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THE HEART OF A COMBINATORIAL MODEL CATEGORY

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Abstract. We show that every small model category that satisfies certain size conditions can be completed to yield a combinatorial model category, and conversely, every combinatorial model category arises in this way. We will also see that these constructions preserve right properness and compatibility with simplicial enrichment. Along the way, we establish some technical results on the index of accessibility of various constructions on accessible categories, which may be of independent interest.

Introduction

Category-theoretic homotopy theory has seen a boom in recent decades. One development was the introduction of the notion of 'combinatorial model categories' by Smith [1998]. These correspond to what Lurie [2009] calls 'presentable ∞-categories' and are therefore a homotopy-theoretic generalisation of the locally presentable categories of Gabriel and Ulmer [1971]. The classification of locally κ-presentable categories says that each one is equivalent to the free κ-ind-completion of a κ-cocomplete small category, and Lurie proved the analogous proposition for presentable ∞-categories, so it should at least seem plausible that every combinatorial model category is generated by a small model category in an appropriate sense.

Indeed, the work of Beke [2000] suggests that more should be true. As stated in the abstract of op. cit.,

> If a Quillen model category can be specified using a certain logical syntax (intuitively, 'is algebraic/combinatorial enough'), so that it can be defined in any category of sheaves, then the satisfaction of Quillen's axioms over any site is a purely formal consequence of their being satisfied over the category of sets.

In the same vein, we can show that the answer to the question of whether a set of generating cofibrations and trivial cofibrations in a locally presentable category really do generate a combinatorial model category depends only on an essentially small full subcategory of small objects, which we may think of as an analogue of the Löwenheim–Skolem theorem in logic. More precisely:

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Theorem. *Let $\mathcal{M}$ be a locally presentable category and let $\mathcal{I}$ and $\mathcal{I}'$ be subsets of* $\operatorname{mor} \mathcal{M}$. *There is a regular cardinal λ such that the weak factorisation systems cofibrantly generated by $\mathcal{I}$ and $\mathcal{I}'$ underlie a model structure on $\mathcal{M}$ if and only if their restrictions to $\mathbf{K}_\lambda(\mathcal{M})$ underlie a model structure on $\mathbf{K}_\lambda(\mathcal{M})$, where $\mathbf{K}_\lambda(\mathcal{M})$ is the full subcategory of λ-presentable objects in $\mathcal{M}$.*

The main difficulty is in choosing a definition of 'weak equivalence in $\mathcal{M}$' for which we can verify the model category axioms. As it turns out, what works is to define 'weak equivalence' to be a morphism such that the right half of its (trivial cofibration, fibration)-factorisation is a trivial fibration. This allows us to apply the theory of accessible categories: the key result needed is a special case of the well-known theorem of Makkai and Paré [1989, §5.1] concerning weighted 2-limits of diagrams of accessible categories. Moreover, by using good estimates for the index of accessibility of the categories obtained in this way, we can establish a stronger result:

Theorem. *Let $\mathcal{M}$ be a locally presentable category and let $\mathcal{I}$ and $\mathcal{I}'$ be subsets of* $\operatorname{mor} \mathcal{M}$. *Suppose κ and λ are regular cardinals that satisfy the following hypotheses:*

- *$\mathcal{M}$ is a locally κ-presentable category, and κ is sharply less than λ.*
- *$\mathbf{K}_\lambda(\mathcal{M})$ is closed under finite limits in $\mathcal{M}$.*
- *There are $< \lambda$ morphisms between any two κ-presentable objects in $\mathcal{M}$.*
- *$\mathcal{I}$ and $\mathcal{I}'$ are λ-small sets of morphisms between κ-presentable objects.*

Then the weak factorisation systems cofibrantly generated by $\mathcal{I}$ and $\mathcal{I}'$ underlie a model structure on $\mathcal{M}$ if and only if their restrictions to $\mathbf{K}_\lambda(\mathcal{M})$ underlie a model structure on $\mathbf{K}_\lambda(\mathcal{M})$.

This is essentially what theorem 5.9 states. Moreover, given $\mathcal{M}$, $\mathcal{I}$, and $\mathcal{I}'$, we can always find regular cardinals κ and λ satisfying the hypotheses above. Thus, if $\mathcal{M}$ is a combinatorial model category, there is a regular cardinal λ such that $\mathbf{K}_\lambda(\mathcal{M})$ not only inherits a model structure from $\mathcal{M}$ but also determines $\mathcal{M}$ as a combinatorial model category—the subcategory $\mathbf{K}_\lambda(\mathcal{M})$ might be called the 'heart' of $\mathcal{M}$. (For details, see proposition 5.12.) When we have explicit sets of generating cofibrations and generating trivial cofibrations, we can also give explicit κ and λ for which this happens:

- If $\mathcal{M}$ is the category of simplicial sets with the Kan–Quillen model structure, then we can take $\kappa = \aleph_0$ and $\lambda = \aleph_1$.
- If $\mathcal{M}$ is the category of unbounded chain complexes of left R-modules, then we can take $\kappa = \aleph_0$ and λ to be the smallest uncountable regular cardinal such that R is λ-small (as a set).
- If $\mathcal{M}$ is the category of symmetric spectra of Hovey et al. [2000] with the stable model structure, then we can take $\kappa = \aleph_1$ and λ to be the cardinal successor of $2^{2^{\aleph_0}}$.

In the converse direction, we obtain a sufficient condition for an essentially small model category $\mathcal{K}$ to arise in this fashion: see theorem 5.14.

The techniques used in the proof of the main theorem are easily generalised, allowing us to make sense of a remark of Dugger [2001]:

> [...] for a combinatorial model category the interesting part of the homotopy theory is all concentrated within some small subcategory—beyond sufficiently large cardinals the homotopy theory is somehow "formal".

For illustration, we will see how to validate the above heuristic in the cases of right properness and axiom SM7.

The structure of this paper is as follows:

- §1 contains some technical results on presentable objects and filtered colimits thereof. In particular, the definition of 'sharply less than' is recalled, in preparation for the statement of the main result.
- §2 is an analysis of some special cases of the theorem of Makkai and Paré on weighted 2-limits of accessible categories (see Theorem 5.1.6 in [Makkai and Paré, 1989], or [Adámek and Rosický, 1994, §2.H]), with a special emphasis on the index of accessibility of the categories and functors involved.

  The results appearing in this section are related to those appearing in a preprint of Ulmer [1977] and probably well known to experts; nonetheless, for the sake of completeness, full proofs are given.
- §3 introduces the notion of accessibly generated category, which is a size-restricted analogue of the notion of accessible category.
- §4 collects together some results about cofibrantly generated weak factorisation systems on locally presentable categories.
- §5 establishes the main result: that every combinatorial model category is generated by a small model category, and conversely, that small model categories satisfying certain size conditions generate combinatorial model categories.

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1. Presentable objects

1.1. Notation. Throughout this section, κ is an arbitrary regular cardinal.

1.2. Definition. Let $\mathcal{C}$ be a locally small category.

- Let λ be a regular cardinal. A (κ, λ)**-presentable object** in $\mathcal{C}$ is an object A in $\mathcal{C}$ such that the representable functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set}$ preserves colimits of all λ-small κ-filtered diagrams.

  We write $\mathbf{K}_\kappa^\lambda(\mathcal{C})$ for the full subcategory of $\mathcal{C}$ spanned by the (κ, λ)-presentable objects.

- A κ**-presentable object** in $\mathcal{C}$ is an object in $\mathcal{C}$ that is (κ, λ)-presentable for all regular cardinals λ.

  We write $\mathbf{K}_\kappa(\mathcal{C})$ for the full subcategory of $\mathcal{C}$ spanned by the κ-presentable objects.

1.3. Example. A set is κ-small if and only if it is a κ-presentable object in **Set**.

1.4. Remark. Although every $\aleph_0$-small (i.e. finite) category is $\aleph_0$-presentable as an object in **Cat**, not every $\aleph_0$-presentable object in **Cat** is $\aleph_0$-small. The difference disappears for uncountable regular cardinals.

1.5. Lemma. *Let $\mathcal{C}$ be a locally small category and let $B : \mathcal{D} \to \mathcal{C}$ be a κ-small diagram. If each Bd is a (κ, λ)-presentable object in $\mathcal{C}$, then the colimit $\varinjlim_{\mathcal{D}} B$, if it exists, is also a (κ, λ)-presentable object in $\mathcal{C}$.*

Proof. This follows from the fact that $\varprojlim_{\mathcal{D}^{\mathrm{op}}} [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Set}$ preserves colimits of small κ-filtered diagrams. ■

1.6. Lemma. *Assume the following hypotheses:*

- *$\mathcal{E}$ is a locally small category with colimits of small κ-filtered diagrams.*
- *$X, Y : \mathcal{I} \to \mathcal{E}$ are two small λ-filtered diagrams whose vertices are λ-presentable objects in $\mathcal{E}$, where $\kappa \le \lambda$.*
- *$\varphi : X \Rightarrow Y$ is a natural transformation.*

Let i_0 be an object in $\mathcal{I}$. If $\varinjlim_{\mathcal{I}} \varphi : \varinjlim_{\mathcal{I}} X \to \varinjlim_{\mathcal{I}} Y$ is an isomorphism in $\mathcal{E}$, then there is a chain $I : \kappa \to \mathcal{I}$ such that $I(0) = i_0$ and $\varinjlim_{\gamma < \kappa} \varphi_{I(\gamma)} : \varinjlim_{\gamma < \kappa} XI(\gamma) \to \varinjlim_{\gamma < \kappa} YI(\gamma)$ is an isomorphism in $\mathcal{E}$.

Proof. Let $C = \varinjlim_{\mathcal{I}} X$ and $D = \varinjlim_{\mathcal{I}} Y$, let $c_i : Xi \to C$ and $d_i : Yi \to D$ are the components of the respective colimiting cocones and let $e = \varinjlim_{\mathcal{I}} \varphi$. We will construct $I : \kappa \to \mathcal{I}$ by transfinite induction.

- Let $I(0) = i_0$.
- Given an ordinal $\alpha < \kappa$ and an object $I(\alpha)$ in $\mathcal{I}$, choose an object $I(\alpha+1)$ in $\mathcal{I}$ and a morphism $I(\alpha \to \alpha+1) : I(\alpha) \to I(\alpha+1)$ in $\mathcal{I}$ for which there is a morphism $YI(\alpha) \to XI(\alpha+1)$ making the diagram in $\mathcal{E}$ shown below commute:

$$\begin{array}{ccccc}
XI(\alpha) & \xrightarrow{XI(\alpha\to\alpha+1)} & XI(\alpha+1) & \xrightarrow{c_{I(\alpha+1)}} & C \\
{\scriptstyle \varphi_{I(\alpha)}}\downarrow & \overset{\psi_\alpha}{\nearrow} & \downarrow{\scriptstyle \varphi_{I(\alpha+1)}} & & \downarrow{\scriptstyle e} \\
YI(\alpha) & \xrightarrow[YI(\alpha\to\alpha+1)]{} & YI(\alpha+1) & \xrightarrow[d_{I(\alpha+1)}]{} & D
\end{array}$$

  Such a choice exists: since $YI(\alpha)$ is a λ-presentable object in $\mathcal{E}$ and $\mathcal{I}$ is λ-filtered, there is an object i' in $\mathcal{I}$ and a commutative diagram in $\mathcal{E}$ of the form below,

$$\begin{array}{ccccc}
XI(\alpha) & & \xrightarrow{\quad c_{I(\alpha)} \quad} & & C \\
 & \overset{s}{\searrow} & & & \| \\
{\scriptstyle \varphi_{I(\alpha)}}\downarrow & & Xi' & \xrightarrow{c_{i'}} & C \\
 & \underset{t}{\nearrow} & & & \uparrow{\scriptstyle e^{-1}} \\
YI(\alpha) & & \xrightarrow[\quad d_{I(\alpha)} \quad]{} & & D
\end{array}$$

  so there exist an object i'' in $\mathcal{I}$ and morphisms $u : I(\alpha) \to i''$ and $v : i' \to i''$ such that the following diagram in $\mathcal{E}$ commutes,

$$\begin{array}{ccccc}
XI(\alpha) & & \xrightarrow{\quad Xu \quad} & & Xi'' \\
 & \underset{s}{\searrow} & & \underset{Xv}{\nearrow} & \\
 & & Xi' & &
\end{array}$$

  and similarly, there exist an object $I(\alpha+1)$ in $\mathcal{I}$ and a morphism $w : i'' \to I(\alpha+1)$ in $\mathcal{I}$ such that the diagram in $\mathcal{E}$ shown below commutes,

$$\begin{array}{ccccccc}
 & & Xi' & \xrightarrow{Xv} & Xi'' & \xrightarrow{Xw} & XI(\alpha+1) \\
 & \overset{t}{\nearrow} & & & & & \downarrow{\scriptstyle \varphi_{I(\alpha+1)}} \\
YI(\alpha) & & \xrightarrow[\quad Yu \quad]{} & & Yi'' & \xrightarrow[Yw]{} & YI(\alpha+1)
\end{array}$$

  so we may take $\psi_\alpha : YI(\alpha) \to XI(\alpha+1)$ to be the composite $Xw \circ Xv \circ t$ and $I(\alpha \to \alpha+1) : I(\alpha) \to I(\alpha+1)$ to be the composite $w \circ u$.

- Given a limit ordinal $\beta < \kappa$, assuming I is defined on the ordinals $\alpha < \beta$, define $I(\beta)$ and $I(\alpha \to \beta)$ (for $\alpha < \beta$) by choosing a cocone over the given α-chain in $\mathcal{I}$.

The above yields a chain $I : \kappa \to \mathcal{I}$. By construction, for every ordinal $\alpha < \kappa$, the following diagram in $\mathcal{E}$ commutes,

$$\begin{array}{ccc} XI(\alpha) & \longrightarrow & \varinjlim_{\gamma<\kappa} XI(\gamma) \\ {\scriptstyle \varphi_{I(\alpha)}}\downarrow & & \downarrow{\scriptstyle \varinjlim_{\gamma<\kappa} \varphi_{I(\gamma)}} \\ YI(\alpha) & \longrightarrow & \varinjlim_{\gamma<\kappa} YI(\gamma) \\ {\scriptstyle \psi_{\alpha}}\downarrow & & \downarrow{\scriptstyle \varinjlim_{\gamma<\kappa} \psi_{\gamma}} \\ XI(\alpha+1) & \longrightarrow & \varinjlim_{\gamma<\kappa} XI(\gamma+1) \end{array}$$

where the horizontal arrows are the respective colimiting cocone components. The composite of the left column is $XI(\alpha \to \alpha+1) : XI(\alpha) \to XI(\alpha+1)$, so $\varinjlim_{\gamma<\kappa} \varphi_{I(\gamma)} : \varinjlim_{\gamma<\kappa} XI(\gamma) \to \varinjlim_{\gamma<\kappa} YI(\gamma)$ is a split monomorphism in $\mathcal{E}$. Similarly, the diagram below commutes,

$$\begin{array}{ccc} YI(\alpha) & \longrightarrow & \varinjlim_{\gamma<\kappa} YI(\gamma) \\ {\scriptstyle \psi_{\alpha}}\downarrow & & \downarrow{\scriptstyle \varinjlim_{\gamma<\kappa} \psi_{\gamma}} \\ XI(\alpha+1) & \longrightarrow & \varinjlim_{\gamma<\kappa} XI(\gamma) \\ {\scriptstyle \varphi_{I(\alpha+1)}}\downarrow & & \downarrow{\scriptstyle \varinjlim_{\gamma<\kappa} \varphi_{I(\gamma)}} \\ YI(\alpha+1) & \longrightarrow & \varinjlim_{\gamma<\kappa} YI(\gamma+1) \end{array}$$

so $\varinjlim_{\gamma<\kappa} \varphi_{I(\gamma)} : \varinjlim_{\gamma<\kappa} XI(\gamma) \to \varinjlim_{\gamma<\kappa} YI(\gamma)$ is also a split epimorphism in $\mathcal{E}$. Thus, $I : \kappa \to \mathcal{I}$ is the desired chain. ■

The following notion is due to Makkai and Paré [1989].

1.7. Definition. Let κ and λ be regular cardinals. We write '$\kappa \lhd \lambda$' and we say 'κ is **sharply less than** λ' for the following condition:

- $\kappa < \lambda$ and, for all λ-small sets X, there is a λ-small cofinal subset of $\mathscr{P}_{\kappa}(X)$, the set of all κ-small subsets of X (partially ordered by inclusion).

1.8. Example. If λ is an uncountable regular cardinal, then $\aleph_0 \lhd \lambda$: indeed, for any λ-small set X, the set $\mathscr{P}_{\aleph_0}(X)$ itself is λ-small.

1.9. Example. If λ is a strongly inaccessible cardinal and $\kappa < \lambda$, then $\kappa \lhd \lambda$: indeed, for any λ-small set X, the set $\mathscr{P}_{\kappa}(X)$ itself is λ-small.

1.10. EXAMPLE. Let κ^+ be the cardinal successor of κ. Then $\kappa \lhd \kappa^+$: every κ^+-small set can be mapped bijectively onto an initial segment α of κ (but possibly all of κ), and it is clear that the subposet

$$\{\beta \mid \beta \leq \alpha\} \subseteq \mathscr{P}_\kappa(\alpha)$$

is a κ^+-small cofinal subposet of $\mathscr{P}_\kappa(\alpha)$: given any κ-small subset $X \subseteq \alpha$, we must have $\sup X \leq \alpha$, and $X \subseteq \sup X$ by definition.

The following is a partial converse to lemma 1.5.

1.11. PROPOSITION. *Let $\mathcal{C}$ be a κ-accessible category. If λ is a regular cardinal and $\kappa \lhd \lambda$, then the following are equivalent for an object C in $\mathcal{C}$:*

(i) *C is a λ-presentable object in $\mathcal{C}$.*

(ii) *There is a λ-small κ-filtered diagram $A : \mathcal{J} \to \mathcal{C}$ such that each Aj is a κ-presentable object in $\mathcal{C}$ and $C \cong \varinjlim_{\mathcal{J}} A$.*

(iii) *There is a λ-small κ-directed diagram $A : \mathcal{J} \to \mathcal{C}$ such that each Aj is a κ-presentable object in $\mathcal{C}$ and C is a retract of $\varinjlim_{\mathcal{J}} A$.*

PROOF. (i) $\Leftrightarrow$ (ii). See Proposition 2.3.11 in [Makkai and Paré, 1989].

(i) $\Leftrightarrow$ (iii). See the proof of Theorem 2.3.10 in [Makkai and Paré, 1989] or Remark 2.15 in [Adámek and Rosický, 1994]. □

1.12. LEMMA. *Let $\mathcal{C}$ be a κ-accessible category, let A be a κ-presentable object in $\mathcal{C}$, and let B be a λ-presentable object in $\mathcal{C}$. If the hom-set $\mathcal{C}(A, A')$ is μ-small for all κ-presentable objects A' in $\mathcal{C}$ and $\kappa \lhd \lambda$, then the hom-set $\mathcal{C}(A, B)$ has cardinality $< \max\{\lambda, \mu\}$.*

PROOF. By proposition 1.11, there is a λ-small κ-filtered diagram $Y : \mathcal{J} \to \mathcal{C}$ such that each Yj is a κ-presentable object in $\mathcal{C}$ and B is a retract of $\varinjlim_{\mathcal{J}} Y$. Since A is a κ-presentable object in $\mathcal{C}$, we have

$$\mathcal{C}\left(A, \varinjlim_{\mathcal{J}} Y\right) \cong \varinjlim_{\mathcal{J}} \mathcal{C}(A, Y)$$

and the RHS is a set of cardinality $< \max\{\lambda, \mu\}$ by lemma 1.5; but $\mathcal{C}(A, B)$ is a retract of the LHS, so we are done. ■

2. Accessible constructions

2.1. NOTATION. Throughout this section, κ is an arbitrary regular cardinal.

2.2. DEFINITION. A **strongly κ-accessible functor** is a functor $F : \mathcal{C} \to \mathcal{D}$ with the following properties:

- Both $\mathcal{C}$ and $\mathcal{D}$ are κ-accessible categories.
- F preserves colimits of small κ-filtered diagrams.
- F sends κ-presentable objects in $\mathcal{C}$ to κ-presentable objects in $\mathcal{D}$.

2.3. Example. Given any functor $F : \mathcal{A} \to \mathcal{B}$, if $\mathcal{A}$ and $\mathcal{B}$ are essentially small categories, then the induced functor $\mathbf{Ind}^\kappa(F) : \mathbf{Ind}^\kappa(\mathcal{A}) \to \mathbf{Ind}^\kappa(\mathcal{B})$ is strongly κ-accessible. If $\mathcal{B}$ is also idempotent-complete, then every strongly κ-accessible functor $\mathbf{Ind}^\kappa(\mathcal{A}) \to \mathbf{Ind}^\kappa(\mathcal{B})$ is of this form (up to isomorphism).

2.4. Proposition. [Products of accessible categories] *If $(\mathcal{C}_i \mid i \in I)$ is a κ-small family of κ-accessible categories, then:*

(i) *The product $\mathcal{C} = \prod_{i \in I} \mathcal{C}_i$ is a κ-accessible category.*

(ii) *Moreover, the projection functors $\mathcal{C} \to \mathcal{C}_i$ are strongly κ-accessible functors.*

Proof. It is clear that $\mathcal{C}$ has colimits of small κ-filtered diagrams: indeed, they can be computed componentwise. Since $\prod : \mathbf{Set}^I \to \mathbf{Set}$ preserves colimits of small κ-filtered diagrams, an object in $\mathcal{C}$ is κ-presentable as soon as its components are κ-presentable objects in their respective categories. The product of a κ-small family of κ-filtered categories is a κ-filtered category, and moreover, the projections are cofinal functors, so it follows that $\mathcal{C}$ is generated under small κ-filtered colimits by a small family of κ-presentable objects, as required of a κ-accessible category. ■

2.5. Lemma. *Let $\mathcal{C}$ and $\mathcal{D}$ be accessible categories and let $F : \mathcal{C} \to \mathcal{D}$ be a κ-accessible functor.*

(i) *There is a regular cardinal λ such that F is a strongly λ-accessible functor.*

(ii) *Moreover, if μ is a regular cardinal such that $\kappa \lhd \mu$ and $\lambda \le \mu$, then F also sends μ-presentable objects in $\mathcal{C}$ to μ-presentable objects in $\mathcal{D}$.*

Proof. (i). See Theorem 2.19 in [Adámek and Rosický, 1994].

(ii). Apply lemma 1.5 and proposition 1.11. □

2.6. Proposition. *If $\mathcal{C}$ is a locally κ-presentable category and $\mathcal{D}$ is any small category, then the functor category $[\mathcal{D}, \mathcal{C}]$ is also a locally κ-presentable category.*

Proof. See Corollary 1.54 in [Adámek and Rosický, 1994]. □

2.7. Proposition. *Let $\mathcal{C}$ be a locally small category and let $\mathcal{D}$ be a κ-small category.*

(i) *If λ is a regular cardinal $\ge \kappa$ such that $\mathcal{C}$ has colimits of small λ-filtered diagrams and $A : \mathcal{D} \to \mathcal{C}$ is a diagram whose vertices are λ-presentable objects in $\mathcal{C}$, then A is a λ-presentable object in $[\mathcal{D}, \mathcal{C}]$.*

(ii) *If $\mathcal{C}$ is a λ-accessible category and has products for κ-small families of objects, then every λ-presentable object in $[\mathcal{D}, \mathcal{C}]$ is componentwise λ-presentable.*

Proof. See (the proof of) Proposition 2.23 in [Low, 2013]. □

2.8. Definition. Given a regular cardinal κ, a **κ-accessible subcategory** of a κ-accessible category $\mathcal{C}$ is a subcategory $\mathcal{B} \subseteq \mathcal{C}$ such that $\mathcal{B}$ is a κ-accessible category and the inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$ is a κ-accessible functor.

2.9. Proposition. *Let $\mathcal{C}$ be a κ-accessible category and let $\mathcal{B}$ be a replete and full κ-accessible subcategory of $\mathcal{C}$.*

(i) *If A is a κ-presentable object in $\mathcal{C}$ and A is in $\mathcal{B}$, then A is also a κ-presentable object in $\mathcal{B}$.*

(ii) *If the inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$ is strongly κ-accessible, then $\mathbf{K}_\kappa(\mathcal{B}) = \mathcal{B} \cap \mathbf{K}_\kappa(\mathcal{C})$.*

Proof. (i). This is clear, since hom-sets and colimits of small κ-filtered diagrams in $\mathcal{B}$ are computed as in $\mathcal{C}$.

(ii). Given (i), it suffices to show that every κ-presentable object in $\mathcal{B}$ is also κ-presentable in $\mathcal{C}$, but this is precisely the hypothesis that the inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$ is strongly κ-accessible. ■

2.10. Lemma. *Let $\mathcal{C}$ be a κ-accessible category and let $\mathcal{B}$ be a full subcategory of $\mathcal{C}$. Assuming $\mathcal{B}$ is closed in $\mathcal{C}$ under colimits of small κ-filtered diagrams, the following are equivalent:*

(i) *The inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$ is a strongly κ-accessible functor.*

(ii) *Given a morphism $f : X \to Y$ in $\mathcal{C}$, if X is a κ-presentable object in $\mathcal{C}$ and Y is an object in $\mathcal{B}$, then $f : X \to Y$ factors through an object in $\mathcal{B}$ that is κ-presentable as an object in $\mathcal{C}$.*

Proof. (i) $\Rightarrow$ (ii). Let $f : X \to Y$ be a morphism in $\mathcal{C}$. The hypothesis implies that Y is a colimit in $\mathcal{C}$ of a small κ-filtered diagram in $\mathcal{B} \cap \mathbf{K}_\kappa(\mathcal{C})$; but X is a κ-presentable object in $\mathcal{C}$, so $f : X \to Y$ must factor through some component of the colimiting cocone.

(ii) $\Rightarrow$ (i). In view of lemma 1.5 and proposition 2.9, it suffices to show that every object in $\mathcal{B}$ is a colimit (in $\mathcal{C}$) of an essentially small κ-filtered diagram in $\mathcal{B} \cap \mathbf{K}_\kappa(\mathcal{C})$.

Let Y be an object in $\mathcal{B}$ and let $\mathcal{J}$ be the full subcategory of the slice category $\mathcal{C}_{/Y}$ spanned by the objects (X, f) where X is an object in $\mathcal{B}$ that is a κ-presentable object in $\mathcal{C}$. Clearly, $\mathcal{J}$ is a full subcategory of $(\mathbf{K}_\kappa(\mathcal{C}) \downarrow Y)$. On the other hand, the evident projection $U : (\mathbf{K}_\kappa(\mathcal{C}) \downarrow Y) \to \mathcal{C}$ is an essentially small κ-filtered diagram and the tautological cocone $U \Rightarrow \Delta Y$ is a colimiting cocone.[1] Moreover, the hypothesis implies that $\mathcal{J}$ is a κ-filtered category and a cofinal subcategory of $(\mathbf{K}_\kappa(\mathcal{C}) \downarrow Y)$. Thus, Y is also a colimit of the diagram obtained by restricting along the inclusion $\mathcal{J} \hookrightarrow (\mathbf{K}_\kappa(\mathcal{C}) \downarrow Y)$. This completes the proof. ■

[1] See Proposition 2.1.5 in [Makkai and Paré, 1989] or Proposition 2.8 in [Adámek and Rosický, 1994].

2.11. Proposition. *Let $F : \mathcal{C} \to \mathcal{D}$ be a strongly κ-accessible functor and let $\mathcal{D}'$ be the full subcategory of $\mathcal{D}$ spanned by the image of F.*

(i) *Every object in $\mathcal{D}'$ is a colimit in $\mathcal{D}$ of some small κ-filtered diagram consisting of objects in $\mathcal{D}'$ that are κ-presentable as objects in $\mathcal{D}$.*

(ii) *Every κ-presentable object in $\mathcal{D}'$ is also κ-presentable as an object in $\mathcal{D}$.*

(iii) *If $\mathcal{D}'$ is closed under colimits of small κ-filtered diagrams in $\mathcal{D}$, then $\mathcal{D}'$ is a κ-accessible subcategory of $\mathcal{D}$.*

Proof. (i). Let D be any object in $\mathcal{D}'$. By definition, there is an object C in $\mathcal{C}$ such that $D = FC$, and since $\mathcal{C}$ is a κ-accessible category, there is a small κ-filtered diagram $X : \mathcal{J} \to \mathcal{C}$ such that each Xj is a κ-presentable object in $\mathcal{C}$ and $C \cong \varinjlim_{\mathcal{J}} X$. Since $F : \mathcal{C} \to \mathcal{D}$ is a strongly κ-accessible functor, each FXj is a κ-presentable object in $\mathcal{D}$ and we have $D \cong \varinjlim_{\mathcal{J}} FX$.

(ii). Moreover, if D is a κ-presentable object in $\mathcal{D}'$, then D must be a retract of FXj for some object j in $\mathcal{J}$, and so D is also κ-presentable as an object in $\mathcal{D}$.

(iii). Any object in $\mathcal{D}'$ that is κ-presentable as an object in $\mathcal{D}$ must be κ-presentable as an object in $\mathcal{D}'$, because $\mathcal{D}'$ is a full subcategory of $\mathcal{D}$ that is closed under colimits of small κ-filtered diagrams. Thus, by (i), $\mathcal{D}'$ is a κ-accessible subcategory of $\mathcal{D}$. ■

2.12. Theorem. [Accessibility of comma categories] *Let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be κ-accessible functors.*

(i) *The comma category $(F \downarrow G)$ has colimits of small κ-filtered diagrams, created by the projection functor $(F \downarrow G) \to \mathcal{C} \times \mathcal{D}$.*

(ii) *Given an object (C, D, e) in $(F \downarrow G)$, if C is a κ-presentable object in $\mathcal{C}$, D is a κ-presentable object in $\mathcal{D}$, and FC is a κ-presentable object in $\mathcal{E}$, then (C, D, e) is a κ-presentable object in $(F \downarrow G)$.*

(iii) *If both F and G are strongly κ-accessible functors, then $(F \downarrow G)$ is a κ-accessible category, and the projection functors $P : (F \downarrow G) \to \mathcal{C}$ and $Q : (F \downarrow G) \to \mathcal{D}$ are strongly κ-accessible.*

Proof. See (the proof of) Theorem 2.43 in [Adámek and Rosický, 1994]. □

2.13. Corollary. *If $\mathcal{C}$ is a κ-accessible category, then so is the functor category $[\mathbf{2}, \mathcal{C}]$. Moreover, the κ-presentable objects in $[\mathbf{2}, \mathcal{C}]$ are precisely the morphisms between κ-presentable objects in $\mathcal{C}$.*

Proof. The functor category $[\mathbf{2}, \mathcal{C}]$ is isomorphic to the comma category $(\mathcal{C} \downarrow \mathcal{C})$, and $\mathrm{id} : \mathcal{C} \to \mathcal{C}$ is certainly a strongly κ-accessible functor, so this is a special case of theorem 2.12. ■

2.14. Theorem. [Accessibility of inverters] *Let $R, S : \mathcal{B} \to \mathcal{E}$ be κ-accessible functors, let $\varphi : R \Rightarrow S$ be a natural transformation, and let $\mathcal{B}'$ be the full subcategory of $\mathcal{B}$ spanned by those objects B in $\mathcal{B}$ such that $\varphi_B : RB \to SB$ is an isomorphism in $\mathcal{E}$.*

(i) *$\mathcal{B}'$ is closed in $\mathcal{B}$ under colimits of small κ-filtered diagrams.*

(ii) *If both R and S are strongly λ-accessible functors and $\kappa < \lambda$, then the inclusion $\mathcal{B}' \hookrightarrow \mathcal{B}$ is strongly λ-accessible.*

Proof. (i). Straightforward.

(ii). By lemma 2.10, it suffices to verify that, for every morphism $f : B \to B'$ in $\mathcal{B}$, if B is a λ-presentable object in $\mathcal{B}$ and B' is in $\mathcal{B}'$, then $f : B \to B'$ factors through some λ-presentable object in $\mathcal{B}$ that is also in $\mathcal{B}'$.

Since $\mathcal{B}$ is a λ-accessible category, we may choose a small λ-filtered diagram $X : \mathcal{I} \to \mathcal{B}$ such that each Xi is a λ-presentable object in $\mathcal{B}$ and $\varinjlim_{\mathcal{I}} X \cong B'$. Since B is a λ-presentable object in $\mathcal{B}$, there is an object i_0 in $\mathcal{I}$ such that $f : B \to B'$ factors as a morphism $B \to Xi_0$ in $\mathcal{B}$ followed by the colimiting cocone component $Xi_0 \to B'$. Then, by lemma 1.6, there is a chain $I : \kappa \to \mathcal{I}$ such that $I(0) = i_0$ and $\hat{B} = \varinjlim_{\gamma < \kappa} XI(\gamma)$ is in $\mathcal{B}'$. Moreover, since $\kappa < \lambda$, $\hat{B}$ is a λ-presentable object in $\mathcal{B}$ (by lemma 1.5). We have thus obtained the required factorisation of $f : B \to B'$. ■

The next theorem is a variation on Proposition 3.1 in [Chorny and Rosický, 2012] and appears as the "pseudopullback theorem" in [Raptis and Rosický, 2015]. Recall that the **iso-comma category** $(F \wr G)$ for functors $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ is the full subcategory of the comma category $(F \downarrow G)$ spanned by those objects (C, D, e) where $e : FC \to GD$ is an isomorphism in $\mathcal{E}$.

2.15. Theorem. [Accessibility of iso-comma categories] *Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories with colimits of small κ-filtered diagrams, and let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors that preserve colimits of small κ-filtered diagrams.*

(i) *The iso-comma category $(F \wr G)$ has colimits of small κ-filtered diagrams, created by the projection functor $(F \wr G) \to \mathcal{C} \times \mathcal{D}$.*

(ii) *Given an object (C, D, e) in $(F \wr G)$, if C is a λ-presentable object in $\mathcal{C}$, D is a λ-presentable object in $\mathcal{D}$, and FC is a λ-presentable object in $\mathcal{E}$, then (C, D, e) is a λ-presentable object in $(F \wr G)$.*

(iii) *If F and G are strongly λ-accessible functors and $\kappa < \lambda$, then the inclusion $(F \wr G)$ is a λ-accessible category, and the projection functors $P : (F \wr G) \to \mathcal{C}$ and $Q : (F \wr G) \to \mathcal{D}$ are strongly λ-accessible.*

Proof. (i). This is a straightforward consequence of the hypothesis that both $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ preserve colimits of small κ-filtered diagrams.

(ii). Apply proposition 2.9 and theorem 2.12.

(iii). By theorem 2.14, the inclusion $(F \wr G) \hookrightarrow (F \downarrow G)$ is a strongly λ-accessible functor. Since the class of strongly λ-accessible functors is closed under composition, it follows that the projections $P : (F \wr G) \to \mathcal{C}$ and $Q : (F \wr G) \to \mathcal{D}$ are also strongly λ-accessible. ■

2.16. Proposition. *Let $\mathcal{C}$ and $\mathcal{E}$ be categories with colimits of small κ-filtered diagrams, let $\mathcal{D}$ be a replete and full subcategory of $\mathcal{E}$ that is closed under colimits of small κ-filtered diagrams, let $F : \mathcal{C} \to \mathcal{E}$ be a functor that preserves colimits of small κ-filtered diagrams, and let $\mathcal{B}$ be the preimage of $\mathcal{D}$ under F, so that we have the following strict pullback diagram:*

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathcal{D} \\ \big\downarrow & & \big\downarrow \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{E} \end{array}$$

(i) *$\mathcal{B}$ is a replete and full subcategory of $\mathcal{D}$ and is closed under colimits of small κ-filtered diagrams in $\mathcal{D}$.*

(ii) *If $F : \mathcal{C} \to \mathcal{E}$ and the inclusion $\mathcal{D} \hookrightarrow \mathcal{E}$ are strongly λ-accessible functors and $\kappa < \lambda$, then $\mathcal{B}$ is a λ-accessible subcategory of $\mathcal{C}$, and moreover, the inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$ is also strongly λ-accessible.*

Proof. (i). Straightforward.

(ii). Consider the iso-comma category $(F \wr \mathcal{D})$ and the induced comparison functor $K : \mathcal{B} \to (F \wr \mathcal{D})$. It is clear that K is fully faithful; but since $\mathcal{D}$ is a replete subcategory of $\mathcal{C}$, for every object (C, D, e) in $(F \wr \mathcal{D})$, there is a canonical isomorphism $KC \to (C, D, e)$, namely the one corresponding to the following commutative diagram in $\mathcal{E}$:

$$\begin{array}{ccc} FC & \xrightarrow{\text{id}} & FC \\ {\scriptstyle \text{id}}\big\downarrow & & \big\downarrow{\scriptstyle e} \\ FC & \xrightarrow[e]{} & D \end{array}$$

Thus, $K : \mathcal{B} \to (F \wr \mathcal{D})$ is (half of) an equivalence of categories. Theorem 2.15 says the projection $P : (F \wr \mathcal{D}) \to \mathcal{C}$ is a strongly λ-accessible functor, so we may deduce that the same is true for the inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$. ■

2.17. Lemma. *Let $\mathcal{C}$ be a locally κ-presentable category and let $\mathbb{T} = (T, \eta, \mu)$ be a monad on $\mathcal{C}$. If the forgetful functor $U : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ is strongly κ-accessible, then so is the functor $T : \mathcal{C} \to \mathcal{C}$.*

Proof. The free $\mathbb{T}$-algebra functor $F : \mathcal{C} \to \mathcal{C}^{\mathbb{T}}$ is strongly κ-accessible if the forgetful functor $U : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ is κ-accessible; but $T = UF$, so T is strongly κ-accessible when U is. ■

The following appears as part of Proposition 4.13 in [Ulmer, 1977].

2.18. Theorem. [The category of algebras for a strongly accessible monad] *Let* $\mathcal{C}$ *be a locally* λ*-presentable category, let* $\mathbb{T} = (T, \eta, \mu)$ *be a monad on* $\mathcal{C}$ *where* $T : \mathcal{C} \to \mathcal{C}$ *preserves colimits of small* κ*-filtered diagrams, and let* $\mathcal{C}^{\mathbb{T}}$ *be the category of algebras for* $\mathbb{T}$*. If* $T : \mathcal{C} \to \mathcal{C}$ *is a strongly* λ*-accessible functor and* $\kappa < \lambda$*, then:*

(i) *Given a coequaliser diagram in* $\mathcal{C}^{\mathbb{T}}$ *of the form below,*

$$(A, \alpha) \rightrightarrows (B, \beta) \longrightarrow (C, \gamma)$$

if A *and* B *are* λ*-presentable objects in* $\mathcal{C}$*, then so is* C*.*

(ii) *Given a* λ*-small family* $((A_i, \alpha_i) \mid i \in I)$ *of* $\mathbb{T}$*-algebras, if each* A_i *is a* λ*-presentable object in* $\mathcal{C}$*, then so is the underlying object of the* $\mathbb{T}$*-algebra coproduct* $\sum_{i \in I} (A_i, \alpha_i)$*.*

(iii) *The forgetful functor* $U : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ *is strongly* λ*-accessible.*

Proof. (i). By referring to the explicit construction of coequalisers in $\mathcal{C}^{\mathbb{T}}$ given in the proof of Proposition 4.3.6 in [Borceux, 1994] and applying lemma 1.5, we see that C is indeed a λ-presentable object in $\mathcal{C}$ when A and B are, provided $T : \mathcal{C} \to \mathcal{C}$ preserves colimits of small κ-filtered diagrams *and* is strongly λ-accessible.

(ii). Let $F : \mathcal{C} \to \mathcal{C}^{\mathbb{T}}$ be a left adjoint for $U : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$. In the proof of Proposition 4.3.4 in [Borceux, 1994], we find that the $\mathbb{T}$-algebra coproduct $\sum_{i \in I} (A_i, \alpha_i)$ may be computed by a coequaliser diagram of the following form:

$$F\big(\textstyle\sum_{i \in I} T A_i\big) \rightrightarrows F\big(\textstyle\sum_{i \in I} A_i\big) \longrightarrow \textstyle\sum_{i \in I} (A_i, \alpha_i)$$

Since $T : \mathcal{C} \to \mathcal{C}$ is strongly λ-accessible, the underlying objects of the $\mathbb{T}$-algebras $F\big(\sum_{i \in I} T A_i\big)$ and $F\big(\sum_{i \in I} A_i\big)$ are λ-presentable objects in $\mathcal{C}$. Thus, by (i), the underlying object of $\sum_{i \in I} (A_i, \alpha_i)$ must also be a λ-presentable object in $\mathcal{C}$.

(iii). It is shown in the proof of Theorem 5.5.9 in [Borceux, 1994] that the full subcategory $\mathcal{F}$ of $\mathcal{C}^{\mathbb{T}}$ spanned by the image of $\mathbf{K}_\lambda(\mathcal{C})$ under $F : \mathcal{C} \to \mathcal{C}^{\mathbb{T}}$ is a dense subcategory. Let $\mathcal{G}$ be the smallest replete full subcategory of $\mathcal{C}^{\mathbb{T}}$ that contains $\mathcal{F}$ and is closed under colimits of λ-small diagrams in $\mathcal{C}$. Observe that (i) and (ii) imply that the underlying object of every $\mathbb{T}$-algebra that is in $\mathcal{G}$ must be a λ-presentable object in $\mathcal{C}$. To show that the forgetful functor $U : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ is strongly λ-accessible, it is enough to verify that every λ-presentable object in $\mathcal{C}^{\mathbb{T}}$ is in $\mathcal{G}$.

It is not hard to see that the comma category $(\mathcal{G} \downarrow (A, \alpha))$ is an essentially small λ-filtered category for any $\mathbb{T}$-algebra (A, α), and moreover, it can be shown that the tautological cocone for the canonical diagram $(\mathcal{G} \downarrow (A, \alpha)) \to \mathcal{C}^{\mathbb{T}}$ is a colimiting cocone. Thus, if (A, α) is a λ-presentable object in $\mathcal{C}^{\mathbb{T}}$, it must be a retract of an object in $\mathcal{G}$. But $\mathcal{G}$ is closed under retracts, so (A, α) is indeed in $\mathcal{G}$. ■

The following result on the existence of free algebras for a pointed endofunctor is a special case of a general construction due to Kelly [1980].

2.19. THEOREM. [Free algebras for a pointed endofunctor] *Let $\mathcal{C}$ be a category with joint coequalisers for κ-small families of parallel pairs and colimits of chains of length $\leq \kappa$, let (J, ι) be a pointed endofunctor on $\mathcal{C}$ such that $J : \mathcal{C} \to \mathcal{C}$ preserves colimits of κ-chains, and let $\mathcal{C}^{(J,\iota)}$ be the category of algebras for (J, ι).*

(i) *The forgetful functor $U : \mathcal{C}^{(J,\iota)} \to \mathcal{C}$ has a left adjoint, say $F : \mathcal{C} \to \mathcal{C}^{(J,\iota)}$.*

(ii) *Let λ be a regular cardinal. If $J : \mathcal{C} \to \mathcal{C}$ sends λ-presentable objects to λ-presentable objects and $\kappa < \lambda$, then the functor $UF : \mathcal{C} \to \mathcal{C}$ has the same property.*

PROOF. Let X be an object in $\mathcal{C}$. We define an object X_α for each ordinal $\alpha \leq \kappa$, a morphism $q_\alpha : JX_\alpha \to X_{\alpha+1}$ for each ordinal $\alpha < \kappa$, and a morphism $s_{\alpha,\beta} : X_\alpha \to X_\beta$ for each pair (α, β) of ordinals such that $\alpha \leq \beta \leq \kappa$ by transfinite recursion as follows:

- We define $X_0 = X$ and $s_{0,0} = \mathrm{id}_{X_0}$.
- For each ordinal $\beta < \kappa$, given X_α for all $\alpha \leq \beta$, q_α for all $\alpha < \beta$, and $s_{\alpha,\beta}$ for all $\alpha \leq \beta$, we define $q_\beta : JX_\beta \to X_{\beta+1}$ to be the joint coequaliser of the parallel pairs

  $$JX_\alpha \underset{\iota_{X_\beta} \circ s_{\alpha+1,\beta} \circ q_\alpha}{\overset{Js_{\alpha,\beta}}{\rightrightarrows}} JX_\beta$$

  for all $\alpha < \beta$. (In particular, $q_0 : JX_0 \to X_1$ is an isomorphism.) We define $s_{\beta+1,\beta+1} = \mathrm{id}_{X_{\beta+1}}$, $s_{\beta,\beta+1} = q_\beta \circ \iota_{X_\beta}$, and $s_{\alpha,\beta+1} = s_{\beta,\beta+1} \circ s_{\alpha,\beta}$ for all $\alpha < \beta$, so that we obtain a chain $X_\bullet : (\beta + 2) \to \mathcal{C}$.
- For each limit ordinal $\gamma \leq \kappa$, given X_α for all $\alpha < \gamma$ and $s_{\alpha,\beta}$ for all $\alpha \leq \beta < \gamma$, we define $X_\gamma = \varinjlim_{\alpha<\gamma} X_\alpha$ and $s_{\gamma,\gamma} = \mathrm{id}_{X_\gamma}$ and, for $\alpha < \gamma$, we define $s_{\alpha,\gamma}$ to be the components of the colimiting cocone.

Let $\bar{X} = X_\kappa$. By construction, for all $\alpha \leq \beta < \kappa$, the diagram in $\mathcal{C}$ shown below commutes,

$$\begin{array}{ccc} JX_\alpha & \xrightarrow{Js_{\alpha,\beta}} & JX_\beta \\ {\scriptstyle q_\alpha}\downarrow & & \downarrow{\scriptstyle q_\beta} \\ X_{\alpha+1} & \xrightarrow[s_{\alpha+1,\beta+1}]{} & X_{\beta+1} \end{array}$$

and by hypothesis, the morphisms $Js_{\alpha,\kappa} : JX_\alpha \to JX_\kappa$ constitute a colimiting cocone for the evident chain $JX_\bullet : \kappa \to \mathcal{C}$, so there is a unique morphism $\bar{q} : J\bar{X} \to \bar{X}$ such that $\bar{q} \circ Js_{\alpha,\kappa} = s_{\alpha+1,\kappa} \circ q_\alpha$ for all $\alpha < \kappa$. Moreover,

$$(\bar{q} \circ \iota_{X_\kappa}) \circ s_{\alpha,\kappa} = \bar{q} \circ Js_{\alpha,\kappa} \circ \iota_{X_\alpha} = s_{\alpha+1,\kappa} \circ q_\alpha \circ \iota_{X_\alpha} = s_{\alpha+1,\kappa} \circ s_{\alpha,\alpha+1} = s_{\alpha,\kappa}$$

so $\bar{q} \circ \iota_{\bar{X}} = \mathrm{id}_{\bar{X}}$, i.e. $\left(\bar{X}, \bar{q}\right)$ is a (J, ι)-algebra.

Define $\eta_X : X \to \bar{X}$ to be $s_{0,\kappa}$. We will now show that $\left(\bar{X}, \bar{q}\right)$ is a *free* (J, ι)-algebra with unit η_X. Let (Y, r) be any (J, ι)-algebra and let $f : X \to Y$ be any morphism in $\mathcal{C}$. We construct a morphism $f_\alpha : X_\alpha \to Y$ for each ordinal $\alpha \le \kappa$ by transfinite recursion:

- We define $f_0 = f$.
- For each ordinal $\beta < \kappa$, given f_α for all $\alpha \le \beta$ such that the following equations are satisfied,

$$\begin{aligned} f_\beta \circ s_{\alpha,\beta} &= f_\alpha && \text{for all } \alpha \le \beta \\ f_{\alpha+1} \circ q_\alpha &= r \circ J f_\alpha && \text{for all } \alpha < \beta \end{aligned}$$

  we also have

$$\begin{aligned} (r \circ J f_\beta) \circ \left(\iota_{X_\beta} \circ s_{\alpha+1,\beta} \circ q_\alpha\right) &= r \circ \iota_Y \circ f_\beta \circ s_{\alpha+1,\beta} \circ q_\alpha \\ &= f_\beta \circ s_{\alpha+1,\beta} \circ q_\alpha \\ &= f_{\alpha+1} \circ q_\alpha \\ &= r \circ J f_\alpha \\ &= (r \circ J f_\beta) \circ J s_{\alpha,\beta} && \text{for all } \alpha < \beta \end{aligned}$$

  so we may define $f_{\beta+1}$ to be the unique morphism $X_{\beta+1} \to Y$ in $\mathcal{C}$ such that $f_{\beta+1} \circ q_\beta = r \circ J f_\beta$. Then,

$$f_{\beta+1} \circ s_{\beta,\beta+1} = f_{\beta+1} \circ q_\beta \circ \iota_{X_\beta} = r \circ J f_\beta \circ \iota_{X_\beta} = r \circ \iota_Y \circ f_\beta = f_\beta$$

  so we have $f_{\beta+1} \circ s_{\alpha,\beta+1} = f_\alpha$ for all $\alpha \le \beta + 1$.
- For each limit ordinal $\gamma \le \kappa$, we define f_γ to be the unique morphism $X_\gamma \to Y$ in $\mathcal{C}$ such that $f_\gamma \circ s_{\alpha,\gamma} = f_\alpha$ for all $\alpha < \gamma$.

By construction, for all ordinals $\alpha < \kappa$,

$$(r \circ J f_\kappa) \circ J s_{\alpha,\kappa} = r \circ J f_\alpha = f_{\alpha+1} \circ q_\alpha = f_\kappa \circ s_{\alpha+1,\kappa} \circ q_\alpha = (f_\kappa \circ \bar{q}) \circ J s_{\alpha,\kappa}$$

so $r \circ J f_\kappa = f_\kappa \circ \bar{q}$, i.e. $f_\kappa : X_\kappa \to Y$ is a (J, ι)-algebra homomorphism $(X_\kappa, \bar{q}) \to (Y, r)$. Moreover, for any homomorphism $\bar{f} : (X_\kappa, \bar{q}) \to (Y, r)$ and any ordinal $\alpha < \kappa$,

$$\left(\bar{f} \circ s_{\alpha+1,\kappa}\right) \circ q_\alpha = \bar{f} \circ \bar{q} \circ J s_{\alpha,\kappa} = r \circ J \bar{f} \circ J s_{\alpha,\kappa}$$

so if $\bar{f} \circ s_{\alpha,\kappa} = f_\alpha$, then $\bar{f} \circ s_{\alpha+1,\kappa} = f_{\alpha+1}$; and for any limit ordinal $\gamma \le \kappa$, if $\bar{f} \circ s_{\alpha,\kappa} = f_\alpha$ for all $\alpha < \gamma$, then $\bar{f} \circ s_{\gamma,\kappa} = f_\gamma$ as well. In particular, if $\bar{f} \circ \eta_X = f$, then $\bar{f} = f_\kappa$ by transfinite induction. Thus, there is a unique homomorphism $\bar{f} : (X_\kappa, \bar{q}) \to (Y, r)$ such that $\bar{f} \circ \eta_X = f$.

The above argument shows that the comma category $(X \downarrow U)$ has an initial object, and it is well known that U has a left adjoint if and only if each comma category $(X \downarrow U)$ has an initial object, so this completes the proof of (i). For (ii), we simply observe that $\mathbf{K}_\lambda(\mathcal{C})$ is closed under colimits of λ-small diagrams in $\mathcal{C}$ (by lemma 1.5), so the above construction can be carried out entirely in $\mathbf{K}_\lambda(\mathcal{C})$. ■

2.20. Theorem. [The category of algebras for a accessible pointed endofunctor] *Let $J : \mathcal{C} \to \mathcal{C}$ be a functor, let $\iota : \mathrm{id}_{\mathcal{C}} \Rightarrow J$ be a natural transformation, and let $\mathcal{C}^{(J,\iota)}$ be the category of algebras for the pointed endofunctor (J, ι).*

(i) *If $\mathcal{C}$ has colimits of small κ-filtered diagrams and $J : \mathcal{C} \to \mathcal{C}$ preserves them, then the forgetful functor $U : \mathcal{C}^{(J,\iota)} \to \mathcal{C}$ creates colimits of small κ-filtered diagrams; and if $\mathcal{C}$ is complete, then $U : \mathcal{C}^{(J,\iota)} \to \mathcal{C}$ also creates limits for all small diagrams.*

(ii) *If $\mathcal{C}$ is an accessible functor, then $\mathcal{C}^{(J,\iota)}$ is an accessible category.*

(iii) *If $\mathcal{C}$ has joint coequalisers for κ-small families of parallel pairs and colimits of chains of length $\leq \kappa$ and $J : \mathcal{C} \to \mathcal{C}$ preserves colimits of κ-chains, then $U : \mathcal{C}^{(J,\iota)} \to \mathcal{C}$ is a monadic functor.*

Proof. (i). This is analogous to the well known fact about monads: cf. Propositions 4.3.1 and 4.3.2 in [Borceux, 1994].

(ii). We may construct $\mathcal{C}^{(J,\iota)}$ using inserters and equifiers, as in the proof of Theorem 2.78 in [Adámek and Rosický, 1994].

(iii). The hypotheses of theorem 2.19 are satisfied, so the forgetful functor $U : \mathcal{C}^{(J,\iota)} \to \mathcal{C}$ has a left adjoint. It is not hard to check that the other hypotheses of Beck's monadicity theorem are satisfied, so U is indeed a monadic functor. □

2.21. Theorem. [The category of algebras for a strongly accessible pointed endofunctor] *Let $\mathcal{C}$ be a locally λ-presentable category, let $J : \mathcal{C} \to \mathcal{C}$ be a functor that preserves colimits of small κ-filtered diagrams, let $\iota : \mathrm{id}_{\mathcal{C}} \Rightarrow J$ be a natural transformation, and let $\mathbb{T} = (T, \eta, \mu)$ be the induced monad on $\mathcal{C}$. If $J : \mathcal{C} \to \mathcal{C}$ is a strongly λ-accessible functor and $\kappa < \lambda$, then:*

(i) *The functor $T : \mathcal{C} \to \mathcal{C}$ preserves colimits of small κ-filtered diagrams and is strongly λ-accessible.*

(ii) *$\mathcal{C}^{(J,\iota)}$ is a locally λ-presentable category.*

(iii) *The forgetful functor $U : \mathcal{C}^{(J,\iota)} \to \mathcal{C}$ is a strongly λ-accessible functor.*

Proof. (i). By theorem 2.20, the forgetful functor $U : \mathcal{C}^{(J,\iota)} \to \mathcal{C}$ creates colimits of small κ-filtered diagrams when $J : \mathcal{C} \to \mathcal{C}$ preserves colimits of small κ-filtered diagrams, so $T : \mathcal{C} \to \mathcal{C}$ must also preserve these colimits. Moreover, theorem 2.19 implies $T : \mathcal{C} \to \mathcal{C}$ is strongly λ-accessible if $J : \mathcal{C} \to \mathcal{C}$ is.

(ii). It is not hard to check that the forgetful functor $\mathcal{C}^{(J,\iota)} \to \mathcal{C}$ is a monadic functor, so the claim reduces to the fact that $\mathcal{C}^{\mathbb{T}}$ is a locally λ-presentable category if $T : \mathcal{C} \to \mathcal{C}$ is a λ-accessible functor.[2]

(iii). Apply theorem 2.18. ■

[2]See Theorem 2.78 and the following remark in [Adámek and Rosický, 1994], or Theorem 5.5.9 in [Borceux, 1994].

3. Accessibly generated categories

3.1. NOTATION. Throughout this section, κ and λ are regular cardinals such that $\kappa \le \lambda$.

3.2. DEFINITION. A (κ, λ)**-accessibly generated category** is an essentially small category $\mathcal{C}$ that satisfies the following conditions:

- Every λ-small κ-filtered diagram in $\mathcal{C}$ has a colimit in $\mathcal{C}$.
- Every object in $\mathcal{C}$ is (the object part of) a colimit of some λ-small κ-filtered diagram of (κ, λ)-presentable objects in $\mathcal{C}$.

REMARK. In the case where λ is a strongly inaccessible cardinal with $\kappa < \lambda$, the concept of (κ, λ)-accessibly generated categories is very closely related to the concept of class-κ-accessible categories (in the sense of Chorny and Rosický [2012]) relative to the universe of hereditarily λ-small sets, though there are some technical differences. For our purposes, we do not need to assume that λ is a strongly inaccessible cardinal.

3.3. REMARK. Every κ-small κ-filtered category has a cofinal idempotent, so every object is automatically (κ, κ)-presentable. Thus, an essentially small category is (κ, κ)-accessibly generated if and only if it is idempotent-complete, i.e. if and only if all idempotent endomorphisms in $\mathcal{C}$ split.

3.4. REMARK. In the definition of '(κ, λ)-accessibly generated category', we can replace 'essentially small category' with 'locally small category such that the full subcategory of (κ, λ)-presentable objects is essentially small'.

3.5. PROPOSITION. *Let $\mathcal{C}$ be a κ-accessible category.*

(i) $\mathbf{K}_\kappa(\mathcal{C})$ *is a (κ, κ)-accessibly generated category, and every object in* $\mathbf{K}_\kappa(\mathcal{C})$ *is (κ, κ)-presentable.*

(ii) *If $\kappa \lhd \lambda$, then* $\mathbf{K}_\lambda(\mathcal{C})$ *is a (κ, λ)-accessibly generated category, and the (κ, λ)-presentable objects in* $\mathbf{K}_\lambda(\mathcal{C})$ *are precisely the κ-presentable objects in $\mathcal{C}$.*

PROOF. Combine lemma 1.5, proposition 1.11, and remark 3.3. ■

3.6. DEFINITION. Let μ be a regular cardinal such that $\lambda \le \mu$. A (κ, λ, μ)**-accessibly generated extension** is a functor $F : \mathcal{A} \to \mathcal{B}$ with the following properties:

- $\mathcal{A}$ is a (κ, λ)-accessibly generated category.
- $\mathcal{B}$ is a (κ, μ)-accessibly generated category.
- $F : \mathcal{A} \to \mathcal{B}$ preserves colimits of λ-small κ-filtered diagrams.
- F sends (κ, λ)-presentable objects in $\mathcal{A}$ to (κ, μ)-presentable objects in $\mathcal{B}$.
- The induced functor $F : \mathbf{K}_\kappa^\lambda(\mathcal{A}) \to \mathbf{K}_\kappa^\mu(\mathcal{B})$ is fully faithful and essentially surjective on objects.

Remark. The concept of accessibly generated extensions is essentially a generalisation of the concept of accessible extensions, as defined in [Low, 2013].

3.7. Remark. Let $\mathcal{C}$ be a (κ, λ)-accessibly generated category. Then, in view of remark 3.3, the inclusion $\mathbf{K}_\kappa^\lambda(\mathcal{C}) \hookrightarrow \mathcal{C}$ is a $(\kappa, \kappa, \lambda)$-accessibly generated extension.

3.8. Lemma. *Let $F : \mathcal{A} \to \mathcal{B}$ be a (κ, λ, μ)-accessibly generated extension and let $G : \mathcal{B} \to \mathcal{C}$ be a (κ, μ, ν)-accessibly generated extension. If $\lambda \le \mu$, then the composite $GF : \mathcal{A} \to \mathcal{C}$ is a (κ, λ, ν)-accessibly generated extension.*

Proof. Straightforward. ♦

3.9. Lemma. *Let $F : \mathcal{A} \to \mathcal{B}$ be a $(\kappa, \kappa, \lambda)$-accessibly generated extension.*

(i) *There is a functor $U : \mathcal{B} \to \mathbf{Ind}^\kappa(\mathcal{A})$ equipped with a natural bijection of the form below,*

$$\mathbf{Ind}^\kappa(\mathcal{A})(A, UB) \cong \mathcal{B}(FA, B)$$

and it is unique up to unique isomorphism.

(ii) *Moreover, the functor $U : \mathcal{B} \to \mathbf{Ind}^\kappa(\mathcal{A})$ is fully faithful and preserves colimits of λ-small κ-filtered diagrams.*

(iii) *In particular, $F : \mathcal{A} \to \mathcal{B}$ is a fully faithful functor.*

(iv) *If $\kappa \lhd \lambda$, then the λ-accessible functor $\bar{U} : \mathbf{Ind}^\lambda(\mathcal{B}) \to \mathbf{Ind}^\kappa(\mathcal{A})$ induced by $U : \mathcal{B} \to \mathbf{Ind}^\kappa(\mathcal{A})$ is fully faithful and essentially surjective on objects.*

(v) *In particular, if $\kappa \lhd \lambda$, then $\mathbf{Ind}^\lambda(\mathcal{B})$ is a κ-accessible category.*

Proof. (i). Let B be an object in $\mathcal{B}$. By hypothesis, there is a λ-small κ-filtered diagram $X : \mathcal{J} \to \mathcal{A}$ such that $B \cong \varinjlim_{\mathcal{J}} FX$. Then, for every object A in $\mathcal{A}$,

$$\mathcal{B}(FA, B) \cong \varinjlim_{\mathcal{J}} \mathcal{B}(FA, FX) \cong \varinjlim_{\mathcal{J}} \mathcal{A}(A, X)$$

so there is an object UB in $\mathbf{Ind}^\kappa(\mathcal{A})$ such that

$$\mathbf{Ind}^\kappa(\mathcal{A})(A, UB) \cong \mathcal{B}(FA, B)$$

for all objects A in $\mathcal{A}$, and an object with such a natural bijection is unique up to unique isomorphism, because $\mathcal{A} \hookrightarrow \mathbf{Ind}^\kappa(\mathcal{A})$ is a dense functor. A similar argument can be used to define Ug for morphisms $g : B_0 \to B_1$ in $\mathcal{B}$, and it is straightforward to check that this indeed defines a functor $U : \mathcal{B} \to \mathbf{Ind}^\kappa(\mathcal{A})$.

(ii). Let $Y : \mathcal{J} \to \mathcal{B}$ be a λ-small κ-filtered diagram in $\mathcal{B}$. Then, for any object A in $\mathcal{A}$,

$$\begin{aligned}\mathcal{B}\Big(FA, \varinjlim_{\mathcal{J}} Y\Big) &\cong \varinjlim_{\mathcal{J}} \mathcal{B}(FA, Y) \\ &\cong \varinjlim_{\mathcal{J}} \mathbf{Ind}^\kappa(\mathcal{A})(A, UY)\end{aligned}$$

$$\cong \mathbf{Ind}^{\kappa}(\mathcal{A})\Big(A, \varinjlim_{\mathcal{J}} UY\Big)$$

so $U : \mathcal{B} \to \mathbf{Ind}^{\kappa}(\mathcal{A})$ indeed preserves colimits of λ-small κ-filtered diagrams. A similar argument can be used to show that $U : \mathcal{B} \to \mathbf{Ind}^{\kappa}(\mathcal{A})$ is fully faithful.

(iii). The composite $UF : \mathcal{A} \to \mathbf{Ind}^{\kappa}(\mathcal{A})$ is clearly fully faithful, so it follows from (ii) that $F : \mathcal{A} \to \mathcal{B}$ is fully faithful.

(iv). Proposition 1.11 implies that $U : \mathcal{B} \to \mathbf{Ind}^{\kappa}(\mathcal{A})$ is essentially surjective onto the full subcategory of λ-presentable objects in $\mathbf{Ind}^{\kappa}(\mathcal{A})$. Moreover, since $\kappa \lhd \lambda$, $\mathbf{Ind}^{\kappa}(\mathcal{A})$ is also a λ-accessible category,[3] and it follows that the induced λ-accessible functor $\mathbf{Ind}^{\lambda}(\mathcal{B}) \to \mathbf{Ind}^{\kappa}(\mathcal{A})$ is fully faithful and essentially surjective on objects.

(v). We know that $\mathbf{Ind}^{\kappa}(\mathcal{A})$ is a κ-accessible category, so it follows from (iv) that $\mathbf{Ind}^{\lambda}(\mathcal{B})$ is also a κ-accessible category. ■

3.10. Proposition. *Let $F : \mathcal{A} \to \mathcal{B}$ be a (κ, λ, μ)-accessibly generated extension. Assuming either $\kappa = \lambda$ or $\kappa \lhd \lambda$:*

(i) *There is a functor $U : \mathcal{B} \to \mathbf{Ind}^{\lambda}(\mathcal{A})$ equipped with a natural bijection of the form below,*

$$\mathbf{Ind}^{\lambda}(\mathcal{A})(A, UB) \cong \mathcal{B}(FA, B)$$

and it is unique up to unique isomorphism.

(ii) *Moreover, the functor $U : \mathcal{B} \to \mathbf{Ind}^{\lambda}(\mathcal{A})$ is fully faithful and preserves colimits of μ-small λ-filtered diagrams.*

(iii) *In particular, $F : \mathcal{A} \to \mathcal{B}$ is a fully faithful functor.*

(iv) *If $\lambda \lhd \mu$, then the μ-accessible functor $\bar{U} : \mathbf{Ind}^{\mu}(\mathcal{B}) \to \mathbf{Ind}^{\lambda}(\mathcal{A})$ induced by $U : \mathcal{B} \to \mathbf{Ind}^{\lambda}(\mathcal{A})$ is fully faithful and essentially surjective on objects.*

(v) *In particular, if $\lambda \lhd \mu$, then $\mathbf{Ind}^{\mu}(\mathcal{B})$ is a κ-accessible category.*

Proof. Remark 3.7 says the inclusion $\mathbf{K}^{\lambda}_{\kappa}(\mathcal{A}) \hookrightarrow \mathcal{A}$ is a $(\kappa, \kappa, \lambda)$-accessibly generated extension, so by lemma 3.8, the composite $\mathbf{K}^{\lambda}_{\kappa}(\mathcal{A}) \hookrightarrow \mathcal{A} \to \mathcal{B}$ is a (κ, κ, μ)-accessible generated extension. Moreover, $\kappa \lhd \mu$,[4] so the claims follow, by (two applications of) lemma 3.9. ■

[3]See Theorem 2.3.10 in [Makkai and Paré, 1989] or Theorem 2.11 in [Adámek and Rosický, 1994].
[4]See Proposition 2.3.2 in [Makkai and Paré, 1989].

3.11. Theorem. *If either $\kappa = \lambda$ or $\kappa \lhd \lambda$, then the following are equivalent for a idempotent-complete category $\mathcal{C}$:*

(i) *$\mathcal{C}$ is a (κ, λ)-accessibly generated category.*

(ii) *$\mathbf{Ind}^{\lambda}(\mathcal{C})$ is a κ-accessible category.*

(iii) *$\mathcal{C}$ is equivalent to $\mathbf{K}_{\lambda}(\mathcal{D})$ for some κ-accessible category $\mathcal{D}$.*

Proof. (i) $\Rightarrow$ (ii). Apply lemma 3.9 to remark 3.7.

(ii) $\Rightarrow$ (iii). It is not hard to check that every λ-presentable object in $\mathbf{Ind}^{\lambda}(\mathcal{C})$ is a retract of some object in the image of the canonical embedding $\mathcal{C} \to \mathbf{Ind}^{\lambda}(\mathcal{C})$. But $\mathcal{C}$ is idempotent-complete, so the canonical embedding is fully faithful and essentially surjective onto the full subcategory of λ-presentable objects in $\mathbf{Ind}^{\lambda}(\mathcal{C})$.

(iii) $\Rightarrow$ (i). See proposition 3.5. ■

3.12. Corollary. *If $\mathcal{C}$ is a (κ, λ)-accessibly generated category, then so is $[\mathbf{2}, \mathcal{C}]$.*

Proof. Combine corollary 2.13 and theorem 3.11. ■

4. Accessible factorisation systems

4.1. Notation. Throughout this section, κ is an arbitrary regular cardinal.

4.2. Lemma. *Let $\mathcal{C}$ be a category with colimits of small κ-filtered diagrams, let $\mathcal{I}$ be a subset of $\operatorname{mor} \mathcal{C}$, and let $\mathcal{I}^{\boxslash}$ be the class of morphisms in $\mathcal{C}$ with the right lifting property with respect to $\mathcal{I}$. If the domains and codomains of the members of $\mathcal{I}$ are κ-presentable objects in $\mathcal{C}$, then $\mathcal{I}^{\boxslash}$ (regarded as a full subcategory of $[\mathbf{2}, \mathcal{C}]$) is closed under colimits of small κ-filtered diagrams in $[\mathbf{2}, \mathcal{C}]$.*

Proof. By proposition 2.7, any element of $\mathcal{I}$ is κ-presentable as an object in $[\mathbf{2}, \mathcal{C}]$. Thus, given any morphism $\varphi : e \to \varinjlim_{\mathcal{J}} f$ in $[\mathbf{2}, \mathcal{C}]$ where e is in $\mathcal{I}$ and $f : \mathcal{J} \to [\mathbf{2}, \mathcal{C}]$ is a small κ-filtered diagram with each vertex in $\mathcal{I}^{\boxslash}$, φ must factor through $f j \to \varinjlim_{\mathcal{J}} f$ for some j in $\mathcal{J}$ (by considering $\varinjlim_{\mathcal{J}} [\mathbf{2}, \mathcal{C}](e, f)$) and so we can construct the required lift. ■

4.3. Lemma. *Let $\mathcal{C}$ be a κ-accessible category and let $\mathcal{R}$ be a κ-accessible full subcategory of $[\mathbf{2}, \mathcal{C}]$. If $g : Z \to W$ is a morphism in $\mathcal{C}$ where both Z and W are κ-presentable objects in $\mathcal{C}$, then:*

(i) *Given a morphism $f : X \to Y$ in $\mathcal{C}$ that is in $\mathcal{R}$, any morphism $g \to f$ in $[\mathbf{2}, \mathcal{C}]$ admits a factorisation of the form $g \to f' \to f$ where f' is in $\mathbf{K}_{\kappa}(\mathcal{R})$.*

(ii) *The morphism $g : Z \to W$ has the left lifting property with respect to $\mathcal{R}$ if and only if it has the left lifting property with respect to $\mathbf{K}_{\kappa}(\mathcal{R})$.*

Proof. (i). Proposition 2.7 says that g is a κ-presentable object in $[\mathbf{2}, \mathcal{C}]$; but every object in $\mathcal{R}$ is the colimit of a small κ-filtered diagram of κ-presentable objects in $\mathcal{R}$, and the inclusion $\mathcal{R} \hookrightarrow [\mathbf{2}, \mathcal{C}]$ is κ-accessible, so any morphism $g \to f$ must factor through some κ-presentable object in $\mathcal{R}$.

(ii). If g has the left lifting property with respect to $\mathcal{R}$, then it certainly has the left lifting property with respect to $\mathbf{K}_\kappa(\mathcal{R})$. Conversely, by factorising morphisms $g \to f$ as in (i), we see that g has the left lifting property with respect to $\mathcal{R}$ as soon as it has the left lifting property with respect to $\mathbf{K}_\kappa(\mathcal{R})$. ■

4.4. Theorem. [Quillen's small object argument] *Let κ be a regular cardinal, let $\mathcal{C}$ be a locally κ-presentable category, and let $\mathcal{I}$ be a small subset of* $\operatorname{mor} \mathcal{C}$.

(i) *There exists a functorial weak factorisation system (L, R) on $\mathcal{C}$ whose right class is $\mathcal{I}^{\boxslash}$; in particular, there is a weak factorisation system on $\mathcal{C}$ cofibrantly generated by $\mathcal{I}$.*

(ii) *If the morphisms that are in $\mathcal{I}$ are κ-presentable as objects in $[\mathbf{2}, \mathcal{C}]$, then (L, R) can be chosen so that the functors $L, R : [\mathbf{2}, \mathcal{C}] \to [\mathbf{2}, \mathcal{C}]$ are κ-accessible.*

(iii) *In addition, if λ is a regular cardinal such that every hom-set of $\mathbf{K}_\kappa(\mathcal{C})$ is λ-small, $\mathcal{I}$ is λ-small, and $\kappa \lhd \lambda$, then (L, R) can be chosen so that the functors $L, R : [\mathbf{2}, \mathcal{C}] \to [\mathbf{2}, \mathcal{C}]$ preserve λ-presentable objects.*

Proof. (i). See e.g. Proposition 10.5.16 in [Hirschhorn, 2003].

(ii) and (iii). These claims can be verified by tracing the construction of L and R and applying lemmas 1.5 and 1.12. □

4.5. Remark. The algebraically free natural weak factorisation system produced by Garner's small object argument [Garner, 2009] satisfy claims (ii) and (iii) of the above theorem (under the same hypotheses). The proof is somewhat more straightforward, because the right half of the resulting algebraic factorisation system can be described in terms of a certain density comonad.

4.6. Proposition. *Let $\mathcal{C}$ be a locally presentable category, let (L, R) be a functorial weak factorisation system on $\mathcal{C}$, and let $\lambda : \mathrm{id}_{[\mathbf{2}, \mathcal{C}]} \Rightarrow R$ be the natural transformation whose component at an object f in $[\mathbf{2}, \mathcal{C}]$ corresponds to the following commutative square in $\mathcal{C}$:*

$$\begin{array}{ccc} \bullet & \xrightarrow{Lf} & \bullet \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Rf} \\ \bullet & = & \bullet \end{array}$$

Let $\mathcal{R}$ be the full subcategory of $[\mathbf{2}, \mathcal{C}]$ spanned by the morphisms in $\mathcal{C}$ that are in the right class of the induced weak factorisation system.

(i) *$\mathcal{R}$ is also the full subcategory of $[\mathbf{2}, \mathcal{C}]$ spanned by the image of the forgetful functor $[\mathbf{2}, \mathcal{C}]^{(R,\lambda)} \to [\mathbf{2}, \mathcal{C}]$, where $[\mathbf{2}, \mathcal{C}]^{(R,\lambda)}$ is the category of algebras for the pointed endofunctor (R, λ).*

(ii) *If $R : [\mathbf{2}, \mathcal{C}] \to [\mathbf{2}, \mathcal{C}]$ is an accessible functor, then $[\mathbf{2}, \mathcal{C}]^{(R,\lambda)}$ is a locally presentable category, and the forgetful functor $[\mathbf{2}, \mathcal{C}]^{(R,\lambda)} \to [\mathbf{2}, \mathcal{C}]$ is monadic.*

(iii) *If $R : [\mathbf{2}, \mathcal{C}] \to [\mathbf{2}, \mathcal{C}]$ is strongly π-accessible and preserves colimits of κ-filtered diagrams, where $\kappa < \pi$, and $\mathcal{R}$ is closed under colimits of small π-filtered diagrams in $[\mathbf{2}, \mathcal{C}]$, then $\mathcal{R}$ is a π-accessible subcategory of $[\mathbf{2}, \mathcal{C}]$.*

Proof. (i). This is essentially the retract argument. See also Theorem 2.4 in [Rosický and Tholen, 2002].

(ii). Apply theorem 2.20.

(iii). By theorem 2.21, $[\mathbf{2}, \mathcal{C}]^{(R,\lambda)}$ is a locally π-presentable category, and the forgetful functor $[\mathbf{2}, \mathcal{C}]^{(R,\lambda)} \to [\mathbf{2}, \mathcal{C}]$ is moreover strongly π-accessible. Thus, we may apply proposition 2.11 to (i) and deduce that $\mathcal{R}$ is a π-accessible subcategory. ■

4.7. Proposition. *Let $\mathcal{C}$ be a locally presentable category, and let $\mathcal{I}$ be a subset of* $\operatorname{mor} \mathcal{C}$. *Then $\mathcal{I}^{\boxslash}$, considered as a full subcategory of $[\mathbf{2}, \mathcal{C}]$, is an accessible subcategory.*

Proof. Combine theorem 4.4 and proposition 4.6. ■

5. Strongly combinatorial model categories

To apply the results of the previous section to the theory of combinatorial model categories, it is useful to collect some convenient hypotheses together as a definition:

5.1. Definition. Let κ and λ be regular cardinals. A **strongly (κ, λ)-combinatorial model category** is a combinatorial model category $\mathcal{M}$ that satisfies these axioms:

- $\mathcal{M}$ is a locally κ-presentable category, and $\kappa \lhd \lambda$.
- $\mathbf{K}_\lambda(\mathcal{M})$ is closed under finite limits in $\mathcal{M}$.
- Each hom-set in $\mathbf{K}_\kappa(\mathcal{M})$ is λ-small.
- There exist λ-small sets of morphisms in $\mathbf{K}_\kappa(\mathcal{M})$ that cofibrantly generate the model structure of $\mathcal{M}$.

5.2. Remark. Let $\mathcal{M}$ be a strongly (κ, λ)-combinatorial model category and let $\lambda \lhd \mu$. Then $\kappa \lhd \mu$, so by lemma 2.5, $\mathbf{K}_\mu(\mathcal{M})$ is also closed under finite limits. Hence, $\mathcal{M}$ is also a strongly (κ, μ)-combinatorial model category.

5.3. Example. Let **sSet** be the category of simplicial sets. **sSet**, equipped with the Kan–Quillen model structure, is a strongly $(\aleph_0, \aleph_1)$-combinatorial model category.

5.4. Example. Let R be a ring, let $\mathbf{Ch}(R)$ be the category of unbounded chain complexes of left R-modules, and let λ be an uncountable regular cardinal such that R is λ-small (as a set).

- It is not hard to verify that $\mathbf{Ch}(R)$ is a locally $\aleph_0$-presentable category where the $\aleph_0$-presentable objects are the bounded chain complexes of finitely presented left R-modules.
- The λ-presentable objects in $\mathbf{Ch}(R)$ are precisely the chain complexes $M_\bullet$ such that $\sum_{n \in \mathbb{Z}} |M_n| < \lambda$, so the full subcategory of λ-presentable objects is closed under finite limits.
- By considering matrices over R, we deduce that the set of chain maps between any two $\aleph_0$-presentable objects in $\mathbf{Ch}(R)$ is λ-small.
- The cofibrations in the projective model structure on $\mathbf{Ch}(R)$ are generated by a countable set of chain maps between $\aleph_0$-presentable chain complexes, as are the trivial cofibrations.

Thus, $\mathbf{Ch}(R)$ is a strongly $(\aleph_0, \lambda)$-combinatorial model category.

5.5. Example. Let $\mathbf{Sp}^{\mathbf{\Sigma}}$ be the category of symmetric spectra of Hovey et al. [2000] and let λ be a regular cardinal such that $\aleph_1 \lhd \lambda$ and $2^{\aleph_0} < \lambda$. (Such a cardinal exists: for instance, we may take λ to be the cardinal successor of $2^{2^{\aleph_0}}$; or, assuming the continuum hypothesis, we may take $\lambda = \aleph_2$.)

- The category of pointed simplicial sets, $\mathbf{sSet}_*$, is locally $\aleph_0$-presentable; hence, so is the category $[\mathbf{\Sigma}, \mathbf{sSet}_*]$ of symmetric sequences of pointed simplicial sets, by proposition 2.6. There is a symmetric monoidal closed structure on $[\mathbf{\Sigma}, \mathbf{sSet}_*]$ such that $\mathbf{Sp}^{\mathbf{\Sigma}}$ is equivalent to the category of S-modules, where S is (the underlying symmetric sequence of) the symmetric sphere spectrum defined in Example 1.2.4 in op. cit.; thus, $\mathbf{Sp}^{\mathbf{\Sigma}}$ is the category of algebras for an $\aleph_0$-accessible monad, hence is itself is a locally $\aleph_0$-presentable category.
- Since (the underlying symmetric sequence of) S is an $\aleph_1$-presentable object in $[\mathbf{\Sigma}, \mathbf{sSet}_*]$, we can apply proposition 2.7 and theorem 2.18 to deduce that the $\aleph_1$-presentable objects in $\mathbf{Sp}^{\mathbf{\Sigma}}$ are precisely the ones whose underlying symmetric sequence consists of countable simplicial sets. Hence, $\mathbf{K}_{\aleph_1}(\mathbf{Sp}^{\mathbf{\Sigma}})$ is closed under finite limits, and the same is true for $\mathbf{K}_\lambda(\mathbf{Sp}^{\mathbf{\Sigma}})$ because $\aleph_1 \lhd \lambda$.
- It is clear that there are $\le 2^{\aleph_0}$ morphisms between two $\aleph_1$-presentable symmetric sequences; in particular, there are $< \lambda$ morphisms between two $\aleph_1$-presentable symmetric spectra.
- The functor $(-)_n : \mathbf{Sp}^{\mathbf{\Sigma}} \to \mathbf{sSet}$ that sends a symmetric spectrum X to the simplicial set X_n preserves filtered colimits, so its left adjoint $F_n : \mathbf{sSet} \to \mathbf{Sp}^{\mathbf{\Sigma}}$ preserves

$\aleph_0$-presentability. Thus, the set of generating cofibrations for the stable model structure on $\mathbf{Sp}^{\mathbf{\Sigma}}$ given by Proposition 3.4.2 in op. cit. is a countable set of morphisms between $\aleph_0$-presentable symmetric spectra.

Using the fact that the mapping cylinder of a morphism between two $\aleph_1$-presentable symmetric spectra is also an $\aleph_1$-presentable symmetric spectrum, we deduce that the set of generating trivial cofibrations given in Definition 3.4.9 in op. cit. is a countable set of morphisms between $\aleph_1$-presentable symmetric spectra.

We therefore conclude that $\mathbf{Sp}^{\mathbf{\Sigma}}$ is a strongly $(\aleph_1, \lambda)$-combinatorial model category.

5.6. Proposition. *For any combinatorial model category $\mathcal{M}$, there exist regular cardinals κ and λ such that $\mathcal{M}$ is a strongly (κ, λ)-combinatorial model category.*

Proof. In view of lemma 2.5, this reduces to the fact that there are arbitrarily large λ such that $\kappa \lhd \lambda$.[5] ■

5.7. Proposition. *Let $\mathcal{M}$ be a strongly (κ, λ)-combinatorial model category.*

(i) *There exist (trivial cofibration, fibration)- and (cofibration, trivial fibration)-factorisation functors that are κ-accessible and strongly λ-accessible.*

(ii) *Let $\mathcal{F}$ (resp. $\mathcal{F}'$) be the full subcategory of $[\mathbf{2}, \mathcal{M}]$ spanned by the fibrations (resp. trivial fibrations). Then $\mathcal{F}$ and $\mathcal{F}'$ are closed under colimits of small κ-filtered diagrams in $[\mathbf{2}, \mathcal{M}]$.*

Proof. (i). Since the weak factorisation systems on $\mathcal{M}$ are cofibrantly generated by λ-small sets of morphisms in $\mathbf{K}_\kappa(\mathcal{M})$ and the hom-sets of $\mathbf{K}_\kappa(\mathcal{M})$ are all λ-small, we may apply theorem 4.4 to obtain the required functorial weak factorisation systems.

(ii). This is a special case of lemma 4.2. ■

5.8. Lemma. *Let $\mathcal{M}$ be a category with limits and colimits of finite diagrams and let $(\mathcal{C}', \mathcal{F})$ and $(\mathcal{C}, \mathcal{F}')$ be weak factorisation systems on $\mathcal{M}$. Assume $\mathcal{W}$ is a class of morphisms in $\mathcal{C}$ with the following property:*

$$\mathcal{W} \subseteq \{ q \circ j \mid j \in \mathcal{C}', q \in \mathcal{F}' \}$$

The following are equivalent:

(i) *$(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a model structure on $\mathcal{M}$.*

(ii) *$\mathcal{W}$ has the 2-out-of-3 property in $\mathcal{M}$, $\mathcal{C}' = \mathcal{C} \cap \mathcal{W}$, and $\mathcal{F}' = \mathcal{W} \cap \mathcal{F}$.*

(iii) *$\mathcal{W}$ has the 2-out-of-3 property in $\mathcal{M}$, $\mathcal{C}' \subseteq \mathcal{W}$, and $\mathcal{F}' = \mathcal{W} \cap \mathcal{F}$.*

[5] See Corollary 2.3.6 in [Makkai and Paré, 1989], or Example 2.13(6) in [Adámek and Rosický, 1994], or Corollary 5.4.8 in [Borceux, 1994].

Proof. (i) ⇒ (ii). Use the retract argument.

(ii) ⇒ (iii). Immediate.

(iii) ⇒ (ii). Suppose $i : X \to Z$ is in $\mathcal{C} \cap \mathcal{W}$; then there must be $j : X \to Y$ in $\mathcal{C}'$ and $q : Y \to Z$ in $\mathcal{F}'$ such that $i = q \circ j$, and so we have the commutative diagram shown below:

$$\begin{array}{ccc} X & \xrightarrow{j} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle q} \\ Z & \xrightarrow[\text{id}]{} & Z \end{array}$$

Since $i \boxslash q$, i must be a retract of j; hence, i is in $\mathcal{C}'$, and therefore $\mathcal{C} \cap \mathcal{W} \subseteq \mathcal{C}'$.

(ii) ⇒ (i). See Lemma 14.2.5 in [May and Ponto, 2012]. ■

5.9. Theorem. *Let (L', R) and (L, R') be functorial weak factorisation systems on a locally presentable category $\mathcal{M}$ and let $\mathcal{F}$ and $\mathcal{F}'$ be the full subcategories of $[\mathbf{2}, \mathcal{M}]$ spanned by the morphisms in the right class of the weak factorisation systems induced by (L', R) and (L, R'), respectively. Suppose κ and λ are regular cardinals satisfying the following hypotheses:*

- *$\mathcal{M}$ is a locally κ-presentable category, and $\kappa \lhd \lambda$.*
- *$\mathcal{F}$ and $\mathcal{F}'$ are closed under colimits of small κ-filtered diagrams in $[\mathbf{2}, \mathcal{M}]$.*
- *$R, R' : [\mathbf{2}, \mathcal{M}] \to [\mathbf{2}, \mathcal{M}]$ are both κ-accessible and strongly λ-accessible.*

Let $\mathcal{C}'$ be the full subcategory of $[\mathbf{2}, \mathcal{M}]$ spanned by the morphisms in the left class of the weak factorisation system induced by (L', R) and let $\mathcal{W}$ be the preimage of $\mathcal{F}'$ under the functor $R : [\mathbf{2}, \mathcal{M}] \to [\mathbf{2}, \mathcal{M}]$. Then:

(i) *The functorial weak factorisation systems (L', R) and (L, R') restrict to functorial weak factorisation systems on $\mathbf{K}_\lambda(\mathcal{M})$.*

(ii) *The inclusions $\mathcal{F} \hookrightarrow [\mathbf{2}, \mathcal{M}]$ and $\mathcal{F}' \hookrightarrow [\mathbf{2}, \mathcal{M}]$ are strongly λ-accessible functors.*

(iii) *$\mathcal{W}$ is closed under colimits of small κ-filtered diagrams in $[\mathbf{2}, \mathcal{M}]$, and the inclusion $\mathcal{W} \hookrightarrow [\mathbf{2}, \mathcal{M}]$ is a strongly λ-accessible functor.*

(iv) *$\mathcal{C}' \subseteq \mathcal{W}$ if and only if the same holds in $\mathbf{K}_\lambda(\mathcal{M})$.*

(v) *$\mathcal{F}' = \mathcal{W} \cap \mathcal{F}$ if and only if the same holds in $\mathbf{K}_\lambda(\mathcal{M})$.*

(vi) *$\mathcal{W}$ (regarded as a class of morphisms in $\mathcal{M}$) has the 2-out-of-3 property in $\mathcal{M}$ if and only if the same is true in $\mathbf{K}_\lambda(\mathcal{M})$.*

(vii) *The weak factorisation systems induced by (L', R) and (L, R') underlie a model structure on $\mathcal{M}$ if and only if their restrictions to $\mathbf{K}_\lambda(\mathcal{M})$ underlie a model structure on $\mathbf{K}_\lambda(\mathcal{M})$.*

Proof. (i). It is clear that we can restrict (L', R) and (L, R') to obtain functorial factorisation systems on $\mathbf{K}_\lambda(\mathcal{M})$, and these are functorial *weak* factorisation systems by Theorem 2.4 in [Rosický and Tholen, 2002].

(ii). Since $R, R' : [\mathbf{2}, \mathcal{M}] \to [\mathbf{2}, \mathcal{M}]$ are both κ-accessible and strongly λ-accessible, we may use proposition 4.6 to deduce that the inclusions $\mathcal{F} \hookrightarrow [\mathbf{2}, \mathcal{M}]$ and $\mathcal{F}' \hookrightarrow [\mathbf{2}, \mathcal{M}]$ are strongly λ-accessible.

(iii). Since $\mathcal{F}'$ is a replete full subcategory of $[\mathbf{2}, \mathcal{M}]$, we may use proposition 2.16 to deduce that $\mathcal{W}$ is closed under colimits of small κ-filtered diagrams in $[\mathbf{2}, \mathcal{M}]$ and that the inclusion $\mathcal{W} \hookrightarrow [\mathbf{2}, \mathcal{M}]$ is a strongly λ-accessible functor.

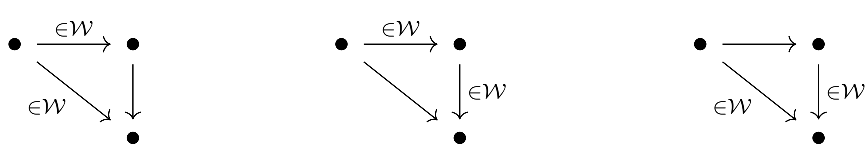
(iv). The endofunctor $L' : [\mathbf{2}, \mathcal{M}] \to [\mathbf{2}, \mathcal{M}]$ is strongly λ-accessible, and $\mathcal{W}$ is closed under colimits of small λ-filtered diagrams, so (recalling propositions 2.6 and 2.7) if L' sends the subcategory $[\mathbf{2}, \mathbf{K}_\lambda(\mathcal{M})]$ to $\mathcal{W}$, then the entirety of the image of L' must be contained in $\mathcal{W}$. The retract argument implies that every object in $\mathcal{C}'$ is a retract of an object in the image of L', and (iii) implies $\mathcal{W}$ is closed under retracts, so we may deduce that $\mathcal{C}' \subseteq \mathcal{W}$ if and only if $\mathcal{C}' \cap [\mathbf{2}, \mathbf{K}_\lambda(\mathcal{M})] \subseteq \mathcal{W} \cap [\mathbf{2}, \mathbf{K}_\lambda(\mathcal{M})]$.

(v). Claims (ii) and (iii) and proposition 2.16 imply the inclusion $\mathcal{W} \cap \mathcal{F} \hookrightarrow [\mathbf{2}, \mathcal{M}]$ is strongly λ-accessible; but by propositions 2.7 and 2.9,

$$\mathbf{K}_\lambda(\mathcal{F}') = \mathcal{F}' \cap [\mathbf{2}, \mathbf{K}_\lambda(\mathcal{M})] \qquad \mathbf{K}_\lambda(\mathcal{W} \cap \mathcal{F}) = (\mathcal{W} \cap \mathcal{F}) \cap [\mathbf{2}, \mathbf{K}_\lambda(\mathcal{M})]$$

so $\mathcal{F}' = \mathcal{W} \cap \mathcal{F}$ if and only if $\mathcal{F}' \cap [\mathbf{2}, \mathbf{K}_\lambda(\mathcal{M})] = (\mathcal{W} \cap \mathcal{F}) \cap [\mathbf{2}, \mathbf{K}_\lambda(\mathcal{M})]$.

(vi). Consider the three full subcategories $\Lambda^2_i(\mathcal{W})$ (where $i \in \{0, 1, 2\}$) of $[\mathbf{3}, \mathcal{M}]$ spanned (respectively) by the diagrams of the form below:



By proposition 2.4, each inclusion $\Lambda^2_i(\mathcal{W}) \hookrightarrow [\mathbf{3}, \mathcal{M}]$ is the pullback of a strongly λ-accessible inclusion of a full subcategory of $[\mathbf{2}, \mathcal{M}]^{\times 3}$ along the evident projection functor $[\mathbf{3}, \mathcal{M}] \to [\mathbf{2}, \mathcal{M}]^{\times 3}$; thus, each inclusion $\Lambda^2_i(\mathcal{W}) \hookrightarrow [\mathbf{3}, \mathcal{M}]$ is a strongly λ-accessible functor. We may then use proposition 2.9 as above to prove the claim.

(vii). Apply lemma 5.8. ■

5.10. Corollary. *Let $\mathcal{M}$ be a strongly (κ, λ)-combinatorial model category. Then the full subcategory $\mathcal{W}$ of $[\mathbf{2}, \mathcal{M}]$ spanned by the weak equivalences is closed under colimits of small κ-filtered diagrams in $[\mathbf{2}, \mathcal{M}]$, and the inclusion $\mathcal{W} \hookrightarrow [\mathbf{2}, \mathcal{M}]$ is a strongly λ-accessible functor.*

Proof. Combine proposition 5.7 and theorem 5.9. ■

Theorem 5.9 suggests that free λ-ind-completions of suitable small model categories are combinatorial model categories. More precisely:

5.11. DEFINITION. Let κ and λ be regular cardinals. A (κ, λ)**-miniature model category** is a model category $\mathcal{M}$ that satisfies these axioms:

- $\mathcal{M}$ is a (κ, λ)-accessible generated category, and $\kappa \lhd \lambda$.
- $\mathcal{M}$ has limits for finite diagrams and colimits of λ-small diagrams.
- Each hom-set in $\mathbf{K}_\kappa^\lambda(\mathcal{M})$ is λ-small.
- There exist λ-small sets of morphisms in $\mathbf{K}_\kappa^\lambda(\mathcal{M})$ that cofibrantly generate the model structure of $\mathcal{M}$.

5.12. PROPOSITION. *If $\mathcal{M}$ is a strongly (κ, λ)-combinatorial model category, then $\mathbf{K}_\lambda(\mathcal{M})$ is a (κ, λ)-miniature model category (with the weak equivalences, cofibrations, and fibrations inherited from $\mathcal{M}$).*

PROOF. By theorem 3.11, $\mathbf{K}_\lambda(\mathcal{M})$ is a (κ, λ)-accessible generated category, and lemma 1.5 implies it is closed under colimits of λ-small diagrams in $\mathcal{M}$. Now, choose a pair of functorial factorisation systems as in proposition 5.7, and recall that a morphism is in the left (resp. right) class of a functorial weak factorisation system if and only if it is a retract of the left (resp. right) half of its functorial factorisation. Since we chose factorisation functors that are strongly λ-accessible, it follows that the weak factorisation systems on $\mathcal{M}$ restricts to weak factorisation systems on $\mathbf{K}_\lambda(\mathcal{M})$. It is then clear that $\mathbf{K}_\lambda(\mathcal{M})$ inherits a model structure from $\mathcal{M}$, and lemma 4.3 implies the model structure on $\mathbf{K}_\lambda(\mathcal{M})$ can be cofibrantly generated by λ-small sets of morphisms in $\mathbf{K}_\kappa(\mathcal{M})$. The remaining axioms for a λ-miniature model category are easily verified. ■

5.13. REMARK. The subcategory $\mathbf{K}_\lambda(\mathcal{M})$ inherits much of the homotopy-theoretic structure of $\mathcal{M}$. For instance, $\mathbf{K}_\lambda(\mathcal{M})$ has simplicial and cosimplicial resolutions and the inclusion $\mathbf{K}_\lambda(\mathcal{M}) \hookrightarrow \mathcal{M}$ preserves them, so $\operatorname{Ho} \mathbf{K}_\lambda(\mathcal{M}) \to \operatorname{Ho} \mathcal{M}$ is a fully faithful $(\operatorname{Ho} \mathbf{sSet})$-enriched functor, where the $(\operatorname{Ho} \mathbf{sSet})$-enrichment is defined as in [Hovey, 1999, Ch. 5]. In particular, the induced functor between the ordinary homotopy categories is fully faithful.

5.14. THEOREM. *Let $\mathcal{K}$ be a (κ, λ)-miniature model category, let $\mathcal{M}$ be the free λ-ind-completion $\mathbf{Ind}^\lambda(\mathcal{K})$, and let $\gamma : \mathcal{K} \to \mathcal{M}$ be the canonical embedding.*

(i) *There is a unique way of making $\mathcal{M}$ into a strongly (κ, λ)-combinatorial model category such that $\gamma : \mathcal{K} \to \mathcal{M}$ preserves and reflects the model structure.*

(ii) *Moreover, for any model category $\mathcal{N}$ with colimits of all small diagrams, restriction along $\gamma : \mathcal{K} \to \mathcal{M}$ induces a functor*

- from *the full subcategory of $[\mathcal{M}, \mathcal{N}]$ spanned by the left Quillen functors*
- to *the full subcategory of $[\mathcal{K}, \mathcal{N}]$ spanned by the functors that preserve cofibrations, trivial cofibrations, and colimits of λ-small diagrams.*

Proof. (i). We will identify $\mathcal{K}$ with the image of $\gamma : \mathcal{K} \to \mathcal{M}$. Note that $\mathcal{M}$ is a locally κ-presentable category, by theorem 3.11. Let $\mathcal{I}$ (resp. $\mathcal{I}'$) be a λ-small set of morphisms in $\mathbf{K}_\kappa^\lambda(\mathcal{K})$ that generate the cofibrations (resp. trivial cofibrations) in $\mathcal{K}$. Let (L', R) and (L, R') be functorial weak factorisation systems cofibrantly generated by $\mathcal{I}'$ and $\mathcal{I}$ respectively; by theorem 4.4, we may assume $R, R' : [\mathbf{2}, \mathcal{M}] \to [\mathbf{2}, \mathcal{M}]$ preserve colimits of small κ-filtered diagrams and are strongly λ-accessible functors.

Let $\mathcal{F}$ and $\mathcal{F}'$ be the full subcategories of $[\mathbf{2}, \mathcal{M}]$ spanned by the right class of the weak factorisation systems induced by (L', R) and (L, R'), respectively. It is not hard to see that any morphism in $\mathcal{K}$ is an object in $\mathcal{F}$ (resp. $\mathcal{F}'$) if and only if it is a fibration (resp. trivial fibration) in $\mathcal{K}$. Lemma 4.2 says $\mathcal{F}$ and $\mathcal{F}'$ are closed under colimits of small κ-filtered diagrams in $[\mathbf{2}, \mathcal{M}]$, so we may now apply theorem 5.9 to deduce that $\mathcal{F}$ and $\mathcal{F}'$ induce a model structure on $\mathcal{M}$. It is clear that $\mathcal{M}$ equipped with this model structure is then a strongly (κ, λ)-combinatorial model category in a way that is compatible with the canonical embedding $\mathcal{K} \to \mathcal{M}$.

Finally, to see that the above construction is the unique way of making $\mathcal{M}$ into a strongly (κ, λ)-combinatorial model category satisfying the given conditions, we simply have to observe that the model structure of a strongly (κ, λ)-combinatorial model category is necessarily cofibrantly generated by the cofibrations and trivial cofibrations in (a small skeleton of) $\mathbf{K}_\kappa(\mathcal{M})$ (independently of the choice of $\mathcal{I}$ and $\mathcal{I}'$).

(ii). Clearly, every left Quillen functor $F : \mathcal{M} \to \mathcal{N}$ restricts to a functor $F\gamma : \mathcal{K} \to \mathcal{N}$ that preserves cofibrations, trivial cofibrations, and colimits of λ-small diagrams. Conversely, given any such functor $F' : \mathcal{K} \to \mathcal{N}$, we may apply the universal property of $\mathcal{M} = \mathbf{Ind}^\lambda(\mathcal{K})$ to obtain a λ-accessible functor $F : \mathcal{M} \to \mathcal{N}$ such that $F\gamma = F'$. Since cofibrations and trivial cofibrations in $\mathcal{M}$ are generated under colimits of λ-filtered diagrams by cofibrations and trivial cofibrations in $\mathcal{K}$, the functor $F : \mathcal{M} \to \mathcal{N}$ preserves cofibrations and trivial cofibrations if $F' : \mathcal{K} \to \mathcal{N}$ does. A similar argument (using proposition 2.6) shows that $F : \mathcal{M} \to \mathcal{N}$ preserves colimits of λ-small diagrams. Thus, $F : \mathcal{M} \to \mathcal{N}$ preserves colimits of all small diagrams,[6] so it has a right adjoint (by e.g. the special adjoint functor theorem) and is indeed a left Quillen functor. ■

5.15. Remark. Let $\mathbf{U}$ and $\mathbf{U}^+$ be universes, with $\mathbf{U} \in \mathbf{U}^+$, let $\mathcal{M}$ be a strongly (κ, λ)-combinatorial model $\mathbf{U}$-category, and let $\mathcal{M} \hookrightarrow \mathcal{M}^+$ be a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-extension in the sense of [Low, 2013]. By combining proposition 5.12 and theorem 5.14, we may deduce that there is a unique way of making $\mathcal{M}^+$ into a strongly (κ, λ)-combinatorial model $\mathbf{U}^+$-category such that the embedding $\mathcal{M} \hookrightarrow \mathcal{M}^+$ preserves and reflects the model structure. In view of proposition 5.6, it follows that every combinatorial model $\mathbf{U}$-category can be canonically extended to a combinatorial model $\mathbf{U}^+$-category; moreover, by Theorem 3.11 in op. cit., the extension does not depend on (κ, λ).

The techniques used in the proof of theorem 5.9 are easily generalised to combinatorial model categories with desirable properties.

[6] See Lemma 2.25 in [Low, 2013].

5.16. Theorem. *Let $\mathcal{M}$ be a strongly (κ, λ)-combinatorial model category. The following are equivalent:*

(i) *$\mathcal{M}$ is a right proper model category.*

(ii) *$\mathbf{K}_\lambda(\mathcal{M})$ is a right proper model category.*

Proof. (i) $\Rightarrow$ (ii). Immediate, because the model structure on $\mathbf{K}_\lambda(\mathcal{M})$ is the restriction of the model structure on $\mathcal{M}$ and $\mathbf{K}_\lambda(\mathcal{M})$ is closed under finite limits in $\mathcal{M}$.

(ii) $\Rightarrow$ (i). Let $\mathcal{D} = \{\bullet \to \bullet \leftarrow \bullet\}$, i.e. the category freely generated by a cospan. Since $\mathcal{D}$ is a finite category and $\mathcal{M}$ is a locally κ-presentable category, proposition 2.6 says $[\mathcal{D}, \mathcal{M}]$ is also a locally κ-presentable category, and proposition 2.7 implies the κ-presentable objects in $[\mathcal{D}, \mathcal{M}]$ are precisely the componentwise κ-presentable objects. Thus, the functor $\Delta : \mathcal{M} \to [\mathcal{D}, \mathcal{M}]$ is strongly κ-accessible, so its right adjoint $\varprojlim_{\mathcal{D}} : [\mathcal{D}, \mathcal{M}] \to \mathcal{M}$ is κ-accessible; moreover, it is strongly λ-accessible because $\mathbf{K}_\lambda(\mathcal{M})$ is closed under finite limits in $\mathcal{M}$.

Consider the full subcategory $\mathcal{P} \subseteq [\mathcal{D}, \mathcal{M}]$ spanned by those diagrams in $\mathcal{M}$ of the form below,

$$\begin{array}{ccc} & & \bullet \\ & & \big\downarrow{\scriptstyle w} \\ \bullet & \xrightarrow[p]{} & \bullet \end{array}$$

where p is a fibration and w is a weak equivalence. Propositions 2.16 and 5.7, theorem 5.9, and corollary 5.10 together imply that $\mathcal{P}$ is closed under colimits of small κ-filtered diagrams in $[\mathcal{D}, \mathcal{M}]$ and that the inclusion $\mathcal{P} \hookrightarrow [\mathcal{D}, \mathcal{M}]$ is a strongly λ-accessible functor. Since $\varprojlim_{\mathcal{D}} : [\mathcal{D}, \mathcal{M}] \to \mathcal{M}$ is strongly λ-accessible and the class of weak equivalences in $\mathcal{M}$ is closed under λ-filtered colimits in $[\mathbf{2}, \mathcal{M}]$, it follows that $\mathcal{M}$ is right proper if $\mathbf{K}_\lambda(\mathcal{M})$ is. ■

5.17. Remark. It is tempting to say that the analogous proposition for left properness follows by duality; unfortunately, the opposite of a combinatorial model category is almost never a combinatorial model category! Nonetheless, the main idea in the proof above can be made to work under the assumption that the category of coalgebras for the left half of the functorial (cofibration, trivial fibration)-factorisation system is generated under colimits of small λ-filtered diagrams of coalgebras whose underlying object in $[\mathbf{2}, \mathcal{M}]$ is a cofibration in $\mathbf{K}_\lambda(\mathcal{M})$. It is not clear whether this hypothesis is always satisfied if we only assume that $\mathcal{M}$ is a strongly (κ, λ)-combinatorial model category, but it is certainly true if λ is sufficiently large, because the category of coalgebras for an accessible copointed endofunctor is always accessible (by an analogue of theorem 2.21) and any accessible functor is strongly λ-accessible for large enough λ (by lemma 2.5).

5.18. Theorem. *Let $\underline{\mathcal{M}}$ be a locally small simplicially enriched category where the underlying ordinary category $\mathcal{M}$ is equipped with a model structure making it a strongly (κ, λ)-combinatorial model category. Assuming the simplicially enriched full subcategory*

$\mathbf{K}_\lambda(\underline{\mathcal{M}}) \subseteq \underline{\mathcal{M}}$ *determined by* $\mathbf{K}_\lambda(\mathcal{M})$ *is closed under cotensor products with finite simplicial sets in* $\underline{\mathcal{M}}$*, the following are equivalent:*

(i) $\underline{\mathcal{M}}$ *is a simplicial model category.*

(ii) *The model structure of* $\mathbf{K}_\lambda(\underline{\mathcal{M}})$ *satisfies axiom SM7.*

PROOF. (i) ⇒ (ii). Immediate, because the model structure of $\mathbf{K}_\lambda(\mathcal{M})$ is the restriction of the model structure of $\mathcal{M}$.

(ii) ⇒ (i). Recalling the fact that **sSet** is a strongly $(\aleph_0, \aleph_1)$-combinatorial model category, this is a consequence of proposition 5.7. ■

5.19. REMARK. In view of the above theorem, it should seem very likely that the free λ-ind-completion of a suitable small simplicial model category will again be a simplicial model category. To prove this, we require the technology of enriched accessibility introduced by Kelly [1982] and Borceux and Quinteriro [1996]; in fact, the only thing we need is to show that the free λ-ind-completion of a λ-cocomplete **sSet**-enriched category is a cocomplete **sSet**-enriched category, and this can be done by mimicking the proof for the case of ordinary categories. The details are left to the reader.

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