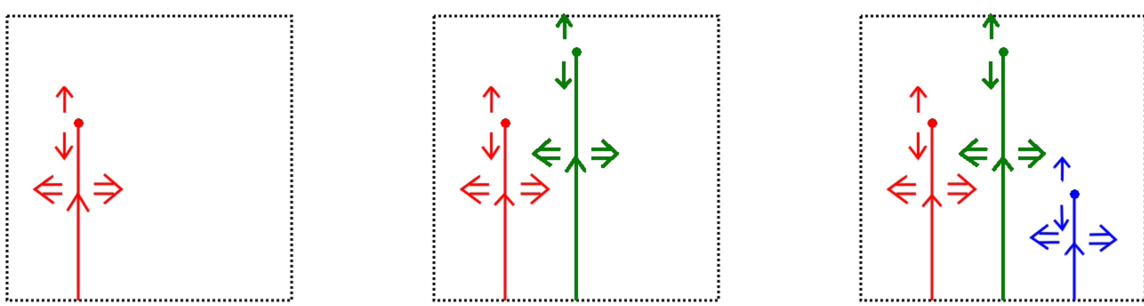


Figure 15: The path-ordered integral $\mathcal{P} \exp \left\{ - \int_0^1 \mathcal{A}_\Sigma \right\}$ is depicted schematically as an infinite sum of terms expressed by placing B at the endpoints of the paths, along which we've computed parallel transport using A making sure to keep the later s-valued terms on the right. The picture is to be interpreted similarly to the one-dimensional case once we've integrated along the t direction (vertical) to obtain $\mathcal{A}_\Sigma$.

3.31. Definition. *The group element $k_{A,B}(\Sigma)$ is called the* surface transport *associated to the bigon Σ and the differential forms A and B.*

$k_{A,B}$ only depends on the thin homotopy class of Σ and therefore factors through a smooth map $F_2 : P^2X \longrightarrow H$ on thin homotopy classes of paths. This map together with F_1 define a strict smooth 2-functor $F : \mathcal{P}_2(X) \longrightarrow \mathcal{B}\mathfrak{G}$ (Proposition 2.17. of [ScWa11]).

3.32. Remark. *Historically, understanding the appropriate generalization of the path-ordered integral to surfaces was a difficult task. It was not obvious which formulas were correct or even what the criteria for correctness should be. The language of functors allows one to make this precise. The criteria for correctness is that surface holonomy should be expressed in terms of a transport 2-functor.* Any *formula that satisfies these functorial properties, has the local constraint given by equation (148), and changes appropriately under gauge transformations (which we have so far only discussed globally but will discuss differentially soon), can be rightfully called surface transport. The specific formula in equation (150) is only one such formula that works. However, there could be many other, potentially simpler formulas, that also describe 2-holonomy. In Section 4 for instance, we prove that for certain structure 2-groups, the formula (150) agrees with one that is easily computable in terms of homotopy classes of paths.*

3.32.1. Local differential cocycles for transport 2-functors. By similar considerations to the previous sections, we can differentiate transport functors and use their properties to obtain relations among all the differential data. All the information in this section is discussed in more detail in [ScWa13]. In particular, the functions, differential forms, and their relations are all derived. We merely reproduce the results here for use in later calculations.

3.33. Definition. *Let $Z^2_X(\mathfrak{G})^\infty$ be the category defined as follows. An object of $Z^2_X(\mathfrak{G})^\infty$ is a pair (A,B) of a 1-form $A \in \Omega^1(X;\underline{G})$ and a 2-form $B \in \Omega^2(X;\underline{H})$ satisfying*

$$\underline{\tau}(B) = dA + \frac{1}{2}[A,A]. \tag{151}$$

A 1-morphism from (A,B) to (A',B') is a pair (h,φ) of a smooth map $h : X \longrightarrow G$ and a 1-form $\varphi \in \Omega^1(X;\underline{H})$ satisfying

$$A' + \underline{\tau}(\varphi) = \mathrm{Ad}_h(A) - h^*\overline{\theta} \tag{152}$$

and

$$B' + \underline{\alpha}_{A'}(\varphi) + d\varphi + \frac{1}{2}[\varphi,\varphi] = \underline{\alpha_g}(B). \tag{153}$$

The composition is defined by

$$(A'',B'') \xleftarrow{(h',\varphi')} (A',B') \xleftarrow{(h,\varphi)} (A,B) := (A'',B'') \xleftarrow{(h'h,\underline{\alpha_{h'}}(\varphi)+\varphi')} (A,B). \tag{154}$$

A 2-morphism from (h,φ) *to* (h',φ')*, which are both 1-morphisms from* (A,B) *to* (A',B')*, is a smooth map* $f : X \longrightarrow H$ *satisfying*

$$h' = \tau(f)h \tag{155}$$

and

$$\varphi' + (\underline{R_f^{-1} \circ \alpha_f})(A') = \mathrm{Ad}_f(\varphi) - f^*\overline{\theta}. \tag{156}$$

The vertical composition is defined by

$$\begin{tikzcd}[column sep=large] (A',B') & & (A,B) \arrow[ll, bend right=50, "{(h,\varphi)}"'] \arrow[ll, "{(h',\varphi')}" description] \arrow[ll, bend left=50, "{(h'',\varphi'')}"] \arrow[ll, phantom, shift right=3.5ex, "\Downarrow f"] \arrow[ll, phantom, shift left=3.5ex, "\Downarrow f'"] \end{tikzcd} \quad := \quad \begin{tikzcd}[column sep=large] (A',B') & & (A,B) \arrow[ll, bend right=40, "{(h,\varphi)}"'] \arrow[ll, bend left=40, "{(h'',\varphi'')}"] \arrow[ll, phantom, "\Downarrow f'f"] \end{tikzcd} . \tag{157}$$

The horizontal composition is defined by

$$\begin{tikzcd} (A'',B'') & (A',B') \arrow[l, bend right=50, "{(h_1,\varphi_1)}"'] \arrow[l, bend left=50, "{(h'_1,\varphi'_1)}"] \arrow[l, phantom, "\Downarrow f_1"] & (A,B) \arrow[l, bend right=50, "{(h_2,\varphi_2)}"'] \arrow[l, bend left=50, "{(h'_2,\varphi'_2)}"] \arrow[l, phantom, "\Downarrow f_2"] \end{tikzcd} \; := \; \begin{tikzcd}[column sep=huge] (A'',B'') & & (A,B) \arrow[ll, bend right=30, "{(h_1h_2,\underline{\alpha_{h_1}}(\varphi_2)+\varphi_1)}"'] \arrow[ll, bend left=30, "{(h'_1h'_2,\underline{\alpha_{h'_1}}(\varphi'_2)+\varphi'_1)}"] \arrow[ll, phantom, "\Downarrow f_1\alpha_{h_1}(f_2)"] \end{tikzcd} . \tag{158}$$

As in Section 2.27.3, these arguments define 2-functors

$$Z^2_X(\mathfrak{G})^\infty \underset{\mathcal{D}_X}{\overset{\mathcal{P}_X}{\rightleftarrows}} \mathrm{Funct}^\infty(X,\mathcal{B}\mathfrak{G}) , \tag{159}$$

which turn out to be strict inverses of each other (Theorem 2.21 of [ScWa11]).

As before, this was for globally defined differential data corresponding to globally trivial transport 2-functors. Transport 2-functors on M are not necessarily of this type, but they are locally trivializable via some surjective submersion $\pi : Y \longrightarrow M$ and a π-local i-trivialization. By similar arguments to the discussion in Section 2.27.3, we are led to the following, rather long and complicated, definition.

3.34. Definition. *Let* $\pi : Y \longrightarrow M$ *be a surjective submersion. Define the 2-category* $Z^2_\pi(\mathfrak{G})^\infty$ *of* differential cocycles subordinate to π *as follows. An object of* $Z^2_\pi(\mathfrak{G})^\infty$ *is a tuple* $((A,B),(g,\varphi),\psi,f)$*, where* (A,B) *is an object in* $Z^2_Y(G)$*,* (g,φ) *is a 1-morphism from* $\pi_1^*(A,B)$ *to* $\pi_2^*(A,B)$ *in* $Z^2_{Y^{[2]}}(\mathfrak{G})$*,* ψ *is a 2-morphism from* $\mathrm{id}_{(A,B)}$ *to* $\Delta^*(g,\varphi)$ *in* $Z^2_Y(\mathfrak{G})$*, and* f *is a 2-morphism from* $\pi_{23}^*(g,\varphi)\circ\pi_{12}^*(g,\varphi)$ *to* $\pi_{13}^*(g,\varphi)$*. A 1-morphism from* $((A,B),(g,\varphi),\psi,f)$ *to* $((A',B'),(g',\varphi'),\psi',f')$ *is tuple* $((h,\phi),\epsilon)$*, where* (h,ϕ) *is a 1-morphism from* (A,B) *to* (A',B') *in* $Z^2_Y(\mathfrak{G})$ *and* ϵ *is a 2-morphism from* $\pi_2^*(h,\phi)\circ(g,\varphi)$ *to* $(g',\varphi')\circ\pi_1^*(h,\phi)$ *in* $Z^2_{Y^{[2]}}(\mathfrak{G})$*. A 2-morphism from* $((h,\phi),\epsilon)$ *to* $((h',\phi'),\epsilon')$ *is a 2-morphism* E *from* (h,ϕ) *to* (h',ϕ') *in* $Z^2_Y(\mathfrak{G})$*.*

The above generalizations produce functors

$$Z^2_\pi(\mathfrak{G})^\infty \underset{\mathcal{D}_\pi}{\overset{\mathcal{P}_\pi}{\rightleftarrows}} \mathfrak{Des}^2_\pi(i)^\infty \tag{160}$$

exhibiting an equivalence of 2-categories whenever $i : \mathcal{B}\mathfrak{G} \longrightarrow T$ is an equivalence.

3.35. Direct limits. In this section, we get rid of the dependence on the surjective submersion in the categories introduced in the prequel. Several of our 2-categories depended on the choice of a surjective submersion. These 2-categories were $\mathrm{Triv}^2_\pi(i)^\infty, \mathfrak{Des}^2_\pi(i)^\infty$, and $Z^2_\pi(\mathfrak{G})^\infty$. Changing the surjective submersion gives a collection of 2-categories dependent on this surjective submersion. One can take a limit over the collection of surjective submersions in this case. This will be a slight generalization of what was done in Section 2.30. However, there are subtle issues in terms of defining the many compositions.

The general construction is done as follows. Let S_π be a family of 2-categories parametrized by surjective submersions $\pi : Y \longrightarrow M$ and let $F(\zeta) : S_\pi \longrightarrow S_{\pi\circ\zeta}$ be a family of 2-functors for every refinement $\zeta : Y' \longrightarrow Y$ of π satisfying the condition that for any iterated refinement $\zeta' : Y'' \longrightarrow Y'$ and $\zeta : Y' \longrightarrow Y$ of $\pi : Y \longrightarrow M$ then $F(\zeta'\circ\zeta) = F(\zeta')\circ F(\zeta)$. In this case, an object of $S_M := \varinjlim_\pi S_\pi$ is given by a pair (π, X) of a surjective submersion $\pi : Y \longrightarrow M$ and an object X of S_π. A 1-morphism from (π_1, X_1) to (π_2, X_2) consists of a common refinement

$$\begin{tikzcd} & Z \arrow[dl, "y_1"'] \arrow[dd, "\zeta"] \arrow[dr, "y_2"] & \\ Y_1 \arrow[dr, "\pi_1"'] & & Y_2 \arrow[dl, "\pi_2"] \\ & M & \end{tikzcd} \tag{161}$$

together with a 1-morphism $f : (F(y_1))(X_1) \longrightarrow (F(y_2))(X_2)$ in S_ζ. It is written as a pair (ζ, f). The composition

$$(\pi_3, X_3) \xleftarrow{(\zeta_{23}, g)} (\pi_2, X_2) \xleftarrow{(\zeta_{12}, f)} (\pi_1, X_1) \tag{162}$$

consists of the pullback refinement

$$\begin{tikzcd} & & Z_{13} \arrow[dl, "\mathrm{pr}_{12}"'] \arrow[dr, "\mathrm{pr}_{23}"] & & \\ & Z_{12} \arrow[dl] \arrow[dr] \arrow[ddr, "\zeta_{12}"'] & & Z_{23} \arrow[dl] \arrow[dr] \arrow[ddl, "\zeta_{23}"] & \\ Y_1 \arrow[drr, "\pi_1"'] & & Y_2 \arrow[d, "\pi_2"] & & Y_3 \arrow[dll, "\pi_3"] \\ & & M & & \end{tikzcd} \tag{163}$$

along with the composition $(F(\mathrm{pr}_{23}))(g) \circ (F(\mathrm{pr}_{12}))(f)$. A 2-morphism from (ζ, f) to (ζ', f') consists of an equivalence class of pairs (ω, α), where ω is a common refinement of ζ and ζ' as in the following diagram

$$
\begin{tikzcd}
& & W \arrow[dl, "z"'] \arrow[dr, "z'"] \arrow[ddd, "\omega"] & & \\
& Z \arrow[dl, "y_1"'] \arrow[drrr, "y_2"] \arrow[ddr, "\zeta"] & & Z' \arrow[dlll, "y_1'"'] \arrow[dr, "y_2'"] \arrow[ddl, "\zeta'"'] & \\
Y_1 \arrow[drr, "\pi_1"'] & & & & Y_2 \arrow[dll, "\pi_2"] \\
& & M & &
\end{tikzcd}
\tag{164}
$$

and α is a 2-morphism $\alpha : F(z)(f) \Rightarrow F(z')(f')$. Two such pairs (ω_1, α_1) and (ω_2, α_2) are equivalent if they agree on the pullback.

After getting rid of the specific choices of the surjective submersions, we can take the limits of all the categories we have introduced. We make the following notation, slightly differing from that of [ScWa13]:

$$\mathrm{Triv}^2_M(i)^\infty := \varinjlim_{\pi} \mathrm{Triv}^2_\pi(i)^\infty \tag{165}$$

$$\mathfrak{Des}^2_M(i)^\infty := \varinjlim_{\pi} \mathfrak{Des}^2_\pi(i)^\infty \tag{166}$$

$$Z^2(M;\mathfrak{G})^\infty := \varinjlim_{\pi} Z^2_\pi(\mathfrak{G})^\infty. \tag{167}$$

Then from our previous discussions, we collect the functors we have introduced relating all these categories to $\mathrm{Trans}^2_{\mathcal{B}\mathfrak{G}}(M,T)$ after taking such limits over surjective submersions:

$$Z^2(M;\mathfrak{G})^\infty \mathrel{\underset{\mathcal{D}}{\overset{\mathcal{P}}{\rightleftarrows}}} \mathfrak{Des}^2_M(i)^\infty \mathrel{\underset{\mathrm{Ex}^2}{\overset{\mathrm{Rec}^2}{\rightleftarrows}}} \mathrm{Triv}^2_M(i)^\infty \mathrel{\underset{c}{\overset{v}{\rightleftarrows}}} \mathrm{Trans}^2_{\mathcal{B}\mathfrak{G}}(M,T) \tag{168}$$

Under the conditions that $i : \mathcal{B}\mathfrak{G} \longrightarrow T$ is an equivalence of categories, all of the above 2-functors are equivalence pairs. Without the smoothness assumptions, a simpler version of some of these equivalences is proven in Proposition 4.2.1. and Theorem 4.2.2. of [ScWa] while the equivalences in (168) are proven in Theorem 3.2.2., Lemma 3.2.3., and Lemma 3.2.4. of [ScWa13]. Completely analogous versions of comments regarding the assumptions on i made before (59) apply here as well.

3.36. Surface transport, 2-holonomy, and gauge invariance. In Section 2.31, we described a procedure that began with a transport functor and produced a group-valued parallel transport operator for thin loops with markings. We discovered that the value of holonomy changed by conjugation depending on the markings for the loops, the choice of a local trivialization procedure, and by using an isomorphic transport functor.

In this section, we will analyze holonomy along surfaces in an analogous manner. The main difference is that bigons have source and target paths so that a closed surface has a marking of one lower dimension, and is therefore not in general just a point as it was for loops. For the examples we give later in this paper, we specialize to spheres with a point marking. Such a surface is depicted as a bigon from the constant loop at a point x to itself (see Figure 16 below and [HoTs93]). However, a sphere can be more generally

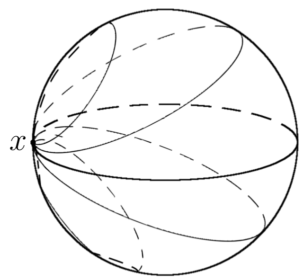


Figure 16: A based sphere viewed as a bigon $\Sigma : \mathrm{id}_x \Rightarrow \mathrm{id}_x$.

described as a bigon from a loop to itself, so we analyze parallel transport for such bigons to cover these extra cases. This analysis is completely independent of what types of Lie 2-groups $\mathcal{B}\mathfrak{G}$ we use. For simplicity, we assume that $i : \mathcal{B}\mathfrak{G} \longrightarrow T$ is a full and faithful 2-functor. This will differ from the presentation in Section 5 of [ScWa13], where surface holonomy was defined using the reduced 2-group. We will *not* be making this restriction.

3.37. Definition. *A* <u>2-group-valued transport extraction</u> *is a composition of functors (starting at the left and moving clockwise)*

$$\begin{tikzcd} & \mathrm{Triv}^2(i)^\infty \arrow[dr, bend left, "\mathrm{Ex}^2"] & \\ \mathrm{Trans}^2_{\mathcal{B}\mathfrak{G}}(M,T) \arrow[ur, bend left, "c"] & & \mathfrak{Des}^2(i)^\infty \arrow[dl, bend left, "\mathrm{Rec}^2"] \\ & \mathrm{Triv}^2(i)^\infty \arrow[ul, bend left, "v"] & \end{tikzcd} . \tag{169}$$

We write the composition (169) as $\mathscr{t}$. By the reconstruction procedure of Section 3.25, $\mathscr{t}$ assigns G-valued elements to thin paths for every transport functor F as well as H-valued elements to thin bigons (more on this below). Technically, thin bigons will be assigned elements in $H \rtimes G$ but as is discussed in Section 3.1, particularly after the proof of Theorem 3.3, such elements are completely determined by their source, an element of G, and their projection in H. $\mathscr{t}$ will also assign G-valued and H-valued gauge transformations for every 1-morphism $\eta : F \longrightarrow F'$ of transport functors. In addition, $\mathscr{t}$ will assign H-valued 2-gauge transformations for every 2-morphism $A : \eta \Rightarrow \eta'$. A pseudo-natural equivalence $\mathscr{s} : \mathrm{id} \Rightarrow \mathscr{t}$ describes how to relate the transport functor to the locally trivialized one. Although modifications of pseudo-natural transformations are allowed, we will not analyze them in this paper. Such modifications are to be interpreted as relating the two different ways of choosing the pseudo-natural transformations that relate the transport functor to the locally trivialized one.

Just as before, we briefly review what the composition of 2-functors defining $\not\ell$ are. For a transport 2-functor F, we choose a local trivialization $c(F) = (\pi, F, \mathrm{triv}, t)$. Then we extract the local descent object $\mathrm{Ex}^2(\pi, F, \mathrm{triv}, t) = (\pi, \mathrm{triv}, g, \psi, f)$. Then, we reconstruct a transport 2-functor $\mathrm{Rec}^2(\pi, \mathrm{triv}, g, \psi, f)$ and then forget the trivialization data keeping just the 2-functor $v(\mathrm{Rec}^2(\pi, \mathrm{triv}, g, \psi, f))$. The resulting transport 2-functor, written as $\not\ell_F$, is defined by (see Section 3.25)

$$\begin{aligned} \mathcal{P}_2(M) &\xrightarrow{\not\ell_F} T \\ M \ni x &\mapsto i(\bullet) =: \mathrm{triv}_i(s^\pi(x)), \\ P^1 M \ni \gamma &\mapsto R_{\mathrm{Ex}^2(c(F))}(s^\pi(\gamma)), \text{ and} \\ P^2 M \ni \Sigma &\mapsto R_{\mathrm{Ex}^2(c(F))}(s^\pi(\Sigma)). \end{aligned} \tag{170}$$

Points in M get sent to $i(\bullet)$ by construction. Because i is full and faithful, the 1-morphisms $R_{\mathrm{Ex}^2(c(F))}(s^\pi(\gamma)) : i(\bullet) \longrightarrow i(\bullet)$ determine unique elements of G. Similarly, the 2-morphisms $R_{\mathrm{Ex}^2(c(F))}(s^\pi(\Sigma))$ determine unique elements in H.

The interested reader can explicitly define the compositor and the unitor for the 2-functor $\not\ell_F$. We won't need the precise details for our analysis when studying surface holonomy. All we need to know is that the 2-functors defining $\not\ell$ are (weakly) invertible.

We'd like to restrict surface holonomy to thin homotopy classes of marked spheres for the purpose of this paper (in general, one would like to restrict to the more general space of thin homotopy classes of marked closed surfaces) and eventually thin free spheres. First we make a definition of the thin marked sphere space, which should be thought of as analogous to the thin marked loop space.

3.38. Definition. *The* <u>marked sphere space of M</u> *is the set*

$$\mathfrak{S}M := \{\Sigma \in BM \mid s(\Sigma) = t(\Sigma) \text{ and } s(s(\Sigma)) = t(t(\Sigma))\} \tag{171}$$

equipped with the subspace smooth structure. Elements of $\mathfrak{S}M$ are called <u>marked spheres</u>. *Similarly, the* <u>thin marked sphere space of M</u> *is the smooth space*

$$\mathfrak{S}^2 M = \{\Sigma \in P^2 M \mid s(\Sigma) = t(\Sigma) \text{ and } s(s(\Sigma)) = t(t(\Sigma))\}. \tag{172}$$

Elements of $\mathfrak{S}^2 M$ are called <u>thin marked spheres</u>.

3.39. Remark. *Note that elements of $\mathfrak{S}^2 M$ need not look like embedded spheres in M. Indeed, they might look like pinched croissants as Figure 17 indicates (or worse). This won't matter in any of our calculations or proofs.*

3.40. Definition. *The* <u>$\not\ell$-2-holonomy of F</u>, *written as $\mathrm{hol}^F_{\not\ell}$, is defined as the projection to H of the restriction of parallel transport of a transport 2-functor F to the thin marked sphere space of M:*

$$\mathrm{hol}^F_{\not\ell} := p_H \circ \not\ell_F \Big|_{\mathfrak{S}^2 M} : \mathfrak{S}^2 M \longrightarrow H. \tag{173}$$

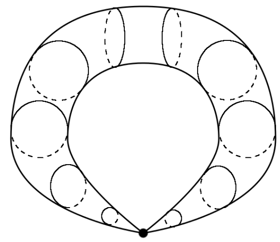


Figure 17: A pinched croissant is an example of a thin marked sphere.

3.41. Remark. *Note that* hol^F_{t} *is the same notation used for thin loops with values in* G. *This should cause no confusion because thin loops are always written using lower case Greek letters such as* γ, δ, *etc. while thin spheres are written using upper case Greek letters such as* Σ, Γ, *etc.*

We now pose three questions analogous to those for 1-holonomy.

i) How does hol^F_{t} depend on the choice of a thin marked sphere? Namely, suppose that two thin marked spheres Σ and Σ', with possibly different markings, are thinly homotopic *without preserving the marking* (see Definition 3.42). Then, how is $\mathrm{hol}^F_{t}(\Sigma)$ related to $\mathrm{hol}^F_{t}(\Sigma')$?

ii) How does hol^F_{t} depend on F? Namely, suppose that $\eta : F \longrightarrow F'$ is a morphism of transport functors. How is hol^F_{t} related to $\mathrm{hol}^{F'}_{t}$ in terms of η?

iii) How does hol^F_{t} depend on t, the choice of trivialization? Namely, suppose that t' is another trivialization. Then how is hol^F_{t} related to $\mathrm{hol}^F_{t'}$?

Due to the fact that we are restricting ourselves to marked spheres instead of arbitrary surfaces, the answer will be closely related to the 1-holonomy case and will be given by a generalized version of conjugation. As before, we need to define what we mean by thin free sphere space and then we'll proceed to answer the above questions. Denote the smooth space of spheres in M by $SM = \{\Sigma : S^2 \longrightarrow M \mid \Sigma \text{ is smooth}\}$.

3.42. Definition. *Two smooth spheres* Σ *and* Σ' *in* M *are* thinly homotopic *if there exists a smooth map* $h : S^2 \times [0,1] \longrightarrow M$ *such that*

i) there exists an $\epsilon > 0$ *with* $h(t,s) = \Sigma(t)$ *for* $s \leqslant \epsilon$ *and* $h(t,s) = \Sigma'(t)$ *for* $s \geqslant \epsilon$ *and for all* $t \in S^2$ *and*

ii) the smooth map h *has rank* $\leqslant 2$.

The space of equivalences classes is denoted by S^2M *and is called the* thin free sphere space of M. *Elements of* S^2M *are called* thin spheres.

3.43. Definition. *Define a function* $f : \mathfrak{S}M \longrightarrow SM$ *by sending a marked sphere* $\Sigma : [0,1] \times [0,1] \longrightarrow M$ *to the associated smooth map* $f(\Sigma) : S^2 \longrightarrow M$ *obtained from identifying the top and bottom of the second interval and then pinching the two ends (see Figure 18).* f *is called the* forgetful map.

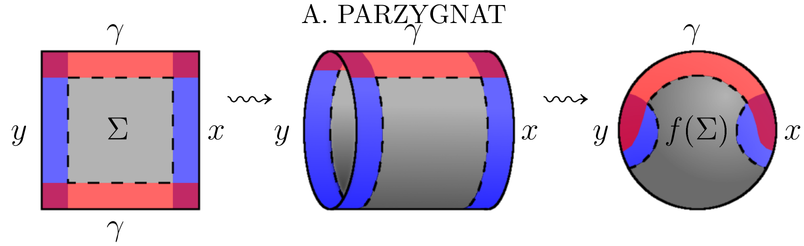


Figure 18: The definition of $f : \mathfrak{S}M \to SM$. This definition makes sense even when $y \neq x$. $y = x$ is a special case.

3.44. Lemma. *There exists a unique map* $f^2 : \mathfrak{S}^2 M \longrightarrow S^2 M$ *such that the diagram*

$$\begin{tikzcd} \mathfrak{S}M \arrow[r] \arrow[d, "f"'] & \mathfrak{S}^2 M \arrow[d, "f^2"] \\ SM \arrow[r] & S^2 M \end{tikzcd} \tag{174}$$

commutes (the horizontal arrows are the projections onto thin homotopy classes).

Proof. The proof is analogous to the case of loops. One chooses a representative, applies f, and then projects. The map is well-defined by the thin homotopy equivalence relation on $S^2 M$. ∎

Note that there is also a function $\mathrm{ev}_1 : \mathfrak{S}^2 M \longrightarrow \mathfrak{L}^1 M$ given by evaluating a thin marked sphere at its source/target. This function forgets the sphere and remembers only the source thin marked loop.

3.45. Definition. *A* marking of thin spheres *is a section* $\mathfrak{m} : S^2 M \longrightarrow \mathfrak{S}^2 M$ *of* $f^2 : \mathfrak{S}^2 M \longrightarrow S^2 M$.

3.46. Lemma. *A marking of thin spheres exists.*

Proof. Let $[\Sigma] \in \mathfrak{S}^2 M$ be a thin sphere and choose representative $\Sigma : S^2 \longrightarrow M$ in SM. Pick a point $\bullet$ on the equator viewed as a loop $\ell : \bullet \longrightarrow \bullet$. The image of ℓ under Σ defines a loop, $\gamma : x \longrightarrow x$, where $x := \Sigma(\bullet)$. There exists a thin homotopy $h : S^2 \times [0,1] \longrightarrow M$ from Σ to a smooth map $\Sigma_\ell : S^2 \longrightarrow M$ such that the family of loops in Figure 19 on the domain of Σ_ℓ define a marked sphere $\tilde{\Sigma} : \gamma \Rightarrow \gamma$. Projecting to thin marked spheres defines $\mathfrak{m}([\Sigma])$. To see that this is well-defined, let $\Sigma' \in SM$ be another representative. Then there exists a thin unmarked homotopy $\tilde{h} : \Sigma' \Rightarrow \Sigma$. Composing this with the thin homotopy h gives $h \circ \tilde{h} : \Sigma' \Rightarrow \Sigma_\ell$. By the thin homotopy equivalence relation on $S^2 M$, this defines a section of f^2. ∎

We now proceed to answering the above questions in order.

i) Let $\mathfrak{m}, \mathfrak{m}' : S^2 M \longrightarrow \mathfrak{S}^2 M$ be two markings for thin spheres in M. Let $[\Sigma] \in S^2 M$ be a thin sphere and let $\Sigma : \gamma \Rightarrow \gamma$ with $\gamma : x \longrightarrow x$ be a representative of $\mathfrak{m}([\Sigma])$ and $\Sigma' : \gamma' \Rightarrow \gamma'$ with $\gamma' : x' \longrightarrow x'$ be a representative of $\mathfrak{m}'([\Sigma])$. Note that these

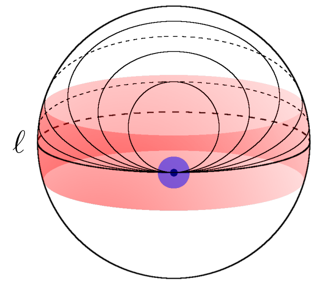


Figure 19: By a thin homotopy, the region around the equator is made to sit at the loop ℓ around the equator so that the nearby loops drawn in the shaded region agree with ℓ. The family of all these loops define a marking.

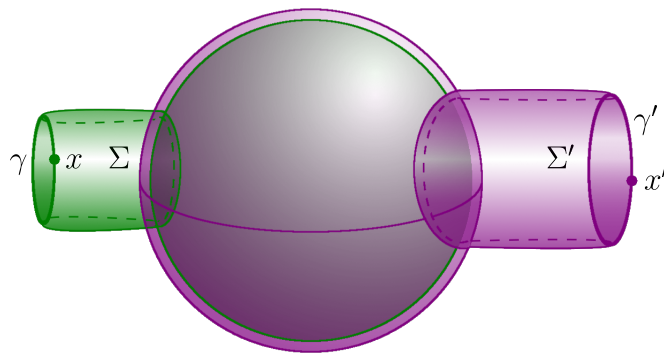


Figure 20: Two different representatives Σ (the 'inner' sphere in green extending left) and Σ' (the 'outer' sphere in purple extending right) of two markings of a thin sphere are shown. The extensions do *not* enclose any volume so that both spheres are thinly homotopic. Their respective sources are $\gamma : x \to x$ and $\gamma' : x' \to x'$, neither of which lie on the other's image. Compare this to Figure 23 where the two marked loops *do* lie on a common sphere.

representatives need not have associated marked loops that lie on some common image. Figure 20 depicts such a possible situation.

As in the case of loops, we can use thin homotopy to draw both marked loops on the same sphere (a more precise statement will be given shortly). First notice that there is a thin homotopy $h : S^2 \times [0,1] \longrightarrow M$ with $h(\,\cdot\,, s) = \Sigma$ for $s \leqslant \epsilon$ and $h(\,\cdot\,, s) = \Sigma'$ for $s \geqslant 1-\epsilon$ for some $\epsilon > 0$. Such a homotopy allows us to choose a sphere $\tilde{\Sigma} \in SM$, a path $\gamma_{x'x} : x \longrightarrow x'$, and three *bigons* $\Sigma_{\gamma x} : \mathrm{id}_x \Rightarrow \gamma$, $\Sigma_{x'\gamma'} : \gamma' \Rightarrow \mathrm{id}_{x'}$, and $\Delta : \gamma_{x'x} \circ \gamma \circ \overline{\gamma_{x'x}}$ with the following properties. First $\tilde{\Sigma}$ can be expressed as either of the compositions

$$f\left(\begin{array}{c} \Sigma_{\gamma' x'} \\ \circ \\ \mathrm{id}_{\gamma_{x'x}} \circ \Sigma_{\gamma x} \circ \mathrm{id}_{\overline{\gamma_{x'x}}} \\ \circ \\ \Delta \end{array}\right) \qquad \text{or} \qquad f\left(\begin{array}{c} \mathrm{id}_{\overline{\gamma_{x'x}}} \circ \Delta \circ \mathrm{id}_{\gamma_{x'x}} \\ \circ \\ \mathrm{id}_{\overline{\gamma_{x'x}}} \circ \Sigma_{\gamma' x'} \circ \mathrm{id}_{\gamma_{x'x}} \\ \circ \\ \Sigma_{\gamma x} \end{array}\right) \tag{175}$$

(in either order vertically). Second, the composition of *bigons*

$$\begin{array}{c} \mathrm{id}_{\overline{\gamma_{x'x}}} \circ \Delta \circ \mathrm{id}_{\gamma_{x'x}} \\ \circ \\ \mathrm{id}_{\overline{\gamma_{x'x}}} \circ \Sigma_{\gamma' x'} \circ \mathrm{id}_{\gamma_{x'x}} \\ \circ \\ \Sigma_{\gamma x} \end{array} \tag{176}$$

is thinly homotopic to Σ preserving the marked loop $\gamma : x \longrightarrow x$. Third, the composition of *bigons*

$$\begin{array}{c} \Sigma_{\gamma' x'} \\ \circ \\ \mathrm{id}_{\gamma_{x'x}} \circ \Sigma_{\gamma x} \circ \mathrm{id}_{\overline{\gamma_{x'x}}} \\ \circ \\ \Delta \end{array} \tag{177}$$

is thinly homotopic to Σ' preserving the marked loop $\gamma' : x' \longrightarrow x'$. This is depicted in Figures 21 and 22.

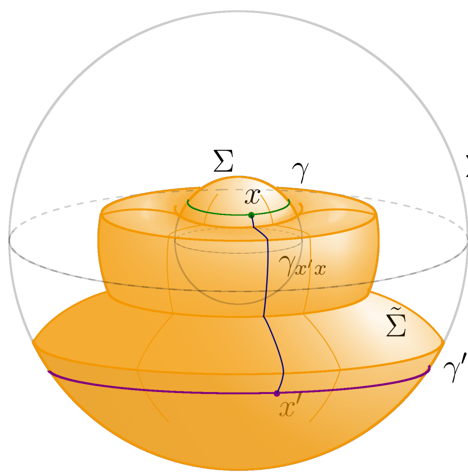


Figure 21: The domain of the homotopy $h : S^2 \times [0, 1] \to M$ is drawn as a solid ball with a smaller solid ball removed from the center. It depicts Σ as the inner sphere and Σ' as the outer sphere. The marked loop $\gamma : x \to x$ of Σ is drawn on the northern hemisphere while the marked loop $\gamma' : x' \to x'$ of Σ' is drawn on the southern hemisphere (by a thin homotopy, one can always position the marked loops in this way). The homotopy h allows us to choose a sphere $\tilde{\Sigma}$, drawn somewhat in the middle (in orange), that contains both based loops γ and γ' and is thinly homotopic to both Σ and Σ'. As a result, there exists a path $\gamma_{x'x} : x \to x'$ on $\tilde{\Sigma}$. We continue this analysis in Figure 22.

These last two equations let us write the bigon Σ in terms of Σ' and vice versa. In fact, we have

$$\Sigma' = \begin{array}{c} \overline{\Delta} \\ \circ \\ \mathrm{id}_{\gamma_{x'x}} \circ \Sigma \circ \mathrm{id}_{\overline{\gamma_{x'x}}} \\ \circ \\ \Delta \end{array} \tag{178}$$

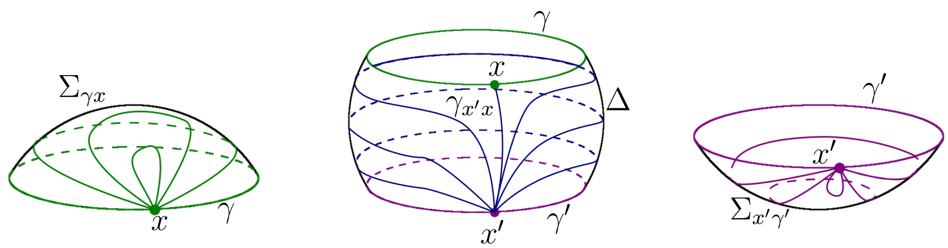


Figure 22: From the sphere $\tilde{\Sigma}$ in Figure 21, the top cap defines a bigon $\Sigma_{\gamma x} : \mathrm{id}_x \Rightarrow \gamma$, drawn on the left in this figure. The path $\gamma_{x'x} : x \to x'$ in Figure 21 defines a bigon $\Delta : \gamma_{x'x} \circ \gamma \circ \overline{\gamma_{x'x}} \Rightarrow \gamma'$ drawn in the middle of this figure. The bottom cap defines a bigon $\Sigma_{x'\gamma'} : \gamma' \Rightarrow \mathrm{id}_{x'}$ drawn on the right.

up to thin homotopy preserving the marked loop $\gamma' : x' \longrightarrow x'$. There is also a similar expression for Σ preserving the marked loop $\gamma : x \longrightarrow x$.

The above argument says that given two marked spheres, with possibly different markings, that are thinly homotopic *without* preserving the markings, one can always choose a representative of such a thin sphere in M with two marked loops so that the associated two *marked* spheres (coming from starting at either marking) are thinly homotopic to the original two with a thin homotopy that preserves the marking. More precisely, we proved the following.

3.47. Lemma. *Let $\mathfrak{m}, \mathfrak{m}' : S^2M \longrightarrow \mathfrak{S}^2M$ be two markings. Let $[\Sigma] \in S^2M$ be a thin sphere in M and write $[\gamma] : x \longrightarrow x$ for $\mathrm{ev}_1(\mathfrak{m}([\Sigma]))$ and $[\gamma'] : x' \longrightarrow x'$ for $\mathrm{ev}_1(\mathfrak{m}'([\Sigma]))$. Then, there exists representatives γ and γ' of $[\gamma]$ and $[\gamma']$, respectively, a path $\gamma_{x'x} : x \longrightarrow x'$ with sitting instants and three bigons $\Sigma_{\gamma x} : \mathrm{id}_x \Rightarrow \gamma$, $\Sigma_{x'\gamma'} : \gamma' \Rightarrow \mathrm{id}_{x'}$, and $\Delta : \gamma_{x'x} \circ \gamma \circ \overline{\gamma_{x'x}} \Rightarrow \gamma'$, such that the following three properties hold (see Figure 23).*

i) The vertical composition of $\Sigma_{\gamma' x'}$, $\mathrm{id}_{\gamma_{x'x}} \circ \Sigma_{\gamma x} \circ \mathrm{id}_{\overline{\gamma_{x'x}}}$, and Δ in the order given (or a cyclic permutation of this order) and forgetting the marking is a representative of $[\Sigma]$.

ii) $\begin{pmatrix} \mathrm{id}_{\overline{\gamma_{x'x}}} \circ \Delta \circ \mathrm{id}_{\gamma_{x'x}} \\ \circ \\ \mathrm{id}_{\overline{\gamma_{x'x}}} \circ \Sigma_{\gamma' x'} \circ \mathrm{id}_{\gamma_{x'x}} \\ \circ \\ \Sigma_{\gamma x} \end{pmatrix}$ *is a representative of $\mathfrak{m}([\Sigma])$ as a bigon.*

iii) $\begin{pmatrix} \Sigma_{\gamma' x'} \\ \circ \\ \mathrm{id}_{\gamma_{x'x}} \circ \Sigma_{\gamma x} \circ \mathrm{id}_{\overline{\gamma_{x'x}}} \\ \circ \\ \Delta \end{pmatrix}$ *is a representative of $\mathfrak{m}'([\Sigma])$ as a bigon.*

Therefore, without loss of generality, we can choose a *single* representative $\tilde{\Sigma}$ of a thin free sphere $[\Sigma]$ with a decomposition as in the Lemma. We use Σ to denote the bigon in ii) of Lemma 3.47 and Σ' to denote the bigon in iii). The two are related by

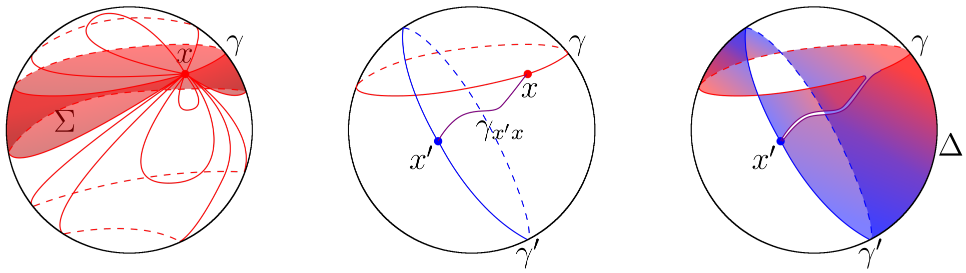


Figure 23: For every thin sphere and two markings, there exists a representative with a decomposition as in Lemma 3.47. On the left is a bigon $\Sigma : \gamma \Rightarrow \gamma$ with $\gamma : x \to x$. The shaded region depicts the surface swept out between $s = 0$ and some small s. In the middle is another bigon $\Sigma' : \gamma' \Rightarrow \gamma'$ with $\gamma' : x' \to x'$ and a path $\gamma_{x'x} : x \to x'$ with sitting instants. On the right is a bigon $\Delta : \gamma_{x'x} \circ \gamma \circ \overline{\gamma_{x'x}} \Rightarrow \gamma'$ relating the two marked loops as in (180).

$$\begin{tikzcd} x' & x' \arrow[l, bend right, "\gamma'"'] \arrow[l, bend left, "\gamma'"] \arrow[l, phantom, "\Downarrow\Sigma'"] \end{tikzcd} \quad = \quad \begin{tikzcd} x' & x \arrow[l, "\gamma_{x'x}"'] & x \arrow[l, bend right, "\gamma"'] \arrow[l, bend left, "\gamma"] \arrow[l, phantom, "\Downarrow\Sigma"] & x' \arrow[l, "\overline{\gamma_{x'x}}"'] \arrow[lll, bend right=60, "\gamma'"'] \arrow[lll, bend left=60, "\gamma'"] \end{tikzcd} \tag{179}$$

i.e.

$$\Sigma_y = \mathrm{id}_{\gamma_{x'x}} \circ \begin{array}{c} \overline{\Delta} \\ \circ \\ \Sigma \\ \circ \\ \Delta \end{array} \circ \mathrm{id}_{\overline{\gamma_{x'x}}}. \tag{180}$$

By functoriality of the transport 2-functor $\not{\ell}_F$, we have

$$\begin{aligned} \mathrm{hol}^F_{\ell}(\Sigma') &= p_H\left(\not{\ell}_F(\Sigma')\right) \\ &= p_H\left(\begin{array}{c} \not{\ell}_F(\overline{\Delta}) \\ C^{-1} \\ \mathrm{id}_{\not{\ell}_F(\gamma_{yx})}\not{\ell}_F(\Sigma_x)\mathrm{id}_{\not{\ell}_F(\overline{\gamma_{yx}})} \\ C \\ \not{\ell}_F(\Delta) \end{array}\right), \end{aligned} \tag{181}$$

where $C : \not{\ell}_F(\gamma_{x'x})\not{\ell}_F(\gamma_x)\not{\ell}_F(\overline{\gamma_{x'x}}) \Rightarrow \not{\ell}_F(\gamma_{x'x} \circ \gamma_x \circ \overline{\gamma_{x'x}})$ is a combination of compositors and associators. Writing out this composition in the 2-group $\mathcal{B}\mathfrak{G}$ gives

$$\begin{array}{c} ((p_H(\not{\ell}_F(\Delta)))^{-1}, \not{\ell}_F(\gamma')) \\ (p_H(C)^{-1}, \not{\ell}_F(\gamma_{x'x} \circ \gamma \circ \overline{\gamma_{x'x}})) \\ (e, \not{\ell}_F(\gamma_{x'x}))\left(\mathrm{hol}^F_{\ell}(\Sigma), \not{\ell}_F(\gamma)\right)(e, \not{\ell}_F(\overline{\gamma_{x'x}})) \\ (p_H(C), \not{\ell}_F(\gamma_{x'x})\not{\ell}_F(\gamma)\not{\ell}_F(\overline{\gamma_{x'x}})) \\ (p_H(\not{\ell}_F(\Delta)), \not{\ell}_F(\gamma_{x'x} \circ \gamma \circ \overline{\gamma_{x'x}})) \end{array}. \tag{182}$$

Multiplying these results out using the rules of 2-group multiplication (see equations (83) and (84)) and taking the H component gives

$$\begin{aligned}\mathrm{hol}^F_{\ell}(\Sigma') &= p_H(\ell_F(\Delta))p_H(C)\alpha_{\ell_F(\gamma_{x'x})}\left(\mathrm{hol}^F_{\ell}(\Sigma)\right)p_H(C)^{-1}\left(p_H\left(\ell_F(\Delta)\right)\right)^{-1} \\ &= \alpha_{\tau(p_H(\ell_F(\Delta))p_H(C))\ell_F(\gamma_{x'x})}\left(\mathrm{hol}^F_{\ell}(\Sigma)\right).\end{aligned} \tag{183}$$

This result says that the 2-holonomy changes by α-conjugation under a change of marking for a thin sphere.

ii) Now suppose that $\eta : F \longrightarrow F'$ is a 1-morphism of transport 2-functors. Then, for every thin path $\gamma : x \longrightarrow y$ we have a 2-isomorphism (remember that $\ell_{F'}(x) = i(\bullet)$ and $\ell_F(x) = i(\bullet)$ for all $x \in M$)

$$\begin{tikzcd} i(\bullet) \arrow[d, "\ell_{F'}(\gamma)"'] & i(\bullet) \arrow[l, "\ell_\eta(x)"'] \arrow[d, "\ell_F(\gamma)"] \\ i(\bullet) & i(\bullet) \arrow[l, "\ell_\eta(y)"] \arrow[ul, Rightarrow, "\ell_\eta(\gamma)"] \end{tikzcd} \tag{184}$$

satisfying the condition that for any thin bigon $\Sigma : \gamma \Rightarrow \delta$, with $\delta : x \longrightarrow y$ another path, the diagram

$$\begin{tikzcd}[column sep=huge, row sep=huge] i(\bullet) \arrow[d, bend right=60, "\ell_{F'}(\delta)"'] \arrow[d, bend left=60, "\ell_{F'}(\gamma)"] \arrow[d, phantom, "\overset{\ell_{F'}(\Sigma)}{\Leftarrow}"] & i(\bullet) \arrow[l, "\ell_\eta(x)"'] \arrow[d, bend right=60, dashed, "\ell_F(\delta)"'] \arrow[d, bend left=60, "\ell_F(\gamma)"] \arrow[d, phantom, "\overset{\ell_F(\Sigma)}{\Leftarrow}"] \\ i(\bullet) & i(\bullet) \arrow[l, "\ell_\eta(y)"] \arrow[ul, Rightarrow, "\ell_\eta(\gamma)"] \end{tikzcd} \tag{185}$$

commutes. In this diagram, the $\ell_\eta(\delta)$ in the back is not shown. This diagram commuting means that

$$\begin{array}{c}\ell_\eta(\gamma) \\ \circ \\ \ell_{F'}(\Sigma)\mathrm{id}_{\ell_\eta(x)}\end{array} = \begin{array}{c}\mathrm{id}_{\ell_\eta(y)}\ell_F(\Sigma) \\ \circ \\ \ell_\eta(\delta)\end{array} \tag{186}$$

and writing this out using group elements gives

$$\begin{array}{c}(p_H(\ell_\eta(\gamma)), \ell_\eta(y)\ell_F(\gamma)) \\ (p_H(\ell_{F'}(\Sigma)), \ell_{F'}(\gamma))(e, \ell_\eta(x))\end{array} = \begin{array}{c}(e, \ell_\eta(y))(p_H(\ell_F(\Sigma)), \ell_F(\gamma)) \\ (p_H(\ell_\eta(\delta)), \ell_\eta(y)\ell_F(\delta))\end{array}, \tag{187}$$

which after evaluating both sides and projecting to H yields

$$p_H(\ell_{F'}(\Sigma))p_H(\ell_\eta(\gamma)) = p_H(\ell_\eta(\delta))\alpha_{\ell_\eta(y)}\left(p_H(\ell_F(\Sigma))\right). \tag{188}$$

Solving for $p_H(\mathscr{T}_{F'}(\Sigma))$ gives

$$p_H(\mathscr{T}_{F'}(\Sigma)) = p_H(\mathscr{T}_\eta(\delta))\alpha_{\mathscr{T}_\eta(y)}\left(p_H(\mathscr{T}_F(\Sigma))\right)p_H(\mathscr{T}_\eta(\gamma))^{-1}. \tag{189}$$

Now, after specializing to the case where the source and targets of Σ are all the same, i.e. $y = x$ and $\delta = \gamma$, so that we are comparing this transport along thin marked spheres, this reduces to

$$\begin{aligned} \mathrm{hol}^{F'}_{\mathscr{T}}(\Sigma) &= p_H(\mathscr{T}_\eta(\gamma))\alpha_{\mathscr{T}_\eta(x)}\left(\mathrm{hol}^{F}_{\mathscr{T}}(\Sigma)\right)p_H(\mathscr{T}_\eta(\gamma))^{-1} \\ &= \alpha_{\tau(p_H(\mathscr{T}_\eta(\gamma)))\mathscr{T}_\eta(x)}\left(\mathrm{hol}^{F}_{\mathscr{T}}(\Sigma)\right). \end{aligned} \tag{190}$$

This says that $\mathrm{hol}^F_{\mathscr{T}}$ when restricted to thin marked spheres changes under α-conjugation when the functor F is changed to a gauge equivalent one F'.

iii) Suppose that another 2-group transport extraction procedure $\mathscr{T}'$ was chosen. Any two such procedures are pseudo-naturally equivalent, i.e. if $\mathscr{T}'$ was another such choice, then there exists a weakly invertible pseudonatural transformation $\mathscr{S} : \mathscr{T}' \Rightarrow \mathscr{T}$. This follows from the fact that each 2-functor in the composition of 2-functors that define $\mathscr{T}$ is an equivalence of 2-categories and weak inverses are unique up to pseudo-natural equivalences. Therefore, for every transport 2-functor F we have a 1-morphism of transport functors $\mathscr{S}_F : \mathscr{T}_F' \longrightarrow \mathscr{T}_F$. Of course, we also have a map assigning to every 1-morphism of transport functors $\eta : F \longrightarrow F'$ a 2-morphism $\mathscr{S}_\eta : \mathscr{S}_F \Rightarrow \mathscr{S}_{F'}$ satisfying naturality, but we will not need this fact for the following observation because we are dealing with strict Lie 2-groups. The 1-morphism of transport functors $\mathscr{S}_F$ assigns to every point $x \in M$ a morphism $\mathscr{S}_F(x) : \mathscr{T}_F'(x) \longrightarrow \mathscr{T}_F(x)$ and to every path $\gamma : x \longrightarrow y$ a 2-isomorphism

$$\begin{tikzcd} i(\bullet) \arrow[d, "\mathscr{T}_F(\gamma)"'] & i(\bullet) \arrow[l, "\mathscr{S}_F(x)"'] \arrow[d, "\mathscr{T}_F'(\gamma)"] \arrow[dl, phantom, "\mathscr{S}_F(\gamma)"] \\ i(\bullet) & i(\bullet) \arrow[l, "\mathscr{S}_F(y)"] \arrow[ul, Rightarrow] \end{tikzcd} \tag{191}$$

satisfying the condition that for a thin bigon $\Sigma : \gamma \longrightarrow \delta$ between two thin paths $\gamma, \delta : x \longrightarrow y$ the diagram

$$\begin{tikzcd}[column sep=huge, row sep=huge] i(\bullet) \arrow[d, bend right=50, "\mathscr{T}_F(\delta)"'] \arrow[d, bend left=50, "\mathscr{T}_F(\gamma)"] \arrow[d, phantom, "\overset{\mathscr{T}_F(\Sigma)}{\Longleftarrow}"] & i(\bullet) \arrow[l, "\mathscr{S}_F(x)"'] \arrow[d, dashed, bend right=50, "\mathscr{T}_F'(\delta)"'] \arrow[d, bend left=50, "\mathscr{T}_F'(\gamma)"] \arrow[d, phantom, "\overset{\mathscr{T}_F'(\Sigma)}{\Longleftarrow}"] \\ i(\bullet) & i(\bullet) \arrow[l, "\mathscr{S}_F(y)"] \arrow[ul, Rightarrow, "\mathscr{S}_F(\gamma)" description] \end{tikzcd} \tag{192}$$

commutes. This result is very similar to the previous one and is given by

$$\mathrm{hol}^{F}_{\ell'}(\Sigma) = \alpha_{\tau(p_H(\jmath_F(\gamma)))\jmath_F(x)}\left(\mathrm{hol}^{F}_{\ell}(\Sigma)\right), \tag{193}$$

which is again just α-conjugation.

In conclusion, when restricted to a sphere, 2-holonomy changes under α-conjugation in each of the three situations described above. This should therefore also be called gauge covariance as in the case for loops. This motivates the following definition.

3.48. Definition. *Let (H, G, τ, α) be a crossed module. The* α-conjugacy classes in H*, denoted by H/α, is defined to be the quotient of H under the equivalence relation*

$$h \sim h' \iff \text{there exists a } g \in G \text{ such that } h = \alpha_g(h'). \tag{194}$$

Denote the quotient map by $q : H \longrightarrow H/\alpha$.

As before, we have a similar theorem for gauge-invariance of 2-holonomy.

3.49. Theorem. *Let M be a smooth manifold, $\mathcal{B}\mathfrak{G}$ a Lie 2-group, T a 2-category, and suppose that $i : \mathcal{B}\mathfrak{G} \longrightarrow T$ is a full and faithful 2-functor. Let F be a transport 2-functor and ℓ a 2-group-valued transport extraction. Let $S^2M, \mathfrak{S}^2M, \mathfrak{m}, \mathrm{hol}^F_{\ell}$ and q be defined as above. Then the composition*

$$H/\alpha \xleftarrow{q} H \xleftarrow{\mathrm{hol}^F_{\ell}} \mathfrak{S}^2M \xleftarrow{\mathfrak{m}} S^2M \tag{195}$$

is

i) *independent of* $\mathfrak{m}$,

ii) *independent of the equivalence class of F,*

iii) *and independent of the equivalence class of ℓ.*

This theorem lets us make the following definition.

3.50. Definition. *Let $[F]$ be an equivalence class of transport 2-functors. The* gauge invariant 2-holonomy *of $[F]$ is defined to be the smooth map in the previous theorem, namely*

$$\mathrm{hol}^{[F]} := q \circ \mathrm{hol}^F_{\ell} \circ \mathfrak{m} : S^2M \longrightarrow H/\alpha \tag{196}$$

where F is a representative of $[F]$, ℓ is a group-valued transport extraction, and $\mathfrak{m} : S^2M \longrightarrow \mathfrak{S}^2M$ is a marking for thin spheres in M. Let $\Sigma \in S^2M$. If $\mathrm{hol}^{[F]}(\Sigma)$ is such that $q^{-1}(\mathrm{hol}^{[F]}(\Sigma))$ is a single element, we will say that $\mathrm{hol}^{[F]}(\Sigma)$ is gauge invariant *and abusively write $\mathrm{hol}^{[F]}(\Sigma)$ instead of $q^{-1}(\mathrm{hol}^{[F]}(\Sigma))$.*

3.51. Remark. *A result analogous to Theorem 3.49 was obtained in the context of a cubical category approach to 2-bundles in [MaPi11].*

We now compare this result to that in [ScWa13], where the *reduced group* associated to a 2-group was introduced in order to obtain a well-defined 2-holonomy independent of the marking as well as the representative of the transport functor used.

3.52. Definition. *Let $\mathcal{B}\mathfrak{G}$ be a 2-group with associated crossed module (H, G, τ, α). The* reduced group of $\mathcal{B}\mathfrak{G}$ *is $\mathfrak{G}_{\mathrm{red}} := H/[G, H]$, where $[G, H] = \langle h^{-1}\alpha_g(h) \mid g \in G, h \in H\rangle$, i.e. the subgroup of H generated by elements of the form $h^{-1}\alpha_g(h)$.*

The analogue of the reduced 2-group in the case of ordinary holonomy for principal G bundles with connection is $G/[G, G]$, the abelianization of G. Recall, $[G, G] = \langle gg'g^{-1}g'^{-1} \mid g, g' \in G\rangle$ is a normal subgroup, called the *commutator subgroup*, of G so the quotient is an abelian group, in fact in a universal sense.

3.53. Lemma. *Let G be a group, $[G, G]$ its commutator subgroup, and $G/\mathrm{Inn}(G)$ conjugacy classes in G. The map $G/\mathrm{Inn}(G) \longrightarrow G/[G, G]$ given by taking a conjugacy class $[g]$, choosing a representative, and projecting to the quotient $G/[G, G]$, is*

i) well-defined,

ii) surjective,

iii) and need not be injective in general.

Proof.

i) The map $G/\mathrm{Inn}(G) \longrightarrow G/[G, G]$ is well-defined because if g' was another representative of $[g]$, then there would be a $\tilde{g} \in G$ such that $\tilde{g}g\tilde{g}^{-1} = g'$, and under the quotient map, the difference between g and g' is $g'g^{-1} = \tilde{g}g\tilde{g}^{-1}g^{-1} \in [G, G]$.

ii) Since $G \longrightarrow G/[G, G]$ is surjective, and the map $G/\mathrm{Inn}(G) \longrightarrow G/[G, G]$ defined by choosing a representative is well-defined, the map $G/\mathrm{Inn}(G) \longrightarrow G/[G, G]$ is surjective.

iii) To see why the map $G/\mathrm{Inn}(G) \longrightarrow G/[G, G]$ is, in general, not injective, consider the following example [DuFo04]. Let S_n be the symmetric group on n letters, i.e. it is the permutation group of n elements. Let A_n be the alternating group on n letters. This group is defined as the kernel of the homomorphism $S_n \longrightarrow \{-1, 1\}$ given by taking the sign of the permutation. It turns out this kernel is also the commutator subgroup of S_n. Furthermore, its index is $[S_n/[S_n, S_n]] = [S_n/A_n] \equiv [S_n : A_n] = 2$. On the other hand, let's compute the conjugacy classes of S_n for some small n. The simplest case actually suffices, although we'll quote some results for higher n to indicate that the difference between conjugacy classes and abelianization gets bigger. For $n = 3$, the set of conjugacy classes in S_3 is given by the following elements. The identity element, written as $(\ )$, is in its own class. The elements $(1, 2), (1, 3)$, and $(2, 3)$ are in their own class. Finally, the elements $(1, 2, 3)$ and $(1, 3, 2)$ are in their own class.

Therefore, the set of conjugacy classes for S_3 is given by a 3-element set whereas the abelianization is a 2-element group. For S_4, the set of conjugacy classes is a set of 5 elements. For S_5, the set of conjugacy classes is a set of 7 elements. The abelianization, however, is always of order 2.

∎

Therefore, conjugacy classes contain at least as much information about ordinary holonomy as do elements of the abelianization, and they are exactly the elements needed to define holonomy in a gauge invariant way due to Theorem 2.47.

In a similar way, the reduced group $\mathfrak{G}_{\mathrm{red}}$ of a 2-group $\mathcal{B}\mathfrak{G}$ is analogous to the abelianization and does not contain the full information of 2-holonomy in general. One needs an analogue of conjugacy classes for 2-holonomy. The candidate, for spheres at least, is α-conjugacy classes, H/α. In fact, we have a similar fact concerning α-conjugacy classes and the reduced group.

3.54. Lemma. *Let (H, G, τ, α) be a crossed module, $\mathcal{B}\mathfrak{G}$ the associated 2-group, $\mathfrak{G}_{\mathrm{red}} := H/[G, H]$ the reduced group of $\mathcal{B}\mathfrak{G}$, and H/α the α-conjugacy classes in H. The map $H/\alpha \longrightarrow \mathfrak{G}_{\mathrm{red}}$ given by taking a conjugacy class $[h]$, choosing a representative, and projecting to the quotient $H/[G, H]$, is*

i) well-defined,

ii) surjective,

iii) and need not be injective in general.

Proof.

i) Let h and h' be two representatives. Then there exists a $g \in G$ such that $\alpha_g(h) = h'$ and so the difference between h and h' is $h^{-1}h' = h^{-1}\alpha_g(h) \in [G, H]$.

ii) Since $H \longrightarrow H/[G, H]$ is surjective, and the map $H/\alpha \longrightarrow \mathfrak{G}_{\mathrm{red}}$ defined by choosing a representative is well-defined, the map $H/\alpha \longrightarrow \mathfrak{G}_{\mathrm{red}}$ is surjective.

iii) To see why the map $H/\alpha \longrightarrow \mathfrak{G}_{\mathrm{red}}$ is, in general, not injective, consider the special case where $H = G$, $\tau = \mathrm{id}$, and α is the ordinary conjugation. Then this case reduces to the previous case of Lemma 3.53.

Although the previous example suffices to show why α-conjugacy classes H/α contain more information than the reduced group in general, holonomy along spheres takes values in $\ker \tau \leqslant H$ by the source-target matching condition. Therefore, it is also important to find an example of a crossed module (H, G, τ, α) such that $\ker \tau = H$ and the map $H/\alpha \longrightarrow \mathfrak{G}_{\mathrm{red}}$ is not injective.

Take $H := \mathbb{Z}_p$, the (additive) cyclic group of order p, where $p \geqslant 3$ is prime. Set $G := \mathrm{Aut}(\mathbb{Z}_p)$, the automorphism group of $\mathbb{Z}_p$. Let τ be the trivial map and $\alpha := \mathrm{id}$ be the identity map. $(\mathbb{Z}_p, \mathrm{Aut}(\mathbb{Z}_p), \tau, \alpha)$ defines a crossed module.

Every element of $\mathrm{Aut}(\mathbb{Z}_p)$ is of the form σ_k with $k \in \{1, 2, \ldots, p-1\}$ and is determined by where it sends the generator: $\sigma_k(1 \mod p) := k \mod p$. For this proof, denote the α-conjugacy class of an element $m \in \mathbb{Z}_p$ by $[m]$. For all k, $\sigma_k(0 \mod p) = 0 \mod p$ so that $0 \mod p$ is fixed under the α action. However, since $\sigma_k(1) = k \mod p$, the set of α-conjugacy classes of $(\mathbb{Z}_p, \mathrm{Aut}(\mathbb{Z}_p), \tau, \alpha)$ is $\mathbb{Z}_p/\alpha = \{[0], [1]\}$, which is just a 2-element set. However, the reduced group is trivial. To see this, consider generators of $[\mathrm{Aut}(\mathbb{Z}_p), \mathbb{Z}_p]$, which are of the form $(\sigma_k(m) - m) \mod p$ with $k \in \{1, 2, \ldots, p-1\}$ and $m \in \{0, 1, 2, \ldots, p-1\}$. Set $m = 1$ and $k = 2$. Then $(\sigma_k(m) - m) \mod p = 1 \mod p$. Therefore, the generator of $\mathbb{Z}_p$ is in the subgroup $[\mathrm{Aut}(\mathbb{Z}_p), \mathbb{Z}_p]$ which means $[\mathrm{Aut}(\mathbb{Z}_p), \mathbb{Z}_p] = \mathbb{Z}_p$. Thus $\mathbb{Z}_p/[\mathrm{Aut}(\mathbb{Z}_p), \mathbb{Z}_p] = \mathbb{Z}_p/\mathbb{Z}_p \cong \{e\}$.

∎

In this case, one can make sense of gauge-invariant quantities coming from 2-holonomy without passing to the reduced group as is done in [ScWa13]. In the case of the examples considered in Section 5, one even gets a fixed point under the α action, in which case one does not need to pass to the α-conjugacy classes.

3.55. Definition. *Let (H, G, τ, α) be a crossed module. Denote the fixed points of H under the α action by*

$$\mathrm{Inv}(\alpha) := \{h \in H \mid \alpha_g(h) = h \text{ for all } g \in G\}. \tag{197}$$

3.56. Lemma. *In the notation of Definition 3.55,* $\mathrm{Inv}(\alpha)$ *is a central subgroup of H.*

Proof. Let $h, h' \in \mathrm{Inv}(\alpha)$. Then

$$\alpha_g(hh') = \alpha_g(h)\alpha_g(h') = hh' \tag{198}$$

for all $g \in G$. Thus, $\mathrm{Inv}(\alpha)$ is closed. $\alpha_g(e) = e$ for all $g \in G$ says $e \in \mathrm{Inv}(\alpha)$. Let $h \in \mathrm{Inv}(\alpha)$, then $\alpha_g(h^{-1}) = (\alpha_g(h))^{-1} = h^{-1}$ showing that $h^{-1} \in \mathrm{Inv}(\alpha)$. Finally, $\mathrm{Inv}(\alpha)$ is central because

$$hkh^{-1} = \alpha_{\tau(h)}(k) = k \tag{199}$$

for all $h \in H$ and $k \in \mathrm{Inv}(\alpha)$. ∎

This will have physical relevance when discussing monopoles, which, as we will show, take values in $\mathrm{Inv}(\alpha)$.

4. The path-curvature 2-functor associated to a transport functor

In this section, given a principal G-bundle with connection and a choice of a subgroup of $\pi_1(G)$, we construct a principal 2-bundle with connection whose structure 2-group is a covering 2-group obtained from G and the subgroup of $\pi_1(G)$. This assignment is functorial. We describe it on all levels introduced in the review, namely as a globally defined transport functor, in terms of descent data, and via differential cocycle data. These constructions respect all of the functors relating these different levels.

4.1. The path-curvature 2-functor. The transport 2-functor defined later in this section is motivated by the study of magnetic monopoles in gauge theories as described in [HoTs93]. Some of the earlier accounts of similar descriptions can be found in the work of Wu and Yang in [WuYa75] under the name 'total circuit' and also in the work of Goddard, Nuyts, and Olive in [GoNuOl77]. Of course, several others worked on understanding the "topological quantum number" due to a magnetic charge in terms of just the magnetic charge alone, but the three references mentioned are the ones that have influenced us. We argue in Section 5 that in the case where the base space is a 3-manifold, this transport 2-functor has 2-holonomy along a sphere which is given by the magnetic flux through that sphere. Therefore, we give a mathematically rigorous description of non-abelian flux for magnetic monopoles in a non-abelian gauge theory. A more detailed description of the physics will be given in that section, but first we explain the mathematical structure.

The starting data consist of (i) a principal G-bundle, where G is a connected Lie group, with connection over a smooth manifold M, and (ii) a subgroup N of $\pi_1(G)$. By the main theorem of [ScWa09], the first part of the data corresponds to a transport functor $\mathrm{tra} : \mathcal{P}_1(M) \longrightarrow G\text{-Tor}$ with $\mathcal{B}G$ structure. From this data, we will construct a transport 2-functor which we call the *path-curvature 2-functor.* We will discuss two interesting cases for the choice of N although other choices are important for applications in physics so we keep this generality for future applications. When $N = \pi_1(G)$, the path-curvature 2-functor coincides with the curvature 2-functor of Schreiber and Waldorf [ScWa13]. The choice $N = \{1\}$, the trivial group, will be more appropriate in the context of gauge theory and computing invariants. This is the case we focus on for all our computations in Section 5.

To set up this example, we introduce the following Lie 2-group associated to any connected Lie group G. Let $\tilde{G}$ be the universal over of G (we will describe what happens for arbitrary covers later) and denote the covering map by $\tau : \tilde{G} \longrightarrow G$. An explicit construction of $\tilde{G}$ in terms of homotopy classes of paths will be useful for our purposes

$$\tilde{G} := \{h : [0,1] \longrightarrow G \mid h(0) = e \text{ and } h \text{ is continuous}\}/_{\sim} \tag{200}$$

where $h \sim h'$ if $h(1) = h'(1)$ and there exists a homotopy $h \Rightarrow h'$ relative the endpoints. $\tilde{G}$ naturally acquires a topology as the quotient space of a subspace of paths. Denote the equivalence class representing a path with square brackets as in $[h]$ or $[t \mapsto h(t)]$, where it is understood that t takes values in $[0,1]$. The multiplication in $\tilde{G}$ is defined by choosing representatives and multiplying them pointwise (later we will show that this multiplication can be described in another way that is sometimes more convenient for our examples). Let $\alpha : G \longrightarrow \mathrm{Aut}(\tilde{G})$ be the conjugation map $\alpha_g([h]) := [ghg^{-1}]$, meaning

$$\alpha_g([h]) := [t \mapsto gh(t)g^{-1}], \tag{201}$$

where the concatenation means multiplication in G. Define $\tau : \tilde{G} \longrightarrow G$ to be evaluation at the endpoint,

$$\tau([h]) := h(1). \tag{202}$$

4.2. Proposition. *$(\tilde{G}, G, \tau, \alpha)$ defined in the previous paragraph is a Lie crossed module.*

Proof. It is useful to recall the definition of a crossed module (Definition 3.2) at this point. Since the equivalence relation involves homotopy relative endpoints, τ is well-defined. α is well-defined because $h \sim h'$ implies $ghg^{-1} \sim gh'g^{-1}$. The topological space $\tilde{G}$ has a unique smooth structure making the map τ a homomorphism and a smooth covering map, i.e. a smooth surjective submersion with the property that for every $g \in G$, there exists an open neighborhood U containing g such that each component of $\tau^{-1}(U)$ maps to U diffeomorphically. This follows from some basic differential topology (see for example Theorem 2.13 of [Le03]). Conjugation in G is a smooth map, and because α is well-defined, α is therefore smooth. The only things left to check are the crossed module identities. First, let $[h], [h'] \in \tilde{G}$ and let h and h' be representatives of $[h]$ and $[h']$ respectively. Then the map

$$[0,1] \times [0,1] \ni (s,t) \mapsto h\Big((1-s) + st\Big)h'(t)h\Big((1-s) + st\Big)^{-1} \tag{203}$$

is a homotopy (relative endpoints) from the path $t \mapsto h(1)h'(t)h(1)^{-1}$ (when $s = 0$) to the path $t \mapsto h(t)h'(t)h(t)^{-1}$ (when $s = 1$). Therefore,

$$\begin{aligned} \alpha_{\tau([h])}([h']) &= [t \mapsto h(1)h'(t)h(1)^{-1}] \\ &= [t \mapsto h(t)h'(t)h(t)^{-1}] \\ &= [h][h'][h^{-1}], \end{aligned} \tag{204}$$

which is the first identity (80). For the second identity, let $g \in G$ and $[h] \in \tilde{G}$ with a representative h. Then

$$\tau(\alpha_g([h])) = \tau[t \mapsto gh(t)g^{-1}] = gh(1)g^{-1} = g\tau([h])g^{-1}, \tag{205}$$

which proves the other identity (81). ∎

4.3. Definition. *The Lie crossed module $(\tilde{G}, G, \tau, \alpha)$ defined above is called the* <u>universal cover crossed module</u> *associated to a Lie group G. The associated Lie 2-group, denoted by $\mathcal{G}_{\{1\}}$, is called the* <u>universal cover 2-group</u> *associated to the Lie group G.*

In fact, the only way to give a smooth covering map a Lie crossed module structure is the way we have done so above. This follows from the following Lemma.

4.4. Lemma. *Let (H, G, τ, α) be a crossed module (not necessarily Lie) with $\tau : H \longrightarrow G$ a surjective homomorphism. Then α* is *conjugation in H by a choice of lift, namely*

$$\alpha_g(h') = hh'h^{-1}, \quad \text{for all } g \in G, h' \in H \tag{206}$$

for some h with $\tau(h) = g$.

Proof. First we prove that conjugating by a lift is well-defined. Let $\tilde{h} \in H$ be another lift with $\tau(\tilde{h}) = g$. Then

$$\begin{aligned} hh'h^{-1}\left(\tilde{h}h'\tilde{h}^{-1}\right)^{-1} &= hh'h^{-1}\tilde{h}h'^{-1}\tilde{h}^{-1} \\ &= \alpha_{\tau(h)}(h')\alpha_{\tau(\tilde{h})}(h'^{-1}) \quad \text{by (80)} \\ &= \alpha_g(h')\alpha_g(h'^{-1}) \\ &= \alpha_g(h'h'^{-1}) \\ &= \alpha_g(e) \\ &= e \end{aligned} \tag{207}$$

since $\alpha_g : H \longrightarrow H$ is a homomorphism. The claim that $\alpha_g(h') = hh'h^{-1}$ for a choice of lift h of g then follows from the identity (80) since $\alpha_g(h') = \alpha_{\tau(h)}(h') = hh'h^{-1}$ for some h because τ is surjective and a lift always exists. ∎

4.5. Lemma. *Let (H, G, τ, α) be a Lie crossed module with $\tau : H \longrightarrow G$ a smooth covering map. Then α* is *conjugation in H by a choice of lift, namely*

$$\alpha_g(h') = hh'h^{-1}, \qquad \textit{for all } g \in G, h' \in H \tag{208}$$

for some h with $\tau(h) = g$.

Proof. The claim holds even if τ is just surjective. The proof follows from Lemma 4.4 viewing H and G as groups (ignoring smooth structure) and using the identity $\alpha_g(h') = \alpha_{\tau(h)}(h')$ for some lift h of g. ∎

Given any subgroup $N \leqslant \pi_1(G)$, we can construct another Lie 2-group in a similar way but by using a different equivalence relation. Define

$$\tilde{G}_N := \{h : [0,1] \longrightarrow G \mid h(0) = e \text{ and } h \text{ is continuous}\}/_{\sim_N}, \tag{209}$$

where $h \sim_N h'$ if $h(1) = h'(1)$ and $\left[\begin{smallmatrix} h \\ \circ \\ \overline{h'} \end{smallmatrix}\right] \in N$, where $\overline{h'}$ denotes the reverse path and we use a vertical representation for the concatenation of paths in this context

$$\begin{smallmatrix} h \\ \circ \\ \overline{h'} \end{smallmatrix}(t) := \begin{cases} h(2t) & \text{for } 0 \leqslant t \leqslant \frac{1}{2} \\ h'(2-2t) & \text{for } \frac{1}{2} \leqslant t \leqslant 1 \end{cases}. \tag{210}$$

4.6. Definition. *An equivalence class of paths under the $\sim_N$ equivalence relation in equation (209) will be denoted by $[h]_N$ or $[t \mapsto h(t)]_N$ and will be called an* <u>N-class</u>.

4.7. Proposition. *Let G be a connected Lie group, $N \leqslant \pi_1(G)$ a subgroup, and $\tilde{G}_N$ as in (209). Then for $[h]_N \in \tilde{G}_N$, the function $\tau : \tilde{G}_N \longrightarrow G$ given by*

$$\tau\left([h]_N\right) := h(1), \tag{211}$$

with h a choice of a representative of $[h]_N$, is a well-defined homomorphism. Furthermore, $\tilde{G}_N$ has a unique smooth structure so that τ is a smooth covering map. Finally, $(\tilde{G}_N, G, \tau, \alpha)$ with $\alpha : G \longrightarrow \mathrm{Aut}(\tilde{G}_N)$ defined by

$$\alpha_g\left([h]_N\right) := [t \mapsto gh(t)g^{-1}]_N \tag{212}$$

is a Lie crossed module.

Proof. τ is well-defined by definition of the equivalence relation $\sim_N$. τ is a homomorphism because $\tau([h]_N[h']_N) = h(1)h'(1) = \tau([h]_N)\tau([h']_N)$. $\tilde{G}_N$ has a natural topology coming from the quotient space of a subspace of paths in G. Because $\pi_1(G)$ is abelian, the conjugacy class of N is N itself. Therefore, by a standard theorem of constructing covering spaces (see for instance Chapter 3 of [Ma99]) $\tau : \tilde{G}_N \longrightarrow G$ is a covering map. By another standard result in differential topology (see Proposition 2.12 of [Le03]), there is a unique smooth structure on $\tilde{G}_N$ making τ a smooth covering map. By construction, $\tilde{G}_N$ has a continuous multiplication making it a topological group. The only things left to prove is that the multiplication and inversion maps in $\tilde{G}_N$ are smooth. This can be done locally using the smoothness of multiplication and inversion in G and the fact that τ is a local diffeomorphism. Therefore, $\tilde{G}_N$ is a Lie group. Since τ is smooth, τ is a Lie group homomorphism. α_g is a well-defined group homomorphism for all $g \in G$ because it can be described as conjugation. It is smooth because for any $g \in G$, there exists an open neighborhood U around g, a diffeomorphism $\varphi : U \longrightarrow V$, with V a component of $\tau^{-1}(U)$, so that $U \ni g' \mapsto \alpha_{g'}$ coincides with conjugation by $\varphi(g')$ by the proof of Lemma 4.5. Since conjugation is smooth for any Lie group, α is smooth. Therefore, $(\tilde{G}_N, G, \tau, \alpha)$ is a Lie crossed module. ∎

Note: We use the same notation τ and α for the maps instead of τ_N and α_N since we typically fix N in any given context.

4.8. Definition. *Let G be a Lie group and N a subgroup of $\pi_1(G)$. Then $(\tilde{G}_N, G, \tau, \alpha)$ as described in Proposition 4.7 is called the* <u>N-cover crossed module</u> *of G. Its associated 2-group is called the* <u>N-covering 2-group</u> *and is denoted by $\mathcal{BG}_N$. We sometimes abusively say* <u>covering crossed module</u> *or* <u>covering 2-group</u> *without referring to N explicitly.*

Let $N \leqslant \pi_1(G)$ be a subgroup of the fundamental group of a Lie group G. We will now construct a 2-category $\widehat{G\text{-Tor}_N}$ whose underlying 1-category $\left(\widehat{G\text{-Tor}_N}\right)_{0,1}$ (see the beginning of Section 3.21) is G-Tor. Although the category G-Tor is not a Lie groupoid, notice that the set of morphisms between any two G-torsors is isomorphic to G and therefore has a unique smooth structure. Furthermore, the composition is a smooth map and is modeled by the group multiplication map $G \times G \longrightarrow G$. By this we mean that by choosing basepoints a, b, and c in G-Torsors A, B, and C respectively, the composition

$$G\text{-Tor}(B, C) \times G\text{-Tor}(A, B) \longrightarrow G\text{-Tor}(A, C) \tag{213}$$

agrees with the multiplication $G \times G \longrightarrow G$ under the isomorphisms specified by the choice of basepoints. Therefore, the composition is smooth. Thus, G-Tor is enriched in smooth manifolds. Using this fact, we can extend G-Tor to an interesting 2-category $\widehat{G\text{-Tor}_N}$ in a non-trivial way.

Let A and B be two G-torsors and let $\varphi, \psi : A \longrightarrow B$ be two morphisms of G-torsors. We define the set of 2-morphisms from φ to ψ, drawn as

$$\begin{tikzcd} B & A \arrow[l, bend right, "\varphi"'] \arrow[l, bend left, "\psi"] \arrow[l, phantom, "\Downarrow"] \end{tikzcd}, \tag{214}$$

to be the set of N-classes of paths from φ to ψ in G-Tor(A, B). This means the following.

4.9. Definition. *Let $N \leqslant \pi_1(G)$ be a subgroup. Two paths $\Sigma : \varphi \longrightarrow \psi$ and $\Sigma' : \varphi \longrightarrow \psi$ in G-*Tor(A,B)*, drawn as*

$$\begin{tikzcd} B & A \arrow[l, bend right, "\varphi"'] \arrow[l, bend left, "\psi"] \arrow[l, phantom, "\Sigma \Downarrow \quad \Downarrow \Sigma'"] \end{tikzcd}, \tag{215}$$

are said to be <u>N-equivalent</u> *if under the diffeomorphism defined by*

$$\begin{aligned} G\text{-Tor}(A,B) &\longrightarrow G \\ \varphi &\mapsto e, \end{aligned} \tag{216}$$

the homotopy class of the loop $\begin{smallmatrix}\Sigma\\ \circ \\ \overline{\Sigma'}\end{smallmatrix} : \varphi \longrightarrow \varphi$*, which gets sent to an element of* $\pi_1(G)$ *under this diffeomorphism, is an element of N. The class associated to Σ is called an* <u>N-class of paths</u> *and is denoted by* $[\Sigma]_N$.

The choice of diffeomorphism (216) where $\varphi \mapsto e$ is merely for convenience. In particular, the element $\left[\begin{smallmatrix}\Sigma\\ \circ \\ \overline{\Sigma'}\end{smallmatrix} : \varphi \longrightarrow \varphi \right]$ is independent of this diffeomorphism. To see this, if any other diffeomorphism was chosen, say sending some other morphism $\varphi' : A \longrightarrow B$ to $e \in G$, then there exists a unique $g \in G$ so that $\varphi \cdot g = \varphi'$ so that $\varphi \mapsto g^{-1}$. In this case, one gets a loop based at g^{-1}. To get one at e, we merely multiply by g to obtain a loop based at $e \in G$. This loop is exactly the same as $\begin{smallmatrix}\Sigma\\ \circ \\ \overline{\Sigma'}\end{smallmatrix}$ under the diffeomorphism defined by $\varphi \mapsto e$. Therefore, the homotopy class is independent of the diffeomorphism chosen.

Vertical composition is defined on representatives as concatenation of paths. Horizontal composition can be defined using the $G \times G \longrightarrow G$ multiplication. More explicitly, for two composable 2-morphisms as in

$$\begin{tikzcd} C & B \arrow[l, bend right, "\varphi'"'] \arrow[l, bend left, "\psi'"] \arrow[l, phantom, "\Downarrow [\Sigma']_N"] & A \arrow[l, bend right, "\varphi"'] \arrow[l, bend left, "\psi"] \arrow[l, phantom, "\Downarrow [\Sigma]_N"] \end{tikzcd}, \tag{217}$$

choose representatives of such paths so that $\Sigma : [0,1] \longrightarrow G\text{-Tor}(A,B)$ and $\Sigma' : [0,1] \to G\text{-Tor}(B,C)$ with $\Sigma(0) = \varphi, \Sigma(1) = \psi, \Sigma'(0) = \varphi'$, and $\Sigma'(1) = \psi'$. Define the horizontal composition to be the N-class of the path $\Sigma' \circ \Sigma$ defined by

$$s \mapsto (\Sigma' \circ \Sigma)(s) := \Sigma'(s) \circ \Sigma(s) \quad \text{for } s \in [0,1], \tag{218}$$

where the composition on the right-hand-side is the usual composition of morphisms in G-Tor. We check that horizontal composition is well-defined. Suppose that $\Sigma \sim_N \Omega$ and $\Sigma' \sim_N \Omega'$. We must show that $\Sigma' \circ \Sigma \sim_N \Omega' \circ \Omega$, i.e.

$$\left[\begin{matrix} \Sigma' \circ \Sigma \\ \circ \\ \hline \Omega' \circ \Omega \end{matrix}\right] \in N \tag{219}$$

but a representative of this is given by

$$\begin{aligned} \begin{matrix} \Sigma' \circ \Sigma \\ \circ \\ \hline \Omega' \circ \Omega \end{matrix}\,(s) &= \begin{cases} \Sigma'(2s) \circ \Sigma(2s) & \text{for } 0 \leqslant s \leqslant \frac{1}{2} \\ \Omega'(2-2s) \circ \Omega(2-2s) & \text{for } \frac{1}{2} \leqslant s \leqslant 1 \end{cases} \\ &= \begin{pmatrix} \Sigma' \\ \circ \\ \Omega' \end{pmatrix}(s) \circ \begin{pmatrix} \Sigma \\ \circ \\ \Omega \end{pmatrix}(s) \end{aligned} \tag{220}$$

which gives two elements of N (after taking the homotopy class) and since N is a subgroup the result is also an element of N. A similar argument is used to show that the interchange law holds. Therefore, $\widehat{G\text{-Tor}_N}$ defines a strict 2-category. We summarize this as a definition.

4.10. Definition. *Let G be a Lie group and $N \leqslant \pi_1(G)$ a subgroup of the fundamental group. The 2-category $\widehat{G\text{-Tor}_N}$ has objects and 1-morphisms that of G-Tor. The composition of 1-morphisms is the same as that in G-Tor. The set of 2-morphisms between G-torsor morphisms φ and ψ in $G\text{-Tor}(A,B)$ are N-classes of paths from φ to ψ. The vertical composition of 2-morphisms is concatenation of representative paths. The horizontal composition of 2-morphisms is the pointwise composition of G-torsor morphisms after choosing representatives.*

4.11. Remark. *When $N = \pi_1(G)$ the 2-categories $\widehat{G\text{-Tor}_N}$ and $\widehat{G\text{-Tor}}$ of [ScWa13] are equivalent because there is a unique $\pi_1(G)$-class of paths between any two morphisms of G-torsors (since every loop is $\pi_1(G)$-equivalent to every other loop).*

We will now start describing the path-curvature 2-functor, the structure 2-groupoid, and prove that it is indeed a transport 2-functor in the sense of Definition 3.24.

4.12. Lemma. *Let $\mathrm{tra} \in \mathrm{Trans}^1_{\mathcal{B}G}(M, G\text{-Tor})$ be a transport functor and let $N \leqslant \pi_1(G)$ be a subgroup. Let $K_N(\mathrm{tra}) : \mathcal{P}_2(M) \longrightarrow \widehat{G\text{-Tor}_N}$ be the following assignment. At the level of objects and 1-morphisms $K_N(\mathrm{tra})$ agrees with $\mathrm{tra} : \mathcal{P}_1(M) \longrightarrow G\text{-Tor}$. For every thin bigon $\Gamma : \gamma \Rightarrow \delta$ in $\mathcal{P}_2(M)$, choose a representative bigon, also denoted by Γ, and let*

$$K_N(\mathrm{tra})(\Gamma) := [s \mapsto \mathrm{tra}(\Gamma(\,\cdot\,, s))]_N\,, \tag{221}$$

i.e. the N-class of the path from $\mathrm{tra}(\gamma)$ *to* $\mathrm{tra}(\delta)$ *going along* $\mathrm{tra}(\Gamma(\,\cdot\,,s))$ *as a function of* $s \in [0,1]$. *The notation means that* $\Gamma(\,\cdot\,,s)$ *is a thin path with respect to the first coordinate for each fixed* s, *and is depicted as a one-parameter family of* G*-torsor morphisms*

$$\begin{tikzcd} B & A \arrow[l, bend right=40, "\varphi"'] \arrow[l, bend right=15, "\mathrm{tra}(\Gamma(\,\cdot\,,s))"'] \arrow[l] \arrow[l, bend left=15] \arrow[l, bend left=40, "\psi"] \end{tikzcd} . \tag{222}$$

This assignment is well-defined, i.e. the function $s \mapsto \mathrm{tra}(\Gamma(\,\cdot\,,s))$ *defines a continuous path and* $K_N(\mathrm{tra})(\Gamma)$ *is independent of the choice of representative bigon.*

K_N is called the path-curvature 2-functor associated to tra and $N \leqslant \pi_1(G)$.

Proof. The assignment in (221) is well-defined since ordinary homotopy is a special case of thin homotopy. More explicitly first notice that for a given bigon $\Gamma : \gamma \Rightarrow \delta$ the function $s \mapsto \mathrm{tra}(\Gamma(\,\cdot\,,s))$ is smooth because tra is a transport functor (this follows for instance from Theorem 3.12 of [ScWa09] and the fact that $G\text{-Tor}(\mathrm{tra}(x), \mathrm{tra}(y))$ is diffeomorphic to G). Now, suppose that Γ' is another representative bigon for the thin bigon Γ. Then there exists a thin homotopy $H : [0,1] \times [0,1] \times [0,1] \longrightarrow M$ with $H(t,s,0) = \Gamma(t,s)$ and $H(t,s,1) = \Gamma'(t,s)$. Thus $(s,r) \mapsto \mathrm{tra}(H(\,\cdot\,,s,r))$ is a smooth homotopy from $s \mapsto \mathrm{tra}(\Gamma(\,\cdot\,,s))$ to $s \mapsto \mathrm{tra}(\Gamma'(\,\cdot\,,s))$, which in particular is a homotopy. Thus $K_N(\mathrm{tra})(\Gamma)$ is well-defined. Similar arguments show that vertical and horizontal compositions are respected under this assignment. Therefore, $K_N(\mathrm{tra})$ defines a strict 2-functor. ∎

We construct a 2-functor $i_N : \mathcal{BG}_N \longrightarrow \widehat{G\text{-Tor}}_N$ as follows. By definition, a 2-morphism in $\mathcal{BG}_N$ is of the form

$$\begin{tikzcd} \bullet & \bullet \arrow[l, bend right=50, "g"', ""{name=U, below}] \arrow[l, bend left=50, "h(1)g", ""{name=D, above}] \arrow[Rightarrow, from=U, to=D, "{([h]_N, g)}"] \end{tikzcd} \tag{223}$$

where $[h]_N$ is viewed as an N-class of a path h in G starting at the identity e in G and ending at a point written as $\tau([h]_N) \equiv h(1)$. The image of this under i_N is defined to be

$$\begin{tikzcd} G & G \arrow[l, bend right=50, "L_g"', ""{name=U, below}] \arrow[l, bend left=50, "L_{h(1)g}", ""{name=D, above}] \arrow[Rightarrow, from=U, to=D, "{[s \mapsto L_{h(s)g}]_N}"] \end{tikzcd} , \tag{224}$$

where $s \mapsto L_{h(s)g}$ is the path in $G\text{-Tor}(G,G) \cong G$ corresponding to the path $s \mapsto h(s)g$ in G under this isomorphism. At this point it is not clear why the vertical composition is respected under i_N.

4.13. Lemma. *Let (H, G, τ, α) be a covering crossed module with elements of H thought of as certain equivalence classes of paths in G starting at the identity $e \in G$. Let h and h' be two representatives of elements $[h], [h'] \in H$. Denote the targets of h and h' by g and g', respectively. Then*

$$[h'][h] = \left[\begin{smallmatrix} h \\ \circ \\ h'g \end{smallmatrix} \right], \tag{225}$$

where $(h'g)(t) := h'(t)g$ for all $t \in [0, 1]$, and the vertical composition is the composition of paths starting with the one on top.

Proof. A homotopy is given by

$$(t, s) \mapsto \begin{cases} h'(st)h\big((2-s)t\big) & \text{for } 0 \leqslant t \leqslant \frac{1}{2} \\ h'\big((2-s)t - 1 + s\big)h(st + 1 - s) & \text{for } \frac{1}{2} \leqslant t \leqslant 1 \end{cases} \tag{226}$$

with $s = 0$ projecting to $\left[\begin{smallmatrix} h \\ \circ \\ h'g \end{smallmatrix} \right]$ and $s = 1$ projecting to $[h'h] = [h'][h]$. ∎

We now come to one of our main theorems.

4.14. Theorem. *The path-curvature 2-functor $K_N(\mathrm{tra})$ defined above is a transport 2-functor with $\mathcal{BG}_N$-structure.*

Proof. To prove this, we must provide a π_N-local i_N-trivialization of $K_N(\mathrm{tra})$ and show that the associated descent object is smooth. This will be done in several steps, outlined as follows.

i) Define $\mathrm{triv}_N : \mathcal{P}_2(Y) \longrightarrow \mathcal{BG}_N$ and show it is a smooth strict 2-functor.

ii) Define a natural equivalence $t_N : \pi^* K_N(\mathrm{tra}) \Rightarrow i_N \circ \mathrm{triv}_N$.

iii) Explicitly construct the associated descent object $(\mathrm{triv}_N, g_N, \psi_N, f_N)$.

iv) Show that the descent object is smooth.

i) To start, $\mathrm{tra} : \mathcal{P}_1(M) \longrightarrow G\text{-Tor}$ is assumed to be a transport functor, so there exists a π-local i-trivialization $(\mathrm{triv} : \mathcal{P}_1(Y) \longrightarrow \mathcal{B}G, t : \pi_1^* \mathrm{triv}_i \Rightarrow \pi_2^* \mathrm{triv}_i)$, where $\pi : Y \longrightarrow M$ is a surjective submersion, and whose associated descent object $\mathrm{Ex}^1_\pi(\mathrm{tra}, \mathrm{triv}, t)$ is smooth. We first define $\pi_N : Y \longrightarrow M$ to be π. Then we define $\mathrm{triv}_N : \mathcal{P}_2(Y) \longrightarrow \mathcal{BG}_N$ by making it agree with triv on the 1-category $\mathcal{P}_1(Y)$ inside $\mathcal{P}_2(Y)$. For a thin bigon $\Gamma : \gamma \Rightarrow \delta$ in Y we define

$$\mathrm{triv}_N(\Gamma) := \Big(\big[s \mapsto \mathrm{triv}(\Gamma(\,\cdot\,, s))\mathrm{triv}(\gamma)^{-1} \big]_N, \mathrm{triv}(\gamma) \Big) \in \tilde{G}_N \rtimes G \tag{227}$$

Note that $[s \mapsto \mathrm{triv}(\Gamma(\,\cdot\,, s))\mathrm{triv}(\gamma)^{-1}]_N$ makes sense as an element of $\tilde{G}_N$ because $\tilde{G}_N$ is precisely defined to be the set of N-classes of paths in G starting at the identity of G. This is well-defined because thin homotopy factors through ordinary homotopy (see Proof of Lemma 4.12).

We first prove that triv_N as defined is a strict 2-functor. It is already a strict 2-functor at the level of objects and 1-morphisms. We first check that vertical composition of bigons goes to vertical composition of bigons. Consider two bigons $\Gamma : \gamma \Rightarrow \delta$ and[6] $\Delta : \delta \Rightarrow \epsilon$. Their respected images under the assignment above gives

$$
\begin{tikzcd}[column sep=large]
\mathrm{triv}(y) & \mathrm{triv}(x) \arrow[l, bend right=60, "\mathrm{triv}(\gamma)"'] \arrow[l, "\mathrm{triv}(\delta)" description] \arrow[l, bend left=60, "\mathrm{triv}(\epsilon)"] \arrow[l, phantom, shift right=4ex, "\Downarrow \mathrm{triv}_N(\Gamma)"] \arrow[l, phantom, shift left=4ex, "\Downarrow \mathrm{triv}_N(\Delta)"]
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge]
\mathrm{triv}(y) & \mathrm{triv}(x) \arrow[l, bend right=50, "\mathrm{triv}(\gamma)"'] \arrow[l, bend left=50, "\mathrm{triv}(\epsilon)"] \arrow[l, phantom, "\substack{\mathrm{triv}_N(\Gamma) \\ \mathrm{triv}_N(\Delta)} \Downarrow"]
\end{tikzcd}
, \tag{228}
$$

which, after taking the $\tilde{G}_N$ component, gives

$$
\left[s \mapsto \mathrm{triv}(\Delta(\,\cdot\,, s))\mathrm{triv}(\delta)^{-1}\mathrm{triv}(\Gamma(\,\cdot\,, s))\mathrm{triv}(\gamma)^{-1}\right]_N \tag{229}
$$

while first composing in $\mathcal{P}_2(Y)$ and then applying triv_N gives[7]

$$
\mathrm{triv}_N\begin{pmatrix}\Gamma \\ \circ \\ \Delta\end{pmatrix} = \left(\left[s \mapsto \mathrm{triv}\left(\begin{matrix}\Gamma \\ \circ \\ \Delta\end{matrix}(\,\cdot\,, s)\right)\mathrm{triv}(\gamma)^{-1}\right]_N, \mathrm{triv}(\gamma)\right). \tag{230}
$$

A homotopy between these two representatives is given by $H(s,r) :=$

$$
\begin{cases}
\mathrm{triv}(\Gamma(\,\cdot\,,(r+1)s))\mathrm{triv}(\gamma)^{-1} & \text{for } 0 \leqslant s \leqslant \frac{r}{2} \\
\mathrm{triv}(\Delta(\,\cdot\,,(r+1)s-r))\mathrm{triv}(\delta)^{-1}\mathrm{triv}(\Gamma(\,\cdot\,,(r+1)s))\mathrm{triv}(\gamma)^{-1} & \text{for } \frac{r}{2} \leqslant s \leqslant 1-\frac{r}{2} \\
\mathrm{triv}(\Delta(\,\cdot\,,(r+1)s-r))\mathrm{triv}(\gamma)^{-1} & \text{for } 1-\frac{r}{2} \leqslant s \leqslant 1
\end{cases} \tag{231}
$$

which indeed satisfies

$$
H(s,0) = \mathrm{triv}(\Delta(\,\cdot\,, s))\mathrm{triv}(\delta)^{-1}\mathrm{triv}(\Gamma(\,\cdot\,, s))\mathrm{triv}(\gamma)^{-1} \tag{232}
$$

and

$$
H(s,1) = \begin{cases}
\mathrm{triv}(\Gamma(\,\cdot\,, 2s))\mathrm{triv}(\gamma)^{-1} & \text{for } 0 \leqslant s \leqslant \frac{1}{2} \\
\mathrm{triv}(\Delta(\,\cdot\,, 2s-1))\mathrm{triv}(\gamma)^{-1} & \text{for } 1-\frac{1}{2} \leqslant s \leqslant 1
\end{cases}. \tag{233}
$$

This proves more than what we needed since all we had to show was that the two elements are in the same N-class. Showing that the two representatives are homotopic is stronger and implies they're in the same N-class.

Now consider the horizontal composition of $\Gamma : \gamma \Rightarrow \delta$ and $\Pi : \alpha \Rightarrow \beta$ written as $\Pi \circ \Gamma : \alpha \circ \gamma \Rightarrow \beta \circ \delta$. First composing the thin bigons and then applying the map triv_N gives

$$
\mathrm{triv}_N(\Pi \circ \Gamma) = \left(\left[s \mapsto \mathrm{triv}\Big((\Pi \circ \Gamma)(\,\cdot\,, s)\Big)\mathrm{triv}(\alpha \circ \gamma)^{-1}\right]_N, \mathrm{triv}(\alpha \circ \gamma)\right) \tag{234}
$$

[6] Technically, $\Delta : \delta' \Rightarrow \epsilon$ and there is a thin homotopy $\Sigma : \delta \Rightarrow \delta'$ but this means $\mathrm{triv}(\delta) = \mathrm{triv}(\delta')$ so the above statement still holds.

[7] Again, this is technically not correct. One has to use a thin homotopy $\Sigma : \delta \Rightarrow \delta'$ but the reader can check that the proof follows through with a slightly modified homotopy.

while first applying the map triv to each thin bigon and then multiplying in $\mathcal{B}\mathcal{G}_N$ gives

$$\begin{aligned} &p_{\tilde{G}_N}\big(\mathrm{triv}_N(\Pi)\mathrm{triv}_N(\Gamma)\big) \\ &= \Big[s \mapsto \mathrm{triv}(\Pi(\,\cdot\,,s))\mathrm{triv}(\alpha)^{-1}\mathrm{triv}(\alpha)\mathrm{triv}(\Gamma(\,\cdot\,,s))\mathrm{triv}(\gamma)^{-1}\mathrm{triv}(\alpha)^{-1}\Big]_N \\ &= \Big[s \mapsto \mathrm{triv}(\Pi(\,\cdot\,,s))\mathrm{triv}(\Gamma(\,\cdot\,,s))\mathrm{triv}(\alpha\circ\gamma)^{-1}\Big]_N \\ &= \Big[s \mapsto \mathrm{triv}\Big((\Pi\circ\Gamma)(\,\cdot\,,s)\Big)\mathrm{triv}(\alpha\circ\gamma)^{-1}\Big]_N \end{aligned} \tag{235}$$

because for every fixed s, parallel transport of paths is a homomorphism. Therefore, triv_N defines a strict 2-functor.

We now show that triv_N is a smooth 2-functor. We already know triv_N is smooth at the level of objects and 1-morphisms. We must therefore show $\mathrm{triv}_N : P^2Y \longrightarrow \tilde{G}_N \rtimes G$ is smooth. At this point, the reader should review Appendix A because we will recall several facts in the proof of this claim. By Definition A.2, triv_N is smooth if and only if for every plot $\varphi : U \longrightarrow P^2Y$, the composition $\mathrm{triv}_N \circ \varphi : U \longrightarrow \tilde{G}_N \rtimes G$ is a plot. By Example A.3, $\mathrm{triv}_N \circ \varphi$ is a plot if and only if it is smooth. By Example A.6, $\mathrm{triv}_N \circ \varphi$ is smooth if and only if both projections $p_G \circ \mathrm{triv}_N \circ \varphi$ and $p_{\tilde{G}_N} \circ \mathrm{triv}_N \circ \varphi$ are smooth. Since we already showed that $p_G \circ \mathrm{triv}_N \circ \varphi = \mathrm{triv} \circ s \circ \varphi$ is smooth (here s is the source of a thin bigon), it remains to show that $p_{\tilde{G}_N} \circ \mathrm{triv}_N \circ \varphi$ is smooth.

For convenience for this proof, set $f := p_{\tilde{G}_N} \circ \mathrm{triv}_N$. By definition, $f \circ \varphi$ is given by

$$U \ni u \mapsto \Big[s \mapsto \mathrm{triv}\big(\varphi(u)(\,\cdot\,,s)\big)\mathrm{triv}\big(\varphi(u)(\,\cdot\,,0)\big)^{-1}\Big]_N, \tag{236}$$

where we've chosen a representative bigon $\varphi(u) : [0,1] \times [0,1] \longrightarrow Y$, fixed s to get a thin path $\varphi(u)(\,\cdot\,,s)$, and then applied triv (unfortunately, there is a lot of abuse of notation to avoid an overabundance of brackets and symbols). The problem with this is that although we know we can always choose bigons $\varphi(u)$, these choices need not form a *smooth* family of bigons in an open neighborhood of $u \in U$. Therefore, proving smoothness this way will not work.

Instead, we use the smooth structures we've defined to *construct* such a smooth family of bigons. P^2Y is the quotient of BY, bigons in Y, under thin homotopy and its smooth structure was defined as such. Therefore, by Example A.5, $\varphi : U \longrightarrow P^2Y$ is a plot if and only if there exists an open cover $\{U_j\}_{j\in J}$ of U and plots $\{\varphi_j : U_j \longrightarrow BY\}_{j\in J}$ such that

$$\begin{tikzcd} BY \arrow[d, "q"'] & U_j \arrow[l, "\varphi_j"'] \arrow[d, hook] \\ P^2Y & U \arrow[l, "\varphi"] \end{tikzcd} \tag{237}$$

commutes for all $j \in J$. For the purposes of this proof, q is the quotient map.

Now, BY itself is a subspace of the space of smooth squares $Y^{[0,1]^2}$ in Y. Denote the inclusion of BY into $Y^{[0,1]^2}$ by k. By Example A.4, $\varphi_j : U_j \longrightarrow BY$ is a plot if and only if $k \circ \varphi_j : U_j \longrightarrow Y^{[0,1]^2}$ are plots. By Example A.7, $k \circ \varphi_j : U_j \longrightarrow Y^{[0,1]^2}$ is a plot if and only if the associated function $\widetilde{k \circ \varphi_j} : U_j \times [0,1]^2 \longrightarrow Y$ defined by $\widetilde{k \circ \varphi_j}(u,t,s) := \Big(k\big(\varphi_j(u)\big)\Big)(t,s)$ is smooth. This gives us our first desired fact: the plot $\varphi : U \longrightarrow P^2Y$ gives a smooth family of bigons $\varphi_j : U_j \longrightarrow BY$ such that $q \circ \varphi_j = \varphi|_{U_j}$. Furthermore, since $\widetilde{k \circ \varphi_j}$ is a smooth map of finite-dimensional manifolds, it is continuous and therefore the smooth family of bigons is also continuous.

By using another adjunction, the smooth map $\widetilde{k \circ \varphi_j}$ can be turned into a plot $\widehat{k \circ \varphi_j} : U_j \times [0,1] \longrightarrow Y^{[0,1]}$ that factors through paths with sitting instants and is defined by $\widehat{k \circ \varphi_j}(u,s)\big(t\big) := \Big(k\big(\varphi_j(u)\big)\Big)(t,s)$. Using this, we get a smooth map $U_j \times [0,1] \longrightarrow G$ given by

$$(u,s) \mapsto \operatorname{triv}\left(\widehat{k \circ \varphi_j}(u,s)\right) \operatorname{triv}\left(\widehat{k \circ \varphi_j}(u,0)\right)^{-1} \tag{238}$$

because triv is smooth on thin paths (we've taken the thin homotopy classes of the paths $\widehat{k \circ \varphi_j}(u,s)$ and $\widehat{k \circ \varphi_j}(u,0)$ in the arguments of triv). For each fixed $u \in U_j \subset U$, this gives a path in G starting at e and the N-class of this path coincides with $f(\varphi(u))$ by commutativity of the diagram in (237). By continuity (which we proved in the previous paragraph), for each u there exists a (sufficiently small) contractible open neighborhood V of u with $u \in V \subset U_j$ together with a (sufficiently small) contractible open neighborhood W of $f(\varphi(u))$ in $\tilde{G}_N$ such that $f\big(\varphi(V)\big) \subset W$ and W maps diffeomorphically to $\tau(W) \subset G$ under the smooth covering map τ. But we just showed that the projection $\tau \circ f \circ \varphi|_V : V \longrightarrow G$ is smooth and since all neighborhoods are small and contractible, a lift is uniquely specified, is smooth, and agrees with $f \circ \varphi|_V$. Therefore, $f \circ \varphi$ is smooth at the point $u \in U$. By applying this argument to all plots at all points, this proves that $f = p_{\tilde{G}_N} \circ \operatorname{triv}_N : P^2Y \longrightarrow \tilde{G}_N$ is smooth.

ii) Our goal now is to define a natural equivalence $t_N : \pi^* K_N(\mathrm{tra}) \Rightarrow i_N \circ \operatorname{triv}_N$. Note that since tra is a transport functor, we have a natural isomorphism $t : \pi^*\mathrm{tra} \Rightarrow i \circ \operatorname{triv}$. Therefore, on points $y \in Y$, i.e. objects of $\mathcal{P}_2(Y)$, define $t_N(y) := t(y)$. For $\gamma \in P^1Y$, since t was a natural transformation for ordinary functors, the required diagram already commutes so we set $t_N(\gamma) := \mathrm{id}$.

iii) Because of our definition of triv_N and t and since our target category is a strict 2-category, the associated descent data will not be too different from the ordinary transport functor case. Namely, the modifications ψ_N and f_N are both trivial, i.e. they are the identity 2-morphisms on objects. g_N is also completely specified by g since t_N is specified by t.

iv) As mentioned above, triv_N is smooth. What's left to show is that $\mathcal{F}(g_N) : \mathcal{P}_1(Y^{[2]}) \to \Lambda\widehat{G\text{-}\mathrm{Tor}}_N$ is a transport functor with $\Lambda\mathcal{BG}_N$-structure. First let's explicitly describe

$\Lambda\mathcal{BG}_N$ and $\Lambda\widehat{G\text{-Tor}_N}$. The objects of $\Lambda\mathcal{BG}_N$ are 1-morphisms of $\mathcal{BG}_N$ which are precisely elements of G. A morphism from g_1 to g_2 in $\Lambda\mathcal{BG}_N$ is a pair of elements g_3 and g_4 of G along with an element $h \in H$ fitting into the diagram

$$\begin{tikzcd} \bullet \arrow[d, "g_2"'] \arrow[dr, phantom, "(h{,}g_2g_3)"] & \bullet \arrow[l, "g_3"'] \arrow[d, "g_1"] \\ \bullet & \bullet \arrow[l, "g_4"] \end{tikzcd} . \tag{239}$$

Similarly an object of $\Lambda\widehat{G\text{-Tor}_N}$ is a pair of objects P and P' in $\widehat{G\text{-Tor}_N}$ and a G-equivariant map $P \xrightarrow{f} P'$. A morphism from $P \xrightarrow{f} Q$ to $P' \xrightarrow{g} Q'$ in $\Lambda\widehat{G\text{-Tor}_N}$ is a pair of G-equivariant maps $p : P \longrightarrow P'$ and $q : Q \longrightarrow Q'$ along with an N-class of a path $\alpha : g \circ p \Rightarrow q \circ f$ as in the diagram

$$\begin{tikzcd} P' \arrow[d, "g"'] \arrow[dr, phantom, "\alpha"] & P \arrow[l, "p"'] \arrow[d, "f"] \\ Q' & Q \arrow[l, "q"] \end{tikzcd} . \tag{240}$$

By applying the general definition of $\mathcal{F}(g_N)$, we have

$$Y^{[2]} \ni y \quad \xmapsto{\mathcal{F}(g_N)} \quad \begin{array}{c} i(\mathrm{triv}(\pi_1(y))) = G \\ \big\downarrow L_{g(y)} \\ i(\mathrm{triv}(\pi_2(y))) = G \end{array} \tag{241}$$

and

$$P^1Y^{[2]} \ni \left(y' \xleftarrow{\gamma} y \right) \quad \xmapsto{\mathcal{F}(g_N)} \quad \begin{tikzcd} G \arrow[d, "L_{g(y')}"'] \arrow[dr, phantom, "\mathrm{id}"] & G \arrow[l, "i(\mathrm{triv}(\pi_1(\gamma)))"'] \arrow[d, "L_{g(y)}"] \\ G & G \arrow[l, "i(\mathrm{triv}(\pi_2(\gamma)))"] \end{tikzcd} . \tag{242}$$

Now, since g is part of the smooth descent object for the functor tra, there exists a smooth natural isomorphism $\tilde{g} : \pi_1^*\mathrm{triv} \Rightarrow \pi_2^*\mathrm{triv}$ such that $g = \mathrm{id}_i \circ \tilde{g}$. Using this fact, one can define $\widetilde{g_N} : \pi_1^*\mathrm{triv}_N \Rightarrow \pi_2^*\mathrm{triv}_N$ in an analogous way to how g_N was defined from g but this time using $\tilde{g}$. Furthermore, $\mathcal{F}(g_N)$ factors through $\Lambda(i_N)$ via $\mathcal{F}(g_N) = \Lambda(i_N) \circ \mathcal{F}(\widetilde{g_N})$ since $g = \mathrm{id}_i \circ \tilde{g}$.

Therefore, this defines a global trivialization with the identity surjective submersion $\mathrm{id} : Y^{[2]} \longrightarrow Y^{[2]}$ with the trivialization functor being $\mathcal{F}(\widetilde{g_N}) : \mathcal{P}_1(Y^{[2]}) \longrightarrow \Lambda\mathcal{BG}_N$. This functor is smooth since $\tilde{g}$ is smooth. Furthermore, the descent object associated to this transport functor is trivial because of the global trivialization. Thus $\mathcal{F}(g_N)$ defines a transport functor.

Thus $K_N(\mathrm{tra})$ defines a transport 2-functor with $\mathcal{BG}_N$ structure. ∎

4.15. Definition. *Let* tra $: \mathcal{P}_1(M) \longrightarrow G$-Tor *be a transport functor over* M *with* $\mathcal{B}G$ *structure and values in* G-Tor *and let* $N \leqslant \pi_1(G)$ *be a subgroup. Then the transport 2-functor* $K_N(\mathrm{tra}) : \mathcal{P}_2(M) \longrightarrow \widehat{G\text{-Tor}}_N$ *defined by*

$$\begin{tikzcd} y & x \arrow[l, bend right, "\gamma"'{name=U}] \arrow[l, bend left, "\delta"{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow\Gamma"] \end{tikzcd} \mapsto \begin{tikzcd} \mathrm{tra}(y) & \mathrm{tra}(x) \arrow[l, bend right, "\mathrm{tra}(\gamma)"'{name=U}] \arrow[l, bend left, "\mathrm{tra}(\delta)"{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow {[s \mapsto \mathrm{tra}(\Gamma(\,\cdot\,, s))]_N}"] \end{tikzcd} \tag{243}$$

is called the path-curvature transport 2-functor *associated to* tra *and* N.

More can be said, although we will not prove the details since the proofs are simple. The above construction is functorial. Namely, for any morphism of parallel transport functors $h : \mathrm{tra} \Rightarrow \mathrm{tra}'$ with $\mathcal{B}G$-structure with values in G-Tor, there is a corresponding 1-morphism of parallel transport 2-functors $h_N : K_N(\mathrm{tra}) \Rightarrow K_N(\mathrm{tra}')$ with $\mathcal{B}\mathcal{G}_N$-structure with values in $\widehat{G\text{-Tor}}_N$. By viewing $\mathrm{Trans}^1_{\mathcal{B}G}(M, G\text{-Tor})$ as a 2-category whose 2-morphisms are all identities, this defines a 2-functor

$$K_N : \mathrm{Trans}^1_{\mathcal{B}G}(M, G\text{-Tor}) \longrightarrow \mathrm{Trans}^2_{\mathcal{B}\mathcal{G}_N}(M, \widehat{G\text{-Tor}}_N). \tag{244}$$

In fact, in the above proof, in steps i) and ii), we have also outlined the definition of a 2-functor (see equation (227) and surrounding text)

$$K_N^{\mathrm{Triv}} : \mathrm{Triv}^1_\pi(i)^\infty \longrightarrow \mathrm{Triv}^2_\pi(i_N)^\infty \tag{245}$$

given by the assignment

$$(\mathrm{tra}, \mathrm{triv}, t) \mapsto (K_N(\mathrm{tra}), \mathrm{triv}_N, t_N := t) \tag{246}$$

on objects (see Definitions 2.13 and 3.15) and

$$\alpha \mapsto \alpha_N := \alpha \tag{247}$$

on morphisms (see Definitions 2.14 and 3.16). In these two assignments, we are viewing a natural transformation as a pseudonatural transformation by assigning the identity 2-morphism to every 1-morphism.

In steps iii) and iv) we have also outlined the definition of a 2-functor

$$K_N^{\mathfrak{Des}} : \mathfrak{Des}^1_\pi(i)^\infty \longrightarrow \mathfrak{Des}^2_\pi(i_N)^\infty \tag{248}$$

given by the assignment

$$(\mathrm{triv}, g) \mapsto (\mathrm{triv}_N, g_N := g, \psi_N := 1, f_N := 1) \tag{249}$$

on objects (see Definitions 2.16 and 3.17) and

$$h \mapsto (h_N := h, \epsilon_N := 1) \tag{250}$$

on morphisms (see Definitions 2.17 and 3.18).

By definition, both squares in the diagram

$$\begin{tikzcd} \mathfrak{Des}^1(i)^\infty \arrow[d, "K_N^{\mathfrak{Des}}"'] & \mathrm{Triv}^1(i)^\infty \arrow[l, "\mathrm{Ex}^1"'] \arrow[d, "K_N^{\mathrm{Triv}}"'] & \mathrm{Trans}^1_{\mathcal{B}G}(M, G\text{-Tor}) \arrow[l, "c"'] \arrow[d, "K_N"] \\ \mathfrak{Des}^2(i_N)^\infty & \mathrm{Triv}^2(i_N)^\infty \arrow[l, "\mathrm{Ex}^2"'] & \mathrm{Trans}^2_{\mathcal{B}\mathcal{G}_N}(M, \widehat{G\text{-Tor}_N}) \arrow[l, "c"'] \end{tikzcd} \tag{251}$$

commute (on the nose).

The path-curvature 2-functor associated to a transport functor is *flat*. To explain this, we first define a modified version of the thin path 2-groupoid.

4.16. Definition. *Let X be a smooth manifold. If one drops condition ii) from Definition 3.13, then one obtains a 2-groupoid $\Pi_2(X)$ that has points of X as objects, thin paths for 1-morphisms, and (ordinary) homotopy classes of bigons for 2-morphisms.*

[ScWa13] call this 2-groupoid the *fundamental 2-groupoid*. Although we prefer to use that terminology for the usual fundamental 2-groupoid (whose 1-morphisms are *also* ordinary homotopy classes of paths), we use this terminology for the purposes of this paper to avoid confusion.

4.17. Definition. *A transport 2-functor $F : \mathcal{P}_2(M) \longrightarrow T$ with* Gr*-structure is said to be* flat *if it factors through the fundamental 2-groupoid $\Pi_2(M)$.*

The curvature 2-functor $K(\mathrm{tra}) \equiv K_{\pi_1(G)}(\mathrm{tra})$ introduced in [ScWa13] is completely determined on bigons by the boundary of the bigon. It is therefore obviously flat, but it is even more restrictive than just that. Not only does it not depend on the homotopy class of the bigon, it doesn't depend on the bigon at all. On the other hand, the path-curvature 2-functor $K_N(\mathrm{tra})$ introduced here depends on the homotopy class of the bigon.

4.18. Corollary. *The path-curvature 2-functor $K_N(\mathrm{tra})$ is flat.*

Proof. Let Γ and Γ' be two bigons that are smoothly homotopic (as opposed to just thinly homotopic). Let $H : [0,1]^3 \longrightarrow Y$ be a smooth homotopy from Γ to Γ' so that $H(t,s,0) = \Gamma(t,s)$ and $H(t,s,1) = \Gamma'(t,s)$. By compactness of $[0,1]^3$, one can choose H so that it has sitting instants around its boundary so that $\mathrm{tra}(H(\,\cdot\,,s,r))$ is well-defined for each $s,r \in [0,1]$. Then

$$\begin{aligned} [0,1] \times [0,1] &\longrightarrow G \\ (s,r) &\mapsto \mathrm{tra}(H(\,\cdot\,,s,r)) \end{aligned} \tag{252}$$

is a smooth homotopy from the path $s \mapsto \mathrm{tra}(\Gamma(\,\cdot\,,s))$ to the path $s \mapsto \mathrm{tra}(\Gamma'(\,\cdot\,,s))$. Therefore, since N-classes of paths is a quotient of the universal cover $\tilde{G}$, the N-classes of these paths agree. ■

4.19. A DESCRIPTION IN TERMS OF DIFFERENTIAL FORM DATA. In this section, we prove several important and useful facts. The first theorem says that *locally* transport functors whose structure 2-group is a covering 2-group can be described in terms of the path-curvature 2-functor. The second part of this section contains a discussion about the relationship between the path-curvature 2-functor specifically and its differential cocycle data. As before, let $\pi : Y \longrightarrow M$ denote a surjective submersion, G a connected Lie group, $N \leqslant \pi_1(G)$ a subgroup, and $\tau : \tilde{G}_N \longrightarrow G$ the cover of G defined by N. It is important to note that $\underline{\tau} : \underline{\tilde{G}_N} \longrightarrow \underline{G}$, the induced map of Lie algebras, is an isomorphism of Lie algebras because τ is a local diffeomorphism. Denote the 2-group associated to the Lie crossed module $(\tilde{G}_N, G, \tau, \alpha)$ by $\mathcal{B}\mathcal{G}_N$.

First, we define a 2-functor $K_N^Z : Z^1_\pi(G) \longrightarrow Z^2_\pi(\mathcal{G}_N)$ by

$$\begin{aligned} (A,g) &\mapsto \left(\left(A, B := \underline{\tau}^{-1}\left(dA + \frac{1}{2}[A,A]\right)\right), (g, \varphi := 1), (\psi := 1, f := 1)\right) \\ h &\mapsto (h, \varphi := 0) \end{aligned} \tag{253}$$

on objects and morphisms, respectively.

Second, notice that specifically for the path-curvature 2-functor $K_N(\mathrm{tra})$, and particularly its associated descent object $K_N^{\mathfrak{Des}}(\mathrm{tra})$, the analysis in Section 3.29 gives the following differential cocycle data associated to $K_N(\mathrm{tra})$. The assignment on thin paths induces a 1-form A with values in $\underline{G}$ since the functor $K_N(\mathrm{tra})$ agrees precisely with tra on thin paths. On thin bigons, the assignment induces a 2-form B with values in $\underline{\tilde{G}_N}$ satisfying $dA + \frac{1}{2}[A,A] = \underline{\tau}(B)$. Since $\underline{\tau}$ is an isomorphism, B is determined by this condition and is given by $B = \underline{\tau}^{-1}\left(dA + \frac{1}{2}[A,A]\right)$. Therefore, the associated differential cocycle data to the path-curvature 2-functor $K_N(\mathrm{tra})$ is

$$\mathcal{D}(K_N^{\mathfrak{Des}}(\mathrm{triv}, g)) = \left(A, B := \underline{\tau}^{-1}\left(dA + \frac{1}{2}[A,A]\right), g, \varphi = 0, f = 1, \psi = 1\right). \tag{254}$$

Therefore, the two descriptions agree showing that the diagram

$$\begin{tikzcd} Z^1_\pi(G) \arrow[d, "K_N^Z"'] & \mathfrak{Des}^1_\pi(i)^\infty \arrow[l, "\mathcal{D}"'] \arrow[d, "K_N^{\mathfrak{Des}}"] \\ Z^2_\pi(\mathcal{G}_N) & \mathfrak{Des}^2_\pi(i_N)^\infty \arrow[l, "\mathcal{D}"] \end{tikzcd} \tag{255}$$

commutes. This analysis is actually a bit more general as the following theorem shows.

4.20. THEOREM. *Let X be a smooth manifold and $F_N : \mathcal{P}_2(X) \longrightarrow \mathcal{B}\mathcal{G}_N$ be* any *smooth strict 2-functor. Then there exists a unique smooth functor $F : \mathcal{P}_1(X) \longrightarrow \mathcal{B}G$ such that $F_N = K_N(F)$.*

Proof. The functor $\mathcal{D}_X : \mathrm{Funct}^\infty(\mathcal{P}_2(X), \mathcal{B}\mathcal{G}_N) \longrightarrow Z^2_X(\mathcal{G}_N)^\infty$ (defined around (159) in Section 3.29) produces $(A \in \Omega^1(X; \underline{G}), B \in \Omega^2(X; \underline{\tilde{G}_N}))$ satisfying $dA + \frac{1}{2}[A,A] = \underline{\tau}(B)$. Since $\underline{\tau} : \underline{\tilde{G}_N} \longrightarrow \underline{G}$ is an isomorphism, $B = \underline{\tau}^{-1}\left(dA + \frac{1}{2}[A,A]\right)$. Restricting F_N to $\mathcal{P}_1(X)$ gives a unique $F : \mathcal{P}_1(X) \longrightarrow \mathcal{B}G$ that satisfies $\mathcal{D}_X(F) = A$. By the same token, we have $\mathcal{D}_X(K_N(F)) = \left(A, \underline{\tau}^{-1}\left(dA + \frac{1}{2}[A,A]\right)\right)$ which coincides with $\mathcal{D}_X(F_N)$. Since $\mathcal{P}_X : Z^2_X(\mathcal{G}_N)^\infty \longrightarrow \mathrm{Funct}^\infty(\mathcal{P}_2(X), \mathcal{B}\mathcal{G}_N)$ is a strict inverse to $\mathcal{D}_X$ by Theorem 2.21 of [ScWa11], we conclude that $F_N = K_N(F)$. ∎

This theorem implies the following interesting and simple explicit formula for local 2-holonomy for transport 2-functors with covering 2-groups as their structure 2-groupoids. This is another one of our main results.

4.21. Corollary. *The formula for local parallel transport for any bigon under* any *smooth 2-functor $F_N : \mathcal{P}_2(X) \longrightarrow \mathcal{B}\mathcal{G}_N$ is given by the formula*

$$F_N\left(\begin{tikzcd} y & x \arrow[l, bend right, "\gamma"'] \arrow[l, bend left, "\delta"] \arrow[l, phantom, "\Downarrow\Gamma"] \end{tikzcd}\right) = \begin{tikzcd}[column sep=huge] \bullet & \bullet \arrow[l, bend right=50, "F(\gamma)"'] \arrow[l, bend left=50, "F(\delta)"] \arrow[l, phantom, "{[s\mapsto F(\Gamma(\ \cdot\ ,s)F(\gamma)^{-1}]_N}"] \end{tikzcd}, \tag{256}$$

where F is the 2-functor F_N restricted to 1-morphisms.

Finally, by Corollary 4.9 of [ScWa09], Theorem 2.21 of [ScWa11], and Proposition 4.1.3 of [ScWa13], the functors $\mathcal{P}$ in each row of

$$\begin{tikzcd} Z^1_\pi(G)^\infty \arrow[r, "\mathcal{P}"] \arrow[d, "K^Z_N"'] & \mathfrak{Des}^1_\pi(i)^\infty \arrow[d, "K^{\mathfrak{Des}}_N"] \\ Z^2_\pi(\mathcal{G}_N)^\infty \arrow[r, "\mathcal{P}"'] & \mathfrak{Des}^2_\pi(i_N)^\infty \end{tikzcd} \tag{257}$$

are (weak) inverses to $\mathcal{D}$ so this diagram commutes weakly.

5. Examples and magnetic monopoles

As briefly mentioned above, the path-curvature transport 2-functor is motivated by constructions in physics. In 1931, Dirac [Di31] studied the charge of a magnetic monopole in $\mathbb{R}^3$ and found it to be quantized and proportional to $\int_{S^2} R$, where S^2 is a sphere enclosing the magnetic monopole and R is the curvature of the $U(1)$ bundle with connection over $\mathbb{R}^3 \setminus \{*\}$ where $\{*\} \subset \mathbb{R}^3$ is the location of the monopole. Of course, the language of bundles and connections was not around at the time, but the ingredients were there. Because R is well-defined globally, the integral $\int_{S^2} R$ is unambiguously defined. Furthermore, it is a topological invariant in the sense that it only depends on the homotopy class of the sphere in $\mathbb{R}^3 \setminus \{*\}$. However, for a non-abelian principal G-bundle with connection, R is not globally defined so it was not clear how to define the magnetic charge. In [WuYa75], [HoTs93], and [GoNuOl77] the authors define the charge of a magnetic monopole in terms of a magnetic flux through a sphere by calculating the holonomy along a family of loops as in Figure 16. This defines a loop at the identity of the group. Taking the homotopy class of this loop was the definition of the magnetic charge in the physics literature. [GoNuOl77] was closer to defining this flux as a double-path-ordered integral, but stopped short and used other means to analyze it.

We want to point out here that it is not obvious that the methods described in the literature make sense. For instance, is it necessary to begin with the constant loop? What should this loop have anything to do with a magnetic flux, which was defined in the abelian case to be $\int_{S^2} R$. Is the resulting quantity gauge invariant? What does gauge invariance even mean? And how does one know that these concepts are even correct?

As we show in this section, the path-curvature transport 2-functor introduced in the previous section describes magnetic flux in terms of surface holonomy. Furthermore, since this magnetic flux is defined using surface holonomy, for which we have *proven* gauge covariance in Section 3.36 (specifically Theorem 3.49), we can meaningfully ask if the magnetic flux is a gauge invariant quantity. This would be the case if it is invariant under α-conjugation. We review the interesting cases considered in the physics literature, those of $U(1)$ monopoles, $SO(3)$ monopoles, and $SU(n)/Z(n)$ for all n. We also consider the cases $U(n)$ for all n. For all of these examples, we take the subgroup $N \leqslant \pi_1(G)$ to be $N = \{1\}$, the trivial subgroup of $\pi_1(G)$. This case is interesting in its own right as the examples will illustrate.

We do this in two ways. We first start with a transport functor, described in terms of its differential cocycle data, and use the methods of Section 3.25 and Section 3.36 to reconstruct a transport functor with group-valued holonomies. We then construct the path-curvature 2-functor and compute surface holonomy. The other method we use, which is equivalent by Theorem 4.20 and Corollary 4.21, is to use the surface-ordered integral of equation (150) from [ScWa11] and the definition of the differential cocycle data of the path-curvature 2-functor discussed in Section 4.19. This is unnecessary due to Corollary 4.21 but we do it anyway for the reader's convenience. In the process, we must choose weak inverses $s^\pi : \mathcal{P}_2(M) \longrightarrow \mathcal{P}_2^\pi(M)$ to the projections $p^\pi : \mathcal{P}_2^\pi(M) \longrightarrow \mathcal{P}_2(M)$ associated to some surjective submersion $\pi : Y \longrightarrow M$. We will define the 2-functor s^π for the paths and bigons of interest to us (rather than defining it for all paths and bigons) in the case of the first example of $U(1)$ monopoles. We then use the same 2-functor s^π for all other examples.

For the following discussions, we will be using the following conventions depicted in Figure 24 for describing coordinates on the sphere.

5.1. Abelian $U(1)$ monopoles. First, we will give an explicit example coming from abelian magnetic monopoles. Let $P[n] \longrightarrow S^2$ be the principal $U(1)$-bundle described by the following local trivialization. Denote the northern and southern hemispheres by U_N and U_S, respectively. We assume that U_N extends a little bit to the southern hemisphere

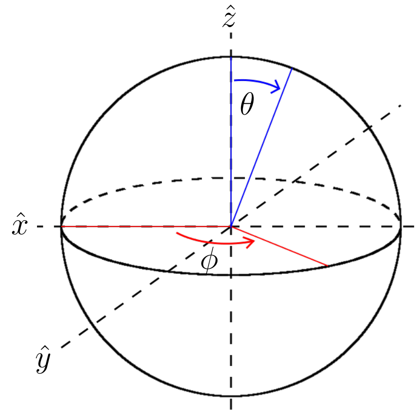


Figure 24: The azimuthal angle ϕ is drawn in red and extends from the x axis (pointing to the left) and goes counterclockwise in the xy-plane. The zenith angle θ is drawn in blue and extends from the z axis (pointing vertically) towards the xy-plane.

so that $U_{NS} \neq \varnothing$ (and similarly for U_S to the northern hemisphere). Let $Y := U_N \coprod U_S$ and $\pi : Y \longrightarrow S^2$ be the projection. Let $s_N : U_N \longrightarrow Y$ and $s_S : U_S \longrightarrow Y$ be the obvious sections. Define the transition function $g_{NS} : U_{NS} \simeq S^1 \longrightarrow U(1)$ along the equator to be

$$S^1 \ni \phi \mapsto g_{NS}(\phi) := e^{in\phi}, \tag{258}$$

where ϕ is the aziumuthal angle and n is an integer. Equip this bundle with a connection $A_N \in \Omega^1(U_N; \underline{U(1)})$ and $A_S \in \Omega^1(U_S; \underline{U(1)})$ given by

$$A_N = \frac{n}{2i}(1 - \cos\theta)d\phi \qquad \& \qquad A_S = -\frac{n}{2i}(1 + \cos\theta)d\phi. \tag{259}$$

These forms satisfy the property

$$A_N = g_{NS} A_S g_{NS}^{-1} - dg_{NS} g_{NS}^{-1} \tag{260}$$

on U_{NS} so that g_{NS}, A_N, and A_S are the local differential cocycle data of a principal $U(1)$-bundle with connection. Since $i : \mathcal{B}U(1) \longrightarrow U(1)$-Tor is an equivalence of categories, this differential cocycle data corresponds to a global transport functor (recall (59)).

We now consider the path-curvature 2-functor where $N = \{1\} \leqslant \pi_1(S^1) \cong \mathbb{Z}$ so that the associated covering 2-group is $(\mathbb{R}, U(1), \tau, \alpha)$ with $\tau : \mathbb{R} \longrightarrow U(1)$ the universal covering map defined by $\phi \mapsto e^{2\pi i\phi}$. The functor $\mathcal{P} : Z^1_\pi(G)^\infty \longrightarrow \mathfrak{Des}^1_\pi(i)$ sends the differential cocycle object (g, A) to triv : $\mathcal{P}_1(U_N \coprod U_S) \longrightarrow \mathcal{B}G$ defined by the path-ordered exponential and the natural transformation $g : \pi_1^*(\mathrm{triv}_i) \Rightarrow \pi_2^*(\mathrm{triv}_i)$ defined on components $\phi \in S^1$ by $i(g_{NS}(\phi))$. We partially define $s^\pi : \mathcal{P}_2(S^2) \longrightarrow \mathcal{P}_2^\pi(S^2)$ as follows. We first make the choice

$$s^\pi(x) := \begin{cases} s_N(x) & \text{if } x \in U_N \\ s_S(x) & \text{if } x \in S^2 \backslash U_N \end{cases} \tag{261}$$

for objects. We'll be a little sloppy now and define a lift of thin paths and thin bigons on representatives of thin homotopy classes. We only lift paths, labelled as γ_θ, of the form

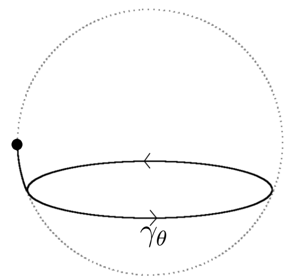


Figure 25: A loop on the sphere is made to always start at the equator at the point $\bullet$. In this figure, the loop is drawn for some θ in the range $\frac{\pi}{2} < \theta < \pi$.

depicted in Figure 25. The reason for this is because we will consider a sequence of such loops starting at the constant loop at the point $\bullet$ on the equator (so that $s^\pi(\bullet) = (\bullet, N)$) enclosing the sphere going from U_N to U_S and finally ending on the constant loop at the point $\bullet$ as depicted in Figure 26. Therefore, we define the assignment on these loops to

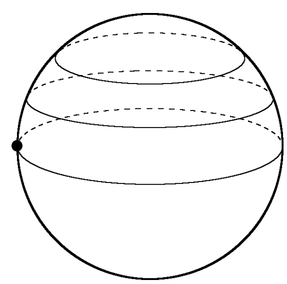


Figure 26: Loops along the ϕ direction on the sphere of constant θ are drawn for $\theta = \frac{\pi}{2}$ and two intermediate values in the range $0 < \theta < \frac{\pi}{2}$. However, each loop is made to start at the point $\bullet$ so that the sphere is thought of as a bigon $S^2 : \mathrm{id}_\bullet \Rightarrow \mathrm{id}_\bullet$.

be

$$s^\pi(\gamma_\theta) := \begin{cases} s_{N*}(\gamma_\theta) & \text{if } 0 \leqslant \theta \leqslant \frac{\pi}{2} \\ \alpha_{NS}(\bullet) * s_{S*}(\gamma_\theta) * \alpha_{SN}(\bullet) & \text{if } \frac{\pi}{2} < \theta \leqslant \pi \end{cases}. \tag{262}$$

We now define the lift on two bigons. The first bigon Σ_N is given by

$$[0, 2\pi] \times [0, \pi/2] \ni (\phi, \theta) \mapsto \Sigma_N(\phi, \theta) := \gamma_\theta(\phi) \tag{263}$$

and is a bigon $\mathrm{id}_\bullet \Rightarrow \gamma_{\pi/2}$ which lands in U_N and covers the northern hemisphere. We send this bigon to $s^{\pi}(\Sigma_N) := s_{N*}(\Sigma_N)$ in $\mathcal{P}_2^{\pi}(S^2)$ because our prescription (140) says

$$
s^{\pi}(\Sigma_N) \quad := \quad
\begin{tikzcd}[column sep=large]
s^{\pi}(\bullet) \arrow[r, "\mathrm{id}"] & s_N(\bullet) \arrow[r, phantom, "\Downarrow s_{N*}(\Sigma_N)"] & s_N(\bullet) \arrow[l, bend right=40, "s_{N*}(\mathrm{id}_{\bullet})"'] \arrow[l, bend left=40, "s_{N*}(\gamma_{\pi/2})"] & s^{\pi}(\bullet) \arrow[l, "\mathrm{id}"'] \arrow[lll, bend right=50, "s^{\pi}(\mathrm{id}_{\bullet})"', ""{name=U}] \arrow[lll, bend left=50, "s^{\pi}(\gamma_{\pi/2})", ""{name=D}]
\end{tikzcd}
\qquad \Downarrow \mathrm{id} \text{ (top)}, \ \Downarrow \mathrm{id} \text{ (bottom)}
\tag{264}
$$

We do a similar thing for the bigon Σ_S given by

$$
[0, 2\pi] \times (\pi/2, \pi] \ni (\phi, \theta) \mapsto \Sigma_S(\phi, \theta) := \gamma_{\theta}(\phi) \tag{265}
$$

which is a bigon $\gamma_{\pi/2} \Rightarrow \mathrm{id}_{\bullet}$ that lands in U_S. This is a bigon covering the southern hemisphere. However, our boundary data need to match up so that we'll be able to compose in $\mathcal{P}_2^{\pi}(S^2)$. Again, following (140)), this is given by

$$
s^{\pi}(\Sigma_S) \quad := \quad
\begin{tikzcd}[column sep=large]
s^{\pi}(\bullet) \arrow[r, "\alpha_{SN}(\bullet)"] & s_S(\bullet) \arrow[r, phantom, "\Downarrow s_{S*}(\Sigma_S)"] & s_S(\bullet) \arrow[l, bend right=40, "s_{S*}(\gamma_{\pi/2})"'] \arrow[l, bend left=40, "s_{S*}(\mathrm{id}_{\bullet})"] & s^{\pi}(\bullet) \arrow[l, "\alpha_{SN}(\bullet)"'] \arrow[lll, bend right=50, "s^{\pi}(\gamma_{\pi/2})"'] \arrow[lll, bend left=50, "s^{\pi}(\mathrm{id}_{\bullet})"]
\end{tikzcd}
\ , \qquad \Downarrow ! \text{ (top)}, \ \Downarrow ! \text{ (bottom)}
\tag{266}
$$

where the ! signifies the unique 2-isomorphisms from Lemma 3.27. For the full bigon $\Sigma : \mathrm{id}_{\bullet} \Rightarrow \mathrm{id}_{\bullet}$ depicting the full sphere as the composition $\underset{\Sigma_S}{\overset{\Sigma_N}{\circ}} : \mathrm{id}_{\bullet} \Rightarrow \gamma_{\pi/2} \Rightarrow \mathrm{id}_{\bullet}$, we break it up into the two pieces defined above and compose vertically. The result of this is

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
s^{\pi}(\bullet) \arrow[r, "\alpha_{SN}(\bullet)"] & s_S(\bullet) \arrow[r, phantom, "\Downarrow s_{S*}(\Sigma_S)"] & s_S(\bullet) \arrow[l, bend right=40, "s_{S*}(\gamma_{\pi/2})"'] \arrow[l, bend left=60, "s_{S*}(\mathrm{id}_{\bullet})"] & s^{\pi}(\bullet) \arrow[l, "\alpha_{SN}(\bullet)"'] \arrow[lll, bend right=70, "s^{\pi}(\mathrm{id}_{\bullet}) := s_{N*}(\mathrm{id}_{\bullet})"'] \arrow[lll, bend right=40, "s^{\pi}(\gamma_{\pi/2}) := s_{N*}(\gamma_{\pi/2})"'] \arrow[lll, bend left=60, "s^{\pi}(c_{\bullet}) := s_{N*}(\mathrm{id}_{\bullet})"]
\end{tikzcd}
\ , \quad \Downarrow s^{\pi}(\Sigma_N) := s_{N*}(\Sigma_N), \ \Downarrow !, \ \Downarrow !
\tag{267}
$$

We rescale our angle θ to $s = \frac{\theta}{\pi}$ to be consistent with our earlier notation. Going from $Z^2_\pi(\mathcal{BG}_{\{1\}})^\infty$ to $\mathrm{Trans}^2_{\mathcal{BG}_{\{1\}}}(M, \widehat{G\text{-}\mathrm{Tor}_{\{1\}}})$ from above to define the global transport functor applied to the sphere, we obtain the following diagram in $\widehat{G\text{-}\mathrm{Tor}_{\{1\}}}$

$$
\begin{tikzcd}[column sep=huge]
G \arrow[r, "\mathrm{id}_G"] & G & G \arrow[l, bend right=50, "L_{\mathrm{triv}(\gamma_{\pi/2})}"'] \arrow[l, bend left=50, "\mathrm{id}_G"] \arrow[l, phantom, "\Downarrow {[s\mapsto L_{\mathrm{triv}(\Sigma_S(\,\cdot\,,s))}]}"] & G \arrow[l, "\mathrm{id}_G"'] \arrow[lll, bend right=70, "\mathrm{id}_G"'] \arrow[lll, bend right=40, "L_{\mathrm{triv}(\gamma_{\pi/2})}"] \arrow[lll, bend left=50, "\mathrm{id}_G"] \arrow[lll, phantom, bend right=55, "\Downarrow {[s\mapsto L_{\mathrm{triv}(\Sigma_N(\,\cdot\,,s))}]}"] \arrow[lll, phantom, bend right=25, "\Downarrow \mathrm{id}_{L_{\mathrm{triv}(\gamma_{\pi/2})}}"] \arrow[lll, phantom, bend left=35, "\Downarrow \mathrm{id}_{\mathrm{id}_G}"]
\end{tikzcd}
\tag{268}
$$

since $i_{\{1\}} \circ \mathrm{triv}(y) = G$ for all y and so on for paths and bigons (see the definition of $R_{(\mathrm{triv},g,\psi,f)}$ in Section 3.25) and $g_{NS}(\phi = 0) \equiv g_{NS}(\bullet) = 1$. Furthermore, g_{NS} on paths is the identity since g_{NS} came from a natural transformation of ordinary functors between ordinary categories. With these simplifications, the composition in (268) is given by

$$
\left[s \mapsto \begin{cases} L_{\mathrm{triv}(\Sigma_N(\,\cdot\,,2s))} & \text{for } 0 \leqslant s \leqslant \frac{1}{2} \\ L_{\mathrm{triv}(\Sigma_S(\,\cdot\,,2s-1))} & \text{for } \frac{1}{2} \leqslant s \leqslant 1 \end{cases} \right]
\tag{269}
$$

which reduces to a computation on the group level. Therefore, all we have to do is compute the homotopy class of the path

$$
s \mapsto \begin{cases} \mathrm{triv}(\Sigma_N(\,\cdot\,,2s)) & \text{for } 0 \leqslant s \leqslant \frac{1}{2} \\ \mathrm{triv}(\Sigma_S(\,\cdot\,,2s-1)) & \text{for } \frac{1}{2} \leqslant s \leqslant 1 \end{cases}
\tag{270}
$$

in the group $U(1)$ thanks to Lemma 4.13. This is easily calculable

$$
\begin{aligned}
\mathrm{triv}(\Sigma_N(\,\cdot\,,2s)) &= \mathrm{triv}\left(\Sigma_N\left(\,\cdot\,,2\frac{\theta}{\pi}\right)\right) \\
&= e^{\frac{n}{2i}\int_0^{2\pi}(1-\cos\theta)d\phi} \\
&= e^{-in\pi(1-\cos\theta)}
\end{aligned}
\tag{271}
$$

since the paths going along θ do not contribute to the parallel transport since the connection form only has a $d\phi$ contribution. Similarly,

$$
\mathrm{triv}(\Sigma_S(\,\cdot\,,2s-1)) = \mathrm{triv}\left(\Sigma_S\left(\,\cdot\,,2\frac{\theta}{\pi}-1\right)\right) = e^{in\pi(1+\cos\theta)}.
\tag{272}
$$

As a sanity check, notice that

$$e^{-in\pi(1-\cos\frac{\pi}{2})} = e^{-in\pi} = e^{in\pi} = e^{in\pi(1+\cos\frac{\pi}{2})} \tag{273}$$

showing that the matching condition (so that our path is continuous) is satisfied. This matching condition was the one used, for instance, in [WuYa75] (see equation (47)).

Notice that $1-\cos\theta$ is a monotonically increasing function of θ for $0 \leqslant \theta \leqslant \frac{\pi}{2}$ starting at 0 when $\theta = 0$ and ending at 1 when $\theta = \frac{\pi}{2}$. Therefore, $e^{-in\pi(1-\cos\theta)}$ winds around the circle starting at 1 and ending at $e^{-in\pi} = (-1)^n$ winding around monotonically $\frac{n}{2}$ times clockwise if n is positive and counterclockwise otherwise. Now, the function $1 + \cos\theta$ is a monotonically decreasing function of θ for $\frac{\pi}{2} \leqslant \theta \leqslant \pi$ starting at $(-1)^n$ when $\theta = 0$ and ending at 1 when $\theta = \pi$. Therefore, $e^{in\pi(1+\cos\theta)}$ winds around the circle starting at $e^{in\pi} = (-1)^n$ and ending at 1 winding around monotonically $\frac{n}{2}$ times clockwise if n is positive and counterclockwise otherwise. In other words, the loop goes a total of n times around clockwise if n is positive and n times counterclockwise if n is negative and the 2-holonomy along S^2 is given by (using the notation of Definition 3.50)

$$\mathrm{hol}^{[n]}(S^2) = -n. \tag{274}$$

If we wanted to, we could have also computed this using differential forms and the formula for 2-transport (150) of Schreiber and Waldorf [ScWa11] locally and pasted the group elements together vertically as above. Of course, by the equivalence between local smooth functors and differential forms, our formula in terms of ordinary holonomy bypasses the rather (a-priori) complicated surface holonomy formula (150) due to Corollary 4.21. It'll actually turn out that the surface holonomy formula (150) is not so complicated in this particular case due to our choice of bigon representing the sphere and the differential forms representing the connection. We will subsequently do this analysis strictly in terms of the differential forms associated to the path-curvature 2-functor discussed in Section 4.19.

The curvature is given by

$$R_N = \frac{n}{2i}\sin\theta d\theta \wedge d\phi \in \Omega^2(U_N; \underline{U(1)}) \tag{275}$$

and similarly for $R_S \in \Omega^2(U_S; \underline{U(1)})$. Therefore, the connection 2-form is given by

$$B_N = \underline{\tau}^{-1}(R_N) = \frac{1}{2\pi i}R_N = -\frac{n}{4\pi}\sin\theta d\theta \wedge d\phi \tag{276}$$

and similarly for B_S. The 1-form $\mathcal{A}_{\Sigma_N}$ (see equation (149)) is given by

$$(\mathcal{A}_{\Sigma_N})_\theta\left(\frac{d}{d\theta}\right) = -\int_0^{2\pi} d\phi\; B_{(\theta,\phi)}\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}\right) = \frac{n}{2}\sin\theta \tag{277}$$

and the 2-transport along Σ_N is given by

$$\begin{aligned} k_{A,B}(\Sigma_N) &= \mathcal{P}\exp\left\{-\int_0^{\pi/2} d\theta\; (\mathcal{A}_{\Sigma_N})_\theta\left(\frac{d}{d\theta}\right)\right\} \\ &= -\int_0^{\pi/2} d\theta\; \frac{n}{2}\sin\theta \end{aligned} \tag{278}$$

because the exponential map $\underline{\mathbb{R}} \longrightarrow \mathbb{R}$ is the identity. The 2-transport along Σ_S is done similarly and is given by

$$k_{A,B}(\Sigma_S) = -\int_{\pi/2}^{\pi} d\theta \, \frac{n}{2} \sin\theta. \tag{279}$$

Vertically composing these results yields

$$\begin{aligned} k_{A,B}(\Sigma_S) + k_{A,B}(\Sigma_N) &= -\int_{\pi/2}^{\pi} d\theta \, \frac{n}{2} \sin\theta - \int_{0}^{\pi/2} d\theta \, \frac{n}{2} \sin\theta \\ &= -\int_{0}^{\pi} d\theta \, \frac{n}{2} \sin\theta \\ &= -n \end{aligned} \tag{280}$$

because the group operation in $\mathbb{R}$ is addition. Therefore, the result obtained in terms of the path-curvature 2-functor in terms of homotopy classes of paths in G agrees with the double path-ordered exponential formula (150) of Schreiber and Waldorf [ScWa11] from the differential cocycle data, which is what we expect due to Corollary 4.21.

5.2. SO(3) MONOPOLES. Now we will give examples for non-abelian magnetic monopoles. The first example will be similar to the abelian case since we will consider the following principal $SO(3)$ bundle over S^2 defined by the two open sets U_N and U_S with transition function $g_{NS} : U_{NS} \simeq S^1 \longrightarrow SO(3)$ to be

$$g_{NS}(\phi) := e^{-\phi J_3} \tag{281}$$

where

$$J_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \& \quad J_3 := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{282}$$

form a set of generators for the Lie algebra $\underline{SO(3)}$. One can give explicit connection forms A_N and A_S on U_N and U_S respectively as follows

$$A_N := \frac{J_3}{2}(1-\cos\theta)d\phi \qquad \& \qquad A_S := -\frac{J_3}{2}(1+\cos\theta)d\phi. \tag{283}$$

These define local curvature 2-forms R_N and R_S. Indeed, the gauge transformation defined above shows that

$$\begin{aligned} g_{NS} A_S g_{NS}^{-1} - dg_{NS} g_{NS}^{-1} &= A_S + J_3 d\phi \\ &= -\frac{J_3}{2}(1+\cos\theta)d\phi + J_3 d\phi \\ &= \frac{J_3}{2}(1-\cos\theta)d\phi \\ &= A_N \end{aligned} \tag{284}$$

because all elements commute. The curvature 2-form is given by

$$R_N = dA_N + \frac{1}{2}[A_N, A_N] = \frac{J_3}{2}\sin\theta\, d\theta \wedge d\phi \tag{285}$$

again because the elements commute. Since $R_N = R_S$ on U_{NS}, this defines a $\underline{SO(3)}$-valued closed 2-form on S^2. Let $\tau : SU(2) \longrightarrow SO(3)$ be the double cover map so that $N = \{1\} \leqslant \pi_1(SO(3)) \cong \mathbb{Z}_2$. Recall that the induced map on the level of Lie algebras $\underline{\tau} : \underline{SU(2)} \longrightarrow \underline{SO(3)}$ is an isomorphism and is given by

$$\underline{\tau}\left(\frac{1}{2i}\sigma_i\right) = J_i, \tag{286}$$

where the σ_i are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \& \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{287}$$

As in the general case, define $B_N := \underline{\tau}^{-1}(R_N)$ and $B_S := \underline{\tau}^{-1}(R_S)$, or explicitly

$$B = \frac{\sigma_3}{4i}\sin\theta\, d\theta \wedge d\phi \tag{288}$$

since $B_N = B_S$ on U_{NS}. By our analysis in Section 4.19, this defines the differential cocycle data of the path-curvature 2-functor. We will compute the 2-holonomy in two different ways. We will follow the same procedure as in the $U(1)$ case and compute 2-holonomy in terms of homotopy classes of paths and then we will use formula (150).

To help us with the first task, we first recall how $SU(2)$ the way described above in terms of the Pauli spin matrices is isomorphic to the universal cover of $SO(3)$ described in terms of homotopy classes of paths starting at the identity in $SO(3)$. An isomorphism $\widetilde{SO(3)} \cong SU(2)$ from the universal cover of $SO(3)$ to $SU(2)$ can be given by using the universal property and the fact that $SU(2)$ is simply connected. Given any path $\gamma : [0,1] \longrightarrow SO(3)$ starting at $\gamma(0) = \mathbb{I}_3$, the 3×3 identity matrix, there exists a unique lift $\tilde{\gamma} : [0,1] \longrightarrow SU(2)$ starting at $\tilde{\gamma}(0) = \mathbb{I}_2$ and such that the diagram

$$\begin{tikzcd} & SU(2) \arrow[d] \\ {[0,1]} \arrow[ur, dashed, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & SO(3) \end{tikzcd} \tag{289}$$

commutes. In this way, we can define a map

$$\begin{aligned} \widetilde{SO(3)} &\longrightarrow SU(2) \\ [\gamma] &\mapsto \tilde{\gamma}(1). \end{aligned} \tag{290}$$

By using the universal property one more time, one can show that this map is well-defined. Finally, it is a smooth diffeomorphism of covering spaces.

We can now check what the value of the path-curvature transport 2-functor is on the sphere by doing the same computations as above but using the new $\underline{SO(3)}$-valued differential forms. The result for the bigon describing the northern hemisphere is given by

$$\begin{aligned}\mathrm{triv}(\Sigma_N(\ \cdot\ ,2s)) &= \mathrm{triv}\left(\Sigma_N\left(\ \cdot\ ,2\frac{\theta}{\pi}\right)\right)\\ &= e^{\frac{J_3}{2}\int_0^{2\pi}(1-\cos\theta)d\phi}\\ &= e^{\pi J_3(1-\cos\theta)}\end{aligned} \tag{291}$$

since the paths going along θ do not contribute to the parallel transport since the connection form only has a $d\phi$ contribution. The path-ordered exponential is reduced to an ordinary exponential of an integral because only J_3 is involved and J_3 commutes with itself. Similarly, the southern hemisphere gives

$$\mathrm{triv}(\Sigma_S(\ \cdot\ ,2s-1)) = \mathrm{triv}\left(\Sigma_S\left(\ \cdot\ ,2\frac{\theta}{\pi}-1\right)\right) = e^{-\pi J_3(1+\cos\theta)}. \tag{292}$$

Again, as a sanity check we show that the boundary values match up between the two hemispheres along the equator:

$$e^{\pi J_3(1-\cos\frac{\pi}{2})} = e^{\pi J_3} = -\mathbb{I}_3 = e^{-\pi J_3} = e^{-\pi J_3(1+\cos\frac{\pi}{2})}. \tag{293}$$

Now we can compute the homotopy class of the path as θ ranges from 0 to π. Using similar arguments, namely that $1-\cos\theta$ is a monotonically increasing function of θ for θ between 0 and $\frac{\pi}{2}$, we see that this defines a nontrivial loop in $SO(3)$ at the identity which agrees with our previous calculation. Therefore, the 2-holonomy along the sphere is

$$\mathrm{hol}(S^2) = -\mathbb{I}_2. \tag{294}$$

Now we will use the differential cocycle data and integrate using formula (150). First, we compute $\mathcal{A}_{\Sigma_N}$ for the northern hemisphere bigon. Because only σ_3 is involved in the computation, everything commutes and conjugation is trivial. Therefore,

$$(\mathcal{A}_{\Sigma_N})_\theta\left(\frac{d}{d\theta}\right) = -\int_0^{2\pi} d\phi\ B_{(\theta,\phi)}\left(\frac{\partial}{\partial\theta},\frac{\partial}{\partial\phi}\right) = -\frac{\pi\sigma_3}{2i}\sin\theta \tag{295}$$

and the 2-transport along Σ_N is given by

$$\begin{aligned}k_{A,B}(\Sigma_N) &= \mathcal{P}\exp\left\{-\int_{\theta=0}^{\theta=\pi/2}(\mathcal{A}_{\Sigma_N})_\theta\left(\frac{d}{d\theta}\right)\right\}\\ &= \exp\left\{\int_{\theta=0}^{\theta=\pi/2}\frac{\pi\sigma_3}{2i}\sin\theta\right\}.\end{aligned} \tag{296}$$

The 2-transport along Σ_S is done similarly and is given by

$$k_{A,B}(\Sigma_S) = \exp\left\{\int_{\theta=\pi/2}^{\theta=\pi} \frac{\pi\sigma_3}{2i}\sin\theta\right\} \tag{297}$$

Vertically composing these results yields

$$k_{A,B}(\Sigma_S)k_{A,B}(\Sigma_N) = \exp\left\{\int_{\theta=0}^{\theta=\pi} \frac{\pi\sigma_3}{2i}\sin\theta\right\} = e^{\pi i\sigma_3} = -\mathbb{I}_{2\times 2} \tag{298}$$

because again every term commutes. We will discuss what these group elements mean after we finish a few more examples.

5.3. $SU(n)/Z(n)$ MONOPOLES. Another collection of non-abelian examples arise from the Lie group $SU(n)$. The center of $SU(n)$ is $Z(n)$ where, in the fundamental representation, elements in $Z(n)$ are of the form

$$\exp\left\{\frac{2\pi ki}{n}\right\}\mathbb{I}_n, \tag{299}$$

where $k \in \{0, 1, \ldots, n-1\}$ and $\mathbb{I}_n$ is the $n \times n$ unit matrix. $SU(n)/Z(n)$ is a Lie group with fundamental group $\pi_1(SU(n)/Z(n))$ isomorphic to $Z(n)$. To see this, recall that the universal cover $\widetilde{SU(n)/Z(n)}$ constructed via paths in $SU(n)/Z(n)$ and modding out by homotopy is naturally isomorphic to $SU(n)$, which is simply connected, by the universal property of universal covers. The isomorphism preserves the fibers over the identity in $SU(n)/Z(n)$ and restricts to the isomorphism between $\pi_1(SU(n)/Z(n))$ and $Z(n)$. The previous example was the special case $n = 2$.

The equivalence relation on $SU(n)/Z(n)$ says that two elements A and B of $SU(n)$ are equivalent if there exists a $k \in \{0, 1, \ldots, n-1\}$ such that

$$AB^{-1} = \exp\left\{\frac{2\pi ki}{n}\right\}\mathbb{I}_n. \tag{300}$$

We denote the elements of equivalence classes with square brackets such as $[A]$.

The possible $SU(n)/Z(n)$ principal bundles over the sphere are determined by the clutching function along the equator, which is a homotopy class of a loop $S^1 \longrightarrow SU(n)/Z(n)$ which by the isomorphism above is precisely an element of $Z(n)$. The quotient map is written as $\tau : SU(n) \longrightarrow SU(n)/Z(n)$ and is a covering map of Lie groups. Therefore, it defines a Lie 2-group.

Let's first consider the case for $n = 3$, which is relevant in the theory of quarks and gluons (see Section 1.4 of [HoTs93]). We fix $k \in \{0, 1, 2\}$. Define X to be the element in the Lie algebra of $SU(3)$ to be

$$X := \frac{i}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{301}$$

The exponential of this matrix is unitary. We define transition functions by

$$g_{NS}(\phi) := \exp\{-k\underline{\tau}(X)\} = [\exp\{-k\phi X\}] = \left[\begin{pmatrix} e^{-\frac{k\phi i}{3}} & 0 & 0 \\ 0 & e^{-\frac{k\phi i}{3}} & 0 \\ 0 & 0 & e^{\frac{2k\phi i}{3}} \end{pmatrix}\right]. \tag{302}$$

The element X is a scalar multiple of the Gell-Mann matrix λ_8. Note we have

$$g_{NS}(0) = g_{NS}(2\pi) = g_{NS}(4\pi) = [\mathbb{I}_3] \in SU(3)/Z(3). \tag{303}$$

The transition function defines a map $\phi \mapsto g_{NS}(\phi)$ whose homotopy class determines a principal $SU(3)/Z(3)$ bundle characterized by the integer $k \in \{0, 1, 2\}$.

We define a connection on this bundle analogously to the $SO(3)$ case by setting

$$A_N := \frac{k\underline{\tau}(X)}{2}(1 - \cos\theta)d\phi \qquad \& \qquad A_S := -\frac{k\underline{\tau}(X)}{2}(1 + \cos\theta)d\phi. \tag{304}$$

A similar computation shows that this collection of 1-forms is consistent with the transition function. The connection 2-form is similarly given by

$$B_N = \frac{kX}{2}\sin\theta\, d\theta \wedge d\phi \tag{305}$$

and likewise for B_S. This defines an $\underline{SU(3)/Z(3)}$-valued closed 2-form on S^2.

Again, we can do the computation for the 2-holonomy in the two ways described earlier. The first case is done by computing the homotopy class of the path of holonomies using the definition of the path-curvature 2-functor of Definition 4.15. The second way is via the differential forms associated to the path-curvature 2-functor described in Section 4.19 and equation (150). The computation is completely analogous to the previous two examples.

For the first case, we have

$$\begin{aligned} \mathrm{hol}^k(S^2) &= \left[\theta \mapsto \begin{cases} \left[e^{\frac{k}{2}X\int_0^{2\pi}(1-\cos\theta)d\phi}\right] & \text{if } 0 \leqslant \theta \leqslant \frac{\pi}{2} \\ \left[e^{-\frac{k}{2}X\int_0^{2\pi}(1+\cos\theta)d\phi}\right] & \text{if } \frac{\pi}{2} \leqslant \theta \leqslant \pi \end{cases}\right] \\ &= \left[\theta \mapsto \begin{cases} \left[e^{k\pi X(1-\cos\theta)}\right] & \text{if } 0 \leqslant \theta \leqslant \frac{\pi}{2} \\ \left[e^{-k\pi X(1+\cos\theta)}\right] & \text{if } \frac{\pi}{2} \leqslant \theta \leqslant \pi \end{cases}\right] \\ &= e^{\frac{2\pi i k}{3}}\mathbb{I}_3. \end{aligned} \tag{306}$$

As for the computation in terms of differential forms, also by analogous computations to previous cases,

$$(\mathcal{A}_{\Sigma_N})_\theta\left(\frac{d}{d\theta}\right) = -\int_0^{2\pi} d\phi \frac{kX}{2}\sin\theta = -k\pi X\sin\theta \tag{307}$$

and likewise for $(\mathcal{A}_{\Sigma_S})_\theta \left(\frac{d}{d\theta}\right)$. Also

$$k_{A,B}(\Sigma_N) = \exp\left\{ \int_0^{\pi/2} k\pi X \sin\theta \, d\theta \right\} \tag{308}$$

and finally the 2-holonomy along the sphere is

$$\mathrm{hol}^{[k]}(S^2) = k_{A,B}(\Sigma_S) k_{A,B}(\Sigma_N) = \exp\{2\pi k X\} = e^{\frac{2\pi i k}{3}} \mathbb{I}_3. \tag{309}$$

For the general case of $SU(n)$, by using the matrix

$$X := \frac{i}{n} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1-n \end{pmatrix} \tag{310}$$

the formulas for the transition function, connection 1-forms, and connection 2-forms are all the same with this new X replacing the old one. Completely analogous computations lead to a 2-holonomy along the sphere given by

$$\mathrm{hol}^{[k]}(S^2) = e^{\frac{2\pi i k}{n}} \mathbb{I}_n, \tag{311}$$

where $k \in \{0, 1, \ldots, n-1\}$. The result is the magnetic charge of a magnetic monopole computed as a non-abelian flux in $SU(n)/Z(n)$ gauge theories.

5.4. $U(n)$ MONOPOLES. We now discuss yet another collection of examples generalizing the $U(1)$ case. Consider the group $U(n)$ of unitary $n \times n$ matrices. The Lie algebra, $\underline{U(n)}$ consists of Hermitian matrices. The universal cover of $U(n)$ is $SU(n) \times \mathbb{R}$. The covering map $\tau : SU(n) \times \mathbb{R} \longrightarrow U(n)$ is defined by $\tau(A, t) := Ae^{2\pi i t}$. The image of τ is clearly a $U(1)$ subgroup of $U(n)$. The fiber of this covering map is given by the kernel which is

$$\begin{aligned} \ker \tau &= \left\{ (A, t) \;\middle|\; A = e^{-2\pi i t} \text{ and } \det A = e^{-2\pi i n t} = 1 \iff t = \frac{k}{n},\ k \in \mathbb{Z} \right\} \\ &= \left\{ \left(e^{\frac{2\pi i k}{n}} \mathbb{I}_n, \frac{k}{n} \right) \;\middle|\; k \in \mathbb{Z} \right\} \\ &\cong \mathbb{Z}. \end{aligned} \tag{312}$$

Consider the Lie algebra element along this real line

$$X := (0_n, 1), \tag{313}$$

where 0_n is the $n \times n$ zero matrix. Then its image in $\underline{U(n)}$ under $\underline{\tau}$ is

$$\underline{\tau}(X) = 2\pi i \mathbb{I}_n. \tag{314}$$

With this, for every integer k, we define the transition function, connection 1-forms, and connection 2-forms completely analogously to the previous examples (specifically the $\mathbb{R} \longrightarrow U(1)$ example), namely

$$g_{NS}(\phi) = e^{ik\phi}\mathbb{I}_n, \tag{315}$$

$$A_N = \frac{k}{2i}(1-\cos\theta)\mathbb{I}_n d\phi \qquad \& \qquad A_S = -\frac{k}{2i}(1+\cos\theta)\mathbb{I}_n d\phi, \tag{316}$$

and

$$B = \underline{\tau}^{-1}\left(\frac{k}{2i}\sin\theta\, \mathbb{I}_n\, d\theta \wedge d\phi\right) = -\frac{k}{4\pi}\sin\theta\, (0_n, 1)\, d\theta \wedge d\phi. \tag{317}$$

In terms of the path of holonomies via the path-curvature 2-functor, the surface holonomy is

$$\begin{aligned} \mathrm{hol}^{[k]}(S^2) &= \left[\theta \mapsto \begin{cases} e^{\frac{k}{2i}\mathbb{I}_n \int_0^{2\pi}(1-\cos\theta)d\phi} & \text{if } 0 \leqslant \theta \leqslant \frac{\pi}{2} \\ e^{-\frac{k}{2i}\mathbb{I}_n \int_0^{2\pi}(1+\cos\theta)d\phi} & \text{if } \frac{\pi}{2} \leqslant \theta \leqslant \pi \end{cases}\right] \\ &= \left[\theta \mapsto \begin{cases} e^{\frac{k\pi}{i}\mathbb{I}_n(1-\cos\theta)} & \text{if } 0 \leqslant \theta \leqslant \frac{\pi}{2} \\ e^{-\frac{k\pi}{i}\mathbb{I}_n(1+\cos\theta)} & \text{if } \frac{\pi}{2} \leqslant \theta \leqslant \pi \end{cases}\right] \\ &= -k \in \mathbb{Z}. \end{aligned} \tag{318}$$

If we want to compute the surface holonomy in terms of formula (150), we first compute

$$(\mathcal{A}_{\Sigma_N})_\theta\left(\frac{d}{d\theta}\right) = \int_0^{2\pi} d\phi\, \frac{k}{4\pi}\sin\theta\, (0_n, 1) = \frac{k}{2}\sin\theta\, (0_n, 1) \tag{319}$$

so that we get

$$k_{A,B}(\Sigma_N) = \mathcal{P}\exp\left\{-\int_0^{\pi/2} d\theta\, \frac{k}{2}\sin\theta\, (0_n, 1)\right\} = \left(\mathbb{I}_n, -\int_0^{\pi/2} d\theta\, \frac{k}{2}\sin\theta\right) \tag{320}$$

and the 2-holonomy along the sphere is

$$\begin{aligned} \mathrm{hol}^k(S^2) &= k_{A,B}(\Sigma_S)k_{A,B}(\Sigma_N) \\ &= \left(\mathbb{I}_n, -\int_{\pi/2}^{\pi} d\theta\, \frac{k}{2}\sin\theta\right)\left(\mathbb{I}_n, -\int_0^{\pi/2} d\theta\, \frac{k}{2}\sin\theta\right) \\ &= \left(\mathbb{I}_n, -\int_0^{\pi} d\theta\, \frac{k}{2}\sin\theta\right) \\ &= (\mathbb{I}_n, -k)\,. \end{aligned} \tag{321}$$

5.5. Magnetic flux is a gauge-invariant quantity. In this section we state a theorem that is trivial to prove in the formalism presented above but gives an interesting physical interpretation. As mentioned earlier, the definition of the magnetic flux in the literature [HoTs93] is given as the homotopy class of a loop of holonomies. However, it was not known [GoNuOl77] how to define it as a surface-ordered integral except in the abelian case. The constructions in this paper use the theory of transport 2-functors as models for 2-bundles with 2-connections to describe this loop of holonomies in terms of a transport 2-functor. The equivalence between this description and the definition in terms of surface holonomy is made precise. This motivates the following definition.

5.6. Definition. *Let $P \longrightarrow M$ be a principal G-bundle with connection over M and denote the associated transport functor by* tra. *Let $\Sigma : S^2 \longrightarrow M$ be the map of a smooth sphere in M. Let $N \leqslant \pi_1(G)$ be a subgroup, $\tilde{G}_N \longrightarrow G$ the associated N-cover, $\mathcal{BG}_N$ the associated Lie 2-group, and $K_N(\mathrm{tra})$ the associated path-curvature transport 2-functor. The 2-holonomy $\mathrm{hol}^{[K_N(\mathrm{tra})]}(\Sigma)$ is the* <u>magnetic flux of any magnetic monopole enclosed by Σ associated to tra and N.</u>

All the previous examples relied on choices for the open cover, paths and bigons used to describe the sphere, and choices of lifts of paths and bigons. It is not immediately clear that the surface holonomy computed is independent of these choices. Theorems 3.49 and 4.20 give us two important results, the first of which tells us the magnetic flux is indeed independent of these choices.

5.7. Corollary. *Under the assumptions of Definition 5.6, the magnetic flux is a gauge-invariant quantity (in terms of the notation of Definition 3.55)*

$$\mathrm{hol}^{[K_N(\mathrm{tra})]}(\Sigma) \in \mathrm{Inv}(\alpha). \tag{322}$$

Proof. Choose a marking for the thin sphere as a thin bigon $\Sigma : \gamma \Rightarrow \gamma$ from a thin loop to itself. Then $K_N(\mathrm{tra})(\Sigma) \in \ker\tau$ by the source-target matching condition (recall comment preceding (148)). By Theorem 3.49, 2-holonomy along a sphere for *any* gauge 2-group is well-defined up to α-conjugation. But α-conjugation for *covering 2-groups* agrees with ordinary conjugation by a lift by Lemma 4.5. Therefore, the α-conjugation action restricted to $G \times \ker\tau$ is trivial because $\ker\tau$ is a central subgroup of $\tilde{G}_N$ by Lemma 3.4. ∎

A corollary of this and Theorem 4.20 is the following which relates the magnetic flux to a surface integral of the magnetic field. This is more of a physics corollary than a math corollary.

5.8. Corollary. *The magnetic flux (Definition 5.6) can be computed as a surface integral by using (150) locally. This surface integral, which lands in the covering group, is the analogue of $\int_{S^2} R$ where in electromagnetism R is the electromagnetic field strength due to the local potential A.*

Therefore, the surface holonomies of transport 2-functors give a mathematically rigorous explanation for the topological quantum number (the magnetic charge) associated to magnetic monopoles for gauge theories with any structure/gauge group in the language of magnetic flux. It is topological in the sense that it only depends on the homotopy class of the sphere by Corollary 4.18. Furthermore, it expresses this quantity as a group element in the center of the universal cover of the gauge group. We emphasize that no Higgs field was introduced to do these computations. This therefore gives a rigorous mathematical result first mentioned by Goddard, Nuyts, and Olive at the end of Section 2 of their paper [GoNuOl77] by using the notion of transport 2-functors introduced by Schreiber and Waldorf in [ScWa13] to describe magnetic flux generalizing the notion from the theory of electromagnetism to non-abelian gauge theories.

A. Smooth spaces

We will briefly state important definitions and smooth structures needed in this paper. The category of finite-dimensional manifolds is not suitable for our purposes, nor is the category of certain infinite-dimensional manifolds. This section reviews diffeological spaces, which constitute one candidate for a notion of smooth spaces. For a review of smooth spaces that also compares several other candidates, please refer to [BaHo11].

A.1. Definition. *A smooth space is a set X together with a collection of plots $\{\varphi : U \longrightarrow X\}$, called its smooth structure, where each U is an open set in some $\mathbb{R}^n$ (n can vary) satisfying the following conditions.*

- *i) If $\varphi : U \longrightarrow X$ is a plot and $\theta : V \longrightarrow U$, where V is an open set of some $\mathbb{R}^m$, is a smooth map, then $\varphi \circ \theta : V \longrightarrow X$ is a plot.*
- *ii) Every map $\mathbb{R}^0 \longrightarrow X$ is a plot.*
- *iii) Let $\varphi : U \longrightarrow X$ be a function and let $\{U_j\}_{j\in I}$ be a collection of open sets covering U with $i_j : U_j \longrightarrow U$ denoting the inclusion. Then if $\varphi \circ i_j : U_j \longrightarrow X$ is a plot for all $j \in I$, then $\varphi : U \longrightarrow X$ is a plot.*

A.2. Definition. *A function $f : X \longrightarrow Y$ between two smooth spaces is smooth if for every plot $\varphi : U \longrightarrow X$ of X, $f \circ \varphi : U \longrightarrow Y$ is a plot of Y.*

A.3. Example. *Let M be a smooth manifold. The manifold smooth structure has as its collection of plots all infinitely differentiable functions $\varphi : U \longrightarrow M$ for various open sets U in Euclidean space. M with this collection of plots forms a smooth space. With this smooth structure, for any two manifolds M and N, a function $M \longrightarrow N$ is smooth if and only if it is differentiable in the usual sense.*

A.4. Example. *Let A be a subset of a smooth space X and denote the inclusion by $i : A \hookrightarrow X$. The subspace smooth structure on A has as its collection of plots all functions $\varphi : U \longrightarrow A$ such that $i \circ \varphi : U \longrightarrow X$ are plots of X. With this smooth structure, the inclusion $i : A \longrightarrow X$ is smooth.*

A.5. Example. *Let X be a smooth space, $\sim$ an equivalence relation on X, and $q : X \longrightarrow X/_\sim$ the quotient map. The* quotient smooth structure *on $X/_\sim$ has as its collection of plots all functions $\varphi : U \longrightarrow X/_\sim$ such that there exists an open cover $\{U_j\}_{j\in J}$ along with plots $\varphi_j : U_j \longrightarrow X$ for X such that*

$$\begin{tikzcd} X \arrow[d] & U_j \arrow[l, "\varphi_j"'] \arrow[d, hook] \\ X/_\sim & U \arrow[l, "\varphi"] \end{tikzcd} \tag{323}$$

commutes for all $j \in J$. With this smooth structure, the quotient map $q : X \longrightarrow X/_\sim$ is smooth.

A.6. Example. *Let X and Y be smooth spaces. The* product smooth structure *on $X \times Y$ has as its collection of plots all functions $\varphi : U \longrightarrow X \times Y$ such that $\pi_X \circ \varphi : U \longrightarrow X$ and $\pi_Y \circ \varphi : U \longrightarrow Y$ are both plots of X and Y, respectively. Here $\pi_X : X \times Y \longrightarrow X$ and $\pi_Y : X \times Y \longrightarrow Y$ are the projection maps and are smooth with respect to this smooth structure.*

A.7. Example. *Let X and Y be two smooth spaces. The* mapping smooth structure *on the set of functions Y^X of X into Y is defined as follows. A function $\varphi : U \longrightarrow Y^X$ is a plot if and only if the associated function $\tilde{\varphi} : U \times X \longrightarrow Y$, defined by $\tilde{\varphi}(u, x) := \varphi(u)(x)$, is smooth. With this smooth structure and the smooth structure on a product, the adjunction $Z^{X\times Y} \cong (Z^Y)^X$ is an isomorphism in the category of smooth spaces for all X, Y, Z.*

Index of (frequently used) notation

Notation	Name/description	Location	Page
G	a Lie group	Def 2.3	1325
$\mathcal{B}G$	a one-object groupoid	Def 2.3	1325
Gr	Lie groupoid/2-groupoid	Def 2.4/3.6	1325/1355
G-Tor	the category of G-torsors	Def 2.8	1327
PX	paths with sitting instants in X	Def 2.9	1328
BX	bigons in X	Def 2.10	1328
P^1X	smooth space of thin paths in X	Def 2.10	1328
$\mathcal{P}_1(X)$	thin path groupoid of X	Def 2.11	1329
L_g	left multiplication by g	Eqn (18)	1329
T	"target" category/2-category	Def 2.13	1330
$i : \mathrm{Gr} \longrightarrow T$	realization of structure groupoid in T	Def 2.13	1330

$\pi : Y \longrightarrow M$	a surjective submersion	Def 2.13	1330
triv	local trivialization functor	Def 2.13	1330
triv_i	$\mathrm{triv}_i := i \circ \mathrm{triv}$	Def 2.13	1330
$\mathrm{Triv}^1_\pi(i)$	category of π-local i-trivializations	Def 2.13, 2.14	1330
$Y^{[n]}$	n-fold fiber product of $\pi : Y \longrightarrow M$	Eqn (20)	1331
$\mathfrak{Des}^1_\pi(i)$	descent category	Def 2.16, 2.17	1331
Ex^1_π	extraction functor	After Def 2.17	1331
$\mathfrak{Des}^1_\pi(i)^\infty$	smooth descent category	After Def 2.20	1332
$\mathrm{Triv}^1_\pi(i)^\infty$	category of smooth π-local i-trivializations	After Def 2.21	1333
$\mathrm{Trans}^1_{\mathrm{Gr}}(M,T)$	category of transport functors	After Def 2.22	1333
$\mathcal{P}^\pi_1(M)$	Čech path groupoid of M	Def 2.25	1333
p^π	canonical projection	Eqn (33)/Lem 3.27	1335/1369
s^π	weak inverse to p^π	Eqn (33)/Lem 3.27	1335/1369
Rec^1_π	Reconstruction functor	Eqn (35)	1336
$\underline{G}$	Lie algebra of G	Sec 2.27	1337
k_A	path transport	Eqn (39)	1338
$\mathcal{P}\exp$	path-ordered exponential	Eqn (40)	1338
$Z^1_\pi(G)^\infty$	category of differential cocycles subordinate to π	Def 2.29	1340
Rec^1 & Ex^1	limit of Rec^1_π & Ex^1_π over π	Eqn (59)	1342
v & c	forgets trivialization & its weak inverse	Eqn (59)	1342
$\ell\!\!\!/$	group-valued transport extraction	Def 2.33/3.37	1343/1379
$\mathfrak{L}^1 M$	thin marked loop space of M	Eqn (67)	1346
$\mathrm{hol}^F_{\ell\!\!\!/}$	$\ell\!\!\!/$-holonomy of a transport functor F	Def 2.37/3.40	1346/1380
$\mathfrak{m}$	thin loop/sphere markings	Def 2.41/3.45	1347/1382
$G/\mathrm{Inn}(G)$	conjugacy classes in G	Before Thm 2.47	1352
$\mathrm{hol}^{[F]}$	gauge-invariant holonomy/2-holonomy	Def 2.48/3.50	1352/1389
(H,G,τ,α)	crossed module	Def 3.2	1353
$\mathcal{P}_2(X)$	path 2-groupoid of X	Def 3.13	1360
P^2X	smooth space of thin bigons in X	Def 3.13	1360
$\mathrm{Triv}^2_\pi(i)$	2-category of π-local i-trivializations	After Def 3.15	1361
$\mathfrak{Des}^2_\pi(i)$	descent 2-category	Def 3.17–3.19	1361

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