GROUPOIDS IN CATEGORIES WITH PRETOPOLOGY

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ABSTRACT. We survey the general theory of groupoids, groupoid actions, groupoid principal bundles, and various kinds of morphisms between groupoids in the framework of categories with pretopology. The categories of topological spaces and finite or infinite dimensional manifolds are examples of such categories. We study extra assumptions on pretopologies that are needed for this theory. We check these extra assumptions in several categories with pretopologies.

Functors between groupoids may be localised at equivalences in two ways. One uses spans of functors, the other bibundles (commuting actions) of groupoids. We show that both approaches give equivalent bicategories. Another type of groupoid morphism, called an actor, is closely related to functors between the categories of groupoid actions. We also generalise actors using bibundles, and show that this gives another bicategory of groupoids.

Contents

1. Introduction

The notion of groupoid has many variants: topological groupoids, étale topological groupoids, Lie groupoids of finite and infinite dimension, algebraic groupoids, and so on. A subtle point is that the definition of a groupoid in a category depends on a notion of "cover"

because the range and source maps are assumed to be covers. A *pretopology* gives a reasonable notion of cover in a category.

The need to assume range and source maps to be covers is plain for Lie groupoids: it ensures that the composable pairs of arrows form a smooth manifold. The covers also influence the notion of principal bundle because their bundle projections are assumed to be covers; this is equivalent to "local triviality" in the sense of the pretopology. If our category is that of topological spaces and the covers are the continuous surjections with local continuous sections, then we get exactly the usual notion of local triviality for principal bundles; this is why many geometers prefer this pretopology on topological spaces. Many operator algebraists prefer the pretopology of open continuous surjections instead.

Some authors (like Goehle [17]) make no assumptions on the range and source maps of a topological groupoid. Since any map with a continuous section is a biquotient map, this amounts to the same as choosing biquotient maps as covers; this is the so called canonical topology, that is, it is the largest subcanonical pretopology on the category of Hausdorff topological spaces (see Section 9.6). But how much of the usual theory remains true in this case?

We will see that some aspects of the theory of groupoid Morita equivalence require extra assumptions on pretopologies. In particular, we will show that for the pretopology of biquotient maps, equivalent groupoids may have non-equivalent categories of actions on spaces. We do not know whether Morita equivalence of groupoids is an equivalence relation when we choose the pretopology of biquotient maps. In addition to the theory of Morita equivalence with bibundles, we also extend the theory of anafunctors. It works without extra assumptions on the pretopology, but is less concrete.

Since we want to work in the abstract setting of groupoids in a category with pretopology, we develop all the general theory of groupoids and their actions from scratch. Most of our results are known for some types of groupoids, of course. Since we develop the theory from scratch, we also take the opportunity to suggest more systematic notation for various kinds of morphisms of groupoids. We study functors, anafunctors, and bibundle functors, actors and bibundle actors; in addition, there are several kinds of equivalences: equivalence functors, ana-isomorphisms, anafunctor equivalences, and bibundle equivalences; and there are the covering bibundle functors, which are the intersection of bibundle functors and bibundle actors. The categories of anafunctors and bibundle functors are equivalent, and ana-isomorphisms and bibundle equivalences are also equivalent notions; all the other types of morphisms are genuinely different and are useful in different situations. Several notions of morphisms have been introduced in the context of Lie groupoids in [41]; our notion of "actor" is unrelated to the one in [41, A.2], however.

Roughly speaking, functors and the related anafunctors and bibundle functors are appropriate if we view groupoids as generalised spaces, whereas actors and bibundle actors are appropriate if we view them as generalised symmetries (groups).

Other new names we introduce are *basic actions* and *basic groupoids*. A groupoid action is basic if, together with some bundle projection, it is a principal action. A groupoid G

is basic if its action on G^0 is a basic action. (The name "principal groupoid" is already used for something else by operator algebraists, so a new name is needed.)

Our initial goal was to generalise the bicategory of topological groupoids with Hilsum–Skandalis morphisms as arrows and isomorphisms of the latter as 2-arrows. It is well-known that this is a bicategory if, say, we use continuous maps with local continuous sections as covers (see [38]). Our bibundle functors are the analogue of Hilsum–Skandalis morphisms in a category with pretopology. To compose bibundle functors or bibundle actors, we need extra assumptions on the pretopology. We introduce these extra assumptions here and check them for some pretopologies.

We now explain the contents of the sections of the article.

Section 2 introduces *pretopologies* on categories with coproducts. Here a pretopology is a family of maps called *covers*, subject to some rather mild axioms. The more established notion of a topos is not useful for our purposes: it does not cover important examples like the category of smooth manifolds because all finite limits are required to exist in a topos.

All pretopologies are required subcanonical. We also introduce three extra assumptions that are only sometimes required. Assumption 2.10 concerns final objects and is needed to define groups and products of groupoids, as opposed to fibre products. Assumption 2.8 requires maps that are "locally covers" to be covers; the stronger Assumption 2.9 requires that if $f \circ p$ and p are covers, then so must be f. We also prove that the property of being an isomorphism is local (this is also noticed by [48, Axiom 4]). Assumptions 2.8 and 2.9 are needed to compose bibundle functors and bibundle actors, respectively.

Section 3 introduces groupoids, functors and anafunctors. We first define *groupoids* in a category with pretopology in two equivalent ways and compare these two definitions. One definition has the unit and inversion maps as data; the other one only has the multiplication, range and source maps as data and characterises groupoids by making sense of the elementwise condition that there should be unique solutions to equations of the form $g \cdot x = h$ or $x \cdot g = h$ for given arrows g and h with equal range or equal source, respectively. We explain in Section 3 how such elementwise formulas should be interpreted in a general category, following ideas from synthetic geometry [26, 34]. We use elementwise formulas throughout because they clarify statements and proofs.

We define *functors* between groupoids and *natural transformations* between such functors in a category with pretopology. We observe that they form a strict 2-category. We define a "base-change" for groupoids: given a cover $p \colon X \twoheadrightarrow \mathsf{G}^0$, we define a groupoid $p^*(\mathsf{G})$ with object space X and a functor $p_* \colon p^*(\mathsf{G}) \to \mathsf{G}$; such functors are called *hypercovers*, and they are the prototypes of equivalences. An *anafunctor* $\mathsf{G} \to \mathsf{H}$ is a triple (X, p, F) with a cover $p \colon X \twoheadrightarrow \mathsf{G}^0$ and a functor $F \colon p^*(\mathsf{G}) \to \mathsf{H}$. An *ana-isomorphism* is an isomorphism $p^*(\mathsf{G}) \cong q^*(\mathsf{H})$ for two covers $p \colon X \to \mathsf{G}^0$ and $q \colon X' \to \mathsf{H}^0$. Anafunctors form a bicategory. We characterise equivalences in this bicategory by two different but equivalent criteria: as those anafunctors that lift to an anafunctor isomorphism, and functors that are almost fully faithful and almost essentially surjective. These constructions all work in any category with a subcanonical pretopology. If we assume that being a cover is a local condition, then a functor is almost fully faithful and almost essentially surjective

if and only if it is fully faithful and essentially surjective.

Section 4 introduces groupoid actions, their transformation groupoids, and actors. Our convention is that the anchor map of a groupoid action need not be a cover; this is needed to associate bibundle functors to functors. We call an action a *sheaf* if its anchor map is a cover. The *transformation groupoid* of a groupoid action combines the action and the groupoid that is acting. An *actor* from G to H is a left action of G on H^1 that commutes with the right translation action of H. We show that such an actor allows to turn an H-action on X into a G-action on X in a natural way. Any such functor from G-actions to H-actions with some mild extra conditions comes from an actor. This is the type of "morphism" that seems most relevant when we think of a groupoid as a general form of symmetry, that is, as something that acts on other objects.

Section 5 introduces principal bundles over groupoids and basic groupoid actions. A *principal G-bundle* is a (right) G-action $\mathsf{m}\colon \mathsf{X} \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \to \mathsf{X}$ with a cover $\mathsf{p}\colon \mathsf{X} \twoheadrightarrow \mathsf{Z}$, such that the map

$$(\mathsf{m},\mathsf{pr}_1)\colon \mathsf{X} \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \to \mathsf{X} \times_{\mathsf{p},\mathsf{Z},\mathsf{p}} \mathsf{X}, \qquad (x,g) \mapsto (x \cdot g, x), \tag{1}$$

is an isomorphism. Then Z is the orbit space of the right G-action, so the bundle projection is unique if it exists. A G-action is *basic* if it is a principal bundle when taken together with its orbit space projection. We construct pull-backs of bundles and relate maps on the base and total spaces of principal bundles; this is crucial for the composition of bibundle functors. We show that the property of being principal is local, that is, a pull-back of a "bundle" along a cover is principal if and only if the original bundle is.

Principal bundles and basic actions allow us to define bibundle equivalences, bibundle functors, and bibundle actors. These are all given by commuting actions of G and H on some object X of the category. For a *bibundle functor*, we require that the right H-action together with the left anchor map $\mathsf{X} \to \mathsf{G}^0$ as bundle projection is a principal bundle; for a *bibundle equivalence*, we require, in addition, that the left G-action together with the right anchor map $\mathsf{X} \to \mathsf{H}^0$ is a principal bundle. For a *bibundle actor*, we require the right action to be basic – with arbitrary orbit space – and the right anchor map $\mathsf{X} \to \mathsf{H}^0$ to be a cover. If X is both a bibundle actor and a bibundle functor, so that both anchor maps are covers, then we call it a *covering bibundle functor*.

We show that functors give rise to bibundle functors and that bibundle functors give rise to anafunctors. The bibundle functor associated to a functor is a bibundle equivalence if and only if the functor is essentially surjective and fully faithful.

The composition of bibundle functors, actors, and equivalences is only introduced in Section 7 because it requires extra assumptions about certain actions being automatically basic. Section 7.1 discusses these extra assumptions. The stronger one, Assumption 7.2, requires all actions of Čech groupoids to be basic; the weaker one, Assumption 7.3, only requires this for sheaves over Čech groupoids, that is, for actions of Čech groupoids with a cover as anchor map. The weaker version of the assumption together with Assumption 2.8 on covers being local suffices to compose bibundle equivalences and bibundle functors; the stronger form together with Assumption 2.9 (the two-out-of-three property for covers) is

needed to compose covering bibundle functors and bibundle actors. Under the appropriate assumptions, all these types of bibundles form bicategories.

We show in Section 7.29 that the equivalences in these bicategories are precisely the bibundle equivalences. Section 7.17 shows that the bicategory of bibundle functors is equivalent to the bicategory of anafunctors. Section 7.24 describes the composite of two bibundle functors by an isomorphism similar to (1). Section 7.27 shows that the assumption needed for the composition of bibundle actors is equivalent to the assumption that the property of being basic is a local property of G-actions.

We show in Section 7.19 that every bibundle actor is a composite of an actor and a bibundle equivalence. Thus the bicategory of bibundle actors is the smallest one that contains bibundle equivalences and actors and is closed under the composition of bibundles. A similar decomposition of bibundle functors into bibundle equivalences and functors is constructed in Section 6.10 when passing from bibundle functors to anafunctors.

Section 8 describes the quasi-categories of bibundle functors, covering bibundle functors, and bibundle equivalences very succinctly. The main point are similarities between the multiplication in a groupoid, a groupoid action, and the map relating the product of two bibundle functors to the composite.

Section 9 considers several examples of categories with classes of "covers" and checks whether these are pretopologies and satisfy our extra assumptions. Section 9.2 considers sets and surjective maps, a rather trivial case. Section 9.6 considers several classes of "covers" between topological spaces; quotient maps do not form a pretopology; biquotient maps form a subcanonical pretopology for which Assumption 7.2 fails; open surjections and several smaller classes of maps are shown to be subcanonical pretopologies satisfying all our assumptions.

Section 9.41 considers smooth manifolds, both of finite and infinite dimension, with submersions as covers. These form a subcanonical pretopology on locally convex manifolds and several subcategories, which satisfies the assumptions needed for the composition of bibundle functors and bibundle equivalences. For Banach manifolds, we also prove the stronger assumptions that are needed for the composition of bibundle actors and covering bibundle functors. It is unclear whether these stronger assumptions still hold for Fréchet or locally convex manifolds. Thus all the theory developed in this article applies to groupoids in the category of Banach manifolds and finite-dimensional manifolds, and most of it, but not all, applies also to Fréchet and locally convex manifolds.

For a category $\mathcal{C}$, we write $X \in\in \mathcal{C}$ for objects and $f \in \mathcal{C}$ for morphisms of $\mathcal{C}$. Our categories need not be small, that is, the objects may form a class only. The set-theoretic issues that this entails are not relevant to our discussions below and can be handled easily using universes. Hence we leave it to the reader to add appropriate remarks where needed.

2. Pretopologies

The notion of a Grothendieck (pre)topology from [1] formalises properties that "covers" in a category should have. Usually, a "cover" of an object X is a family of maps $f_\alpha \colon U_\alpha \to X$,

$\alpha \in A$. If our category is extensive (see [10]), we may replace such a family of maps by a single map

$$(f_\alpha)_{\alpha \in A} \colon \bigsqcup_{\alpha \in A} U_\alpha \to X.$$

This reduces notational overhead, and is ideal for our purposes because in the following definitions, we never meet covering families but only single maps that are covers.

2.1. DEFINITION. *Let $\mathcal{C}$ be a category with coproducts. A* pretopology *on $\mathcal{C}$ is a collection $\mathcal{T}$ of arrows, called* covers, *with the following properties:*

(1) *isomorphisms are covers;*

(2) *the composite of two covers is a cover;*

(3) *if $f \colon Y \to X$ is an arrow in $\mathcal{C}$ and $g \colon U \twoheadrightarrow X$ is a cover, then the fibre product $Y \times_X U$ exists in $\mathcal{C}$ and the coordinate projection $\mathsf{pr}_1 \colon Y \times_X U \twoheadrightarrow Y$ is a cover. Symbolically,*

$$
\begin{array}{ccc}
Y & & Y \times_{f,X,g} U \xrightarrow{\ \mathsf{pr}_1\ } Y \\
\downarrow{\scriptstyle f} & \Longrightarrow & \ \downarrow{\scriptstyle \mathsf{pr}_2} \qquad\qquad \downarrow{\scriptstyle f} \\
U \xrightarrow{\ g\ } X & & U \xrightarrow{\quad g \quad} X
\end{array}
\qquad (2)
$$

We use double-headed arrows $\twoheadrightarrow$ to denote covers.

We always require pretopologies to be subcanonical:

2.2. DEFINITION AND LEMMA. *A pretopology $\mathcal{T}$ is* subcanonical *if it satisfies the following equivalent conditions:*

(1) *each cover $f \colon U \twoheadrightarrow X$ in $\mathcal{T}$ is a coequaliser, that is, it is the coequaliser of some pair of parallel maps $g_1, g_2 \colon Z \rightrightarrows U$;*

(2) *each cover $f \colon U \twoheadrightarrow X$ in $\mathcal{T}$ is the coequaliser of $\mathsf{pr}_1, \mathsf{pr}_2 \colon U \times_{f,X,f} U \rightrightarrows U$;*

(3) *for any object W and any cover $f \colon U \twoheadrightarrow X$, we have a bijection*

$$\mathcal{C}(X, W) \xrightarrow{\sim} \{ h \in \mathcal{C}(U, W) \mid h \circ \mathsf{pr}_1 = h \circ \mathsf{pr}_2 \text{ in } \mathcal{C}(U \times_{f,X,f} U, W) \}, \quad g \mapsto g \circ f;$$

(4) *all the representable functors $\mathcal{C}(\_, W)$ on $\mathcal{C}$ are sheaves.*

PROOF. Condition (3) makes explicit what it means for $\mathcal{C}(\_, W)$ to be a sheaf and for f to be the coequaliser of $\mathsf{pr}_1, \mathsf{pr}_2$, so (2) $\iff$ (3) $\iff$ (4). (2) implies (1) by taking $Z = U \times_{f,X,f} U$ and $g_i = \mathsf{pr}_i$ for $i = 1, 2$. It remains to prove (1)$\Rightarrow$(3).

Let $g_1, g_2 \colon Z \rightrightarrows U$ be as in (1). Then $f g_1 = f g_2$, so that (g_1, g_2) is a map $Z \to U \times_{f,X,f} U$. Let $\mathsf{pr}_i \colon U \times_{f,X,f} U \to U$ for $i = 1, 2$ be the coordinate projections. Since $\mathsf{pr}_i \circ (g_1, g_2) = g_i$ for $i = 1, 2$, a map $h \in \mathcal{C}(U, W)$ with $h \circ \mathsf{pr}_1 = h \circ \mathsf{pr}_2$ also satisfies $h \circ g_1 = h \circ g_2$. Since we assume f to be a coequaliser of g_1 and g_2, such a map h is of the form $\tilde{h} \circ f$ for a unique $\tilde{h} \in \mathcal{C}(X, W)$. Conversely, any map of the form $\tilde{h} \circ f$ satisfies $(\tilde{h} \circ f) \circ \mathsf{pr}_1 = (\tilde{h} \circ f) \circ \mathsf{pr}_2$. Thus (1) implies (3). $\blacksquare$

2.3. Definition. *Let $\mathcal{T}$ be a pretopology on $\mathcal{C}$ and let P be a property that arrows in $\mathcal{C}$ may or may not have. An arrow $f \colon Y \to X$ has P* locally *if there is a cover $g \colon U \twoheadrightarrow Y$ such that $\mathsf{pr}_2 \colon Y \times_{f,X,g} U \to U$ has P.*

The property P is local *if an arrow has P if and only if it has P locally, that is, in the situation of (2), $f \colon Y \to X$ has P if and only if $\mathsf{pr}_2 \colon Y \times_{f,X,g} U \to U$ has P.*

2.4. Isomorphisms are local.

2.5. Proposition. *If the pretopology is subcanonical, then the property of being an isomorphism is local.*

Proof. Giorgi Arabidze pointed out a slightly stronger version of this result that works in any category $\mathcal{C}$, not necessarily with a pretopology. Namely, let $Y \xrightarrow{f} X$ and $U \xrightarrow{g} X$ be arrows in $\mathcal{C}$ such that the fibre product $Y \times_{f,X,g} U$ exists in $\mathcal{C}$ and the projection $\mathsf{pr}_2 \colon Y \times_{f,X,g} U \to U$ is an isomorphism. If g is a coequaliser in $\mathcal{C}$ (see Definition and Lemma 2.2) and the projection $\mathsf{pr}_1 \colon Y \times_{f,X,g} U \to Y$ is an epimorphism, then f is an isomorphism. If $\mathcal{C}$ is a category with a subcanonical pretopology and g is a cover, then the assumptions above are satisfied.

To prove the more general statement above, take two arrows $u_1, u_2 \colon A \rightrightarrows U$ that have g as their coequaliser. Form $y := \mathsf{pr}_1 \circ \mathsf{pr}_2^{-1} \circ u_2 \colon A \to U \to Y \times_{f,X,g} U \to Y$. Since $f \circ y = g \circ \mathsf{pr}_2 \circ \mathsf{pr}_2^{-1} \circ u_2 = g \circ u_1$, there is a unique $b \colon A \to Y \times_{f,X,g} U$ with $\mathsf{pr}_1 \circ b = y$ and $\mathsf{pr}_2 \circ b = u_1$. Since pr_2 is invertible, $b = \mathsf{pr}_2^{-1} \circ u_1$, so $\mathsf{pr}_1 \circ \mathsf{pr}_2^{-1} \circ u_1 = y = \mathsf{pr}_1 \circ \mathsf{pr}_2^{-1} \circ u_2$. Since g is the coequaliser of u_1 and u_2, there is a unique arrow $c \colon X \to Y$ with $c \circ g = \mathsf{pr}_1 \circ \mathsf{pr}_2^{-1}$. This satisfies $f \circ c \circ g = g$. Since g is a coequaliser, it is epic, so $f \circ c \circ g = g$ implies $f \circ c = \mathrm{id}_X$. Furthermore, $c \circ f \circ \mathsf{pr}_1 = c \circ g \circ \mathsf{pr}_2 = \mathsf{pr}_1$ gives $c \circ f = \mathrm{id}_Y$ because we assumed pr_1 to be epic. Thus c is inverse to f. ∎

2.6. Remark. Example 9.12 provides Hausdorff topological spaces (X, τ_1), (X, τ_2) and Y such that the identity map $(X, \tau_1) \to (X, \tau_2)$ is a continuous bijection but not a homeomorphism, and a quotient map $f \colon (X, \tau_2) \to Y$ such that the map $f \colon (X, \tau_1) \to Y$ is a quotient map as well, and such that the identical map

$$(X, \tau_1) \times_{f,Y,f} (X, \tau_1) \to (X, \tau_1) \times_{f,Y,f} (X, \tau_2)$$

is a homeomorphism. Thus the pull-back of a non-homeomorphism along a quotient map may become a homeomorphism. Quotient maps on topological spaces do not form a pretopology, so this does not contradict Proposition 2.5.

2.7. Simple extra assumptions on pretopologies. Consider the fibre-product situation of (2). Then g is a cover by assumption and pr_1 is one by the definition of a pretopology. If f is a cover as well, then so is pr_2 by the definition of a pretopology. That the converse holds is an extra assumption on the pretopology, which means that the property of being a cover is local.

2.8. ASSUMPTION. *The property of being a cover is local, that is, in the fibre-product situation of* (2)*, if g and pr_2 in* (2) *are covers, then so is f.*

If g and pr_2 are covers, then so is $g \circ \mathsf{pr}_2 = f \circ \mathsf{pr}_1$ as a composite of covers, and pr_1 is a cover by definition of a pretopology. Hence Assumption 2.8 is weaker than the following *two-out-of-three property*:

2.9. ASSUMPTION. *Let $f \in \mathcal{C}(Y, Z)$ and $p \in \mathcal{C}(X, Y)$ be composable. If $f \circ p$ and p are covers, then so is f.*

A pretopology is called *saturated* if f is a cover whenever $f \circ p$ is one, without requiring p to be a cover as well (see [44, Definition 3.11]). This stronger assumption fails in many cases where Assumption 2.9 holds (see Example 9.33), and Assumption 2.9 suffices for all our applications. For the categories of locally convex and Fréchet manifolds with surjective submersions as covers, Assumption 2.8 holds, but we cannot prove Assumption 2.9. The other examples we consider in Section 9 all verify the stronger Assumption 2.9.

2.10. ASSUMPTION. *There is a final object $\star$ in $\mathcal{C}$, and all maps to it are covers.*

2.11. LEMMA. *Under Assumption* 2.10*, $\mathcal{C}$ has finite products. If $f_1 \colon U_1 \twoheadrightarrow X_1$ and $f_2 \colon U_2 \twoheadrightarrow X_2$ are covers, then so is $f_1 \times f_2 \colon U_1 \times U_2 \to X_1 \times X_2$.*

PROOF. The first statement is clear because products are fibre products over the final object $\star$. The second one is [22, Lemma 2.7]. ∎

Assumption 2.10 has problems with initial objects. For instance, the unique map from the empty set to the one-point set is not surjective, so Assumption 2.10 fails for any subcanonical pretopology on the category of sets. We must exclude the empty set for Assumption 2.10 to hold. We will not use Assumption 2.10 much.

3. Groupoids in a category with pretopology

Let $\mathcal{C}$ be a category with coproducts and let $\mathcal{T}$ be a subcanonical pretopology on $\mathcal{C}$. We define groupoids in $(\mathcal{C}, \mathcal{T})$ in two equivalent ways, with more or less data.

3.1. FIRST DEFINITION. A groupoid in $(\mathcal{C}, \mathcal{T})$ consists of

- objects $\mathsf{G}^0 \in\in \mathcal{C}$ (*objects*) and $\mathsf{G}^1 \in\in \mathcal{C}$ (*arrows*),

- arrows $\mathsf{r} \in \mathcal{C}(\mathsf{G}^1, \mathsf{G}^0)$ (*range*), $\mathsf{s} \in \mathcal{C}(\mathsf{G}^1, \mathsf{G}^0)$ (*source*), $\mathsf{m} \in \mathcal{C}(\mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1, \mathsf{G}^1)$ (*multiplication*), $\mathsf{u} \in \mathcal{C}(\mathsf{G}^0, \mathsf{G}^1)$ (*unit*), and $\mathsf{i} \in \mathcal{C}(\mathsf{G}^1, \mathsf{G}^1)$ (*inversion*),

such that

(1) r and s are covers;

(2) m is associative, that is, $\mathsf{r} \circ \mathsf{m} = \mathsf{r} \circ \mathsf{pr}_1$ and $\mathsf{s} \circ \mathsf{m} = \mathsf{s} \circ \mathsf{pr}_2$ for the coordinate projections $\mathsf{pr}_1, \mathsf{pr}_2 \colon \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \rightrightarrows \mathsf{G}^1$, and the following diagram commutes:

$$
\begin{array}{ccc}
\mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 & \xrightarrow{\ \mathsf{m} \times_{\mathsf{G}^0,\mathsf{r}} \mathrm{id}_{\mathsf{G}^1}\ } & \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \\
{\scriptstyle \mathrm{id}_{\mathsf{G}^1} \times_{\mathsf{s},\mathsf{G}^0} \mathsf{m}} \downarrow & & \downarrow {\scriptstyle \mathsf{m}} \\
\mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 & \xrightarrow{\qquad \mathsf{m} \qquad} & \mathsf{G}^1
\end{array}
\tag{3}
$$

(3) the following equations hold:

$$
\begin{aligned}
\mathsf{r} \circ \mathsf{u} &= \mathrm{id}_{\mathsf{G}^0}, & \mathsf{r}(1_x) &= x & \forall x &\in \mathsf{G}^0, \\
\mathsf{s} \circ \mathsf{u} &= \mathrm{id}_{\mathsf{G}^0}, & \mathsf{s}(1_x) &= x & \forall x &\in \mathsf{G}^0, \\
\mathsf{m} \circ (\mathsf{u} \circ \mathsf{r}, \mathrm{id}_{\mathsf{G}^1}) &= \mathrm{id}_{\mathsf{G}^1}, & 1_{\mathsf{r}(g)} \cdot g &= g & \forall g &\in \mathsf{G}^1, \\
\mathsf{m} \circ (\mathrm{id}_{\mathsf{G}^1}, \mathsf{u} \circ \mathsf{s}) &= \mathrm{id}_{\mathsf{G}^1}, & g \cdot 1_{\mathsf{s}(g)} &= g & \forall g &\in \mathsf{G}^1, \\
\mathsf{s} \circ \mathsf{i} &= \mathsf{r}, & \mathsf{s}(g^{-1}) &= \mathsf{r}(g) & \forall g &\in \mathsf{G}^1, \\
\mathsf{r} \circ \mathsf{i} &= \mathsf{s}, & \mathsf{r}(g^{-1}) &= \mathsf{s}(g) & \forall g &\in \mathsf{G}^1, \\
\mathsf{m} \circ (\mathsf{i}, \mathrm{id}_{\mathsf{G}^1}) &= \mathsf{u} \circ \mathsf{s}, & g^{-1} \cdot g &= 1_{\mathsf{s}(g)} & \forall g &\in \mathsf{G}^1, \\
\mathsf{m} \circ (\mathrm{id}_{\mathsf{G}^1}, \mathsf{i}) &= \mathsf{u} \circ \mathsf{r}, & g \cdot g^{-1} &= 1_{\mathsf{r}(g)} & \forall g &\in \mathsf{G}^1.
\end{aligned}
$$

The right two columns interpret the equalities of maps in the left column in terms of elements. The conditions above imply

$$
\begin{aligned}
\mathsf{m} \circ (\mathsf{u}, \mathsf{u}) &= \mathsf{u}, & 1_x \cdot 1_x &= 1_x & \forall x &\in \mathsf{G}^0, \\
\mathsf{i}^2 &= \mathrm{id}_{\mathsf{G}^1}, & (g^{-1})^{-1} &= g & \forall g &\in \mathsf{G}^1, \\
\mathsf{m} \circ (\mathsf{i} \times_{\mathsf{r},\mathsf{G}^0,\mathsf{s}} \mathsf{i}) &= \mathsf{i} \circ \mathsf{m} \circ \sigma, & g^{-1} \cdot h^{-1} &= (h \cdot g)^{-1} & \forall g, h &\in \mathsf{G}^1, \mathsf{r}(g) = \mathsf{s}(h);
\end{aligned}
$$

here $\sigma \colon \mathsf{G}^1 \times_{\mathsf{r},\mathsf{G}^0,\mathsf{s}} \mathsf{G}^1 \xrightarrow{\sim} \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1$ denotes the flip of the two factors.

Most statements and proofs are much clearer in terms of elements. It is worthwhile, therefore, to introduce the algorithm that interprets equations in terms of elements as in the middle and right columns above as equations of maps as in the left column.

The variables $x \in \mathsf{G}^0$, $g, h \in \mathsf{G}^1$ are interpreted as maps in $\mathcal{C}$ from some object $? \in\in \mathcal{C}$ to G^0 and G^1, respectively. Thus an element x of $X \in\in \mathcal{C}$ is interpreted as a map $x \colon ? \to X$, and denoted by $x \in X$. The elements of X form a category, which determines X by the Yoneda Lemma.

If an elementwise expression A is already interpreted as a map $A \colon ? \to \mathsf{G}^1$, then $\mathsf{r}(A)$ means $\mathsf{r} \circ A \colon ? \to \mathsf{G}^0$, $\mathsf{s}(A)$ means $\mathsf{s} \circ A \colon ? \to \mathsf{G}^0$, and A^{-1} means $\mathsf{i} \circ A \colon ? \to \mathsf{G}^1$; if an elementwise expression A is interpreted as $A \colon ? \to \mathsf{G}^0$, then 1_A means $\mathsf{u} \circ A \colon ? \to \mathsf{G}^1$. If A and B are elementwise expressions that translate to maps $A, B \colon ? \to \mathsf{G}^1$ with $\mathsf{s}(A) = \mathsf{r}(B)$, that is, $\mathsf{s} \circ A = \mathsf{r} \circ B \colon ? \to \mathsf{G}^0$, then $A \cdot B$ means the composite map

$$
\mathsf{m} \circ (A, B) \colon ? \xrightarrow{(A,B)} \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \xrightarrow{\mathsf{m}} \mathsf{G}^1.
$$

This algorithm turns the conditions in the middle and right column above into the conditions in the left column if we let ? vary through all objects of $\mathcal{C}$. More precisely, the equation in the left column implies the interpretation of the condition in the other two columns for any choice of ?, and the converse holds for a suitable choice of ?.

For instance, in the last condition $g^{-1} \cdot h^{-1} = (h \cdot g)^{-1}$, we may choose ? $= \mathsf{G}^1 \times_{\mathsf{r},\mathsf{G}^0,\mathsf{s}} \mathsf{G}^1$, $g = \mathsf{pr}_1 \colon ? \to \mathsf{G}^1$, and $h = \mathsf{pr}_2 \colon ? \to \mathsf{G}^1$. The interpretation of the formula $g^{-1} \cdot h^{-1} = (h \cdot g)^{-1}$ for these choices is $\mathsf{m} \circ (\mathsf{i} \times_{\mathsf{r},\mathsf{G}^0,\mathsf{s}} \mathsf{i}) = \mathsf{i} \circ \mathsf{m} \circ \sigma$.

As another example, the associativity diagram (3) is equivalent to the elementwise statement

$$(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3) \qquad \forall g_1, g_2, g_3 \in \mathsf{G}^1, \ \mathsf{s}(g_1) = \mathsf{r}(g_2), \ \mathsf{s}(g_2) = \mathsf{r}(g_3). \qquad (4)$$

To go from the elementwise statement to (3), choose ? $= \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1$, $g_1 = \mathsf{pr}_1$, $g_2 = \mathsf{pr}_2$, $g_3 = \mathsf{pr}_3$ (we take one factor for each of the free variables g_1, g_2, g_3, and implement the assumptions $\mathsf{s}(g_1) = \mathsf{r}(g_2)$, $\mathsf{s}(g_2) = \mathsf{r}(g_3)$ by fibre-product conditions). Then $A := g_1 \cdot g_2$ is interpreted as $\mathsf{m} \circ (\mathsf{pr}_1, \mathsf{pr}_2) \colon \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \to \mathsf{G}^1$, so $(g_1 \cdot g_2) \cdot g_3 = A \cdot g_3$ becomes the map

$$\mathsf{m} \circ (\mathsf{m} \circ (\mathsf{pr}_1, \mathsf{pr}_2), \mathsf{pr}_3) = \mathsf{m} \circ (\mathsf{m} \times_{\mathsf{G}^0,\mathsf{r}} \mathsf{id}_{\mathsf{G}^1}) \circ (\mathsf{pr}_1, \mathsf{pr}_2, \mathsf{pr}_3) = \mathsf{m} \circ (\mathsf{m} \times_{\mathsf{G}^0,\mathsf{r}} \mathsf{id}_{\mathsf{G}^1})$$

from $\mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1$ to G^1. Similarly, $g_1 \cdot (g_2 \cdot g_3)$ is interpreted as $\mathsf{m} \circ (\mathsf{id}_{\mathsf{G}^1} \times_{\mathsf{s},\mathsf{G}^0} \mathsf{m})$. Thus $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$ implies that (3) commutes. Conversely, if (3) commutes, then $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$ for any choice of ? $\in\in \mathcal{C}$ and $g_1, g_2, g_3 \colon ? \to \mathsf{G}^1$ with $\mathsf{s} \circ g_1 = \mathsf{r} \circ g_2$ and $\mathsf{s} \circ g_2 = \mathsf{r} \circ g_3$. Thus (3) is equivalent to the elementwise statement above.

We may also use elementwise formulas to define maps. For instance, suppose that we have already defined maps $f \colon X \to \mathsf{G}^1$ and $g \colon Y \to \mathsf{G}^1$. Then there is a unique map $F \colon X \times_{\mathsf{s} \circ f, \mathsf{G}^0, \mathsf{r} \circ g} Y \to \mathsf{G}^1$ given elementwise by $F(x, y) := f(x) \cdot g(y)$ for all $x \in X$, $y \in Y$ with $\mathsf{s} \circ f(x) = \mathsf{r} \circ g(y)$. If a map $f \in \mathcal{C}(X, Z)$ is given in terms of elements, then it is an isomorphism if and only if every element z of Z may be written as $f(x)$ for a unique $x \in X$: the interpretation of this statement in terms of maps is exactly the Yoneda Lemma.

3.2. REMARK. Why do we need the range and source maps to be covers? Of course, we need some assumption for the fibre product $\mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1$ to exist, but there are deeper reasons. Many results in the theory of principal bundles depend on the locality of isomorphisms (Proposition 2.5), which holds only for pull-backs along covers. The composition of bibundle equivalences cannot work unless we require their anchor maps to be covers. The range and source maps of a groupoid must be covers because they are the anchor maps for the unit bibundle equivalence G^1 on G.

3.3. SECOND DEFINITION. For groupoids in sets, the existence of units and inverses is equivalent to the existence of unique solutions $x \in \mathsf{G}^1$ to equations of the form $x \cdot g = h$ for $g, h \in \mathsf{G}^1$ with $\mathsf{s}(g) = \mathsf{s}(h)$ and $g \cdot x = h$ for $g, h \in \mathsf{G}^1$ with $\mathsf{r}(g) = \mathsf{r}(h)$. This leads us to the following equivalent definition of a groupoid in $(\mathcal{C}, \mathcal{T})$:

3.4. DEFINITION. *A groupoid* G *in* $(\mathcal{C}, \mathcal{T})$ *consists of* $\mathsf{G}^0 \in\in \mathcal{C}$ *(objects),* $\mathsf{G}^1 \in\in \mathcal{C}$ *(arrows),* $\mathsf{r} \in \mathcal{C}(\mathsf{G}^1, \mathsf{G}^0)$ *(range),* $\mathsf{s} \in \mathcal{C}(\mathsf{G}^1, \mathsf{G}^0)$ *(source),* *and* $\mathsf{m} \in \mathcal{C}(\mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1, \mathsf{G}^1)$ *(multiplication),* *such that*

(1) *the maps* r *and* s *are covers;*

(2) *the maps*

$$(\mathsf{pr}_2, \mathsf{m}) \colon \mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1 \xrightarrow{\sim} \mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{s}} \mathsf{G}^1, \qquad (x, g) \mapsto (g, x \cdot g), \qquad (5)$$

$$(\mathsf{pr}_1, \mathsf{m}) \colon \mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1 \xrightarrow{\sim} \mathsf{G}^1 \times_{\mathsf{r}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1, \qquad (g, x) \mapsto (g, g \cdot x), \qquad (6)$$

are well-defined isomorphisms;

(3) m *is associative in the sense of* (3) *or, equivalently,* (4).

The first condition implies that the fibre products $\mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1$, $\mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{s}} \mathsf{G}^1$ and $\mathsf{G}^1 \times_{\mathsf{r}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1$ used above exist in $\mathcal{C}$. The maps in (5) and (6) are well-defined if and only if $\mathsf{s} \circ \mathsf{m} = \mathsf{s} \circ \mathsf{pr}_2$ and $\mathsf{r} \circ \mathsf{m} = \mathsf{r} \circ \mathsf{pr}_1$, so these two equations are assumed implicitly.

If $(\mathsf{G}^1, \mathsf{G}^0, \mathsf{r}, \mathsf{s}, \mathsf{m}, \mathsf{u}, \mathsf{i})$ is a groupoid in the first sense above, then it is one in the sense of Definition 3.4: the map $(g, h) \mapsto (h \cdot g^{-1}, g)$ is inverse to (5), and the map $(g, h) \mapsto (g, g^{-1} \cdot h)$ is inverse to (6). The converse is proved in Proposition 3.9.

3.5. LEMMA. *The invertibility of the map in* (5) *is equivalent to the elementwise statement that for all* $g, h \in \mathsf{G}^1$ *with* $\mathsf{s}(g) = \mathsf{s}(h)$ *there is a unique* $x \in \mathsf{G}^1$ *with* $\mathsf{s}(x) = \mathsf{r}(g)$ *and* $x \cdot g = h$. *The invertibility of the map in* (6) *is equivalent to the elementwise statement that for all* $g, h \in \mathsf{G}^1$ *with* $\mathsf{r}(g) = \mathsf{r}(h)$ *there is a unique* $x \in \mathsf{G}^1$ *with* $\mathsf{s}(g) = \mathsf{r}(x)$ *and* $g \cdot x = h$.

PROOF. We only prove the first statement, the second one is similar.

We interpret g, h, x as maps $g, h, x \colon ? \to \mathsf{G}^1$ in $\mathcal{C}$. The elementwise statement says that for $? \in\in \mathcal{C}$ and $g, h \colon ? \to \mathsf{G}^1$ with $\mathsf{s} \circ g = \mathsf{s} \circ h$ there is a unique $x \in \mathcal{C}(?, \mathsf{G}^1)$ with $\mathsf{s} \circ x = \mathsf{r} \circ g$ and $\mathsf{m} \circ (x, g) = h$. The maps g and h with $\mathsf{s} \circ g = \mathsf{s} \circ h$ are equivalent to a single map $(g, h) \colon ? \to \mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{s}} \mathsf{G}^1$, and the conditions $\mathsf{s} \circ x = \mathsf{r} \circ g$ and $\mathsf{m} \circ (x, g) = h$ together are equivalent to (x, g) being a map $? \to \mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1$ with $(\mathsf{pr}_2, \mathsf{m}) \circ (x, g) = (g, h)$. Hence the elementwise statement is equivalent to the statement that for each object $?$ of $\mathcal{C}$ and each map $(g, h) \colon ? \to \mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{s}} \mathsf{G}^1$ there is a unique map $A \colon ? \to \mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1$ with $(\mathsf{pr}_2, \mathsf{m}) \circ A = (g, h)$. This statement means that $(\mathsf{pr}_2, \mathsf{m})$ is invertible by the Yoneda Lemma. ∎

To better understand (5) and (6), we view m as a *ternary relation* on G^1.

3.6. DEFINITION. *An n-ary relation* *between* $X_1, \ldots, X_n \in\in \mathcal{C}$ *is* $R \in\in \mathcal{C}$ *with maps* $p_j \colon R \to X_j$ *for* $j = 1, \ldots, n$ *such that given* $x_j \in X_j$ *for* $j = 1, \ldots, n$, *there is at most one* $r \in R$ *with* $p_j(r) = x_j$ *for* $j = 1, \ldots, n$.

The *multiplication relation* defined by m has

$$R := \mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1, \qquad p_1 := \mathsf{pr}_1, \quad p_2 := \mathsf{pr}_2, \quad p_3 := \mathsf{m}.$$

3.7. LEMMA. *The maps in* (5) *and* (6) *are well-defined isomorphisms if and only if the following three maps are well-defined isomorphisms:*

$$(p_1, p_2)\colon R \to \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1,$$
$$(p_2, p_3)\colon R \to \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,\mathsf{s}} \mathsf{G}^1,$$
$$(p_1, p_3)\colon R \to \mathsf{G}^1 \times_{\mathsf{r},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1.$$

PROOF. The first isomorphism is the definition of R, the second one is (5), the third one is (6). ∎

When we view the multiplication as a relation, then the isomorphisms (5) and (6) become similar to the statement that the multiplication is a partially defined map.

3.8. EQUIVALENCE OF BOTH DEFINITIONS OF A GROUPOID.

3.9. PROPOSITION. *Let* $(\mathsf{G}^0, \mathsf{G}^1, \mathsf{r}, \mathsf{s}, \mathsf{m})$ *be a groupoid in* $(\mathcal{C}, \mathcal{T})$ *as in Definition* 3.4, *where* $(\mathcal{C}, \mathcal{T})$ *is a category with a subcanonical pretopology. Then there are unique maps* $\mathsf{u}\colon \mathsf{G}^0 \to \mathsf{G}^1$ *and* $\mathsf{i}\colon \mathsf{G}^1 \to \mathsf{G}^1$ *with the properties of unit map and inversion listed above. Moreover, the multiplication* m *is a cover.*

PROOF. We write down the proof in terms of elements. Interpreting this as explained above gives a proof in a general category with pretopology.

First we construct the unit map. For any $g \in \mathsf{G}^1$, the elementwise interpretation of (5) in Lemma 3.5 gives a unique map $\bar{\mathsf{u}}\colon \mathsf{G}^1 \to \mathsf{G}^1$ with $\mathsf{s}(\bar{\mathsf{u}}(g)) = \mathsf{r}(g)$ and $\bar{\mathsf{u}}(g) \cdot g = g$. Hence $\mathsf{r}(\bar{\mathsf{u}}(g)) = \mathsf{r}(\bar{\mathsf{u}}(g) \cdot g) = \mathsf{r}(g)$. Associativity implies

$$\bar{\mathsf{u}}(g) \cdot (g \cdot h) = (\bar{\mathsf{u}}(g) \cdot g) \cdot h = g \cdot h = \bar{\mathsf{u}}(g \cdot h) \cdot (g \cdot h)$$

for all $g, h \in \mathsf{G}^1$ with $\mathsf{s}(g) = \mathsf{r}(h)$. Since $\bar{\mathsf{u}}(g \cdot h)$ is unique, this gives $\bar{\mathsf{u}}(g \cdot h) = \bar{\mathsf{u}}(g)$ for all $g, h \in \mathsf{G}^1$ with $\mathsf{s}(g) = \mathsf{r}(h)$, that is, $\bar{\mathsf{u}} \circ \mathsf{m} = \bar{\mathsf{u}} \circ \mathsf{pr}_1$. The map $(\mathsf{pr}_1, \mathsf{m})$ is the isomorphism in (6), so we get $\bar{\mathsf{u}} \circ \mathsf{pr}_1 = \bar{\mathsf{u}} \circ \mathsf{pr}_2$ on $\mathsf{G}^1 \times_{\mathsf{r},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1$. Since the pretopology $\mathcal{T}$ is subcanonical, the cover r is the coequaliser of $\mathsf{pr}_1, \mathsf{pr}_2\colon \mathsf{G}^1 \times_{\mathsf{r},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \rightrightarrows \mathsf{G}^1$. Hence there is a unique map $\mathsf{u}\colon \mathsf{G}^0 \to \mathsf{G}^1$ with $\bar{\mathsf{u}} = \mathsf{u} \circ \mathsf{r}$. Since r is an epimorphism, the equations $\mathsf{s} \circ \bar{\mathsf{u}} = \mathsf{r}$ and $\mathsf{r} \circ \bar{\mathsf{u}} = \mathsf{r}$ imply $\mathsf{s} \circ \mathsf{u} = \mathrm{id}_{\mathsf{G}^0}$ and $\mathsf{r} \circ \mathsf{u} = \mathrm{id}_{\mathsf{G}^0}$. Interpreting $1_x = \mathsf{u}(x)$ for $x \in \mathsf{G}^0$, this becomes $\mathsf{s}(1_x) = x = \mathsf{r}(1_x)$ for all $x \in \mathsf{G}^0$. The defining condition $\bar{\mathsf{u}}(g) \cdot g = g$ for $g \in \mathsf{G}^1$ becomes $1_{\mathsf{r}(g)} \cdot g = g$ for all $g \in \mathsf{G}^1$. If $g, h \in \mathsf{G}^1$ satisfy $\mathsf{s}(g) = \mathsf{r}(h)$, then

$$(g \cdot 1_{\mathsf{s}(g)}) \cdot h = g \cdot (1_{\mathsf{s}(g)} \cdot h) = g \cdot (1_{\mathsf{r}(h)} \cdot h) = g \cdot h.$$

Since g is the unique element of G^1 with $x \cdot h = g \cdot h$ by (5), we also get $g \cdot 1_{\mathsf{s}(g)} = g$ for all $g \in \mathsf{G}^1$. Hence the unit map has all required properties.

The inverse map $\mathsf{i}\colon \mathsf{G}^1 \to \mathsf{G}^1$ is defined as the unique map with $\mathsf{s}(\mathsf{i}(g)) = \mathsf{r}(g)$ and $\mathsf{i}(g) \cdot g = 1_{\mathsf{s}(g)}$ for all $g \in \mathsf{G}^1$. Since $\mathsf{s}(g) = \mathsf{s}(1_{\mathsf{s}(g)})$, (5) provides a unique map with this property. We write g^{-1} instead of $\mathsf{i}(g)$ in the following. Then $\mathsf{s}(g^{-1}) = \mathsf{r}(g)$, $g^{-1} \cdot g = 1_{\mathsf{s}(g)}$ by definition, and $\mathsf{r}(g^{-1}) = \mathsf{r}(g^{-1} \cdot g) = \mathsf{r}(1_{\mathsf{s}(g)}) = \mathsf{s}(g)$. Associativity gives

$$(g \cdot g^{-1}) \cdot g = g \cdot (g^{-1} \cdot g) = g \cdot 1_{\mathsf{s}(g)} = g = 1_{\mathsf{r}(g)} \cdot g.$$

Since the map $x\colon \mathsf{G}^1 \to \mathsf{G}^1$ with $x \cdot g = 1_{\mathsf{r}(g)} \cdot g$ is unique by (5), this implies $g \cdot g^{-1} = 1_{\mathsf{r}(g)}$. Similarly, associativity gives

$$(g \cdot h)^{-1} \cdot (g \cdot h) = 1_{\mathsf{s}(g \cdot h)} = 1_{\mathsf{s}(h)} = (h^{-1} \cdot g^{-1}) \cdot (g \cdot h)$$

for all $g, h \in \mathsf{G}^1$ with $\mathsf{s}(g) = \mathsf{r}(h)$. Hence the uniqueness of the solution of $x \cdot (g \cdot h) = 1_{\mathsf{s}(h)}$ implies $(g \cdot h)^{-1} = h^{-1} \cdot g^{-1}$ for all $g, h \in \mathsf{G}^1$ with $\mathsf{s}(g) = \mathsf{r}(h)$. Similarly,

$$(g^{-1})^{-1} \cdot g^{-1} = 1_{\mathsf{s}(g^{-1})} = 1_{\mathsf{r}(g)} = g \cdot g^{-1},$$

and the uniqueness of the solution to $x \cdot g^{-1} = 1_{\mathsf{r}(g)}$ gives $(g^{-1})^{-1} = g$ for all $g \in \mathsf{G}^1$. Thus the inversion has all expected properties.

Since s is a cover, so is the coordinate projection $\mathsf{pr}_2\colon \mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{s}} \mathsf{G}^1 \to \mathsf{G}^1$. Composing pr_2 with the invertible map in (5) gives that $\mathsf{m}\colon \mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1 \to \mathsf{G}^1$ is a cover. ∎

3.10. EXAMPLES. The following examples play an important role for the general theory.

3.11. EXAMPLE. An object X of $\mathcal{C}$ is viewed as a groupoid by taking $\mathsf{G}^1 = \mathsf{G}^0 = X$, $\mathsf{r} = \mathsf{s} = \mathrm{id}_X$, and letting m be the canonical isomorphism $X \times_X X \xrightarrow{\sim} X$. A groupoid is isomorphic to one of this form if and only if its range or source map is an isomorphism. Such groupoids are called 0-*groupoids*.

3.12. EXAMPLE. Let $p\colon X \twoheadrightarrow Y$ be a cover. Its *Čech groupoid* is the groupoid defined by $\mathsf{G}^0 = X$, $\mathsf{G}^1 = X \times_{p, Y, p} X$, $\mathsf{r}(x_1, x_2) = x_1$, $\mathsf{s}(x_1, x_2) = x_2$ for all $x_1, x_2 \in X$ with $p(x_1) = p(x_2)$, and $(x_1, x_2) \cdot (x_2, x_3) := (x_1, x_3)$ for all $x_1, x_2, x_3 \in X$ with $p(x_1) = p(x_2) = p(x_3)$. The range and source maps are covers by construction. The multiplication is clearly associative. The canonical isomorphisms

$$
\begin{aligned}
\mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1 &\xrightarrow{\sim} X \times_{p, Y, p} X \times_{p, Y, p} X, & (x_1, x_2, x_2, x_3) &\mapsto (x_1, x_2, x_3), \\
\mathsf{G}^1 \times_{\mathsf{r}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1 &\xrightarrow{\sim} X \times_{p, Y, p} X \times_{p, Y, p} X, & (x_1, x_2, x_1, x_3) &\mapsto (x_1, x_2, x_3), \\
\mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{s}} \mathsf{G}^1 &\xrightarrow{\sim} X \times_{p, Y, p} X \times_{p, Y, p} X, & (x_1, x_2, x_3, x_2) &\mapsto (x_1, x_2, x_3),
\end{aligned}
$$

show that (5) and (6) are isomorphisms. Thus we have a groupoid in $(\mathcal{C}, \mathcal{T})$. Its unit and inversion maps are given by $1_x = (x, x)$ and $(x_1, x_2)^{-1} = (x_2, x_1)$.

3.13. EXAMPLE. Let G be a groupoid and let $p\colon X \twoheadrightarrow \mathsf{G}^0$ be a cover. We define a groupoid $p^*\mathsf{G} = \mathsf{G}(X)$ with objects X and arrows $X \times_{p, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, p} X$, range and source maps pr_1 and pr_3, and with the multiplication map defined elementwise by

$$(x_1, g_1, x_2) \cdot (x_2, g_2, x_3) := (x_1, g_1 \cdot g_2, x_3)$$

for all $x_1, x_2, x_3 \in X$, $g_1, g_2 \in \mathsf{G}^1$ with $p(x_1) = \mathsf{r}(g_1)$, $p(x_2) = \mathsf{s}(g_1) = \mathsf{r}(g_2)$, $p(x_3) = \mathsf{s}(g_2)$; this multiplication is associative. Since the maps s, r, and p are covers, the fibre product $X \times_{p, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, p} X$ exists and the source and range maps of $\mathsf{G}(X)$ are covers. Straightforward computations show that (5) and (6) are isomorphisms. Units and inverses

are given by $1_x = (x, 1_{p(x)}, x)$ and $(x_1, g, x_2)^{-1} = (x_2, g^{-1}, x_1)$ for all $x, x_1, x_2 \in X$, $g \in \mathsf{G}^1$ with $p(x_1) = \mathsf{r}(g)$, $p(x_2) = \mathsf{s}(g)$.

If G is just a space Y viewed as a groupoid (Example 3.11), then $\mathsf{G}(X)$ is the Čech groupoid defined in Example 3.12.

Let $p_1 \colon X_1 \twoheadrightarrow X_2$ and $p_2 \colon X_2 \twoheadrightarrow \mathsf{G}^0$ be covers. Then $p_2 \circ p_1$ is a cover and there is a natural groupoid isomorphism $p_1^*(p_2^*\mathsf{G}) \cong (p_2 \circ p_1)^*\mathsf{G}$.

3.14. EXAMPLE. Assume Assumption 2.10 about a final object $\star$. Then a *group* in $(\mathcal{C}, \mathcal{T})$ is a groupoid G with $\mathsf{G}^0 = \star$. The range and source maps $\mathsf{G}^1 \to \mathsf{G}^0 = \star$ are automatically covers by Assumption 2.10. The multiplication is now defined on the full product $\mathsf{G}^1 \times_\star \mathsf{G}^1 = \mathsf{G}^1 \times \mathsf{G}^1$, and the isomorphisms (5) and (6) also take place on the product $\mathsf{G}^1 \times \mathsf{G}^1$. The unit in a groupoid is a map $\mathsf{u} \colon \star \to \mathsf{G}^1$. For any object $? \in\in \mathcal{C}$, let $1 \colon ? \to \mathsf{G}^1$ be the composite of u with the unique map $? \to \star$. These maps give the *unit element* in G^1. It satisfies $1 \cdot g = g = g \cdot 1$ for any map $g \colon ? \to \mathsf{G}^1$.

3.15. FUNCTORS AND NATURAL TRANSFORMATIONS.

3.16. DEFINITION. *Let G and H be groupoids in $(\mathcal{C}, \mathcal{T})$. A functor from G to H is given by arrows $F^i \in \mathcal{C}(\mathsf{G}^i, \mathsf{H}^i)$ in $\mathcal{C}$ for $i = 0, 1$ with $\mathsf{r}_\mathsf{H}(F^1(g)) = F^0(\mathsf{r}_\mathsf{G}(g))$ and $\mathsf{s}_\mathsf{H}(F^1(g)) = F^0(\mathsf{s}_\mathsf{G}(g))$ for all $g \in \mathsf{G}^1$ and $F^1(g_1 \cdot g_2) = F^1(g_1) \cdot F^1(g_2)$ for all $g_1, g_2 \in \mathsf{G}^1$ with $\mathsf{s}_\mathsf{G}(g_1) = \mathsf{r}_\mathsf{G}(g_2)$.*

The identity functor $\mathrm{id}_\mathsf{G} \colon \mathsf{G} \to \mathsf{G}$ *on a groupoid G has $F^i = \mathrm{id}_{\mathsf{G}^i}$ for $i = 0, 1$.*

The product *of two functors $F_1 \colon \mathsf{G} \to \mathsf{H}$ and $F_2 \colon \mathsf{H} \to \mathsf{K}$ is the functor $F_2 \circ F_1 \colon \mathsf{G} \to \mathsf{K}$ given by the composite maps $F_2^i \circ F_1^i \colon \mathsf{G}^i \to \mathsf{K}^i$ for $i = 0, 1$.*

Let $F_1, F_2 \colon \mathsf{G} \rightrightarrows \mathsf{H}$ be two parallel functors. A natural transformation $F_1 \Rightarrow F_2$ is $\Phi \in \mathcal{C}(\mathsf{G}^0, \mathsf{H}^1)$ with $\mathsf{s}(\Phi(x)) = F_1^0(x)$, $\mathsf{r}(\Phi(x)) = F_2^0(x)$ for all $x \in \mathsf{G}^0$ and $\Phi(\mathsf{r}(g)) \cdot F_1^1(g) = F_2^1(g) \cdot \Phi(\mathsf{s}(g))$ for all $g \in \mathsf{G}^1$:

$$
\begin{array}{ccc}
F_1^0(\mathsf{s}(g)) & \xrightarrow{\;F_1^1(g)\;} & F_1^0(\mathsf{r}(g)) \\
{\scriptstyle\Phi(\mathsf{s}(g))}\downarrow & & \downarrow{\scriptstyle\Phi(\mathsf{r}(g))} \\
F_2^0(\mathsf{s}(g)) & \xrightarrow[\;F_2^1(g)\;]{} & F_2^0(\mathsf{r}(g))
\end{array}
$$

The identity transformation $1_F \colon F \Rightarrow F$ *on a functor $F \colon \mathsf{G} \to \mathsf{H}$ is given by $\Phi(x) = 1_{F(x)}$ for all $x \in \mathsf{G}^0$. If $\Phi \colon F_1 \Rightarrow F_2$ is a natural transformation, then its* inverse Φ^{-1} *is the natural transformation $F_2 \Rightarrow F_1$ given by $\mathsf{i} \circ \Phi \colon x \mapsto \Phi(x)^{-1}$.*

The vertical product *of natural transformations $\Phi \colon F_1 \Rightarrow F_2$ and $\Psi \colon F_2 \Rightarrow F_3$ for three functors $F_1, F_2, F_3 \colon \mathsf{G} \rightrightarrows\!\!\!\!\rightarrow \mathsf{H}$ is the natural transformation $\Psi \cdot \Phi \colon F_1 \Rightarrow F_3$ defined by $(\Psi \cdot \Phi)(x) := \Psi(x) \cdot \Phi(x)$ for all $x \in \mathsf{G}^0$.*

Let $F_1, F_1' \colon \mathsf{G} \rightrightarrows \mathsf{H}$ and $F_2, F_2' \colon \mathsf{H} \rightrightarrows \mathsf{K}$ be composable pairs of parallel functors and let $\Phi \colon F_1 \Rightarrow F_1'$ and $\Psi \colon F_2 \Rightarrow F_2'$ be natural transformations. Their horizontal product *is the natural transformation $\Psi \circ \Phi \colon F_2 \circ F_1 \Rightarrow F_2' \circ F_1'$ defined by*

$$
(\Psi \circ \Phi)(x) := \Psi((F_1')^0(x)) \cdot F_2^1(\Phi(x)) = (F_2')^1(\Phi(x)) \cdot \Psi(F_1^0(x)),
$$

$$F_2^0 \circ F_1^0(x) \xrightarrow{\quad F_2^1(\Phi(x)) \quad} F_2^0 \circ (F_1')^0(x)$$

$$\Psi(F_1^0(x)) \downarrow \qquad (\Psi \circ \Phi)(x) \qquad \downarrow \Psi((F_1')^0(x))$$

$$(F_2')^0 \circ F_1^0(x) \xrightarrow[\quad (F_2')^1(\Phi(x)) \quad]{} (F_2')^0 \circ (F_1')^0(x)$$

for all $x \in \mathsf{G}^0$. This diagram commutes because of the naturality of Ψ applied to $\Phi(x)$.

If $\Phi \colon F_1 \Rightarrow F_2$ is a natural transformation, then

$$\Phi^{-1} \cdot \Phi = 1_{F_1} \colon F_1 \Rightarrow F_1, \qquad \Phi \cdot \Phi^{-1} = 1_{F_2} \colon F_2 \Rightarrow F_2,$$

justifying the name inverse. Thus all natural transformations between groupoids are *natural isomorphisms.*

3.17. PROPOSITION. *Groupoids, functors, and natural transformations in $(\mathcal{C}, \mathcal{T})$ with the products defined above form a strict 2-category in which all 2-arrows are invertible.*

PROOF. See [2, 28, 39] for the definition of a strict 2-category. The proof is straightforward. ∎

3.18. EXAMPLE. Let G be a groupoid and let $p \colon X \twoheadrightarrow \mathsf{G}^0$ be a cover. Then p and the map $\mathsf{pr}_2 \colon X \times_{p,\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,p} X \to \mathsf{G}^1$ form a functor $p_* \colon \mathsf{G}(X) \to \mathsf{G}$. Functors of this form are called *hypercovers.*

3.19. EXAMPLE. Let G and H be groupoids, $F \colon \mathsf{G} \to \mathsf{H}$ a functor, and $p \colon X \twoheadrightarrow \mathsf{H}^0$ a cover. Let $\tilde{X} := \mathsf{G}^0 \times_{F^0,\mathsf{H}^0,p} X$; this exists and $\mathsf{pr}_1 \colon \tilde{X} \twoheadrightarrow \mathsf{G}^0$ is a cover because p is a cover. We define a functor $p^*(F) = F(X) \colon \mathsf{G}(\tilde{X}) \to \mathsf{H}(X)$ by $\mathsf{pr}_2 \colon \tilde{X} \to X$ on objects and by $(x_1, g, x_2) \mapsto (\mathsf{pr}_2(x_1), F^1(g), \mathsf{pr}_2(x_2))$ for all $x_1, x_2 \in \tilde{X}$, $g \in \mathsf{G}^1$ with $\mathsf{pr}_1(x_1) = \mathsf{r}(g)$ and $\mathsf{pr}_1(x_2) = \mathsf{s}(g)$ on arrows.

3.20. EXAMPLE. An equivalence from the identity functor on G to a functor $F \colon \mathsf{G} \to \mathsf{G}$ is $\Phi \in \mathcal{C}(\mathsf{G}^0, \mathsf{G}^1)$ with $\mathsf{s} \circ \Phi = \mathrm{id}_{\mathsf{G}^0}$, $\mathsf{r} \circ \Phi = F^0$, and

$$F^1(g) = \Phi(\mathsf{r}(g)) \cdot g \cdot \Phi(\mathsf{s}(g))^{-1} \qquad \text{for all } g \in \mathsf{G}^1. \tag{7}$$

Conversely, let Φ be any section for $\mathsf{s} \colon \mathsf{G}^1 \twoheadrightarrow \mathsf{G}^0$. Then $F^0 = \mathsf{r} \circ \Phi$ and (7) define an endomorphism $\mathrm{Ad}(\Phi) = F \colon \mathsf{G} \to \mathsf{G}$ such that Φ is a natural transformation $\mathrm{id}_{\mathsf{G}^0} \Rightarrow F$. Functors $F \colon \mathsf{G} \to \mathsf{G}$ that are naturally equivalent to the identity functor are called *inner.*

The horizontal product of $\Phi_i \colon \mathrm{id}_{\mathsf{G}^0} \Rightarrow \mathrm{Ad}(\Phi_i)$ for $i = 1, 2$ is a natural transformation

$$\Phi_1 \circ \Phi_2 \colon \mathrm{id}_{\mathsf{G}^0} \Rightarrow \mathrm{Ad}(\Phi_1) \circ \mathrm{Ad}(\Phi_2).$$

Here

$$\Phi_1 \circ \Phi_2(x) := \Phi_1(\mathsf{r} \circ \Phi_2(x)) \cdot \Phi_2(x) \qquad \text{for all } x \in \mathsf{G}^0. \tag{8}$$

This product on sections for s gives a monoid with unit $\mathsf{u} \colon x \mapsto 1_x$. The map Ad from this monoid to the submonoid of inner endomorphisms of G is a unital homomorphism, that is, $\mathrm{Ad}(\Phi_1 \circ \Phi_2) = \mathrm{Ad}(\Phi_1) \circ \mathrm{Ad}(\Phi_2)$ and $\mathrm{Ad}(1) = \mathrm{id}_\mathsf{G}$.

3.21. DEFINITION AND LEMMA. *The following are equivalent for a section Φ of* s*:*

(1) Φ *is invertible for the horizontal product* $\circ$*;*

(2) $\mathrm{Ad}(\Phi)$ *is an automorphism;*

(3) $\mathsf{r} \circ \Phi$ *is invertible in* $\mathcal{C}$*.*

Sections of s *with these equivalent properties are called* bisections*.*

PROOF. Let Φ be invertible for $\circ$ with inverse Φ^{-1}. Then $\mathrm{Ad}(\Phi^{-1})$ is inverse to $\mathrm{Ad}(\Phi)$, so $\mathrm{Ad}(\Phi)$ is an automorphism of G. Then $\mathsf{r} \circ \Phi = \mathrm{Ad}(\Phi)^0 \in \mathcal{C}(\mathsf{G}^0, \mathsf{G}^0)$ is invertible in $\mathcal{C}$. Conversely, if $\alpha := \mathsf{r} \circ \Phi$ is invertible in $\mathcal{C}$, then $\Phi^{-1}(x) := \Phi(\alpha^{-1}(x))^{-1} \colon \mathsf{G}^0 \to \mathsf{G}^1$ has $\mathsf{s}(\Phi^{-1}(x)) = \mathsf{r}(\Phi(\alpha^{-1}(x))) = \alpha\alpha^{-1}(x) = x$, so it is a section, and both $\Phi \circ \Phi^{-1} = 1_x$ and $\Phi^{-1} \circ \Phi = 1_x$. We have shown $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$. ∎

We may think of a section as a subobject $\Phi(\mathsf{G}^0) \subseteq \mathsf{G}^1$ such that s restricts to an isomorphism $\Phi(\mathsf{G}^0) \overset{\sim}{\to} \mathsf{G}^0$. Being a bisection means that *both* s and r restrict to isomorphisms $\Phi(\mathsf{G}^0) \overset{\sim}{\to} \mathsf{G}^0$. This explains the name "bisection."

Isomorphisms of groupoids with natural transformations between them form a strict 2-groupoid. Thus the automorphisms of G with natural transformations between them form a strict 2-group or, equivalently, a crossed module. The crossed module combines automorphisms of G and bisections because the latter are the natural transformations from the identity functor to another automorphism of G. The crossed module involves the horizontal product $\circ$ on bisections and the composition $\circ$ on automorphisms, and the group homomorphism Ad from bisections to automorphisms; another piece of structure is the conjugation action of automorphisms on bisections, given by conjugation:

$$F \bullet \Phi := 1_F \circ \Phi \circ 1_{F^{-1}} \colon \mathrm{id}_\mathsf{G} = F \circ \mathrm{id}_\mathsf{G} \circ F^{-1} \Rightarrow F \, \mathrm{Ad}(\Phi) F^{-1}.$$

By the definition of the horizontal product, we have

$$F \bullet \Phi(x) = F^1(\Phi(x)) \qquad \text{for all } x \in \mathsf{G}^0.$$

The crossed module conditions $\mathrm{Ad}(F \bullet \Phi) = F \, \mathrm{Ad}(\Phi) F^{-1}$ and $\mathrm{Ad}(\Phi_1) \bullet \Phi_2 = \Phi_1 \circ \Phi_2 \circ \Phi_1^{-1}$ are easy to check.

3.22. ANAFUNCTORS. The functor $p_* \colon \mathsf{G}(X) \to \mathsf{G}$ from Example 3.18 is not an isomorphism, but should be an equivalence of groupoids. When we formally invert these functors, we arrive at the following definition:

3.23. DEFINITION. *Let* G *and* H *be groupoids in* $\mathcal{C}$*. An* anafunctor *from* G *to* H *is a triple* (X, p, F)*, where* $X \in\in \mathcal{C}$*,* $p \colon X \twoheadrightarrow \mathsf{G}^0$ *is a cover, and* $F \colon \mathsf{G}(X) \to \mathsf{H}$ *is a functor, with* $\mathsf{G}(X)$ *defined in Example* 3.13*.*

An isomorphism *between two anafunctors* (X_i, p_i, F_i)*,* $i = 1, 2$*, is an isomorphism* $\varphi \in \mathcal{C}(X_1, X_2)$ *with* $p_2 \circ \varphi = p_1$ *and* $F_2 \circ \varphi_* = F_1$*, where* $\varphi_* \colon \mathsf{G}(X_1) \to \mathsf{G}(X_2)$ *is the*

functor given on objects by φ and on arrows by $\varphi_*^1(x_1, g, x_2) = (\varphi(x_1), g, \varphi(x_2))$ for all $x_1, x_2 \in X_1$, $g \in \mathsf{G}^1$ with $p_1(x_1) = \mathsf{r}(g)$, $p_1(x_2) = \mathsf{s}(g)$.

The functor $F\colon \mathsf{G}(X) \to \mathsf{H}$ is a pair of maps

$$F^0\colon X \to \mathsf{H}^0, \qquad F^1\colon X \times_{p,\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,p} X \to \mathsf{H}^1$$

with $\mathsf{r}(F^1(x_1, g, x_2)) = F^0(x_1)$, $\mathsf{s}(F^1(x_1, g, x_2)) = F^0(x_2)$ for all $x_1, x_2 \in X$, $g \in \mathsf{G}^1$ with $p(x_1) = \mathsf{r}(g)$, $\mathsf{s}(g) = p(x_2)$, and $F^1(x_1, g_1, x_2) \cdot F^1(x_2, g_2, x_3) = F^1(x_1, g_1 \cdot g_2, x_3)$ for all $x_1, x_2, x_3 \in X$, $g_1, g_2 \in \mathsf{G}^1$ with $p(x_1) = \mathsf{r}(g_1)$, $p(x_2) = \mathsf{s}(g_1) = \mathsf{r}(g_2)$, and $p(x_3) = \mathsf{s}(g_2)$.

Pronk [42] and Carchedi [11] and several other authors use spans of functors like we do, but with weak equivalences instead of hypercovers. As a consequence, they use the weak instead of the strong pull-back to compose spans. An equivalent notion of anafunctor is used by Roberts [44], who also shows that we get an equivalent bicategory if we invert all weak equivalences. The following construction of the bicategory of anafunctors may also be found in [44].

We are going to compose anafunctors. Let G_1, G_2 and G_3 be groupoids in $(\mathcal{C}, \mathcal{T})$ and let (X_{ij}, p_{ij}, F_{ij}) for $ij = 12$ and $ij = 23$ be anafunctors from G_i to G_j. Their product (X_{13}, p_{13}, F_{13}) is the anafunctor from G_1 to G_3 given by

$$X_{13} := X_{12} \times_{F_{12}^0, \mathsf{G}_2^0, p_{23}} X_{23},$$
$$p_{13} := p_{12} \circ \mathsf{pr}_1\colon X_{13} \twoheadrightarrow X_{12} \twoheadrightarrow \mathsf{G}^0,$$
$$F_{13} := F_{23} \circ F_{12}(X_{23})\colon \mathsf{G}_1(X_{13}) \cong \mathsf{G}_1(X_{12})(X_{23}) \to \mathsf{G}_2(X_{23}) \to \mathsf{G}_3.$$

Thus $F_{13}^0(x_1, x_2) = F_{23}^0(x_2)$ for all $x_1 \in X_{12}$, $x_2 \in X_{23}$ with $F_{12}^0(x_1) = p_{23}(x_2)$ and $F_{13}^1(x_1, x_2, g, x_3, x_4) = F_{23}^1(x_2, F_{12}^1(x_1, g, x_3), x_4)$ for all $x_1, x_3 \in X_{12}$, $x_2, x_4 \in X_{23}$, $g \in \mathsf{G}^1$ with $F_{12}^0(x_1) = p_{23}(x_2)$, $F_{12}^0(x_3) = p_{23}(x_4)$, $p_{12}(x_1) = \mathsf{r}(g)$, $p_{12}(x_3) = \mathsf{s}(g)$. The map $\mathsf{pr}_1\colon X_{13} \twoheadrightarrow X_{12}$ is a cover because p_{23} is one. Hence p_{13} is a cover as a composite of covers.

3.24. LEMMA. *The composition of anafunctors is associative up to isomorphism; the associator comes from the canonical isomorphism*

$$(X_{12} \times_{F_{12}^0, \mathsf{G}_2^0, p_{23}} X_{23}) \times_{F_{23}^0 \mathsf{pr}_2, \mathsf{G}_3^0, p_{34}} X_{34} \xrightarrow{\sim} X_{12} \times_{F_{12}^0, \mathsf{G}_2^0, p_{23}\mathsf{pr}_1} (X_{23} \times_{F_{23}^0, \mathsf{G}_3^0, p_{34}} X_{34}).$$

The identity functor viewed as an anafunctor with $X = \mathsf{G}^0$ and $p = \mathrm{id}_{\mathsf{G}^0}$ is a unit for this composition, up to the natural isomorphisms $Y \times_{\mathsf{G}^0} \mathsf{G}^0 \cong Y \cong \mathsf{G}^0 \times_{\mathsf{G}^0} Y$.

PROOF. The proof is routine. ∎

Since fibre products are, in general, only associative up to the canonical isomorphism in the above lemma, anafunctors do not form a category; together with their isomorphisms, they give a bicategory. We now incorporate natural transformations into this bicategory.

To simplify notation, we usually ignore the associators in Lemma 3.24 from now on, assuming that fibre products are strictly associative.

3.25. Definition. *Let* G *and* H *be groupoids in* $\mathcal{C}$ *and let* (X_1, p_1, F_1) *and* (X_2, p_2, F_2) *be anafunctors from* G *to* H. *Let* $X := X_1 \times_{p_1, \mathsf{G}^0, p_2} X_2$ *and let* $p := p_1 \circ \mathsf{pr}_1 = p_2 \circ \mathsf{pr}_2 \colon X \twoheadrightarrow \mathsf{G}^0$. *We get functors* $F_1 \circ (\mathsf{pr}_1)_* \colon \mathsf{G}(X) \to \mathsf{H}$ *and* $F_2 \circ (\mathsf{pr}_2)_* \colon \mathsf{G}(X) \to \mathsf{H}$. *An* anafunctor natural transformation *from* (X_1, p_1, F_1) *to* (X_2, p_2, F_2) *is a natural transformation* $\Phi \colon F_1 \circ (\mathsf{pr}_1)_* \Rightarrow F_2 \circ (\mathsf{pr}_2)_*$:

$$
\begin{array}{ccc}
X \xrightarrow{\ \mathsf{pr}_1\ } X_1 & \qquad & \mathsf{G}(X) \xrightarrow{\ (\mathsf{pr}_1)_*\ } \mathsf{G}(X_1) \\[2pt]
{\scriptstyle \mathsf{pr}_2}\Big\downarrow \ \ \searrow^{p} \ \Big\downarrow{\scriptstyle p_1} & & {\scriptstyle (\mathsf{pr}_2)_*}\Big\downarrow \quad \Phi \quad \Big\downarrow{\scriptstyle F_1} \\[2pt]
X_2 \xrightarrow[\ p_2\ ]{} \mathsf{G}^0 & & \mathsf{G}(X_2) \xrightarrow[\ F_2\ ]{} \mathsf{H}
\end{array}
$$

The natural transformation Φ *is a map* $\Phi \colon X \to \mathsf{H}^1$ *with* $\mathsf{s} \circ \Phi = F_1^0 \circ \mathsf{pr}_1 \colon X \to \mathsf{H}^0$, $\mathsf{r} \circ \Phi = F_2^0 \circ \mathsf{pr}_2 \colon X \to \mathsf{H}^0$, *and*

$$
\Phi(x_1, x_2) \cdot F_1^1(x_1, g, x_3) = F_2^1(x_2, g, x_4) \cdot \Phi(x_3, x_4)
$$

for all $x_1, x_3 \in X_1$, $x_2, x_4 \in X_2$, $g \in \mathsf{G}^1$ *with* $p_1(x_1) = p_2(x_2) = \mathsf{r}(g)$ *and* $p_1(x_3) = p_2(x_4) = \mathsf{s}(g)$.

3.26. Example. Let (X_i, p_i, F_i) for $i = 1, 2$ be anafunctors from G to H and let $\varphi \colon X_1 \xrightarrow{\sim} X_2$ be an isomorphism between them. Then

$$
\Phi(x_1, x_2) := F_1^1(\varphi^{-1}(x_2), 1_{p_1(x_1)}, x_1) = F_2^1(x_2, 1_{p_2(x_2)}, \varphi(x_1))
$$

for all $x_1 \in X_1$, $x_2 \in X_2$ with $p_1(x_1) = p_2(x_2)$ is a natural transformation from (X_1, p_1, F_1) to (X_2, p_2, F_2).

3.27. Lemma. *Let* $F_1, F_2 \colon \mathsf{G} \to \mathsf{H}$ *be functors and let* $p \colon X \twoheadrightarrow \mathsf{G}^0$ *be a cover. Let* $p_* \colon \mathsf{G}(X) \to \mathsf{G}$ *be the induced hypercover. Then natural transformations* $F_1 \circ p_* \Rightarrow F_2 \circ p_*$ *are in canonical bijection with natural transformations* $F_1 \Rightarrow F_2$.

Proof. A natural transformations $\Phi \colon F_1 \Rightarrow F_2$ induces a natural transformation $\Phi \circ 1_{p_*} \colon F_1 \circ p_* \Rightarrow F_2 \circ p_*$ by the horizontal product; explicitly, $\Phi \circ 1_{p_*}$ is the map $\Phi \circ p \colon X \twoheadrightarrow \mathsf{G}^0 \to \mathsf{H}^1$. To show that this canonical map $\Phi \mapsto \Phi \circ 1_{p_*}$ is bijective, we must prove that any natural transformation $\Psi \colon F_1 \circ p_* \Rightarrow F_2 \circ p_* \colon \mathsf{G}(X) \rightrightarrows \mathsf{H}^1$ factors uniquely through p and a map $\Phi \colon \mathsf{G}^0 \to \mathsf{H}^1$; the latter is then automatically a natural transformation $F_1 \Rightarrow F_2$ because $p_*^1 \colon \mathsf{G}(X)^1 \twoheadrightarrow \mathsf{G}^1$ is a cover, hence an epimorphism, and the conditions for a natural transformation hold after composing with p_*^1.

Since Ψ is natural with respect to all arrows $(x_1, 1_{p(x_1)}, x_2)$ in $\mathsf{G}(X)$ for $x_1, x_2 \in X$ with $p(x_1) = p(x_2)$, we get $\Psi(x_1) = \Psi(x_2)$ if $p(x_1) = p(x_2)$. Since the pretopology is subcanonical and p is a cover, Ψ factors uniquely through $p \colon X \twoheadrightarrow \mathsf{G}^0$. ∎

3.28. THEOREM. *Groupoids in $(\mathcal{C}, \mathcal{T})$, anafunctors and natural transformations of ana-functors with the composition of anafunctors described above form a bicategory with in-vertible 2-arrows.*

PROOF. We must compose natural transformations of anafunctors vertically and horizon-tally. We reduce this to the same constructions for ordinary natural transformations: the reduction explains why the conditions for a bicategory hold.

Let (X_i, p_i, F_i) be three anafunctors from G to H and let $\Phi_{ij} \colon (X_i, p_i, F_i) \Rightarrow (X_j, p_j, F_j)$ for $ij = 12, 23$ be natural transformations to compose vertically. Let $X_{ij} := X_i \times_{p_i, \mathsf{G}^0, p_j} X_j$ and $p_{ij} := p_i \circ \mathsf{pr}_1 = p_j \circ \mathsf{pr}_2 \colon X_{ij} \twoheadrightarrow \mathsf{G}^0$ for $ij = 12, 13, 23$. Let $X_{123} := X_1 \times_{p_1, \mathsf{G}^0, p_2} X_2 \times_{p_2, \mathsf{G}^0, p_3} X_3$ and $p_{123} := p_1 \circ \mathsf{pr}_1 = p_2 \circ \mathsf{pr}_2 = p_3 \circ \mathsf{pr}_3 \colon X \twoheadrightarrow \mathsf{G}^0$.

The given natural transformations are maps $\Phi_{12} \colon X_{12} \to \mathsf{H}^1$ and $\Phi_{23} \colon X_{23} \to \mathsf{H}^1$ with, among others, $\mathsf{s}(\Phi_{12}(x_1, x_2)) = F_1^0(x_1)$, $\mathsf{r}(\Phi_{12}(x_1, x_2)) = F_2^0(x_2) = \mathsf{s}(\Phi_{23}(x_2, x_3))$, $\mathsf{r}(\Phi_{23}(x_2, x_3)) = F_3^0(x_3)$ for all $x_i \in X_i$ with $p_i(x_i) = p_j(x_j)$ for $i, j \in \{1, 2, 3\}$. We define

$$\Phi \colon X_{123} \to \mathsf{H}^1, \qquad \Phi(x_1, x_2, x_3) := \Phi_{23}(x_2, x_3) \cdot \Phi_{12}(x_1, x_2),$$

which is well-defined because $\mathsf{r}(\Phi_{12}(x_1, x_2)) = F_2^0(x_2) = \mathsf{s}(\Phi_{23}(x_2, x_3))$; moreover,

$$\mathsf{s}(\Phi(x_1, x_2, x_3)) = \mathsf{s}(\Phi_{12}(x_1, x_2)) = F_1^0(x_1)$$

and

$$\mathsf{r}(\Phi(x_1, x_2, x_3)) = \mathsf{r}(\Phi_{23}(x_2, x_3)) = F_3^0(x_3).$$

The naturality of Φ_{ij} implies that Φ is a natural transformation from $F_1 \circ (\mathsf{pr}_1)_* \colon \mathsf{G}(X_{123}) \to \mathsf{H}$ to $F_3 \circ (\mathsf{pr}_3)_* \colon \mathsf{G}(X_{123}) \to \mathsf{H}$. Lemma 3.27 shows that Φ factors uniquely through $\mathsf{pr}_{13} \colon X_{123} \twoheadrightarrow X_{13}$ and thus gives a natural transformation $\Phi_{13} \colon (X_1, p_1, F_1) \Rightarrow (X_3, p_3, F_3)$. This Φ_{13} is the vertical product of Φ_{23} and Φ_{12}.

If Φ_{12} comes from an isomorphism of anafunctors $\varphi_{12} \colon X_1 \to X_2$ as in Example 3.26, then the composite Φ_{13} above is

$$\Phi_{13} \colon X_1 \times_{p_1, \mathsf{G}^0, p_3} X_3 \xrightarrow[\cong]{\varphi_{12} \times_{\mathsf{G}^0, p_3} X_3} X_2 \times_{p_2, \mathsf{G}^0, p_3} X_3 \xrightarrow{\Phi_{23}} \mathsf{H}^1. \tag{9}$$

Similarly, if Φ_{23} comes from an isomorphism of anafunctors $\varphi_{23} \colon X_2 \to X_3$, then the vertical product Φ_{13} is

$$\Phi_{13} \colon X_1 \times_{p_1, \mathsf{G}^0, p_3} X_3 \xrightarrow[\cong]{X_1 \times_{p_1, \mathsf{G}^0} \varphi_{23}^{-1}} X_1 \times_{p_1, \mathsf{G}^0, p_2} X_2 \xrightarrow{\Phi_{12}} \mathsf{H}^1. \tag{10}$$

As a consequence, the natural transformation corresponding to the identity isomorphism of an anafunctor is a unit for the vertical product.

Since arrows in groupoids are invertible, any natural transformation of anafunctors has an inverse: exchange the order of the factors in the fibre product and compose Φ with the inversion map on H^1. This is an inverse for the vertical product above. Thus all 2-arrows are invertible.

When we further compose with a natural transformation

$$\Phi_{34} \colon (X_3, p_3, F_3) \Rightarrow (X_4, p_4, F_4),$$

then we may first construct a natural transformation on the fibre product X_{1234} of all four X_i. Then by Lemma 3.27 this factors uniquely through the covers $X_{1234} \twoheadrightarrow X_{124}$ and $X_{1234} \twoheadrightarrow X_{134}$, and then further through the covers $X_{124} \twoheadrightarrow X_{14}$ and $X_{134} \twoheadrightarrow X_{14}$. The resulting map on X_{14} does not depend on whether we factor through X_{124} or X_{134} first. Thus the vertical product is associative.

Now we turn to the horizontal product. Let K be a third groupoid in $(\mathcal{C}, \mathcal{T})$; let (X_i, p_i, F_i) for $i = 1, 2$ be anafunctors $\mathsf{G} \to \mathsf{H}$ and let (Y_i, q_i, E_i) for $i = 1, 2$ be anafunctors $\mathsf{H} \to \mathsf{K}$; let $\Phi \colon (X_1, p_1, F_1) \Rightarrow (X_2, p_2, F_2)$ and $\Psi \colon (Y_1, q_1, E_1) \Rightarrow (Y_2, q_2, E_2)$ be anafunctor natural transformations. The horizontal product $\Psi \circ \Phi$ must be a natural transformation of anafunctors $E_1 \circ F_1 \Rightarrow E_2 \circ F_2$. We are going to reduce it to an ordinary horizontal product of natural transformations of composable functors. To make our functors composable, we pass to larger fibre products.

First, let

$$Y := Y_1 \times_{q_1, \mathsf{H}^0, q_2} Y_2, \qquad \bar{E}_i := E_i \circ (\mathsf{pr}_i)_* \colon \mathsf{H}(Y) \to \mathsf{H}(Y_i) \to \mathsf{K};$$

by definition, Ψ is a natural transformation of functors $\Psi \colon \bar{E}_1 \Rightarrow \bar{E}_2$. Secondly, let

$$Y_i X_j := Y_i \times_{q_i, \mathsf{H}^0, F_j^0} X_j$$

for $i, j \in \{1, 2\}$ and

$$YX := Y_1 X_1 \times_{\mathsf{G}^0} Y_2 X_1 \times_{\mathsf{G}^0} Y_1 X_2 \times_{\mathsf{G}^0} Y_2 X_2 \cong (Y \times_{q, \mathsf{H}^0, F_1^0} X_1) \times_{\mathsf{G}^0} (Y \times_{q, \mathsf{H}^0, F_2^0} X_2).$$

Let $\bar{F}_i \colon \mathsf{G}(YX) \to \mathsf{H}(Y)$ be the composite functors $\mathsf{G}(YX) \to \mathsf{G}(Y \times_{\mathsf{G}^0} X_i) \to \mathsf{H}(Y)$, where the first functor is associated to the coordinate projection $YX \twoheadrightarrow Y \times_{\mathsf{G}^0} X_i$ and the second functor is $F_i(Y)$. The natural transformation of anafunctors Φ induces a natural transformation of ordinary functors $\bar{\Phi} \colon \bar{F}_1 \Rightarrow \bar{F}_2$.

Now the natural transformations of ordinary functors Ψ and $\bar{\Phi}$ may be composed horizontally in the usual way, giving a natural transformation $\bar{E}_1 \circ \bar{F}_1 \Rightarrow \bar{E}_2 \circ \bar{F}_2$. The composite functor $\bar{E}_i \circ \bar{F}_j \colon \mathsf{G}(YX) \to \mathsf{K}$ and the composite anafunctor $E_i \circ F_j$ are related as follows: the latter is given by a functor $E_i \circ F_j \colon \mathsf{G}(Y_i X_j) \to \mathsf{K}$, and we get $\bar{E}_i \circ \bar{F}_j$ by composing $E_i \circ F_j$ with the functor $\mathsf{G}(YX) \to \mathsf{G}(Y_i X_j)$ associated to the coordinate projection $YX \to Y_i X_j$. Now Lemma 3.27 shows that the natural transformation $\Psi \circ \bar{\Phi}$ descends to a natural transformation of anafunctors $E_1 \circ F_1 \Rightarrow E_2 \circ F_2$. This defines the horizontal product.

It is routine to see that the horizontal product is associative and commutes with the vertical product (exchange law). For each proof, we replace anafunctors by ordinary functors defined over suitable fibre products, and first get the desired equality of natural transformations on this larger fibre product. Then Lemma 3.27 shows that it holds as an equality of natural transformations of anafunctors.

The composition of anafunctors is associative and unital up to canonical isomorphisms of anafunctors by Lemma 3.27. These isomorphisms give natural transformations by Example 3.26. These are natural with respect to natural transformations of anafunctors and satisfy the usual coherence conditions; this follows from the simplifications of the vertical product with isomorphisms in (9) and (10). ∎

3.29. THEOREM. *The bicategory of anafunctors is the localisation of the 2-category of groupoids, functors and natural transformations at the class of hypercovers, and it is also the localisation at the class of weak equivalences.*

This theorem clarifies in which sense the anafunctor bicategory is the right bicategory of groupoids to work with. It follows from results in [44]: first, [44, Theorem 7.2] says that the anafunctor bicategory is the localisation at the class of all weak equivalences; secondly, [44, Proposition 6.4] implies that it makes no difference whether we localise at hypercovers or weak equivalences. Instead of a proof, we make the statement of Theorem 3.29 more concrete.

Let $\mathfrak{Fun}$ be the strict 2-category of groupoids, functors and transformations. Let $\mathfrak{Ana}$ be the bicategory of groupoids, anafunctors and anafunctor natural transformations. Let $\mathcal{D}$ be another bicategory. We embed $\mathfrak{Fun} \to \mathfrak{Ana}$ in the obvious way, sending a functor $F \colon \mathsf{G} \to \mathsf{H}$ to the anafunctor $(\mathsf{G}^0, \mathrm{id}_{\mathsf{G}^0}, F)$. We assume now that fibre products in $\mathcal{C}$ of the form $X \times_X Y$ or $Y \times_X X$ are *equal* to Y, not just canonically *isomorphic* via the coordinate projection; this can be arranged by redefining the fibre product in these cases. This choice ensures that the map from functors to anafunctors above is part of a *strict* homomorphism of bicategories. Furthermore, it implies that the identity functors are strict units in the bicategory of anafunctors, that is, the unit transformations $\mathrm{id}_{\mathsf{G}} \circ (X, p, F) \cong (X, p, F)$ and $(X, p, F) \circ \mathrm{id}_{\mathsf{H}} \cong (X, p, F)$ for an anafunctor $(X, p, F) \colon \mathsf{G} \to \mathsf{H}$ are identity 2-arrows.

Since the embedding $\mathfrak{Fun} \hookrightarrow \mathfrak{Ana}$ is a strict homomorphism, we may restrict a morphism of bicategories $E \colon \mathfrak{Ana} \to \mathcal{D}$ to a morphism $E|_{\mathfrak{Fun}} \colon \mathfrak{Fun} \to \mathcal{D}$; restrict a transformation $\Phi \colon E_1 \to E_2$ between two morphisms $E_1, E_2 \colon \mathfrak{Ana} \rightrightarrows \mathcal{D}$ to a transformation $\Phi|_{\mathfrak{Fun}} \colon E_1|_{\mathfrak{Fun}} \to E_2|_{\mathfrak{Fun}}$; and restrict a modification $\psi \colon \Phi_1 \to \Phi_2$ between two such transformations $\Phi_1, \Phi_2 \colon E_1 \rightrightarrows E_2$ to a modification $\psi|_{\mathfrak{Fun}} \colon \Phi_1|_{\mathfrak{Fun}} \to \Phi_2|_{\mathfrak{Fun}}$. Here and below we use the notation of [28] regarding bicategories, morphisms, transformations, and modifications.

Theorem 3.29 turns out to be equivalent to the combination of the following three more concrete statements:

(1) Any homomorphism $\mathfrak{Fun} \to \mathcal{D}$ that maps hypercovers to equivalences is the restriction of a homomorphism $\mathfrak{Ana} \to \mathcal{D}$.

(2) Let $\bar{E}_1, \bar{E}_2 \colon \mathfrak{Ana} \rightrightarrows \mathcal{D}$ be homomorphisms. Any strong transformation $\bar{E}_1|_{\mathfrak{Fun}} \to \bar{E}_2|_{\mathfrak{Fun}}$ is the restriction of a strong transformation $\bar{E}_1 \to \bar{E}_2$.

(3) Let $\bar{E}_1, \bar{E}_2 \colon \mathfrak{Ana} \rightrightarrows \mathcal{D}$ be homomorphisms and let $\bar{\Phi}_1, \bar{\Phi}_2 \colon \bar{E}_1 \rightrightarrows \bar{E}_2$ be strong transformations. Restriction to $\mathfrak{Fun}$ gives a bijection from modifications $\bar{\Phi}_1 \Rightarrow \bar{\Phi}_2$ to modifications $\bar{\Phi}_1|_{\mathfrak{Fun}} \Rightarrow \bar{\Phi}_2|_{\mathfrak{Fun}}$.

We restrict attention to *homo*morphisms and *strong* transformations, as usual in the theory of localisations [42, 44]. That is, we require the 2-arrows in $\mathcal{D}$ that appear in the definition of a morphism or a transformation to be invertible.

Actually, instead of (2), the localisation theorem in [44] only says that any strong transformation $\Phi\colon \bar{E}_1|_{\mathfrak{Fun}} \to \bar{E}_2|_{\mathfrak{Fun}}$ is *equivalent to* the restriction of a strong transformation $\bar{\Phi}\colon \bar{E}_1 \to \bar{E}_2$ by an invertible modification $t\colon \bar{\Phi}|_{\mathfrak{Fun}} \to \Phi$. We may, however, get rid of t because $\mathfrak{Fun}$ and $\mathfrak{Ana}$ have the same objects. Thus we may conjugate $\bar{\Phi}$ by the invertible modification t to get a strong transformation that restricts to Φ *exactly*. Hence the usual form of the localisation theorem gives (2). Similarly, instead of (1), the localisation theorem in [44] only says that for any homomorphism $E\colon \mathfrak{Fun} \to \mathcal{D}$ there is a homomorphism $\bar{E}\colon \mathfrak{Ana} \to \mathcal{D}$ such that $\bar{E}|_{\mathfrak{Fun}}$ is *equivalent to* E; that is, there are strong transformations between E and $\bar{E}|_{\mathfrak{Fun}}$ that are inverse to each other up to invertible modifications. In particular, this strong transformation gives equivalences $E(\mathsf{G}) \cong \bar{E}(\mathsf{G})$ for any groupoid G; we may conjugate $\bar{E}$ by these to produce an equivalent homomorphism $\bar{E}'$ so that all the arrows $\bar{E}'(\mathsf{G}) \to E(\mathsf{G})$ involved in the resulting strong transformation $\bar{E}'|_{\mathfrak{Fun}} \cong E$ become identities. Hence this strong transformation only consists of invertible 2-arrows $\bar{E}'(F) \cong E(F)$ for all functors F, subject to some conditions. We use these invertible 2-arrows together with the identity 2-arrows on $\bar{E}'(X, p, F)$ for anafunctors with non-identical p to change $\bar{E}'$ once more to an equivalent homomorphism $\bar{E}''\colon \mathfrak{Ana} \to \mathcal{D}$. Now the strong transformation $\bar{E}''|_{\mathfrak{Fun}} \cong E$ becomes the identity transformation; that is, $\bar{E}''|_{\mathfrak{Fun}} = E$. Thus (1)–(3) are equivalent to the usual form of the localisation theorem.

If the bicategory $\mathcal{D}$ is strictly unital as well, then it makes sense to restrict attention to strictly unital homomorphisms. A homomorphism on $\mathfrak{Ana}$ is strictly unital if and only if its restriction to $\mathfrak{Fun}$ is. Hence assertions (1)–(2) continue to hold for strictly unital homomorphisms and strictly unital strong transformations as well.

3.30. ANAFUNCTOR EQUIVALENCES AND ANA-ISOMORPHISMS. An arrow $f\colon \mathsf{G} \to \mathsf{H}$ in a bicategory is an *equivalence* if there are an arrow $g\colon \mathsf{H} \to \mathsf{G}$ and invertible 2-arrows $g \circ f \Rightarrow \mathrm{id}_{\mathsf{G}}$ and $f \circ g \Rightarrow \mathrm{id}_{\mathsf{H}}$.

3.31. DEFINITION. *Equivalences in the bicategory of anafunctors are called* anafunctor equivalences.

Anafunctor natural transformations may be far from being "isomorphisms." This makes it non-trivial to decide whether a given anafunctor is an anafunctor equivalence. We are going to provide necessary and sufficient conditions for this that are easier to check.

3.32. DEFINITION. *An* ana-isomorphism *between two groupoids* G *and* H *in* $(\mathcal{C}, \mathcal{T})$ *is given by an object* $X \in\in \mathcal{C}$, *covers* $p\colon X \twoheadrightarrow \mathsf{G}^0$ *and* $q\colon X \twoheadrightarrow \mathsf{H}^0$, *and a groupoid isomorphism* $F\colon p^*(\mathsf{G}) \xrightarrow{\sim} q^*(\mathsf{H})$ *with* $F^0 = \mathrm{id}_X$. *An anafunctor from* G *to* H *lifts to an ana-isomorphism if it is equivalent* (*through an anafunctor natural transformation*) *to* $(X, p, q_* \circ \bar{F})$ *for an ana-isomorphism* $(X, p, q, \bar{F})$.

3.33. DEFINITION. *A functor $F\colon \mathsf{G} \to \mathsf{H}$ is* essentially surjective *if the map*

$$\mathsf{r_X}\colon \mathsf{G}^0 \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1 \to \mathsf{H}^0, \qquad (g,x) \mapsto \mathsf{s}(x), \tag{11}$$

is a cover. It is almost essentially surjective *if the map* (11) *is a cover locally (see Definition* 2.3*). It is* fully faithful *if the fibre product $\mathsf{G}^0 \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1 \times_{\mathsf{s},\mathsf{H}^0,F^0} \mathsf{G}^0$ exists in $\mathcal{C}$ and the map*

$$\mathsf{G}^1 \to \mathsf{G}^0 \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1 \times_{\mathsf{s},\mathsf{H}^0,F^0} \mathsf{G}^0, \qquad g \mapsto (\mathsf{r}(g), F^1(g), \mathsf{s}(g)), \tag{12}$$

is an isomorphism in $\mathcal{C}$. It is almost fully faithful *if there is some cover $q\colon Y \twoheadrightarrow \mathsf{H}^0$ for which the functor $q^*(F)$ is fully faithful.*

It is trivial that essentially surjective functors are almost essentially surjective and fully faithful functors are almost fully faithful. The converse holds under Assumption 2.8:

3.34. LEMMA. *If Assumption* 2.8 *holds, then any almost essentially surjective functor is essentially surjective. An essentially surjective and almost fully faithful functor is fully faithful.*

PROOF. Assumption 2.8 says that a map that is locally a cover is a cover, so an almost essentially surjective functor is essentially surjective.

The fibre product $\mathsf{G}^0 \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1$ always exists because r is a cover. If F is essentially surjective, then $\mathsf{s}\colon \mathsf{G}^0 \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1 \twoheadrightarrow \mathsf{H}^0$ is a cover, so the fibre product $(\mathsf{G}^0 \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1) \times_{\mathsf{s},\mathsf{H}^0,F^0} \mathsf{G}^0$ exists. Let $q\colon Y \twoheadrightarrow \mathsf{H}^0$ be a cover such that $q^*(F)$ is fully faithful. The projection

$$\mathsf{pr}_{234}\colon Y \times_{q,\mathsf{H}^0,F^0} \mathsf{G}^0 \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1 \times_{\mathsf{s},\mathsf{H}^0,F^0} \mathsf{G}^0 \times_{F^0,\mathsf{H}^0,q} Y \to \mathsf{G}^0 \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1 \times_{\mathsf{s},\mathsf{H}^0,F^0} \mathsf{G}^0$$

is a cover because q is a cover. The pull-back of the map (12) for the functor F along this cover gives the map (12) for the functor $q^*(F)$, which is assumed to be an isomorphism. Since being an isomorphism is a local property by Proposition 2.5, the functor F is fully faithful if $q^*(F)$ is fully faithful. (This conclusion holds whenever the fibre product $\mathsf{G}^0 \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1 \times_{\mathsf{s},\mathsf{H}^0,F^0} \mathsf{G}^0$ exists.) ∎

3.35. THEOREM. *Let (X,p,F) be an anafunctor from G to H. Then the following are equivalent:*

(1) *(X,p,F) is an anafunctor equivalence;*

(2) *the functor $F\colon \mathsf{G}(X) \to \mathsf{H}$ is almost fully faithful and almost essentially surjective;*

(3) *(X,p,F) lifts to an ana-isomorphism.*

If Assumption 2.8 *holds, then we may replace* (2) *by the condition that F is fully faithful and essentially surjective – the usual definition of a weak equivalence.*

The proof of the theorem will occupy the remainder of this subsection. The last statement about replacing (2) is Lemma 3.34. We will show the implications (1)⇒(2)⇒(3)⇒(1).

We start with the easiest implication: (3)⇒(1). The main point here is the following example:

3.36. EXAMPLE. Let G be a groupoid in $(\mathcal{C}, \mathcal{T})$ and let $p \colon X \twoheadrightarrow \mathsf{G}^0$ be a cover. The hypercover $p_* \colon \mathsf{G}(X) \to \mathsf{G}$ is an anafunctor equivalence. Its quasi-inverse is the anafunctor $(X, p, \mathrm{id}_{\mathsf{G}(X)})$ from G to $\mathsf{G}(X)$. The composite functor $\mathsf{G} \to \mathsf{G}(X) \to \mathsf{G}$ is the anafunctor (X, p, p_*). The unit section $X \to \mathsf{G}(X)^1$ gives an anafunctor natural transformation between (X, p, p_*) and the identity anafunctor $(\mathsf{G}^0, \mathrm{id}_{\mathsf{G}^0}, \mathrm{id}_{\mathsf{G}})$ on G. The other composite functor $\mathsf{G}(X) \to \mathsf{G} \to \mathsf{G}(X)$ is the anafunctor $(X \times_{\mathsf{G}^0} X, \mathsf{pr}_1, (\mathsf{pr}_2)_*)$. It is naturally equivalent to the identity on $\mathsf{G}(X)$ because the functors $(\mathsf{pr}_2)_*$ and $(\mathsf{pr}_1)_*$ from $\mathsf{G}(X \times_{\mathsf{G}^0} X)$ to $\mathsf{G}(X)$ are naturally equivalent.

If $F \cong q_* \circ \bar{F}$ for a cover $q \colon X \twoheadrightarrow \mathsf{H}^0$ and an isomorphism $\bar{F}$, then the functors $F \colon \mathsf{G}(X) \to \mathsf{H}$ and $p_* \colon \mathsf{G}(X) \to \mathsf{G}$ are anafunctor equivalences by Example 3.36. Since $(X, p, F) \circ p_* \simeq F \colon \mathsf{G}(X) \to \mathsf{H}$, we have $(X, p, F) \simeq F \circ (p_*)^{-1}$, so (X, p, F) is an anafunctor equivalence as well; here $\simeq$ denotes equivalence of anafunctors. Thus (3) implies (1) in Theorem 3.35.

Now we show (2)$\Rightarrow$(3). The following lemma explains why (almost) fully faithful functors are related to hypercovers.

3.37. LEMMA. *Let* $F \colon \mathsf{G}(X) \to \mathsf{H}$ *be a functor. The following are equivalent:*

(1) *F is isomorphic to a hypercover;*

(2) *F is fully faithful and $F^0 \colon X \twoheadrightarrow \mathsf{H}^0$ is a cover;*

(3) *F is almost fully faithful and $F^0 \colon X \twoheadrightarrow \mathsf{H}^0$ is a cover.*

Here "isomorphic" means that there is an isomorphism $\mathsf{G}(X) \cong q^*(\mathsf{H})$ intertwining F and the hypercover $q_* \colon q^*(\mathsf{H}) \to \mathsf{H}$.

PROOF. If F^0 is a cover, then F is essentially surjective, so F is almost fully faithful if and only if F is fully faithful by Lemma 3.34. Thus we only need to show the equivalence between (1) and (2).

Assume (2). The map $(\mathsf{r}, F^1, \mathsf{s}) \colon \mathsf{G}(X)^1 \xrightarrow{\sim} X \times_{F^0, \mathsf{H}^0, \mathsf{r}} \mathsf{H}^1 \times_{\mathsf{s}, \mathsf{H}^0, F^0} X$ is an isomorphism because F is fully faithful. This isomorphism together with the identity map on X is a groupoid isomorphism from $\mathsf{G}(X)$ to $(F^0)^*(\mathsf{H})$ that intertwines F and q_*, giving (1). Conversely, assume (1). If $F \cong q_*$, then F is fully faithful and F^0 is a cover because q is one. ∎

Now assume that $F \colon \mathsf{G}(X) \to \mathsf{H}$ is almost fully faithful and almost essentially surjective. Then there is a cover $q \colon Y \twoheadrightarrow \mathsf{H}^0$ such that the map

$$\mathsf{pr}_3 \colon XHY := X \times_{F^0, \mathsf{H}^0, \mathsf{r}} \mathsf{H}^1 \times_{\mathsf{s}, \mathsf{H}^0, q} Y \twoheadrightarrow Y$$

is a cover. Then so is $q \circ \mathsf{pr}_3 \colon XHY \twoheadrightarrow \mathsf{H}^0$. The projection $\mathsf{pr}_1 \colon XHY \twoheadrightarrow X$ is a cover because q and r are covers. Hence the anafunctor (X, p, F) is equivalent to $(XHY, p \circ \mathsf{pr}_1, F \circ (\mathsf{pr}_1)_*)$. The functor $F \circ (\mathsf{pr}_1)_*$ maps $(x, h, y) \mapsto F^0(x)$ on objects and $(x_1, h_1, y_1, g, x_2, h_2, y_2) \mapsto F^1(x_1, g, x_2)$ on arrows. It is equivalent to the functor E given by

$$E^0(x, h, y) := q(y) = \mathsf{s}(h), \qquad E^1(x_1, h_1, y_1, g, x_2, h_2, y_2) := h_1^{-1} \cdot F^1(x_1, g, x_2) \cdot h_2;$$

the natural transformation $E \Rightarrow F \circ (\mathsf{pr}_1)_*$ is $(x, h, y) \mapsto h$. Thus (X, p, F) is equivalent to $(X\mathsf{H}Y, p \circ \mathsf{pr}_1, E)$. The map $E^0 = q \circ \mathsf{pr}_3 \colon X\mathsf{H}Y \twoheadrightarrow \mathsf{H}^0$ is a cover.

3.38. LEMMA. *The functor E is almost fully faithful.*

PROOF. By assumption, there is a cover $r \colon Z \twoheadrightarrow \mathsf{H}^0$ such that $r^*(F)$ is fully faithful; that is, the map

$$Z \times_{r,\mathsf{H}^0,F^0} X \times_{p,\mathsf{G}^0,r} \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,p} X \times_{F^0,\mathsf{H}^0,r} Z$$
$$\to Z \times_{r,\mathsf{H}^0,F^0} X \times_{F^0,\mathsf{H}^0,r} \mathsf{H}^1 \times_{\mathsf{s},\mathsf{H}^0,F^0} X \times_{F^0,\mathsf{H}^0,r} Z,$$
$$(z_1, x_1, g, x_2, z_2) \mapsto (z_1, x_1, F^1(x_1, g, x_2), x_2, z_2),$$

is an isomorphism. Adding extra variables $y_1, y_2 \in Y$, $h_1, h_2 \in \mathsf{H}^1$ with $(x_i, h_i, y_i) \in X\mathsf{H}Y$ for $i = 1, 2$ gives an isomorphism

$$(z_1, x_1, h_1, y_1, g, x_2, h_2, y_2, z_2) \mapsto (z_1, x_1, h_1, y_1, F^1(x_1, g, x_2), x_2, h_2, y_2, z_2)$$

between the resulting fibre products. The conjugation map

$$(z_1, x_1, h_1, y_1, h, x_2, h_2, y_2, z_2) \mapsto (z_1, x_1, h_1, y_1, h_1^{-1} h h_2, x_2, h_2, y_2, z_2)$$

is an isomorphism between suitable fibre products as well, with inverse

$$(z_1, x_1, h_1, y_1, h, x_2, h_2, y_2, z_2) \mapsto (z_1, x_1, h_1, y_1, h_1 h h_2^{-1}, x_2, h_2, y_2, z_2).$$

Composing these isomorphisms gives the map

$$(z_1, x_1, h_1, y_1, g, x_2, h_2, y_2, z_2)$$
$$\mapsto (z_1, x_1, h_1, y_1, E^1(x_1, h_1, y_1, g, x_2, h_2, y_2), x_2, h_2, y_2, z_2).$$

Since this is an isomorphism, E is almost fully faithful. ∎

Since E is almost fully faithful and E^0 is a cover, E is isomorphic to a hypercover by Lemma 3.37. Thus (X, p, F) lifts to an ana-isomorphism. This finishes the proof that (2)$\Rightarrow$(3) in Theorem 3.35.

Finally, we show that (1) implies (2) in Theorem 3.35; this is the most difficult part of the theorem. An anafunctor equivalence is given by anafunctors $(X, p, F) \colon \mathsf{G} \to \mathsf{H}$ and $(Y, q, E) \colon \mathsf{H} \to \mathsf{G}$ and natural transformations $\Psi \colon (X, p, F) \circ (Y, q, E) \Rightarrow \mathrm{id}_\mathsf{H}$ and $\Phi \colon (Y, q, E) \circ (X, p, F) \Rightarrow \mathrm{id}_\mathsf{G}$, which are automatically invertible. More explicitly, $\Psi \colon Y \times_{E^0,\mathsf{G}^0,p} X \to \mathsf{H}^1$ is an anafunctor natural transformation from $F \circ p^* E$ to id_H; that is, $\Psi(y, x) \in \mathsf{H}^1$ is defined for all $y \in Y$, $x \in X$ with $E^0(y) = p(x)$ and is an arrow in H^1 with $\mathsf{s}(\Psi(y, x)) = F^0(x)$ and $\mathsf{r}(\Psi(y, x)) = q(y)$; and for all $(x_1, y_1, h, y_2, x_2) \in X \times_{p,\mathsf{G}^0,E^0} Y \times_{q,\mathsf{H}^0,r} \mathsf{H}^1 \times_{\mathsf{s},\mathsf{H}^0,q} Y \times_{E^0,\mathsf{G}^0,p} X$,

$$\Psi(y_1, x_1) \cdot F^1(x_1, E^1(y_1, h, y_2), x_2) = h \cdot \Psi(y_2, x_2); \tag{13}$$

and $\Phi\colon X \times_{F^0,\mathsf{H}^0,q} Y \to \mathsf{G}^1$ is a natural transformation from $E \circ q^*F$ to id_G, that is, $\Phi(x,y) \in \mathsf{G}^1$ is defined for all $x \in X$, $y \in Y$ with $F^0(x) = q(y)$ and is an arrow in G^1 with $\mathsf{s}(\Psi(x,y)) = E^0(y)$ and $\mathsf{r}(\Psi(x,y)) = p(x)$; and for all $(y_1, x_1, g, x_2, y_2) \in Y \times_{q,\mathsf{H}^0,F^0} X \times_{p,\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,p} X \times_{F^0,\mathsf{H}^0,q} Y$,

$$\Phi(x_1, y_1) \cdot E^1(y_1, F^1(x_1, g, x_2), y_2) = g \cdot \Phi(x_2, y_2). \tag{14}$$

3.39. LEMMA. *There is a unique map* $a\colon X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1 \times_{\mathsf{s},\mathsf{H}^0,q} Y \to \mathsf{G}^1$ *with*

$$a(x_1, h, y_2) = \Phi(x_1, y_1) \cdot E^1(y_1, h, y_2) \tag{15}$$

for all $x_1 \in X$, $h \in \mathsf{H}^1$, $y_1, y_2 \in Y$ *with* $F^0(x_1) = \mathsf{r}(h) = q(y_1)$, $\mathsf{s}(h) = q(y_2)$.
The following map is an isomorphism:

$$(\mathsf{pr}_1, a, \mathsf{pr}_3)\colon X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1 \times_{\mathsf{s},\mathsf{H}^0,q} Y \xrightarrow{\sim} X \times_{p,\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,E^0} Y.$$

PROOF. The fibre products $X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1 \times_{\mathsf{s},\mathsf{H}^0,q} Y$ and $Y \times_{q,\mathsf{H}^0,F^0} X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1 \times_{\mathsf{s},\mathsf{H}^0,q} Y$ exist because r and q are covers. The fibre product $X \times_{p,\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,E^0} Y$ exists because p and s are covers. Equation (15) defines a map $\hat{a}\colon Y \times_{q,\mathsf{H}^0,F^0} X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1 \times_{\mathsf{s},\mathsf{H}^0,q} Y \to \mathsf{G}^1$. We must check that $\hat{a}$ factors through the projection pr_{234}. Since q is a cover, so is pr_{234}. Since our pretopology is subcanonical, $\hat{a}$ factors through this cover if and only if $\hat{a} \circ \pi_1 = \hat{a} \circ \pi_2$, where π_1, π_2 are the two canonical projections

$$(Y \times_{q,\mathsf{H}^0,F^0} X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1 \times_{\mathsf{s},\mathsf{H}^0,q} Y) \times_{\mathsf{pr}_{234},\mathsf{pr}_{234}} (Y \times_{q,\mathsf{H}^0,F^0} X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1 \times_{\mathsf{s},\mathsf{H}^0,q} Y)$$
$$\rightrightarrows Y \times_{q,\mathsf{H}^0,F^0} X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1 \times_{\mathsf{s},\mathsf{H}^0,q} Y.$$

We may identify π_1, π_2 with the two coordinate projections

$$\mathsf{pr}_{2345}, \mathsf{pr}_{1345}\colon Y \times_{q,\mathsf{H}^0,q} Y \times_{q,\mathsf{H}^0,F^0} X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1 \times_{\mathsf{s},\mathsf{H}^0,q} Y$$
$$\rightrightarrows Y \times_{q,\mathsf{H}^0,F^0} X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1 \times_{\mathsf{s},\mathsf{H}^0,q} Y.$$

Hence $\hat{a}$ factors through pr_{234} if and only if $\hat{a}(y_1, x_1, h, y_2) = \hat{a}(y_1', x_1, h, y_2)$ for all $y_1, y_1', y_2 \in Y$, $x_1 \in X$, $h \in \mathsf{H}^1$ with $q(y_1) = q(y_1') = F^0(x_1) = \mathsf{r}(h)$, $\mathsf{s}(h) = q(y_2)$. Equivalently,

$$\Phi(x_1, y_1) \cdot E^1(y_1, h, y_2) = \Phi(x_1, y_1') \cdot E^1(y_1', h, y_2)$$

for y_1, y_1', x_1, h, y_2 as above. Since $E^1(y_1', h, y_2) = E^1(y_1', 1_{q(y_1)}, y_1) \cdot E^1(y_1, h, y_2)$, and $q(y_1) = q(y_1') = F^0(x_1)$, this is equivalent to

$$\Phi(x_1, y_1) = \Phi(x_1, y_1') \cdot E^1(y_1', 1_{q(y_1)}, y_1) = \Phi(x_1, y_1') \cdot E^1(y_1', F^1(x_1, 1_{p(x_1)}, x_1), y_1).$$

This equation is a special case of the naturality condition (14) for Φ. This finishes the proof that $\hat{a}$ factors through a unique map $a\colon X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1 \times_{\mathsf{s},\mathsf{H}^0,q} Y \to \mathsf{G}^1$. Since $\mathsf{r}(a(x_1, h, y_2)) = \mathsf{r}(\Phi(x_1, y_2)) = p(x_1)$ and $\mathsf{s}(a(x_1, h, y_2)) = \mathsf{s}(E^1(y_1, h, y_2)) = E^0(y_2)$, the map $(\mathsf{pr}_1, a, \mathsf{pr}_3)$ maps $X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1 \times_{\mathsf{s},\mathsf{H}^0,q} Y$ to $X \times_{p,\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,E^0} Y$.

The property of being an isomorphism is local by Proposition 2.5. Thus the map $(\mathsf{pr}_1, a, \mathsf{pr}_3)$ is an isomorphism if and only if the following pull-back along a cover induced by q is an isomorphism:

$$(\mathsf{pr}_1, \mathsf{pr}_2, \hat{a}, \mathsf{pr}_4) \colon Y \times_{q, \mathsf{H}^0, F^0} X \times_{F^0, \mathsf{H}^0, r} \mathsf{H}^1 \times_{\mathsf{s}, \mathsf{H}^0, q} Y$$
$$\to Y \times_{q, \mathsf{H}^0, F^0} X \times_{p, \mathsf{G}^0, r} \mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, E^0} Y.$$

Here we have simplified the pull-back using the construction of a through $\hat{a}$. Actually, to construct a map in the opposite direction, we add even more variables and study the map

$$\beta \colon X \times_{p, \mathsf{H}^0, E^0} Y \times_{q, \mathsf{H}^0, F^0} X \times_{F^0, \mathsf{H}^0, r} \mathsf{H}^1 \times_{\mathsf{s}, \mathsf{H}^0, q} Y \times_{E^0, \mathsf{G}^0, p} X \times_{F^0, \mathsf{H}^0, q} Y$$
$$\to X \times_{p, \mathsf{H}^0, E^0} Y \times_{q, \mathsf{H}^0, F^0} X \times_{p, \mathsf{G}^0, r} \mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, E^0} Y \times_{E^0, \mathsf{G}^0, p} X \times_{F^0, \mathsf{H}^0, q} Y,$$
$$(x_3, y_1, x_1, h, y_2, x_2, y_3) \mapsto (x_3, y_1, x_1, \Phi(x_1, y_1) \cdot E^1(y_1, h, y_2), y_2, x_2, y_3).$$

Since p and q are covers, the fibre products above exist and β is a pull-back of $(\mathsf{pr}_1, a, \mathsf{pr}_3)$ along a cover. By Proposition 2.5, it suffices to prove that β is invertible.

In the opposite direction, we have the map defined elementwise by

$$\gamma(x_3, y_1, x_1, g, y_2, x_2, y_3) := (x_3, y_1, x_1, F^1(x_1, g, x_2) \cdot \Psi(y_2, x_2)^{-1}, y_2, x_2, y_3);$$

the product in the fourth entry is well-defined because $\mathsf{s}(F^1(x_1, g, x_2)) = F^0(x_2) = \mathsf{s}(\Psi(y_2, x_2))$, and it has range and source

$$\mathsf{r}(F^1(x_1, g, x_2)) = F^0(x_1), \qquad \mathsf{r}(\Psi(y_2, x_2)) = q(y_2),$$

respectively, so $(x_3, y_1, x_1, F^1(x_1, g, x_2) \cdot \Psi(y_2, x_2)^{-1}, y_2, x_2, y_3)$ belongs to the domain of β. In the following computations, we only consider the fourth entry for simplicity because this is the only one that is touched by β. We compute

$$\begin{aligned}
\mathsf{pr}_4 &\circ \beta \circ \gamma(x_3, y_1, x_1, g, y_2, x_2, y_3) \\
&= \Phi(x_1, y_1) \cdot E^1(y_1, F^1(x_1, g, x_2) \cdot \Psi^{-1}(y_2, x_2), y_2) \\
&= \Phi(x_1, y_1) \cdot E^1(y_1, F^1(x_1, g, x_2), y_3) \cdot E^1(y_3, \Psi(y_2, x_2)^{-1}, y_2) \\
&= g \cdot \Phi(x_2, y_3) \cdot E^1(y_3, \Psi(y_2, x_2)^{-1}, y_2), \\
\mathsf{pr}_4 &\circ \gamma \circ \beta(x_3, y_1, x_1, h, y_2, x_2, y_3) \\
&= F^1(x_1, \Phi(x_1, y_1) \cdot E^1(y_1, h, y_2), x_2) \cdot \Psi(y_2, x_2)^{-1} \\
&= F^1(x_1, \Phi(x_1, y_1), x_3) \cdot F^1(x_3, E^1(y_1, h, y_2), x_2) \cdot \Psi(y_2, x_2)^{-1} \\
&= F^1(x_1, \Phi(x_1, y_1), x_3) \cdot \Psi(y_1, x_3)^{-1} \cdot h.
\end{aligned}$$

The first map multiplies g from the right with some complicated element ξ of G^1 with $\mathsf{r}(\xi) = \mathsf{s}(\xi) = \mathsf{s}(g)$; the crucial point is that ξ does not depend on g, so the map $g \mapsto g \cdot \xi^{-1}$ is a well-defined inverse for $\beta \circ \gamma$. Similarly, the map $\gamma \circ \beta$ is invertible because it multiplies h on the left with some element η of H^1 with $\mathsf{s}(\eta) = \mathsf{r}(\eta) = \mathsf{r}(h)$ that does not depend on h. Thus β is both left and right invertible, so it is invertible. ∎

3.40. LEMMA. *The projection* $\mathsf{pr}_3\colon X\mathsf{H}Y := X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1 \times_{\mathsf{s},\mathsf{H}^0,q} Y \to Y$ *is a cover.*

PROOF. The isomorphism in Lemma 3.39 allows us to replace $\mathsf{pr}_3\colon X\mathsf{H}Y \to Y$ by $\mathsf{pr}_3\colon X\mathsf{G}Y$ $:= X \times_{p,\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,E^0} Y \to Y$. Since $p\colon X \twoheadrightarrow \mathsf{G}^0$ and $\mathsf{s}\colon \mathsf{G}^1 \twoheadrightarrow \mathsf{G}^0$ are covers, so are the induced maps $\mathsf{pr}_{23}\colon X\mathsf{G}Y \twoheadrightarrow \mathsf{G}Y$ and $\mathsf{pr}_2\colon \mathsf{G}Y \twoheadrightarrow Y$. Hence so is their composite $\mathsf{pr}_3\colon X\mathsf{G}Y \twoheadrightarrow Y$. ∎

3.41. LEMMA. *The functor* $F\colon \mathsf{G}(X) \to \mathsf{H}$ *is almost essentially surjective.*

PROOF. We consider the following diagram

$$
\begin{array}{ccc}
X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1 \times_{\mathsf{s},\mathsf{H},q} Y & \xrightarrow{\;\mathsf{pr}_Y\;} & Y \\[2pt]
{\scriptstyle \mathsf{pr}_{12}} \downarrow & & \downarrow {\scriptstyle q} \\[2pt]
X \times_{F^0,\mathsf{G}^0,\mathsf{r}} \mathsf{H}^1 & \xrightarrow{\;\mathsf{s}\,\circ\,\mathsf{pr}_{\mathsf{H}^1}\;} & \mathsf{H}^0
\end{array}
$$

Lemma 3.40 shows that pr_Y is a cover. The map q is a cover by assumption. Thus $\mathsf{s}\circ\mathsf{pr}_{\mathsf{H}^1}$ is locally a cover, meaning that F is almost essentially surjective. ∎

3.42. LEMMA. *The functor* $F\colon \mathsf{G}(X) \to \mathsf{H}$ *is almost fully faithful.*

PROOF. We claim that $q^*(F)$ is fully faithful for the given cover $q\colon Y \twoheadrightarrow \mathsf{H}^0$, that is, the map

$$
\begin{aligned}
\delta\colon\, & Y \times_{q,\mathsf{H}^0,F^0} X \times_{p,\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,p} X \times_{F^0,\mathsf{H}^0,q} Y \\
& \qquad \to Y \times_{q,\mathsf{H}^0,F^0} X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1 \times_{\mathsf{s},\mathsf{H}^0,q} Y \times_{q,\mathsf{H}^0,F^0} X, \\
& \qquad\qquad (y_1, x_1, g, x_2, y_2) \mapsto (y_1, x_1, F^1(x_1, g, x_2), y_2, x_2),
\end{aligned}
$$

is an isomorphism. The fibre product on the right exists by Lemma 3.40. An element of the codomain of δ is given by $y_1, y_2 \in Y$, $x_1, x_2 \in X$, $h \in \mathsf{H}^1$ with $q(y_1) = F^0(x_1) = \mathsf{r}(h)$, $q(y_2) = F^0(x_2) = \mathsf{s}(h)$. Hence $E^1(y_1, h, y_2)$, $\Phi(x_1, y_1)$ and $\Phi(x_2, y_2)$ are well-defined arrows in G^1, and their ranges and sources match so that

$$
\epsilon(y_1, x_1, h, x_2, y_2) := \Phi(x_1, y_1) \cdot E^1(y_1, h, y_2) \cdot \Phi(x_2, y_2)^{-1} \in \mathsf{G}^1
$$

is well-defined and has range $\mathsf{r}(\Phi(x_1, y_1)) = p(x_1)$ and source $\mathsf{s}(\Phi(x_2, y_2)) = p(x_2)$. We get a map

$$
\begin{aligned}
\bar\epsilon\colon\, & Y \times_{q,\mathsf{H}^0,F^0} X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1 \times_{\mathsf{s},\mathsf{H}^0,F^0} X \times_{F^0,\mathsf{H}^0,q} Y \\
& \qquad \to Y \times_{q,\mathsf{H}^0,F^0} X \times_{p,\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,p} X \times_{F^0,\mathsf{H}^0,q} Y, \\
& \qquad\qquad (y_1, x_1, h, x_2, y_2) \mapsto (y_1, x_1, \epsilon(y_1, x_1, h, x_2, y_2), x_2, y_2).
\end{aligned}
$$

The map $\bar\epsilon \circ \delta$ is the identity on $Y \times_{q,\mathsf{H}^0,F^0} X \times_{p,\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,p} X \times_{F^0,\mathsf{H}^0,q} Y$ by the naturality condition (14). We are going to show that the composite $\delta \circ \bar\epsilon$ is invertible (and hence

the identity because $\bar{\epsilon} \circ \delta$ is the identity). Elementwise, $\delta \circ \bar{\epsilon}$ maps (y_1, x_1, h, x_2, y_2) to $(y_1, x_1, \eta(y_1, x_1, h, x_2, y_2), x_2, y_2)$ with

$$\eta(y_1, x_1, h, x_2, y_2) = F^1(x_1, \Phi(x_1, y_1) \cdot E^1(y_1, h, y_2) \cdot \Phi(x_2, y_2)^{-1}, x_2).$$

To show that $\delta \circ \bar{\epsilon}$ is an isomorphism, we pull $\delta \circ \bar{\epsilon}$ back to the map on

$$X \times_{p, \mathsf{G}^0, E^0} Y \times_{q, \mathsf{H}^0, F^0} X \times_{F^0, \mathsf{H}^0, r} \mathsf{H}^1 \times_{s, \mathsf{H}^0, F^0} X \times_{F^0, \mathsf{H}^0, q} Y \times_{E^0, \mathsf{G}^0, p} X$$

that sends $(x_3, y_1, x_1, h, x_2, y_2, x_4)$ to $(x_3, y_1, x_1, \eta(y_1, x_1, h, x_2, y_2), x_2, y_2, x_4)$. Since $p \colon X \to \mathsf{G}^0$ is a cover, Proposition 2.5 shows that $\delta \circ \bar{\epsilon}$ is an isomorphism if and only if this new map is an isomorphism. Using the extra variables x_3, x_4, we may simplify η:

$$\begin{aligned}
\eta(y_1, x_1, h, x_2, y_2) &= F^1(x_1, \Phi(x_1, y_1) \cdot E^1(y_1, h, y_2) \cdot \Phi(x_2, y_2)^{-1}, x_2) \\
&= F^1(x_1, \Phi(x_1, y_1), x_3) \cdot F^1(x_3, E^1(y_1, h, y_2), x_4) \cdot F^1(x_4, \Phi(x_2, y_2)^{-1}, x_2) \\
&= F^1(x_1, \Phi(x_1, y_1), x_3) \cdot \Psi(y_1, x_3)^{-1} \cdot h \cdot \Psi(y_2, x_4) \cdot F^1(x_4, \Phi(x_2, y_2)^{-1}, x_2).
\end{aligned}$$

This map is invertible because η multiplies h on the left and right by expressions that do not depend on h: the inverse maps $(x_3, y_1, x_1, h, x_2, y_2, x_4)$ to

$$\begin{aligned}
(x_3, y_1, x_1, \Psi(y_1, x_3) \cdot F^1(x_1, \Phi(x_1, y_1), x_3)^{-1} \cdot h \\
\times F^1(x_2, \Phi(x_2, y_2), x_4) \cdot \Psi(y_2, x_4)^{-1}, x_2, y_2, x_4).
\end{aligned}$$

Thus $\delta \circ \bar{\epsilon}$ is invertible. It follows that δ is invertible, so that F is almost fully faithful. ∎

   This finishes the proof that (1) implies (2) in Theorem 3.35, and hence the proof of the theorem.

4. Groupoid actions

Let $\mathcal{C}$ be a category with coproducts and $\mathcal{T}$ a subcanonical pretopology on $\mathcal{C}$.

4.1. DEFINITION AND LEMMA. *Let* $\mathsf{G} = (\mathsf{G}^0, \mathsf{G}^1, r, s, m)$ *be a groupoid in* $(\mathcal{C}, \mathcal{T})$. *A (right)* G-*action in* $\mathcal{C}$ *is* $X \in\in \mathcal{C}$ *with* $s \in \mathcal{C}(X, \mathsf{G}^0)$ *(anchor) and* $m \in \mathcal{C}(X \times_{s, \mathsf{G}^0, r} \mathsf{G}^1, X)$ *(action), denoted multiplicatively as* $\cdot$, *such that*

   (1) $s(x \cdot g) = s(g)$ *for all* $x \in X$, $g \in \mathsf{G}^1$ *with* $s(x) = r(g)$;

   (2) $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 \cdot g_2)$ *for all* $x \in X$, $g_1, g_2 \in \mathsf{G}^1$ *with* $s(x) = r(g_1)$, $s(g_1) = r(g_2)$;

   (3) $x \cdot 1_{s(x)} = x$ *for all* $x \in X$.

This definition does not depend on $\mathcal{T}$. *In the presence of (1) and (2), the third condition is equivalent to*

   (3′) $m \colon X \times_{s, \mathsf{G}^0, r} \mathsf{G}^1 \to X$ *is an epimorphism;*

   (3″) $m \colon X \times_{s, \mathsf{G}^0, r} \mathsf{G}^1 \to X$ *is a cover;*

   (3‴) $(m, pr_2) \colon X \times_{s, \mathsf{G}^0, r} \mathsf{G}^1 \to X \times_{s, \mathsf{G}^0, s} \mathsf{G}^1$, $(x, g) \mapsto (x \cdot g, g)$, *is invertible.*

An action where the anchor map is a cover is called a sheaf *over* G.

PROOF. Conditions (1)–(3) imply $(x \cdot g^{-1}) \cdot g = x \cdot (g^{-1} \cdot g) = x \cdot 1_{\mathsf{s}(x)} = x$ for all $x \in \mathsf{X}$, $g \in \mathsf{G}^1$ with $\mathsf{s}(x) = \mathsf{s}(g)$ and $(x \cdot g) \cdot g^{-1} = x$ for $x \in \mathsf{X}$, $g \in \mathsf{G}^1$ with $\mathsf{s}(x) = \mathsf{r}(g)$. Hence the elementwise formula $(x, g) \mapsto (x \cdot g^{-1}, g)$ gives an inverse for the map in (3‴). Thus (3) implies (3‴) in the presence of (1) and (2).

Assume (3‴). The map $\mathsf{pr}_1 \colon \mathsf{X} \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{s}} \mathsf{G}^1 \twoheadrightarrow \mathsf{X}$ is a cover because $\mathsf{s} \colon \mathsf{G}^1 \twoheadrightarrow \mathsf{G}^0$ is a cover. Composing with the isomorphism in (3‴) shows that m is a cover. Covers are epimorphisms because the pretopology is subcanonical. Thus

$$(1)\text{–}(3) \Rightarrow (3''') \Rightarrow (3'') \Rightarrow (3').$$

Since $(x \cdot g) \cdot 1_{\mathsf{s}(g)} = x \cdot (g \cdot 1_{\mathsf{s}(g)}) = x \cdot g$ for all $x \in \mathsf{X}$, $g \in \mathsf{G}^1$ with $\mathsf{s}(x) = \mathsf{r}(g)$, the map $f \colon \mathsf{X} \to \mathsf{X}$, $x \mapsto x \cdot 1_{\mathsf{s}(x)}$, satisfies $f \circ \mathsf{m} = \mathsf{m}$. If m is an epimorphism, this implies $f = \mathrm{id}_\mathsf{X}$. Thus (3′) implies (3). ∎

4.2. REMARK. Sheaves over a Lie groupoid are equivalent to groupoid actions with an étale anchor map (see [33, Section 3.1]). Sheaf theory only works well for étale groupoids, however. If we work in the category of smooth manifolds with étale maps as covers, then the above definition of a sheaf is an equivalent way of defining the concept of a sheaf via local sections [33, Section 3.1]. We also require the anchor map to be surjective, which restricts to sheaves for which all stalks are non-empty; this seems a rather mild restriction. We have not yet tried to carry over the usual constructions with sheaves to non-étale groupoids using the above definition.

The class of groupoid actions with a cover as anchor map certainly deserves special consideration, but it is not a good idea to require this for all actions. Then we would also have to require this for bibundles and, in particular, for bibundle functors. Hence we could only treat covering bibundle functors, where both anchor maps are covers. But the bibundle functor associated to a functor need not be covering: this happens if and only if the functor is essentially surjective. As a result, anafunctors and bibundle functors would no longer be equivalent.

4.3. DEFINITION. *Let* X *and* Y *be right* G-*actions. A* G-equivariant map *or briefly* G-map $\mathsf{X} \to \mathsf{Y}$ *is* $f \in \mathcal{C}(\mathsf{X}, \mathsf{Y})$ *with* $\mathsf{s}(f(x)) = \mathsf{s}(x)$ *for all* $x \in \mathsf{X}$ *and* $f(x \cdot g) = f(x) \cdot g$ *for all* $x \in \mathsf{X}$, $g \in \mathsf{G}^1$ *with* $\mathsf{s}(x) = \mathsf{r}(g)$.

The G-*actions and* G-*maps form a category, which we denote by* $\mathcal{C}(\mathsf{G})$. *Let* $\mathcal{C}_\mathcal{T}(\mathsf{G}) \subseteq \mathcal{C}(\mathsf{G})$ *be the full subcategory of* G-*sheaves.*

4.4. DEFINITION. *Let* X *be a* G-*action and* $\mathsf{Z} \in\in \mathcal{C}$. *A map* $f \in \mathcal{C}(\mathsf{X}, \mathsf{Z})$ *is* G-invariant *if* $f(x \cdot g) = f(x)$ *for all* $x \in \mathsf{X}$, $g \in \mathsf{G}^1$ *with* $\mathsf{s}(x) = \mathsf{r}(g)$.

4.5. EXAMPLES.

4.6. EXAMPLE. Any groupoid G acts on G^0 by $\mathsf{s} = \mathrm{id}_{\mathsf{G}^0}$ and $\mathsf{r}(g) \cdot g = \mathsf{s}(g)$ for all $g \in \mathsf{G}^1$.

4.7. PROPOSITION. G^0 *is a final object in* $\mathcal{C}(\mathsf{G})$ *and* $\mathcal{C}_\mathcal{T}(\mathsf{G})$.

PROOF. For any G-action X, the source map s is a G-map, and it is the only G-map $\mathsf{X} \to \mathsf{G}^0$. Since identity maps are covers, G^0 belongs to $\mathcal{C}_\mathcal{T}(\mathsf{G})$. ∎

4.8. EXAMPLE. View $Y \in\in \mathcal{C}$ as a groupoid as in Example 3.11. A Y-action on X is equivalent to a map $X \to Y$, namely, the anchor map of the action; the multiplication map $X \times_Y Y \to X$ must be the canonical isomorphism. A Y-map between actions $s_i \colon X_i \to Y$, $i = 1, 2$, of Y is a map $f \colon X_1 \to X_2$ with $s_2 \circ f = s_1$. Thus $\mathcal{C}(Y)$ is the slice category $\mathcal{C} \downarrow Y$ of objects in $\mathcal{C}$ over Y.

4.9. EXAMPLE. Let G be a group in $(\mathcal{C}, \mathcal{T})$, that is, G^0 is a final object in $\mathcal{C}$. Then there is a unique map $X \to G^0$ for any $X \in\in \mathcal{C}$. Since this unique map is the only choice for the anchor map, a G-action is given by the multiplication map $X \times G^1 \to X$ alone. This defines a group action if and only if $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 \cdot g_2)$ for all $x \in X$, $g_1, g_2 \in G^1$, and $x \cdot 1 = x$ for all $x \in X$; the unit element 1 is defined in Example 3.14.

4.10. TRANSFORMATION GROUPOIDS.

4.11. DEFINITION AND LEMMA. *Let X be a right G-action. The* transformation groupoid $X \rtimes G$ *in $(\mathcal{C}, \mathcal{T})$ is the groupoid with objects X, arrows $X \times_{s,G^0,r} G^1$, range pr_1, source m, and multiplication defined by*

$$(x_1, g_1) \cdot (x_2, g_2) := (x_1, g_1 \cdot g_2)$$

for all $x_1, x_2 \in X$, $g_1, g_2 \in G^1$ with $s(x_1) = r(g_1)$, $s(x_2) = r(g_2)$, and $x_1 \cdot g_1 = x_2$, that is, $s(x_1, g_1) = r(x_2, g_2)$ in X; then $s(x_1) = r(g_1 \cdot g_2)$, so that $(x_1, g_1 \cdot g_2) \in X \times_{s,G^0,r} G^1$.
This is a groupoid in $(\mathcal{C}, \mathcal{T})$ with unit map and inversion given by $1_x := (x, 1_{s(x)})$ and $(x, g)^{-1} := (x \cdot g, g^{-1})$. It acts on X with anchor map $s = \mathrm{id}_X \colon X \to X$, by $x \cdot (x, g) := x \cdot g$ for all $x \in X$, $g \in G^1$ with $s(x) = r(g)$.

PROOF. The range map $\mathsf{pr}_1 \colon X \times_{s,G^0,r} G^1 \twoheadrightarrow X$ of $X \rtimes G$ is a cover because $\mathsf{r} \colon G^1 \twoheadrightarrow G^0$ is a cover; Lemma 4.1 shows that the source map m of $X \rtimes G$ is a cover as well. The remaining properties are routine computations. The action of $X \rtimes G$ on X is a special case of Example 4.6. ∎

4.12. PROPOSITION. *Let X be a G-action. An action of the transformation groupoid $X \rtimes G$ on an object $Y \in\in \mathcal{C}$ is equivalent to an action of G on Y together with a G-map $f \colon Y \to X$. Furthermore, the following two groupoids are isomorphic:*

$$Y \rtimes (X \rtimes G) \cong Y \rtimes G.$$

A map $Y \to Z$ is $X \rtimes G$-invariant if and only if it is G-invariant, and a map between two $X \rtimes G$-actions is $X \rtimes G$-equivariant if and only if it is G-equivariant and over X.

PROOF. An action of $X \rtimes G$ on Y is given by an anchor map $f \colon Y \to X$ and a multiplication map

$$Y \times_{f,X,\mathsf{pr}_1} (X \times_{s,G^0,r} G^1) \to Y.$$

We compose f with $s_X \colon X \to G^0$ to get an anchor map $s_Y \colon Y \to G^0$, and we compose the multiplication map with the canonical isomorphism

$$Y \times_{s_Y,G^0,r} G^1 \xrightarrow{\sim} Y \times_{f,X,\mathsf{pr}_1} (X \times_{s_X,G^0,r} G^1) \tag{16}$$

to get a G-action on Y: $y \cdot g := y \cdot (f(y), g)$ for all $y \in \mathsf{Y}$, $g \in \mathsf{G}^1$ with $\mathsf{s}_\mathsf{Y}(y) = \mathsf{r}(g)$. It is routine to check that this is an action of G. The map f is G-equivariant: $\mathsf{s}_\mathsf{X} \circ f = \mathsf{s}_\mathsf{Y}$ by construction, and $f(y \cdot g) = \mathsf{s}_{\mathsf{X} \rtimes \mathsf{G}}(f(y), g) = f(y) \cdot g$.

Conversely, given a G-action on Y and a G-equivariant map $f \colon \mathsf{X} \to \mathsf{Y}$, we define an action of $\mathsf{X} \rtimes \mathsf{G}$ by taking f as the anchor map and $y \cdot (x, g) := y \cdot g$ for all $y \in \mathsf{Y}$, $x \in \mathsf{X}$, $g \in \mathsf{G}^1$ with $f(y) = x$, $\mathsf{s}(x) = \mathsf{r}(g)$. These two processes are inverse to each other. Since both constructions are natural, a map is $\mathsf{X} \rtimes \mathsf{G}$-equivariant if and only if it is both G-equivariant and a map over X.

The transformation groupoids $\mathsf{Y} \rtimes (\mathsf{X} \rtimes \mathsf{G})$ and $\mathsf{Y} \rtimes \mathsf{G}$ have the same objects Y, and isomorphic arrows by (16). This isomorphism also intertwines their range, source and multiplication maps, so both transformation groupoids for Y are isomorphic. The isomorphism (16) also implies that $\mathsf{X} \rtimes \mathsf{G}$- and G-invariance are equivalent. ∎

4.13. LEFT ACTIONS AND SEVERAL COMMUTING ACTIONS. *Left G-actions* and G-maps between them are defined like right actions; but we denote anchor maps for left actions by $\mathsf{r} \colon \mathsf{X} \to \mathsf{G}^0$ instead of s.

4.14. LEMMA. *The categories of left and right G-actions are isomorphic.*

PROOF. We turn a right G-action (s, m) on X into a left G-action by $\mathsf{r} = \mathsf{s}$ and $g \cdot x := x \cdot g^{-1}$ for $g \in \mathsf{G}^1$, $x \in \mathsf{X}$ with $\mathsf{r}(g^{-1}) = \mathsf{s}(g) = \mathsf{r}(x)$. A left G-action gives a right one by $\mathsf{s} = \mathsf{r}$ and $x \cdot g := g^{-1} \cdot x$ for $x \in \mathsf{X}$, $g \in \mathsf{G}^1$ with $\mathsf{s}(x) = \mathsf{r}(g) = \mathsf{s}(g^{-1})$. A map is equivariant for left actions if and only if it is equivariant for the corresponding right actions, so this is an isomorphism of categories. ∎

4.15. DEFINITION. *Let G and H be groupoids in $(\mathcal{C}, \mathcal{T})$. A G, H-bibundle is $\mathsf{X} \in\in \mathcal{C}$ with a left G-action and a right H-action such that $\mathsf{s}(g \cdot x) = \mathsf{s}(x)$, $\mathsf{r}(x \cdot h) = \mathsf{r}(x)$, and $(g \cdot x) \cdot h = g \cdot (x \cdot h)$ for all $g \in \mathsf{G}^1$, $x \in \mathsf{X}$, $h \in \mathsf{H}^1$ with $\mathsf{s}(g) = \mathsf{r}(x)$, $\mathsf{s}(x) = \mathsf{r}(h)$. Let $\mathcal{C}(\mathsf{G}, \mathsf{H})$ be the category with G, H-bibundles as objects and G, H-maps – maps $\mathsf{X} \to \mathsf{Y}$ that are equivariant for both actions – as arrows.*

A G, H-bibundle also has a *transformation groupoid* $\mathsf{G} \ltimes \mathsf{X} \rtimes \mathsf{H}$ with objects X and arrows
$$(\mathsf{G} \ltimes \mathsf{X} \rtimes \mathsf{H})^1 := \mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{X} \times_{\mathsf{s}, \mathsf{H}^0, \mathsf{r}} \mathsf{H}^1;$$
$\mathsf{r}(g, x, h) := g \cdot x$, $\mathsf{s}(g, x, h) := x \cdot h$, and
$$(g_1, x_1, h_1) \cdot (g_2, x_2, h_2) := (g_1 \cdot g_2, g_2^{-1} \cdot x_1, h_1 \cdot h_2) = (g_1 \cdot g_2, x_2 \cdot h_1^{-1}, h_1 \cdot h_2)$$
for all $g_1, g_2 \in \mathsf{G}^1$, $x_1, x_2 \in \mathsf{X}$, $h_1, h_2 \in \mathsf{H}^1$ with $\mathsf{s}(g_i) = \mathsf{r}(x_i)$, $\mathsf{s}(x_i) = \mathsf{r}(h_i)$ for $i = 1, 2$ and $x_1 \cdot h_1 = g_2 \cdot x_2$, so that $g_2^{-1} \cdot x_1 = x_2 \cdot h_1^{-1}$.

4.16. REMARK. Assume Assumption 2.10. Then the product of two groupoids in $(\mathcal{C}, \mathcal{T})$ is again a groupoid in $(\mathcal{C}, \mathcal{T})$; the range and source maps are again covers by Lemma 2.11. The category $\mathcal{C}(\mathsf{G}, \mathsf{H})$ is isomorphic to $\mathcal{C}(\mathsf{G} \times \mathsf{H})$: actions of G and H determine an action of $\mathsf{G} \times \mathsf{H}$ by $x \cdot (g, h) := g^{-1} \cdot x \cdot h$ for $x \in \mathsf{X}$, $g \in \mathsf{G}^1$, $h \in \mathsf{H}^1$ with $\mathsf{r}(g) = \mathsf{s}(x)$, $\mathsf{r}(x) = \mathsf{s}(h)$. Here we use that $(g, h) \in \mathsf{G}^1 \times \mathsf{H}^1$ is equivalent to $g \in \mathsf{G}^1$ and $h \in \mathsf{H}^1$. The transformation

groupoid $\mathsf{G} \ltimes \mathsf{X} \rtimes \mathsf{H}$ is naturally isomorphic to the transformation groupoid $\mathsf{X} \rtimes (\mathsf{G} \times \mathsf{H})$ via $(g, x, h) \mapsto (g \cdot x, g^{-1}, h)$.

More generally, we may consider an object of $\mathcal{C}$ with several groupoids acting on the left and several on the right, with all actions commuting. We single out G, H-bibundles because they are the basis of several bicategories of groupoids.

4.17. FIBRE PRODUCTS OF GROUPOID ACTIONS. Let X_1, X_2 and Y be G-actions and let $f_i \colon \mathsf{X}_i \to \mathsf{Y}$ for $i = 1, 2$ be G-maps. Assume that the fibre product $\mathsf{X} := \mathsf{X}_1 \times_{f_1, \mathsf{Y}, f_2} \mathsf{X}_2$ exists in $\mathcal{C}$ (this happens, for instance, if f_1 or f_2 is a cover).

4.18. LEMMA. *There is a unique G-action on X for which both coordinate projections* $\mathsf{pr}_i \colon \mathsf{X} \to \mathsf{X}_i$, $i = 1, 2$, *are equivariant. With this G-action, X becomes a fibre product in the category of G-actions.*

PROOF. We define a G-action on X as follows. The anchor map $\mathsf{s} \colon \mathsf{X} \to \mathsf{G}^0$ is defined by

$$\mathsf{s}(x_1, x_2) := \mathsf{s}(x_1) = \mathsf{s}(f_1(x_1)) = \mathsf{s}(f_2(x_2)) = \mathsf{s}(x_2),$$

the multiplication by $(x_1, x_2) \cdot g := (x_1 \cdot g, x_2 \cdot g)$ for all $x_1, x_2 \in \mathsf{X}_1$, $g \in \mathsf{G}^1$ with $f_1(x_1) = f_2(x_2)$ and $\mathsf{s}(x_1, x_2) = \mathsf{r}(g)$. This elementwise formula defines a map $(\mathsf{X}_1 \times_{f_1, \mathsf{Y}, f_2} \mathsf{X}_2) \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1 \to \mathsf{X}_1 \times_{f_1, \mathsf{Y}, f_2} \mathsf{X}_2$ by taking $? = (\mathsf{X}_1 \times_{f_1, \mathsf{Y}, f_2} \mathsf{X}_2) \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1$ and x_1, x_2, g the three coordinate projections on $?$.

Routine computations show that this defines a G-action on X. The coordinate projections $\mathsf{X} \to \mathsf{X}_i$ for $i = 1, 2$ are G-equivariant by construction, and the above formulas are clearly the only ones that make this happen. Finally, a map $h \colon \mathsf{W} \to \mathsf{X}$ is a G-map if and only if $h_i := \mathsf{pr}_i \circ h$ for $i = 1, 2$ are G-maps, and two G-maps $h_i \colon \mathsf{W} \to \mathsf{X}_i$ combine to a G-map $\mathsf{W} \to \mathsf{X}$ if and only if $f_1 \circ h_1 = f_2 \circ h_2$. Thus X has the property of a fibre product in the category $\mathcal{C}(\mathsf{G})$. $\blacksquare$

4.19. ACTORS: ANOTHER CATEGORY OF GROUPOIDS. A functor $\mathsf{G} \to \mathsf{H}$ between two groupoids does *not* induce a functor $\mathcal{C}(\mathsf{H}) \to \mathcal{C}(\mathsf{G})$. (The only general result is Proposition 7.8, which gives a functor $\mathcal{C}_{\mathcal{T}}(\mathsf{H}) \to \mathcal{C}(\mathsf{G})$ on the subcategory of sheaves.) Here we introduce actors, a different type of groupoid morphisms that are equivalent to functors $\mathcal{C}(\mathsf{H}) \to \mathcal{C}(\mathsf{G})$ with some extra properties. In the context of locally compact groupoids, these were studied by Buneci (see [7,8]) because they induce morphisms between groupoid C^*-algebras. In the context of Lie groupoids and Lie algebroids, these were studied under the name of comorphisms (see [13], [29, Definition 4.3.16] and the references there). Our notion of "actor" is unrelated to the one in [41, A.2].

4.20. DEFINITION. *Let G and H be groupoids in $(\mathcal{C}, \mathcal{T})$. An* actor *from G to H is a left G-action on H^1 that commutes with the right multiplication action of H on H^1.*

Other authors exchange left and right in the above definition. The following generalises [9, Lemma 4.3]:

4.21. PROPOSITION. *An actor from G to H is equivalent to a pair consisting of a left action of G on H^0 and a functor $\mathsf{G} \ltimes \mathsf{H}^0 \to \mathsf{H}$ that is the identity on objects.*

PROOF. Let $r_{\mathsf{G}^0}\colon \mathsf{H}^1 \to \mathsf{G}^0$ and $m\colon \mathsf{G}^1 \times_{s,\mathsf{G}^0,r} \mathsf{H}^1 \to \mathsf{H}^1$ define an actor from G to H. Since the left and right actions on H^1 commute, we have $r_{\mathsf{G}^0}(h_1 \cdot h_2) = r_{\mathsf{G}^0}(h_1)$ for all $h_1, h_2 \in \mathsf{H}^1$ with $s(h_1) = r_{\mathsf{H}^0}(h_2)$. Thus $r_{\mathsf{G}^0}(h) = r_{\mathsf{G}^0}(h \cdot h^{-1}) = r_{\mathsf{G}^0}(1_{r_{\mathsf{H}^0}(h)})$. With the map $r^0\colon \mathsf{H}^0 \to \mathsf{G}^0$ defined by $r^0(x) := r_{\mathsf{G}^0}(1_x)$ for $x \in \mathsf{H}^0$, this becomes $r_{\mathsf{G}^0} = r^0 \circ r_{\mathsf{H}^0}$.

The map r^0 is the anchor map of a G-action on H^0. The action $\mathsf{G}^1 \times_{s,\mathsf{G}^0,r^0} \mathsf{H}^0 \to \mathsf{H}^0$ is defined elementwise by $g \cdot x := r_{\mathsf{H}^0}(g \cdot 1_x)$. We have $r^0(g \cdot x) = r_{\mathsf{G}^0}(1_{g\cdot x}) = r_{\mathsf{G}^0}(1_{r_{\mathsf{H}^0}(g \cdot 1_x)}) = r_{\mathsf{G}^0}(g \cdot 1_x) = r(g)$ and $1_{r^0(x)} \cdot x = x$ for all $x \in \mathsf{H}^0$. Furthermore, $r_{\mathsf{H}^0}(g \cdot h) = r_{\mathsf{H}^0}(g \cdot h \cdot h^{-1}) = g \cdot r_{\mathsf{H}^0}(h)$ for all $g \in \mathsf{G}^1$, $h \in \mathsf{H}^1$ with $s(g) = r_{\mathsf{G}^0}(h)$. Hence $(g_1 \cdot g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ for all $g_1, g_2 \in \mathsf{G}^1$, $x \in \mathsf{H}^0$ with $s(g_1) = r(g_2)$, $s(g_2)(x) = r^0(x)$.

The map

$$F^1\colon \mathsf{G}^1 \times_{s,\mathsf{G}^0,r^0} \mathsf{H}^0 \to \mathsf{H}^1, \qquad (g,x) \mapsto g \cdot 1_x,$$

together with the identity map on H^0 is a functor $F\colon \mathsf{G} \ltimes \mathsf{H}^0 \to \mathsf{H}$. This functor determines the actor because $g \cdot h = (g \cdot 1_{r_{\mathsf{H}^0}(h)}) \cdot h = F(g, r_{\mathsf{H}^0}(h)) \cdot h$ for all $g \in \mathsf{G}^1$, $h \in \mathsf{H}^1$ with $s(g) = r_{\mathsf{G}^0}(h)$.

Conversely, a left G-action on H^0 and a functor $F\colon \mathsf{G} \ltimes \mathsf{H}^0 \to \mathsf{H}$ yield an actor by taking the anchor map $r^0 \circ r_{\mathsf{H}^0}\colon \mathsf{H}^1 \to \mathsf{H}^0 \to \mathsf{G}^0$ and the multiplication $g \cdot h := F(g, r_{\mathsf{H}^0}(h)) \cdot h$. ∎

Proposition 4.21 shows that actors $\mathsf{G} \to \mathsf{H}$ are usually not functors $\mathsf{G} \to \mathsf{H}$, and vice versa. The intersection of both types of morphisms is described in the following example:

4.22. EXAMPLE. Let $F\colon \mathsf{G} \to \mathsf{H}$ be a functor with invertible $F^0 \in \mathcal{C}(\mathsf{G}^0, \mathsf{H}^0)$. Then we define an associated actor by $r_{\mathsf{G}}\colon \mathsf{H}^1 \to \mathsf{G}^0$, $g \mapsto (F^0)^{-1}(r_{\mathsf{H}}(g))$, and $g \cdot h := F^1(g) \cdot h$ for all $g \in \mathsf{G}^1$, $h \in \mathsf{H}^1$ with $r(h) = s(F^1(g)) = F^0(s(g))$. Conversely, if an actor has the property that the associated anchor map $r\colon \mathsf{H}^0 \to \mathsf{G}^0$ is invertible, then we may identify G with the transformation groupoid $\mathsf{G} \ltimes \mathsf{H}^0$ and get a functor $\mathsf{G} \to \mathsf{H}$ by Proposition 4.21. The actor associated to this functor is the one we started with.

4.23. PROPOSITION. *An actor from G to H induces a functor $\mathcal{C}(\mathsf{H}) \to \mathcal{C}(\mathsf{G})$ that does not change the underlying objects in $\mathcal{C}$, that is, we have a commutative diagram*

$$
\begin{array}{ccc}
\mathcal{C}(\mathsf{H}) & \longrightarrow & \mathcal{C}(\mathsf{G}), \\
& \searrow^{F_{\mathsf{H}}} \quad \swarrow_{F_{\mathsf{G}}} & \\
& \mathcal{C} &
\end{array}
$$

where F_{G} and F_{H} are the functors forgetting the G- and H-actions, respectively. Furthermore, H-invariant maps are also G-invariant. Any functor $\mathcal{C}(\mathsf{H}) \to \mathcal{C}(\mathsf{G})$ with these extra properties comes from an actor. Thus groupoids as objects and actors as arrows form a category, and $\mathsf{G} \mapsto \mathcal{C}(\mathsf{G})$ is a contravariant functor on this category.

PROOF. We use categories of left actions during the proof. Describe an actor as in Proposition 4.21 as a right G-action on H^0 with a functor $F\colon \mathsf{H}^0 \rtimes \mathsf{G} \to \mathsf{H}$. Let X be a left H-action. Then we define a left G-action on X by taking the anchor map $\mathsf{r}^0 \circ \mathsf{r}\colon \mathsf{X} \to \mathsf{H}^0 \to \mathsf{G}^0$ and the multiplication $g \cdot x := F^1(g, \mathsf{r}(x)) \cdot x$ for all $g \in \mathsf{G}^1$, $x \in \mathsf{X}$ with $\mathsf{s}(g) = \mathsf{r}^0(\mathsf{r}(x))$. This action is the unique one with $\mathsf{r}_{\mathsf{G}^0}(h \cdot x) = \mathsf{r}_{\mathsf{G}^0}(h)$ and $g \cdot (h \cdot x) = (g \cdot h) \cdot x$ for all $g \in \mathsf{G}^1$, $h \in \mathsf{H}^1$, $x \in \mathsf{X}$ with $\mathsf{s}(g) = \mathsf{r}_{\mathsf{G}^0}(h)$, $\mathsf{s}(h) = \mathsf{r}(x)$.

Routine computations show that the above formulas define a G-action on X in a natural way so as to give a functor $\mathcal{C}(\mathsf{H}) \to \mathcal{C}(\mathsf{G})$, and that H-invariant maps $f\colon \mathsf{X} \to \mathsf{Z}$ remain G-invariant.

Conversely, let $F\colon \mathcal{C}(\mathsf{H}) \to \mathcal{C}(\mathsf{G})$ be a functor that does not change the underlying object of $\mathcal{C}$ and that preserves invariant maps. That is, F equips any left H-action X with a left G-action, such that H-maps remain G-maps and H-invariant maps remain G-invariant. When we apply F to the left multiplication action on H^1, we get a left G-action on H^1. We claim that this commutes with the right multiplication action of H^1, so that we have an actor, and that this actor induces the functor F.

Let X be any left H-action. Let H act on $\mathsf{HX} := \mathsf{H}^1 \times_{\mathsf{s},\mathsf{H}^0,\mathsf{r}} \mathsf{X}$ by $h_1 \cdot (h_2, x) := (h_1 \cdot h_2, x)$ for all $h_1, h_2 \in \mathsf{H}^1$, $x \in \mathsf{X}$ with $\mathsf{s}(h_1) = \mathsf{r}(h_2)$, $\mathsf{s}(h_2) = \mathsf{r}(x)$. Equip HX with the G-action from $F(\mathsf{HX})$. Routine computations show that the maps $\mathsf{pr}_1\colon \mathsf{HX} \to \mathsf{H}^1$ and $\mathsf{m}\colon \mathsf{HX} \to \mathsf{X}$ are H-equivariant and the map $\mathsf{pr}_2\colon \mathsf{HX} \to \mathsf{X}$ is H-invariant. By assumption, pr_1 and m are G-equivariant and pr_2 is G-invariant. The statements for pr_1 and pr_2 mean that $g \cdot (h, x) = (g \cdot h, x)$ for all $g \in \mathsf{G}^1$, $h \in \mathsf{H}^1$, $x \in \mathsf{X}$ with $\mathsf{s}(g) = \mathsf{r}_{\mathsf{G}^0}(h)$, $\mathsf{s}(h) = \mathsf{r}(x)$. The G-equivariance of m gives $g \cdot \mathsf{m}(h, x) = \mathsf{m}(g \cdot (h, x)) = \mathsf{m}(g \cdot h, x)$ or briefly $g \cdot (h \cdot x) = (g \cdot h) \cdot x$ for g, h, x as above. For $\mathsf{X} = \mathsf{H}^1$, this says that the left G-action on G^1 commutes with the right H-action by multiplication, so that we have an actor. For general X, we get $g \cdot x = g \cdot (1_{\mathsf{r}(x)} \cdot x) = (g \cdot 1_{\mathsf{r}(x)}) \cdot x$, so the G-action given by F is the one from this actor. We also see that the functor F determines the actor uniquely.

Since the composite of two functors $\mathcal{C}(\mathsf{H}) \to \mathcal{C}(\mathsf{G})$ and $\mathcal{C}(\mathsf{K}) \to \mathcal{C}(\mathsf{H})$ that does not change the underlying object of $\mathcal{C}$ and preserves invariant maps again has the same properties, it must come from an actor. Thus actors may be composed. More explicitly, given actors $\mathsf{G} \to \mathsf{H}$ and $\mathsf{H} \to \mathsf{K}$, there is a unique right G-action on K^1 with $(g \cdot h) \cdot k = g \cdot (h \cdot k)$ for all $k \in \mathsf{K}^1$, $h \in \mathsf{H}^1$, $g \in \mathsf{G}^1$ with $\mathsf{s}(g) = \mathsf{r}_{\mathsf{G}^0}(h)$ and $\mathsf{s}(h) = \mathsf{r}_{\mathsf{H}^0}(k)$. This left action gives the actor from G to K that corresponds to the composite functor $\mathcal{C}(\mathsf{K}) \to \mathcal{C}(\mathsf{G})$. ∎

4.24. PROPOSITION. *An actor $\mathsf{G} \to \mathsf{H}$ is invertible if and only if it comes from an isomorphism between G and H.*

PROOF. The anchor maps $\mathsf{H}^0 \to \mathsf{G}^0$ are composed when we compose actors. Since this anchor map is the identity for the identity actor, the anchor map $\mathsf{H}^0 \to \mathsf{G}^0$ for an invertible actor must be invertible in $\mathcal{C}$. The identification between actors $\mathsf{G} \to \mathsf{H}$ with invertible anchor map $\mathsf{H}^0 \to \mathsf{G}^0$ and functors acting identically on objects in Example 4.22 is compatible with the compositions on both sides. Thus invertible actors are the same as invertible functors, that is, isomorphisms of groupoids. ∎

In order to enrich actors to a 2-category, we study automorphisms of H^1 that preserve the right H-action:

4.25. PROPOSITION. *The right H-maps $\varphi\colon \mathsf{H}^1 \to \mathsf{H}^1$ are exactly the maps of the form $h \mapsto \Phi(\mathsf{r}(h)) \cdot h$ for a section $\Phi\colon \mathsf{H}^0 \to \mathsf{H}^1$ of s. This map is invertible if and only if Φ is a bisection.*

PROOF. In the following, "section" always means a section of s. It is clear that left multiplication with a section defines an H-map. Furthermore, the map from sections to H-maps is a homomorphism for the horizontal product (8) of sections, so a bisection gives an invertible map on H^1. It remains to prove the converse. Let $\varphi\colon \mathsf{H}^1 \to \mathsf{H}^1$ be an H-map and let $h \in \mathsf{H}^1$. Then $\mathsf{s}(\varphi(h)) = \mathsf{s}(h)$, so Lemma 3.5 gives a unique ω_h with $\varphi(h) = \omega_h \cdot h$. This defines a map $\omega\colon \mathsf{H}^1 \to \mathsf{H}^1$ with $\mathsf{s} \circ \omega = \mathsf{r}$ and $\omega(h) \cdot h = \varphi(h)$ for all $h \in \mathsf{H}^1$. Since $\varphi(h_1 \cdot h_2) = \varphi(h_1) \cdot h_2$ for all $h_1, h_2 \in \mathsf{H}^1$ with $\mathsf{s}(h_1) = \mathsf{r}(h_2)$, we get $\omega(h_1 \cdot h_2) = \omega(h_1)$. This implies $\omega = \Phi \circ \mathsf{r}$ for a map $\Phi\colon \mathsf{H}^0 \to \mathsf{H}^1$, compare the proof of Proposition 3.9. We have $\mathsf{s} \circ \Phi = \mathrm{id}_{\mathsf{H}^0}$, so Φ is a section, and $\varphi(h) = \Phi(\mathsf{r}(h)) \cdot h$ for all $h \in \mathsf{H}^1$ by construction. The section Φ is uniquely determined by φ.

If φ is invertible, its inverse is also an H-map and hence associated to some section Ψ. Since the map $\Phi \mapsto \varphi$ is injective, Φ is invertible in the monoid of sections, hence it is a bisection by Lemma 3.21. ∎

Let $\mathsf{m}_1, \mathsf{m}_2$ be two actors from G to H, that is, left actions of G on H^1 that commute with the right multiplication action. A 2-arrow from m_1 to m_2 is a G, H-bibundle map $(\mathsf{H}^1, \mathsf{m}_1) \to (\mathsf{H}^1, \mathsf{m}_2)$. By Proposition 4.25, this is the same as a section Φ of H with

$$\Phi(\mathsf{r}(g \cdot_1 h)) \cdot (g \cdot_1 h) = g \cdot_2 (\Phi(\mathsf{r}(h)) \cdot_1 h)$$

for all $g \in \mathsf{G}^1$, $h \in \mathsf{H}^1$ with $\mathsf{s}(g) = \mathsf{r}_{\mathsf{G},1}(\mathsf{r}(h))$, where $\cdot_1$ and $\cdot_2$ come from m_1 and m_2, respectively. Composing the G, H-maps associated to sections Φ_1 and Φ_2 gives the G, H-map associated to the section $\Phi_1 \circ \Phi_2$ defined in (8).

4.26. PROPOSITION. *There is a strict 2-category with groupoids in $(\mathcal{C}, \mathcal{T})$ as objects, actors as arrows, sections Φ as above as 2-arrows, the usual composition of actors and the usual unit actors, and with the product $\circ$ for sections as vertical product. The equivalences in this 2-category are precisely the isomorphisms of groupoids in $(\mathcal{C}, \mathcal{T})$.*

PROOF. It remains to construct the horizontal product. Let G, H and K be groupoids, let $\mathsf{m}_1, \mathsf{m}_2\colon \mathsf{G} \rightrightarrows \mathsf{H}$ and $\mathsf{m}'_1, \mathsf{m}'_2\colon \mathsf{H} \rightrightarrows \mathsf{K}$ be actors, let $\Phi\colon \mathsf{H}^0 \to \mathsf{H}^1$ and $\Psi\colon \mathsf{K}^0 \to \mathsf{K}^1$ be sections with

$$\Phi(\mathsf{r}(g \cdot_1 h)) \cdot (g \cdot_1 h) = g \cdot_2 (\Phi(\mathsf{r}(h)) \cdot_1 h), \qquad \Psi(\mathsf{r}(h \cdot_1 k)) \cdot (h \cdot_1 k) = h \cdot_2 (\Psi(\mathsf{r}(k)) \cdot_1 k)$$

for all $g \in \mathsf{G}^1$, $h \in \mathsf{H}^1$, $k \in \mathsf{K}^1$ with $\mathsf{s}(g) = \mathsf{r}_{\mathsf{G},1}(\mathsf{r}(h))$ and $\mathsf{s}(h) = \mathsf{r}_{\mathsf{H},1}(\mathsf{r}(k))$, respectively. Then $\Psi \bullet \Phi\colon \mathsf{K}^0 \to \mathsf{K}^1$, $x \mapsto \Phi(\mathsf{r}_{\mathsf{H},1}(x)) \cdot \Psi(x)$, is a section for K that gives a 2-arrow from the composite $\mathsf{m}'_1 \circ \mathsf{m}_1$ to $\mathsf{m}'_2 \circ \mathsf{m}_2$.

An equivalence $\alpha\colon \mathsf{G} \to \mathsf{H}$ in this 2-category has a quasi-inverse $\beta\colon \mathsf{H} \to \mathsf{G}$ and invertible 2-arrows $\mathrm{id}_\mathsf{G} \Rightarrow \beta \circ \alpha$ and $\mathrm{id}_\mathsf{H} \Rightarrow \alpha \circ \beta$. Being invertible, they correspond to bisections. Then it follows that the actors $\beta \circ \alpha$ and $\alpha \circ \beta$ come from inner automorphisms of G and H, respectively, as in Example 4.22. Thus $\beta \circ \alpha$ and $\alpha \circ \beta$ are isomorphisms in the 1-category of actors. This implies that α is an isomorphism in the 1-category of actors. Now Proposition 4.24 shows that α is an isomorphism of categories. ∎

5. Principal bundles

5.1. DEFINITION. *Let G be a groupoid in $(\mathcal{C}, \mathcal{T})$. A G-bundle (over Z) is a G-action* $(\mathsf{X}, \mathsf{s}, \mathsf{m})$ *with a G-invariant map $\mathsf{p}\colon \mathsf{X} \to \mathsf{Z}$ (bundle projection). A G-bundle is* principal *if*

(1) p *is a cover;*

(2) *the following map is an isomorphism:*

$$(\mathsf{pr}_1, \mathsf{m})\colon \mathsf{X} \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \xrightarrow{\sim} \mathsf{X} \times_{\mathsf{p},\mathsf{Z},\mathsf{p}} \mathsf{X}, \qquad (x, g) \mapsto (x, x \cdot g). \qquad (17)$$

The isomorphism (17) is equivalent to the elementwise statement that for $x_1, x_2 \in \mathsf{X}$ with $\mathsf{p}(x_1) = \mathsf{p}(x_2)$, there is a unique $g \in \mathsf{G}^1$ with $\mathsf{s}(x_1) = \mathsf{r}(g)$ and $x_1 \cdot g = x_2$; this translation is proved like Lemma 3.5.

5.2. DEFINITION. *Let G be a groupoid and X a G-action in $(\mathcal{C}, \mathcal{T})$. The* orbit space projection $\pi_\mathsf{X}\colon \mathsf{X} \to \mathsf{X}/\mathsf{G}$ *is the coequaliser of the two maps $\mathsf{pr}_1, \mathsf{m}\colon \mathsf{X} \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \rightrightarrows \mathsf{X}$.*

While the coequaliser need not exist in $\mathcal{C}$, it always exists in the category of presheaves over $\mathcal{C}$.

5.3. LEMMA. *Let X with bundle projection $\mathsf{p}\colon \mathsf{X} \twoheadrightarrow \mathsf{Z}$ be a principal G-action. Then p is equivalent to the orbit space projection $\mathsf{X} \to \mathsf{X}/\mathsf{G}$, which therefore exists in $\mathcal{C}$ and is a cover. The orbit space projection is always G-invariant, and condition $(3'')$ in Definition 4.1 follows from* (17).

PROOF. Since the pretopology is subcanonical, the cover $\mathsf{p}\colon \mathsf{X} \twoheadrightarrow \mathsf{Z}$ is the coequaliser of $\mathsf{pr}_1, \mathsf{pr}_2\colon \mathsf{X} \times_{\mathsf{p},\mathsf{Z},\mathsf{p}} \mathsf{X} \rightrightarrows \mathsf{X}$, and these two maps are covers. By the isomorphism (17), p is the coequaliser of $\mathsf{pr}_1, \mathsf{m}\colon \mathsf{X} \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \rightrightarrows \mathsf{X}$ as well, and both pr_1 and m are covers. Thus p is an orbit space projection and m is automatically a cover; the latter is Definition 4.1.$(3'')$. ∎

5.4. EXAMPLES.

5.5. EXAMPLE. Let Y be a 0-groupoid (see Example 3.11), let $\mathsf{X} \to \mathsf{Y}$ be a Y-action, and let $\mathsf{p} = \mathrm{id}_\mathsf{X}$. This is a principal Y-bundle.

5.6. EXAMPLE. Let $p\colon X \twoheadrightarrow Z$ be a cover and let G be its Čech groupoid (see Example 3.12). The canonical action of G on its objects X is given by $s = \mathrm{id}_X$ and $x_1 \cdot (x_1, x_2) := x_2$ for all $x_1, x_2 \in X$ with $p(x_1) = p(x_2)$ (see Example 4.6). Together with the bundle projection $p\colon X \to Z$, this is a principal bundle: the map $X \times_{\mathrm{id}_X, X, \mathrm{pr}_1} (X \times_{p,Z,p} X) \to X \times_{p,Z,p} X$ defined elementwise by $(x_1, (x_1, x_2)) \mapsto (x_1, x_2)$ is an isomorphism.

5.7. PULL-BACKS OF PRINCIPAL BUNDLES.

5.8. PROPOSITION. *Let $f\colon \tilde{Z} \to Z$ be a map in $\mathcal{C}$ and let (X, s, p, m) be a principal G-bundle over Z. Then there is a principal G-bundle $(\tilde{X}, \tilde{s}, \tilde{p}, \tilde{m})$ over $\tilde{Z}$ with a G-map $\hat{f}\colon \tilde{X} \to X$ with $p \circ \hat{f} = f \circ \tilde{p}$. These are unique in the following sense: for any principal G-bundle $(\tilde{X}', \tilde{s}', \tilde{p}', \tilde{m}')$ over $\tilde{Z}$ and G-map $\hat{f}'\colon \tilde{X}' \to X$, there is a unique isomorphism of principal G-bundles $\varphi\colon \tilde{X} \xrightarrow{\sim} \tilde{X}'$ with $\hat{f}' \circ \varphi = \hat{f}$.*

This principal G-bundle is called the *pull-back* of (X, s, p, m) along f.

PROOF. Let $\tilde{X} := \tilde{Z} \times_{f,Z,p} X$, $\tilde{s} := s \circ \mathrm{pr}_2\colon \tilde{X} \to G^0$ and $\tilde{p} := \mathrm{pr}_1\colon \tilde{X} \to \tilde{Z}$. The fibre product $\tilde{X}$ exists and $\tilde{p}$ is a cover because p is a cover. Let

$$\tilde{m}\colon \tilde{X} \times_{\tilde{s},G^0,r} G^1 = \tilde{Z} \times_{f,Z,p} X \times_{s,G^0,r} G^1 \xrightarrow{\mathrm{id}_{\tilde{Z}} \times_{f,Z} m} \tilde{Z} \times_{f,Z,p} X = \tilde{X};$$

this map is well-defined because $p \circ m = p \circ \mathrm{pr}_1$.

Routine computations show that $(\tilde{X}, \tilde{s}, \tilde{m})$ is a G-action and that $\tilde{p} \circ \tilde{m} = \tilde{p} \circ \mathrm{pr}_1$. The map (17) with tildes is the pull-back along f of the same map without tildes, where we use the canonical maps from both sides to Z. Since pull-backs of isomorphisms remain isomorphisms, $(\tilde{X}, \tilde{Z}, \tilde{s}, \tilde{m}, \tilde{p})$ is a principal G-bundle over Z. The second coordinate projection $\hat{f}\colon \tilde{X} \to X$ is a G-map with $p \circ \hat{f} = f \circ \tilde{p}$.

Let $(\tilde{X}', \tilde{s}', \tilde{p}', \tilde{m}')$ be another principal G-bundle over $\tilde{Z}$ with a G-map $\hat{f}'\colon \tilde{X}' \to X$ with $p \circ \hat{f}' = f \circ \tilde{p}'$. Then $\varphi := (\tilde{p}', \hat{f}')\colon \tilde{X}' \to \tilde{X}$ is a G-map with $\tilde{p} \circ \varphi = \tilde{p}'$, and it is the only G-map with this extra property. The proof will be finished by showing that φ is an isomorphism.

Since $\tilde{p}'\colon \tilde{X}' \twoheadrightarrow \tilde{Z}$ is a cover, so is its pull-back along $\tilde{p}\colon \tilde{X} \to \tilde{Z}$. Identifying $\tilde{X}' \times_{\tilde{p}', \tilde{Z}, \tilde{p}} \tilde{X} \cong \tilde{X}' \times_{f \circ \tilde{p}', Z, p} X$, this cover becomes the map

$$\tilde{p}' \times_Z \mathrm{id}_X\colon \tilde{X}' \times_{f\tilde{p}', Z, p} X \twoheadrightarrow \tilde{X} = \tilde{Z} \times_{f,Z,p} X, \qquad (x_1, x_2) \mapsto (\tilde{p}'(x_1), x_2).$$

Since $\tilde{p}' \times_Z \mathrm{id}_X$ is a cover, the fibre product of the following two maps exists:

$$\tilde{X}' \times_{f\tilde{p}', Z, p} X \xrightarrow{\tilde{p}' \times_Z \mathrm{id}_X} \tilde{X} \xleftarrow{\varphi} \tilde{X}'.$$

An element of this fibre product is a triple $x_1 \in \tilde{X}'$, $x_2 \in X$, $x_3 \in \tilde{X}'$ with $f(\tilde{p}'(x_1)) = p(x_2)$, $\hat{f}'(x_3) = x_2$, $\tilde{p}'(x_3) = \tilde{p}'(x_1)$. Then $p(x_2) = p(\hat{f}'(x_3)) = f(\tilde{p}'(x_3)) = f(\tilde{p}'(x_1)) = p(\hat{f}'(x_1))$. Since X is a principal G-bundle over Z, there is a unique $g \in G$ with $s(x_2) = r(g)$ and $\hat{f}'(x_1) = x_2 \cdot g$. Since $\tilde{X}'$ is a principal G-bundle over $\tilde{Z}$ and $\tilde{p}'(x_3) = \tilde{p}'(x_1)$, there is also a unique $g' \in G^1$ with $\tilde{s}'(x_1) = r(g')$ and $x_1 \cdot g' = x_3$. Then $x_2 = \hat{f}'(x_3) = \hat{f}'(x_1 \cdot g') =$

$\hat{f}'(x_1) \cdot g'$, so $g = g'$. Thus $x_3 = \hat{f}'(x_1) \cdot g$ is uniquely determined by x_1 and x_2. That is, the pull-back of φ along the cover $\tilde{\mathsf{p}}' \times_\mathsf{Z} \mathrm{id}_\mathsf{X}$ is an isomorphism. This implies that φ is an isomorphism by Proposition 2.5. ∎

5.9. PROPOSITION. *Let X_1 and X_2 be principal G-bundles over Z_1 and Z_2, respectively. Then a G-map $f \colon \mathsf{X}_1 \to \mathsf{X}_2$ induces a map $f/\mathsf{G} \colon \mathsf{Z}_1 \to \mathsf{Z}_2$. The map f is an isomorphism if and only if f/G is. If f/G is a cover, so is f. Conversely, under Assumption 2.8, f is a cover if and only if f/G is.*

PROOF. Since Z_i is the orbit space of X_i and orbit spaces are constructed naturally, the G-map f induces a map $f/\mathsf{G} \colon \mathsf{Z}_1 \to \mathsf{Z}_2$. Since $\mathsf{p}_2 \circ f = f/\mathsf{G} \circ \mathsf{p}_1$, X_1 must be the pull-back of X_2 along f/G by Proposition 5.8. That is, the diagram

$$
\begin{array}{ccc}
\mathsf{X}_1 & \xrightarrow{\ f\ } & \mathsf{X}_2 \\
{\scriptstyle \mathsf{p}_1}\downarrow & & \downarrow{\scriptstyle \mathsf{p}_2} \\
\mathsf{Z}_1 & \xrightarrow[\ f/\mathsf{G}\ ]{} & \mathsf{Z}_2
\end{array}
$$

is a fibre-product diagram. Proposition 2.5 shows that f is an isomorphism if and only if f/G is one. By the pretopology axioms, f is a cover if f/G is, and the converse holds under Assumption 2.8. ∎

5.10. PROPOSITION. *Consider a commuting square of G-maps between principal G-bundles X_i and the corresponding maps between their base spaces Z_i:*

$$
\begin{array}{ccc}
\mathsf{X}_1 & \xrightarrow{\ \gamma_2\ } & \mathsf{X}_2 \\
{\scriptstyle \gamma_3}\downarrow & & \downarrow{\scriptstyle \delta_2} \\
\mathsf{X}_3 & \xrightarrow[\ \delta_3\ ]{} & \mathsf{X}_4
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathsf{Z}_1 & \xrightarrow{\ \beta_2\ } & \mathsf{Z}_2 \\
{\scriptstyle \beta_3}\downarrow & & \downarrow{\scriptstyle \alpha_2} \\
\mathsf{Z}_3 & \xrightarrow[\ \alpha_3\ ]{} & \mathsf{Z}_4
\end{array}
$$

The first square is a fibre-product square if and only if the second one is.

PROOF. Assume first that the square of Z_i is a fibre-product square. Let $x_2 \in \mathsf{X}_2$, $x_3 \in \mathsf{X}_3$ satisfy $x_4 := \delta_2(x_2) = \delta_3(x_3)$. We must show that there is a unique $x_1 \in \mathsf{X}_1$ with $\gamma_i(x_1) = x_i$ for $i = 2, 3$. Let $z_i := \mathsf{p}_i(x_i) \in \mathsf{Z}_i$, then $\alpha_2(z_2) = \mathsf{p}_4(\delta_2(x_2)) = \mathsf{p}_4(\delta_3(x_3)) = \alpha_3(z_3)$. Since Z_1 is the pull-back of α_2, α_3 by assumption, we get a unique $z_1 \in \mathsf{Z}_1$ with $\beta_i(z_1) = z_i$ for $i = 2, 3$. Proposition 5.8 shows that $\mathsf{X}_1 \twoheadrightarrow \mathsf{Z}_1$ is the pull-back of $\mathsf{X}_4 \twoheadrightarrow \mathsf{Z}_4$ along $\alpha_i \circ \beta_i \colon \mathsf{Z}_1 \to \mathsf{Z}_4$, and similarly for $\mathsf{X}_i \twoheadrightarrow \mathsf{Z}_i$ for $i = 2, 3$. Thus $x_4 \in \mathsf{X}_4$ and $z_1 \in \mathsf{Z}_1$ determine a unique element $x_1 \in \mathsf{X}_1$ with $\mathsf{p}_1(x_1) = z_1$ and $\delta_i \gamma_i(x_1) = x_4$ for $i = 2, 3$. We claim that this is the unique element of X_1 with $\gamma_i(x_1) = x_i$ for $i = 2, 3$.

Any element of X_1 with $\gamma_i(x_1) = x_i$ for $i = 2, 3$ will also satisfy $\mathsf{p}_1(x_1) = z_1$ and $\delta_i \gamma_i(x_1) = x_4$ for $i = 2, 3$, so uniqueness is clear. Since $\mathsf{X}_i \twoheadrightarrow \mathsf{Z}_i$ is the pull-back of $\mathsf{X}_4 \twoheadrightarrow \mathsf{Z}_4$ for $i = 2, 3$, the element $x_i \in \mathsf{X}_i$ is uniquely determined by $z_i = \mathsf{p}_i(x_i)$ and

$x_4 = \delta_i(x_i)$. Since $\mathsf{p}_i(\gamma_i(x_1)) = \beta_i\mathsf{p}_1(x_1) = \beta_i(z_1) = z_i = \mathsf{p}_i(x_i)$ and $\delta_i(\gamma_i(x_1)) = x_4 = \delta_i(x_i)$, we get $x_i = \gamma_i(x_1)$ as desired. Hence we get a bijection between $x_1 \in \mathsf{X}_1$ and pairs $x_2 \in \mathsf{X}_2$, $x_3 \in \mathsf{X}_3$ with $\delta_2(x_2) = \delta_3(x_3)$. This is the elementwise statement corresponding to $\mathsf{X}_1 \cong \mathsf{X}_2 \times_{\mathsf{X}_4} \mathsf{X}_3$.

Now assume, conversely, that $\mathsf{X}_1 \cong \mathsf{X}_2 \times_{\mathsf{X}_4} \mathsf{X}_3$. We are going to show that $\mathsf{Z}_1 \cong \mathsf{Z}_2 \times_{\mathsf{Z}_4} \mathsf{Z}_3$. This means that, for all $? \in\in \mathcal{C}$, the map

$$\mathcal{C}(?, \mathsf{Z}_1) \to \{(z_2, z_3) \in \mathcal{C}(?, \mathsf{Z}_2) \times \mathcal{C}(?, \mathsf{Z}_3) \mid \alpha_2 z_2 = \alpha_3 z_3\}$$

is a bijection. Let $z_i\colon ? \to \mathsf{Z}_i$ for $i = 2, 3, 4$ with $\alpha_2 z_2 = z_4 = \alpha_3 z_3$ be given. The G-maps $\mathsf{X}_i \to \mathsf{X}_4$ for $i = 2, 3$ give isomorphisms between $\mathsf{X}_i \twoheadrightarrow \mathsf{Z}_i$ and the pull-back of $\mathsf{X}_4 \twoheadrightarrow \mathsf{Z}_4$ along α_i. Pulling back further along the maps z_i shows that the three pull-backs of the principal G-bundles $\mathsf{X}_i \twoheadrightarrow \mathsf{Z}_i$ along z_i for $i = 2, 3, 4$ are canonically isomorphic principal G-bundles over $?$. Let $?? \twoheadrightarrow ?$ be this unique principal G-bundle over $?$. By Proposition 5.8, the maps z_i for $i = 2, 3, 4$ lift uniquely to $x_i\colon ?? \to \mathsf{X}_i$. These liftings still satisfy $\gamma_2 x_2 = x_4 = \gamma_3 x_3$ because the lifting of z_4 is unique.

Since $\mathsf{X}_1 \cong \mathsf{X}_2 \times_{\mathsf{X}_4} \mathsf{X}_3$ is a fibre product in the category of G-actions by Lemma 4.18, the unique map $x_1\colon ?? \to \mathsf{X}_1$ with $\delta_i x_1 = x_i$ for $i = 2, 3$ is a G-map. Thus it induces a map $z_1\colon ? \to \mathsf{Z}_1$ by Proposition 5.9. The naturality of this construction implies $\beta_i z_1 = z_i$ for $i = 2, 3$. Any map $z'_1\colon ? \to \mathsf{Z}_1$ with $\beta_i z'_1 = z_i$ for $i = 2, 3$ lifts to a map $x'_1 = (z'_1, x_i)\colon ?? \to \mathsf{Z}_1 \times_{\mathsf{Z}_i} \mathsf{X}_i \cong \mathsf{X}_1$ with $\delta_i x'_1 = x_i$ for $i = 2, 3$ by Proposition 5.8. Since $\mathsf{X}_1 \cong \mathsf{X}_2 \times_{\mathsf{X}_4} \mathsf{X}_3$, we must have $x'_1 = x_1$, which gives $z'_1 = z_1$. Hence there is a unique map $z_1\colon ? \to \mathsf{Z}_1$ with $\beta_i z_1 = z_i$ for $i = 2, 3$, as desired. ∎

5.11. LOCALITY OF PRINCIPAL BUNDLES. We formulate now what it means for principal bundles to be a "local" notion. Let G be a groupoid in $(\mathcal{C}, \mathcal{T})$. Let $\mathsf{X}, \mathsf{Z} \in\in \mathcal{C}$ and let $\mathsf{s}\colon \mathsf{X} \to \mathsf{G}^0$, $\mathsf{p}\colon \mathsf{X} \to \mathsf{Z}$, and $\mathsf{m}\colon \mathsf{X} \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1 \to \mathsf{X}$ be maps. We pull back this data along a cover $f\colon \tilde{\mathsf{Z}} \twoheadrightarrow \mathsf{Z}$ as follows. Let $\tilde{\mathsf{X}} := \tilde{\mathsf{Z}} \times_{f, \mathsf{Z}, \mathsf{p}} \mathsf{X}$, $\tilde{\mathsf{s}} := \mathsf{s} \circ \mathrm{pr}_2\colon \tilde{\mathsf{X}} \to \mathsf{G}^0$, $\tilde{\mathsf{p}} := \mathrm{pr}_1\colon \tilde{\mathsf{X}} \to \tilde{\mathsf{Z}}$ and

$$\tilde{\mathsf{m}}\colon \tilde{\mathsf{X}} \times_{\tilde{\mathsf{s}}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1 = \tilde{\mathsf{Z}} \times_{f, \mathsf{Z}, \mathsf{p}} \mathsf{X} \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1 \xrightarrow{\mathrm{id}_{\tilde{\mathsf{Z}}} \times_{f, \mathsf{Z}} \mathsf{m}} \tilde{\mathsf{Z}} \times_{f, \mathsf{Z}, \mathsf{p}} \mathsf{X} = \tilde{\mathsf{X}};$$

this map is well-defined if $\mathsf{p} \circ \mathsf{m} = \mathsf{p} \circ \mathrm{pr}_1$.

5.12. PROPOSITION. *The data $(\tilde{\mathsf{X}}, \tilde{\mathsf{Z}}, \tilde{\mathsf{s}}, \tilde{\mathsf{m}}, \tilde{\mathsf{p}})$ is a well-defined principal G-bundle over $\tilde{\mathsf{Z}}$ if and only if $(\mathsf{X}, \mathsf{Z}, \mathsf{s}, \mathsf{m}, \mathsf{p})$ is a principal G-bundle over Z.*

Put in a nutshell, principality for G-bundles is a local property.

PROOF. If $(\mathsf{X}, \mathsf{s}, \mathsf{m}, \mathsf{p})$ is a principal G-bundle over Z, then Proposition 5.8 shows that $(\tilde{\mathsf{X}}, \tilde{\mathsf{s}}, \tilde{\mathsf{m}}, \tilde{\mathsf{p}})$ is a principal G-bundle over $\tilde{\mathsf{Z}}$.

Now assume that $(\tilde{\mathsf{X}}, \tilde{\mathsf{s}}, \tilde{\mathsf{m}}, \tilde{\mathsf{p}})$ is a well-defined principal G-bundle over $\tilde{\mathsf{Z}}$. We are going to show that $(\mathsf{X}, \mathsf{s}, \mathsf{m}, \mathsf{p})$ is a principal G-bundle over Z. For $\tilde{\mathsf{m}}$ to be well-defined, we need $\mathsf{p} \circ \mathsf{m} = \mathsf{p} \circ \mathrm{pr}_1$ or $\mathsf{p}(x \cdot g) = \mathsf{p}(x)$ for all $x \in \mathsf{X}$, $g \in \mathsf{G}^1$ with $\mathsf{s}(x) = \mathsf{r}(g)$. Since f is a cover, so is the map $\tilde{\mathsf{X}} \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1 \twoheadrightarrow \mathsf{X} \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1$ it induces. Since $\mathcal{T}$ is subcanonical, this map is an epimorphism. Now we get $\mathsf{s} \circ \mathsf{m} = \mathsf{s} \circ \mathrm{pr}_2\colon \mathsf{X} \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1 \to \mathsf{G}^0$ because composing $\mathsf{s} \circ \mathsf{m}$ and

$s \circ pr_2$ with the cover $\tilde{X} \times_{s,G^0,r} G^1 \twoheadrightarrow X \times_{s,G^0,r} G^1$ gives the same map $\tilde{s} \circ m = \tilde{s} \circ pr_2$. A similar, more complicated argument shows that m is associative because $\tilde{m}$ is associative and both multiplications are intertwined by covers $\tilde{X} \times_{s,G^0,r} G^1 \times_{s,G^0,r} G^1 \twoheadrightarrow X \times_{s,G^0,r} G^1 \times_{s,G^0,r} G^1$ and $\tilde{X} \twoheadrightarrow X$. Similarly, we get the unitality condition for the multiplication, so (X, s, m) is a G-action.

It remains to verify that the map

$$(m, pr_1) : X \times_{s,G^0,r} G^1 \to X \times_{p,Z,p} X$$

is an isomorphism. Since the coordinate projection $\tilde{X} \twoheadrightarrow X$ is a cover, so is the induced map $X \times_{p,Z,f \circ \tilde{p}} \tilde{X} \twoheadrightarrow X \times_{p,Z,p} X$. The pull-back of (m, pr_1) along this map is equivalent to the map

$$(\tilde{m}, pr_1) : \tilde{X} \times_{\tilde{s},G^0,r} G^1 \to \tilde{X} \times_{\tilde{p},\tilde{Z},\tilde{p}} \tilde{X}, \tag{18}$$

where we identify

$$\tilde{X} \times_{\tilde{p},\tilde{Z},\tilde{p}} \tilde{X} \cong X \times_{p,\tilde{Z},f \circ \tilde{p}} \tilde{X},$$

$$(X \times_{s,G^0,r} G^1) \times_{X \times_{p,Z,p} X} (X \times_{p,Z,f \circ \tilde{p}} \tilde{X}) \cong \tilde{X} \times_{\tilde{s},G^0,r} G^1$$

in the obvious way. The map (18) is an isomorphism because $\tilde{X}$ is principal. This implies that (m, pr_1) is an isomorphism because the property of being an isomorphism is local by Proposition 2.5. ∎

5.13. Basic actions and Čech groupoids.

5.14. Definition. *A groupoid action or sheaf that, together with some bundle projection, is part of a principal bundle is called* basic. *A groupoid is called* basic *if its canonical action on G^0 in Example 4.6 is basic.*

We call such actions "basic" because they have a well-behaved base and because "principal groupoid" already has a different meaning for topological groupoids (it means that the action on G^0 is free).

5.15. Proposition. *A G-action X is basic if and only if the transformation groupoid $X \rtimes G$ is isomorphic to a Čech groupoid of a certain cover $p : X \twoheadrightarrow Z$.*

Proof. Let X be basic with bundle projection $p : X \twoheadrightarrow Z$. Then (17) provides an isomorphism $(X \rtimes G)^1 \xrightarrow{\sim} X \times_{p,Z,p} X$. Together with the identity on objects, this is an isomorphism of groupoids from $X \rtimes G$ to the Čech groupoid of p. Conversely, let $X \rtimes G$ be isomorphic to a Čech groupoid. We may assume that the isomorphism is the identity on objects, so that the cover whose Čech groupoid we take is a map $p : X \twoheadrightarrow Z$. Since the maps pr_1 and m in (17) are the range and source maps of $X \rtimes G$, the isomorphism of groupoids from $X \rtimes G$ to the Čech groupoid of p must be given by $(x, g) \mapsto (x, x \cdot g)$ on arrows; thus (17) is an isomorphism. ∎

5.16. COROLLARY. *A* G-*action on* X *is basic if and only if its transformation groupoid* X ⋊ G *is basic.*

PROOF. The criterion in Proposition 5.15 depends only on X ⋊ G. ∎

5.17. COROLLARY. *An action of a transformation groupoid* X ⋊ G *is basic if and only if the restriction of the action to* G *is basic.*

PROOF. Proposition 4.12 shows that Y ⋊ (X ⋊ G) ≅ Y ⋊ G, so the assertion follows from Corollary 5.16. ∎

5.18. LEMMA. *Let* Y *carry an action of a transformation groupoid* X ⋊ G, *equip it with the resulting* G-*action. Then* Y/(X ⋊ G) ≅ Y/G.

PROOF. Proposition 4.12 shows that the invariant maps for both actions are the same, hence so are the orbit spaces. ∎

6. Bibundle functors, actors, and equivalences

Let G and H be groupoids in $(\mathcal{C}, \mathcal{T})$. We describe several important classes of G, H-bibundles:

6.1. DEFINITION. *A* bibundle equivalence *from* G *to* H *is a* G, H-*bibundle* X *such that both the left and right actions are principal bundles with bundle projections* s: X ↠ H⁰ *and* r: X ↠ G⁰, *respectively. We call* G *and* H equivalent *if there is a bibundle equivalence from* G *to* H.

A bibundle functor *from* G *to* H *is a* G, H-*bibundle* X *such that the right* H-*action is a principal bundle with bundle projection* r: X ↠ G⁰. *A bibundle functor is* covering *if the anchor map* s: X ↠ H⁰ *is a cover.*

A bibundle actor *from* G *to* H *is a* G, H-*bibundle* X *such that the right* H-*action is basic and a sheaf, that is,* s: X ↠ H⁰ *is a cover.*

Bibundle functors are also called generalised morphisms[33, Section 2.5] or Hilsum–Skandalis morphisms [35], and bibundle equivalences just equivalences or Morita equivalences. We will show later that for sufficiently nice pretopologies, bibundle functors and actors are precisely the products of bibundle equivalences with functors and actors, respectively (see Sections 6.10 and 7.19). This justifies the names above.

The anchor map r: X ↠ G⁰ of a bibundle functor is always a cover because it is the bundle projection of a principal action. For an equivalence X, both anchor maps r: X ↠ G⁰ and s: X ↠ H⁰ are covers for the same reason. Thus bibundle equivalences are covering bibundle functors. A bibundle functor is a bibundle actor as well if and only if

it is covering. The following diagram illustrates the relations between these notions:

	bibundle functor	
	covering bibundle functor	bibundle actor
	bibundle equivalence	
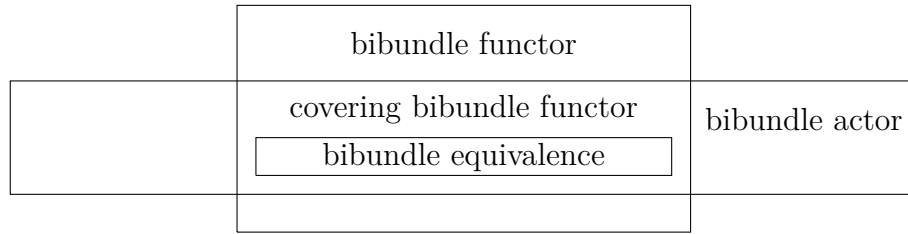
6.2. EXAMPLE. Let G be a groupoid in $(\mathcal{C}, \mathcal{T})$. Then G acts on G^1 on the left and right by multiplication. These actions turn G^1 into a bibundle equivalence from G to itself, so equivalence of groupoids is a reflexive relation. First, the two actions commute by associativity, so they form a G, G-bibundle; secondly, the right multiplication action gives a principal G-bundle with bundle projection $\mathsf{s}\colon \mathsf{G}^1 \twoheadrightarrow \mathsf{G}^0$ by (5); thirdly, the left action gives a principal G-bundle with bundle projection $\mathsf{r}\colon \mathsf{G}^1 \twoheadrightarrow \mathsf{G}^0$ by (6).

The assumptions that r and s be covers and the conditions (5) and (6) in the definition of a groupoid are necessary and sufficient for this unit bibundle equivalence to work.

Left and right actions of groupoids are equivalent by Lemma 4.14. Left and right principal bundles are also equivalent in the same way. Thus a G, H-bibundle X gives an H, G-bibundle X^*, which is the same object of $\mathcal{C}$ with the two anchor maps exchanged and $h \cdot x \cdot g := g^{-1} \cdot x \cdot h^{-1}$; this is a bibundle equivalence if and only if X is one. Thus equivalence of groupoids is a symmetric relation. Transitivity seems to require an extra assumption in order to compose bibundle equivalences, see Section 7.

6.3. EXAMPLE. Let $\mathsf{p}\colon \mathsf{X} \twoheadrightarrow \mathsf{Z}$ be a cover in $(\mathcal{C}, \mathcal{T})$. View Z as a groupoid with only identity arrows and let G be the Čech groupoid of p. Then G and Z are equivalent.

The equivalence bibundle is X, with the canonical right action of G on its object space and with the left Z-action given by the anchor map p. The right G-action on X with p as bundle projection gives a principal G-bundle by Example 5.6. The left Z-action is given simply by its anchor map p, and any such action gives a principal bundle with the identity map $\mathsf{X} \to \mathsf{X}$ as bundle projection; the identity is also the right anchor map.

Conversely, if X is a bibundle equivalence from a groupoid G to a 0-groupoid Z, then G is isomorphic to the Čech groupoid of the anchor map $\mathsf{p}\colon \mathsf{X} \twoheadrightarrow \mathsf{Z}$.

6.4. EXAMPLE. More generally, let $\mathsf{p}_i\colon \mathsf{X}_i \twoheadrightarrow \mathsf{Z}$ for $i = 1, 2$ be two covers and let G_1 and G_2 be their Čech groupoids. Let $\mathsf{X} := \mathsf{X}_1 \times_{\mathsf{p}_1, \mathsf{Z}, \mathsf{p}_2} \mathsf{X}_2$, $\mathsf{r} := \mathsf{pr}_1\colon \mathsf{X} \twoheadrightarrow \mathsf{X}_1 = \mathsf{G}_1^0$ and $\mathsf{s} := \mathsf{pr}_2\colon \mathsf{X} \twoheadrightarrow \mathsf{X}_2 = \mathsf{G}_2^0$, and define the multiplication maps by $(x_1, y_1) \cdot (y_1, y_2) := (x_1, y_2)$, $(x_1, x_2) \cdot (x_2, y_2) := (x_1, y_2)$ for $x_1, y_1 \in \mathsf{X}_1$, $x_2, y_2 \in \mathsf{X}_2$ with $\mathsf{p}_1(x_1) = \mathsf{p}_1(y_1) = \mathsf{p}_2(x_2) = \mathsf{p}_2(y_2)$. Then X is a bibundle equivalence from G_1 to G_2. This shows that equivalence of groupoids is a transitive relation among Čech groupoids.

6.5. EXAMPLE. Let Y be an object of $\mathcal{C}$ viewed as a groupoid. An action of Y is the same as a map to Y by Example 4.8. Thus a G, Y-bibundle is the same as a (left) G-action X with a G-invariant map $f\colon \mathsf{X} \to \mathsf{Y}$.

A bibundle functor $G \to Y$ is, up to isomorphism, the same as a G-invariant map $G^0 \to Y$ because of the assumption that r induces an isomorphism $X = X/Y \xrightarrow{\sim} G^0$. A bibundle functor $Y \to G$ is the same as a principal G-bundle over Y.

A bibundle actor $G \to Y$ is the same as a G-action X with a G-invariant cover $X \twoheadrightarrow Y$. A bibundle actor $Y \to G$ is the same as a principal G-bundle X over some space Z with a map $Z \to Y$, such that the anchor map $s\colon X \twoheadrightarrow G^0$ is a cover.

Let Y_1 and Y_2 be two objects of $\mathcal{C}$ viewed as groupoids. Then a Y_1, Y_2-bibundle is a span $Y_1 \leftarrow X \to Y_2$ in $\mathcal{C}$. This is

- a bibundle functor if and only if the map $Y_1 \leftarrow X$ is invertible;

- a bibundle equivalence if and only if both maps $Y_1 \leftarrow X \to Y_2$ are isomorphisms;

- a bibundle actor if and only if the map $X \twoheadrightarrow Y_2$ is a cover.

Example 6.3 shows that a groupoid is equivalent to a 0-groupoid if and only if it is basic. By Proposition 5.15, a groupoid is basic if and only if it is isomorphic to a Čech groupoid.

6.6. FROM FUNCTORS TO BIBUNDLE FUNCTORS. Let G and H be groupoids in $(\mathcal{C}, \mathcal{T})$ and let $F\colon G \to H$ be a functor, given by $F^i \in \mathcal{C}(G^i, H^i)$ for $i = 0, 1$. We are going to define an associated bibundle functor X_F from G to H (see also [33]).

Since $r\colon H^1 \twoheadrightarrow H^0$ is a cover, the fibre product $X_F = X := G^0 \times_{F^0, H^0, r} H^1$ exists and the coordinate projection $pr_1\colon X \twoheadrightarrow G^0$ is a cover. This map is the anchor map for a left G-action on X defined elementwise by $g \cdot (x, h) := (r(g), F^1(g) \cdot h)$ for all $g \in G^1$, $x \in G^0$, $h \in H^1$ with $s(g) = x$, $r(h) = F^0(x) = F^0(s(g)) = s(F^1(g))$. The map $s\colon X \to H^0$, $(x, h) \mapsto s(h)$, for $x \in G^0$, $h \in H^1$ with $F^0(x) = r(h)$ is the anchor map for a right H-action defined elementwise by $(x, h_1) \cdot h_2 := (x, h_1 \cdot h_2)$ for all $x \in G^0$, $h_1, h_2 \in H^1$ with $F^0(x) = r(h_1)$ and $s(h_1) = r(h_2)$.

6.7. LEMMA. *The G, H-bibundle X is a bibundle functor.*

PROOF. The left and right actions on X commute. For $x \in G^0$, $h_1, h_2, y \in H^1$ with $F^0(x) = r(h_1) = r(h_2)$ we have $s(h_1) = r(y)$ and $(x, h_1) \cdot y = (x, h_2)$ if and only if $y = h_1^{-1} \cdot h_2$. Hence there is a unique such y, proving the isomorphism (17) for the H-bundle $r\colon X \to G^0$. Since $r\colon X \twoheadrightarrow G^0$ is a cover as well, the right H-action together with r is a principal H-bundle. ∎

We may generalise the construction above as follows. Let $F\colon G \to H$ be a functor and let Y be a bibundle functor from H to K. We may then construct a bibundle functor $F^*(Y)$ from G to K. The bibundle functor X_F is the special case where $Y = H^1$ is the identity bibundle functor on H. The only change in the construction is to replace H^1 by Y everywhere. Thus the underlying object of $\mathcal{C}$ is $F^*(Y) := G^0 \times_{F^0, H^0, r} Y$; this exists in $\mathcal{C}$ and the coordinate projection $pr_1 = s\colon F^*(Y) \twoheadrightarrow G^0$ is a cover because $r\colon Y \twoheadrightarrow H^0$ is a cover.

The same formulas as above define a left G-action and a right K-action on Y. The right K-action together with $\mathsf{pr}_1 \colon F^*(\mathsf{Y}) \twoheadrightarrow \mathsf{G}^0$ is a principal K-bundle because for $x \in \mathsf{G}^0$, $y_1, y_2 \in \mathsf{Y}$, $k \in \mathsf{K}^1$ with $F^0(x) = \mathsf{r}(y_1) = \mathsf{r}(y_2)$, we have $\mathsf{s}(y_1) = \mathsf{r}(k)$ and $(x, y_1) \cdot k = (x, y_2)$ if and only if $\mathsf{s}(y_1) = \mathsf{r}(k)$ and $y_1 \cdot k = y_2$, and this has a unique solution if $\mathsf{r}(y_1) = \mathsf{r}(y_2)$ because Y is a bibundle functor.

6.8. PROPOSITION. *The bibundle functor associated to a functor F is covering if and only if F is essentially surjective, and an equivalence bibundle if and only if F is essentially surjective and fully faithful.*

PROOF. The right anchor map of X_F is given by $(x, h) \mapsto \mathsf{s}(h)$ for $x \in \mathsf{G}^0$, $h \in \mathsf{H}^1$ with $F^0(x) = \mathsf{r}(h)$; this is exactly the map that is required to be a cover in order for F to be essentially surjective. The bibundle X_F is a bibundle equivalence if and only if the right anchor map is a cover and the following map is invertible:

$$\varphi \colon \mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{X}_F \to \mathsf{X}_F \times_{\mathsf{s}, \mathsf{H}^0, \mathsf{s}} \mathsf{X}_F, \qquad (g, x) \mapsto (g \cdot x, x).$$

The domain and codomain of φ are naturally isomorphic to $\mathsf{G}^1 \times_{F^0 \mathsf{os}, \mathsf{H}^0, \mathsf{r}} \mathsf{H}^1$ and $\mathsf{G}^0 \times_{F^0, \mathsf{H}^0, \mathsf{r}} \mathsf{H}^1 \times_{\mathsf{s}, \mathsf{H}^0, \mathsf{s}} \mathsf{H}^1 \times_{\mathsf{r}, \mathsf{H}^0, F^0} \mathsf{G}^0$, respectively, and φ is given elementwise by $\varphi(g, h) = (\mathsf{r}(g), F^1(g) \cdot h, h, \mathsf{s}(g))$ for all $g \in \mathsf{G}^1$, $h \in \mathsf{H}^1$ with $F^0(\mathsf{s}(g)) = \mathsf{r}(h)$. If φ is invertible, then the first component of φ^{-1} applied to $(x_1, h, 1_{\mathsf{s}(h)}, x_2)$ for $x_1, x_2 \in \mathsf{G}^0$, $h \in \mathsf{H}^1$ with $F^0(x_1) = \mathsf{r}(h)$, $F^0(x_2) = \mathsf{s}(h)$, gives an inverse to the map ψ in (12). Conversely, if ψ is invertible, then so is φ with $\varphi^{-1}(x_1, h_1, h_2, x_2) = (\psi^{-1}(x_1, h_1 \cdot h_2^{-1}, x_2), h_2)$. ∎

6.9. EXAMPLE. Let $p \colon X \twoheadrightarrow \mathsf{G}^0$ be a cover. The hypercover $p_* \colon \mathsf{G}(X) \to \mathsf{G}$ is an equivalence functor.

6.10. FROM BIBUNDLE FUNCTORS TO ANAFUNCTORS. Let X be a bibundle functor between two groupoids G and H in $(\mathcal{C}, \mathcal{T})$. We are going to turn it into an anafunctor $\mathsf{G} \to \mathsf{H}$. We take $\mathsf{r} \colon \mathsf{X} \twoheadrightarrow \mathsf{G}^0$ as the cover that is part of an anafunctor. It remains to define a functor $F \colon \mathsf{G}(\mathsf{X}) \to \mathsf{H}$ using the G, H-bibundle X.

6.11. PROPOSITION. *Let X be a bibundle functor $\mathsf{G} \to \mathsf{H}$. There is a natural isomorphism of groupoids $\mathsf{G}(\mathsf{X}) \cong \mathsf{G} \ltimes \mathsf{X} \rtimes \mathsf{H}$ that acts identically on objects.*

PROOF. Both groupoids $\mathsf{G} \ltimes \mathsf{X} \rtimes \mathsf{H}$ and $\mathsf{G}(\mathsf{X})$ have object space X. Their arrow spaces are $\mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{X} \times_{\mathsf{s}, \mathsf{H}^0, \mathsf{r}} \mathsf{H}^1$ and $\mathsf{X} \times_{\mathsf{r}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{X}$, respectively. We define

$$F^1 \colon \mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{X} \times_{\mathsf{s}, \mathsf{H}^0, \mathsf{r}} \mathsf{H}^1 \to \mathsf{X} \times_{\mathsf{r}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{X}$$

elementwise by $F^1(g, x, h) := (g \cdot x, g, x \cdot h)$ for all $g \in \mathsf{G}^1$, $x \in \mathsf{X}$, $h \in \mathsf{H}^1$ with $\mathsf{s}(g) = \mathsf{r}(x)$, $\mathsf{s}(x) = \mathsf{r}(h)$; this is well-defined because then $g \cdot x$ and $x \cdot h$ are defined and $\mathsf{r}(g \cdot x) = \mathsf{r}(g)$ and $\mathsf{s}(g) = \mathsf{r}(x) = \mathsf{r}(x \cdot h)$. Furthermore, $\mathsf{r}(F^1(g, x, h)) = g \cdot x = \mathsf{r}(g, x, h)$, $\mathsf{s}(F^1(g, x, h)) = x \cdot h = \mathsf{s}(g, x, h)$ and

$$F^1(g_1, x_1, h_1) \cdot F^1(g_2, x_2, h_2) = (g_1 \cdot x_1, g_1 \cdot g_2, x_2 \cdot h_2) = F^1(g_1 \cdot g_2, g_2^{-1} \cdot x_1, h_1 \cdot h_2)$$

for all $g_1, g_2 \in \mathsf{G}^1$, $x_1, x_2 \in \mathsf{X}$, $h_1, h_2 \in \mathsf{H}^1$ with $\mathsf{s}(g_i) = \mathsf{r}(x_i)$, $\mathsf{s}(x_i) = \mathsf{r}(h_i)$ and $x_1 \cdot h_1 = g_2 \cdot x_2$. Thus F^1 together with the identity on arrows is a functor. It remains to show that F^1 is an invertible map in $\mathcal{C}$.

Let $x_1, x_2 \in \mathsf{X}$, $g \in \mathsf{G}^1$ satisfy $\mathsf{r}(x_1) = \mathsf{r}(g)$, $\mathsf{s}(g) = \mathsf{r}(x_2)$. Then $x := g^{-1} \cdot x_1$ is well-defined and $\mathsf{r}(x) = \mathsf{s}(g) = \mathsf{r}(x_2)$. Since X with bundle projection r is a principal H-bundle, there is a unique $h \in \mathsf{H}^1$ with $\mathsf{s}(x) = \mathsf{r}(h)$ and $x_2 = x \cdot h$. Thus $F^1(g, x, h) = (x_1, g, x_2)$. Furthermore, the element (g, x, h) is unique with this property, so F^1 is invertible by the Yoneda Lemma.   ∎

We compose the isomorphism $\mathsf{G}(\mathsf{X}) \xrightarrow{\sim} \mathsf{G} \ltimes \mathsf{X} \rtimes \mathsf{H}$ in Proposition 6.11 with the functor

$$\mathsf{G} \ltimes \mathsf{X} \rtimes \mathsf{H} \to \mathsf{H}$$

that is $\mathsf{s} \colon \mathsf{X} \to \mathsf{H}$ on objects and the coordinate projection $\mathsf{pr}_3 \colon (\mathsf{G} \ltimes \mathsf{X} \rtimes \mathsf{H})^1 \to \mathsf{H}^1$ on arrows. This yields a functor $F_\mathsf{X} \colon \mathsf{G}(\mathsf{X}) \to \mathsf{H}$, given on $\mathsf{G}(\mathsf{X})^0 = \mathsf{X}$ by s and on $\mathsf{G}(\mathsf{X})^1 = \mathsf{X} \times_{\mathsf{r},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{X}$ by

$$F_\mathsf{X}^1(g \cdot x, g, x \cdot h) := h$$

for all $g \in \mathsf{G}^1$, $x \in \mathsf{X}$, $h \in \mathsf{H}^1$ with $\mathsf{s}(g) = \mathsf{r}(x)$, $\mathsf{s}(x) = \mathsf{r}(h)$; Proposition 6.11 says that any $(x_1, g, x_2) \in \mathsf{X} \times_{\mathsf{r},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{X}$ may be rewritten as $x_1 = g \cdot x$, $x_2 = x \cdot h$ for unique g, x, h as above.

The triple $(\mathsf{X}, \mathsf{r}, F_\mathsf{X})$ is an anafunctor from G to H.

6.12. COROLLARY. *An equivalence bibundle* X *from* G *to* H *induces an isomorphism* $\mathsf{G}(\mathsf{X}) \xrightarrow{\sim} \mathsf{H}(\mathsf{X})$, *that is, an ana-isomorphism between* G *and* H.

PROOF. If X is an equivalence bibundle from G to H, then we may exchange left and right and get another equivalence bibundle X^* from H to G. Proposition 6.11 applied to X and X^* gives groupoid isomorphisms

$$\mathsf{r}^*(\mathsf{G}) = \mathsf{G}(\mathsf{X}) \xrightarrow{\sim} \mathsf{G} \ltimes \mathsf{X} \rtimes \mathsf{H} \xrightarrow{\sim} \mathsf{H}(\mathsf{X}) = \mathsf{s}^*(\mathsf{H}).$$

∎

6.13. LEMMA. *Let* $F \colon \mathsf{G} \to \mathsf{H}$ *be a functor. This gives rise first to a bibundle functor* $\mathsf{X}_F \colon \mathsf{G} \to \mathsf{H}$, *secondly to an anafunctor* $(\mathsf{X}_F, \mathsf{r}_F, \tilde{F})$ *from* G *to* H, *where* $\mathsf{r}_F \colon \mathsf{X}_F \twoheadrightarrow \mathsf{G}^0$ *is the left anchor map of* X_F. *This anafunctor is equivalent to* $(\mathsf{G}^0, \mathrm{id}_{\mathsf{G}^0}, F)$, *that is, to* F *viewed as an anafunctor.*

PROOF. We have $\mathsf{X}_F = \mathsf{G}^0 \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1$ with $\mathsf{r}_F = \mathsf{pr}_1$. The functor $\tilde{F} \colon \mathsf{G}(\mathsf{X}_F) \to \mathsf{H}$ is given elementwise by

$$\tilde{F}^0(x, h) := \mathsf{s}(h), \qquad \tilde{F}^1\big((x_1, h_1), g, (x_2, h_2)\big) = h_1^{-1} \cdot F^1(g) \cdot h_2$$

for all $x, x_1, x_2 \in \mathsf{G}^0$, $h, h_1, h_2 \in \mathsf{H}^1$, $g \in \mathsf{G}^1$ with $F^0(x) = \mathsf{r}(h)$, $F^0(x_1) = \mathsf{r}(h_1)$, $F^0(x_2) = \mathsf{r}(h_2)$, $x_1 = \mathsf{r}(g)$, $x_2 = \mathsf{s}(g)$. The map $\Phi \colon \mathsf{X}_F \to \mathsf{H}^1$, $(x, h) \mapsto h$, is a natural transformation $\tilde{F} \Rightarrow F$ because of the following commuting diagram in H^1:

$$
\begin{array}{ccc}
\mathsf{r}(h_2) = F^0(x_2) = F^0(\mathsf{s}(g)) & \xrightarrow{\ F^1(g)\ } & F^0(\mathsf{r}(g)) = F^0(x_1) = \mathsf{r}(h_1) \\
h_2 \uparrow & & h_1 \uparrow \\
\mathsf{s}(h_2) & \xrightarrow[\tilde{F}^1\big((x_1, h_1), g, (x_2, h_2)\big) = h_1^{-1} \cdot F^1(g) \cdot h_2]{} & \mathsf{s}(h_1)
\end{array}
$$

∎

7. Composition of bibundles

The composition of bibundle functors and actors requires an extra assumption on the pretopology. We first formulate this assumption in several closely related ways and then use it to compose the various types of bibundles.

7.1. ASSUMPTIONS ON ČECH GROUPOID ACTIONS. Let $\mathcal{T}$ be a subcanonical pretopology on $\mathcal{C}$.

7.2. ASSUMPTION. *Any action of a Čech groupoid of a cover in $(\mathcal{C}, \mathcal{T})$ is basic.*

7.3. ASSUMPTION. *Any sheaf over a Čech groupoid of a cover in $(\mathcal{C}, \mathcal{T})$ is basic.*

Assumption 7.2 is obviously stronger than Assumption 7.3. We will use Assumptions 2.8 and 7.3 to compose bibundle *functors* and *equivalences*. The stronger Assumptions 2.9 and 7.2 are necessary and sufficient to compose bibundle *actors*.

The following two lemmas reformulate the assumptions above. Recall that $\mathcal{C}(\mathsf{G})$ and $\mathcal{C}_{\mathcal{T}}(\mathsf{G})$ are the categories of G-actions and G-sheaves for a groupoid G in $(\mathcal{C}, \mathcal{T})$.

7.4. PROPOSITION. *The following are equivalent:*

(1) *Let G be a groupoid, X_1 and X_2 G-actions and $f \colon \mathsf{X}_1 \to \mathsf{X}_2$ a G-map. If X_2 is basic, then so is X_1.*

(2) *Any action of a basic groupoid is basic.*

(3) *Any action of a Čech groupoid is basic.*

(4) *Let $\mathsf{p} \colon \mathsf{X} \to \mathsf{Z}$ be a cover, let G be its Čech groupoid, and let Y be a G-action. Then there are $\tilde{\mathsf{Z}} \in\in \mathcal{C}$ and a map $f \colon \tilde{\mathsf{Z}} \to \mathsf{Z}$ such that $\mathsf{Y} \cong \tilde{\mathsf{Z}} \times_{f, \mathsf{Z}, \mathsf{p}} \mathsf{X}$ with G acting on $\tilde{\mathsf{Z}} \times_{f, \mathsf{Z}, \mathsf{p}} \mathsf{X}$ by $(z, x_1) \cdot (x_1, x_2) := (z, x_2)$, as in Proposition 5.8.*

(5) *Let $\mathsf{p} \colon \mathsf{X} \to \mathsf{Z}$ be a cover and let G be its Čech groupoid. Then the functor $\mathcal{C}(\mathsf{Z}) \to \mathcal{C}(\mathsf{G})$ induced by the equivalence bibundle X between G and the 0-groupoid Z is an equivalence of categories.*

PROOF. $(1)\Rightarrow(2)$: Let G be a basic groupoid and let X be a G-action. Let $\mathsf{X}_2 = \mathsf{G}^0$ and let $f = \mathsf{s}\colon \mathsf{X} \to \mathsf{X}_2$ be the anchor map. Since f is a G-map and the G-action on G^0 is basic by assumption, (1) gives that the G-action on X is also basic.

(2) implies (3) because Čech groupoids are basic (Example 5.6).

(3) implies (1): the G-action on X_1 and the map f combine to an action of $\mathsf{X}_2 \rtimes \mathsf{G}$ on X_1. The groupoid $\mathsf{X}_2 \rtimes \mathsf{G}$ is isomorphic to a Čech groupoid of a cover by Proposition 5.15. Hence its action on X_1 is basic by (3). Corollary 5.17 shows that the action of G on X_1 is basic as well.

$(4) \Longleftrightarrow (3)$: The action of the Čech groupoid of p on X is basic by Example 5.6. Hence its pull-back along $f\colon \tilde{\mathsf{Z}} \to \mathsf{Z}$ remains a basic action by Proposition 5.8. Thus the actions described in (4) are basic, so (4) implies (3). Conversely, let $\tilde{\mathsf{X}}$ be a basic G-action for the Čech groupoid of $\mathsf{p}\colon \mathsf{X} \twoheadrightarrow \mathsf{Z}$. Let $\tilde{\mathsf{p}}\colon \tilde{\mathsf{X}} \to \tilde{\mathsf{Z}}$ be the bundle projection. The anchor map $\hat{f} := \tilde{\mathsf{s}}\colon \tilde{\mathsf{X}} \to \mathsf{X} = \mathsf{G}^0$ of the G-action on $\tilde{\mathsf{X}}$ is a G-map. It induces a map $f = \hat{f}/\mathsf{G}\colon \tilde{\mathsf{Z}} \to \mathsf{Z}$ by Proposition 5.9. Proposition 5.8 shows that $\tilde{\mathsf{X}}$ is isomorphic to $\tilde{\mathsf{Z}} \times_{f,\mathsf{Z},\mathsf{p}} \mathsf{X}$ with the canonical action. Thus (3) implies (4).

A Z-action is the same as a map $\tilde{\mathsf{Z}} \to \mathsf{Z}$. The Čech groupoid G of $\mathsf{p}\colon \mathsf{X} \to \mathsf{Z}$ is equivalent to the 0-groupoid Z by Example 6.3. The functor $\mathcal{C}(\mathsf{Z}) \to \mathcal{C}(\mathsf{G})$ induced by this equivalence is, by definition, the pull-back construction described in (4). (This is indeed a special case of the composition of bibundle equivalences with actions defined below.) Propositions 5.8 and 5.9 yield a bijection between maps $\tilde{\mathsf{Z}}_1 \to \tilde{\mathsf{Z}}_2$ and G-maps $\tilde{\mathsf{Z}}_1 \times_\mathsf{Z} \mathsf{X} \to \tilde{\mathsf{Z}}_2 \times_\mathsf{Z} \mathsf{X}$. This means that the functor $\mathcal{C}(\mathsf{Z}) \to \mathcal{C}(\mathsf{G})$ is fully faithful. Condition (4) means that it is essentially surjective. A fully faithful functor is an equivalence if and only if it is essentially surjective. ∎

Proposition 7.4.(5) shows that we need Assumption 7.2 if we want equivalent groupoids to have equivalent action categories.

7.5. PROPOSITION. *The following are equivalent:*

(1) *Let G be a groupoid, X_1 and X_2 G-actions and $f\colon \mathsf{X}_1 \twoheadrightarrow \mathsf{X}_2$ a cover that is also a G-map. If X_2 is basic, then so is X_1.*

(2) *Any sheaf over a basic groupoid is basic.*

(3) *Any sheaf over a Čech groupoid is basic.*

(4) *Let $\mathsf{p}\colon \mathsf{X} \twoheadrightarrow \mathsf{Z}$ be a cover, let G be its Čech groupoid. Let Y be a G-sheaf. Then there are $\tilde{\mathsf{Z}} \in\in \mathcal{C}$ and a map $f\colon \tilde{\mathsf{Z}} \to \mathsf{Z}$ such that $\mathsf{Y} \cong \tilde{\mathsf{Z}} \times_{f,\mathsf{Z},\mathsf{p}} \mathsf{X}$ with G acting on $\tilde{\mathsf{Z}} \times_{f,\mathsf{Z},\mathsf{p}} \mathsf{X}$ by $(z, x_1) \cdot (x_1, x_2) := (z, x_2)$, as in Proposition 5.8.*

Assumption 2.8 is equivalent to the statement that the map f in (4) is automatically a cover.

PROOF. The proof is the same as for Proposition 7.4. Assumption 2.8 is necessary and sufficient for the map f in (4) to be a cover because each fibre-product situation (2) in which g is a cover gives rise to a situation as in (4). ∎

Let $p\colon X \twoheadrightarrow Z$ be a cover and let G be its Čech groupoid. Then the equivalence X induces a functor $\mathcal{C}_{\mathcal{T}}(Z) \to \mathcal{C}_{\mathcal{T}}(G)$ between the 0-groupoid Z and G. By Proposition 7.5, Assumptions 2.8 and 2.9 together are equivalent to the statement that this functor $\mathcal{C}_{\mathcal{T}}(Z) \to \mathcal{C}_{\mathcal{T}}(G)$ is an equivalence of categories.

7.6. PROPOSITION. *Let $\mathcal{T}$ and $\mathcal{T}'$ be pretopologies on $\mathcal{C}$ with $\mathcal{T} \subseteq \mathcal{T}'$. If $\mathcal{T}'$ is sub-canonical, then so is $\mathcal{T}$. If $\mathcal{T}'$ satisfies Assumption 7.2, then so does $\mathcal{T}$. If $\mathcal{T}'$ satisfies Assumption 7.3, then so does $\mathcal{T}$.*

PROOF. Let $p\colon X \twoheadrightarrow Z$ be a cover in $\mathcal{T}$ and let G be its Čech groupoid. Let Y be a G-action. Since $\mathcal{T} \subseteq \mathcal{T}'$, p is also a cover in $\mathcal{T}'$. Proposition 7.4.(4) for the pretopology $\mathcal{T}'$ shows that there is a map $f\colon \tilde{Z} \to Z$ such that $Y \cong \tilde{Z} \times_{f,Z,p} X$ with G acting by $(z, x_1)\cdot(x_1, x_2) := (z, x_2)$. This action is basic also in $(\mathcal{C}, \mathcal{T})$. The proof of the second statement is similar. ∎

Often a category admits many different pretopologies. Proposition 7.6 shows that we do not have to check Assumptions 7.2 and 7.3 for all pretopologies, it suffices to look at a maximal one among the interesting pretopologies.

7.7. COMPOSITION OF BIBUNDLE FUNCTORS AND ACTORS.

7.8. PROPOSITION. *Let G and H be groupoids in $(\mathcal{C}, \mathcal{T})$. Under Assumption 7.2, a bibundle actor X from G to H induces a functor*

$$\mathcal{C}(H) \to \mathcal{C}(G), \qquad Y \mapsto X \times_H Y.$$

Under Assumption 7.3, a G, H-bibundle X with basic action of H induces a functor

$$\mathcal{C}_{\mathcal{T}}(H) \to \mathcal{C}(G), \qquad Y \mapsto X \times_H Y.$$

Both constructions above are natural in X, that is, a G, H-map $X_1 \to X_2$ induces a natural G-map $X_1 \times_H Y \to X_2 \times_H Y$. Assume Assumption 2.8. If the G, H-map $f\colon X_1 \twoheadrightarrow X_2$ and the H-map $g\colon Y_1 \twoheadrightarrow Y_2$ are covers, then so is the induced map $f \times_H g\colon X_1 \times_H Y_1 \to X_2 \times_H Y_2$.

If Assumptions 7.3 and 2.9 hold, then $X \times_H Y \in\in \mathcal{C}_{\mathcal{T}}(G)$ if X is a G, H-bibundle functor and $Y \in\in \mathcal{C}_{\mathcal{T}}(H)$.

PROOF. The proofs for bibundle functors and actors are almost the same. The first difference is that the fibre product

$$XY := X \times_{s,H^0,s} Y$$

exists in $\mathcal{C}$ for different reasons: if X is a bibundle actor, then $s\colon X \twoheadrightarrow H^0$ is a cover, and otherwise $Y \in\in \mathcal{C}_{\mathcal{T}}(H)$ means that $s\colon Y \twoheadrightarrow H^0$ is a cover.

There is an obvious left action of G on XY with anchor map $r_{XY} := r \circ pr_X\colon XY \to X \to G^0$ and multiplication map

$$G^1 \times_{s,G^0,r_{XY}} XY \to XY$$

given elementwise by $g \cdot (x, y) := (g \cdot x, y)$ for $g \in \mathsf{G}^1$, $x \in \mathsf{X}$, $y \in \mathsf{Y}$ with $\mathsf{s}(g) = \mathsf{r}(x)$, $\mathsf{s}(x) = \mathsf{s}(y)$. We equip XY with the unique right H-action for which the coordinate projections $\mathsf{pr_X} \colon \mathsf{XY} \to \mathsf{X}$ and $\mathsf{pr_Y} \colon \mathsf{XY} \to \mathsf{Y}$ are H-maps (see Lemma 4.18). Elementwise, we have $\mathsf{s}(x, y) = \mathsf{s}(x) = \mathsf{s}(y)$ and $(x, y) \cdot h = (x \cdot h, y \cdot h)$ for all $x \in \mathsf{X}$, $y \in \mathsf{Y}$, $h \in \mathsf{H}^1$ with $\mathsf{s}(x) = \mathsf{s}(y) = \mathsf{r}(h)$.

The actions of G and H on XY clearly commute, and the map on objects

$$\mathcal{C}(\mathsf{G}, \mathsf{H}) \times \mathcal{C}(\mathsf{H}) \to \mathcal{C}(\mathsf{G}, \mathsf{H}), \qquad \mathsf{X}, \mathsf{Y} \mapsto \mathsf{XY},$$

is part of a bifunctor; that is, a G, H-map $\mathsf{X}_1 \to \mathsf{X}_2$ and an H-map $\mathsf{Y}_1 \to \mathsf{Y}_2$ induce a G, H-map $\mathsf{X}_1 \mathsf{Y}_1 \to \mathsf{X}_2 \mathsf{Y}_2$.

The coordinate projection $\mathsf{pr_X} \colon \mathsf{XY} \to \mathsf{X}$ is an H-map and X is a basic H-action by assumption. In the first case, Proposition 7.4.(1) shows that XY is a basic H-action. In the second case, the map $\mathsf{pr_X} \colon \mathsf{XY} \twoheadrightarrow \mathsf{X}$ is a cover because $\mathsf{s} \colon \mathsf{Y} \twoheadrightarrow \mathsf{H}^0$ is one; thus Proposition 7.5.(1) shows that XY is a basic H-action. Let $\mathsf{p} \colon \mathsf{XY} \twoheadrightarrow \mathsf{Z}$ be the bundle projection of this basic action. We also write $\mathsf{Z} = \mathsf{X} \times_{\mathsf{H}} \mathsf{Y}$.

Recall that a G, H-map $f \colon \mathsf{X}_1 \to \mathsf{X}_2$ and an H-map $g \colon \mathsf{Y}_1 \to \mathsf{Y}_2$ induce a G, H-map $f \times_{\mathsf{H}^0} g \colon \mathsf{X}_1 \mathsf{Y}_1 \to \mathsf{X}_2 \mathsf{Y}_2$. By Proposition 5.9, this induces a map $f \times_{\mathsf{H}} g \colon \mathsf{X}_1 \times_{\mathsf{H}} \mathsf{Y}_1 \to \mathsf{X}_2 \times_{\mathsf{H}} \mathsf{Y}_2$. Thus the construction of $\mathsf{X} \times_{\mathsf{H}} \mathsf{Y}$ is bifunctorial. If f and g are covers, then so is the induced map $f \times_{\mathsf{H}^0} g$; this is a general property of pretopologies. By Proposition 5.9, the map $f \times_{\mathsf{H}} g$ induced by this on the base spaces of principal bundles is a cover as well if Assumption 2.8 holds.

It remains to push the G-action on XY down to a natural G-action on Z. Let $\mathsf{W} := \mathsf{X}/\mathsf{H}$. The anchor map $\mathsf{s} \colon \mathsf{X} \to \mathsf{G}^0$ descends to a map $\mathsf{s_W} \colon \mathsf{W} \to \mathsf{G}^0$ because it is H-invariant. Proposition 5.9 gives a unique map $\mathsf{pr_X}/\mathsf{H} \colon \mathsf{Z} \to \mathsf{W}$, which we compose with $\mathsf{s_W}$ to get a map $\mathsf{r_Z} \colon \mathsf{Z} \to \mathsf{G}^0$. This is the anchor map of the desired action on Z. It is the unique map with $\mathsf{r_X} \circ \mathsf{pr_X} = \mathsf{r_Z} \circ \mathsf{p} \colon \mathsf{XY} \to \mathsf{G}^0$.

Pulling back the principal H-bundle $\mathsf{XY} \twoheadrightarrow \mathsf{Z}$ along $\mathsf{pr}_2 \colon \mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r_Z}} \mathsf{Z} \twoheadrightarrow \mathsf{Z}$ gives

$$\mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r_Z}} \mathsf{Z} \times_{\mathsf{Z}} \mathsf{XY} \cong \mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{XY}$$

with its usual H-action. This is a principal bundle over $\mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r_Z}} \mathsf{Z}$ by Proposition 5.8. By Proposition 5.9, the left G^1-action $\mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{XY} \to \mathsf{XY}$ induces a map $\mathsf{m_Z} \colon \mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r_Z}} \mathsf{Z} \to \mathsf{Z}$ on the bases of these principal bundles. It is routine to see that $\mathsf{r_Z}$ and $\mathsf{m_Z}$ define a left G-action on Z, using the corresponding facts for XY and passing to orbit spaces by Proposition 5.9. Thus $\mathsf{X} \times_{\mathsf{H}} \mathsf{Y}$ carries a left G-action. This G-action is natural in the sense that the maps $f \times_{\mathsf{H}} g$ defined above are G-equivariant.

If X is a G, H-bibundle functor and $\mathsf{Y} \in \in \mathcal{C}_{\mathcal{T}}(\mathsf{H})$, then $\mathsf{pr_X} \colon \mathsf{XY} \twoheadrightarrow \mathsf{X}$ is a cover because $\mathsf{s} \colon \mathsf{Y} \twoheadrightarrow \mathsf{H}^0$ is one. Since $\mathsf{r} \colon \mathsf{X} \twoheadrightarrow \mathsf{G}^0$ for bibundle functors, the composite map $\mathsf{XY} \twoheadrightarrow \mathsf{G}^0$ is a cover as well. The map $\mathsf{XY} \twoheadrightarrow \mathsf{X} \times_{\mathsf{H}} \mathsf{Y}$ is a cover as the bundle projection of a principal bundle. Assumption 2.9 shows that the induced map $\mathsf{X} \times_{\mathsf{H}} \mathsf{Y} \to \mathsf{G}^0$ is a cover, so $\mathsf{X} \times_{\mathsf{H}} \mathsf{Y} \in \in \mathcal{C}_{\mathcal{T}}(\mathsf{G})$. ∎

7.9. Proposition. *The quotient* X/H *for a* G, H-*bibundle* X *with basic action of* H *inherits a natural* G-*action. If* X *carries an action of a third groupoid* K *that commutes with the actions of* G *and* H, *then the induced actions of* G *and* K *on* X/H *commute.*

Proof. The first assertion is shown during the proof of Proposition 7.8. We are dealing with the special case $Y = H^0$, where $XY \cong X$, so that $X \times_H H^0 \cong X/H$. No extra assumption about actions of Čech groupoids is needed in this particular case. A (left or right) action of K commuting with those of G and H induces an action on X/H for the same reasons as for the action of G. The proof that the induced actions of G and K on X/H still commute is routine and omitted. We only remark that, under Assumption 2.10, we could replace a pair of commuting actions of G and K by a single action of $G \times K$, which would clearly still commute with the action of H. This action of $G \times K$ descends to X/H and may then be turned back into two commuting actions of G and K on X/H.   ∎

7.10. Proposition. *Assume Assumptions* 7.3 *and* 2.8.

  Let G, H *and* K *be groupoids in* $(\mathcal{C}, \mathcal{T})$. *Let* X *and* Y *be bibundle functors from* G *to* H *and from* H *to* K, *respectively. Then* $X \times_H Y$ *is a bibundle functor from* G *to* K *in a natural way with respect to* G, H-*maps* $X_1 \to X_2$ *and* H, K-*maps* $Y_1 \to Y_2$. *If both* X *and* Y *are bibundle equivalences, then so is* $X \times_H Y$.

  Under Assumption 2.9, *if both bibundle functors* X *and* Y *are covering, so is* $X \times_H Y$.

  Assume Assumptions 7.2 *and* 2.9. *Let* X *and* Y *be bibundle actors from* G *to* H *and from* H *to* K, *respectively. Then* $X \times_H Y$ *is a bibundle actor from* G *to* K *in a natural way with respect to* G, H-*maps* $X_1 \to X_2$ *and* H, K-*maps* $Y_1 \to Y_2$.

Proof. Since right and left actions of H are equivalent, Proposition 7.8 also works for a left H-action on Y. All cases of composition considered fall into one of the two cases of Proposition 7.8, which provides $X \times_H Y \in\in \mathcal{C}$ with a natural G-action.

  A right K-action on $X \times_H Y$ is constructed like the left G-action in the proof of Proposition 7.8. First, XY carries a right K-action that commutes with the actions of G and H: the anchor map is $s \circ pr_Y \colon XY \to Y \to K^0$ and the multiplication is $id_X \times_{s, H^0} m_Y \colon XY \times_{s \circ pr_Y, K^0, r} K^1 \to XY$. Proposition 7.9 shows that this K-action descends from XY to an action on $X \times_H Y$ that commutes with the induced action of G.

  It remains to show that the G, K-bibundle $X \times_H Y$ is a bibundle functor, bibundle equivalence, covering bibundle functor, or bibundle actor if both X and Y are, under appropriate assumptions on $(\mathcal{C}, \mathcal{T})$. We consider the case of bibundle functors first. We must show that $r \colon X \times_H Y \to G^0$ is a principal K-bundle.

  It is clear that the anchor map $r \colon X \times_H Y \to G^0$ is K-invariant. We claim that it is a cover. The map $pr_X \colon XY \twoheadrightarrow X$ is a cover because $r \colon Y \twoheadrightarrow H^0$ is one. Hence so is the induced map $pr_X/H \colon X \times_H Y \twoheadrightarrow X/H \cong G^0$ by Proposition 5.9; here we need Assumption 2.8. This map is the anchor map $r \colon X \times_H Y \to G^0$.

  We must also show that the map

$$(m, pr_1) \colon (X \times_H Y) \times_{s, K^0, r} K^1 \to (X \times_H Y) \times_{r, G^0, r} (X \times_H Y) \tag{19}$$

is an isomorphism. Since $r\colon Y \twoheadrightarrow H^0$ is a principal K-bundle, its pull-back along $s\colon X \twoheadrightarrow H^0$ is a principal K-bundle $\mathsf{pr}_X\colon XY \twoheadrightarrow X$. This means that the map

$$XY \times_{s,K^0,r} K^1 \xrightarrow{(m,\mathsf{pr}_1)} XY \times_{\mathsf{pr}_X,X,\mathsf{pr}_X} XY \tag{20}$$

is an isomorphism.

Pulling $XY \twoheadrightarrow X \times_H Y$ back along $r\colon K^1 \twoheadrightarrow K^0$ gives a principal H-bundle $XY \times_{s,K^0,r} K^1 \twoheadrightarrow (X \times_H Y) \times_{s,K^0,r} K^1$. Proposition 5.10 shows that $XY \times_{\mathsf{pr}_X,X,\mathsf{pr}_X} XY$ is a principal H-bundle over $(X \times_H Y) \times_{r,G^0,r} (X \times_H Y)$. Hence (19) is the map on the base spaces induced by the H-equivariant isomorphism (20). Thus (19) is an isomorphism as well by Proposition 5.9. This finishes the proof that $X \times_H Y$ is a bibundle functor if X and Y are. We have used Assumptions 7.3 and 2.8.

If both X and Y are bibundle equivalences, then the same arguments as above for the right K-action apply to the left G-action on $X \times_H Y$. They show that $s\colon X \times_H Y \twoheadrightarrow K^0$ is a principal G-bundle. Thus $X \times_H Y$ is a bibundle equivalence if X and Y are, under Assumptions 7.3 and 2.8.

Assume now that $s\colon X \twoheadrightarrow H^0$ and $s\colon Y \twoheadrightarrow K^0$ are covers. Then so are $\mathsf{pr}_Y\colon XY \twoheadrightarrow Y$ and the composite map $XY \twoheadrightarrow Y \twoheadrightarrow K^0$. Since the bundle projection $XY \twoheadrightarrow X \times_H Y$ is a cover as well, Assumption 2.9 gives that the anchor map $X \times_H Y \to K^0$ is a cover. Thus a product of covering bibundle functors is again a covering bibundle functor, under Assumptions 2.9 and 7.3. Furthermore, the right anchor map for a product of two bibundle actors is a cover. It remains to show that the right action on the product of two bibundle actors is basic.

Since the K-action on Y is basic, we may form its base $W \cong Y/K$ and get a principal K-bundle $p\colon Y \twoheadrightarrow W$. Proposition 7.9 shows that the H-action on Y descends to W.

Now we may also form $X \times_H W$, and the cover p induces a cover $q\colon X \times_H Y \twoheadrightarrow X \times_H W$ by Proposition 7.8; here we only need Assumption 2.8. The map

$$(m,\mathsf{pr}_1)\colon XY \times_{s,K^0,r} K^1 \to XY \times_{XW} XY \tag{21}$$

is an isomorphism because $p\colon Y \twoheadrightarrow W$ is a principal K-bundle, which we have pulled back to XW. The spaces in (21) are total spaces of principal H-bundles over $(X \times_H Y) \times_{s,K^0,r} K^1$ and $(X \times_H Y) \times_{X \times_H W} (X \times_H Y)$, respectively. Proposition 7.8 shows that the induced map $(m,\mathsf{pr}_1)\colon (X \times_H Y) \times_{s,K^0,r} K^1 \to (X \times_H Y) \times_{X \times_H W} (X \times_H Y)$ on the bases is also an isomorphism. Hence the induced K-action on $X \times_H Y$ is basic with bundle projection q. ∎

7.11. REMARK. The proof above shows that $(X \times_H Y)/K \cong X \times_H (Y/K)$ whenever the actions of H on X and of K on Y are basic and suitable assumptions about $(\mathcal{C}, \mathcal{T})$ ensure that $X \times_H Y$ and $X \times_H (Y/K)$ exist and $q\colon X \times_H Y \to X \times_H (Y/K)$ is a cover.

7.12. PROPOSITION. *The compositions defined above under suitable assumptions are associative in the following sense:*

(1) *For bibundle functors or bibundle actors $X\colon G \to H$, $Y\colon H \to K$, and $Z\colon K \to L$, there is a natural bibundle isomorphism $(X \times_H Y) \times_K Z \xrightarrow{\sim} X \times_H (Y \times_K Z)$; these associators*

> *for four composable bibundle functors or actors make the usual pentagon diagram commute.*

(2) *For bibundle actors* $X \colon G \to H$ *and* $Y \colon H \to K$, *and a* K-*action* Z, *there is a natural* G-*equivariant isomorphism* $(X \times_H Y) \times_K Z \xrightarrow{\sim} X \times_H (Y \times_K Z)$.

PROOF. Literally the same argument proves both statements. The usual fibre product $XYZ := X \times_{s,H^0,r} Y \times_{s,K^0,r} Z$ is associative up to very canonical isomorphisms, which we drop from our notation to simplify. This space carries commuting actions of H and K by $(x, y, z) \cdot h = (x \cdot h, h^{-1} \cdot y, z)$ and $(x, y, z) \cdot k = (x, y \cdot k, k^{-1} \cdot z)$ for $x \in X$, $y \in Y$, $z \in Z$, $h \in H^1$, $k \in K^1$ with $s(x) = r(y) = r(h)$, $s(y) = r(z) = r(k)$. We get $X \times_H (Y \times_K Z)$ from XYZ by first taking the base space $X \times_{H^0} (Y \times_K Z)$ for the basic K-action on XYZ and then taking the base space of the resulting basic H-action on $X \times_{H^0} (Y \times_K Z)$. Remark 7.11 implies that $X \times_H (Y \times_K Z)$ is naturally isomorphic to $(X \times_H (Y \times_{K^0} Z))/K$. The latter is, in turn, naturally isomorphic to $((X \times_H Y) \times_{K^0} Z)/K$, which is $(X \times_H Y) \times_K Z$ by definition. This provides the desired associator.

The associator is characterised uniquely by the statement that it is lifted by the associator $(X \times_{s,H^0,r} Y) \times_{s,K^0,r} Z \to X \times_{s,H^0,r} (Y \times_{s,K^0,r} Z)$. Since the latter associators clearly make the associator pentagon commute, the same holds for the induced maps on the quotients. ∎

7.13. REMARK. Assume Assumption 2.10 about a final object $\star$. Then a left G-action X is the same as a bibundle actor from G to $\star$. The functor $\mathcal{C}(H) \to \mathcal{C}(G)$ defined by a bibundle actor X from G to H in Proposition 7.8 is equivalent to the composition of bibundle actors $G \to H \to \star$ under this identification. Thus the second statement in Proposition 7.12 becomes a special case of the first one under Assumption 2.10.

7.14. PROPOSITION. *The bibundle equivalence* G^1 *from* G *to itself is a unit both for the composition of bibundle functors and bibundle actors, up to natural isomorphisms* $X \times_G G^1 \cong X$ *and* $G^1 \times_G Y \cong Y$ *for a bibundle functor or actor* X *to* G *and a bibundle functor or actor* Y *out of* G. *The functor* $\mathcal{C}(G) \to \mathcal{C}(G)$ *induced by the bibundle equivalence* G^1 *is naturally isomorphic to the identity functor. It is routine to check that this natural construction is also equivariant with respect to the appropriate left or right actions.*

PROOF. The bibundle equivalence G^1 is constructed in Example 6.2. Pulling the principal left G-bundle $s \colon G^1 \twoheadrightarrow G^0$ back along the anchor map $s \colon X \twoheadrightarrow G^0$ gives a principal left G-bundle $X \times_{s,G^0,s} G^1 \twoheadrightarrow X$; here the action of G is by $g_1 \cdot (x, g_2) := (x, g_1 \cdot g_2)$. The map $X \times_{s,G^0,s} G^1 \to X \times_{s,G^0,r} G^1$, $(x, g) \mapsto (x \cdot g^{-1}, g)$, is an isomorphism with inverse $(x, g) \mapsto (x \cdot g, g)$. It intertwines the G-action on $X \times_{s,G^0,s} G^1$ with the G-action on $X \times_{s,G^0,r} G^1$ by $g_1 \cdot (x, g_2) := (x \cdot g_1^{-1}, g_1 \cdot g_2)$. These isomorphic G-actions have isomorphic orbit spaces, and the orbit space for the latter is $X \times_G G^1$. Hence the multiplication map $X \times_{s,G^0,r} G^1 \to X$, $(x, g) \mapsto x \cdot g$, induces an isomorphism $X \times_G G^1 \xrightarrow{\sim} X$. Similarly, the multiplication map $G^1 \times_{s,G^0,r} Y \to Y$, $(g, y) \mapsto g \cdot y$, induces an isomorphism $G^1 \times_G Y \xrightarrow{\sim} Y$. ∎

Putting together the results above gives the following theorem:

7.15. Theorem. *Assume Assumptions* 2.8 *and* 7.2. *Then there is a bicategory with groupoids in* $(\mathcal{C}, \mathcal{T})$ *as objects, bibundle functors from* G *to* H *as arrows, maps of* G, H-*bibundles as* 2-*arrows between arrows* G $\to$ H, $\times_\mathsf{G}$ *in reverse order as composition,* G^1 *as unit arrow on* G, *the associator and unitors from Propositions* 7.12 *and* 7.14, *and composition of maps as vertical product of* 2-*arrows; the horizontal product of* 2-*arrows* $f\colon \mathsf{X}_1 \Rightarrow \mathsf{X}_2$ *for* $\mathsf{X}_1, \mathsf{X}_2\colon \mathsf{G} \rightrightarrows \mathsf{H}$ *and* $g\colon \mathsf{Y}_1 \Rightarrow \mathsf{Y}_2$ *for* $\mathsf{Y}_1, \mathsf{Y}_2\colon \mathsf{H} \rightrightarrows \mathsf{K}$ *is* $f \times_\mathsf{H} g\colon \mathsf{X}_1 \times_\mathsf{H} \mathsf{Y}_1 \to \mathsf{X}_2 \times_\mathsf{H} \mathsf{Y}_2$. *All* 2-*arrows in this bicategory are invertible.*

Assume Assumptions 2.9 *and* 7.2. *Then almost all of the above holds for bibundle actors, except that* 2-*arrows need no longer be invertible. Furthermore, the covering bibundle functors form a sub-bicategory of both.*

Proof. It is a direct consequence of the propositions above that bibundle functors, bibundle equivalences, bibundle actors, and covering bibundle functors are the arrows in bicategories, with the 2-arrows and compositions as asserted.

We show that 2-arrows between bibundle functors are equivalences. Let X_1 and X_2 be bibundle functors from G to H and let $f\colon \mathsf{X}_1 \Rightarrow \mathsf{X}_2$ be a G, H-map. Then the induced map $f/\mathsf{H}\colon \mathsf{X}_1/\mathsf{H} \to \mathsf{X}_2/\mathsf{H}$ is the identity map on G^0 by G-equivariance and because $\mathsf{X}_i/\mathsf{H} \cong \mathsf{G}^0$ by assumption. Since f/H is an isomorphism, so is f by Proposition 5.9. This argument fails for bibundle actors because their definition does not determine X_i/G. ∎

7.16. Remark. Assume Assumption 2.10. Then Assumptions 7.2 and 2.9 are necessary to compose bibundle actors.

By Remark 7.13, we may identify $\mathcal{C}(\mathsf{G})$ for a groupoid G with the category of bibundle actors G $\to \star$ and 2-arrows between them. Let $\mathsf{p}\colon \mathsf{X} \twoheadrightarrow \mathsf{Z}$ be a cover and let G be its Čech groupoid. The functor $\mathcal{C}(\mathsf{Z}) \to \mathcal{C}(\mathsf{G})$, $\mathsf{Y} \mapsto \mathsf{Y} \times_\mathsf{Z} \mathsf{X}$, in Proposition 7.4.5 composes with the bibundle equivalence X from Z to G. Exchanging left and right in X gives a bibundle equivalence X^* from G to Z, such that $\mathsf{X}^* \times_\mathsf{Z} \mathsf{X} \cong \mathsf{G}^1$ and $\mathsf{X} \times_\mathsf{G} \mathsf{X}^* \cong \mathsf{Z}$. That is, X^* is inverse to X. If we can compose bibundle actors, we can compose them with bibundle equivalences as well, and the latter composition must be an equivalence. Hence the existence of a composition of bibundle actors implies Proposition 7.4.5 and thus Assumption 7.2.

Now consider also a map $f\colon \mathsf{Z} \to \mathsf{Y}$ such that $f \circ \mathsf{p}\colon \mathsf{X} \twoheadrightarrow \mathsf{Y}$ is a cover. Then X with its usual left action of G and right action of Y by $f \circ \mathsf{p}$ is a covering bibundle functor from G to Y, and hence a bibundle actor. Composing it with the equivalence X^* gives the bibundle functor Z $\to$ G $\to$ Y from the map f. This is a bibundle actor if and only if it is a covering bibundle functor if and only if f is a cover. Thus Assumption 2.9 is necessary for composites of bibundle actors to exist, and also for composites of covering bibundle functors to remain covering bibundle functors.

7.17. Bibundle functors versus anafunctors.

7.18. THEOREM. *The construction of anafunctors from bibundle functors in Section* 6.10 *is part of an equivalence from the bicategory of bibundle functors to the bicategory of anafunctors; this equivalence is the identity on objects.*

This theorem requires Assumptions 2.8 and 7.3 in order for the bicategory of bibundle functors to be defined. Related results for Lie groupoids and topological groupoids (with suitable covers) are proved by Pronk [42] and Carchedi [11], see also [33]. Pronk develops a general calculus of fractions in bicategories for such purposes.

PROOF. Let G and H be groupoids in $(\mathcal{C}, \mathcal{T})$. We show first that the groupoid of anafunctors from G to H with natural transformations of anafunctors as arrows is equivalent to the groupoid of bibundle functors from G to H with G, H-maps between them as arrows. Secondly, we check that this equivalence is compatible with the composition of arrows in the appropriate weak sense.

The anafunctor associated to a bibundle functor X from G to H is $(\mathsf{X}, \mathsf{r}, F_\mathsf{X})$, where $F_\mathsf{X}^0 := \mathsf{s} \colon \mathsf{X} \to \mathsf{H}^0$ on objects and $F_\mathsf{X}^1 \colon \mathsf{X} \times_{\mathsf{r},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{X} \to \mathsf{H}^1$ is determined by the elementwise formula $F_\mathsf{X}^1(g \cdot x, g, x \cdot h) = h$ for all $x \in \mathsf{X}$, $g \in \mathsf{G}^1$, $h \in \mathsf{H}^1$ with $\mathsf{s}(g) = \mathsf{r}(x)$, $\mathsf{s}(x) = \mathsf{r}(h)$ (see the proof of Proposition 6.11).

Conversely, let (X, p, F) be an anafunctor from G to H. We have built bibundle functors from functors in Section 6.6. In particular, $p_* \colon \mathsf{G}(X) \to \mathsf{G}$ gives a bibundle functor $X \times_{p,\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1$ from $\mathsf{G}(X)$ to G, and $F \colon \mathsf{G}(X) \to \mathsf{H}$ gives a bibundle functor $X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1$ from $\mathsf{G}(X)$ to H. The bibundle functor $X \times_{p,\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1$ from $\mathsf{G}(X)$ to G is a bibundle equivalence (Example 6.9); exchanging the left and right actions on $X \times_{p,\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1$ gives a quasi-inverse bibundle functor $(X \times_{p,\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1)^*$ from G to $\mathsf{G}(X)$. We map the anafunctor (X, p, F) to the bibundle functor

$$\beta(X, p, F) := (X \times_{p,\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1)^* \times_{\mathsf{G}(X)} (X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1).$$

This is the orbit space of the right action of $\mathsf{G}(X)$ on the fibre product

$$(X \times_{p,\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1)^* \times_X (X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1) \cong \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,p} X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1$$

with anchor map $\mathsf{s}(g, x, h) := x$ and

$$(g_1, x_1, h) \cdot (x_1, g_2, x_2) := (g_1 \cdot g_2, x_2, F^1(x_1, g_2, x_2)^{-1} \cdot h)$$

for all $g_1, g_2 \in \mathsf{G}^1$, $x_1, x_2 \in \mathsf{X}$, $h \in \mathsf{H}^1$ with $\mathsf{s}(g_1) = p(x_1) = \mathsf{r}(g_2)$, $F^0(x_1) = \mathsf{r}(h)$, $p(x_2) = \mathsf{s}(g_2)$; the isomorphism above involves the inversion in G^1, which exchanges the source and range maps. The G, H-action on $\beta(X, p, F)$ is induced by the obvious G, H-action

$$g_1 \cdot (g_2, x, h_1) \cdot h_2 := (g_1 \cdot g_2, x, h_1 \cdot h_2)$$

on $\mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,p} X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1$.

We claim that these two maps between bibundle functors and anafunctors are inverse to each other up to natural 2-arrows. First we start with a bibundle functor X, turn it

into an anafunctor $(\mathsf{X}, \mathsf{r}, F_{\mathsf{X}})$ and then turn that into a bibundle functor $(\mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{X} \times_{\mathsf{s},\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1)/\mathsf{G}(X)$. We claim that the map

$$\mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{X} \times_{\mathsf{s},\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1 \to \mathsf{X}, \qquad (g, x, h) \mapsto g \cdot x \cdot h,$$

descends to an isomorphism $\beta(\mathsf{X}, \mathsf{r}, F_{\mathsf{X}}) \xrightarrow{\sim} \mathsf{X}$; it is clear that this isomorphism is a G, H-map. We must show that $(g_1, x_1, h_1), (g_2, x_2, h_2) \in \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{X} \times_{\mathsf{s},\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1$ satisfy $g_1 \cdot x_1 \cdot h_1 = g_2 \cdot x_2 \cdot h_2$ if and only if there is $(x_3, g_3, x_4) \in \mathsf{G}(X)^1$ with $\mathsf{s}(g_1, x_1, h_1) = \mathsf{r}(x_3, g_3, x_4)$ and $(g_1, x_1, h_1) \cdot (x_3, g_3, x_4) = (g_2, x_2, h_2)$. First, $\mathsf{s}(g_i, x_i, h_i) = x_i$, $\mathsf{r}(x_3, g_3, x_4) = x_3$ and $\mathsf{s}((g_1, x_1, h_1) \cdot (x_3, g_3, x_4)) = \mathsf{s}(x_3, g_3, x_4) = x_4$, so we must have $x_1 = x_3$ and $x_2 = x_4$. Moreover, the G^1-entry in $(g_1, x_1, h_1) \cdot (x_3, g_3, x_4)$ is $g_1 \cdot g_3$, so we must have $g_3 = g_1^{-1} \cdot g_2$. Since $\mathsf{s}(g_1^{-1}) = \mathsf{r}(g_1) = \mathsf{r}(g_1 \cdot x_1 \cdot h_1)$ and $\mathsf{r}(g_2) = \mathsf{r}(g_2 \cdot x_2 \cdot h_2)$, this product is well-defined if $g_1 \cdot x_1 \cdot h_1 = g_2 \cdot x_2 \cdot h_2$. Furthermore, $\mathsf{r}(g_1^{-1} \cdot g_2) = \mathsf{s}(g_1) = \mathsf{r}(x_1)$ and $\mathsf{s}(g_1^{-1} \cdot g_2) = \mathsf{s}(g_2) = \mathsf{r}(x_2)$, so that $(x_1, g_1^{-1} \cdot g_2, x_2) \in \mathsf{G}(X)^1$. Unravelling the definition of F_{X}, we see that $(g_1, x_1, h_1) \cdot (x_1, g_1^{-1} \cdot g_2, x_2) = (g_2, x_2, h_2)$ if and only if $g_1 \cdot x_1 \cdot h_1 = g_2 \cdot x_2 \cdot h_2$. This proves the natural isomorphism of bibundle functors $\beta(\mathsf{X}, \mathsf{r}, F_{\mathsf{X}}) \cong \mathsf{X}$.

Next we start with an anafunctor (X, p, F), take the associated bibundle functor

$$\mathsf{Y} := (\mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,p} X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1)/\mathsf{G}(X),$$

and construct an anafunctor from that. This anafunctor contains the cover $q \colon \mathsf{Y} \twoheadrightarrow \mathsf{G}^0$ induced by the map $\mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,p} X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1 \to \mathsf{G}^0$, $(g, x, h) \mapsto \mathsf{r}(g)$, and a functor $E \colon \mathsf{G}(\mathsf{Y}) \to \mathsf{H}$. This functor is described elementwise by $E^0([g, x, h]) = \mathsf{s}(h)$ for $g \in \mathsf{G}^1$, $h \in \mathsf{H}^1$, $x \in X$ with $\mathsf{s}(g) = p(x)$, $F^0(x) = \mathsf{r}(h)$ and

$$E^1([g_1, x_1, h_1], g, [g_2, x_2, h_2]) := h_1^{-1} \cdot F^1(x_1, g_1^{-1} \cdot g \cdot g_2, x_2) \cdot h_2$$

for $g, g_1, g_2 \in \mathsf{G}^1$, $h_1, h_2 \in \mathsf{H}^1$, $x_1, x_2 \in X$ with $\mathsf{s}(g_i) = p(x_i)$, $F^0(x_i) = \mathsf{r}(h_i)$ for $i = 1, 2$ and $\mathsf{r}(g) = \mathsf{r}(g_1)$, $\mathsf{s}(g) = \mathsf{r}(g_2)$; here $[g, x, h]$ stands for the image of $(g, x, h) \in \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,p} X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1$ under the quotient map to Y. The constructions above show that there is a unique functor E with these properties.

We describe a canonical natural transformation Ψ from (Y, q, E) to (X, p, F); this is a map from $X \times_{p,\mathsf{G}^0,q} \mathsf{Y}$ to H^1 with suitable properties. Elementwise, Ψ is determined uniquely by

$$\Psi(\tilde{x}, [g, x, h]) = F^1(\tilde{x}, g, x) \cdot h \colon \mathsf{s}(h) \xrightarrow{h} \mathsf{r}(h) = F^0(x) \xrightarrow{F^1(\tilde{x}, g, x)} F^0(\tilde{x})$$

for $\tilde{x}, x \in X$, $g \in \mathsf{G}^1$, $h \in \mathsf{H}^1$ with $\mathsf{s}(g) = p(x)$, $F^0(x) = \mathsf{r}(h)$, $p(\tilde{x}) = \mathsf{r}(g)$. We must check that this is well-defined and that it gives a natural transformation.

For well-definedness, we use that $X \times_{p,\mathsf{G}^0,q} \mathsf{Y}$ is the orbit space of the $\mathsf{G}(X)$-action on $X \times_{p,\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,p} X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1$ on the last three legs; this follows as in the construction of the left G-action on $\mathsf{X} \times_{\mathsf{G}} \mathsf{Y}$ in the proof of Proposition 7.8. It is clear that Ψ is well-defined as a map $X \times_{p,\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,p} X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1 \to \mathsf{H}^1$. This map is $\mathsf{G}(X)$-invariant because

$$F^1(\tilde{x}, g_1 \cdot g_2, x_2) \cdot F^1(x_1, g_2, x_2)^{-1} \cdot h = F^1(\tilde{x}, g_1, x_1) \cdot h$$

for $\tilde{x}, x_1, x_2 \in X$, $g_1, g_2 \in \mathsf{G}^1$, $h \in \mathsf{H}^1$ with $p(\tilde{x}) = \mathsf{r}(g_1)$, $\mathsf{s}(g_1) = p(x_1) = \mathsf{r}(g_2)$, $F^0(x_1) = \mathsf{r}(h)$, $\mathsf{s}(g_2) = p(x_2)$, so that $(x_1, g_2, x_2) \in \mathsf{G}(X)$ and $(\tilde{x}, g_1, x_1, h) \in X \times_{p,\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,p} X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1$.

Naturality of Ψ follows because for $(\tilde{x}_i, g_i, x_i, h_i) \in X \times_{p,\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \times_{\mathsf{s},\mathsf{G}^0,p} X \times_{F^0,\mathsf{H}^0,\mathsf{r}} \mathsf{H}^1$ for $i = 1, 2$, $g \in \mathsf{G}^1$, with $\mathsf{r}(g) = p(\tilde{x}_1)$, $\mathsf{s}(g) = p(\tilde{x}_2)$, the following diagram in H commutes:

$$
\begin{array}{ccccc}
\mathsf{s}(h_1) & \xrightarrow{\ h_1\ } & \mathsf{r}(h_1) = F^0(x_1) & \xrightarrow{\ F^1(\tilde{x}_1, g_1, x_1)\ } & F^0(\tilde{x}_1) \\[4pt]
{\scriptstyle h_1^{-1} F^1(x_1, g_1^{-1} g g_2, x_2) h_2} \Big\uparrow & & \Big\uparrow {\scriptstyle F^1(x_1, g_1^{-1} g g_2, x_2)} & & \Big\uparrow {\scriptstyle F^1(\tilde{x}_1, g, \tilde{x}_2)} \\[4pt]
\mathsf{s}(h_2) & \xrightarrow[\ h_2\ ]{} & \mathsf{r}(h_2) = F^0(x_2) & \xrightarrow[\ F^1(\tilde{x}_2, g_2, x_2)\ ]{} & F^0(\tilde{x}_2)
\end{array}
$$

the two rows give Ψ and the left and right vertical maps give E^1 and F^1 for an arrow in $\mathsf{G}(X \times_{\mathsf{G}^0} \mathsf{Y})$.

We have now constructed an equivalence between the groupoids of bibundle functors and anafunctors from G to H. To check that we have an equivalence of bicategories, it remains to show that the map from anafunctors to bibundle functors is compatible with composition of anafunctors up to natural G, H-maps. The identity (ana)functor on a groupoid G is clearly mapped to the unit bibundle functor on G.

The first step is to show that the composition of bibundle functors gives the usual composition for ordinary functors. Let $F_2 \colon \mathsf{G} \to \mathsf{H}$ and $F_1 \colon \mathsf{H} \to \mathsf{K}$ be functors. Let $\beta(F_2)$ and $\beta(F_1)$ be the associated bibundle functors. We claim that $\beta(F_1 \circ F_2)$ is naturally isomorphic to $\beta(F_2) \times_{\mathsf{H}} \beta(F_1)$ as a G, K-action. The underlying objects are $\mathsf{G}^0 \times_{F_1^0 \circ F_2^0, \mathsf{K}^0, \mathsf{r}} \mathsf{K}^1$ for $\beta(F_1 \circ F_2)$ and

$$(\mathsf{G}^0 \times_{F_2^0, \mathsf{H}^0, \mathsf{r}} \mathsf{H}^1) \times_{\mathsf{H}} (\mathsf{H}^0 \times_{F_1^0, \mathsf{K}^0, \mathsf{r}} \mathsf{K}^1)$$

for $\beta(F_2) \times_{\mathsf{H}} \beta(F_1)$; the latter is the orbit space of a canonical H-action on

$$(\mathsf{G}^0 \times_{F_2^0, \mathsf{H}^0, \mathsf{r}} \mathsf{H}^1) \times_{\mathsf{H}^0} (\mathsf{H}^0 \times_{F_1^0, \mathsf{K}^0, \mathsf{r}} \mathsf{K}^1) \cong \mathsf{G}^0 \times_{F_2^0, \mathsf{H}^0, \mathsf{r}} \mathsf{H}^1 \times_{F_1^0 \circ \mathsf{s}, \mathsf{K}^0, \mathsf{r}} \mathsf{K}^1.$$

The isomorphism is induced by the map

$$\alpha \colon \mathsf{G}^0 \times_{F_2^0, \mathsf{H}^0, \mathsf{r}} \mathsf{H}^1 \times_{F_1^0 \circ \mathsf{s}, \mathsf{K}^0, \mathsf{r}} \mathsf{K}^1 \to \mathsf{G}^0 \times_{F_1^0 \circ F_2^0, \mathsf{K}^0, \mathsf{r}} \mathsf{K}^1, \qquad (x, h, k) \mapsto (x, F_1^1(h) \cdot k),$$

for $x \in \mathsf{G}^0$, $h \in \mathsf{H}^1$, $k \in \mathsf{K}^1$ with $F_2^0(x) = \mathsf{r}(h)$, $F_1^0(\mathsf{s}(h)) = \mathsf{s}(F_1^1(h)) = \mathsf{r}(k)$. Since $\mathsf{r} \colon \mathsf{H}^1 \twoheadrightarrow \mathsf{H}^0$ is a cover, so is the coordinate projection $\mathsf{G}^0 \times_{F_2^0, \mathsf{H}^0, \mathsf{r}} \mathsf{H}^1 \twoheadrightarrow \mathsf{G}^0$. The pull-back of this cover along the coordinate projection $\mathsf{G}^0 \times_{F_1^0 \circ F_2^0, \mathsf{K}^0, \mathsf{r}} \mathsf{K}^1 \to \mathsf{G}^0$ is a cover as well, and this pull-back is exactly α. Thus α is a cover. We must also show that α induces an isomorphism

$$(\mathsf{G}^0 \times_{F_2^0, \mathsf{H}^0, \mathsf{r}} \mathsf{H}^1) \times_{\mathsf{H}} (\mathsf{H}^0 \times_{F_1^0, \mathsf{K}^0, \mathsf{r}} \mathsf{K}^1) \to \mathsf{G}^0 \times_{F_1^0 \circ F_2^0, \mathsf{K}^0, \mathsf{r}} \mathsf{K}^1.$$

We show that α is the bundle projection of a principal H-bundle. This means that if $(x_i, h_i, k_i) \in \mathsf{G}^0 \times_{F_2^0, \mathsf{H}^0, \mathsf{r}} \mathsf{H}^1 \times_{F_1^0 \circ \mathsf{s}, \mathsf{K}^0, \mathsf{r}} \mathsf{K}^1$ for $i = 1, 2$, then there is $h \in \mathsf{H}^1$ with

$$(x_2, h_2, k_2) = (x_1, h_1, k_1) \cdot h := (x_1, h_1 \cdot h, F_1^1(h)^{-1} \cdot k_1)$$

if and only if $\alpha(x_1, h_1, k_1) = \alpha(x_2, h_2, k_2)$. Indeed, the only possible choice is $h = h_1^{-1} \cdot h_2$, and this does the trick if and only if $x_1 = x_2$ and $F_1^1(h_1) \cdot k_1 = F_1^1(h_2) \cdot k_2$. This provides an isomorphism

$$\beta(F_2) \times_{\mathsf{H}} \beta(F_1) \cong \beta(F_1 \circ F_2) \tag{22}$$

for two functors F_1 and F_2.

It is straightforward to see that the isomorphisms (22) turn β into a homomorphism from the bicategory of functors to the bicategory of bibundle functors (see [28] for the definition of homomorphisms between bicategories). Theorem 3.29 implies that β extends to a homomorphism from the bicategory of anafunctors to the bicategory of bibundle functors. This homomorphism acts on arrows and 2-arrows by the equivalences described above, so these are part of a homomorphism. Thus we have got an equivalence of bicategories. ∎

7.19. Decomposing bibundle actors. Let G and H be groupoids and let X be a bibundle actor from G to H. We will decompose X into an ordinary actor from G to an auxiliary groupoid K and a bibundle equivalence from K to H:

7.20. Proposition. *Assume Assumptions 2.8 and 7.3. Any bibundle actor is a composite of an actor and a bibundle equivalence.*

The groupoid K is defined using only the right H-action on X, in such a way that X is a bibundle equivalence from K to G. The construction goes back to Ehresmann; for locally compact groupoids, this construction is contained in [36, p. 5].

Proof. The fibre product $\mathsf{X} \times_{\mathsf{s},\mathsf{H}^0,\mathsf{s}} \mathsf{X}$ exists and the coordinate projections

$$\mathsf{pr}_1, \mathsf{pr}_2 \colon \mathsf{X} \times_{\mathsf{s},\mathsf{H}^0,\mathsf{s}} \mathsf{X} \rightrightarrows \mathsf{X}$$

are covers because $\mathsf{s} \colon \mathsf{X} \twoheadrightarrow \mathsf{H}^0$ is a cover by assumption. We let $\mathsf{K}^0 := \mathsf{X}/\mathsf{H}$ and $\mathsf{K}^1 := (\mathsf{X} \times_{\mathsf{s},\mathsf{H}^0,\mathsf{s}} \mathsf{X})/\mathsf{H}$, where H acts diagonally on $\mathsf{X} \times_{\mathsf{s},\mathsf{H}^0,\mathsf{s}} \mathsf{X}$, that is, $(x_1, x_2) \cdot h := (x_1 \cdot h, x_2 \cdot h)$ if $x_1, x_2 \in \mathsf{X}$, $h \in \mathsf{H}^1$ with $\mathsf{s}(x_1) = \mathsf{s}(x_2) = \mathsf{r}(h)$. Assumption 7.3 ensures that this action is basic because pr_1 and pr_2 are H-maps and the H-action on X is basic. The range and source maps $\mathsf{r}, \mathsf{s} \colon \mathsf{K}^1 \rightrightarrows \mathsf{K}^0$ are induced by the coordinate projections pr_1 and pr_2 above and are covers by Proposition 5.9; here we need Assumption 2.8. Using Proposition 5.10, we may identify

$$\mathsf{K}^1 \times_{\mathsf{s},\mathsf{K}^0,\mathsf{r}} \mathsf{K}^1 \cong ((\mathsf{X} \times_{\mathsf{s},\mathsf{H}^0,\mathsf{s}} \mathsf{X}) \times_{\mathsf{X}} (\mathsf{X} \times_{\mathsf{s},\mathsf{H}^0,\mathsf{s}} \mathsf{X}))/\mathsf{H}$$
$$\cong (\mathsf{X} \times_{\mathsf{s},\mathsf{H}^0,\mathsf{s}} \mathsf{X} \times_{\mathsf{s},\mathsf{H}^0,\mathsf{s}} \mathsf{X})/\mathsf{H}.$$

The multiplication map of K^1 is the map on the base induced by the H-map

$$(\mathsf{pr}_1, \mathsf{pr}_3) \colon \mathsf{X} \times_{\mathsf{s},\mathsf{H}^0,\mathsf{s}} \mathsf{X} \times_{\mathsf{s},\mathsf{H}^0,\mathsf{s}} \mathsf{X} \to \mathsf{X} \times_{\mathsf{s},\mathsf{H}^0,\mathsf{s}} \mathsf{X}.$$

Proposition 5.10 also implies the isomorphisms (5) and (6) for K, so that K is a groupoid in $(\mathcal{C}, \mathcal{T})$.

We are going to construct a canonical actor from G to K. Let the groupoid G act on $\mathsf{X} \times_{\mathsf{s},\mathsf{H}^0,\mathsf{s}} \mathsf{X}$ by $g \cdot (x_1, x_2) := (g \cdot x_1, x_2)$ for all $g \in \mathsf{G}^1$, $x_1, x_2 \in \mathsf{X}$ with $\mathsf{s}(g) = \mathsf{r}(x_1)$ and $\mathsf{s}(x_1) = \mathsf{s}(x_2)$; this is well-defined because $\mathsf{s}(g \cdot x_1) = \mathsf{s}(x_1)$. This G-action commutes with the diagonal H-action and hence descends to a G-action on the H-orbit space K^1 by Proposition 7.9. The resulting G-action on K^1 commutes with the right multiplication action because $g \cdot [x_1, x_3] = [g \cdot x_1, x_3] = (g \cdot [x_1, x_2]) \cdot [x_2, x_3]$ for all $g \in \mathsf{G}^1$, $x_1, x_2, x_3 \in \mathsf{X}$ with $\mathsf{s}(g) = \mathsf{r}(x_1)$, $\mathsf{s}(x_1) = \mathsf{s}(x_2) = \mathsf{s}(x_3)$.

We are going to construct a left K-action on X such that X becomes a bibundle equivalence from K to H. Its anchor map is the bundle projection $\mathsf{p} \colon \mathsf{X} \twoheadrightarrow \mathsf{X}/\mathsf{H}$. The action $(\mathsf{X} \times_{\mathsf{s},\mathsf{H}^0,\mathsf{s}} \mathsf{X})/\mathsf{H} \times_{\mathsf{p},\mathsf{X}/\mathsf{H},\mathsf{p}} \mathsf{X} \to \mathsf{X}$ is induced by the map $\mathsf{X} \times_{\mathsf{s},\mathsf{H}^0,\mathsf{s}} \mathsf{X} \times_{\mathsf{p},\mathsf{X}/\mathsf{H},\mathsf{p}} \mathsf{X} \to \mathsf{X}$ that is given elementwise by $(x_1, x_2) \cdot (x_2 \cdot h) := x_1 \cdot h$ for all $x_1, x_2 \in \mathsf{X}$, $h \in \mathsf{H}^1$ with $\mathsf{s}(x_1) = \mathsf{s}(x_2) = \mathsf{r}(h)$; this defines a map on $\mathsf{X} \times_{\mathsf{s},\mathsf{H}^0,\mathsf{s}} \mathsf{X} \times_{\mathsf{p},\mathsf{X}/\mathsf{H},\mathsf{p}} \mathsf{X}$ because there is $h \in \mathsf{H}^1$ with $x_3 = x_2 \cdot h$ whenever $x_2, x_3 \in \mathsf{X}$ satisfy $\mathsf{p}(x_2) = \mathsf{p}(x_3)$. We have $(\mathsf{X} \times_{\mathsf{s},\mathsf{H}^0,\mathsf{s}} \mathsf{X})/\mathsf{H} \times_{\mathsf{p},\mathsf{X}/\mathsf{H},\mathsf{p}} \mathsf{X} \cong (\mathsf{X} \times_{\mathsf{s},\mathsf{H}^0,\mathsf{s}} \mathsf{X} \times_{\mathsf{p},\mathsf{X}/\mathsf{H},\mathsf{p}} \mathsf{X})/\mathsf{H}$, where H acts on $\mathsf{X} \times_{\mathsf{s},\mathsf{H}^0,\mathsf{s}} \mathsf{X} \times_{\mathsf{p},\mathsf{X}/\mathsf{H},\mathsf{p}} \mathsf{X}$ by $(x_1, x_2, x_3) \cdot h := (x_1 \cdot h, x_2 \cdot h, x_3)$. The multiplication map defined above is H-invariant for this action, hence it descends to a map $\mathsf{K}^1 \times_{\mathsf{s},\mathsf{X}/\mathsf{H},\mathsf{p}} \mathsf{X} \to \mathsf{X}$.

Let H act on $\mathsf{K}^1 \times_{\mathsf{s},\mathsf{K}^0,\mathsf{p}} \mathsf{X}$ by $(k, x) \cdot h := (k, x \cdot h)$. This action is basic with

$$(\mathsf{K}^1 \times_{\mathsf{s},\mathsf{K}^0,\mathsf{p}} \mathsf{X})/\mathsf{H} \cong \mathsf{K}^1 \times_{\mathsf{s},\mathsf{K}^0,\mathsf{p}} (\mathsf{X}/\mathsf{H}) \cong \mathsf{K}^1,$$

by Remark 7.11. Thus the H-map

$$\mathsf{K}^1 \times_{\mathsf{s},\mathsf{K}^0,\mathsf{p}} \mathsf{X} \to \mathsf{X} \times_{\mathsf{s},\mathsf{H}^0,\mathsf{s}} \mathsf{X}, \qquad (k, x) \mapsto (k \cdot x, x), \tag{23}$$

induces an isomorphism $(\mathsf{K}^1 \times_{\mathsf{s},\mathsf{K}^0,\mathsf{p}} \mathsf{X})/\mathsf{H} \cong (\mathsf{X} \times_{\mathsf{s},\mathsf{H}^0,\mathsf{s}} \mathsf{X})/\mathsf{H}$ on the bases of principal H-bundles. Hence the map in (23) is an isomorphism as well by Proposition 5.9. Thus $\mathsf{s} \colon \mathsf{X} \twoheadrightarrow \mathsf{H}^0$ and the K-action on X form a principal K-bundle. Thus we have turned X into a bibundle equivalence from K to H.

The actor from G to K is also a bibundle actor. Its composite with the bibundle equivalence X from K to H is X as a right H-action because $\mathsf{K}^1 \times_{\mathsf{K}} \mathsf{X} \cong \mathsf{X}$ for any bibundle actor X from K to H. The induced action of G on the composite is the given one because $g \cdot [x, x] \cdot x = [g \cdot x, x] \cdot x = g \cdot x$ for all $g \in \mathsf{G}^1$, $x \in \mathsf{X}$, where we interpret $[x, x] \in \mathsf{K}^1$. This shows that the actor and the bibundle equivalence constructed above compose to the given bibundle actor X. ∎

We conclude that the category of bibundle actors is the smallest one that contains both bibundle equivalences and actors and where the composition is given by $\times_{\mathsf{H}}$.

Now let X be a covering bibundle functor. Equivalently, X is a bibundle actor and the range map induces an isomorphism $\mathsf{X}/\mathsf{H} \xrightarrow{\sim} \mathsf{G}^0$. In the above construction, this means that $\mathsf{K}^0 = \mathsf{G}^0$, so that the actor from G to K acts identically on objects. Such actors are exactly the ones that are also functors, by Proposition 4.21.

7.21. THE SYMMETRIC IMPRIMITIVITY THEOREM.

7.22. THEOREM. *Let* G *and* H *be groupoids and* X *a* G, H-*bibundle with basic actions of* G *and* H. *This induces a left action of* G *on* X/H *and a right action of* H *on* G\X, *and* X *becomes a bibundle equivalence from* G ⋉ (X/H) *to* (G\X) ⋊ H.

PROOF. The induced G-action on X/H exists by Proposition 7.9. The bundle projection $s': X \twoheadrightarrow X/H$ is a G-map. Hence the G-action on X and s' combine to a left action of G ⋉ (X/H) on X with anchor map s' by Proposition 4.12. This left action is basic with the same orbit space G\X by Corollary 5.17 and Lemma 5.18 because the G-action on X is basic. Exchanging left and right, the same arguments give an H-action on G\X and a basic right action of (G\X) ⋊ H on X with anchor map $r': X \twoheadrightarrow$ G\X and orbit space X/H. The actions of G ⋉ (X/H) and (G\X) ⋊ H on X commute because the actions of G and H commute, s' is H-invariant, and r' is G-invariant. Hence they form a bibundle equivalence. ∎

This theorem generalises the symmetric imprimitivity theorem for actions of locally compact groups by Green and Rieffel [43] and provides many important examples of bibundle equivalences between transformation groupoids.

7.23. EXAMPLE. Assume Assumption 2.10 and let G be a group in $\mathcal{C}$. Let H ↪ G and K ↪ G be "closed subgroups" of G; we mean by this that the restrictions of the multiplication actions on G to H and K are basic. Since the left and right multiplication actions commute, G is an H, K-bibundle. We get an induced bibundle equivalence H ⋉ (G/K) ∼ (H\G) ⋊ K.

7.24. CHARACTERISING COMPOSITES OF BIBUNDLE FUNCTORS.

7.25. PROPOSITION. *Let* G, H, K *be groupoids in* $\mathcal{C}$. *Let* X: G → H, Y: H → K, *and* W: G → K *be bibundle functors. Then isomorphisms* $X \times_H Y \xrightarrow{\sim} W$ *are in canonical bijection with* G, K-*maps* $m: X \times_{s,H^0,r} Y \to W$ *with* $m(x \cdot h, y) = m(x, h \cdot y)$ *for all* $x \in X$, $h \in H^1$, $y \in Y$ *with* $s(x) = r(h)$, $s(h) = r(y)$. *If* m *is such a map, then* m *is a cover and the following map is an isomorphism:*

$$(\mathsf{pr}_1, \mathsf{m}): X \times_{s,H^0,r} Y \to X \times_{r,G^0,r} W, \qquad (x, y) \mapsto (x, \mathsf{m}(x, y)). \qquad (24)$$

PROOF. Any G, K-map $X \times_H Y \to W$ is an isomorphism by Theorem 7.15. Since $X \times_H Y$ is the orbit space of an H-action on $X \times_{s,H^0,r} Y$, a G, K-map $X \times_H Y \to W$ is equivalent to an H-invariant G, K-map $X \times_{s,H^0,r} Y \to W$. It remains to show that (24) is always an isomorphism.

Since X is a principal H-bundle over G^0, its pull-back $X \times_{r,G^0,r} W$ along $r: W \twoheadrightarrow G^0$ is a principal H-bundle over W with bundle projection pr_2 by Proposition 5.8; here H acts on $X \times_{r,G^0,r} W$ by $s(x, w) = s(x)$ and $(x, w) \cdot h := (x \cdot h, w)$ for all $x \in X$, $w \in W$, $h \in H^1$ with $r(x) = r(w)$, $s(x) = r(h)$. By definition of the composition, $X \times_{s,H^0,r} Y \twoheadrightarrow X \times_H Y$ is a principal H-bundle as well. We compute that $(\mathsf{pr}_1, \mathsf{m})$ is an H-map:

$$\begin{aligned}
(\mathsf{pr}_1, \mathsf{m})((x, y) \cdot h) &= (\mathsf{pr}_1, \mathsf{m})(x \cdot h, h^{-1} \cdot y) \\
&= (x \cdot h, \mathsf{m}(x \cdot h, h^{-1} \cdot y)) = (x \cdot h, \mathsf{m}(x, y)) = (x, \mathsf{m}(x, y)) \cdot h
\end{aligned}$$

for all $x \in \mathsf{X}$, $y \in \mathsf{Y}$, $h \in \mathsf{H}^1$ with $\mathsf{s}(x) = \mathsf{r}(y) = \mathsf{r}(h)$. Proposition 5.9 shows that (24) is an isomorphism if and only if the induced map $\mathsf{X} \times_{\mathsf{H}} \mathsf{Y} \to \mathsf{W}$ is an isomorphism.

Since $\mathsf{r} \colon \mathsf{X} \twoheadrightarrow \mathsf{G}^0$ is a cover, so is the induced map $\mathsf{pr}_2 \colon \mathsf{X} \times_{\mathsf{r},\mathsf{G}^0,\mathsf{r}} \mathsf{W} \twoheadrightarrow \mathsf{W}$. Composing this with the isomorphism (24) shows that m is a cover. $\blacksquare$

We have seen some special cases of the isomorphism (24) before. First, the condition (6) in the definition of a groupoid is equivalent to $\mathsf{G}^1 \times_{\mathsf{G}} \mathsf{G}^1 \cong \mathsf{G}^1$. Secondly, the condition (17) for a principal bundle is equivalent to $\mathsf{X} \times_{\mathsf{G}} \mathsf{G}^1 \cong \mathsf{X}$ for a principal G-bundle $\mathsf{X} \twoheadrightarrow \mathsf{Z}$ viewed as a bibundle functor from Z to G. We will use this similarity to simplify the description of the quasi-category of groupoids and bibundle functors in Section 8. Condition (3‴) in Definition 4.1 is also similar and corresponds to a variant of Proposition 7.25 where we characterise the composite of a covering bibundle functor with a groupoid action. This variant is also true, and the isomorphism in (3‴) in Definition 4.1 is equivalent to $\mathsf{G}^1 \times_{\mathsf{G}} \mathsf{X} \cong \mathsf{X}$ for any left G-action X.

7.26. REMARK. The proof of Proposition 7.25 does not use Assumption 7.2. For any subcanonical pretopology $(\mathcal{C}, \mathcal{T})$, if G, H, K are groupoids, $\mathsf{X} \colon \mathsf{G} \to \mathsf{H}$, $\mathsf{Y} \colon \mathsf{H} \to \mathsf{K}$ and $\mathsf{W} \colon \mathsf{G} \to \mathsf{K}$ bibundle functors, and $\mathsf{m} \colon \mathsf{X} \times_{\mathsf{s},\mathsf{H}^0,\mathsf{r}} \mathsf{Y} \to \mathsf{W}$ an H-invariant G, K-map and a cover such that (24) is an isomorphism, then m is the bundle projection of a principal H-bundle, so that $\mathsf{X} \times_{\mathsf{H}} \mathsf{Y}$ exists and is isomorphic to W.

7.27. LOCALITY OF BASIC ACTIONS. We now reformulate Assumption 7.2 in a way similar to the locality of principal bundles in Proposition 5.12.

Let G be a groupoid, X a right G-action, $f \colon \tilde{\mathsf{Z}} \twoheadrightarrow \mathsf{Z}$ a cover, $\mathsf{Z}, \tilde{\mathsf{Z}} \in\in \mathcal{C}$, and let $\varphi \colon \mathsf{X} \to \mathsf{Z}$ be some G-invariant map. Then we may pull back X along f to a G-action $\tilde{\mathsf{X}}$: let $\tilde{\mathsf{X}} := \tilde{\mathsf{Z}} \times_{f,\mathsf{Z},\varphi} \mathsf{X}$ (this exists because f is a cover) and define the G-action on $\tilde{\mathsf{X}}$ by $\mathsf{s}(z,x) := \mathsf{s}(x)$ and $(z,x) \cdot g := (z, x \cdot g)$ for all $z \in \tilde{\mathsf{Z}}$, $x \in \mathsf{X}$, $g \in \mathsf{G}^1$ with $f(z) = \varphi(x)$, $\mathsf{s}(x) = \mathsf{r}(g)$.

7.28. PROPOSITION. *In the above situation, if* X *is a basic* G-*action, then so is* $\tilde{\mathsf{X}}$*. Under Assumption 2.9, Assumption 7.2 is equivalent to the following converse statement: in the above situation,* X *is basic if* $\tilde{\mathsf{X}}$ *is.*

PROOF. If X is a basic G-action, let Z_0 be its base and $\mathsf{p} \colon \mathsf{X} \twoheadrightarrow \mathsf{Z}_0$ its bundle projection. There is a unique map $\varphi_0 \colon \mathsf{Z}_0 \to \mathsf{Z}$ with $\varphi_0 \circ \mathsf{p} = \varphi$ because φ is G-invariant and p is the orbit space projection by Lemma 5.3. Let $\tilde{\mathsf{Z}}_0 := \tilde{\mathsf{Z}} \times_{f,\mathsf{Z},\varphi_0} \mathsf{Z}_0$. Then $\tilde{\mathsf{X}}$ is naturally isomorphic to the pull-back $\tilde{\mathsf{Z}}_0 \times_{\mathsf{pr}_2,\mathsf{Z}_0,\mathsf{p}} \mathsf{X}$. The induced G-action on this together with the projection to $\tilde{\mathsf{Z}}_0$ is a principal bundle by Proposition 5.8. Thus $\tilde{\mathsf{X}}$ is basic if X is.

Conversely, assume that $\tilde{\mathsf{X}}$ is a basic G-action. First, we also assume that Assumptions 2.9 and 7.2 hold. Since f is a cover, so is $\mathsf{pr}_2 \colon \tilde{\mathsf{X}} \twoheadrightarrow \mathsf{X}$. The map pr_2 is a G-map as well, so the G-action and pr_2 give a right $\mathsf{X} \rtimes \mathsf{G}$-action on $\tilde{\mathsf{X}}$ by Proposition 4.12. This action is basic by Corollary 5.17 and because the G-action on $\tilde{\mathsf{X}}$ is assumed basic. The advantage of the right $\mathsf{X} \rtimes \mathsf{G}$-action over the right G-action is that its anchor map pr_2 is a cover.

Let H be the Čech groupoid of f. It acts on $\tilde{\mathsf{X}}$ on the left with anchor map pr_1 and action $(z_1, z_2) \cdot (z_2, x) := (z_1, x)$ for all $z_1, z_2 \in \tilde{\mathsf{Z}}$, $x \in \mathsf{X}$ with $f(z_1) = f(z_2) = \varphi(x)$. This

action commutes with the right $\mathsf{X} \rtimes \mathsf{G}$-action. Thus $\tilde{\mathsf{X}}$ is a bibundle actor from H to G. Proposition 7.10 allows us to compose bibundle actors. In particular, we may compose $\tilde{\mathsf{X}}$ with the bibundle equivalence $\tilde{\mathsf{Z}}$ from Z to H (see Example 6.3). The composite is $\mathsf{H} \backslash \tilde{\mathsf{X}} \cong \mathsf{X}$ because the pull-back of the principal H-bundle $\tilde{\mathsf{Z}} \twoheadrightarrow \mathsf{Z}$ along φ gives a principal H-bundle $\tilde{\mathsf{X}} \twoheadrightarrow \mathsf{X}$.

Since the isomorphism is the coordinate projection, the induced $\mathsf{X} \rtimes \mathsf{G}$-action on $\tilde{\mathsf{X}} \twoheadrightarrow \mathsf{X}$ is the one we started with. Since this composite of bibundle actors is again a bibundle actor from Z to $\mathsf{X} \rtimes \mathsf{G}$, we conclude that X is a basic right $\mathsf{X} \rtimes \mathsf{G}$-action. Hence it is a basic right G-action by Corollary 5.17. Thus being basic is a local property of groupoid actions.

Now assume that basic actions are local. Let $f \colon \tilde{\mathsf{Z}} \twoheadrightarrow \mathsf{Z}$ be a cover and let G be its Čech groupoid. Let X be a G-action. We want to show that X is basic as well, by proving that it is locally basic. Composing the anchor map $\mathsf{s} \colon \mathsf{X} \to \tilde{\mathsf{Z}}$ with f gives a G-invariant map $\varphi := f \circ \mathsf{s} \colon \mathsf{X} \to \mathsf{Z}$. The pull-back of the G-action X along f gives $\tilde{\mathsf{X}} := \mathsf{X} \times_{\varphi, \mathsf{Z}, f} \tilde{\mathsf{Z}}$ with G-action by $(x, z) \cdot (z_1, z_2) = (x \cdot (z_1, z_2), z)$ for all $x \in \mathsf{X}$, $z, z_1, z_2 \in \tilde{\mathsf{Z}}$ with $\varphi(x) = f(z) = f(z_1) = f(z_2)$, $z_1 = \mathsf{s}(x)$.

The multiplication $\mathsf{m} \colon \mathsf{X} \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1 \twoheadrightarrow \mathsf{X}$ is a cover by $(3'')$ in Definition 4.1. Identifying

$$\mathsf{X} \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1 := \mathsf{X} \times_{\mathsf{s}, \tilde{\mathsf{Z}}, \mathsf{pr}_1} (\tilde{\mathsf{Z}} \times_{f, \mathsf{Z}, f} \tilde{\mathsf{Z}}) \cong \mathsf{X} \times_{f \circ \mathsf{s}, \mathsf{Z}, f} \tilde{\mathsf{Z}},$$

this cover becomes $q \colon \tilde{\mathsf{X}} \twoheadrightarrow \mathsf{X}$, $(x, z) \mapsto x \cdot (\mathsf{s}(x), z)$. We claim that q is the bundle projection of a principal G-bundle. If $(x_1, z_1), (x_2, z_2) \in \tilde{\mathsf{X}}$ with $x_1 \cdot (\mathsf{s}(x_1), z_1) = x_2 \cdot (\mathsf{s}(x_2), z_2)$, then $f(\mathsf{s}(x_1)) = f(z_1)$, $f(\mathsf{s}(x_2)) = f(z_2)$, and $z_1 = \mathsf{s}(x_1 \cdot (\mathsf{s}(x_1), z_1)) = \mathsf{s}(x_2 \cdot (\mathsf{s}(x_2), z_2)) = z_2$. Thus

$$\begin{aligned}
(x_1, z_1) \cdot (\mathsf{s}(x_1), \mathsf{s}(x_2)) &= (x_1, z_1) \cdot (\mathsf{s}(x_1), z_1) \cdot (z_2, \mathsf{s}(x_2)) \\
&= (x_2, z_2) \cdot (\mathsf{s}(x_2), z_2) \cdot (z_2, \mathsf{s}(x_2)) = (x_2, z_2).
\end{aligned}$$

Furthermore, if $(x_1, z_1) \cdot (z_3, z_4) = (x_2, z_2)$, then $z_3 = \mathsf{s}(x_1)$ and $z_4 = \mathsf{s}(x_2)$, so $(\mathsf{s}(x_1), \mathsf{s}(x_2))$ is the unique element of G^1 with this property. Thus the map (17) is an isomorphism and $\tilde{\mathsf{X}} \twoheadrightarrow \mathsf{X}$ is a principal G-bundle. Since $\tilde{\mathsf{X}}$ is the pull-back of the G-action X along the cover f, any G-action is locally basic. Therefore, if all locally basic actions are basic, then Assumption 7.2 follows. ∎

7.29. Equivalences in bibundle functors and actors.

7.30. Proposition. *Let G and H be groupoids and let $\mathsf{X} \colon \mathsf{G} \to \mathsf{H}$ be a bibundle equivalence in $(\mathcal{C}, \mathcal{T})$. Then there are canonical isomorphisms of bibundle equivalences $\mathsf{X} \times_{\mathsf{H}} \mathsf{X}^* \cong \mathsf{G}^1$ and/ $\mathsf{X}^* \times_{\mathsf{G}} \mathsf{X} \cong \mathsf{H}^1$.*

Proof. Since $\mathsf{r} \colon \mathsf{X} \twoheadrightarrow \mathsf{H}^0$ is a left principal G-bundle, the map $\mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{X} \to \mathsf{X} \times_{\mathsf{s}, \mathsf{H}^0, \mathsf{s}} \mathsf{X}$, $(g, x) \mapsto (x, g \cdot x)$, is an isomorphism. The inverse is of the form $(x_1, x_2) \mapsto (x_1, \mathsf{m}(x_1, x_2))$ for a map $\mathsf{m} \colon \mathsf{X} \times_{\mathsf{s}, \mathsf{H}^0, \mathsf{s}} \mathsf{X} \to \mathsf{G}^1$. Identifying $\mathsf{X} \times_{\mathsf{s}, \mathsf{H}^0, \mathsf{s}} \mathsf{X} \cong \mathsf{X} \times_{\mathsf{s}, \mathsf{H}^0, \mathsf{r}} \mathsf{X}^*$, the criterion of Proposition 7.25 applied to m shows that $\mathsf{X} \times_{\mathsf{H}} \mathsf{X}^* \cong \mathsf{G}^1$. Explicitly, the isomorphism is

induced by m, so it maps the class of $(x_1, x_2) \in \mathsf{X} \times_{\mathsf{s}, \mathsf{H}^0, \mathsf{r}} \mathsf{X}^*$ to the unique $g \in \mathsf{G}^1$ with $g \cdot x_2 = x_1$.

The same reasoning for X^* instead of X gives $\mathsf{X}^* \times_{\mathsf{G}} \mathsf{X} \cong \mathsf{H}^1$. ∎

7.31. Theorem. *The equivalences in the bicategory of bibundle functors and bibundle actors are exactly the bibundle equivalences. Furthermore, if $\mathsf{Y} \colon \mathsf{H} \to \mathsf{G}$ is quasi-inverse to $\mathsf{X} \colon \mathsf{G} \to \mathsf{H}$, then $\mathsf{Y} \cong \mathsf{X}^*$ is obtained from X by exchanging left and right actions.*

We require Assumptions 2.8 and 7.3 for the bibundle functor case and Assumptions 2.9 and 7.2 for the bibundle actor case because otherwise the bicategories in question need not be defined (Theorem 7.15).

Proof. The bicategories of bibundle functors and anafunctors are equivalent by Theorem 7.18. By Theorem 3.35, the equivalences in the bicategory of anafunctors are exactly those anafunctors that lift to ana-isomorphisms. Since the equivalence to bibundle functors maps ana-isomorphisms to bibundle equivalences, the equivalences in the bicategory of bibundle functors are exactly the bibundle equivalences.

Without Assumption 2.8 something goes wrong at this point because a fully faithful functor that is only almost essentially surjective but not essentially surjective is an equivalence in the bicategory of anafunctors, but the associated bibundle functor is not a bibundle equivalence by Proposition 6.8.

It is clear that bibundle equivalences are equivalences in the bicategory of bibundle actors. Conversely, let X be an equivalence in this bicategory. Proposition 7.20 decomposes X as a composite of a bibundle equivalence and an actor. Thus it suffices to show that an actor that is an equivalence in the bicategory of bibundle actors is already an isomorphism of categories.

Let $\mathsf{m} \colon \mathsf{G}^1 \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}_{\mathsf{G}}} \mathsf{H}^1 \to \mathsf{H}^1$ be an actor that is an equivalence in the bicategory of bibundle actors. We write ${}_{\mathsf{m}}\mathsf{H}^1$ for the bibundle actor described by m. Let Y be its quasi-inverse. Thus ${}_{\mathsf{m}}\mathsf{H}^1 \times_{\mathsf{H}} \mathsf{Y} \cong \mathsf{G}^1$ as G, G-bibundles. Since there is a natural isomorphism of right G-actions $\mathsf{H}^1 \times_{\mathsf{H}} \mathsf{Y} \cong \mathsf{Y}$, we get $\mathsf{Y} \cong \mathsf{G}^1$ as a right G-action. Thus Y is an actor. When we view actors as bibundle actors, the 2-arrows are exactly the G, H-maps $\mathsf{H}^1 \to \mathsf{H}^1$. These are the same 2-arrows that we already used in the bicategory of actors. Any equivalence in this bicategory is an isomorphism of categories by Proposition 4.26. Since the 2-arrows are the same in both bicategories, we conclude that our original actor is an isomorphism of categories. ∎

8. The quasi-category of bibundle functors

Let $\mathcal{C}$ be a category with a pretopology $\mathcal{T}$ that satisfies Assumptions 2.8 and 7.3. Theorem 7.15 shows that groupoids in $(\mathcal{C}, \mathcal{T})$ with bibundle functors as arrows and G, H-maps as 2-arrows form a bicategory with invertible 2-arrows, that is, a $(2, 1)$-category. Its nerve is a quasi-category (an $(\infty, 1)$-category) in the sense of Boardman–Vogt [3] and André Joyal [25], that is, a simplicial set satisfying all inner Kan conditions; furthermore, coming

from a bicategory, it satisfies inner unique Kan conditions in dimensions above 3. This quasi-category contains essentially the same information as the bicategory.

We are going to describe this quasi-category in a more elementary way, without mentioning groupoids and bibundle functors. The main point is the similarity between the conditions (17), (6) and (24) in the definitions of groupoids, principal bundles, and the composition of bibundle functors. An n-simplex in $\mathfrak{Gr}$ consists of

- $X_i \in\in \mathcal{C}$ for $0 \le i \le n$;

- $X_{ij} \in\in \mathcal{C}$ for $0 \le i \le j \le n$;

- $\mathsf{r}_{ij} \in \mathcal{C}(X_{ij}, X_i)$ and $\mathsf{s}_{ij} \in \mathcal{C}(X_{ij}, X_j)$ for $0 \le i \le j \le n$;

- $\mathsf{m}_{ijk} \in \mathcal{C}(X_{ij} \times_{\mathsf{s}_{ij}, X_j, \mathsf{r}_{jk}} X_{jk}, X_{ik})$ for $0 \le i \le j \le k \le n$;

such that the following conditions hold:

(1) r_{ij} is a cover for all $0 \le i \le j \le n$; hence the domain $X_{ij} \times_{\mathsf{s}_{ij}, X_j, \mathsf{r}_{jk}} X_{jk}$ of m_{ijk} is well-defined;

(2) s_{ii} is a cover for all $0 \le i \le n$;

(3) $\mathsf{r}_{ik} \circ \mathsf{m}_{ijk} = \mathsf{r}_{ij} \circ \mathsf{pr}_1$ and $\mathsf{s}_{ik} \circ \mathsf{m}_{ijk} = \mathsf{s}_{jk} \circ \mathsf{pr}_2$ on $X_{ij} \times_{X_j} X_{jk}$ for all $0 \le i \le j \le k \le n$; briefly, $\mathsf{r}(x \cdot y) = \mathsf{r}(x)$ and $\mathsf{s}(x \cdot y) = \mathsf{s}(y)$;

(4) (associativity) for $0 \le i \le j \le k \le l \le n$, the following diagram commutes:

$$
\begin{array}{ccc}
X_{ij} \times_{X_j} X_{jk} \times_{X_k} X_{kl} & \xrightarrow{\ \mathrm{id}_{X_{ij}} \times_{X_j} \mathsf{m}_{jkl}\ } & X_{ij} \times_{X_j} X_{jl} \\
{\scriptstyle \mathsf{m}_{ijk} \times_{X_k} \mathrm{id}_{X_{kl}}} \downarrow & & \downarrow {\scriptstyle \mathsf{m}_{ijl}} \\
X_{ik} \times_{X_k} X_{kl} & \xrightarrow{\ \mathsf{m}_{ikl}\ } & X_{il}
\end{array}
$$

briefly, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;

(5) the following maps are isomorphisms for all $0 \le i \le j \le k \le n$ with $i = j$ or $j = k$:

$$(\mathsf{pr}_1, \mathsf{m}_{ijk}) \colon X_{ij} \times_{\mathsf{s}_{ij}, X_j, \mathsf{r}_{jk}} X_{jk} \to X_{ij} \times_{\mathsf{r}_{ij}, X_i, \mathsf{r}_{ik}} X_{ik}, \qquad (x, y) \mapsto (x, x \cdot y),$$

(6) the following maps are isomorphisms if $0 \le i = j = k \le n$:

$$(\mathsf{m}_{ijk}, \mathsf{pr}_2) \colon X_{ij} \times_{\mathsf{s}_{ij}, X_j, \mathsf{r}_{jk}} X_{jk} \to X_{ik} \times_{\mathsf{s}_{ik}, X_k, \mathsf{s}_{jk}} X_{jk}, \qquad (x, y) \mapsto (x \cdot y, y).$$

An order-preserving map $\varphi\colon \{0,\ldots,n\} \to \{0,\ldots,m\}$ induces $\varphi^*\colon \mathfrak{Gr}_m \to \mathfrak{Gr}_n$: take $X_{\varphi(i)}$, $X_{\varphi(i)\varphi(j)}$, $\mathsf{r}_{\varphi(i)\varphi(j)}$, $\mathsf{s}_{\varphi(i)\varphi(j)}$, $\mathsf{m}_{\varphi(i)\varphi(j)\varphi(k)}$. Thus $\mathfrak{Gr}_\bullet$ is a simplicial set.

Why is this the nerve of the bicategory of bibundle functors?

To begin with, for each $0 \le i \le n$, there is a groupoid G_i with objects X_i and arrows X_{ii}, range r_{ii}, source s_{ii}, and multiplication m_{iii}. The conditions above for equal indices amount to the conditions of Definition 3.4.

Next, X_{il} with left anchor map $\mathsf{r}_{il}\colon X_{il} \to X_i$, right anchor map $\mathsf{s}_{il}\colon X_{il} \to X_l$, left action m_{iil} and right action m_{ill} is a $\mathsf{G}_i,\mathsf{G}_l$-bibundle; the associativity condition (4) for $i = j = k \le l$ says that the left action is associative, the one for $i \le j = k = l$ says that the right action is associative, the one for $i = j \le k = l$ says that both actions commute; (5) for $i = j \le k$ says that the left G_i-action satisfies (3‴) in Definition 4.1; (5) for $i \le j = k$ implies that the right G_j-action satisfies condition (3″) in Definition 4.1, compare the proof of Lemma 5.3.

The conditions (1) for $i \le k$ and (5) for $i \le j = k$ say that the right G_k-action on X_{ik} with bundle projection $\mathsf{r}_{ik}\colon X_{ik} \twoheadrightarrow X_i$ is a principal bundle; thus X_{ik} is a bibundle functor from G_i to G_k. By construction, X_{ii} is the identity bibundle functor on G_i.

The associativity conditions (4) for $0 \le i = j \le k \le l \le n$, $0 \le i \le j = k \le l \le n$, and $0 \le i \le j \le k = l \le n$ and (3) say that the maps $m_{ijk}\colon X_{ij} \times_{X_j} X_{jk} \to X_{ik}$ are G_j-invariant $\mathsf{G}_i,\mathsf{G}_k$-maps. Hence they induce isomorphisms of $\mathsf{G}_i,\mathsf{G}_k$-bibundles

$$\dot{\mathsf{m}}_{ijk}\colon X_{ij} \times_{\mathsf{G}_j} X_{jk} \xrightarrow{\sim} X_{ik}$$

by Proposition 7.25. Proposition 7.25 also shows that the maps m_{ijk} are covers for all $0 \le i \le j \le k \le n$ and that the map in (5) is an isomorphism also for $i < j < k$.

By construction, $\dot{\mathsf{m}}_{iik}\colon \mathsf{G}_i^1 \times_{\mathsf{G}_i} X_{ik} \to X_{ik}$ and $\dot{\mathsf{m}}_{ikk}\colon X_{ik} \times_{\mathsf{G}_k} \mathsf{G}_k^1 \to X_{ik}$ are the canonical isomorphisms. Finally, for $0 \le i \le j \le k \le l \le n$, the associativity condition (4) is equivalent to the following commutative diagram of isomorphisms of bibundle functors:

$$
\begin{array}{ccc}
X_{ij} \times_{\mathsf{G}_j} X_{jk} \times_{\mathsf{G}_k} X_{kl} & \xrightarrow{\ \mathrm{id}_{X_{ij}} \times_{\mathsf{G}_j} \dot{\mathsf{m}}_{jkl}\ } & X_{ij} \times_{\mathsf{G}_j} X_{jl} \\
{\scriptstyle \dot{\mathsf{m}}_{ijk} \times_{\mathsf{G}_k} \mathrm{id}_{X_{kl}}} \downarrow & & \downarrow {\scriptstyle \dot{\mathsf{m}}_{ijl}} \\
X_{ij} \times_{\mathsf{G}_j} X_{jl} & \xrightarrow{\ \dot{\mathsf{m}}_{ikl}\ } & X_{il}
\end{array}
$$

Rewriting our n-simplices in terms of the groupoids G_i, bibundle functors $X_{ij}\colon \mathsf{G}_i \to \mathsf{G}_j$ and isomorphisms of bibundle functors $\dot{\mathsf{m}}_{ijk}\colon X_{ij} \times_{\mathsf{G}_j} X_{jk} \to X_{ik}$ gives us precisely the definition of the nerve of the bicategory of bibundle functors. It is well-known that the nerve of a bicategory satisfies the Kan condition $\mathrm{Kan}(2,1)$ and the unique Kan conditions $\mathrm{Kan}!(n,j)$ for $n \ge 3$ and $1 \le j \le n-1$ (and also for $n \ge 5$ and $j = 0$ or $j = n$). Of course, this only holds under Assumptions 2.8 and 7.2 because otherwise we do not have a bicategory.

The quasi-category associated to the sub-bicategory of covering bibundle functors is defined similarly; in addition, we require the maps s_{ij} to be covers for all $0 \le i \le j \le n$.

The quasi-category associated to the sub-bicategory of bibundle equivalences is defined similarly, with the following extra conditions:

- the maps s_{ij} should be covers for all $0 \leq i \leq j \leq n$;

- the maps in (5) and (6) should be isomorphisms if $0 \leq i \leq j \leq k \leq n$ and $i = j$ or $j = k$.

With this symmetric form of the above axioms, we may reverse the order of $0, \ldots, n$ and thus show that $\mathsf{s}_{ij} \colon X_{ij} \twoheadrightarrow X_j$ is a principal G_i-bundle.

Proposition 7.25 shows that the maps in (5) and (6) are automatically isomorphisms if $0 \leq i < j < k \leq n$ if this is assumed for $i = j$ or $j = k$. So for the quasi-category of bibundle equivalences, we may require (5) and (6) for all $0 \leq i \leq j \leq k \leq n$. This stronger condition is less technical and provides a very satisfactory description of the quasi-category of groupoids in $(\mathcal{C}, \mathcal{T})$ with bibundle equivalences. Since bibundle equivalences are invertible up to 2-arrows, this is a weak 2-groupoid. Thus the corresponding quasi-category also satisfies the outer Kan conditions $\mathrm{Kan}(2, j)$ for $j = 0, 2$ and $\mathrm{Kan}!(n, j)$ for $n \geq 3$, $j = 0, n$.

9. Examples of categories with pretopology

In this section, we discuss pretopologies on different categories and check whether they satisfy our extra assumptions. We also describe basic actions in each case. We begin with a trivial pretopology on an arbitrary category.

9.1. EXAMPLE. Let $\mathcal{C}$ be any category and let $\mathcal{T}$ be the class of all isomorphisms in $\mathcal{C}$. This is a subcanonical pretopology satisfying Assumptions 2.8 and 2.9. All groupoids in $(\mathcal{C}, \mathcal{T})$ are 0-groupoids by Example 3.11, so there are no interesting examples of groupoids. Assumptions 7.2 and 7.3 are trivial. Assumption 2.10 on final objects usually fails.

In this section, unlike the previous ones, *elementwise statements are meant naively, with elements of sets in the usual sense.* Since all our examples are concrete categories, these naive elements suffice.

9.2. SETS AND SURJECTIONS. Let Sets be the category of sets.

9.3. PROPOSITION. *The class $\mathcal{T}_{\mathrm{surj}}$ of surjective maps in Sets is a subcanonical, saturated pretopology satisfying Assumptions 2.8, 2.9, 7.2, and 7.3. The subcategory of non-empty sets satisfies the same assumptions and also Assumption 2.10.*

Thus all the theory developed above applies to groupoids in $(\mathsf{Sets}, \mathcal{T}_{\mathrm{surj}})$. We are going to prove this and characterise basic actions in $(\mathsf{Sets}, \mathcal{T}_{\mathrm{surj}})$.

The category Sets is complete and therefore closed under fibre products. In the fibre-product situation (2), if $g \colon U \twoheadrightarrow X$ is surjective, then so is the induced map $\mathsf{pr}_1 \colon Y \times_{f,X,g} U \twoheadrightarrow Y$. The isomorphisms in Sets are the bijections and belong to $\mathcal{T}_{\mathrm{surj}}$. Let $f_1 \colon X \to Y$

and $f_2 \colon Y \to Z$ be composable maps. If f_1 and f_2 are surjective, then so is $f_2 \circ f_1$. Thus surjective maps in Sets form a pretopology.

If $f_2 \circ f_1$ is surjective, then so is f_2. Hence the pretopology $\mathcal{T}_{\mathrm{surj}}$ is saturated, and Assumptions 2.9 and 2.8 follow. A map in Sets is a coequaliser if and only if it is surjective. Thus the pretopology $\mathcal{T}_{\mathrm{surj}}$ is subcanonical, and it is the largest subcanonical pretopology on Sets. Any set with a single element is a final object, and any map to it from a non-empty set is surjective, hence a cover. Hence Assumption 2.10 holds for non-empty sets, but not if we include the empty set. Since a fibre-product of non-empty sets along a surjective map is never empty, surjections still form a pretopology on the subcategory of non-empty sets, and it still satisfies the same assumptions. Since Assumption 2.10 does not play an important role for our theory, it is a matter of taste whether one should remove the empty set or not.

A groupoid in $(\mathsf{Sets}, \mathcal{T}_{\mathrm{surj}})$ is just a groupoid in the usual sense; the surjectivity of the range and source maps is no condition because the unit map is a one-sided inverse. Since Sets has arbitrary colimits, any G-action X has an orbit set X/G (Definition 5.2). The canonical map $\mathsf{X} \twoheadrightarrow \mathsf{X}/\mathsf{G}$ is always surjective.

9.4. DEFINITION. *A groupoid action in* Sets *is* free *if for each* $x \in \mathsf{X}$, *the only* $g \in \mathsf{G}^1$ *with* $\mathsf{s}(x) = \mathsf{r}(g)$ *and* $x \cdot g = x$ *is* $g = 1_{\mathsf{s}(x)}$.

9.5. PROPOSITION. *A groupoid action in* $(\mathsf{Sets}, \mathcal{T}_{\mathrm{surj}})$ *is basic if and only if it is free.*

PROOF. Let G be a groupoid and X a G-action in $(\mathsf{Sets}, \mathcal{T}_{\mathrm{surj}})$. The canonical map $\mathsf{p} \colon \mathsf{X} \twoheadrightarrow \mathsf{X}/\mathsf{G}$ is a cover. We have $\mathsf{p}(x_1) = \mathsf{p}(x_2)$ for $x_1, x_2 \in \mathsf{X}$ if and only if there is $g \in \mathsf{G}^1$ with $\mathsf{s}(x_1) = \mathsf{r}(g)$ and $x_1 \cdot g = x_2$. Thus the map $\mathsf{X} \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1 \to \mathsf{X} \times_{\mathsf{p}, \mathsf{X}/\mathsf{G}, \mathsf{p}} \mathsf{X}$ is surjective. It is injective if and only if the action is free. ∎

Hence a groupoid G in $(\mathsf{Sets}, \mathcal{T}_{\mathrm{surj}})$ is basic if and only if all $g \in \mathsf{G}^1$ with $\mathsf{s}(g) = \mathsf{r}(g)$ are units. Equivalently, G is an equivalence relation on G^0.

We may now finish the proof of Proposition 9.3 by showing that any action of a Čech groupoid in Sets is basic. Let G be a Čech groupoid and let X be a G-action. If $x \in \mathsf{X}$, $g \in \mathsf{G}^1$ satisfy $\mathsf{s}(x) = \mathsf{r}(g)$ and $x \cdot g = x$, then $\mathsf{s}(g) = \mathsf{s}(x \cdot g) = \mathsf{s}(x) = \mathsf{r}(g)$. Then $g = 1_{\mathsf{s}(x)}$ because G is a Čech groupoid. Thus the G-action X is free; it is basic by Proposition 9.5. This verifies Assumptions 7.2 and 7.3 for $(\mathsf{Sets}, \mathcal{T}_{\mathrm{surj}})$.

9.6. PRETOPOLOGIES ON THE CATEGORY OF TOPOLOGICAL SPACES. Let Top be the category of topological spaces and continuous maps. This category is complete and co-complete. In particular, all fibre products exist and any groupoid action has an orbit space. There are several classes of maps in Top that give candidates for pretopologies:

(1) quotient maps

(2) biquotient maps (also called limit lifting maps)

(3) maps with global continuous sections

(4) maps with local continuous sections

(5) maps with local continuous sections and partitions of unity

(6) closed surjections

(7) proper surjections

(8) open surjections

(9) surjections with many continuous local sections

(10) étale surjections

We shall see that quotient maps and closed surjections do not form pretopologies. The other classes of maps all give subcanonical pretopologies that satisfy Assumptions 2.8 and 2.9. Assumption 2.10 about final objects fails for proper maps and étale maps and holds for the remaining pretopologies, if we remove the empty topological space. The pretopologies (2)–(5) are saturated, (7)–(10) are not saturated. Assumption 7.2 and hence also Assumption 7.3 hold in cases (3)–(5) and (8)–(10). Assumption 7.2 fails for biquotient maps; we do not know whether Assumption 7.3 holds for biquotient maps.

QUOTIENT MAPS.

9.7. LEMMA. *A continuous map $f\colon X \to Y$ is a coequaliser in* Top *if and only if it is a quotient map, that is, f is surjective and $A \subseteq Y$ is open if and only if $f^{-1}(A)$ is open.*

PROOF. Let $g_1, g_2\colon Z \rightrightarrows X$ be two maps. Let $\sim$ be the equivalence relation on X generated by $g_1(z) \sim g_2(z)$ for all $z \in Z$. Equip $X/\sim$ with the quotient topology. The coequaliser of g_1, g_2 is the canonical map $X \to X/\sim$. Thus coequalisers are quotient maps. Conversely, let $f\colon X \to Y$ be a quotient map and define an equivalence relation $\sim$ on X by $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$. Since f is a quotient map, the induced map $X/\sim \to Y$ is a homeomorphism. Viewed as a subset of $X \times X$, we have $\sim\; = X \times_{f,Y,f} X$. Thus f is a coequaliser of $\mathsf{pr}_1, \mathsf{pr}_2\colon X \times_{f,Y,f} X \rightrightarrows X$. ∎

There are a quotient map $f\colon X \to Y$ and a topological space Z such that $f \times \mathrm{id}_Z$ is no longer a quotient map, see [30, Example 8.4]. Thus the pull-back of f along the quotient map $\mathsf{pr}_1\colon Y \times Z \to Y$ is not a quotient map any more. This shows that quotient maps do *not* form a pretopology on Top. The spaces in [30, Example 8.4] are Hausdorff spaces, so it does not help to restrict from Top to the full subcategory Haus-Top of Hausdorff spaces.

BIQUOTIENT MAPS.

9.8. DEFINITION. [30] *Let $f\colon X \to Y$ be a continuous surjection. It is a* biquotient *map if for every $y \in Y$ and every open covering $\mathcal{U}$ of $f^{-1}(y)$ in X, there are finitely many $U \in \mathcal{U}$ for which the subsets $f(U)$ cover some neighbourhood of y in Y.*

9.9. DEFINITION. [[19, 20]] *The map f is* limit lifting *if every convergent net in Y lifts to a convergent net in X. More precisely, let $(I, \leq)$ be a directed set and let $(y_i)_{i \in I}$ be a net in Y converging to some $y \in Y$. A* lifting *of this convergent net is a directed set $(J, \leq)$ with a surjective order-preserving map $\varphi \colon J \to I$ and a net $(x_j)_{j \in J}$ in X with $f(x_j) = y_{\varphi(j)}$ for all $j \in J$, converging to some $x \in X$ with $f(x) = y$.*

9.10. PROPOSITION. [30] *Biquotient maps are the same as limit lifting maps.*

We will use both characterisations of these maps where convenient.

9.11. PROPOSITION. *The biquotient maps $\mathcal{T}_{\mathrm{biqu}}$ form a subcanonical, saturated pretopology that satisfies Assumptions 2.8, 2.9, but not Assumption 7.2. It satisfies Assumption 2.10 if we remove the empty space from the category.*

PROOF. It is clear that isomorphisms are limit lifting. Let $f_1 \colon X \to Y$ and $f_2 \colon Y \to Z$ be composable maps. If f_1 and f_2 are limit lifting, then so is $f_2 \circ f_1$ by lifting in two steps; and if $f_2 \circ f_1$ is limit lifting, then so is f_2 by first lifting a convergent net along $f_2 \circ f_1$ and then applying the continuous map f_1. Thus we have a saturated pretopology, and Assumptions 2.8 and 2.9 follow.

Next we claim that pull-backs inherit the property of being limit lifting. In the fibre-product situation (2), assume that $g \colon U \twoheadrightarrow X$ is limit lifting. Let $(y_i)_{i \in I}$ be a net in Y that converges to some $y \in Y$. Then $f(y_i)$ converges to $f(y)$ in X. Since g is limit lifting, there is an order-preserving surjection $\varphi \colon J \to I$ and a net $(u_j)_{j \in J}$ in U with $g(u_j) = f(y_{\varphi(j)})$ for all $j \in J$, converging to some $u \in U$ with $g(u) = f(y)$. Then $(y_{\varphi(j)}, u_j)$ is a net in $Y \times_{f,X,g} U$ that converges towards (y, u). Thus $\mathsf{pr}_1 \colon Y \times_{f,X,g} U \twoheadrightarrow Y$ is limit lifting. This completes the proof that $\mathcal{T}_{\mathrm{biqu}}$ is a pretopology. Limit lifting maps are quotient maps, hence coequalisers, so this pretopology is subcanonical.

A space with a single element is a final object in Top, and any map from a non-empty space to it is limit lifting. Thus Assumption 2.10 holds if we exclude the empty space. The following counterexample to Assumption 7.2 will finish the proof of Proposition 9.11. ∎

9.12. EXAMPLE. We first recall [30, Example 8.4]. Let X be the disjoint union of $X_1 = (0, 1)$ and $X_2 = \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. Let τ_1 be the obvious topology on X: X_1 and X_2 are open and carry the subspace topologies from $\mathbb{R}$. We will later introduce another topology τ_2 on X.

Let Y be the quotient space of X by the relation that identifies $\frac{1}{n}$ in X_1 and X_2, and let $f \colon X \to Y$ be the quotient map. This is a quotient map that is not limit lifting. As a set, $Y = [0, 1)$. The topology on Y is the usual one on the subset $(0, 1)$; a subset U of Y with $0 \in U$ is a neighbourhood of 0 if and only if there are $n_0 \in \mathbb{N}$ and $\epsilon_n > 0$ for $n \geq n_0$ with $\bigcup_{n \geq n_0}(\frac{1}{n} - \epsilon_n, \frac{1}{n} + \epsilon_n) \subseteq U$.

We now define another topology τ_2 on X for which the map $f \colon (X, \tau_2) \to Y$ is limit lifting. The subset $X_1 \sqcup X_2 \setminus \{0\}$ is open in both topologies τ_1 and τ_2, and both restrict to the same topology on $X_1 \sqcup X_2 \setminus \{0\}$. A subset $U \subseteq X$ is a τ_2-neighbourhood of 0 if $U \cap X_2$ is a neighbourhood of 0 and there are $n_0 \in \mathbb{N}$ and $\epsilon_n > 0$ for all $n \geq n_0$ such that $\bigcup_{n \geq n_0}\left(\frac{1}{n} - \epsilon_n, \frac{1}{n} + \epsilon_n\right) \setminus \left\{\frac{1}{n}\right\} \subseteq U$. This uniquely determines the topology τ_2.

On the open subset $(0,1) \subseteq Y$, the inclusion of X_1 into X gives a continuous section for $f|_{f^{-1}(0,1)}$; hence a net in Y that converges to some element of $(0,1)$ lifts to a net in $X_1 \subseteq X$ that converges in both (X, τ_1) and (X, τ_2). Let now $(y_i)_{i \in I}$ be a net in Y converging to 0. We lift it to a net in X by lifting y_i to $y_i \in X_2$ if $y_i \in X_2$, and to $y_i \in X_1$ if $y_i \notin X_2$. This net in X converges to $0 \in X_2$ in the topology τ_2.

The topology τ_1 is finer than τ_2, that is, the identity map on X is a continuous map $(X, \tau_1) \to (X, \tau_2)$. We claim that the identity map

$$(X, \tau_1) \times_{f,Y,f} (X, \tau_1) \to (X, \tau_1) \times_{f,Y,f} (X, \tau_2) \tag{25}$$

is a homeomorphism. That is, the topologies τ_1 and τ_2 pull back to the same topology on $(X, \tau_1) \times_{f,Y,f} X$ along the quotient map $f \colon (X, \tau_1) \to Y$. This also shows that the locality of isomorphisms, Proposition 2.5, fails for quotient maps in Top; this is okay because they do not form a pretopology.

To prove that the map in (25) is a homeomorphism, we show that any net in $X \times_{f,Y,f} X \subseteq X \times X$ that converges for $\tau_1 \times \tau_2$ also converges for the topology $\tau_1 \times \tau_1$. A net in $X \times_Y X$ is a pair of nets (x_{1i}) and (x_{2i}) with $f(x_{1i}) = f(x_{2i})$. We assume that (x_{1i}, x_{2i}) is $\tau_1 \times \tau_2$-convergent to (x_1, x_2). Equivalently, x_{1i} has τ_1-limit x_1 and (x_{2i}) has τ_2-limit x_2. We must show that x_{2i} also converges to x_2 in the topology τ_1. This is clear if $x_2 \neq 0 \in X_2$ because the topologies τ_1 and τ_2 agree away from 0. Thus we may assume $x_1 = x_2 = 0$. Since X_2 is τ_1-open, we have $x_{1i} \in X_2$ for almost all i; hence also $f(x_{2i}) = f(x_{1i}) \in X_2$. Since the points $\frac{1}{n}$ in X_1 are excluded from the τ_2-neighbourhoods of 0 and x_{2i} converges to 0 in τ_2, this implies $x_{2i} \in X_2$ for almost all i as well. Since $f|_{X_2}$ is injective, we get $x_{1i} = x_{2i}$ for almost all i, so (x_{2i}) converges in τ_1.

Now let G be the Čech groupoid of the limit lifting map $(X, \tau_2) \to Y$. The action of G on its orbit space (X, τ_2) is basic in $(\mathsf{Top}, \mathcal{T}_{\mathrm{biqu}})$ by Example 5.6. The same action on (X, τ_1) is still a continuous action: the anchor map $(X, \tau_1) \to (X, \tau_2)$ is continuous, and the action map

$$(X, \tau_1) \times_{f,X,f} (X, \tau_2) \cong (X, \tau_1) \times_X \left((X, \tau_2) \times_{f,X,f} (X, \tau_2) \right) \xrightarrow{\mathsf{pr}_3} (X, \tau_1)$$

is continuous because of the homeomorphism (25). This action is, however, not basic because the orbit space projection $(X, \tau_1) \to (X, \tau_1)/\mathsf{G} \cong Y$ is not in $\mathcal{T}_{\mathrm{biqu}}$. If this action were basic, it would contradict Proposition 5.9 because the G-actions on (X, τ_1) and (X, τ_2) have the same orbit space Y.

We do not know whether Assumption 7.3 holds for $(\mathsf{Top}, \mathcal{T}_{\mathrm{biqu}})$. A counterexample would have to be of different nature because a continuous bijection that is also a (bi)quotient map is already a homeomorphism.

9.13. PROPOSITION. *The pretopology of biquotient maps is the largest subcanonical pretopology on* Top.

PROOF. We claim that a continuous map $f\colon X \to Y$ is biquotient if and only if $\mathsf{pr}_1\colon Z\times_{g,Y,f} X \to Z$ is a quotient map for any map $g\colon Z \to Y$. We have already seen that biquotient maps form a pretopology; in particular, if f is biquotient, then $\mathsf{pr}_1\colon Z \times_{g,Y,f} X \to Z$ is biquotient and thus quotient. The converse remains to be proved. It implies the proposition because the covers of a subcanonical pretopology on Top must be quotient maps by Lemma 9.7, hence biquotient by our claim.

Assume that pr_1 is a quotient map for any map $g\colon Z \to Y$, although f is not biquotient; we want to arrive at a contradiction. By assumption, there are $y_\infty \in Y$ and an open covering $\mathcal{U}$ of $f^{-1}(y_\infty)$ such that for any finite set $F \subseteq \mathcal{U}$, $\bigcup_{U\in F} f(U)$ is not a neighbourhood of y_∞. Let I be the set of pairs (F,V) for a finite subset $F \subseteq \mathcal{U}$ and an open neighbourhood V of y_∞ in Y. We order I by $(F_1, V_1) \leq (F_2, V_2)$ if $F_1 \subseteq F_2$ and $V_1 \supseteq V_2$; this gives a directed set. By assumption, $V \setminus \bigcup_{U\in F} f(U) \neq \emptyset$ for all $(F, V) \in I$; we choose $y_{F,V}$ in this difference. Since $y_{F,V} \in V$, $\lim y_{F,V} = y_\infty$.

Let $I^+ := I \sqcup \{\infty\}$, topologised so that any subset of I is open and a subset W containing ∞ is open if and only if there is $i \in I$ with $j \in W$ for all $j \geq i$. Let

$$Z := \{(i,y) \in I^+ \times Y \mid y = y_i\}$$

be the graph of the function $I^+ \ni i \mapsto y_i \in Y$, equipped with the subspace topology; let $g\colon Z \to Y$ be the second coordinate projection. We identify

$$Z \times_{g,Y,f} X \cong \{(i,x) \in I^+ \times X \mid f(x) = y_i\}, \qquad (i,y,x) \mapsto (i,x),$$

for $(i,y) \in Z$, $x \in X$ with $f(x) = g(i,y) = y$. This bijection is a homeomorphism for the subspace topologies from $Z \times X \subseteq I^+ \times Y \times X$ and $I^+ \times X$.

The subset $S := Z \setminus \{(\infty, y_\infty)\}$ in Z is not closed because $\lim_{i\in I}(i, y_i) = (\infty, y_\infty)$. Since the coordinate projection $Z \times_{g,Y,f} X \to Z$ is assumed to be a quotient map, the preimage $\hat{S}$ of S is not closed in $Z \times_{g,Y,f} X$. Hence there is a net $(i_j, x_j)_{j\in J}$ in $\hat{S}$ that converges in $Z \times_{g,Y,f} X$ towards some point in the complement of $\hat{S}$, that is, of the form (∞, x_∞) with $x_\infty \in X$ such that $f(x_\infty) = y_\infty$. That is, $\lim i_j = \infty$ in I^+ and $\lim x_j = x_\infty$ in X. We have $f(x_j) = y_{i_j}$ because $(i_j, x_j) \in Z \times_{g,Y,f} X$.

Since $\mathcal{U}$ is an open covering of $f^{-1}(y_\infty)$ and $f(x_\infty) = y_\infty$, there is $U \in \mathcal{U}$ with $x_\infty \in U$. Then also $x_j \in U$ for sufficiently large $j \in J$. Since $i_j \to \infty$, there is $j_0 \in J$ with $i_j \geq (\{U\}, Y)$ for all $j \geq j_0$; thus $y_{i_j} \notin f(U)$ for $j \geq j_0$. Then $x_j \notin U$ for $j \geq j_0$, a contradiction. Hence the coordinate projection $Z \times_{g,Y,f} X \to Z$ cannot be a quotient map. ∎

9.14. REMARK. There are several other notions of improved quotient maps, such as triquotient maps (see [15, 32]). Triquotient maps also form a subcanonical pretopology satisfying Assumptions 2.8 and 2.9. The biquotient map $(X, \tau_2) \to Y$ is a triquotient map as well, however. Thus the same counterexample as above shows that Assumption 7.2 fails for the pretopology of triquotient maps. The surjective open maps form the largest pretopology on Top for which we know Assumption 7.2.

Now we consider groupoids in $(\mathsf{Top}, \mathcal{T}_{\mathrm{biqu}})$. Any map with a continuous section is limit lifting. Since the unit map is a section for the range and source maps in a groupoid, the condition that r and s be limit lifting is redundant in the first definition of a groupoid. In the second definition of a groupoid, it is enough to assume that r and s are both quotient maps: this suffices to construct a continuous unit map, which then implies that r and s are limit lifting.

The category Top has arbitrary colimits, so X/G exists for any G-action X; it is the set of orbits X/G with the quotient topology.

9.15. PROPOSITION. *Let $\mathcal{T}$ be any subcanonical pretopology on Top. Let G be a groupoid and X a G-action in $(\mathsf{Top}, \mathcal{T})$.*

The G-action X is basic if and only if it satisfies the following conditions:

(1) *the G-action X is free;*

(2) *the map $\mathsf{X} \times_{\mathsf{p}, \mathsf{X}/\mathsf{G}, \mathsf{p}} \mathsf{X} \to \mathsf{G}^1$ that maps $x_1, x_2 \in \mathsf{X}$ in the same orbit to the unique $g \in \mathsf{G}^1$ with $\mathsf{s}(x_1) = \mathsf{r}(g)$ and $x_1 \cdot g = x_2$ is continuous;*

(3) *the orbit space projection $\mathsf{X} \to \mathsf{X}/\mathsf{G}$ is in $\mathcal{T}$.*

The first two conditions hold for any G-action of a Čech groupoid of a cover in $(\mathsf{Top}, \mathcal{T})$.

PROOF. By Lemma 5.3, the bundle projection for a basic action must be the orbit space projection $\mathsf{p} \colon \mathsf{X} \twoheadrightarrow \mathsf{X}/\mathsf{G}$. For a basic action, this must be a cover. The continuous map $\mathsf{X} \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1 \to \mathsf{X} \times_{\mathsf{p}, \mathsf{X}/\mathsf{G}, \mathsf{p}} \mathsf{X}$, $(x, g) \mapsto (x, x \cdot g)$, is surjective by the definition of X/G. It is bijective if and only if it is injective if and only if the action is free (see Proposition 9.5). In this case, the inverse map sends $(x_1, x_2) \in \mathsf{X}^2$ with $\mathsf{p}(x_1) = \mathsf{p}(x_2)$ to (x_1, g) for the unique $g \in \mathsf{G}^1$ with $\mathsf{s}(x_1) = \mathsf{r}(g)$ and $x_1 \cdot g = x_2$. This inverse map is continuous if and only if the map $(x_1, x_2) \mapsto g$ in (2) is continuous. Hence the action is basic if and only if (1)–(3) hold.

Let G be the Čech groupoid of a cover $f \colon \mathsf{Y} \to \mathsf{Z}$. Let $(x_1, x_2) \in \mathsf{X}$ be in the same G-orbit. Then any $g \in \mathsf{G}^1$ with $\mathsf{s}(x_1) = \mathsf{r}(g)$ and $x_1 \cdot g = x_2$ will satisfy $\mathsf{s}(g) = \mathsf{s}(x_1 \cdot g) = \mathsf{s}(x_2)$, hence $g = (\mathsf{s}(x_1), \mathsf{s}(x_2))$. This is unique and depends continuously on x_1, x_2, giving (1) and (2) above. ∎

Hence the only way that Čech groupoid actions in $(\mathsf{Top}, \mathcal{T})$ may fail to be basic is by the orbit space projection not being a cover.

9.16. EXAMPLE. We construct a free groupoid action in $(\mathsf{Top}, \mathcal{T}_{\mathrm{biqu}})$ that satisfies (3) in Proposition 9.15, but not (2). Let $\mathbb{R}_d$ be $\mathbb{R}$ with the discrete topology. The translation action of $\mathbb{R}_d$ on $\mathbb{R}$ with the usual topology is free. Its orbit space has only one point, so the map $\mathbb{R} \to \mathbb{R}/\mathbb{R}_d$ is a cover. Condition (2) is violated, so the action is not basic.

The transformation groupoid of the non-basic action in Example 9.12 satisfies (1) and (2) in Proposition 9.15, but not (3).

HAUSDORFF ORBIT SPACES. We now replace Top by its full subcategory Haus-Top of Hausdorff topological spaces. The proof of Proposition 9.13 still works in this subcategory, showing that the biquotient maps form the largest subcanonical pretopology on Haus-Top. A groupoid in $(\mathsf{Haus\text{-}Top}, \mathcal{T}_{\mathrm{biqu}})$ is the same as a groupoid in $(\mathsf{Top}, \mathcal{T}_{\mathrm{biqu}})$ with Hausdorff spaces G^0 and G^1. A G-action for a Hausdorff groupoid on a Hausdorff space is basic in $(\mathsf{Haus\text{-}Top}, \mathcal{T}_{\mathrm{biqu}})$ if and only if it is basic in $(\mathsf{Top}, \mathcal{T}_{\mathrm{biqu}})$ *and the orbit space is Hausdorff.* This is not automatic:

9.17. EXAMPLE. Let $X := [0,1] \sqcup [0,1]$ and let Y be the quotient of X by the relation that identifies the two copies of $(0,1]$. This is a non-Hausdorff, locally Hausdorff space. The quotient map $X \to Y$ is open and hence biquotient. Therefore, its Čech groupoid is basic in $(\mathsf{Top}, \mathcal{T}_{\mathrm{biqu}})$ (Example 5.6). This Čech groupoid is also a groupoid in $(\mathsf{Haus\text{-}Top}, \mathcal{T}_{\mathrm{biqu}})$ because its object space $\mathsf{G}^0 = X$ and arrow space $\mathsf{G}^1 = X \times_Y X \subseteq X \times X$ are both Hausdorff. Since its orbit space is non-Hausdorff, it is *not* basic in $(\mathsf{Haus\text{-}Top}, \mathcal{T}_{\mathrm{biqu}})$.

9.18. PROPOSITION. *Let $f \colon X \to Y$ be a biquotient map. The space Y is Hausdorff if and only if the subset $X \times_{f,Y,f} X \subseteq X \times X$ is closed.*

PROOF. If Y is Hausdorff and $f \colon X \to Y$ is continuous, then

$$X \times_{f,Y,f} X = \{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\}$$

is closed: if $(x_1, x_2) \in X \times X$ with $f(x_1) \neq f(x_2)$, then there are neighbourhoods U_i of $f(x_i)$ in Y for $i = 1, 2$ with $U_1 \cap U_2 = \emptyset$ because Y is Hausdorff; then $f^{-1}(U_1) \times f^{-1}(U_2)$ is a neighbourhood of (x_1, x_2) in $X \times X$ that does not intersect $X \times_{f,Y,f} X$.

It remains to show that Y is Hausdorff if $X \times_{f,Y,f} X$ is closed in $X \times X$ and f is biquotient. We choose $y, y' \in Y$ with $y \neq y'$ and try to separate them by neighbourhoods. If $x, x' \in X$ satisfy $f(x) = y$ and $f(x') = y'$, then $(x, x') \notin X \times_{f,Y,f} X$, so there are open neighbourhoods $U_{x,x'}$ and $U'_{x,x'}$ of x and x', respectively, with $U_{x,x'} \times U'_{x,x'} \cap X \times_{f,Y,f} X = \emptyset$; that is, $f(U_{x,x'}) \cap f(U'_{x,x'}) = \emptyset$. Choose such neighbourhoods for all $(x, x') \in f^{-1}(y) \times f^{-1}(y')$. For fixed x, the open sets $U'_{x,x'}$ for $x' \in f^{-1}(y')$ cover $f^{-1}(y')$. Since f is biquotient, there is a finite set $A_x \subseteq f^{-1}(y')$ such that $V_x := \bigcup_{x' \in A_x} f(U'_{x,x'})$ is a neighbourhood of y' in Y.

Let $U_x := \bigcap_{x' \in A_x} U_{x,x'}$. This is an open neighbourhood of x because A_x is finite. We have $f(U_x) \cap f(U'_{x,x'}) = \emptyset$ for all $x' \in A_x$ and hence $f(U_x) \cap V_x = \emptyset$. Since f is biquotient and the open subsets U_x cover $f^{-1}(y)$, there is a finite subset $B \subseteq f^{-1}(y)$ such that $\bigcup_{x \in B} f(U_x)$ is a neighbourhood of y in Y. The finite intersection $\bigcap_{x \in B} V_x$ is a neighbourhood of y' in Y. These two neighbourhoods separate y and y'. ■

9.19. COROLLARY. *A topological space X is Hausdorff if and only if the diagonal is a closed subset in $X \times X$.*

PROOF. Apply Proposition 9.18 to the identity map. ■

The following definitions are needed to characterise when the orbit space of a basic action in $(\mathsf{Top}, \mathcal{T}_{\text{biqu}})$ is Hausdorff.

9.20. DEFINITION. [4, I.10.1], "application propre" *A map $f\colon X \to Y$ of topological spaces is* proper *if $f \times \mathrm{id}_Z\colon X \times Z \to Y \times Z$ is closed for any topological space Z.*

9.21. DEFINITION. *An action of a topological groupoid G on a topological space X is* proper *if the map*

$$\mathsf{X} \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \to \mathsf{X} \times \mathsf{X}, \qquad (x,g) \mapsto (x, x \cdot g), \tag{26}$$

is proper. *A groupoid G is* proper *if its canonical action on G^0 is proper, that is, the map $(\mathsf{r},\mathsf{s})\colon \mathsf{G}^1 \to \mathsf{G}^0 \times \mathsf{G}^0$ is proper.*

A map from a Hausdorff space X to a Hausdorff, locally compact space Y is proper if and only if preimages of compact subsets are compact ([4, Proposition 7 in I.10.3]).

9.22. PROPOSITION. *Let $\mathcal{T}$ be any subcanonical pretopology on $\mathsf{Haus\text{-}Top}$. Let G be a groupoid and X a G-action in $(\mathsf{Haus\text{-}Top}, \mathcal{T})$. The G-action X is basic in $(\mathsf{Haus\text{-}Top}, \mathcal{T})$ if and only if it satisfies the following conditions:*

(1) *the G-action X is free;*

(2) *the G-action X is proper;*

(3) *the orbit space projection $\mathsf{X} \to \mathsf{X}/\mathsf{G}$ belongs to $\mathcal{T}$.*

The first two conditions hold for any action of a Čech groupoid in $(\mathsf{Haus\text{-}Top}, \mathcal{T})$.

PROOF. Since $\mathcal{T}_{\text{biqu}}$ is the largest subcanonical pretopology on $\mathsf{Haus\text{-}Top}$, all covers in $\mathcal{T}$ are biquotient. Thus a basic G-action in $(\mathsf{Haus\text{-}Top}, \mathcal{T})$ is still basic in $(\mathsf{Top}, \mathcal{T}_{\text{biqu}})$; furthermore, its orbit space X/G is Hausdorff, and the orbit space projection $\mathsf{p}\colon \mathsf{X} \to \mathsf{X}/\mathsf{G}$ is in $\mathcal{T}$. Conversely, these three conditions imply that the action is basic in $(\mathsf{Haus\text{-}Top}, \mathcal{T})$.

Proposition 9.15 describes when the action is basic in $(\mathsf{Top}, \mathcal{T}_{\text{biqu}})$. An injective continuous map is proper if and only if it is closed, if and only if it is a homeomorphism onto a closed subset (see [4, I.10.1, Proposition 2]). Thus a free action is proper if and only if the bijection $\mathsf{X} \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \to \mathsf{X} \times_{\mathsf{p},\mathsf{X}/\mathsf{G},\mathsf{p}} \mathsf{X}$ is a homeomorphism and $\mathsf{X} \times_{\mathsf{p},\mathsf{X}/\mathsf{G},\mathsf{p}} \mathsf{X}$ is closed in $\mathsf{X} \times \mathsf{X}$. The first part of this is the second condition in Proposition 9.15, and the closedness of $\mathsf{X} \times_{\mathsf{p},\mathsf{G}^0,\mathsf{p}} \mathsf{X}$ in $\mathsf{X} \times \mathsf{X}$ is equivalent to X/G being Hausdorff by Proposition 9.18 because p is biquotient. Now it is routine to see that (1)–(3) above characterise basic actions in $(\mathsf{Haus\text{-}Top}, \mathcal{T})$.

The first two conditions in Proposition 9.15 are automatic for actions of Čech groupoids in $(\mathsf{Top}, \mathcal{T}_{\text{biqu}})$. It remains to show that $\mathsf{X} \times_{\mathsf{p},\mathsf{X}/\mathsf{G},\mathsf{p}} \mathsf{X}$ is closed in $\mathsf{X} \times \mathsf{X}$ for any action of a Čech groupoid in $(\mathsf{Haus\text{-}Top}, \mathcal{T}_{\text{biqu}})$. Let Y and Z be Hausdorff spaces and let $f\colon Y \twoheadrightarrow Z$ be a biquotient map with Čech groupoid G. Thus $Y = \mathsf{G}^0$ and the anchor map on X is a map $\mathsf{s}\colon \mathsf{X} \to Y$. Since Z is Hausdorff,

$$\mathsf{X} \times_Z \mathsf{X} = \{(x_1, x_2) \in \mathsf{X} \times \mathsf{X} \mid f(\mathsf{s}(x_1)) = f(\mathsf{s}(x_2))\}$$

is a closed subset of $\mathsf{X} \times \mathsf{X}$. If $(x_1, x_2) \in \mathsf{X} \times_Z \mathsf{X}$, then $(\mathsf{s}(x_1), \mathsf{s}(x_2)) \in \mathsf{G}^1$, so $x_2' := x_1 \cdot (\mathsf{s}(x_1), \mathsf{s}(x_2)) \in \mathsf{X}$ is defined. This is the unique element in the orbit of x_1 with $\mathsf{s}(x_2') = \mathsf{s}(x_2)$. Thus

$$\mathsf{X} \times_{\mathsf{X}/\mathsf{G}} \mathsf{X} = \{(x_1, x_2) \in \mathsf{X} \times_Z \mathsf{X} \mid x_1 \cdot (\mathsf{s}(x_1), \mathsf{s}(x_2)) = x_2\}.$$

Since X is Hausdorff and the two maps sending (x_1, x_2) to $x_1 \cdot (\mathsf{s}(x_1), \mathsf{s}(x_2))$ and x_2 are continuous, this subset is closed in $\mathsf{X} \times_Z \mathsf{X}$ and hence in $\mathsf{X} \times \mathsf{X}$. ∎

We may also restrict to other subcategories of topological spaces, still with biquotient maps as covers:

9.23. PROPOSITION. *Let $f\colon X \to Y$ be biquotient. The following properties are inherited by Y if X has them:*

(1) *locally quasi-compact;*

(2) *having a countable base;*

(3) *sequential: a subset A is closed if and only if it is closed under taking limits of convergent sequences in A;*

(4) *Fréchet: the closure of a subset A is the set of all points that are limits of convergent sequences in A;*

(5) *k-space: a subset is closed if and only if $A \cap K$ is relatively compact in K for any quasi-compact subset K.*

PROOF. That biquotient images inherit the first two properties is stated in [30, Proposition 3.4]. The remaining statements follow from [31], which characterises the images of various classes of "nice" spaces under five classes of "quotient maps." If a property P characterises the biquotient images of some particular class of "nice" spaces, then P is inherited by biquotient images because composites of biquotient maps are again biquotient. Composites of biquotient maps with quotient, hereditarily quotient, or countably biquotient maps are also again of the same sort. Hence all the properties in rows 3–6 of the table in [31, p. 93] are preserved under biquotient images. This includes the properties of being sequential, Fréchet, or a k-space given above, and many less familiar properties. ∎

The classes of spaces in the first row in the table in [31, p. 93] are, however, not closed under biquotient images. This includes the class of locally compact, paracompact spaces and the class of metrisable spaces. [31] shows that any locally compact space is a biquotient image of a locally compact paracompact space (it is a biquotient image of the disjoint union of its compact subsets), while any "bisequential" space is a biquotient image of a metrisable space.

Since orbit space projections of basic actions in $(\mathsf{Top}, \mathcal{T})$ are biquotient, the orbit space X/G of a basic action in $(\mathsf{Top}, \mathcal{T})$ will inherit the properties listed in Proposition 9.23 from X. Let $\mathcal{C} \subseteq \mathsf{Top}$ be the full subcategory described by this property, say, the full

subcategory of k-spaces, and let $\mathcal{T}$ be some pretopology on Top. It follows that a groupoid action in $(\mathcal{C}, \mathcal{T} \cap \mathcal{C})$ is basic if and only if it is basic in $(\text{Top}, \mathcal{T})$; the latter is characterised by Proposition 9.15. If we add Hausdorffness, then a groupoid action in $(\mathcal{C} \cap \text{Haus-Top}, \mathcal{T} \cap \mathcal{C} \cap \text{Haus-Top})$ is basic if and only if it is basic in $(\text{Haus-Top}, \mathcal{T} \cap \text{Haus-Top})$, which is characterised by Proposition 9.22.

What if we want to restrict to a subcategory that is not closed under biquotient images, say, the category of metrisable topological spaces? An action of a metrisable topological groupoid on a metrisable topological space is basic with respect to a pretopology $\mathcal{T}$ contained in $\mathcal{T}_{\mathrm{biqu}}$ if and only if it satisfies the conditions in Proposition 9.22 or 9.15 and, in addition, X/G is metrisable.

9.24. EXAMPLE. Let Y be bisequential and not metrisable. Let $f \colon X \twoheadrightarrow Y$ be a biquotient map with X metrisable. Then $X \times_{f,Y,f} X \subseteq X \times X$ is again metrisable. Hence the Čech groupoid of f is a groupoid in the category of metrisable topological spaces with biquotient maps as covers. It is not basic in the category of metrisable spaces, however, because its orbit space is not metrisable.

COVERS DEFINED BY CONTINUOUS SECTIONS. Let $f \colon X \to Y$ be a continuous map.

9.25. DEFINITION. *A* global continuous section *for f is a continuous map $\sigma \colon Y \to X$ with $f \circ \sigma = \mathrm{id}_Y$. Let $\mathcal{T}_{\mathrm{split}}$ be the class of all continuous maps with a global continuous section.*

A local continuous section *for f at $y \in Y$ is a pair (U, σ) consisting of a neighbourhood U of y and a continuous map $\sigma \colon U \to X$ with $f \circ \sigma = \mathrm{id}_U$. We call f* locally split *if such local continuous sections exist at all $y \in Y$. Let $\mathcal{T}_{\mathrm{loc.split}}$ be the class of all locally split continuous maps.*

We call f numerably split *if there is a* numerable *open covering $\mathcal{U}$ of Y with local sections defined on all elements of $\mathcal{U}$; an open covering is numerable if it has a subordinate partition of unity (see [16]). Let $\mathcal{T}_{\mathrm{num.split}}$ be the class of numerably split maps.*

By definition,
$$\mathcal{T}_{\mathrm{split}} \subseteq \mathcal{T}_{\mathrm{num.split}} \subseteq \mathcal{T}_{\mathrm{loc.split}}.$$

A local continuous section $\sigma \colon U \to X$ for $f \colon X \to Y$ at $y \in Y$ is determined by its image $S := \sigma(U) \subseteq X$. This is a subset on which f restricts to a homeomorphism onto a neighbourhood of y in Y, and any such subset gives a unique local continuous section for f near y. The subset S is also called a *slice*.

9.26. PROPOSITION. $\mathcal{T}_{\mathrm{split}}$, $\mathcal{T}_{\mathrm{num.split}}$ *and* $\mathcal{T}_{\mathrm{loc.split}}$ *are subcanonical, saturated pretopologies on* Top *and* Haus-Top *satisfying Assumptions 2.8, 2.9, 7.2, and 7.3. They satisfy 2.10 if we exclude the empty topological space.*

PROOF. Let $f \colon Y \to X$ be a continuous map and let $g \colon U \twoheadrightarrow X$ belong to $\mathcal{T}_{\mathrm{split}}$, $\mathcal{T}_{\mathrm{num.split}}$ or $\mathcal{T}_{\mathrm{loc.split}}$, respectively. Let $W \subseteq X$ be the domain of a local continuous section $\sigma \colon W \to U$ for g. Then $(\mathrm{id}, \sigma \circ f) \colon f^{-1}(W) \to Y \times_X U$ is a local continuous section for the coordinate projection $\mathsf{pr}_1 \colon Y \times_X U \twoheadrightarrow Y$. If $W = X$, then we get a global continuous section. If the

domains of such local sections cover X, then the subsets $f^{-1}(W)$ cover Y. If $(\psi_W)_{W \in \mathcal{U}}$ is a partition of unity on X subordinate to a cover by domains of local continuous sections, then $(\psi_W \circ f)_{W \in \mathcal{U}}$ provides such a partition of unity for the domains of the induced local continuous sections for $\mathsf{pr}_1 \colon Y \times_X U \twoheadrightarrow Y$. Thus the map pr_1 belongs to $\mathcal{T}_{\mathrm{split}}$, $\mathcal{T}_{\mathrm{num.split}}$ or $\mathcal{T}_{\mathrm{loc.split}}$, respectively, if g does.

It is clear that homeomorphisms are in $\mathcal{T}_{\mathrm{loc.split}}$ and hence in the other classes as well. If $f_2 \circ f_1$ is defined and belongs to $\mathcal{T}_{\mathrm{loc.split}}$, then we may get local sections for f_2 by composing local sections for $f_2 \circ f_1$ with the continuous map f_1. Since this does not change the domain, we may use the same partition of unity for f_2 as for $f_2 \circ f_1$ in the numerably split case. Hence all three pretopologies are saturated, which implies Assumptions 2.8 and 2.9. It is also clear that the constant map from any non-empty space to the one-point space has global continuous sections, which gives Assumption 2.10 for all three pretopologies if we exclude the empty space. A map with local continuous sections must be surjective, and we may use local continuous sections to lift convergent nets. Hence $\mathcal{T}_{\mathrm{loc.split}} \subseteq \mathcal{T}_{\mathrm{biqu}}$, so that the pretopologies are subcanonical.

Let $f \colon X \twoheadrightarrow Z$ be a cover in one of our three pretopologies, let G be its Čech groupoid, and let Y be a G-action. Let $\mathsf{p} \colon \mathsf{Y} \to \mathsf{Y}/\mathsf{G}$ be the orbit space projection. By Proposition 9.15, the G-action is basic if and only if p is a cover. The anchor map $\mathsf{s} \colon \mathsf{Y} \to \mathsf{X}$ of the G-action induces a continuous map $\mathsf{s}/\mathsf{G} \colon \mathsf{Y}/\mathsf{G} \to \mathsf{X}/\mathsf{G} = \mathsf{Z}$.

Let $\mathcal{U}$ be an open cover of Z by domains of local sections $\sigma_U \colon U \to \mathsf{X}$ for $U \in \mathcal{U}$; in the case of the pretopology $\mathcal{T}_{\mathrm{num.split}}$, we assume a partition of unity $(\psi_U)_{U \in \mathcal{U}}$ subordinate to this cover, and in the case of the pretopology $\mathcal{T}_{\mathrm{split}}$, we assume $\mathcal{U} = \{\mathsf{Z}\}$. The subsets $\mathsf{s}^{-1}(U)$ for $U \in \mathcal{U}$ form an open cover $\mathsf{s}^*(\mathcal{U})$ of Y/G because s/G is continuous; in the case of $\mathcal{T}_{\mathrm{num.split}}$, $(\psi_U \circ (\mathsf{s}/\mathsf{G}))_{U \in \mathcal{U}}$ is a partition of unity on Y/G subordinate to $\mathsf{s}^*(\mathcal{U})$, and in the case of $\mathcal{T}_{\mathrm{split}}$, $\mathsf{s}^*(\mathcal{U}) = \{\mathsf{Y}/\mathsf{G}\}$. It remains to construct local continuous sections $\sigma'_U \colon \mathsf{s}^{-1}(U) \to \mathsf{Y}$ for $U \in \mathcal{U}$.

Let $y \in \mathsf{Y}$ represent $[y] \in \mathsf{s}^{-1}(U) \subseteq \mathsf{Y}/\mathsf{G}$. Then $\mathsf{s}(y) \in f^{-1}(U) \subseteq \mathsf{X}$ and $g_y := (\mathsf{s}(y), \sigma_U(f(\mathsf{s}(y)))) \in \mathsf{G}^1$ has $\mathsf{r}(g_y) = \mathsf{s}(y)$, so that $y \cdot g_y$ is defined in Y. This is the unique element in the G-orbit of y with $\mathsf{s}(y \cdot g_y) \in \sigma_U(U)$. Hence $y \cdot g_y$ depends only on $[y] \in \mathsf{Y}/\mathsf{G}$, so that we get a continuous map $\sigma'_U \colon \mathsf{s}^{-1}(U) \to \mathsf{Y}$, $[y] \mapsto y \cdot g_y$. This is a continuous section for p on $\mathsf{s}^{-1}(U)$ as needed. ∎

Thus our whole theory applies to groupoids in Top for the three pretopologies above. The range and source maps of a groupoid are automatically in $\mathcal{T}_{\mathrm{split}}$ because the unit map provides a global continuous section. Hence our pretopologies do not restrict the class of topological groupoids. They give different notions of principal bundles, however, and thus different bicategories of bibundle functors, equivalences, and actors.

Let G be a topological groupoid (without condition on the range and source maps) and let X be a G-action that satisfies (1) and (2) in Proposition 9.15. Let $\mathsf{p} \colon \mathsf{X} \to \mathsf{X}/\mathsf{G}$ be its orbit space projection. This is a principal G-bundle for $\mathcal{T}$ if and only if $\mathsf{p} \in \mathcal{T}$. We now discuss what this means.

9.27. Lemma. *The map p is in $\mathcal{T}_{\mathrm{split}}$ if and only if X is isomorphic to a pull-back of the*

principal G-*bundle* $\mathsf{r}\colon \mathsf{G}^1 \twoheadrightarrow \mathsf{G}^0$ *along some map* $\mathsf{X}/\mathsf{G} \to \mathsf{G}^0$.

PROOF. By definition, $\mathsf{p} \in \mathcal{T}_{\mathrm{split}}$ if and only if it has a global continuous section $\sigma\colon \mathsf{X}/\mathsf{G} \to \mathsf{X}$. We claim that the map

$$\mathsf{X}/\mathsf{G} \times_{\mathsf{s}\circ\sigma,\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1 \to \mathsf{X}, \qquad ([x],g) \mapsto \sigma[x] \cdot g, \tag{27}$$

is a homeomorphism; it is clearly a continuous bijection, and the inverse map sends $y \in \mathsf{X}$ to $([y],g)$ where g is the unique element of G^1 with $\mathsf{s}(\sigma[y]) = \mathsf{r}(g)$ and $\sigma[y] \cdot g = y$; the map $y \mapsto ([y],g)$ is continuous by (2) in Proposition 9.15.

The homeomorphism (27) is G-equivariant for the G-action on $\mathsf{X}/\mathsf{G} \times_{\mathsf{s}\circ\sigma,\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1$ defined by $([x],g_1) \cdot g_2 = ([x], g_1 \cdot g_2)$. Hence the principal G-bundle $\mathsf{X} \twoheadrightarrow \mathsf{X}/\mathsf{G}$ is isomorphic to the pull-back of the principal G-bundle $\mathsf{r}\colon \mathsf{G}^1 \twoheadrightarrow \mathsf{G}^0$ along the continuous map $\mathsf{s}\circ\sigma\colon \mathsf{X}/\mathsf{G} \to \mathsf{G}^0$ (Proposition 5.8). Conversely, such pull-backs are principal G-bundles in $(\mathsf{Top}, \mathcal{T}_{\mathrm{split}})$ because $\mathsf{r}\colon \mathsf{G}^1 \twoheadrightarrow \mathsf{G}^0$ is a principal G-bundle in $(\mathsf{Top}, \mathcal{T}_{\mathrm{split}})$. ∎

Thus the principal G-bundles in $(\mathsf{Top}, \mathcal{T}_{\mathrm{split}})$ are precisely the *trivial* G-bundles in the following sense:

9.28. DEFINITION. *A* G-*bundle* $\mathsf{p}\colon \mathsf{X} \twoheadrightarrow \mathsf{Z}$ *is* trivial *if it is isomorphic to a pull-back of the* G-*bundle* $\mathsf{r}\colon \mathsf{G}^1 \twoheadrightarrow \mathsf{G}^0$ *along some map* $\mathsf{Z} \to \mathsf{G}^0$.

A G-*bundle* $\mathsf{p}\colon \mathsf{X} \twoheadrightarrow \mathsf{Z}$ *is* locally trivial *if there is an open covering* $\mathcal{U}$ *of* Z *such that the restriction* $\mathsf{p}|_U\colon \mathsf{p}^{-1}(U) \twoheadrightarrow U$ *is a trivial* G-*bundle for each* $U \in \mathcal{U}$.

If $\mathcal{T} = \mathcal{T}_{\mathrm{loc.split}}$, then there is an open covering $\mathcal{U}$ of X/G such that trivialisations as in (27) are defined on the G-invariant subsets $\mathsf{p}^{-1}(U)$ for $U \in \mathcal{U}$. Lemma 9.27 shows that the principal G-bundles in $(\mathsf{Top}, \mathcal{T}_{\mathrm{loc.split}})$ are precisely the *locally trivial* G-bundles.

Let $\mathsf{p}\colon \mathsf{X} \twoheadrightarrow \mathsf{Z}$ be a locally trivial bundle. We have an open covering $\mathcal{U}$ of Z and continuous maps $\varphi_U\colon U \to \mathsf{G}^0$ and local trivialisations $\tau_U\colon \mathsf{p}^{-1}(U) \to U \times_{\varphi_U,\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1$ for all $U \in \mathcal{U}$. These local trivialisations are not unique and therefore do not agree on intersections $U_1 \cap U_2$ for $U_1, U_2 \in \mathcal{U}$. Since any G-map between fibres of $\mathsf{r}\colon \mathsf{G}^1 \to \mathsf{G}^0$ is given by left multiplication with some $g \in \mathsf{G}^1$, we get continuous maps $g_{U_1,U_2}\colon U_1 \cap U_2 \to \mathsf{G}^1$ with $\varphi_{U_2}(z) = \mathsf{r}(g_{U_1,U_2}(z))$, $\varphi_{U_1}(z) = \mathsf{s}(g_{U_1,U_2}(z))$, and $\tau_{U_2} \circ \tau_{U_1}^{-1}(z,g) = (z, g_{U_1,U_2}(z) \cdot g)$ for all $(z,g) \in (U_1 \cap U_2) \times_{\varphi_{U_1},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1$. These maps g_{U_1,U_2} satisfy the usual cocycle conditions.

9.29. REMARK. If $\mathcal{T} = \mathcal{T}_{\mathrm{num.split}}$, then principal bundles are locally trivial and, in addition, the open covering by trivialisation charts is numerable. This suffices to construct a continuous *classifying map* $\varphi\colon \mathsf{X}/\mathsf{G} \to B\mathsf{G}$ such that $\mathsf{X} \twoheadrightarrow \mathsf{X}/\mathsf{G}$ is the pull-back of the universal principal bundle $E\mathsf{G} \twoheadrightarrow B\mathsf{G}$ along φ, even if the base space X/G is not paracompact. Classifying spaces for topological groupoids were introduced by Haefliger [18], following Segal [45]; Bracho [6] proves that G-principal bundles over locally compact topological spaces are classified by continuous maps to $B\mathsf{G}$. The local compactness assumption should not be needed here, as shown by the example of principal bundles over topological groups, but we have not found a proof of this in the literature.

If X is Hausdorff, then continuous local sections for the orbit space projection $X \to X/G$ imply that X/G is locally Hausdorff, that is, every point has a Hausdorff open neighbourhood; but it does not yet imply that X/G is Hausdorff. Indeed, Example 9.17 is also a basic groupoid in $(\mathsf{Top}, \mathcal{T}_{\mathrm{loc.split}})$.

CLOSED AND PROPER MAPS. A map between topological spaces is *closed* if it maps closed subsets to closed subsets. A map $f \colon X \to Y$ is proper if and only if it is closed and $f^{-1}(y) \subseteq X$ is quasi-compact for all $y \in Y$; this is shown in [4, I.10.2, Théorème 1]. In particular, there are closed maps that are not proper. Thus closed maps are not hereditary for pull-backs and do not form a pretopology.

9.30. PROPOSITION. *The proper maps form a subcanonical pretopology* $\mathcal{T}_{\mathrm{prop}}$ *on* Top. *It satisfies Assumptions* 2.8 *and* 2.9, *but not Assumption* 2.10.

PROOF. It is clear that homeomorphisms are proper and that composites of proper or closed maps are again proper or closed, respectively. Let $g \colon U \to X$ be proper and let $f \colon Y \to X$ be any map. Let Z be another topological space. Then $g_* := \mathrm{id}_Y \times g \times \mathrm{id}_Z \colon Y \times U \times Z \to Y \times X \times Z$ is closed. So is its restriction to any subset of the form $g_*^{-1}(S)$ for $S \subseteq Y \times X \times Z$. If S is the product of the graph of f and Z, then this restriction is the map $\mathsf{pr}_1 \times \mathrm{id}_Z \colon Y \times_{f,X,g} U \times Z \to Y \times Z$. Since this is closed for any Z, $\mathsf{pr}_1 \colon Y \times_{f,X,g} U \to Y$ is proper.

Proper surjections are biquotient maps by [30, Proposition 3.2]. Hence they are co-equalisers, so the pretopology of proper maps is subcanonical.

Let $f_1 \colon X \to Y$ and $f_2 \colon Y \to Z$ be continuous surjections. If $f_2 \circ f_1$ is closed, then so is f_2: for a closed subset $S \subseteq Y$, $f_2(S) = (f_2 \circ f_1) f_1^{-1}(S)$ because f_1 is surjective, and this is closed because f_1 is continuous and $f_2 \circ f_1$ is closed. Hence the proper maps satisfy Assumption 2.9 and thus Assumption 2.8. The map from a space X to the one-point space is proper if and only if X is quasi-compact. Hence Assumption 2.10 fails even if we exclude the empty space. ∎

For a groupoid in $(\mathsf{Top}, \mathcal{T}_{\mathrm{prop}})$, the source and range maps must be proper. This restriction is so strong that we seem to get a rather useless class of groupoids. In particular, proper topological groupoids need not be groupoids in $(\mathsf{Top}, \mathcal{T}_{\mathrm{prop}})$.

OPEN SURJECTIONS. This pretopology is used for topological groupoids in [42], and is implicitly used by most of the literature on groupoid C*-algebras for locally compact groupoids.

A map between topological spaces is *open* if it maps open subsets to open subsets. The following criterion for open maps is similar to but subtly different from the definition of limit lifting maps:

9.31. PROPOSITION. [46, Proposition 1.15] *A continuous surjection* $f \colon X \to Y$ *between topological spaces is open if and only if, for any* $x \in X$, *a convergent net* (y_i) *in* Y *with* $\lim y_i = f(x)$ *lifts to a net in* X *converging to* x.

9.32. Proposition. *The class $\mathcal{T}_{\text{open}}$ of surjective, open maps in* Top *is a subcanonical pretopology that satisfies Assumptions* 2.8, 2.9, 7.2 *and* 7.3, *but it is not saturated. It satisfies Assumption* 2.10 *if we exclude the empty space.*

The proof that $\mathcal{T}_{\text{open}}$ is a subcanonical pretopology satisfying Assumptions 2.8, 2.9 and 2.10 is similar to the proof for $\mathcal{T}_{\text{biqu}}$, using Proposition 9.31, and left to the reader. The following simple counterexample shows that it is not saturated:

9.33. Example. Let $f\colon X \to Y$ be a continuous map that is not open. Let $f_2 := (f, \text{id}_X)\colon X \sqcup Y \to Y$ and let $f_1\colon Y \to X \sqcup Y$ be the inclusion map. Then $f_2 \circ f_1$ is the identity map on Y, hence open. But f_2 is not open because f is not open. Hence the pretopology $\mathcal{T}_{\text{open}}$ is not saturated.

Groupoids in $(\mathsf{Top}, \mathcal{T}_{\text{open}})$ must have open range and source maps. This is no serious restriction for operator algebraists because they need Haar systems to construct groupoid C^*-algebras, and Haar systems cannot exist unless the range and source maps are open. The benefit of this assumption is that it forces the orbit space projections $\mathsf{X} \to \mathsf{X}/\mathsf{G}$ for all G-actions to be open:

9.34. Proposition. *Let* G *be a groupoid and* X *a* G-*action in* $(\mathsf{Top}, \mathcal{T}_{\text{open}})$. *The orbit space projection* $\mathsf{p}\colon \mathsf{X} \to \mathsf{X}/\mathsf{G}$ *is open. All actions of Čech groupoids in* $(\mathsf{Top}, \mathcal{T}_{\text{open}})$ *are basic.*

Proof. Let $U \subseteq \mathsf{X}$ be open. Then $\mathsf{p}^{-1}(\mathsf{p}(U))$ is the set of all $x \cdot g$ with $x \in U$, $g \in \mathsf{G}^1$, $\mathsf{s}(x) = \mathsf{r}(g)$. This is $\mathsf{m}\big((U \times \mathsf{G}^1) \cap (\mathsf{X} \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1)\big)$, which is open because $(U \times \mathsf{G}^1) \cap (\mathsf{X} \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1)$ is open in $\mathsf{X} \times_{\mathsf{s},\mathsf{G}^0,\mathsf{r}} \mathsf{G}^1$ and m is a cover (open surjection) by Proposition 3.9. Thus $\mathsf{p}(U)$ is open in X/G, and p is open. Proposition 9.15 characterises basic actions by three conditions, of which two are automatic for actions of Čech groupoids. The third condition is that p should be a cover, which *is* automatic for the pretopology $\mathcal{T}_{\text{open}}$. Thus all Čech groupoid actions in $(\mathsf{Top}, \mathcal{T}_{\text{open}})$ are basic, verifying Assumptions 7.2 and 7.3. ∎

This finishes the proof of Proposition 9.32.

9.35. Corollary. *Let* G *be a groupoid and* X *a* G-*action in* $(\mathsf{Top}, \mathcal{T}_{\text{open}})$. *The following are equivalent:*

(1) *the action is basic with Hausdorff base space* X/G;

(2) *the action is free and proper.*

Proof. This follows from the proof of Proposition 9.22, which still works even if G or X are non-Hausdorff. ∎

The proof of Proposition 9.22 shows that an action is free and proper if and only if it satisfies the first two conditions in Proposition 9.15 and $\mathsf{X} \times_{\mathsf{p},\mathsf{X}/\mathsf{G},\mathsf{p}} \mathsf{X}$ is closed in $\mathsf{X} \times \mathsf{X}$. This is exactly how Henri Cartan defines principal fibre bundles for a topological group G in [12, condition (FP) on page 6-05]:

> Un espace fibré principal E est un espace topologique E, où opère un groupe topologique G (appelé *groupe structural*), de manière que soit rempli l'axiome suivant :
>
> (FP) : le graphe C de la relation d'équivalence définie par G est une partie *fermée* de $E \times E$; pour chaque point $(x, y) \in C$ il existe un $s \in G$ et un seul tel que $x \cdot s = y$; en outre, en désignant par u l'application de C dans G ainsi définie, on suppose que u est une fonction *continue*.

Thus our notion of principal bundle in $(\mathsf{Top}, \mathcal{T}_{\mathrm{open}})$ is the same as Henri Cartan's. Palais [40] and later authors ([14, 23]) call a locally compact group(oid) action *Cartan* if it satisfies a condition that, for free actions, is equivalent to the continuity of the map $(x, x \cdot g) \mapsto g$, allowing non-Hausdorff orbit spaces. Cartan himself, however, requires the orbit space to be Hausdorff because he requires the orbit equivalence relation to be closed (Proposition 9.18).

COVERS WITH MANY LOCAL CONTINUOUS SECTIONS.

9.36. DEFINITION. *A continuous map $f \colon X \to Y$ has many local continuous sections if it is surjective and for all $x \in X$ there is an open neighbourhood $U \subseteq Y$ of $f(x)$ and a continuous map $\sigma \colon U \to X$ with $\sigma(f(x)) = x$ and $f(\sigma(y)) = y$ for all $y \in U$. Let $\mathcal{T}_{\mathrm{loc.sect.}}$ be the class of continuous maps with many local continuous sections.*

The difference between this pretopology and $\mathcal{T}_{\mathrm{loc.split}}$ is that we require local sections with a given $x \in X$ in the image. This forces the map f to be open, so $\mathcal{T}_{\mathrm{loc.sect.}} \subseteq \mathcal{T}_{\mathrm{open}}$, whereas $\mathcal{T}_{\mathrm{loc.split}} \nsubseteq \mathcal{T}_{\mathrm{open}}$.

9.37. PROPOSITION. *The class $\mathcal{T}_{\mathrm{loc.sect.}}$ is a subcanonical pretopology on Top that satisfies Assumptions 2.8, 2.9, 7.2, and 7.3, but is not saturated. It satisfies Assumption 2.10 if we exclude the empty space.*

PROOF. The proof is very similar to the proof for the pretopology $\mathcal{T}_{\mathrm{loc.split}}$ (Proposition 9.26) and therefore left to the reader. The idea of Example 9.33 also shows that $\mathcal{T}_{\mathrm{loc.sect.}}$ is not saturated. Since $\mathcal{T}_{\mathrm{loc.sect.}}$ is contained both in $\mathcal{T}_{\mathrm{loc.split}}$ and $\mathcal{T}_{\mathrm{open}}$ and both are subcanonical and have the property that actions of Čech groupoids are basic, Assumptions 7.2 and hence 7.3 follow from Proposition 7.6. ∎

The topological groupoids in $(\mathsf{Top}, \mathcal{T}_{\mathrm{loc.sect.}})$ are those topological groupoids for which r or, equivalently, s has many local continuous sections. This happens for both maps once it happens for one because of the continuous inversion on G^1.

9.38. PROPOSITION. *Let G be a groupoid and X a G-action in $(\mathsf{Top}, \mathcal{T}_{\mathrm{loc.sect.}})$. Then X is basic in $(\mathsf{Top}, \mathcal{T}_{\mathrm{loc.sect.}})$ if and only if it is basic in $(\mathsf{Top}, \mathcal{T}_{\mathrm{loc.split}})$.*

PROOF. Since $r\colon G^1 \twoheadrightarrow G^0$ is a principal G-bundle in $(\mathsf{Top}, \mathcal{T}_{\mathrm{loc.sect.}})$, so are its pull-backs, that is, all trivial bundles. Since the existence of many local continuous sections is a local condition, this extends to locally trivial G-bundles. Since the principal G-bundle in $(\mathsf{Top}, \mathcal{T}_{\mathrm{loc.split}})$ are precisely the locally trivial G-bundles, they are principal in $(\mathsf{Top}, \mathcal{T}_{\mathrm{loc.sect.}})$ as well. The same then holds for basic actions. ∎

Roughly speaking, if the map $r\colon G^1 \twoheadrightarrow G^0$ has many local continuous sections, then we may shift local continuous sections for the quotient map $X \to X/G$ so that a particular point in a fibre is in the image of the local section.

Proposition 9.38 implies that the bicategories of bibundle functors, equivalences, and actors for $(\mathsf{Top}, \mathcal{T}_{\mathrm{loc.sect.}})$ are full sub-bicategories of the corresponding bicategories for $(\mathsf{Top}, \mathcal{T}_{\mathrm{loc.split}})$; we only restrict from all topological groupoids to those where the range and source maps have many local continuous sections.

ÉTALE SURJECTIONS. A continuous map $f\colon X \to Y$ is called *étale* (or local homeomorphism) if for all $x \in X$ there is an open neighbourhood U such that $f(U)$ is open and $f|_U\colon U \to f(U)$ is a homeomorphism for the subspace topologies on U and $f(U)$ from X and Y, respectively. This implies that f is open. Let $\mathcal{T}_{\mathrm{ét}}$ be the class of étale surjections.

9.39. PROPOSITION. *The class $\mathcal{T}_{\mathrm{ét}}$ is a subcanonical pretopology on Top that satisfies Assumptions 2.8, 2.9, 7.2 and 7.3; it is not saturated and Assumption 2.10 fails.*

PROOF. The proof is very similar to that of Proposition 9.26 and therefore left to the reader. Unless X is discrete, the constant map from X to a point is not étale, so Assumption 2.10 fails for $\mathcal{T}_{\mathrm{ét}}$ even if we exclude the empty space. The idea of Example 9.33 shows that $\mathcal{T}_{\mathrm{ét}}$ is not saturated. ∎

The groupoids in $(\mathsf{Top}, \mathcal{T}_{\mathrm{ét}})$ are precisely the étale topological groupoids.

9.40. PROPOSITION. *An action of an étale topological groupoid is basic in $(\mathsf{Top}, \mathcal{T}_{\mathrm{ét}})$ if and only if it satisfies the first two conditions in Proposition 9.15. A groupoid action in $(\mathsf{Haus\text{-}Top}, \mathcal{T}_{\mathrm{ét}})$ is basic if and only if it is free and proper.*

PROOF. Let G be an étale topological groupoid and let X be a G-action in Top that satisfies the first two conditions in Proposition 9.15. We must prove that the canonical map $p\colon X \to X/G$ is étale, that is, the third condition in Proposition 9.15 is automatic for the pretopology $\mathcal{T}_{\mathrm{ét}}$. The characterisation of basic actions in $(\mathsf{Haus\text{-}Top}, \mathcal{T}_{\mathrm{ét}})$ then follows from Proposition 9.22.

The first two conditions in Proposition 9.15 say that the map

$$X \times_{\mathsf{s}, G^0, \mathsf{r}} G^1 \to X \times_{\mathsf{p}, X/G, \mathsf{p}} X, \qquad (x, g) \mapsto (x, x \cdot g),$$

is a homeomorphism. Since G is étale, the set of units $u(G^0)$ is open in G^1. Its image in $X \times_{\mathsf{p}, X/G, \mathsf{p}} X$ is the diagonal $\{(x_1, x_2) \in X \times X \mid x_1 = x_2\}$. Hence any $x \in X$ has an open neighbourhood $U \subseteq X$ such that $(U \times U) \cap (X \times_{\mathsf{p}, X/G, \mathsf{p}} X)$ is the diagonal in U. This means that for $x_1, x_2 \in U$, $p(x_1) = p(x_2)$ only if $x_1 = x_2$. Thus p is injective on the open subset $U \subseteq X$. Since G is an étale groupoid, its range and source maps are open. Hence p is

open by Proposition 9.34. Its restriction to U is injective, open and continuous, hence a homeomorphism onto an open subset of X/G.   ∎

9.41. SMOOTH MANIFOLDS OF FINITE AND INFINITE DIMENSION. We consider the following categories of smooth manifolds of increasing generality:

$$\begin{array}{ll} \mathsf{Mfd}_{\mathrm{fin}} & \text{finite-dimensional manifolds;} \\ \mathsf{Mfd}_{\mathrm{Hil}} & \text{Hilbert manifolds;} \\ \mathsf{Mfd}_{\mathrm{Ban}} & \text{Banach manifolds;} \\ \mathsf{Mfd}_{\mathrm{Fré}} & \text{Fréchet manifolds;} \\ \mathsf{Mfd}_{\mathrm{lcs}} & \text{locally convex manifolds.} \end{array}$$

Such manifolds are Hausdorff topological spaces that are locally homeomorphic to finite-dimensional vector spaces, Hilbert spaces, Banach spaces, Fréchet spaces, or locally convex topological vector spaces, respectively. Paracompactness is a standard assumption for finite-dimensional manifolds, but not for infinite-dimensional ones. The morphisms between all these types of manifolds are smooth maps, meaning maps given in local charts by smooth maps between topological vector spaces. Banach manifolds are treated in Lang's textbook [27], Fréchet manifolds in [21, Section I.4], and locally convex manifolds in [37, Appendix A] and [47, Appendix B]. The covers are, in each case, the *surjective submersions* in the following sense:

9.42. DEFINITION. [see [21, Definition 4.4.8] and [37, Appendix A]] *Let X and Y be locally convex manifolds. A smooth map is a* submersion *if for each $x \in X$, there is an open neighbourhood V of x in X such that $U := f(V)$ is open in Y, and there are a smooth manifold W and a diffeomorphism $V \cong U \times W$ that intertwines f and the coordinate projection $\mathsf{pr}_1 \colon U \times W \to U$.*

Choosing U small enough, we may achieve that V, U and W are locally convex topological vector spaces. So submersions are maps that locally look like projections onto direct summands in locally convex topological vector spaces.

9.43. PROPOSITION. *Surjective submersions form a subcanonical pretopology on* $\mathsf{Mfd}_{\mathrm{lcs}}$, $\mathsf{Mfd}_{\mathrm{Fré}}$, $\mathsf{Mfd}_{\mathrm{Ban}}$, $\mathsf{Mfd}_{\mathrm{Hil}}$ *and* $\mathsf{Mfd}_{\mathrm{fin}}$. *In each case, Assumptions 2.8 and 7.3 hold, and Assumption 2.10 holds if we exclude the empty manifold. The pretopology is not saturated.*

PROOF. That surjective submersions form a pretopology on locally convex manifolds is shown in [37, 47]. We briefly recall the argument below. It also works in the subcategories $\mathsf{Mfd}_{\mathrm{fin}}$, $\mathsf{Mfd}_{\mathrm{Hil}}$, $\mathsf{Mfd}_{\mathrm{Ban}}$, and $\mathsf{Mfd}_{\mathrm{Fré}}$; more generally, we may use manifolds based on any class $\mathcal{C}$ of topological vector spaces (not necessarily locally convex) that is closed under taking finite products and closed subspaces (this ensures that fibre products of manifolds locally modelled on topological vector spaces in $\mathcal{C}$ still have local models in $\mathcal{C}$).

It is routine to check that isomorphisms are surjective submersions and that composites of surjective submersions are again surjective submersions. If $f_i \colon M_i \to N$ for $i = 1, 2$ are smooth maps and f_1 is a submersion, then the fibre-product $M_1 \times_N M_2$ is a smooth submanifold of $M_1 \times M_2$, which satisfies the universal property of a fibre product. With

our definition of smooth submersion, the proof is rather trivial, see [21, Theorem 4.4.10] for the case of Fréchet manifolds and [37, Proposition A.3] for the general case. The proof also shows that $\mathsf{pr}_2 \colon M_1 \times_N M_2 \to M_2$ is a (surjective) submersion if f_1 is one, so $\mathcal{T}_{\mathrm{subm}}$ is a pretopology. Our categories of smooth manifolds do not admit fibre products in general, so we must be careful about their representability.

Next we show that $\mathcal{T}_{\mathrm{subm}}$ is subcanonical. A submersion $f \colon X \to Y$ is open because it is locally open. So our covers are open surjections and hence quotient maps of the underlying topological spaces, since open surjections form a subcanonical pretopology by Proposition 9.32. Thus any smooth map $X \to Z$ that equalises the coordinate projections $X \times_Y X \rightrightarrows X$ factors through a continuous map $f' \colon Y \to Z$. Since any submersion f admits smooth local sections, this map f' is smooth if and only if f is smooth. Hence f is the equaliser of the coordinate projections $X \times_Y X \rightrightarrows X$ also in the category of smooth manifolds, so the pretopology $\mathcal{T}_{\mathrm{subm}}$ is subcanonical.

The final object is the zero-dimensional one-point manifold, and any map to it from a non-empty manifold is a surjective submersion. So Assumption 2.10 is trivial for all our categories of smooth manifolds if we exclude the empty manifold. The idea of Example 9.33 shows that $\mathcal{T}_{\mathrm{subm}}$ is not saturated on any of our categories.

Now we check Assumption 2.8. Let $f \colon X \twoheadrightarrow U$ be a surjective submersion and let $g \colon Y \to U$ be a smooth map. We have already shown that surjective submersions form a pretopology. So the fibre product $W := X \times_{f,U,g} Y$ exists and the projection $\mathsf{pr}_2 \colon W \twoheadrightarrow Y$ is a surjective submersion. Assume that $\mathsf{pr}_1 \colon W \twoheadrightarrow X$ is a surjective submersion as well. We are going to show that g is a surjective submersion.

It is clear that g is surjective. The submersion condition is local, so we must check it near some $y \in Y$. Since f is surjective, there is some $x \in X$ with $f(x) = g(y)$, that is, $(x, y) \in W$. Since f is a submersion, there are neighbourhoods X_0 of x in X and U_0 of $f(x)$ in U, a smooth manifold Ξ_0, and a diffeomorphism $\alpha \colon \Xi_0 \times U_0 \xrightarrow{\sim} X_0$ such that $f \circ \alpha(\xi, u) = u$ for all $\xi \in \Xi_0$, $u \in U_0$. Let $(\xi_0, u_0) = \alpha^{-1}(x)$. The preimage $W_0 := \mathsf{pr}_1^{-1}(X_0)$ in W is an open submanifold. It is diffeomorphic to the fibre product $(\Xi_0 \times U_0) \times_{U_0} Y \cong \Xi_0 \times Y_0$ with $Y_0 := g^{-1}(U_0)$ because W is a fibre product. The restriction of pr_1 to W_0 is still a surjective submersion $W_0 \to X_0$ because pr_1 is one and the condition for submersions is local. Hence we have improved our original fibre-product situation to one where f is the projection map $\mathsf{pr}_1 \colon \Xi_0 \times U_0 \to U_0$. Thus $W_0 = \Xi_0 \times Y_0$ is an ordinary product and $\mathsf{pr}_1 \colon W_0 \to X_0$ becomes the map

$$\mathrm{id} \times g_0 \colon \Xi_0 \times Y_0 \to \Xi_0 \times U_0, \qquad (\xi, y_0) \mapsto (\xi, g_0(y_0)),$$

where $g_0 \colon Y_0 \to U_0$ is the restriction of g.

The map $\mathrm{id} \times g_0$ is a submersion by assumption. Hence there are neighbourhoods W_1 around (ξ_0, y_0) in $\Xi_0 \times Y_0$ and X_1 around (ξ_0, u_0) in $\Xi_0 \times U_0$, a smooth manifold Ω_1, and a diffeomorphism $\beta \colon W_1 \xrightarrow{\sim} \Omega_1 \times X_1$ that intertwines $\mathrm{id} \times g_0$ and the coordinate projection $\Omega_1 \times X_1 \to X_1$. We may shrink X_1 to be of product type: $X_1 = \Xi_1 \times U_1$ for open submanifolds $\Xi_1 \subseteq \Xi_0$ and $U_1 \subseteq U_0$ and replace W_1 by the preimage of Ω_1 times the new X_1, so we may assume without loss of generality that X_1 is of product type. If

$(\xi_1, y_1) \in W_1$, then

$$\beta(\xi_1, y_1) = (\omega_1(\xi_1, y_1), \xi_1, g_0(y_1))$$

for some function $\omega_1 \colon W_1 \to \Omega_1$: the second two components of β are $\mathrm{id} \times g_0$ by construction. The subset

$$Y_1 := \{y \in Y_0 \mid (\xi_0, y) \in W_1\}$$

is open in Y_0. We claim that $\gamma(y_1) := (\omega_1(\xi_0, y_1), g_0(y_1))$ defines a diffeomorphism from Y_1 onto $\Omega_1 \times U_1$; indeed, if $(\omega_1, u_1) \in \Omega_1 \times U_1$, then $\beta^{-1}(\omega_1, \xi_0, u_1)$ must be of the form (ξ_0, y_1) for some $y_1 \in Y_0$ with $(\xi_0, y_1) \in W_1$ because the Ξ_0-component of $\beta(\xi_1, y_1)$ is ξ_1. Hence the smooth map $\beta^{-1}|_{\Omega_1 \times \{\xi_0\} \times U_1}$ is inverse to γ. The restriction of g_0 to Y_1 becomes the coordinate projection $\Omega_1 \times U_1 \to U_1$ by construction. Hence $g \colon Y \to U$ satisfies the submersion condition near y. This finishes the proof that Assumption 2.8 holds for $\mathcal{T}_{\mathrm{subm}}$.

Now we check Assumption 7.3. Let $\mathsf{p} \colon X \twoheadrightarrow Z$ be a surjective submersion. Let G be its Čech groupoid. Let Y be a sheaf over G, that is, a G-action that has a surjective submersion $\mathsf{s} \colon Y \twoheadrightarrow X$ as anchor map. We claim that the G-action on Y is basic. That is, the orbit space Y/G is a smooth manifold, the orbit space projection $Y \to Y/\mathsf{G}$ is a surjective submersion, and the shear map $Y \times_{\mathsf{s}, \mathsf{G}^0, \mathsf{r}} \mathsf{G}^1 \to Y \times_{Y/\mathsf{G}} Y$, $(y, g) \mapsto (y, y \cdot g)$, is a diffeomorphism. Since surjective submersions are open, G becomes a Čech groupoid in $(\mathsf{Haus\text{-}Top}, \mathcal{T}_{\mathrm{open}})$. We know that Assumption 7.2 holds in that case, so Y/G is a Hausdorff topological space, the orbit space projection $Y \to Y/\mathsf{G}$ is a continuous open surjection, and the shear map is a homeomorphism.

Since p and s are covers, so is $\mathsf{p} \circ \mathsf{s}$, so the fibre product $Y \times_Z Y$ is a smooth submanifold of Y. If $(y_1, y_2) \in Y \times_Z Y$, then $(\mathsf{s}(y_1), \mathsf{s}(y_2)) \in X \times_Z X = \mathsf{G}^1$, with range and source $\mathsf{s}(y_1)$ and $\mathsf{s}(y_2)$, respectively. Thus there is a well-defined smooth map

$$\psi \colon Y \times_Z Y \to Y \times_X \mathsf{G}^1, \qquad (y_1, y_2) \mapsto (y_1, (\mathsf{s}(y_1), \mathsf{s}(y_2))).$$

Composing it with the shear map gives the map

$$\varphi \colon Y \times_Z Y \to Y \times_Z Y, \qquad (y_1, y_2) \mapsto (y_1, y_1 \cdot (\mathsf{s}(y_1), \mathsf{s}(y_2))).$$

Since $\mathsf{s}(y_1 \cdot (\mathsf{s}(y_1), \mathsf{s}(y_2))) = \mathsf{s}(y_1)$ and $(\mathsf{s}(y_1), \mathsf{s}(y_1))$ is an identity arrow, we get $\varphi^2 = \varphi$. If y_1, y_2 are in the same G-orbit, that is, $y_1 \cdot g = y_2$ for some $g \in \mathsf{G}^1 = X \times_Z X$, then $\mathsf{r}(g) = \mathsf{s}(y_1)$ and $\mathsf{s}(g) = \mathsf{s}(y_2)$, so $g = (\mathsf{s}(y_1), \mathsf{s}(y_2))$. Thus $\mathsf{p}(\mathsf{s}(y_1)) = \mathsf{p}(\mathsf{s}(y_2))$ is necessary for y_1 and y_2 to have the same orbit (we write $y_1 \mathsf{G} = y_2 \mathsf{G}$). If this necessary condition is satisfied, then the only arrow in G that has a chance to map y_1 to y_2 is $(\mathsf{s}(y_1), \mathsf{s}(y_2))$, so $y_1 \mathsf{G} = y_2 \mathsf{G}$ if and only if $y_1 \cdot (\mathsf{s}(y_1), \mathsf{s}(y_2)) = y_2$. This is equivalent to $\varphi(y_1, y_2) = (y_1, y_2)$. Thus the image of φ is the subspace $Y \times_{Y/\mathsf{G}} Y$ of $Y \times_Z Y$, and the restriction of ψ to $Y \times_{Y/\mathsf{G}} Y$ is a smooth inverse for the shear map. More precisely, this is a smooth inverse once we know that $Y \times_{Y/\mathsf{G}} Y$ is a smooth submanifold of $Y \times Y$ and hence a smooth submanifold of $Y \times_Z Y$. Thus the condition on the shear map is automatic in our case, and we only have to construct a smooth structure on Y/G such that the projection $\pi \colon Y \to Y/\mathsf{G}$ is a submersion.

We first restrict to small open subsets where p and s are of product type. Fix $y \in Y$. We may choose neighbourhoods Y_0 of y in Y, X_0 of $\mathsf{s}(y)$ in X and Z_0 of $\mathsf{p}(\mathsf{s}(y))$ in Z, smooth manifolds Ω_0 and Ξ_0, and diffeomorphisms $Y_0 \cong \Omega_0 \times X_0$, $X_0 \cong \Xi_0 \times Z_0$, such that the maps s and p become the projections to the second coordinate, respectively. We identify $Y_0 = \Omega_0 \times \Xi_0 \times Z_0$ and $X_0 = \Xi_0 \times Z_0$, so that $\mathsf{s}(\omega_0, \xi_0, z_0) = (\xi_0, z_0)$ and $\mathsf{p}(\xi_0, z_0) = z_0$ for all $\omega_0 \in \Omega_0$, $\xi_0 \in \Xi_0$, $z_0 \in Z_0$. Let (ω, ξ, z) be the coordinates of the chosen point $y \in Y$.

What happens to the G-action on Y in these coordinates? Let $(\omega_0, \xi_0, z_0) \in Y_0$. Then $\mathsf{s}(\omega_0, \xi_0, z_0) = (\xi_0, z_0)$. An arrow in $G \cap (X_0 \times_{Z_0} X_0)$ with range (ξ_0, z_0) is a pair $((\xi_0, z_0), (\xi_1, z_0))$, which we identify with the triple (ξ_0, z_0, ξ_1). If ξ_1 is sufficiently close to ξ_0, then $(\omega_0, \xi_0, z_0) \cdot (\xi_0, z_0, \xi_1)$ belongs to Y_0 and may be written in our coordinates. Since $\mathsf{s}((\omega_0, \xi_0, z_0) \cdot (\xi_0, z_0, \xi_1)) = (\xi_1, z_0)$, we get

$$(\omega_0, \xi_0, z_0) \cdot (\xi_0, z_0, \xi_1) = (\varphi(\omega_0, \xi_0, z_0, \xi_1), \xi_1, z_0)$$

for a smooth function φ defined on some neighbourhood W_1 of $(\omega_0, \xi_0, z_0, \xi_0)$ in $\Omega_0 \times \Xi_0 \times Z_0 \times \Xi_0$. Let $Y_1 := \{y \in Y_0 \mid (\omega_0, \xi_0, z_0, \xi) \in W_1\}$, where ξ is the Ξ_0-coordinate of our chosen point y. Define

$$\alpha \colon Y_1 \to \Omega_0 \times \Xi_0 \times Z_0, \qquad (\omega_0, \xi_0, z_0) \mapsto (\varphi(\omega_0, \xi_0, z_0, \xi), \xi_0, z_0).$$

In other words, $\alpha(\omega_0, \xi_0, z_0)$ takes the original Ξ_0-coordinate and takes the other coordinates from $(\omega_0, \xi_0, z_0) \cdot (\xi_0, z_0, \xi)$. Since $(\xi_1, z_0, \xi_2) \cdot (\xi_2, z_0, \xi_1) = (\xi_1, z_0, \xi_1)$ is a unit arrow and thus acts identically on Y for any $\xi_1, \xi_2 \in \Xi_0$, $z_0 \in Z_0$, we get

$$\alpha^{-1}(\omega_0, \xi_0, z_0) = (\varphi(\omega_0, \xi, z_0, \xi_0), \xi_0, z_0)$$

on some neighbourhood of (ω, ξ, z). Thus α is a diffeomorphism from some neighbourhood Y_1 of y in Y_0 onto a neighbourhood $\Omega_1 \times \Xi_1 \times Z_1$ of $\alpha(y) = (\omega, \xi, z)$ in $\Omega_0 \times \Xi_0 \times Z_0$.

When are $y_1, y_2 \in Y_1$ in the same G-orbit? Write $\alpha(y_i) = (\omega_i, \xi_i, z_i)$. Then $\mathsf{s}(y_i) = (\xi_i, z_i)$ and $\mathsf{ps}(y_i) = z_i$ in local coordinates. Since $\mathsf{ps}(y_1) = \mathsf{ps}(y_2)$ is necessary for $y_1\mathsf{G} = y_2\mathsf{G}$, we have $z_1 = z_2$ if $y_1\mathsf{G} = y_2\mathsf{G}$. Assume this, then $(\mathsf{s}(y_1), \mathsf{s}(y_2)) \in X \times_Z X = \mathsf{G}^1$, so we have $y_1\mathsf{G} = y_2\mathsf{G}$ if and only if $y_1 \cdot (\mathsf{s}(y_1), \mathsf{s}(y_2)) = y_2$ by the discussion above. This is equivalent to $z_1 = z_2$ and

$$y_1 \cdot (\mathsf{s}(y_1), (z_1, \xi)) = y_2 \cdot (\mathsf{s}(y_2), (z_2, \xi)),$$

where ξ is the Ξ_0-coordinate of our fixed point y. And this is equivalent to $\mathsf{pr}_{13} \circ \alpha(y_1) = \mathsf{pr}_{13} \circ \alpha(y_2)$.

As a consequence, the restriction of the quotient map $Y \to Y/G$ to $\alpha^{-1}(\Omega_1 \times \{\xi\} \times U_1)$ is a homeomorphism onto its image; this image is the same as the image of Y_1, which is an open subset in Y/G because Y_1 is open and the quotient map $Y \to Y/G$ is open. Furthermore, with this coordinate chart, the restriction of the projection $Y \to Y/G$ to Y_1 becomes the coordinate projection $\mathsf{pr}_{13} \colon \Omega_1 \times \Xi_1 \times Z_1 \to \Omega_1 \times Z_1$. Doing the above

construction for all $y \in Y$, we get a covering of Y by open subsets like our Y_1 with corresponding charts on $\pi(Y_1)$. We claim that these local charts on Y/G have smooth coordinate change maps. Thus Y/G becomes a smooth manifold. The charts are defined in such a way that the projection $\pi \colon Y \to Y/G$ is a smooth submersion. We have already seen above that the shear map is a diffeomorphism. Hence the proof that the G-action on Y is basic will be finished once we show that coordinate change maps between our local charts on Y/G are smooth.

The charts on Y/G are defined in such a way that the projection map $\pi \colon Y \to Y/G$ is smooth for each chart and there are smooth sections $Y/G \supseteq \Omega_1 \times U_1 \to Y$ on the chart neighbourhoods, defined by $(\omega_1, u_1) \mapsto (\omega_1, \xi, u_1)$ in local coordinates. Now take two overlapping charts on Y/G defined on open subsets $W_1, W_2 \subseteq Y/G$ with their smooth local sections $\sigma_i \colon W_i \to Y$. If $w \in W_1 \cap W_2$, then $\sigma_1(w)$ and $\sigma_2(w)$ are two representatives of the same G-orbit. By the discussion above, this means that $\mathsf{ps}(\sigma_1(w)) = \mathsf{ps}(\sigma_2(w))$ and $\sigma_2(w) = \sigma_1(w) \cdot (\mathsf{s}(\sigma_1(w)), \mathsf{s}(\sigma_2(w)))$. Thus the coordinate change map from W_1 to W_2 is of the form

$$w \mapsto \pi\Big(\sigma_1(w) \cdot (\mathsf{s}(\sigma_1(w)), \mathsf{s}(\sigma_2(w)))\Big).$$

Since the maps π, σ_i and s are all smooth in our local coordinates, so is the composite map. This finishes the proof that Assumption 7.3 holds for $\mathcal{T}_{\mathrm{subm}}$. The proof works for all our categories of infinite-dimensional manifolds.   ∎

It is unclear whether Assumptions 2.9 and 7.2 hold for Fréchet and locally convex manifolds. The problem is that we lack a general implicit function theorem. Such a theorem is available for Banach manifolds and gives the following equivalent characterisations of submersions:

9.44. PROPOSITION. *Let X and Y be Banach manifolds and let $f \colon X \to Y$ be a smooth map. The following are equivalent:*

(1) *f is a submersion in the sense of Definition 9.42;*

(2) *f has many smooth local sections, that is, for each $x \in X$, there is an open neighbourhood U of $f(x)$ and a smooth map $\sigma \colon U \to X$ with $\sigma(f(x)) = x$ and $f \circ \sigma = \mathrm{id}_U$;*

(3) *for each $x \in X$, the derivative $D_x f \colon T_x X \to T_{f(x)} Y$ is split surjective, that is, there is a continuous linear map $s \colon T_{f(x)} Y \to T_x X$ with $D_x f \circ s = \mathrm{id}_{T_{f(x)}Y}$.*

If X and Y are Hilbert manifolds, then (3) is equivalent to $D_x f$ being surjective.

PROOF. It is trivial that (1) implies (2). Using $D_{f(x)}\sigma$ as continuous linear section for $D_x f$, we see that (2) implies (3). The implication from (3) to (1) is [27, Proposition 2.2 in Chapter II]. If X and Y are Fréchet manifolds, then the derivative $D_x f$ is open once it is surjective by the Open Mapping Theorem. If $D_x f$ is surjective and X and Y are Hilbert manifolds, then the orthogonal projection onto $\ker D_x f$ splits $X \cong \ker(D_x f) \oplus D_{f(x)} Y$, so (3) follows if $D_x f$ is surjective and X and Y are Hilbert manifolds.   ∎

9.45. PROPOSITION. *On the categories* $\mathsf{Mfd}_{\mathrm{Ban}}$, $\mathsf{Mfd}_{\mathrm{Hil}}$ *and* $\mathsf{Mfd}_{\mathrm{fin}}$, *the pretopology* $\mathcal{T}_{\mathrm{subm}}$ *also satisfies Assumption* 2.9.

PROOF. Since we are working with Banach manifolds or smaller categories, we may use Proposition 9.44 to redefine our covers as surjective smooth maps with many smooth local sections. With this alternative definition of our covers, Assumption 2.9 holds for the same reason as for the pretopology of continuous surjections with many continuous local sections on the category of topological spaces. Let X, Y and Z be Banach manifolds and let $f\colon X \to Y$ and $g\colon Y \to Z$ be smooth maps. We assume that $g \circ f$ and f are surjective submersions and want to show that g is so to. Of course, g must be surjective if $g \circ f$ is. Given $y \in Y$, there is $x \in X$ with $f(x) = y$ because g is surjective, and there is a smooth local section σ for $g \circ f$ near $g(y)$ with $\sigma(g(y)) = x$. Then $f \circ \sigma$ is a smooth local section for g with $f \circ \sigma(g(y)) = y$. ∎

If we redefined submersions of locally convex or Fréchet manifolds using one of the alternative criteria in Proposition 9.44, then the existence of pull-backs along submersions would become unclear.

The next lemma is needed to verify Assumption 7.2 for Banach manifolds.

9.46. LEMMA. *Let Y be a Banach manifold and let $\varphi\colon Y \to Y$ be a smooth map with $\varphi^2 = \varphi$. The image $\mathrm{im}(\varphi)$ is a closed submanifold of Y and $\varphi\colon Y \to \mathrm{im}(\varphi)$ is a surjective submersion.*

PROOF. The image is closed because Y is Hausdorff and

$$\mathrm{im}(\varphi) = \{x \in Y \mid \varphi(x) = x\}.$$

We need a submanifold chart for $\mathrm{im}(\varphi)$ near $y \in \mathrm{im}(\varphi)$. We use a chart $\gamma\colon T_yY \to Y$, that is, γ is a diffeomorphism onto an open neighbourhood of y in Y whose derivative at 0 is the identity map on the Banach space T_yY. It suffices to find a submanifold chart for $\varphi' := \gamma^{-1} \circ \varphi \circ \gamma$, which is an idempotent smooth map on a neighbourhood of 0 in T_yY with $\varphi'(0) = 0$. Let $D\colon T_yY \to T_yY$ be the derivative of φ' at 0, which is a linear map with $D^2 = D$. We may assume that φ' is defined on all of T_yY to simplify notation.

Let $X = \mathrm{im}(D)$, $W = \ker(D)$. The map

$$\Psi\colon X \oplus W \to T_yY, \qquad (\xi, \eta) \mapsto \varphi'(\xi) + \eta,$$

has the identity map as derivative near $(0,0)$. By the Implicit Function Theorem for Banach manifolds, Ψ is a diffeomorphism between suitable open neighbourhoods of 0 in $X \oplus W$ and T_yY. We have $\Psi(\xi, 0) = \varphi'(\xi) \in \mathrm{im}(\varphi')$ for all $\xi \in X$, and

$$\begin{aligned}
\varphi'(\Psi(\xi,\eta)) - \Psi(\xi,\eta) &= \varphi'(\varphi'(\xi) + \eta) - \varphi'(\xi) - \eta \\
&= D(D\xi + \eta) - D\xi - \eta + O(\|\xi\|^2 + \|\eta\|^2) = -\eta + O(\|\xi\|^2 + \|\eta\|^2).
\end{aligned}$$

This is non-zero if $\eta \neq 0$ and η, ξ are small enough. Hence, for sufficiently small ξ, η, we have $\Psi(\xi, \eta) \in \mathrm{im}(\varphi')$ if and only if $\eta = 0$. Thus Ψ is a submanifold chart for $\mathrm{im}(\varphi')$

near 0. Its composite with γ gives the required submanifold chart for $\mathrm{im}(\varphi)$ near y. It is clear from the construction that the map $\varphi\colon Y \to \mathrm{im}(\varphi)$ has surjective derivative at y, hence it is a smooth submersion by Proposition 9.44. ∎

9.47. PROPOSITION. *Every action of a Čech groupoid of a cover in $(\mathsf{Mfd}_{\mathrm{Ban}}, \mathcal{T}_{\mathrm{subm}})$ is basic.*

PROOF. Let $\mathsf{p}\colon M \to N$ be a surjective submersion and let $\mathsf{G} = (M \times_{\mathsf{p},N,\mathsf{p}} M \rightrightarrows N)$ be its Čech groupoid (see Example 3.12). Let G act on a smooth manifold Y with anchor map $\mathsf{s}\colon Y \to \mathsf{G}^0$.

Since p has many smooth local sections, we may cover N by open subsets U_i, $i \in I$, for which there are smooth local sections $\sigma_i\colon U_i \to M$ with $\mathsf{p} \circ \sigma_i = \mathrm{id}_{U_i}$. Then $Y_i := (\mathsf{ps})^{-1}(U_i) \subseteq Y$ is open (and thus an open submanifold of Y) and G-invariant.

The map
$$\varphi_i\colon Y_i \to Y_i, \qquad y \mapsto y \cdot \big(\mathsf{s}(y), \sigma_i \circ \mathsf{p} \circ \mathsf{s}(y)\big),$$
is smooth and maps y to a point y' in the same G-orbit with $\mathsf{s}(y') \in \sigma_i(U_i)$. If y' and y'' belong to the same G-orbit, then $\mathsf{s}(y') = \mathsf{s}(y'')$ implies $y' = y''$ because the only element of G that may map y' to y'' is $(\mathsf{s}(y'), \mathsf{s}(y''))$. Thus $\varphi_i(y)$ is the unique element in the G-orbit of y that belongs to $\mathsf{s}^{-1}(\sigma_i(U_i))$. Thus $\varphi_i^2 = \varphi_i$. By Lemma 9.46, $\mathrm{im}(\varphi)$ is a smooth submanifold of Y and $\varphi_i\colon Y_i \to \mathrm{im}(\varphi_i)$ is a surjective submersion.

If $y_1, y_2 \in Y_i$ satisfy $\varphi_i(y_1) = \varphi_i(y_2)$, then $y_2 = y_1 \cdot (\mathsf{s}(y_1), \mathsf{s}(y_2))$. Hence the map
$$Y_i \times_{\mathsf{s},M,\mathsf{pr}_1} (M \times_N M) \to Y_i \times_{\varphi_i,Y_i,\varphi_i} Y_i, \qquad (y, (m_1, m_2)) \mapsto (y, y \cdot (m_1, m_2)),$$
is a diffeomorphism with inverse $(y_1, y_2) \mapsto (y_1, (\mathsf{s}(y_1), \mathsf{s}(y_2)))$. As a consequence, the restriction of the G-action to Y_i with the bundle projection $\varphi_i\colon Y_i \to \mathrm{im}(\varphi_i)$ is a principal bundle.

It remains to glue together these local constructions. Let Y/G be the quotient space with the quotient topology. The quotient map $\pi\colon Y \to Y/\mathsf{G}$ is open by Proposition 9.34 and because submersions are open. Thus the subsets $\pi(Y_i)$ form an open cover of Y/G. The above argument shows that π restricts to a homeomorphism from $\mathrm{im}(\varphi_i)$ onto $\pi(Y_i)$. We claim that there is a unique smooth manifold structure on Y/G for which the maps $\pi\colon \mathrm{im}(\varphi_i) \to Y/\mathsf{G}$ become diffeomorphisms onto open subsets of Y/G. We only have to check that the coordinate change maps on Y/G from $\mathrm{im}(\varphi_i)$ to $\mathrm{im}(\varphi_j)$ are smooth maps. This is so because the map
$$Y_i \cap \mathrm{im}(\varphi_j) \to \mathrm{im}(\varphi_i) \cap Y_j, \qquad y \mapsto y \cdot (\sigma_i \circ \mathsf{s}(y), \sigma_j \circ \mathsf{s}(y)),$$
is a diffeomorphism between submanifolds of Y_i, with inverse given by a similar formula. The orbit space projection $Y \to Y/\mathsf{G}$ is a surjective submersion for this smooth manifold structure on Y/G because this holds locally on each Y_i. The same argument as above shows that the map
$$Y \times_{\mathsf{s},M,\mathsf{pr}_1} (M \times_N M) \to Y \times_{Y/\mathsf{G}} Y, \qquad (y, (m_1, m_2)) \mapsto (y, y \cdot (m_1, m_2)),$$
is a diffeomorphism with inverse $(y_1, y_2) \mapsto (y_1, (\mathsf{s}(y_1), \mathsf{s}(y_2)))$. Hence $Y \twoheadrightarrow Y/\mathsf{G}$ is a principal G-bundle. ∎

As a result, most of our theory works for locally convex manifolds with surjective submersions as covers. This includes the bicategories of anafunctors, bibundle functors and bibundle equivalences, but *not* the bicategories of covering bibundle functors and bibundle actors. We only know that these bicategories exist for Banach, Hilbert and finite-dimensional manifolds.

9.48. REMARK. An action is basic in the category of finite-dimensional manifolds if and only if it is basic in the category of Hausdorff spaces, if and only if it is free and proper (see Proposition 9.22). This remains true for Hilbert manifolds. For group actions, this is the main result of [24]; groupoid actions may be reduced to group actions using the proof of [49, Lemma 3.11].

For Banach manifolds and, even more generally, for locally convex manifolds, it remains true that basic actions are free and proper, but the converse fails. Here is a counterexample (see also [5, Chapter 3]). Let E be a Banach space and let F be a closed subspace without complement, for instance, $c_0(\mathbb{N}) \subset \ell^\infty(\mathbb{N})$. Let F act on E by inclusion and linear addition. This action is free and proper, and the orbit space projection is the quotient map $E \to E/F$. The derivative of this map is the same map $E \to E/F$, which has no linear section by assumption. Hence the orbit space projection is not a cover, so the action is not basic.

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