CONSTRUCTION OF CATEGORICAL BUNDLES FROM LOCAL DATA

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ABSTRACT. A categorical principal bundle is a structure comprised of categories that is analogous to a classical principal bundle; examples arise from geometric contexts involving bundles over path spaces. We show how a categorical principal bundle can be constructed from local data specified through transition functors and natural transformations.

1. Introduction

A categorical principal bundle $\pi : \mathbf{P} \longrightarrow \mathbf{B}$ is a structure analogous to a classical principal bundle, but with all the spaces involved replaced by categories and maps by functors. Of interest to us is the case where these categories have a geometric significance; for example, there is an underlying classical principal bundle $\pi : P \longrightarrow B$ and the objects of the 'base category' $\mathbf{B}$ are the points of B while the morphisms arise from paths on B. In the 'bundle category' $\mathbf{P}$ the objects are the points of P and morphisms are of the form $(\overline{\gamma}, h)$, where $\overline{\gamma}$ comes from a path on P that is horizontal with respect to a connection form on $\pi : P \longrightarrow B$ and h is a 'decoration' drawn from a Lie group H. There are different notions of local triviality for such structures. In this paper we construct a categorical bundle from local data. The local data do not come as a traditional cocycle of transition functions but rather as functors that fall short of a cocycle relation. We describe a quotient procedure that leads to "functorial cocycles" and then construct a categorical bundle from such cocycles.

The literature in category theoretic geometry has grown rapidly in recent years. We mention here the works of Abbaspour and Wagemann [1], Aschieri et al. [2], Attal [3,4], Baez et al. [5,6], Barrett [7], Bartels [8], Breen and Messing [9], Mackaay [17], Parzygnat [22], Picken et al. [10,19,20], Soncini and Zucchini [25], Schreiber and Waldorf [23,24], Viennot [26] and Wang [27,28]. Among others, the works [1, sec. 3], [5, Propn 2.2; 6], [22, sec. 2] study the relationship between gerbe local data and 2-bundles, as well as connections on such structures. Our framework and structures are closely related in spirit but there are differences. In particular we have two categorical groups $\mathbf{G}$ and $\mathbf{H}$ in terms of which the local data are specified. The technical nature of what a categorical bundle

is also specified differently in our approach. A theory of parallel transport for categorical principal bundles is developed in our earlier work [12]; briefly, a *connection* on such a bundle $\pi : \mathbf{P} \longrightarrow \mathbf{B}$ is specified through horizontal lifts of morphisms (viewed as paths) of the base category $\mathbf{B}$ to morphisms of $\mathbf{P}$ such that the lifting is functorial in a suitable sense and the categorical group action on the bundle carries such horizontal morphisms to horizontal morphisms.

1.1. RESULTS AND ORGANIZATION. We begin in section 2 with a summary of essential notions and notation concerning categorical groups, categorical bundles, and categories arising from points and paths on manifolds. All through this paper the base space of the bundle is a manifold B, and local data is specified relative to an open covering $\{U_i\}_{i \in \mathcal{I}}$ of B. Associated to these sets are categories $\mathbf{U}_i$ (objects are points of U_i and morphisms arise from paths on B that lie inside U_i) and overlap categories such as $\mathbf{U}_{ik} = \mathbf{U}_i \cap \mathbf{U}_k$.

In section 3 we work with two categorical groups $\mathbf{G}$ and $\mathbf{H}$, and introduce *gerbal cocycles* (subsection 3.1), which are analogous to classical cocycles except that they fall short of satisfying the exact identities needed for classical cocycles. Then we construct *functorial* forms of these gerbal cocycles; briefly put they are given by functors

$$\theta_{ik} : \mathbf{U}_{ik} \longrightarrow \mathbf{G} \tag{1.1}$$

and natural isomorphisms

$$\mathbb{T}_{ikm} : \theta_{ik}\theta_{km} \longrightarrow \theta_{im}. \tag{1.2}$$

This is explained in subsection 3.3. We establish several properties of the natural transformations (1.1).

In order to obtain genuine cocycles we take quotients to form a category $\overline{\mathbf{G}}$ and a categorical group $\overline{\mathbf{G}}_\tau$ (these are constructed in subsections 3.10 and 3.12).

We turn next in section 4 to the construction of a globally defined categorical bundle over $\mathbf{B}$ (points forming the base manifold B and morphisms arising from paths on B). The classical construction of a principal bundle $\pi : X \longrightarrow B$ from a cocycle of transition functions may be viewed as the construction of a projective limit from a family of trivial bundles

$$U_i \times G \longrightarrow U_i.$$

Using this viewpoint we construct in section 4 a categorical principal bundle

$$\mathbf{X} \longrightarrow \mathbf{B},$$

by sewing together the trivial categorical bundles $\mathbf{U}_i \times \overline{\mathbf{G}}_\tau$ using the functorial transition data in (1.1) and (1.2). Here we use the formalism of *quivers*, explained in subsection 4.5.

Using techniques similar to those described in this paper it is possible to construct a categorical principal $\mathbf{G}$-bundle (instead of $\bar{\mathbf{G}}_\tau$-bundle), which enjoys a *weak* local trivialization property, where local triviality is understood in terms of equivalences rather than isomorphisms. For this it suffices that the target maps $\tau' : J \longrightarrow H$ and $\tau : H \longrightarrow G$ satisfy the condition that $\mathrm{Ker}(\tau) \cap \mathrm{Im}(\tau')$ is trivial, where the notation is as explained at the beginning of section 3. We will not explore this line of investigation in the present paper.

2. Basic notions

In this section we summarize the essentials of terminology and notions that we use. The categories we work with are all small categories, the objects and morphisms forming sets. In fact the object sets of the categories we work with are smooth manifolds and functors are, at the level of objects, given by smooth functions.

By a *categorical group* $\mathbf{G}$ we mean a small category along with a functor

$$\mathbf{G} \times \mathbf{G} \longrightarrow \mathbf{G}$$

that makes both the object set $\mathrm{Obj}(\mathbf{G})$ and the morphism set $\mathrm{Mor}(\mathbf{G})$ groups. The source and target maps

$$s, t : \mathrm{Mor}(\mathbf{G}) \longrightarrow \mathrm{Obj}(\mathbf{G})$$

are homomorphisms. We say that the categorical group $\mathbf{G}$ is a categorical Lie group if $\mathrm{Obj}(\mathbf{G})$ and $\mathrm{Mor}(\mathbf{G})$ are Lie groups and s and t are smooth mappings. Associated to a categorical group $\mathbf{G}$ is a *crossed module* (G, H, α, τ), where G and H are groups, and

$$\tau : H \longrightarrow G \qquad \text{and} \qquad \alpha : G \longrightarrow \mathrm{Aut}(H) : g \mapsto \alpha_g \tag{2.1}$$

are homomorphisms satisfying the Peiffer identities

$$\begin{aligned} \tau\big(\alpha_g(h)\big) &= g\tau(h)g^{-1} \\ \alpha_{\tau(h)}(h') &= hh'h^{-1} \end{aligned} \tag{2.2}$$

for all $g \in G$ and $h \in H$. As a consequence of the first Peiffer identity the image $\tau(H)$ is a normal subgroup of G:

$$g\tau(h)g^{-1} = \tau\big(\alpha_g(h)\big) \in \tau(H) \text{ for all } h \in H \text{ and } g \in G. \tag{2.3}$$

The relationship between $\mathbf{G}$ and the crossed module is given by

$$G = \mathrm{Obj}(\mathbf{G}) \quad \text{and} \quad H = \ker s \subset \mathrm{Mor}(\mathbf{G}).$$

The morphism group $\mathrm{Mor}(\mathbf{G})$ can be identified with the semidirect product $H \rtimes_\alpha G$:

$$\mathrm{Mor}(\mathbf{G}) \simeq H \rtimes_\alpha G,$$

with $(h, g) \in H \rtimes_\alpha G$ having source g and target $\tau(h)g$:

$$s(h, g) = g \qquad \text{and} \qquad t(h, g) = \tau(h)g. \tag{2.4}$$

Composition of morphisms is given in $H \rtimes_\alpha G$ by

$$(h_2, g_2) \circ (h_1, g_1) = (h_2 h_1, g_1), \tag{2.5}$$

in contrast to the product operation in $\mathrm{Mor}(\mathbf{G})$ which is given by the semidirect product operation

$$(h_2, g_2)(h_1, g_1) = \big(h_2 \alpha_{g_2}(h_1), g_2 g_1\big). \tag{2.6}$$

The categorical group $\mathbf{G}$ is a Lie group if and only if G and H are Lie groups and the mappings $\tau : h \mapsto \tau(h)$ and $(h, g) \mapsto \alpha_g(h)$ are smooth.

It will often, but not always, be convenient to identify H and G with the subgroups $H \times \{e\}$ and $\{e\} \times G$ in $H \rtimes_\alpha G$, so that (h, g) can be written simply as a product:

$$hg = (h, g). \tag{2.7}$$

With this notation, conjugation by g (that is, (e, g)) is simply α_g:

$$\begin{aligned}
ghg^{-1} = (e, g)(h, e)(e, g^{-1}) &= \big(e\alpha_g(h), g\big)(e, g^{-1}) \\
&= \big(\alpha_g(h)\alpha_g(e), gg^{-1}\big) \\
&= \big(\alpha_g(h), e\big) \\
&= \alpha_g(h).
\end{aligned} \tag{2.8}$$

Let us also note how inverses work with this notation:

$$\begin{aligned}
(hg)^{-1} = (h, g)^{-1} &= \big(\alpha_{g^{-1}}(h^{-1}), g^{-1}\big) \\
&= \alpha_{g^{-1}}(h^{-1})g^{-1} \\
&= g^{-1}h^{-1}g \cdot g^{-1} \\
&= g^{-1}h^{-1},
\end{aligned} \tag{2.9}$$

wherein the second equality in the first line can be verified by working out the product $(h, g)\big(\alpha_{g^{-1}}(h^{-1}), g^{-1}\big)$.

By a *categorical principal bundle* with *structure categorical group* $\mathbf{G}$ we mean a functor

$$\pi : \mathbf{P} \longrightarrow \mathbf{B}$$

that is surjective both on the level of objects and on the level of morphisms, along a functor

$$\mathbf{P} \times \mathbf{G} \longrightarrow \mathbf{P}$$

that is a free right action both on objects and on morphisms, such that $\pi(pg) = \pi(p)$ for all objects/morphisms p of $\mathbf{P}$ and all objects/morphisms g of $\mathbf{G}$. (For more on categorical principal bundles we refer to [12].) This is a 'bare bones' definition; in practice we are only concerned with those examples in which $\mathbf{G}$ is a categorical Lie group, $\mathrm{Obj}(\mathbf{P})$ and $\mathrm{Obj}(\mathbf{B})$ are smooth manifolds, and the object bundle

$$\mathrm{Obj}(\mathbf{P}) \longrightarrow \mathrm{Obj}(\mathbf{B})$$

is a principal G-bundle, where $G = \mathrm{Obj}(\mathbf{G})$. The morphisms of $\mathbf{B}$ arise from paths on $B = \mathrm{Obj}(\mathbf{B})$, as we now discuss.

2.1. CATEGORIES FROM POINTS AND PATHS. We turn now to categories of points and paths. Associated to a smooth manifold M there is a category $\mathbf{M}$ whose objects are the points of M, and whose morphisms are all piecewise smooth paths on M. Let us specify this in more detail. The paths we use are smooth mappings of the form $[a, b] \longrightarrow M$, where $a, b \in \mathbb{R}$ with $a < b$, and the paths are assumed to be constant near the initial time a and the terminal time b. Paths $\gamma_1 : [a, b] \longrightarrow M$ and $\gamma_2 : [c, d] \longrightarrow M$ are identified if there is a constant r such that $[c, d] = [a, b] + r$ and

$$\gamma_2(t) = \gamma_1(t - r) \qquad \text{for all } t \in [c, d]. \tag{2.10}$$

The source of γ is the initial point and the target of γ is the terminating point; often we will find it notationally convenient to write γ_0 or even $\gamma(0)$ to denote the source $s(\gamma)$, and γ_1 or $\gamma(1)$ to denote the target $t(\gamma)$. Composition of morphisms is defined by composition of paths; the requirement that paths be constant near their initial and terminal times ensures that the composition of two such paths is smooth and has the same property.
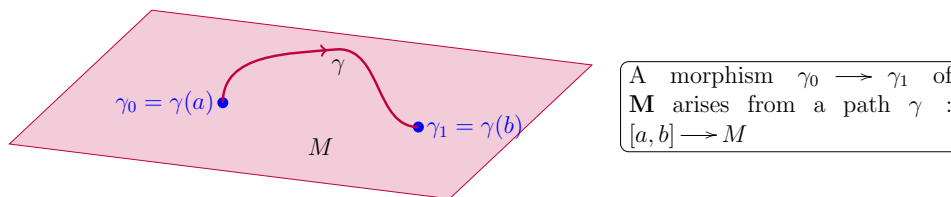

Figure 1: The category $\mathbf{M}$

2.2. TRIVIALITY AND LOCAL TRIVIALITY. There are different notions of triviality that are of interest for categorical bundles. The product categorical bundle over a base category $\mathbf{U}$ and having structure categorical group $\mathbf{G}$ is given by the projection functor

$$\mathbf{U} \times \mathbf{G} \longrightarrow \mathbf{U}$$

and the obvious right action of $\mathbf{G}$ on $\mathbf{U} \times \mathbf{G}$. The simplest and strongest notion of triviality of a categorical principal $\mathbf{G}$-bundle $\mathbf{P} \longrightarrow \mathbf{U}$ to require that there be an isomorphism of categories $\Phi : \mathbf{P} \longrightarrow \mathbf{U} \times \mathbf{G}$, of appropriate smoothness, that respects the action of $\mathbf{G}$ as well as the projection functor. An alternate notion, which allows a richer geometric structure, is explored in [15]. An even weaker notion is to require that Φ be an equivalence, rather than an isomorphism. Corresponding to these notions of global triviality there are notions of local triviality for categorical bundles.

2.3. COVERING SUBCATEGORIES. We will find it convenient to work with subcategories of $\mathbf{B}$ for which paths initiate and terminate in specified sets of the open covering $\{U_i\}_{i \in \mathcal{I}}$ of a manifold B. To this end let $\mathbf{U}_i$ be the category whose object set is U_i and whose morphisms are the morphisms γ of $\mathbf{B}$ that lie entirely inside U_i.

It is also useful to introduce the overlap category

$$\mathbf{U}_{ik} = \mathbf{U}_i \cap \mathbf{U}_k, \tag{2.11}$$

the object set and morphism sets being just the intersections of the corresponding sets for $\mathbf{U}_i$ and $\mathbf{U}_k$. Of course, this overlap category is defined only if $U_i \cap U_k \neq \emptyset$. Analogously, we also have triple overlap categories $\mathbf{U}_{ikm}$, if $U_j \cap U_k \cap U_m$ is nonempty, and, more generally, $\mathbf{U}_I$, for any finite subset $I \subset \mathcal{I}$ for which the intersection

$$U_I \overset{\text{def}}{=} \cap_{i \in I} U_i \tag{2.12}$$

is nonempty. We denote by $S_{\mathcal{I}}$ the set of all nonempty subsets I of $\mathcal{I}$:

$$S_{\mathcal{I}} = \{I \subset \mathcal{I} \,:\, U_I \neq \emptyset\}. \tag{2.13}$$

If $I, J \in S_{\mathcal{I}}$ with $I \subset J$ then $U_J \subset U_I$ and so we have a functor

$$\mathbf{U}_J \longrightarrow \mathbf{U}_I \tag{2.14}$$

induced by the inclusion $U_J \longrightarrow U_I$. Thus, if we denote by $\mathbf{S}_{\mathcal{I}}$ the category whose object set is $S_{\mathcal{I}}$ and whose morphisms $I \longrightarrow I$ are the inclusion maps $J \longrightarrow I$, then

$$\mathbf{U}_. : I \mapsto \mathbf{U}_I \tag{2.15}$$

specifies a functor from the category $\mathbf{S}_{\mathcal{I}}$ to the category $\mathbf{CMan}$ whose objects are categories $\mathbf{M}$ arising from manifolds and whose morphisms arise from smooth mappings between manifolds. The functor $\mathbf{U}_.$ is specified on morphisms in the obvious way: it carries the morphism $I \longrightarrow J$ in $\mathbf{S}_{\mathcal{I}}$ to the 'inclusion' functor $\mathbf{U}_I \longrightarrow \mathbf{U}_J$.

The set-theoretic union of the morphism sets $\mathrm{Mor}(\mathbf{U}_i)$ is not generally equal to $\mathrm{Mor}(\mathbf{B})$, and so the categories $\mathbf{U}_i$ do not 'cover' $\mathbf{B}$ in any literal sense. For example, a path that initiates and terminates in a set U_i but does not lie entirely inside U_i would not appear in $\cup_i \mathrm{Mor}(\mathbf{U}_i)$. Thus a construction is needed that implements the idea of sewing together paths γ_k, lying inside U_k, for different values of k, to form a more general path γ. This construction is carried out, in the context of bundles, in subsection 4.2.

To gain further perspective into this matter, let us note that we have the functors

$$\mathrm{inc}_I : \mathbf{U}_I \longrightarrow \mathbf{B} \tag{2.16}$$

induced by the inclusion maps $U_I \longrightarrow B$. If a morphism γ_i in U_i and a morphism γ_j in U_j happen to arise from the same path, lying inside $U_i \cap U_j$, then there is a morphism $\gamma_{ij} \in \mathrm{Mor}(\mathbf{U}_{ij})$ (arising from the same path again, but viewed as lying in $U_i \cap U_j$), that is carried to $\gamma_i \in \mathrm{Mor}(\mathbf{U}_i)$ and to $\gamma_j \in \mathrm{Mor}(\mathbf{U}_j)$ by the 'inclusion' functors $\mathbf{U}_{\{i,j\}} \longrightarrow \mathbf{U}_i$ and $\mathbf{U}_{\{i,j\}} \longrightarrow \mathbf{U}_j$. Then $\mathbf{B}$ along with the 'inclusion' functors $\mathbf{U}_I \longrightarrow \mathbf{B}$ is a co-limit for the functor $\mathbf{U}_.$ given in (2.15).

Suppose for instance the manifold B is the union of just two of the sets, U_i and U_k, with $U_i \cap U_k$ being nonempty. A path γ, for example one that begins and ends in $U_i \cap U_k$ but does not lie entirely inside this intersection, can be broken up into pieces that lie inside one or the other of these sets:

$$\gamma = \gamma_m \circ \ldots \circ \gamma_1,$$

where each γ_j lies either entirely inside U_i or inside U_k. Thus we could view γ, as a morphism of the colimit category $\mathbf{B}$, as being specified by the sequence of morphisms $\gamma_1, \ldots, \gamma_m$, each in either $\mathrm{Mor}(\mathbf{U}_i)$ or $\mathrm{Mor}(\mathbf{U}_k)$, with two such sequences being identified when they have the same composite as paths.

We present this formalism mainly to motivate the thinking behind the method we use later in section 4 to construct a global categorical bundle from local trivial ones.

3. Cocycles: from gerbal to functorial

In this section we construct a cocycle with values in a categorical group, starting with some group-valued locally-defined functions that need not form a cocycle.

All through this paper we will work with a categorical Lie group $\mathbf{G}$, associated with a Lie crossed module (G, H, α, τ), and a categorical Lie group $\mathbf{H}$, with associated Lie crossed module (H, J, α', τ').

3.1. GERBAL COCYCLES. We work with a manifold B, and an open covering

$$\{U_i\}_{i \in \mathcal{I}}.$$

By a *gerbal cocycle* associated with this covering and the Lie crossed module (H, J, α', τ') we mean a collection of smooth functions

$$h_{ik} : U_i \cap U_k \longrightarrow H \qquad \text{and} \qquad j_{ikm} : U_i \cap U_k \cap U_m \longrightarrow J, \tag{3.1}$$

with h_{ik} defined when $U_i \cap U_k \neq \emptyset$ and j_{ikm} defined when $U_i \cap U_k \cap U_m \neq \emptyset$, such that

$$h_{im}(u) = \tau'\big(j_{ikm}(u)\big) h_{ik}(u) h_{km}(u) \qquad \text{for all } u \in U_i \cap U_k \cap U_m. \tag{3.2}$$

The pattern here is that on the right the effect of $\tau'\big(j_{ikm}(u)\big)$ is to 'combine' the subscripts ik and km into im.

We will use the notation U_{ik} and U_{ikm} for intersections:

$$U_{ik} = U_i \cap U_k \qquad \text{and} \qquad U_{ikm} = U_i \cap U_k \cap U_m. \tag{3.3}$$

In terms of the categorical group $\mathbf{H}$ we have a morphism

$$\psi_{ikm}(u) : h_{ik}(u) h_{km}(u) \longrightarrow h_{im}(u) \qquad \text{for all } u \in U_{ikm}, \tag{3.4}$$

where $\psi_{ikm}(u) \in \mathrm{Mor}(\mathbf{H}) \simeq J \rtimes_{\alpha'} H$ is given by

$$\psi_{ikm}(u) = \big(j_{ikm}(u), h_{ik}(u) h_{km}(u)\big).$$

Now let

$$\begin{aligned} g_{ik} &= \tau(h_{ik}) : U_{ik} \longrightarrow G \qquad \text{if } U_{ik} \neq \emptyset; \\ h_{ikm} &= \tau'(j_{ikm}) : U_{ikm} \longrightarrow H \qquad \text{if } U_{ikm} \neq \emptyset. \end{aligned} \tag{3.5}$$

Then by (3.2) we have

$$\begin{aligned}
h_{im}(u) &= h_{ikm}(u)h_{ik}(u)h_{km}(u) \\
g_{im}(u) &= \tau\big(h_{ikm}(u)\big)g_{ik}(u)g_{km}(u)
\end{aligned} \tag{3.6}$$

for all $u \in U_{ikm}$. Thus the system of functions $\{g_{ik}\}$ and $\{h_{ikm}\}$ is a gerbal cocycle associated with the covering $\{U_i\}_{i \in \mathcal{I}}$ and the Lie crossed module (G, H, α, τ).

There are different notational conventions in specifying gerbe data. Ours is consistent with [1, Definition 7].

3.2. A SECOND GERBE RELATION. The conditions (3.6) imply another relation satisfied by the h_{ik} and h_{ikm} (suppressing the point u for notational ease):

$$\begin{aligned}
h_{ijm}\alpha_{g_{ij}}(h_{jkm}) &= h_{ijm}h_{ij}h_{jkm}h_{ij}^{-1} \\
&\qquad \text{(by the second Peiffer identity (2.2))} \\
&= h_{im}h_{jm}^{-1}h_{ij}^{-1} \cdot h_{ij} \cdot h_{jm}h_{km}^{-1}h_{jk}^{-1} \cdot h_{ij}^{-1} \\
&\qquad \text{(using the first relation in (3.6))} \\
&= h_{im}h_{km}^{-1}h_{jk}^{-1}h_{ij}^{-1} \\
&= h_{ikm}h_{ik}h_{km} \cdot h_{km}^{-1}h_{jk}^{-1}h_{ij}^{-1} \quad \text{(again by (3.6))} \\
&= h_{ikm}h_{ik}h_{jk}^{-1}h_{ij}^{-1}.
\end{aligned} \tag{3.7}$$

Using the first relation in (3.6) once again, we obtain

$$h_{ijm}\alpha_{g_{ij}}(h_{jkm}) = h_{ikm}h_{ijk}. \tag{3.8}$$

This along with the second relation in (3.6) ensure that the data $\{g_{ij}\}$ and $\{h_{ijk}\}$ are the local data for a gerbe structure [1, Definition 7].

3.3. CONSTRUCTION OF FUNCTORIAL COCYCLES. We continue to work with an open covering $\{U_i\}_{i \in \mathcal{I}}$ of B, and the corresponding path categories $\mathbf{U}_i$ and overlap categories $\mathbf{U}_{ik}$ as discussed in subsection 2.1.

Starting with a given gerbal cocycle as in (3.1) let us define

$$\theta_{ik} : \mathbf{U}_{ik} \longrightarrow \mathbf{G} \tag{3.9}$$

on objects by

$$\theta_{ik}(u) = g_{ik}(u) \overset{\text{def}}{=} \tau\big(h_{ik}(u)\big) \tag{3.10}$$

for all $u \in U_{ik}$, assumed nonempty, and on morphisms by

$$\mathrm{Mor}(\mathbf{U}_{ik}) \longrightarrow \mathrm{Mor}(\mathbf{G})$$
$$\theta_{ik}(\gamma) = \Big(h_{ik}(\gamma), g_{ik}(\gamma_0)\Big) : g_{ik}(\gamma_0) \longrightarrow \tau\big(h_{ik}(\gamma)\big)g_{ik}(\gamma_0), \tag{3.11}$$

where $\gamma_0 = s(\gamma)$ and

$$h_{ik}(\gamma) = h_{ik}(\gamma_1)h_{ik}(\gamma_0)^{-1}. \tag{3.12}$$

Let us note that in (3.11) the target of the morphism $\theta_{ik}(\gamma)$ is $g_{ik}(\gamma_1)$:

$$t\big(\theta_{ik}(\gamma)\big) = g_{ik}(\gamma_1). \tag{3.13}$$

In this subsection we show that θ_{ik} is a functor and work out some of the significant properties of this system of functors.

Let us verify functoriality of θ_{ik}. If γ and γ' are morphisms in $\mathbf{U}_{ik}$ and $\gamma'_0 = \gamma_1$ then

$$\begin{aligned}
\theta_{ik}(\gamma' \circ \gamma) &= \big(h_{ik}(\gamma'), g_{ik}(\gamma'_0)\big) \circ \big(h_{ik}(\gamma), g_{ik}(\gamma_0)\big) \\
&= \big(h_{ik}(\gamma')h_{ik}(\gamma), g_{ik}(\gamma_0)\big) \\
&= \big(h_{ik}(\gamma'_1)h_{ik}(\gamma_0)^{-1}, g_{ik}(\gamma_0)\big) \\
&= \big(h_{ik}(\gamma' \circ \gamma), g_{ik}(\gamma_0)\big) \\
&= \theta_{ik}(\gamma') \circ \theta_{ik}(\gamma)
\end{aligned} \tag{3.14}$$

furthermore, θ_{ik} clearly maps any identity morphism $i_u : u \longrightarrow u$ in $\mathbf{U}_{ik}$ to the identity morphism

$$(e, g_{ik}(u)) : g_{ik}(u) \longrightarrow g_{ik}(u)$$

in $\mathrm{Mor}(\mathbf{G})$.

If the triple overlap category $\mathbf{U}_{ikm}$ is defined then we can restrict the transition functor to obtain a functor

$$\theta_{i\,k\,|m|} = \theta_{ik}|_{\mathbf{U}_{ikm}} : \mathbf{U}_{ikm} \longrightarrow \mathbf{G}. \tag{3.15}$$

To minimize notational clutter we will drop the subscript $|m|$ and write θ_{ik} for the restricted functor as well, leaving it to the context to make the intended meaning clear.

Let us now verify that on $\mathbf{U}_{ikm}$ there is a natural transformation

$$\mathbb{T}_{ikm} : \theta_{ik}\theta_{km} \Rightarrow \theta_{im} \tag{3.16}$$

given on any object $u \in U_{ikm}$ by

$$\mathbb{T}_{ikm}(u) = \Big(h_{ikm}(u),\, g_{ik}(u)g_{km}(u)\Big) \in H \rtimes_\alpha G \simeq \mathrm{Mor}(\mathbf{G}), \tag{3.17}$$

where h_{ikm} is as given in (3.5). The source of this morphism is the product $g_{ik}(u)g_{km}(u)$ and the target is

$$t\big(\mathbb{T}_{ikm}(u)\big) = \tau\big(h_{ikm}(u)\big)g_{ik}(u)g_{km}(u) = g_{im}(u) = \theta_{im}(u)$$

by the second equation of the gerbal relations (3.6).

3.4. PROPOSITION. *With notation as above, $\mathbb{T}_{ikm}$ is a natural transformation.*

Thus $\{\theta_{ik}\}$, along with $\{\mathbb{T}_{ikm}\}$, is a *functorial cocycle*, by which we mean a system of functors $\theta_{ik} : \mathbf{U}_{ik} \longrightarrow \mathbf{G}$ satisfying the relation

$$\mathbb{T}_{ikm} : \theta_{ik}\theta_{km} \Rightarrow \theta_{im}, \tag{3.18}$$

where $\mathbb{T}_{ikm}$ are natural transformations.

PROOF. The main task is to verify that for any morphism $\gamma : u \longrightarrow v$ in $\mathbf{U}_{ikm}$ the diagram

$$\theta_{ik}(u)\theta_{km}(u) \xrightarrow{\mathbb{T}_{ikm}(u)} \theta_{im}(u) \tag{3.19}$$

$$\theta_{ik}(\gamma)\theta_{km}(\gamma) \downarrow \qquad\qquad \downarrow \theta_{im}(\gamma)$$

$$\theta_{ik}(v)\theta_{km}(v) \xrightarrow[\mathbb{T}_{ikm}(v)]{} \theta_{im}(v)$$

commutes. To this end let us first compute $\mathbb{T}_{ikm}(v) \circ \big(\theta_{ik}(\gamma)\theta_{km}(\gamma)\big)$.

Let us recall the way composition works in $\mathrm{Mor}(\mathbf{G}) \simeq H \rtimes_\alpha G$:

$$(h', g') \circ (h, g) = (hh', g) \qquad \text{if } \tau(h)g = g', \tag{3.20}$$

bearing in mind that $t(h, g) = \tau(h)g$ and $s(h', g') = g'$. Multiplication is given by

$$(h_2, g_2)(h_1, g_1) = \big(h_2 g_2 h_1 g_2^{-1}, g_2 g_1\big). \tag{3.21}$$

Thus, recalling $\theta_{ik}(\gamma)$ from (3.11), we have

$$\theta_{ik}(\gamma)\theta_{km}(\gamma) = \big(h_{ik}(\gamma)g_{ik}(\gamma_0)h_{km}(\gamma)g_{ik}(\gamma_0)^{-1},\ g_{ik}(\gamma_0)g_{km}(\gamma_0)\big). \tag{3.22}$$

and so

$$
\begin{aligned}
&\mathbb{T}_{ikm}(v) \circ \big(\theta_{ik}(\gamma)\theta_{km}(\gamma)\big) \\
&= \big(h_{ikm}(v), g_{ik}(v)g_{km}(v)\big) \circ \big(h_{ik}(\gamma)g_{ik}(\gamma_0)h_{km}(\gamma)g_{ik}(\gamma_0)^{-1}, \\
&\hspace{6cm} g_{ik}(\gamma_0)g_{km}(\gamma_0)\big) \\
&= \big(h_{ikm}(v)h_{ik}(\gamma)g_{ik}(\gamma_0)h_{km}(\gamma)g_{ik}(\gamma_0)^{-1},\ g_{ik}(\gamma_0)g_{km}(\gamma_0)\big).
\end{aligned}
\tag{3.23}
$$

Focusing on the H-component we have:

$$
\begin{aligned}
&h_{ikm}(v)h_{ik}(\gamma)g_{ik}(\gamma_0)h_{km}(\gamma)g_{ik}(\gamma_0)^{-1} \\
&= h_{ikm}(v)h_{ik}(\gamma)h_{ik}(\gamma_0)h_{km}(\gamma)h_{ik}(\gamma_0)^{-1},
\end{aligned}
\tag{3.24}
$$

upon using the second Peiffer identity (2.2):

$$\tau(h)h_1\tau(h)^{-1} = hh_1h^{-1} \qquad \text{for all } h, h_1 \in H. \tag{3.25}$$

Continuing, we have

$$
\begin{aligned}
&h_{ikm}(v)h_{ik}(\gamma)g_{ik}(\gamma_0)h_{km}(\gamma)g_{ik}(\gamma_0)^{-1} \\
&= h_{ikm}(v)h_{ik}(\gamma_1)h_{ik}(\gamma_0)^{-1} \cdot h_{ik}(\gamma_0) \cdot h_{km}(\gamma_1)h_{km}(\gamma_0)^{-1} \cdot h_{ik}(\gamma_0)^{-1} \\
&= h_{ikm}(v)h_{ik}(\gamma_1)h_{km}(\gamma_1)h_{km}(\gamma_0)^{-1}h_{ik}(\gamma_0)^{-1} \\
&= \Big(h_{ikm}(v)h_{ik}(v)h_{km}(v)\Big) \Big(h_{ikm}(u)h_{ik}(u)h_{km}(u)\Big)^{-1} h_{ikm}(u) \\
&= h_{im}(v)h_{im}(u)^{-1}h_{ikm}(u)
\end{aligned}
\tag{3.26}
$$

where, in the last equality, we have used the first gerbal relation in (3.6). Thus

$$
\begin{aligned}
\mathbb{T}_{ikm}(v) &\circ \big(\theta_{ik}(\gamma)\theta_{km}(\gamma)\big) \\
&= \big(h_{im}(v)h_{im}(u)^{-1}h_{ikm}(u),\ g_{ik}(\gamma_0)g_{km}(\gamma_0)\big).
\end{aligned}
\tag{3.27}
$$

On the other hand, $\theta_{im}(\gamma) \circ \mathbb{T}_{ikm}(u)$ is given by:

$$
\theta_{im}(\gamma) \circ \mathbb{T}_{ikm}(u) = \big(h_{im}(\gamma_1)h_{im}(\gamma_0)^{-1}h_{ikm}(u),\ g_{ik}(u)g_{km}(u)\big),
\tag{3.28}
$$

on using the expression for $\mathbb{T}_{ikm}(u)$ from (3.17). Comparing with (3.27) we conclude that the diagram (3.19) commutes. ∎

Next let us make an observation about the product $\theta_{ik}(\gamma)\theta_{km}(\gamma)$:

3.5. PROPOSITION. *With notation as above,*

$$
\Theta_{ikm}(\gamma)\theta_{ik}(\gamma)\theta_{km}(\gamma) = \theta_{im}(\gamma)
\tag{3.29}
$$

for all $\gamma \in \mathrm{Mor}(\mathbf{U}_{ikm})$, where

$$
\Theta_{ikm}(\gamma) = \big(h_{ikm}(\gamma), g_{ikm}(\gamma_0)\big),
\tag{3.30}
$$

with

$$
h_{ikm}(\gamma) = h_{ikm}(\gamma_1)h_{ikm}(\gamma_0)^{-1},
$$

where on the right we have the function h_{ikm} on U_{ikm} as in (3.5), and $g_{ikm}(u) = \tau\big(h_{ikm}(u)\big)$ for all objects u of $\mathbf{U}_{ikm}$.

PROOF. The H-component of $\theta_{ik}(\gamma)\theta_{km}(\gamma)$ is

$$
\begin{aligned}
\Big(\theta_{ik}(\gamma)&\theta_{km}(\gamma)\Big)_H \\
&= h_{ik}(\gamma)g_{ik}(\gamma_0)h_{km}(\gamma)g_{ik}(\gamma_0)^{-1} \\
&= h_{ik}(\gamma)h_{ik}(\gamma_0)h_{km}(\gamma)h_{ik}(\gamma_0)^{-1} \\
&= h_{ik}(\gamma_1)h_{ik}(\gamma_0)^{-1} \cdot h_{ik}(\gamma_0) \cdot h_{km}(\gamma_1)h_{km}(\gamma_0)^{-1} \cdot h_{ik}(\gamma_0)^{-1} \\
&= h_{ik}(\gamma_1)h_{km}(\gamma_1)\big(h_{ik}(\gamma_0)h_{km}(\gamma_0)\big)^{-1}.
\end{aligned}
\tag{3.31}
$$

On the other hand, switching notation and writing hg for $(h,g) \in H \times_\alpha G$, and $g_{ikm} = \tau(h_{ikm})$, we have:

$$
\begin{aligned}
\Theta_{ikm}(\gamma)&^{-1}\theta_{im}(\gamma) \\
&= \big(h_{ikm}(\gamma_1)h_{ikm}(\gamma_0)^{-1}g_{ikm}(\gamma_0)\big)^{-1}\theta_{im}(\gamma) \\
&= g_{ikm}(\gamma_0)^{-1}h_{ikm}(\gamma_0)h_{ikm}(\gamma_1)^{-1}\,h_{im}(\gamma_1)h_{im}(\gamma_0)^{-1}g_{im}(\gamma_0).
\end{aligned}
\tag{3.32}
$$

(Here we have used the validity of working with inverses in the notation $hg = (h, g)$, as explained in (2.9).) Inserting $g_{ikm}(\gamma_0)g_{ikm}(\gamma_0)^{-1}$ before the term $g_{im}(\gamma_0)$, we have:

$$
\begin{aligned}
\Theta_{ikm}&(\gamma)^{-1}\theta_{im}(\gamma) \\
&= g_{ikm}(\gamma_0)^{-1}h_{ikm}(\gamma_0) \cdot h_{ikm}(\gamma_1)^{-1} h_{im}(\gamma_1) \cdot h_{im}(\gamma_0)^{-1}g_{ikm}(\gamma_0)\cdot \\
&\qquad \cdot g_{ikm}(\gamma_0)^{-1}g_{im}(\gamma_0)
\end{aligned}
\tag{3.33}
$$

$$
\begin{aligned}
&= h_{ikm}(\gamma_0)^{-1}h_{ikm}(\gamma_0) \cdot h_{ikm}(\gamma_1)^{-1} h_{im}(\gamma_1) \cdot h_{im}(\gamma_0)^{-1}h_{ikm}(\gamma_0)\cdot \\
&\qquad \cdot g_{ikm}(\gamma_0)^{-1}g_{im}(\gamma_0)
\end{aligned}
$$

where we have used the second Peiffer identity (2.2) to switch the conjugation by g_{ikm} to conjugation by h_{ikm}. Next, using the gerbal relations (3.6) we conclude that

$$
\begin{aligned}
\Theta_{ikm}&(\gamma)^{-1}\theta_{im}(\gamma) \\
&= h_{ik}(\gamma_1)h_{km}(\gamma_1)\big(h_{ik}(\gamma_0)h_{km}(\gamma_0)\big)^{-1} \cdot g_{ikm}(\gamma_0)^{-1}g_{im}(\gamma_0)
\end{aligned}
\tag{3.34}
$$

The H-component of this clearly matches the right hand side of (3.31):

$$
\big(\Theta_{ikm}(\gamma)^{-1}\theta_{im}(\gamma)\big)_H = \big(\theta_{ik}(\gamma)\theta_{km}(\gamma)\big)_H.
\tag{3.35}
$$

The G-component of $\theta_{ik}(\gamma)\theta_{km}(\gamma)$ is

$$
g_{ik}(\gamma)g_{km}(\gamma)
$$

which again matches the G-component $g_{ikm}(\gamma_0)^{-1}g_{im}(\gamma_0)$ on the right hand side of (3.34) again by the gerbal relations (3.6).

We have thus shown that $\Theta_{ikm}(\gamma)^{-1}\theta_{im}(\gamma)$ equals $\theta_{ik}(\gamma)\theta_{km}(\gamma)$.   ∎

3.6. QUOTIENTS FOR COCYCLES. We continue with the same notation and framework. In particular, (G, H, α, τ) and (H, J, α', τ') are Lie crossed modules. The image $\tau'(J)$ is a normal subgroup of H, as noted earlier in (2.3). Then $\tau\tau'(J)$ is a subgroup of G. We assume that $\tau'(J)$ and $\tau\tau'(J)$ are *closed* subgroups of H and of G, respectively. Let us observe that $\tau\tau'(J)$ is normal inside $\tau(H)$:

$$
\tau(h)\tau\tau'(j)\tau(h)^{-1} = \tau\big(h\tau'(j)h^{-1}\big) = \tau\tau'\big(\alpha'_h(j)\big)
\tag{3.36}
$$

for all $h \in H$ and $j \in J$. Let $\overline{G}$ be the quotient set

$$
\overline{G} = G/\tau\tau'(J),
\tag{3.37}
$$

and $\overline{G}_\tau$ the quotient group

$$
\overline{G}_\tau = \tau(H)/\tau\tau'(J).
\tag{3.38}
$$

We recall from (3.5) the functions

$$
g_{ik} = \tau(h_{ik}) : U_{ik} \longrightarrow \tau(H) \subset G.
$$

Thus, by the normality observation (3.36),

$$g_{ik}(u)\tau\tau'(J)g_{ik}(u)^{-1} = \tau\tau'(J),$$

for all $u \in U_{ik}$. Let us recall from (3.6) the gerbal relation

$$g_{im}(u) = \tau\big(h_{ikm}(u)\big)g_{ik}(u)g_{km}(u) \tag{3.39}$$

for all $u \in U_{ikm}$. Thus the functions g_{ik} form a cocycle modulo $\tau\tau'(J)$. Working with the $\overline{G}_\tau$-valued functions

$$\overline{g}_{ik} : U_{ik} \longrightarrow \overline{G}_\tau : u \mapsto \overline{g}_{ik}(u) = g_{ik}(u)\tau\tau'(J) \qquad \text{for } u \in U_{ikm}, \tag{3.40}$$

we have then

$$\overline{g}_{ik}(u)\overline{g}_{km}(u) = \overline{g}_{im}(u) \qquad \text{for all } u \in U_{ikm}. \tag{3.41}$$

Thus, $\{\overline{g}_{ik}\}$ is a genuine $\overline{G}_\tau$-valued cocycle, associated to the covering $\{U_i\}_{i\in\mathcal{I}}$, in the traditional sense.

3.7. A BUNDLE FROM THE COCYCLE. Let

$$X \longrightarrow B$$

be the bundle that is specified by the open covering $\{U_i\}_{i\in\mathcal{I}}$ and the $\overline{G}$-valued transition functions $\{\overline{g}_{ik}\}$. A point of $\mathbf{X}$ is an equivalence class

$$[i, u, \overline{g}],$$

where $(i, u, \overline{g}), (j, v, \overline{g}')$, with $u \in U_i$, $v \in U_j$, and $\overline{g}, \overline{g}' \in \overline{G}_\tau$, are 'equivalent' if $v = u$ and

$$\overline{g}' = \overline{g}_{ji}(u)\overline{g}. \tag{3.42}$$

That this is in fact an equivalence relation on the set

$$\cup_{i\in\mathcal{I}}\{i\} \times U_i \times \overline{G}_\tau$$

is readily checked. We will return to more on this in section 4.

3.8. MORPHISM COCYCLES. Turning now to morphisms, let us recall the relation (3.29):

$$\Theta_{ikm}(\gamma)\theta_{ik}(\gamma)\theta_{km}(\gamma) = \theta_{im}(\gamma) \tag{3.43}$$

for all $\gamma \in \text{Mor}(\mathbf{U}_{ikm})$, where

$$\begin{aligned}
\Theta_{ikm}(\gamma) &= \Big(h_{ikm}(\gamma), g_{ikm}(\gamma_0)\Big) = \Big(\tau'\big(j_{ikm}(\gamma)\big), \tau\big(h_{ikm}(\gamma_0)\big)\Big) \\
&= \Big(\tau'\big(j_{ikm}(\gamma_1)j_{ikm}(\gamma_0)^{-1}\big), \tau\tau'\big(j_{ikm}(\gamma_0)\big)\Big),
\end{aligned} \tag{3.44}$$

and
$$\theta_{im}(\gamma) = \big(h_{im}(\gamma), g_{im}(\gamma_0)\big). \tag{3.45}$$

Thus the $H \rtimes_\alpha G$-valued functions θ_{ij} form a cocycle modulo the elements of the form $\big(\tau'(j), \tau\tau'(j')\big)$ in $H \rtimes_\alpha G$, where $j, j' \in J$. Let us denote the set of all such elements by J_H:

$$J_H = \{\big(\tau'(j), \tau\tau'(j')\big) \;:\; j, j' \in J\} \subset H \rtimes_\alpha G. \tag{3.46}$$

We now verify that J_H is a normal subgroup of $H \rtimes_\alpha \tau(H)$, so that there is a useful notion of equality 'modulo' J_H.

3.9. PROPOSITION. *The mapping*

$$\overline{\tau} : J \rtimes_{\alpha'} H \longrightarrow H \rtimes_\alpha G : (j, h) \mapsto \big(\tau'(j), \tau(h)\big) \tag{3.47}$$

is a homomorphism. The set J_H forms a normal subgroup of $H \rtimes_\alpha \tau(H) \subset \mathrm{Mor}(\mathbf{G})$.

PROOF. First let us check that $\overline{\tau}$ is a homomorphism:

$$\begin{aligned}
\big(\tau'(j_2), \tau(h_2)\big)\big(\tau'(j_1), \tau(h_1)\big) &= \Big(\tau'(j_2)\alpha_{\tau(h_2)}\big(\tau'(j_1)\big),\; \tau(h_2)\tau(h_1)\Big) \\
&= \Big(\tau'(j_2)h_2\tau'(j_1)h_2^{-1},\; \tau(h_2h_1)\Big) \\
&\quad \text{by the second Peiffer identity in (2.2)} \\
&= \Big(\tau'(j_2)\tau'\big(\alpha'_{h_2}(j_1)\big),\; \tau(h_2h_1)\Big) \\
&\quad \text{by the first Peiffer identity in (2.2)} \\
&= \Big(\tau'\big(j_2\alpha'_{h_2}(j_1)\big), \tau(h_2h_1)\Big).
\end{aligned} \tag{3.48}$$

Hence J_H, being the image under $\overline{\tau}$ of the subgroup $J \rtimes_{\alpha'} \tau'(J) \subset J \rtimes_{\alpha'} H$, is a subgroup of $H \rtimes_\alpha G$. This image is clearly contained inside $H \rtimes_\alpha \tau(H)$.

Let us now work out how elements of

$$\mathrm{Im}(\overline{\tau}) = \tau'(J) \rtimes_\alpha \tau(H)$$

behave under conjugation by elements of $H \rtimes_\alpha \tau(H)$. The advantage of working with the smaller subgroup $H \rtimes_\alpha \tau(H)$ rather than $H \rtimes_\alpha G$ is that we have the relation

$$\alpha_g\big(\tau'(j)\big) = h\tau'(j)h^{-1} = \tau'\big(\alpha'_h(j)\big) \in \tau'(J),$$

if $g = \tau(h)$, $h \in H$ and $j \in J$. Now let $h, h_0, h_1 \in H$ and $j \in J$; then, working in the

group $H \rtimes_\alpha G$, we have the conjugation

$$
\begin{aligned}
\big(h, \tau(h_0)\big)&\big(\tau'(j), \tau(h_1)\big)\big(h, \tau(h_0)\big)^{-1} \\
&= \Big(h\alpha_{\tau(h_0)}\big(\tau'(j)\big), \, \tau(h_0 h_1)\Big)\big(h, \tau(h_0)\big)^{-1} \\
&= \Big(h\alpha_{\tau(h_0)}\big(\tau'(j)\big), \, \tau(h_0 h_1)\Big)\Big(\alpha_{\tau(h_0)^{-1}}(h^{-1}), \tau(h_0)^{-1}\Big) \\
&= \Big(h\alpha_{\tau(h_0)}\big(\tau'(j)\big)\alpha_{\tau(h_0 h_1)}\alpha_{\tau(h_0)^{-1}}(h^{-1}), \, \tau(h_0 h_1 h_0^{-1})\Big) \\
&= \Big(h \cdot h_0\tau'(j)h_0^{-1} \cdot (h_0 h_1 h_0^{-1})h^{-1}(h_0 h_1 h_0^{-1})^{-1}, \, \tau(h_0 h_1 h_0^{-1})\Big) \\
&= \Big(\tau'\big(\alpha'_{hh_0}(j)\big) \cdot hh_0 h_1 h_0^{-1}h^{-1} \cdot h_0 h_1^{-1}h_0^{-1}, \, \tau(h_0 h_1 h_0^{-1})\Big).
\end{aligned}
\tag{3.49}
$$

As it stands it is not apparent if this is an element of J_H; however, we need to insert the condition that the second component of $\big(\tau'(j), \tau(h_1)\big)$ is in fact in $\tau\tau'(J)$, by requiring that

$$
h_1 = \tau'(j_1),
$$

for some $j_1 \in J$. Then we have

$$
\begin{aligned}
\big(h, \tau(h_0)\big)&\big(\tau'(j), \tau(h_1)\big)\big(h, \tau(h_0)\big)^{-1} \\
&= \Big(\tau'\big(\alpha'_{hh_0}(j)\big) \cdot hh_0 h_1 h_0^{-1}h^{-1} \cdot h_0 h_1^{-1}h_0^{-1}, \, \tau(h_0 h_1 h_0^{-1})\Big) \\
&= \Big(\tau'\big(\alpha'_{hh_0}(j)\alpha'_{hh_0}(j_1)\alpha'_{h_0}(j_1^{-1})\big), \, \tau\tau'\big(\alpha'_{h_0}(j_1)\big)\Big),
\end{aligned}
\tag{3.50}
$$

which is indeed an element of J_H. ∎

3.10. The quotient category $\overline{\mathbf{G}}$.

We denote by $\overline{\mathbf{G}}$ the pair comprised of the object set

$$
\mathrm{Obj}(\overline{\mathbf{G}}) = G/\tau\tau'(J)
\tag{3.51}
$$

and the morphism set

$$
\mathrm{Mor}(\overline{\mathbf{G}}) = \big(H \rtimes_\alpha G\big)/J_H,
\tag{3.52}
$$

where, as before,

$$
J_H = \tau'(J) \times \tau\tau'(J) \subset H \rtimes_\alpha \tau(H) \subset H \rtimes_\alpha G = \mathrm{Mor}(\mathbf{G}).
$$

Let us note that J_H *is closed under composition*: if $f'_2 = \big(\tau'(j_2), \tau\tau'(j'_2)\big)$ and $f'_1 = \big(\tau'(j_1), \tau\tau'(j'_1)\big)$ are such that the composition $f'_2 \circ f'_1$ is defined as a morphism of $\mathbf{G}$ then

$$
f'_2 \circ f'_1 = \big(\tau'(j_2 j_1), \tau\tau'(j_1)\big) \in J_H.
\tag{3.53}
$$

We define source and target maps

$$
\begin{aligned}
s &: \mathrm{Mor}(\overline{\mathbf{G}}) \longrightarrow \mathrm{Obj}(\overline{\mathbf{G}}) \\
t &: \mathrm{Mor}(\overline{\mathbf{G}}) \longrightarrow \mathrm{Obj}(\overline{\mathbf{G}})
\end{aligned}
\tag{3.54}
$$

to be the maps induced by the source and target maps in the category $\mathbf{G}$. The following result verifies that these maps are well-defined and lead to a category $\overline{\mathbf{G}}$.

3.11. PROPOSITION. $\overline{\mathbf{G}}$, *as specified above, is a category under composition, source and target as inherited from* $\mathbf{G}$. *The quotient mappings*

$$\begin{aligned}
\mathrm{Obj}(\mathbf{G}) &\longrightarrow \mathrm{Obj}(\overline{\mathbf{G}}) : g \mapsto g\tau\tau'(J) \\
\mathrm{Mor}(\mathbf{G}) &\longrightarrow \mathrm{Mor}(\overline{\mathbf{G}}) : (h,g) \mapsto (h,g)J_H
\end{aligned} \tag{3.55}$$

specify a functor

$$q : \mathbf{G} \longrightarrow \overline{\mathbf{G}}. \tag{3.56}$$

PROOF. The source map

$$s : \mathrm{Mor}(\mathbf{G}) \longrightarrow \mathrm{Obj}(\mathbf{G})$$

maps an element $\left(\tau'(j_1), \tau\tau'(j_2)\right)$ of the subgroup

$$J_H = \tau'(J) \times \tau\tau'(J) \subset H \rtimes_\alpha \tau(H) \subset H \rtimes_\alpha G$$

to $\tau\tau'(j_2)$, which lies in $\tau\tau'(J)$. The target map

$$t : \mathrm{Mor}(\mathbf{G}) \longrightarrow \mathrm{Obj}(\mathbf{G})$$

carries $\left(\tau'(j_1), \tau\tau'(j_2)\right)$ to $\tau\tau'(j_1 j_2)$, which is also in $\tau\tau'(J)$. Hence s and t induce well-defined maps

$$\left(H \rtimes_\alpha G\right)/J_H \longrightarrow G/\tau\tau'(J) \tag{3.57}$$

where the set on the left is the quotient set of cosets.

Let $f_1', f_2' \in J_H$ be such that the composition

$$(f_2 f_2') \circ (f_1 f_1')$$

in $\mathrm{Mor}(\mathbf{G})$ is meaningful. Then

$$(f_2 f_2') \circ (f_1 f_1') = (f_2 \circ f_1)(f_2' \circ f_1') \equiv (f_2 \circ f_1) \qquad \mathrm{mod}\ J_H, \tag{3.58}$$

by (3.53). Hence the composition law in $\mathrm{Mor}(\mathbf{G})$ induces a well-defined composition law in the quotient $\left(H \rtimes_\alpha G\right)/J_H$.

It is clear that for each $x = g\tau\tau'(J) \in \overline{G}$ the morphism $(e,g) \in H \rtimes_\alpha G$ induces the identity morphism 1_x.

Thus $\overline{\mathbf{G}}$ is a category, and the way we have defined source, target, identity morphisms, and composition in $\overline{\mathbf{G}}$ ensures that $q : \mathbf{G} \longrightarrow \overline{\mathbf{G}}$ is a functor.   ■

3.12. THE CATEGORICAL GROUP $\overline{\mathbf{G}}_\tau$. Let us recall that J_H is a normal subgroup inside $H \rtimes_\alpha \tau(H)$ rather than in $H \rtimes_\alpha G$. Thus the quotient

$$(H \rtimes_\alpha \tau(H))/J_H$$

is actually a group. Moreover, the subgroup $H \rtimes_\alpha \tau(H)$ is closed under the composition law in $H \rtimes_\alpha G$; in fact if

$$f_1 = \big(h_1, \tau(h_1')\big), f_2 = \big(h_2, \tau(h_2')\big) \in H \rtimes_\alpha \tau(H) \subset \mathrm{Mor}(\mathbf{G}),$$

are morphisms for which the composition $f_2 \circ f_1$ is meaningful, that is

$$\tau(h_2') = \tau(h_1 h_1'),$$

we have

$$f_2 \circ f_1 = \big(h_2 h_1, \tau(h_1')\big) \in H \rtimes_\alpha \tau(H). \tag{3.59}$$

As we have seen in (3.53) J_H is also closed under composition and by (3.58) the composition law in $H \rtimes_\alpha G$ then induces a well-defined operation on the quotient

$$\big(H \rtimes_\alpha \tau(H)\big)/J_H.$$

In summary, for the subcategory $\overline{\mathbf{G}}_\tau$ of $\overline{\mathbf{G}}$ specified by

$$\begin{aligned}
\mathrm{Obj}(\overline{\mathbf{G}}_\tau) &= \tau(H)/\tau\tau'(J) \\
\mathrm{Mor}(\overline{\mathbf{G}}_\tau) &= \big(H \rtimes_\alpha \tau(H)\big)/J_H,
\end{aligned} \tag{3.60}$$

both object set and morphism set are groups, with the obvious quotient group structures. Moreover, since s and t are clearly group homomorphisms, $\overline{\mathbf{G}}_\tau$ is in fact a categorical group. Furthermore, the 'quotient' functor $q : \mathbf{G} \longrightarrow \overline{\mathbf{G}}$ defined in (3.56) restricts to a functor

$$q_\tau : \mathbf{G}_\tau \longrightarrow \overline{\mathbf{G}}_\tau, \tag{3.61}$$

where $\mathbf{G}_\tau$ is the subcategory of $\mathbf{G}$ whose object set is $\tau(H)$ and whose morphisms are those morphism of $\mathbf{G}$ whose sources lie in $\tau(H)$. It is readily checked that q_τ is a homomorphism both on objects and on morphisms.

3.13. CATEGORICAL GROUP EXTENSIONS. Another way to interpret the results of preceding section is in terms of *extensions of categorical groups*. We work, as before, with the categorical Lie groups $\mathbf{H}$ and $\mathbf{G}$, with associated Lie crossed modules (H, J, α', τ') and (G, H, α, τ), respectively. By a *homomorphism* $\lambda : \mathbf{H} \longrightarrow \mathbf{G}$ we mean a functor

$$\lambda : \mathbf{H} \longrightarrow \mathbf{G}$$

such that $\lambda : \mathrm{Obj}(\mathbf{H}) = H \longrightarrow \mathrm{Obj}(\mathbf{G}) = G$ and $\lambda : \mathrm{Mor}(\mathbf{H}) = H \rtimes_\alpha J \longrightarrow \mathrm{Mor}(\mathbf{G}) = H \rtimes_{\alpha'} G$ both are Lie-group homomorphisms.

Let $\mathbf{H}_{\tau'}$ be the categorical group, with object group $\tau'(J) \subset H$ and morphism group $J \rtimes_{\alpha'} \tau'(J) \subset J \rtimes_{\alpha'} H$:

$$
\begin{aligned}
\mathrm{Obj}(\mathbf{H}_{\tau'}) &:= \tau'(J) \subset H, \\
\mathrm{Mor}(\mathbf{H}_{\tau'}) &:= J \rtimes_{\alpha'} \tau'(J) \subset J \rtimes_{\alpha'} H.
\end{aligned}
\tag{3.62}
$$

Likewise, let $\mathbf{G}_\tau$ be the categorical group with

$$
\begin{aligned}
\mathrm{Obj}(\mathbf{G}_\tau) &:= \tau(H) \subset G, \\
\mathrm{Mor}(\mathbf{G}_\tau) &:= H \rtimes_\alpha \tau(H) \subset H \rtimes_\alpha G.
\end{aligned}
\tag{3.63}
$$

Restricting the group homomorphism

$$
\overline{\tau} : J \rtimes_{\alpha'} H \longrightarrow H \rtimes_\alpha G
$$

in Proposition 3.9 to $J \rtimes_{\alpha'} \tau'(J)$ we obtain a homomorphism

$$
\overline{\tau}|_{J \rtimes_{\alpha'} \tau'(J)} : J \rtimes_{\alpha'} \tau'(J) \longrightarrow H \rtimes_\alpha \tau(H) \subset H \rtimes_\alpha G,
\tag{3.64}
$$

whereas $\tau : H \longrightarrow G$ restricts to a group homomorphism

$$
\tau|_{\tau'(J)} : \tau'(J) \longrightarrow \tau(H).
\tag{3.65}
$$

Moreover the pair $\left(\overline{\tau}|_{J \rtimes_{\alpha'} \tau'(J)}, \tau|_{\tau'(J)}\right)$ specifies a functor $\widehat{\tau}$

$$
\begin{aligned}
\widehat{\tau} &: \mathbf{H}_{\tau'} \longrightarrow \mathbf{G}_\tau, \\
\tau|_{\tau'(J)} &: \mathrm{Obj}(\mathbf{H}_{\tau'}) \longrightarrow \mathrm{Obj}(\mathbf{G}_\tau), \\
\overline{\tau}|_{J \rtimes_{\alpha'} \tau'(J)} &: \mathrm{Mor}(\mathbf{H}_{\tau'}) \longrightarrow \mathrm{Mor}(\mathbf{G}_\tau).
\end{aligned}
\tag{3.66}
$$

The diagram

$$
\begin{array}{ccccccc}
J \rtimes H & \longleftarrow & J \rtimes \tau'(J) & \xrightarrow{\ \overline{\tau}|_{J \rtimes \tau'(J)}\ } & H \rtimes \tau(H) & \lhook\joinrel\longrightarrow & H \rtimes G \\
{\scriptstyle t}\big\downarrow{\scriptstyle s} & & {\scriptstyle t}\big\downarrow{\scriptstyle s} & & {\scriptstyle t}\big\downarrow{\scriptstyle s} & & {\scriptstyle t}\big\downarrow{\scriptstyle s} \\
H & \longleftarrow & \tau'(J) & \xrightarrow[\ \tau|_{\tau'(J)}\ ]{} & \tau(H) & \lhook\joinrel\longrightarrow & G
\end{array}
$$

illustrates the morphisms of interest.

3.14. PROPOSITION. *Equation* (3.66) *specifies a functor; moreover,*

$$
\widehat{\tau} : \mathbf{H}_{\tau'} \longrightarrow \mathbf{G}_\tau
$$

is a homomorphism between categorical Lie groups.

PROOF. Recalling the definition of $\overline{\tau}$ in (3.47), we write for composable morphisms $(j_2, \tau'(\widetilde{j_2})), (j_1, \tau'(\widetilde{j_1})) \in J \rtimes_{\alpha'} \tau'(J)$,

$$
\begin{aligned}
\overline{\tau}&\big((j_2, \tau'(\widetilde{j_2})) \circ (j_1, \tau'(\widetilde{j_1}))\big) \\
&= \overline{\tau}\big(j_2 j_1, \tau'(\widetilde{j_1})\big) \\
&= \big(\tau'(j_2 j_1), \tau\tau'(\widetilde{j_1})\big) \\
&= \big(\tau'(j_2)\tau'(j_1), \tau\tau'(\widetilde{j_1})\big) \\
&= \big(\tau'(j_2), \tau\tau'(\widetilde{j_2})\big) \circ \big(\tau'(j_1), \tau\tau'(\widetilde{j_1})\big) \\
&= \overline{\tau}\big(j_2, \tau'(\widetilde{j_2})\big) \circ \overline{\tau}\big(j_1, \tau'(\widetilde{j_1})\big).
\end{aligned}
\tag{3.67}
$$

■

Note (3.46) and (3.66) imply

$$
\begin{aligned}
\mathrm{Obj}\big(\mathrm{Im}(\widehat{\tau})\big) &= \tau\tau'(J) \subset \tau(H) \subset G, \\
\mathrm{Mor}\big(\mathrm{Im}(\widehat{\tau})\big) &= J_H \subset H \rtimes_\alpha \tau(H) \subset H \rtimes_\alpha G.
\end{aligned}
\tag{3.68}
$$

On the other hand according to Proposition 3.9 and (3.36) J_H, $\tau\tau'(J)$ are respectively normal subgroups of $H \rtimes_\alpha \tau(H)$ and $\tau(H)$. Thus corresponding to the functor $\widehat{\tau}$, we have the quotient categorical group $\overline{\mathbf{G}}_\tau$ (constructed in subsection 3.12)

$$
\overline{\mathbf{G}}_\tau = \mathbf{G}_\tau / \widehat{\tau}(\mathbf{H}_{\tau'}),
$$

and the quotient functor (see (3.61))

$$
q_\tau : \mathbf{G}_\tau \longrightarrow \mathbf{G}_\tau / \widehat{\tau}(\mathbf{H}_{\tau'}) = \overline{\mathbf{G}}_\tau.
$$

Consequently we have a short exact sequence of categorical groups (crossed modules). In other words the categorical group $\mathbf{G}_\tau$ is an extension of the categorical group $\overline{\mathbf{G}}_\tau$ by $\mathrm{Im}(\widehat{\tau})$:

$$
\begin{array}{ccccccc}
\{e_H\} \times \{e_G\} \longrightarrow & J_H \hookrightarrow & H \rtimes \tau(H) \xrightarrow{q_\tau} & H \rtimes \tau(H)/J_H \rightarrow \{e_H\} \times \{e_G\} \\
& {\scriptstyle t}\downarrow{\scriptstyle s} & {\scriptstyle t}\downarrow{\scriptstyle s} & {\scriptstyle t}\downarrow{\scriptstyle s} \\
\{e_G\} \longrightarrow & \tau\tau'(J) \hookrightarrow & \tau(H) \xrightarrow{q_\tau} & \tau(H)/\tau\tau'(J) \longrightarrow \{e_G\}
\end{array}
$$

Let us turn our attention to functorial cocycles constructed in subsection 3.3. As before let $\{U_i\}_{i\in\mathcal{I}}$ be an open covering of B, and $\mathbf{U}_i$, $\mathbf{U}_{ik}$ respectively be the corresponding path categories and overlap categories as discussed in subsection 2.1. We have shown that a gerbal cocycle given by $\{h_{ik}\}$ and $\{j_{ikm}\}$, as in (3.1), leads to functorial cocycles

$$
\theta_{ik} : \mathbf{U}_{ik} \longrightarrow \mathbf{G}_\tau \subset \mathbf{G},
$$

constructed in (3.9)–(3.11). Furthermore by Proposition 3.4, on nonempty triple overlaps $\mathbf{U}_{ikm}$ we have natural transformations

$$\mathbb{T}_{ikm} : \theta_{ik}\theta_{km} \Longrightarrow \theta_{im}.$$

More conveniently, using Proposition 3.5, for $\gamma \in \mathrm{Mor}(\mathbf{U}_{ikm})$ we can write

$$\Theta_{ikm}(\gamma)\theta_{ik}(\gamma)\theta_{km}(\gamma) = \theta_{im}(\gamma)$$

and for any $u \in \mathrm{Obj}(\mathbf{U}_{ikm})$ we have (see (3.39))

$$g_{im}(u) = \tau\big(h_{ikm}(u)\big)g_{ik}(u)g_{km}(u).$$

In fact by (3.44) and (3.5) we have

$$\begin{aligned}
\Theta_{ikm}(\gamma) &\in J_H \subset H \rtimes \tau(H) \\
\tau\big(h_{ikm}(u)\big) &\in \tau\tau'(J) \subset \tau(H)
\end{aligned} \tag{3.69}$$

and in turn we obtain a functor $\widehat{\Theta}_{ikm} : \mathbf{U}_{ikm} \longrightarrow \mathrm{Im}(\widehat{\tau})$

$$\begin{aligned}
\widehat{\Theta}_{ikm} :\mathbf{U}_{ikm} &\longrightarrow \mathrm{Im}(\widehat{\tau}) \\
\mathrm{Obj}(\mathbf{U}_{ikm}) &\longrightarrow \mathrm{Obj}(\mathrm{Im}(\widehat{\tau})) = \tau\tau'(J) \\
u &\mapsto \tau(h_{ikm}(u)) = \tau\tau'(j_{ikm}(u)) \\
\mathrm{Mor}(\mathbf{U}_{ikm}) &\longrightarrow \mathrm{Mor}(\mathrm{Im}(\widehat{\tau})) = J_H \\
\gamma &\mapsto \Theta_{ikm}(\gamma)
\end{aligned} \tag{3.70}$$

satisfying

$$\widehat{\Theta}_{ikm}\theta_{ik}\theta_{km} = \theta_{im}. \tag{3.71}$$

Let us note that $q_\tau \circ \widehat{\Theta}_{ikm} : \mathbf{U}_{ikm} \longrightarrow \overline{\mathbf{G}}_\tau$ is the trivial functor, taking the constant value identity on both objects and morphisms.

Defining $\overline{\theta}_{ik} := q_\tau \circ \theta_{ik} : \mathbf{U}_{ik} \longrightarrow \overline{\mathbf{G}}_\tau$ by composition of functors

$$\begin{array}{ccc}
\mathbf{U}_{ik} & \xrightarrow{\ \theta_{ik}\ } & \mathbf{G}_\tau \\
 & \searrow{\scriptstyle \overline{\theta}_{ik}} & \downarrow{\scriptstyle q_\tau} \\
 & & \overline{\mathbf{G}}_\tau
\end{array}$$

we have then the functors

$$\overline{\theta}_{ik} : \mathbf{U}_{ik} \longrightarrow \overline{\mathbf{G}}_\tau, \tag{3.72}$$

satisfying

$$\overline{\theta}_{ik}\overline{\theta}_{km} = \overline{\theta}_{im} \tag{3.73}$$

on the triple overlap categories $\mathbf{U}_{ikm}$. This means that both the collections $\big\{\overline{\theta}_{ik}\big|_{\mathrm{Obj}(\mathbf{U}_{ik})}\big\}_{i,k\in\mathcal{I}}$ and $\big\{\overline{\theta}_{ik}\big|_{\mathrm{Mor}(\mathbf{U}_{ik})}\big\}_{i,k\in\mathcal{I}}$ satisfy genuine cocycle relations respectively for the groups $\mathrm{Obj}(\overline{\mathbf{G}}_\tau) =$

$\tau(H)/\tau\tau'(J)$ and $\mathrm{Mor}(\overline{\mathbf{G}}_\tau) = H \rtimes \tau(H)/J_H$. Our objective henceforth will be to show how this leads to a global categorical $\overline{\mathbf{G}}_\tau$-principal bundle over $\mathbf{B}$.

Let us note that construction of a categorical principal bundle is more than the mere construction of an $\mathrm{Obj}(\overline{\mathbf{G}}_\tau) = \tau(H)/\tau\tau'(J)$-principal bundle over $\mathrm{Obj}(\mathbf{B}) = B$ and a $\mathrm{Mor}(\overline{\mathbf{G}}_\tau) = \big(H \rtimes \tau(H)\big)/J_H$-principal bundle over $\mathrm{Mor}(\mathbf{B})$.

In this context we note that non-abelian gerbes have been described in terms of fibration over a group extension [2, 17]. A description of non-abelian differentiable gerbes in terms of the "G-extension" of Lie groupoids can be found in [16]. It is not our objective in this paper to give a new or alternate description of gerbes, but rather to study the kind of global categorical structures that arise out of non-abelian gerbes, and their geometric interpretation.

3.15. SUMMARY. Starting with Lie crossed modules (G, H, α, τ), associated with a categorical Lie group $\mathbf{G}$, and (H, J, α', τ'), associated with a categorical Lie group $\mathbf{H}$, and a manifold B, we introduced local data (h_{ik}, j_{ikl}), named 'gerbal cocycle', associated with an open covering $\{U_i\}_{i\in\mathcal{I}}$. From this data we constructed categories $\mathbf{U}_i$ (with objects being the points of U_i and morphisms coming from the paths lying in U_i) and functors $\theta_{ik} : \mathbf{U}_{ik} \longrightarrow \mathbf{G}$ that, along with certain natural transformations $\mathbb{T}_{ikm} : \theta_{ik}\theta_{km} \Rightarrow \theta_{im}$, form a 'functorial cocycle'. We introduced a category $\overline{\mathbf{G}}$, by quotienting the object and morphism groups of $\mathrm{Obj}(\mathbf{G})$ by subgroups specified by τ and τ'. Inside $\overline{\mathbf{G}}$ is a subcategory $\overline{\mathbf{G}}_\tau$ that is a categorical group and contains all the geometrically relevant data. Our results in this section show that the system of function $\{\overline{g}_{ik}\}$, where $\overline{g}_{ik} = \tau(h_{ik})$, and $\{\overline{\theta}_{ik}\}$, where

$$\overline{\theta}_{ik}(\gamma) = \theta_{ik}(\gamma)J_H \in \big(H \rtimes_\alpha \tau(H)\big)/J_H, \tag{3.74}$$

form cocycles in the traditional sense.

4. Construction of bundles from local data

In this section we start with a gerbal cocycle (3.1) as described in the preceding section and construct a categorical bundle

$$\mathbf{X} \longrightarrow \mathbf{B}$$

as a limit of 'local' categorical bundles $\mathbf{X}_\alpha \longrightarrow \mathbf{U}_\alpha$.

We take as given a manifold B, an open covering $\{U_i\}_{i\in\mathcal{I}}$ of B, and a gerbal cocycle $\{h_{ik}, j_{ikl}\}$ as in (3.1). We use the notation and constructions from the preceding section. We also work with the categorical group

$$\overline{\mathbf{G}}_\tau,$$

whose objects form the quotient group

$$\tau(H)/\tau\tau'(J),$$

and whose morphisms form the quotient group

$$\bigl(H \rtimes_\alpha \tau(H)\bigr)/J_H,$$

where J_H is the normal subgroup defined in (3.46). We will make the standing assumption that $\tau(H)$, $\tau'(J)$ and $\tau\tau'(J)$ are *closed* subgroups; this ensures that the quotients are Lie groups.

For the sake of practical convenience, mainly notational, it would be useful to assume that τ is surjective:

$$\tau(H) = G. \tag{4.1}$$

This would make it possible for us to work with $\overline{\mathbf{G}}$ instead of $\overline{\mathbf{G}}_\tau$ as the structure categorical group for the resulting categorical principal bundle $\mathbf{P}$.

4.1. A LOCAL SYSTEM OF CATEGORICAL BUNDLES. As in subsection 2.3, we denote by $S_{\mathcal{I}}$ the set of all finite nonempty subsets I of $\mathcal{I}$ for which the intersection

$$U_I = \cap_{k \in I} U_k \tag{4.2}$$

is nonempty. We can visualize I itself as a simplex whose vertices are the points $i \in I$. We also have, for every $I \in S_{\mathcal{I}}$, the category

$$\mathbf{U}_I$$

whose object set is U_I and whose morphisms arise from the paths on B that lie entirely within each U_i for $i \in I$:

$$\mathrm{Mor}(\mathbf{U}_I) = \cap_{i \in I} \mathrm{Mor}(\mathbf{U}_i). \tag{4.3}$$

Source, target, and composition are all inherited from any of the $\mathbf{U}_i$. It will be useful to keep track of the index i and the collection of indices I.

We consider also the categories

$$\{(i,I)\} \times \mathbf{U}_I \times \overline{\mathbf{G}}_\tau,$$

where the objects are of the form $(i, I, u, \overline{g})$, with $u \in U_I$ and $\overline{g} \in \overline{G}_\tau = \tau(H)/\tau\tau'(J)$, and morphisms are of the form (i, I, γ, ϕ), where $\gamma \in \mathrm{Mor}(\mathbf{U}_I)$ and $\phi \in \mathrm{Mor}(\overline{\mathbf{G}}_\tau) = \bigl(H \rtimes_\alpha \tau(H)\bigr)/J_H$. Source and target maps are given by

$$\begin{aligned}
s(i,I,\gamma,\phi) &= \bigl(i,I,s(\gamma),s(\phi)\bigr) \\
t(i,I,\gamma,\phi) &= \bigl(i,I,t(\gamma),t(\phi)\bigr).
\end{aligned} \tag{4.4}$$

Consider a pair of indices $i, k \in I$, where $I \in S_{\mathcal{I}}$. We think of the open set U_I and two trivializations of a bundle over U_I, with transition function $\overline{g}_{ik}$. At the categorical level we have a functor

$$\Phi_{ki} : \{(i,I)\} \times \mathbf{U}_I \times \overline{\mathbf{G}}_\tau \longrightarrow \{(k,I)\} \times \mathbf{U}_I \times \overline{\mathbf{G}}_\tau, \tag{4.5}$$

which is given on objects by

$$(i, I, u, \overline{g}) \mapsto \big(k, I, u, \overline{g}_{ki}(u)\overline{g}\big)$$

and is given on morphisms by

$$(i, I, \gamma, \overline{\phi}) \mapsto \big(k, I, \gamma, \overline{\theta}_{ki}(\gamma)\overline{\phi}\big). \tag{4.6}$$

Moreover, if $I \subset J$ (so that U_I is a larger set than U_J) and $i \in I$ then we have an 'inclusion' morphism

$$\{(i, J)\} \times \mathbf{U}_J \times \overline{\mathbf{G}}_\tau \longrightarrow \{(i, I)\} \times \mathbf{U}_I \times \overline{\mathbf{G}}_\tau \tag{4.7}$$

that takes any object $(i, J, u, \overline{g})$ to $(i, I, u, \overline{g})$ and that takes any morphism $(i, J, \gamma, \overline{\phi})$ to $(i, I, \gamma, \overline{\phi})$.

4.2. FROM LOCAL DATA TO THE GLOBAL CATEGORY. Our goal is to splice together the local data into a base category and a 'bundle' category $\mathbf{X}$. The base category is simply $\mathbf{B}$, with object set the manifold B and morphisms being smooth paths on B with suitable identifications (as discussed in the context of (2.10)). We think of the category $\mathbf{X}$ as being a limit of the system of categories $\{(i, I)\} \times \mathbf{U}_I \times \overline{\mathbf{G}}_\tau$ along with the functors discussed in (4.5) and (4.7). The role played by I in $(i, I, u, \overline{g})$ is not essential and is meant only to help track the intersection sets U_I.

4.3. THE OBJECT SET OF $\mathbf{X}$. The object set of the category $\mathbf{X}$ is the bundle space X for the bundle obtained from the cocycles $\{\overline{g}_{ik} : i, k \in I\}$. Thus a point of X is an equivalence class

$$[i, u, \overline{g}],$$

where $i \in I$, $u \in U_i$, $\overline{g} \in \overline{G}_\tau = \tau(H)$, and the equivalence relation is defined by requiring that $[i, u, \overline{g}]$ and $[k, u', \overline{g}']$ be equal if and only if $u' = u \in U_i \cap U_k$ and

$$\overline{g}' = \overline{g}_{ki}(u)\overline{g}.$$

We have then injections

$$\{(i, I)\} \times U_I \times \overline{G}_\tau \longrightarrow X : (i, I, u, \overline{g}) \mapsto [i, u, \overline{g}]. \tag{4.8}$$

We have essentially seen this construction in subsection 3.7.

4.4. TOWARDS MORPHISMS. Proceeding towards morphisms, we need first the quadruples

$$(i, I, \gamma, \overline{\phi}),$$

where $i \in \mathcal{I}$, $I \in S_\mathcal{I}$, $\gamma \in \mathrm{Mor}(\mathbf{U}_I)$, $\overline{\phi} \in \mathrm{Mor}(\overline{\mathbf{G}}_\tau)$. Thus we have here a path γ that lies inside the open set

$$U_I = \cap_{j \in I} U_j,$$

a particular index choice $i \in I$ that indicates a trivialization over U_i, and a morphism $\overline{\phi} = (\overline{h}, \overline{g})$ of $\overline{\mathbf{G}}_\tau$ that we think of as indicating, through $\overline{g}$, the 'location' of the path in the bundle X along with a decoration $\overline{h}$ on it. A difficulty arises when we try to form a composition

$$(j, J, \gamma_j, \phi_j) \circ (i, I, \gamma_i, \phi_i)$$

where the composition $s(\gamma_j) = t(\gamma_i)$. It is not apparent what this composition ought to be. The 'coarsest' solution to this difficulty is to define the composite simply to be the sequence of two quadruples

$$\big((i, I, \gamma_i, \phi_i), (j, J, \gamma_j, \phi_j)\big).$$

4.5. QUIVERS. The language of quivers helps with the structure here. By a *quiver Q* we mean a set E of (directed) 'edges' and a nonempty set V of vertices, along with source and target maps $s, t : E \longrightarrow V$. This gives rise to a category $\mathbf{C}_Q$, called the *free category* for Q: the object set is just V, and a morphism is a sequence $(e_1, \ldots, e_n)$ of edges with the target of each edge e_i equal to the source of the next edge e_{i+1}; in addition, we also include an identity morphism for each object. Composition is defined by concatenation of sequences.

4.6. THE QUIVER OF DECORATED LOCAL PATHS. Returning to our context, the vertex set V for our quiver Q is X, consisting of all equivalence classes $[i, I, u, \overline{g}]$; edges are the quadruples

$$(i, I, \gamma_i, \phi_i),$$

with source and target given by

$$
\begin{aligned}
s(i, I, \gamma_i, \phi_i) &= [i, I, s(\gamma_i), s(\phi_i)] \\
t(i, I, \gamma_i, \phi_i) &= [i, I, t(\gamma_i), t(\phi_i)].
\end{aligned}
\tag{4.9}
$$

However, in the resulting quiver category $\mathbf{C}_Q$ certain morphisms need to be identified. It is convenient to do this at an algebraic level rather than 'geometric', and so we turn first to the notion of the quiver algebra.

4.7. THE QUIVER ALGEBRA. The *quiver algebra* $\mathcal{A}_Q$ of a quiver Q is the free algebra on the set of non-identity morphisms of the quiver category $\mathbf{C}_Q$ quotiented so that the product of two morphisms that have non-matching source-target relation is zero and in other cases the product is given by concatenation; thus an element of $\mathcal{A}_Q$ can be expressed uniquely as a linear combination of sequences of paths of the form

$$(e_1, \ldots, e_n),$$

where $n \geq 1$, each $e_i \in E$, and $t(e_i) = s(e_{i+1})$ for all $i \in \{1, \ldots, n\}$. The product of two such morphisms is given by

$$(e_1, \ldots, e_n) \cdot (e_{n+1}, \ldots, e_m) = (e_1, \ldots, e_m) \quad \text{if } t(e_n) = s(e_{n+1}),$$

and is 0 in all other cases. In particular, if $(e_1, \ldots, e_n)$ is above, with source-target matching for successive edges, then

$$e_1 e_2 \ldots e_n = (e_1, \ldots, e_n). \tag{4.10}$$

4.8. A QUOTIENT OF THE QUIVER ALGEBRA. For the quiver algebra of our path quiver $\mathbf{C}_Q$, let N_0 be the ideal in $\mathcal{A}_Q$ generated by all elements of the form

$$(i, I, \gamma, \overline{\phi}) - (j, J, \gamma, \overline{\theta}_{ij}(\gamma)\overline{\phi}), \tag{4.11}$$

where $I, J \in S_{\mathcal{I}}$, $i \in I$, $j \in J$, γ is any morphism in $\mathbf{U}_{\{i,j\}}$, and $\overline{\phi} \in \mathrm{Mor}(\overline{\mathbf{G}}_\tau)$. Moreover, from a geometric viewpoint, if the sets U_I and U_J lie inside U_K then, for any $\gamma_i \in \mathrm{Mor}(\mathbf{U}_i)$ and $\gamma_j \in \mathrm{Mor}(\mathbf{U}_j)$ for which $s(\gamma_j) = t(\gamma_i)$, the composition

$$(j, J, \gamma_j, \overline{\phi}_j) \circ (i, I, \gamma_i, \overline{\phi}_i)$$

ought to be obtainable by transforming both to the (k, K) system:

$$\begin{aligned}
&\big(k, K, \gamma_j, \overline{\theta}_{kj}(\gamma_j)\overline{\phi}_j)\big) \circ \big(k, K, \gamma_i, \overline{\theta}_{ki}(\gamma_i)\overline{\phi}_i\big) \\
&= \Big(k, K, \gamma_j \circ \gamma_i, \big(\overline{\theta}_{kj}(\gamma_j)\overline{\phi}_j\big) \circ \big(\overline{\theta}_{ki}(\gamma_i)\overline{\phi}_i\big)\Big).
\end{aligned} \tag{4.12}$$

With this in mind, we consider the ideal N generated by N_0 along with all elements of $\mathcal{A}_Q$ of the form

$$(k, K, \gamma_k, \overline{\phi}_k) - (i, I, \gamma_i, \overline{\phi}_i) \cdot (j, J, \gamma_j, \overline{\phi}_j), \tag{4.13}$$

where $I, J, K \in S_{\mathcal{I}}$ are such that

$$K \subset I \cap J \qquad \text{(which implies } U_I, U_J \subset U_K) \tag{4.14}$$

and

$$\begin{aligned}
\gamma_j \circ \gamma_i &= \gamma_k \\
\big(\overline{\theta}_{kj}(\gamma_j)\overline{\phi}_j\big) \circ \big(\overline{\theta}_{ki}(\gamma_i)\overline{\phi}_i\big) &= \overline{\phi}_k.
\end{aligned} \tag{4.15}$$

Thus in the quotient algebra $\mathcal{A}_Q/N$ we have

$$[k, K, \gamma_k, \overline{\phi}_k] = [i, I, \gamma_i, \overline{\phi}_i][j, J, \gamma_j, \overline{\phi}_j]. \tag{4.16}$$

4.9. EQUIVALENT MORPHISMS IN THE QUOTIENT. We define two non-identity morphisms of the category $\mathbf{C}_Q$ to be equivalent if they have the same image in the quotient algebra $\mathcal{A}_Q/N$. We denote by

$$[i, I, \gamma, \overline{\phi}]$$

the equivalence class in $\mathcal{A}_Q/N$ corresponding to the element $(i, I, \gamma, \overline{\phi})$ in $\mathcal{A}_Q$. Thus,

$$[i, I, \gamma, \overline{\phi}] = [j, J, \gamma, \overline{\theta}_{ij}(\gamma)\overline{\phi}] \tag{4.17}$$

if $\gamma \in \mathrm{Mor}(\mathbf{U}_{\{i,j\}})$.

We denote the set of equivalence classes of non-identity morphisms of $\mathbf{C}_Q$ by

$$\mathrm{Mor}(\mathbf{X})_0, \tag{4.18}$$

with notation anticipating our objective of showing that $\mathbf{X}$ is a category. The subscript 0 here is a 'temporary hold' pending inclusion of identity morphisms.

4.10. THE MOR-SET $\mathrm{Mor}(\mathbf{X})$. Now we include an identity morphism 1_x for each object $x \in \mathrm{Obj}(\mathbf{X})$, and define

$$\mathrm{Mor}(\mathbf{X}) = \{1_x \; : \; x \in \mathrm{Obj}(\mathbf{X})\} \cup \mathrm{Mor}(\mathbf{X})_0. \tag{4.19}$$

4.11. SOURCE AND TARGET. The source and target maps

$$s, t : E \longrightarrow V$$

of a quiver Q give rise to linear maps

$$s, t : \mathcal{A}_Q \longrightarrow \mathbb{R}[V], \tag{4.20}$$

where the latter is the free real vector space over the set V of vertices (thus every element of $\mathbb{R}[V]$ is uniquely a (finite) linear combination of the form $\sum_j a_j v_j$, with $a_j \in \mathbb{R}$ and $v_j \in V$). These maps do not respect multiplication since

$$s(e_1 e_2) = s(e_1) \qquad \text{and} \qquad t(e_1 e_2) = t(e_2),$$

for example. Now returning to our context, let us note that, with notation as in (4.13),

$$\begin{aligned} s(\overline{\phi}_k) = s\big(\overline{\theta}_{ki}(\gamma_i)\overline{\phi}_i\big) &= s\big(\overline{\theta}_{ki}(\gamma_i)\big) s(\overline{\phi}_i) \\ &= \overline{\theta}_{ki}\big(s(\gamma_i)\big) s(\overline{\phi}_i), \end{aligned} \tag{4.21}$$

and similarly for the targets. Consequently we have the following source/target consistency:

$$\begin{aligned} s(k, K, s(\gamma_k), \overline{\phi}_k) &= \big(k, K, s(\gamma_i), \overline{\theta}_{ki}\big(s(\gamma_i)\big) s(\overline{\phi}_i)\big) = s(i, I, s(\gamma_i), \overline{\phi}_i); \\ t(k, K, \gamma_k, \overline{\phi}_k) &= t(j, J, t(\gamma_j), \overline{\phi}_j). \end{aligned} \tag{4.22}$$

Moreover, for $\gamma \in \mathbf{U}_{\{i,j\}}$, where $i \in I$ and $j \in J$, and any $\overline{\phi} \in \mathrm{Mor}(\overline{\mathbf{G}}_\tau)$, we have

$$\begin{aligned} s(i, I, \gamma, \overline{\phi}) &= s\big(j, J, s(\gamma), \overline{\theta}_{ij}(\gamma)\overline{\phi}\big) \\ t(i, I, \gamma, \overline{\phi}) &= t\big(j, J, t(\gamma), \overline{\theta}_{ij}(\gamma)\overline{\phi}\big). \end{aligned} \tag{4.23}$$

Because of these relations and (4.22), the value of s on a quiver algebra element of the form $e_1 \ldots e_n$, being just $s(e_1)$, remains unchanged if any of the e_i is replaced by either a single-edge element e or a two-edge morphism ee' related to e_i in any of the ways described above in (4.17). An analogous statement holds for the target map t. Hence the source and target maps descend to well-defined linear maps

$$s, t : \mathcal{A}_Q/N \longrightarrow \mathbb{R}[V]. \tag{4.24}$$

In particular, s and t are well-defined maps when restricted to $\mathrm{Mor}(\mathbf{X})_0$ (the values of s and t on elements of $\mathrm{Mor}(\mathbf{X})_0$ are elements of V). For the identity morphism 1_x we define both source and target to be x.

4.12. COMPOSITION OF MORPHISMS IN $\mathbf{X}$. The multiplication operation on $\mathcal{A}_Q$ induces a well-defined operation on the quotient $\mathcal{A}_Q/N$ and this restricts to a well-defined operation on $\mathrm{Mor}(\mathbf{X})_0$. We use this to define composition of elements in $\mathrm{Mor}(\mathbf{X})_0$. Concretely, composition is given by concatenation:

$$(e_{n+1} \dots e_{n+m}) \circ (e_1 \dots e_n) = e_1 \dots e_n e_{n+1} \dots e_{n+m}, \tag{4.25}$$

where each e_j is of the form

$$e_j = [j, J, \gamma_j, \overline{\phi}_j]$$

and

$$t(e_n) = s(e_{n+1}).$$

(This last condition ensures that the product on the right in (4.25) is not 0 but an element of $\mathrm{Mor}(\mathbf{X})_0$ instead.) The significance of (4.25) is that the value of the right hand side as an element of $\mathrm{Mor}(\mathbf{X})_0$ is independent of the specific choices of the e_j used in representing the elements of $\mathrm{Mor}(\mathbf{X})$ being composed on the left. The definition (4.25) makes it clear that if $f, g \in \mathrm{Mor}(\mathbf{X})$ for which $t(f) = s(g)$ then $g \circ f$ is defined an

$$s(g \circ f) = s(f) \qquad \text{and} \qquad t(g \circ f) = t(g). \tag{4.26}$$

Of course, we define composition with identity morphisms by

$$f \circ 1_x = f \qquad \text{and} \qquad 1_y \circ f = f \tag{4.27}$$

if $x = s(f)$ and $y = t(f)$. Associativity of the composition law on $\mathrm{Mor}(\mathbf{X})$ follows from associativity of the concatenation process.

We have thus constructed the category $\mathbf{X}$.

4.13. PROJECTION TO $\mathbf{B}$. There is a well-defined projection functor

$$\pi : \mathbf{X} \longrightarrow \mathbf{B}$$

given by

$$[i, I, u, \overline{g}] \mapsto u$$
$$[i_1, I_1, \gamma_{i_1}, \phi_{i_1}] \dots [i_n, I_n, \gamma_{i_n}, \phi_{i_n}] \mapsto \gamma_{i_n} \circ \dots \gamma_{i_1} \tag{4.28}$$

That this is well-defined on morphisms follows from the first relation in (4.15).

4.14. LOCAL TRIVIALITY. It is clear that at the object-level we have traditional local triviality of the principal $\overline{G}_\tau$-bundle $X \longrightarrow B$. At the level of morphisms any morphism of $\mathbf{X}$ that projects to a morphism γ that lies entirely inside U_I arises from some

$$(i, I, \gamma, \phi) \in \mathrm{Mor}\big(\{(i, I)\} \times \mathbf{U}_I \times \overline{\mathbf{G}}_\tau\big).$$

Thus the category $\mathbf{X}_I$, with objects in $\pi^{-1}(U_I)$ and morphism projecting to $\mathbf{U}_I$, is isomorphic to $\mathbf{U}_I \times \overline{\mathbf{G}}_\tau$:

$$\Phi_{i,I} : \mathbf{U}_I \times \overline{\mathbf{G}}_\tau \longrightarrow \mathbf{X}_I : \begin{cases} (u, \overline{g}) & \mapsto [i, I, u, \overline{g}]; \\ (\gamma, \overline{\phi}) & \mapsto [i, I, \gamma, \overline{\phi}]. \end{cases} \tag{4.29}$$

4.15. ACTION OF THE GROUP $\overline{\mathbf{G}}_\tau$. There is a well-defined right action

$$\mathbf{X} \times \overline{\mathbf{G}}_\tau \longrightarrow \mathbf{X} : \begin{cases} ([i, I, u], \overline{g}) & \mapsto [i, I, u\overline{g}]; \\ ([i, I, \gamma, \overline{\phi}], \overline{\psi}) & \mapsto [i, I, \gamma, \overline{\phi}\overline{\psi}]. \end{cases} \tag{4.30}$$

That the action is well-defined on morphisms may be seen from

$$(\overline{\phi}_2 \circ \overline{\phi}_1)\overline{\psi} = (\overline{\phi}_2\overline{\psi}) \circ (\overline{\phi}_1\overline{\psi}). \tag{4.31}$$

Using local triviality it follows that this action is free.

We have thus shown that

$$\mathbf{X} \longrightarrow \mathbf{B}$$

is a categorical principal $\overline{\mathbf{G}}_\tau$-bundle. With the simplifying assumption $\tau(H) = G$ mentioned in (4.1), $\mathbf{X} \longrightarrow \mathbf{B}$ is a categorical principal $\overline{\mathbf{G}}$-bundle.

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