

THE FERMAT FUNCTORS

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ABSTRACT. In this paper, we use some basic quasi-topos theory to study two functors: one adding infinitesimals of Fermat reals to diffeological spaces (which generalize smooth manifolds including singular spaces and infinite-dimensional spaces), and the other deleting infinitesimals on Fermat spaces. We study the properties of these functors, and calculate some examples. These serve as fundamentals for developing differential geometry on diffeological spaces using infinitesimals in a future paper.

1. Introduction

Using infinitesimals to study geometry goes back to I. Newton or even earlier, as one of the motivations for developing calculus, and hence the start of the modern mathematics. Although infinitesimal theory was not rigorous at the beginning, the intuitive idea behind it was so enlightening that a great amount of work at that time by mathematicians like L. Euler, J.-L. Lagrange, etc, were influenced by that. It was A.-L. Cauchy who made the definition of limit rigorous using the epsilon-delta language. Since then infinitesimal theory gradually left the main stream of mathematics.

On the other hand, many concepts in geometry came from intuitive infinitesimal considerations, for example, tangent vectors, vector fields, Lie groups, Lie algebras, connections, curvature, etc. Many modern formulations of these concepts leave very little trace of their original ideas, but they are very convenient for doing computations. In other words, there is a step from translating geometric ideas using infinitesimals to the modern formulations, and most of time, this step is left as a gap in most literature, especially for students start to learn this field. It is always a great hope that infinitesimal theory could be made rigorous and enter differential geometry for the real content.

Going back to the rigor of infinitesimal theory, nowadays, there are a few such theories available on the market. Two of the most developed ones are Non-Standard Analysis (NSA; see for example [15]) and Synthetic Differential Geometry (SDG; see for example [12, 13]). The infinitesimals in SDG are nilpotent¹, while those in NSA are not. In the smooth manifold² case, a tangent space at a point is the first-order approximation of the

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¹More precisely, the square of any infinitesimal number in SDG is 0.

²By a smooth manifold in this paper, we always mean it to be finite-dimensional, second countable,

manifold. That is, if we embed the smooth manifold in some Euclidean space, and assume that the local defining function of the manifold around that point is f , then the Taylor expansion of f up to order 1 is the tangent space there. This example implies that we can use nilpotent infinitesimals to do differential geometry, i.e., if the infinitesimal increment of the variables of f already has the property that the multiplication of any two of them is 0, then the tangent space is exactly the Taylor expansion at that point. One can also imagine that the higher order geometric structures such as jets can be characterized using higher order infinitesimals. In order to axiomatize first-order approximation, SDG has a very strong axiom called the Kock-Lawvere axiom. This axiom requires the (commutative unital) ring with infinitesimals to satisfy an affine condition for every function from infinitesimals to the ring, and in the framework of classical logic, the only such ring is the trivial ring. In other words, the whole theory of SDG is built upon a new world called intuitionistic logic.

The theory of Fermat reals introduced by P. Giordano in [6] is another infinitesimal theory, where every infinitesimal is nilpotent, and the theory is compatible with classical logic; see Section 2 for a brief summary of the basics of this theory. It is not hard to redo many classical constructions of differential geometry on smooth manifolds using Fermat reals; some of them have already been explored in [5], and more will be presented systematically in a following paper.

Note that many spaces other than smooth manifolds arise naturally and frequently in geometry, for example, smooth manifolds with boundary or corners, singular orbit spaces of Lie groups acting on smooth manifolds (in particular, orbifolds), function spaces between smooth manifolds, diffeomorphism groups of smooth manifolds, etc. These spaces are usually studied separately in the literature. There are generalizations of smooth manifolds which contain (some of) them. Diffeology is one of such generalizations, introduced by J.-M. Souriau in [16, 17]. A standard textbook is [10]. Briefly, a diffeological space is a set together with specified functions from open subsets of \mathbb{R}^n for all n to this set, satisfying three simple axioms. These axioms declare when a function from an open subset of a Euclidean space to this set is “smooth”. A typical non-trivial and important example is an irrational torus, which cannot be characterized by (continuous) maps from this space to Euclidean spaces. Moreover, there is a quasi-topos approach to diffeology ([1]), and this idea has been extended to Fermat reals ([7]), called Fermat spaces.

An approach of adding infinitesimals on diffeological spaces has been tried in [5], which uses maps from diffeological spaces to Euclidean spaces to identify little-oh polynomials (a pre-model for infinitesimals) on this diffeological space; see [5, Chapter 8]. The theory goes well with smooth manifolds, but not with general diffeological spaces³. This leads the author to think of a very different approach of adding infinitesimals. More precisely, since diffeological spaces are concrete sheaves over the Souriau site, and Fermat spaces are concrete sheaves over the Fermat site (Example 3.2), to find natural relationship between

Hausdorff, and without boundary.

³For example, it is easy to check by definition that it turns an irrational torus to a trivial Fermat space.

them, it is enough to find natural functors between the two sites. In this way, we not only get a definition of the adding infinitesimal functor (Proposition 4.2) which is different from the one presented in [5] (Proposition 4.3), but also obtain its left inverse, called the deleting infinitesimal functor (Propositions 4.16 and 4.17). Almost every property of the adding infinitesimal functor in [5] holds in this new definition, with most restrictive conditions removed (see Subsection 4.1). In Subsection 4.20, we discuss the comparison of another natural adding infinitesimal functor⁴ with the current one, and explain why the current one is better. Finally in Subsection 4.27, we do a few calculations, and show that in general the calculation is not easy⁵. All of these will serve as fundamentals for developing differential geometry on diffeological spaces in a future paper.

I would like to thank P. Giordano for suggesting this project.

2. Basic of Fermat reals

Fermat reals were introduced by P. Giordano in [5, 6, 7, 8]. Let us review the basic theory here; see these references for detailed proof of these results.

Let U be an open subset of \mathbb{R}^n . We define $U_0[t]$, the *little-oh polynomials* on U , to be the set of functions $x : [0, \epsilon) \rightarrow U$ for some (not fixed) $\epsilon \in \mathbb{R}_{>0}$ with the property that

$$\|x(t) - r - \sum_{i=1}^k \alpha_i t^{a_i}\| = o(t) \quad \text{i.e.,} \quad \lim_{t \rightarrow 0} \frac{\|x(t) - r - \sum_{i=1}^k \alpha_i t^{a_i}\|}{t} = 0$$

for some $r \in U$, $k \in \mathbb{N}$, $\alpha_i \in \mathbb{R}^n$ and $a_i > 0$. Two little-oh polynomials x and y are called equivalent if $x(0) = y(0)$ and $x(t) - y(t) = o(t)$. This is an equivalence relation on $U_0[t]$, and the quotient set is denoted by $\bullet U$. As a consequence, every element in $\bullet U$ has a unique representing little-oh polynomial of the form

$$y(t) = \circ y + \sum_{i=1}^l \beta_i t^{b_i} \tag{1}$$

for some $\circ y (= y(0)) \in U$, $l \in \mathbb{N}$, $\beta_i \in (\mathbb{R}^n \setminus \{0\})$ and $0 < b_1 < b_2 < \dots < b_l \leq 1$, defined on $[0, \delta)$ for some maximum $\delta \in \mathbb{R}_{>0} \cup \{\infty\}$.⁶ We call this the *decomposition* of

⁴It uses another canonical concrete site; see Remark 3.3. Indeed, we used this concrete site in the definition of the deleting infinitesimal functor (Proposition 4.16).

⁵More precisely, we show in Example 4.33 that the adding infinitesimal functor does not commute with arbitrary colimits.

⁶Careful reader will notice that there are two main differences between the presentation here and the one in the existing references. One is, we use germs at 0 for little-oh polynomials, since sometimes, such functions are not necessarily globally defined, and another one is the expression of the unique representing little-oh polynomials without using the notation changes: $t^b \leftrightarrow dt_{1/b}$, since from my opinion, the use of t^b is closer to the traditional way of viewing such functions as a kind of “polynomials”, and much easier for doing computations, etc.

the element $[y]$, ${}^\circ y$ the *standard part*, and we define $\omega([y]) := \frac{1}{b_1}$ the *order* of $[y]$. For convenience, we sometimes use a similar form of $y(t)$ as (1) but allowing $\beta_i = 0$, and we call such a form a *quasi-decomposition* of $[y]$. From now on, we write elements in $\bullet U$ by y instead of $[y]$ whenever there is no confusion.

Given a finite set of open subsets $\{U_i\}_{i \in I}$ of Euclidean spaces, $\bullet(\prod_{i \in I} U_i)$ naturally bijects $\prod_{i \in I} \bullet U_i$. Therefore, we do not distinguish $\bullet(\mathbb{R}^n)$ and $(\bullet\mathbb{R})^n$, and write it as $\bullet\mathbb{R}^n$. We can also identify $\bullet U$ as a subset of $\bullet\mathbb{R}^n$ by $\bullet U = \{x \in \bullet\mathbb{R}^n \mid {}^\circ x \in U\}$ when U is an open subset of \mathbb{R}^n .

There are canonical functions $i_U : U \rightarrow \bullet U$ and $ev_0 : \bullet U \rightarrow U$ defined by $i_U(u)(t) = u$ and $ev_0(x) = {}^\circ x$, and we have $ev_0 \circ i_U = 1_U$. Therefore, $\bullet U$ is an extension of U , and for $x \in \bullet U$, we call $\delta x := x - {}^\circ x$ the *infinitesimal part* of x . The meaning is clear when $U = \mathbb{R}$: we can give a well ordering on $\bullet\mathbb{R}$ ⁷ by $x \leq y$ if $x = {}^\circ x + \sum_{i=1}^n \alpha_i t^{a_i}$ and $y = {}^\circ y + \sum_{i=1}^n \beta_i t^{a_i}$, both in the quasi-standard form, with $({}^\circ x, \alpha_1, \dots, \alpha_n) \leq ({}^\circ y, \beta_1, \dots, \beta_n)$ in the dictionary order, and then $D_\infty := \{x \in \bullet\mathbb{R} \mid {}^\circ x = 0\} = \{x \in \bullet\mathbb{R} \mid -r < x < r \text{ for all } r \in \mathbb{R}_{>0}\}$. Moreover, every infinitesimal part δx of $x \in \bullet U$ is nilpotent, i.e., there exists some $m = m(x) \in \mathbb{N}$ such that $(\delta x)^m = 0$.

D_∞ is the unique maximal (prime) ideal of $\bullet\mathbb{R}$. The subsets $\{0\}$, $D_a = \{x \in D_\infty \mid \omega(x) < a + 1\}$ for all $a \in \mathbb{R}_{>0} \cup \{\infty\}$ and $I_b = \{x \in D_\infty \mid \omega(x) \leq b\}$ for all $b \in \mathbb{R}_{\geq 1}$ are all the ideals of $\bullet\mathbb{R}$. We simply write D for D_1 , called the set of *first-order infinitesimals*.

On $\bullet\mathbb{R}^n$, define $\tau := \{\bullet U \mid U \text{ is an open subset of } \mathbb{R}^n\}$. Then τ is a topology on $\bullet\mathbb{R}^n$, called the *Fermat topology*, since $\bullet(U \cap V) = \bullet U \cap \bullet V$ and $\bullet(\cup_i U_i) = \cup_i \bullet U_i$. Without specification, for every subset A of $\bullet\mathbb{R}^n$, we always equip it with the sub-topology of the Fermat topology of $\bullet\mathbb{R}^n$.

Let $f : U \rightarrow V$ be a smooth map between open subsets of Euclidean spaces. Then $\bullet f : \bullet U \rightarrow \bullet V$ by $\bullet f(x) = f \circ x$ is a well-defined map extending f (called the *Fermat extension of f*), i.e., we have the following commutative diagram in **Set**:

$$\begin{array}{ccccc} U & \xrightarrow{i_U} & \bullet U & \xrightarrow{ev_0} & U \\ f \downarrow & & \bullet f \downarrow & & \downarrow f \\ V & \xrightarrow{i_V} & \bullet V & \xrightarrow{ev_0} & V, \end{array}$$

The calculation of $\bullet f(x) = \bullet f({}^\circ x + \delta x)$ can be done by Taylor's expansion of f at the point ${}^\circ x$, using the nilpotency of δx . More precisely, if the $(m + 1)^{th}$ power of each component of δx is 0 for some $m \in \mathbb{N}$, then we have

$$\bullet f(x) = \bullet f({}^\circ x + \delta x) = \sum_{i \in \mathbb{N}^m, |i| \leq m} \frac{1}{i!} \frac{\partial^{|i|} f}{\partial x^i}({}^\circ x) \cdot (\delta x)^i.$$

Therefore, for any open subset W of V , we have $\bullet(f^{-1}(W)) = (\bullet f)^{-1}(\bullet W)$, i.e., $\bullet f$ is continuous with respect to the Fermat topology.

⁷It is a commutative unital ring under pointwise addition and pointwise multiplication, called the ring of *Fermat reals*.

Note that when $U \neq \emptyset$ and $\dim(V) > 0$, not every constant map $\bullet U \rightarrow \bullet V$ is of the form $\bullet f$ for some smooth map $f : U \rightarrow V$, since otherwise $\bullet f(u) \in V \subset \bullet V$ for every $u \in U \subseteq \bullet U$. In order to get a concrete site (see next section), we introduce the following definition:

2.1. DEFINITION. *Let $A \subseteq \bullet \mathbb{R}^n$ and $B \subseteq \bullet \mathbb{R}^m$ be arbitrary subsets. A function $f : A \rightarrow B$ is called quasi-standard smooth if for every $a \in A$, there exist an open neighborhood U of $\circ a$ in \mathbb{R}^n , an open subset P of some Euclidean space, a smooth map $\alpha : P \times U \rightarrow \mathbb{R}^m$ and some fixed point $p \in \bullet P$, such that for every $x \in A \cap \bullet U$, we have*

$$f(x) = \bullet \alpha(p, x).$$

In particular, every constant map $A \rightarrow B$ and $i_U : U \rightarrow \bullet U$ are quasi-standard smooth. Moreover, every quasi-standard smooth map is continuous with respect to the Fermat topology.

3. Concrete sites and concrete sheaves

The notion of concrete sites and concrete sheaves goes back to [4]. A review of the categories of concrete sheaves, with special attention to smooth spaces is in [1]. We collect some essential results here and review two examples related to this paper, in order to unify notations for the following sections. For explicit definitions and detailed proof of the properties, see [4], [1], [11] and [18, Subsections 1.2 and 2.1], but we will not need any of them in this paper.

To be brief, a concrete site is a site with a terminal object, such that there is a faithful functor from the site to the category **Set** of sets and functions (defined using the terminal object), every cover is jointly surjective on the underlying sets, and every representable presheaf is actually a sheaf. A concrete sheaf over a concrete site is a sheaf over this site with an underlying set (as sections over the terminal object), such that every section is a function between the underlying sets. Given a concrete site \mathcal{A} , the category $\mathbf{CSh}(\mathcal{A})$ of all concrete sheaves over this site forms a quasi-topos:

3.1. DEFINITION. [1, Definition 3.1] *A quasi-topos is a locally Cartesian closed category with finite colimits and a weak subobject classifier.*

In other words, it is like a topos, but with a weak subobject classifier (that is, it only classifies strong subobjects instead of all subobjects). See [11] for more discussions on quasi-topos.

Here are some basic properties of $\mathbf{CSh}(\mathcal{A})$:

- It is complete and cocomplete.
- It is (locally) Cartesian closed.
- It is locally presentable.

We will make use of the following corollaries a lot in the following sections:

1. The concrete site is canonically a full subcategory of the category of concrete sheaves over it. By abuse of notation, we use the same notations to denote objects and morphisms in these categories.
2. Every subset (or quotient set) of a concrete sheaf is canonically a concrete sheaf.
3. The faithful underlying set functor $|-| : \mathbf{CSh}(\mathcal{A}) \rightarrow \mathbf{Set}$ has both left and right adjoints. Therefore, (co)limits in $\mathbf{CSh}(\mathcal{A})$ are the (co)lifting of the corresponding (co)limits for the underlying sets.
4. Let \mathcal{A} be a concrete site. For any concrete sheaf X over \mathcal{A} , write \mathcal{A}/X (called the *plot category of X*) for the overcategory with objects all sections $p : A \rightarrow X$ and morphisms commutative triangles

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ & \searrow p & \swarrow p' \\ & & X \end{array}$$

where both p and p' are sections and f is a morphism in \mathcal{A} . There is a canonical functor $\mathcal{A}/X \rightarrow \mathbf{CSh}(\mathcal{A})$ sending the above triangle to $f : A \rightarrow A'$, and the colimit of this functor is X . In other words, every concrete sheaf is a colimit of the representing (concrete) sheaves indexed by the plot category over it, written as $X = \mathbf{colim}_{A \in \mathcal{A}/X} A$.

We will mainly focus on the following two examples in this paper:

3.2. EXAMPLE.

1. [1, Lemma 4.14 and Proposition 4.15] Let \mathcal{S} be the site (called the *Souriau site*) with objects all open subsets of \mathbb{R}^n for all $n \in \mathbb{N}$, morphisms smooth maps between them, and covers the usual open coverings. Then \mathcal{S} is a concrete site with terminal object \mathbb{R}^0 . The category $\mathbf{CSh}(\mathcal{S})$ of concrete sheaves over \mathcal{S} is equivalent to the category \mathbf{Diff} of *diffeological spaces* and smooth maps. Isomorphisms in \mathbf{Diff} are called *diffeomorphisms*. For more discussions of diffeological spaces, see the standard textbook [10]; for a three-page concise introduction together with basic notation and terminology, see [3, Section 2].
2. [5, Section 8.3] Let \mathcal{F} be the site (called the *Fermat site*) with objects all subsets of $\bullet\mathbb{R}^n$ for all $n \in \mathbb{N}$, morphisms quasi-standard smooth maps between them, and covers the Fermat open coverings. Then \mathcal{F} is a concrete site with terminal object $\bullet\mathbb{R}^0 = \mathbb{R}^0$.⁸ The category of concrete sheaves over \mathcal{F} is denoted by $\bullet C^\infty$, called the category of *Fermat spaces* and Fermat maps.

⁸The reference didn't prove this fact using the language of quasi-topos, and instead introduced a new terminology called "a category of figures". This fact is indeed an easy consequence of the results proved there.

3.3. REMARK. We will relate the category **Diff** of diffeological spaces and the category $\bullet C^\infty$ of Fermat spaces in next section. If we define \mathcal{F}' to be the full subcategory of \mathcal{F} consisting of objects of the form $\bullet U$ with U an open subset of a Euclidean space, then by Example 3.2(2), \mathcal{F}' is also a concrete site. It seems more natural to relate the category **CSh**(\mathcal{F}') to the category **Diff**. We will show in Subsection 4.20 in what sense the category $\bullet C^\infty$ is better than the category **CSh**(\mathcal{F}').

In the above two examples, note that every object in the concrete site \mathcal{S} or \mathcal{F} is not just a set, but a topological space, and every morphism is continuous. More generally, assume that a concrete site \mathcal{A} is a subcategory of the category **Top** of topological spaces and continuous maps, with covers the open coverings. Then every concrete sheaf X over \mathcal{A} has a canonical topology, which is the final topology with respect to all sections $A \rightarrow X$, i.e., the largest topology on the set $|X|$ making all sections continuous. This defines a functor **CSh**(\mathcal{A}) \rightarrow **Top**. This functor sends every object in \mathcal{A} to the same topological space. When $\mathcal{A} = \mathcal{S}$, this topology is called the *D-topology*⁹ on diffeological spaces (see [3] for detailed discussion), and when $\mathcal{A} = \mathcal{F}$, this topology is called the *Fermat topology* on Fermat spaces. Moreover, this functor **CSh**(\mathcal{A}) \rightarrow **Top** has a right adjoint, sending every topological space Y to a concrete sheaf over \mathcal{A} with sections over an object A in \mathcal{A} the set of all continuous maps $A \rightarrow Y$.

4. Extending diffeological spaces with infinitesimals

We use the following notations as in Examples 3.2(1) and 3.2(2) throughout this section: \mathcal{S} is the Souriau site, \mathcal{F} is the Fermat site, **Diff** is the category of diffeological spaces and smooth maps, and $\bullet C^\infty$ is the category of Fermat spaces and Fermat maps.

From Examples 3.2(1) and 3.2(2), we know that both categories **Diff** and $\bullet C^\infty$ are concrete sheaves over concrete sites \mathcal{S} and \mathcal{F} , respectively. In order to find relationship between categories of concrete sheaves, we only need to find “good” functors between the two sites. There are already some candidates for such functors introduced in Section 2, and we will use them to build the adding and the deleting infinitesimal functors.

4.1. THE ADDING INFINITESIMAL FUNCTOR $\bullet(-)$. In [5, Chapters 7-10], an attempt of adding infinitesimals on smooth spaces has been made, by using smooth functions from diffeological spaces to \mathbb{R} . The theory goes well for smooth manifolds, or more generally for separated diffeological spaces, i.e., diffeological spaces whose smooth functions to \mathbb{R} separate points. But if we take the diffeological space to be a 1-dimensional irrational torus, then after that procedure of adding infinitesimals, we get a trivial Fermat space (i.e., a single point), since the *D-topology* on any irrational torus is indiscrete – the only open subsets are the empty set and the whole space. In other words, that procedure of adding infinitesimals turns an important and highly non-trivial diffeological space into a trivial Fermat space. In this subsection, we introduce a new approach to extend diffeological

⁹The letter “D” in “D-topology” refers to “diffeology”, not the first-order infinitesimals D introduced in the previous section. We use the same convention for the terminology “D-open” in the following sections.

spaces with infinitesimals to overcome this problem, and still keep all the nice properties as stated in [5, Chapters 7-10] for general diffeological spaces, instead of separated ones.

We introduce the following functor from diffeological spaces to Fermat spaces, using Fermat extension of smooth functions:

4.2. PROPOSITION. *The assignment $\mathcal{S} \rightarrow \mathcal{F}$ by*

$$f : U \rightarrow V \mapsto \bullet f : \bullet U \rightarrow \bullet V$$

is a functor between the two sites, and hence induces a functor $\bullet(-) : \mathbf{Diff} \rightarrow \bullet C^\infty$ by

$$X = \mathbf{colim}_{U \in \mathcal{S}/X} U \in \mathbf{Diff} \mapsto \bullet X = \mathbf{colim}_{U \in \mathcal{S}/X} \bullet U \in \bullet C^\infty.$$

Note that although the above two colimits have the same indexing category, the colimits are taken in different categories. We call the functor $\bullet(-) : \mathbf{Diff} \rightarrow \bullet C^\infty$ the *adding infinitesimal functor*. Since $\bullet C^\infty$ is a category of concrete sheaves, every point in the Fermat space $\bullet X$ can be thought of as a point in $\bullet U$ for some plot $p : U \rightarrow X$. Two such points in $\bullet X$ are equal if and only if they are connected by the Fermat extension of a zig-zag diagram of plots of X , instead of using smooth functions $X \rightarrow \mathbb{R}$. We will see in next proposition that the adding infinitesimal functor $\bullet(-)$ is different in general from the one introduced in [5, Chapter 9], although we use the same notation. In particular, this functor sends $U \in \mathcal{S}$ to $\bullet U$, which coincides with the notation introduced in Section 2, since the indexing category \mathcal{S}/U has a terminal object $1_U : U \rightarrow U$.

PROOF. This is straightforward. Indeed, this is the left Kan extension (see [14, X.3]) of the composite of functors $\mathcal{S} \rightarrow \mathcal{F} \hookrightarrow \bullet C^\infty$ along the inclusion functor $\mathcal{S} \hookrightarrow \mathbf{Diff}$. ■

Here is the relationship between the underlying sets of X and $\bullet X$:

4.3. PROPOSITION. *The adding infinitesimal functor $\bullet(-) : \mathbf{Diff} \rightarrow \bullet C^\infty$ makes every diffeological space a subset of the corresponding Fermat space.*

In particular, if X is a 1-dimensional irrational torus, then $|X|$ is a subset of $|\bullet X|$, which implies that $\bullet X$ is not a trivial Fermat space; see Example 4.32 for the final answer of $\bullet X$. Therefore, the adding infinitesimal functor $\bullet(-)$ is different from the one introduced in [5, Chapter 9].

PROOF. Since the functor $|-| : \mathbf{CSh}(\mathcal{A}) \rightarrow \mathbf{Set}$ has a right adjoint for any concrete site \mathcal{A} , it preserves colimits, i.e., for any diffeological space X , we have

$$\begin{aligned} |X| &= |\mathbf{colim}_{U \in \mathcal{S}/X} U| = \mathbf{colim}_{U \in \mathcal{S}/X} |U| \text{ and} \\ |\bullet X| &= |\mathbf{colim}_{U \in \mathcal{S}/X} \bullet U| = \mathbf{colim}_{U \in \mathcal{S}/X} |\bullet U| \text{ in } \mathbf{Set}. \end{aligned}$$

Recall that for any smooth map $f : U \rightarrow V$ between open subsets of Euclidean spaces, we have the following commutative diagram in **Set**:

$$\begin{array}{ccccc} |U| & \xrightarrow{i_U} & |\bullet U| & \xrightarrow{ev_0} & |U| \\ |f| \downarrow & & |\bullet f| \downarrow & & \downarrow |f| \\ |V| & \xrightarrow{i_V} & |\bullet V| & \xrightarrow{ev_0} & |V|, \end{array}$$

and the composites of the two horizontal maps are identities. Therefore, we have maps $i_X : |X| \rightarrow |\bullet X|$ and $ev_0 : |\bullet X| \rightarrow |X|$ such that $ev_0 \circ i_X = 1_{|X|}$. This implies that i_X is injective, and hence $|X|$ is a subset of $|\bullet X|$. ■

Moreover, for any smooth map $f : X \rightarrow Y$ between diffeological spaces, we have the following commutative diagram in **Set**:

$$\begin{array}{ccccc} |X| & \xrightarrow{i_X} & |\bullet X| & \xrightarrow{ev_0} & |X| \\ |f| \downarrow & & |\bullet f| \downarrow & & \downarrow |f| \\ |Y| & \xrightarrow{i_Y} & |\bullet Y| & \xrightarrow{ev_0} & |Y|. \end{array}$$

(Actually this holds in **Top**, where X and Y are equipped with the D-topology, and $\bullet X$ and $\bullet Y$ are equipped with the Fermat topology. But we will not need this fact in this paper.) In other words, $|\bullet f|$ is always a retract of $|f|$. Therefore, if $|\bullet f|$ is injective (resp. surjective or bijective), then so is $|f|$.¹⁰ When X and Y are open subsets of Euclidean spaces, $\bullet f$ coincides with the notation introduced in Section 2.

The adding infinitesimal functor behaves nicely with respect to D-open subsets:

4.4. PROPOSITION. *Let A be a D-open subset of a diffeological space X , equipped with the subset diffeology. Then $\bullet A$ is a Fermat open subset of $\bullet X$.*

PROOF. Let $i : A \hookrightarrow X$ be the inclusion map, which induces a Fermat map $\bullet i : \bullet A \rightarrow \bullet X$. Since A is a D-open subset of X , for any plot $p : U \rightarrow X$, $p^{-1}(A) \subseteq U$ is open and $p|_{p^{-1}(A)} : p^{-1}(A) \rightarrow A$ is a plot of A . So we get a functor $\mathcal{S}/X \rightarrow \mathcal{S}/A$, such that the composite $\mathcal{S}/A \hookrightarrow \mathcal{S}/X \rightarrow \mathcal{S}/A$ is identity. This does not mean that we always have a Fermat map $\bullet X \rightarrow \bullet A$, but from this it follows that $\bullet i$ is injective.

For any Fermat plot $q : B \rightarrow \bullet X = \mathbf{colim}_{U \in \mathcal{S}/X} \bullet U$ and any point $b \in B$, there exist a Fermat open neighborhood C of b in B , some plot $r : U \rightarrow X$, and a quasi-standard smooth map $f : C \rightarrow \bullet U$ such that the following square commutes in $\bullet C^\infty$:

$$\begin{array}{ccc} C & \hookrightarrow & B \\ f \downarrow & & \downarrow q \\ \bullet U & \xrightarrow{\bullet r} & \bullet X. \end{array}$$

¹⁰Note that, the converse is not necessarily true.

Since every quasi-standard smooth map is continuous with respect to the Fermat topology, it is enough to prove that $(\bullet r)^{-1}(\bullet A) = \bullet(r^{-1}(A))$, which is the statement of the next proposition. ■

4.5. PROPOSITION. *Let $f : X \rightarrow Y$ be a smooth map between diffeological spaces, and let A be a D -open subset of Y , equipped with the subset diffeology of Y . Also equip $f^{-1}(A)$ with the subset diffeology of X . Then*

$$(\bullet f)^{-1}(\bullet A) = \bullet(f^{-1}(A)).$$

PROOF. Since $f : X \rightarrow Y$ is smooth and A is D -open in Y , $f^{-1}(A)$ is D -open in X . From what we have proved in the previous proposition, we know that the inclusion map $f^{-1}(A) \hookrightarrow X$ induces an injective map $\bullet(f^{-1}(A)) \rightarrow \bullet X$. So both $(\bullet f)^{-1}(\bullet A)$ and $\bullet(f^{-1}(A))$ are subsets of $\bullet X$.

For any plot $p : U \rightarrow f^{-1}(A)$, we have the following commutative diagram in **Diff**:

$$\begin{array}{ccc} U & \xrightarrow{p} & f^{-1}(A) \hookrightarrow X \\ & & \downarrow f|_{f^{-1}(A)} \quad \downarrow f \\ & & A \hookrightarrow Y, \end{array}$$

which induces a commutative square in $\bullet C^\infty$:

$$\begin{array}{ccc} \bullet U & \longrightarrow & \bullet X \\ \downarrow & & \downarrow \bullet f \\ \bullet A & \hookrightarrow & \bullet Y. \end{array}$$

Therefore, $\mathbf{colim}_{U \in \mathcal{S}/f^{-1}(A)} \bullet U = \bullet(f^{-1}(A)) \subseteq (\bullet f)^{-1}(\bullet A)$.

For the converse inclusion, assume that

$$\bullet f(x) \in \bullet A = \mathbf{colim}_{V \in \mathcal{S}/A} \bullet V$$

for some $x \in \bullet X = \mathbf{colim}_{U \in \mathcal{S}/X} \bullet U$. So there exist plots $p : U \rightarrow X$ and $q : V \rightarrow A$, and points $u \in \bullet U$ and $v \in \bullet V$ such that $\bullet p : \bullet U \rightarrow \bullet X$ sends u to x and $\bullet q : \bullet V \rightarrow \bullet A$ sends v to $\bullet f(x)$. That is, $\bullet q(v) = \bullet(f \circ p)(u) \in \bullet Y = \mathbf{colim}_{W \in \mathcal{S}/Y} \bullet W$.

Since $|-| : \bullet C^\infty \rightarrow \mathbf{Set}$ is faithful and has a right adjoint, there exist finitely many plots $r_i : W_i \rightarrow Y$, points $w_i \in \bullet W_i$ and zig-zag morphisms in \mathcal{S}/Y connecting $f \circ p$ and $j \circ q$ via these r_i 's, where $j : A \hookrightarrow Y$, so that u and v are connected via these w_i 's when applying the adding infinitesimal functor on the zig-zag. Let us do the following to “shorten” the length of the zig-zag:

(1) If we have the following commutative triangle in **Diff**:

$$\begin{array}{ccc} W & \xrightarrow{g} & V \\ & \searrow r & \swarrow q \\ & & Y, \end{array}$$

then r can also be viewed as a plot of A , so we switch to consider the pair (r, w) with $w \in \bullet W$ given (so $\bullet g(w) = v$) instead of (q, v) ;

(2) If we have the following commutative triangle in **Diff**:

$$\begin{array}{ccc} W & \xleftarrow{g} & V \\ & \searrow r & \swarrow q \\ & & Y, \end{array}$$

then $r^{-1}(A) \neq \emptyset$, and the given $w \in \bullet W$ is actually in $r^{-1}(A)$. So we switch to consider the pair $(r|_{r^{-1}(A)}, w)$ instead of (q, v) . In this case, we might need to shrink one W_i next to W or U a bit to keep the zig-zag in \mathcal{S}/Y , but without changing the given points w_i .

After finitely many steps of switching pairs, we know that there exists an open neighborhood U' of $\circ u$ in U such that $f(p(U')) \subseteq A$. Therefore, $(\bullet f)^{-1}(\bullet A) \subseteq \bullet(f^{-1}(A))$. ■

In the next two results, we are going to connect the D-topology on a diffeological space X and the Fermat topology on $\bullet X$.

4.6. PROPOSITION. *Let X be a diffeological space, and let A be a Fermat open subset of $\bullet X$. Then $X \cap A$ ¹¹ is a D-open subset of X , and $A = \bullet(X \cap A)$.*

PROOF. Let $p : U \rightarrow X$ be an arbitrary plot. Using the commutative square

$$\begin{array}{ccc} |U| & \xrightarrow{i_U} & |\bullet U| \\ |p| \downarrow & & \downarrow |\bullet p| \\ |X| & \xrightarrow{i_X} & |\bullet X|, \end{array}$$

it is straightforward to check that $p^{-1}(X \cap A) = U \cap (\bullet p)^{-1}(A)$, and hence $X \cap A$ is a D-open subset of X . So both A and $\bullet(X \cap A)$ (Proposition 4.4) are Fermat open subsets of $\bullet X$.

Note that every point in $\bullet(X \cap A)$ can be represented by $v_q \in \bullet V$, where $q : V \rightarrow X$ is a plot whose image is in $X \cap A$. Since $V \subseteq q^{-1}(X \cap A) = V \cap (\bullet q)^{-1}(A)$, we have $\bullet q(v_q) \in A$. Hence, $\bullet(X \cap A) \subseteq A$.

On the other hand, assume that $w_r \in \bullet W$ with $r : W \rightarrow X$ a plot represents a point in A , i.e., $\bullet r(w_r) \in A$. Since A is Fermat open in X , $(\bullet r)^{-1}(A)$ is Fermat open in $\bullet W$, which implies that $r(\circ w_r) \in X \cap A$, and hence $\bullet r(w_r) \in \bullet(X \cap A)$. Therefore, $A \subseteq \bullet(X \cap A)$.

As a result, we have $A = \bullet(X \cap A)$. ■

In conclusion, we have:

4.7. THEOREM. *Let X be a diffeological space. Then there is a bijection between the D-open subsets of X and the Fermat open subsets of $\bullet X$.*

¹¹By Proposition 4.3, X is canonically a subset of $\bullet X$.

PROOF. The maps between these sets are given by sending a D-open subset A of X to $\bullet A$, and by sending a Fermat open subset B of $\bullet X$ to $X \cap B$, respectively. To prove that these maps are inverse to each other, by Propositions 4.4 and 4.6, we are left to show that $X \cap \bullet A = A$. Assume that $u_p \in U$ with plot $p : U \rightarrow X$ and $v_q \in \bullet V$ with plot $q : V \rightarrow X$ whose image is in A represent the same element in $\bullet X$. By using ev_0 , it is clear that $\circ v_q$ and u_p represent the same element in X , and the former actually represents an element in A . Hence, $X \cap \bullet A \subseteq A$. The converse inclusion is clear. ■

The next two results are easy applications:

4.8. COROLLARY. *Let X be a diffeological space, and let $\{A_i\}_{i \in I}$ be a set of D-open subsets of X . Then we have*

$$\begin{aligned} \bullet(A_1 \cap A_2) &= \bullet A_1 \cap \bullet A_2, \\ \bullet\left(\bigcup_{i \in I} A_i\right) &= \bigcup_{i \in I} \bullet A_i, \end{aligned}$$

and

$$\bullet(\text{int}(X \setminus A_1)) = \text{int}(\bullet X \setminus \bullet A_1),$$

where int denotes the interior.

4.9. PROPOSITION. *Let $f : X \rightarrow Y$ be a smooth map between diffeological spaces, which is an open map with respect to the D-topology. Let A be a D-open subset of X . Then $\bullet(f(A)) = (\bullet f)(\bullet A)$, where $f(A)$ and A are equipped with the subset diffeology of Y and X , respectively.*

As a consequence, the adding infinitesimal functor $\bullet(-)$ preserves open maps.

PROOF. Since A is a D-open subset of X and $f : X \rightarrow Y$ is an open map, $f(A)$ is a D-open subset of Y . Then $(\bullet f)(\bullet A) \subseteq \bullet(f(A))$ follows from applying the functor $\bullet(-)$ to the commutative square

$$\begin{array}{ccc} A & \hookrightarrow & X \\ f|_A \downarrow & & \downarrow f \\ f(A) & \hookrightarrow & Y \end{array}$$

together with Proposition 4.4. Since $f(A) \subseteq (\bullet f)(\bullet A)$, we have $\bullet(f(A)) \subseteq (\bullet f)(\bullet A)$ by Proposition 4.6. Therefore, $\bullet(f(A)) = (\bullet f)(\bullet A)$. ■

For quotient spaces, we have:

4.10. PROPOSITION. *If Y is a quotient space of a diffeological space X , then $\bullet Y$ is a quotient space of the Fermat space $\bullet X$.*

PROOF. Since Y is a quotient space of X , every plot of Y locally factors through a plot of X , which implies that the quotient map $X \rightarrow Y$ induces a surjective map $|\bullet X| \rightarrow |\bullet Y|$, and moreover, every Fermat plot of $\bullet Y$ also locally factors through a Fermat plot of $\bullet X$. ■

The adding infinitesimal functor preserves finite products:

4.11. PROPOSITION. *For any diffeological spaces X_1 and X_2 , we have a natural isomorphism $\bullet(X_1 \times X_2) \cong \bullet X_1 \times \bullet X_2$ in $\bullet C^\infty$.*

PROOF. Note that

$$\bullet(X_1 \times X_2) = \mathbf{colim}_{U \in \mathcal{S}/(X_1 \times X_2)} \bullet U,$$

and

$$\begin{aligned} \bullet X_1 \times \bullet X_2 &= (\mathbf{colim}_{V \in \mathcal{S}/X_1} \bullet V) \times (\mathbf{colim}_{W \in \mathcal{S}/X_2} \bullet W) \\ &= \mathbf{colim}_{V \in \mathcal{S}/X_1} (\bullet V \times \mathbf{colim}_{W \in \mathcal{S}/X_2} \bullet W) \\ &= \mathbf{colim}_{(V \in \mathcal{S}/X_1) \times (W \in \mathcal{S}/X_2)} (\bullet V \times \bullet W) \\ &= \mathbf{colim}_{(V \in \mathcal{S}/X_1) \times (W \in \mathcal{S}/X_2)} \bullet(V \times W) \end{aligned}$$

where the second and the third equalities follow from Cartesian closedness of $\bullet C^\infty$.

We can define a functor $(\mathcal{S}/X_1) \times (\mathcal{S}/X_2) \rightarrow \mathcal{S}/(X_1 \times X_2)$ sending $(f, g) : (q : V \rightarrow X_1, r : W \rightarrow X_2) \rightarrow (q' : V' \rightarrow X_1, r' : W' \rightarrow X_2)$ to $(f \times g) : (q \times r : V \times W \rightarrow X_1 \times X_2) \rightarrow (q' \times r' : V' \times W' \rightarrow X_1 \times X_2)$. It is straightforward to check that this functor is final ([14, Section IX.3]), and hence $\bullet X_1 \times \bullet X_2 \rightarrow \bullet(X_1 \times X_2)$ is an isomorphism¹² in $\bullet C^\infty$ ([14, Theorem IX.3.1]).

The naturality means that if $f_1 : X_1 \rightarrow X'_1$ and $f_2 : X_2 \rightarrow X'_2$ are smooth maps between diffeological spaces, then we have a commutative square in $\bullet C^\infty$:

$$\begin{array}{ccc} \bullet X_1 \times \bullet X_2 & \xrightarrow{\bullet f_1 \times \bullet f_2} & \bullet X'_1 \times \bullet X'_2 \\ \downarrow & & \downarrow \\ \bullet(X_1 \times X_2) & \xrightarrow{\bullet(f_1 \times f_2)} & \bullet(X'_1 \times X'_2). \end{array}$$

This follows easily from the canonical map $\bullet X_1 \times \bullet X_2 \rightarrow \bullet(X_1 \times X_2)$ described above. ■

4.12. REMARK. More generally, we have the following result by a similar proof. Let \mathcal{A} and \mathcal{B} be concrete sites with finite products, and let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a natural finite-product-preserving functor. Then the induced functor $F : \mathbf{CSh}(\mathcal{A}) \rightarrow \mathbf{CSh}(\mathcal{B})$ defined by $X = \mathbf{colim}_{A \in \mathcal{A}/X} A \mapsto F(X) := \mathbf{colim}_{A \in \mathcal{A}/X} F(A)$ naturally preserves finite products. This result will be used in Proposition 4.19.

For infinite products, the situation is different:

4.13. REMARK. Let I be an infinite index set. Then the natural map $\bullet(\prod_{i \in I} X_i) \rightarrow \prod_{i \in I} \bullet X_i$ is not necessarily surjective. For example, consider the natural map $\bullet(\prod_{\mathbb{N}} \mathbb{R}) \rightarrow \prod_{\mathbb{N}} \bullet \mathbb{R}$. For any element in $\bullet(\prod_{\mathbb{N}} \mathbb{R})$ represented by a plot $p : U \rightarrow \prod_{\mathbb{N}} \mathbb{R}$ and a fixed $u \in \bullet U$, its image $(\bullet p_i(u))_{i \in \mathbb{N}}$ in $\prod_{\mathbb{N}} \bullet \mathbb{R}$ has the property that the set $\{\omega(\bullet p_i(u))\}_{i \in \mathbb{N}}$ has

¹²The inverse of this isomorphism is induced by the projections $\pi_i : X_1 \times X_2 \rightarrow X_i$ for $i = 1, 2$.

a finite upper bound by Taylor’s expansion, while $\prod_{\mathbb{N}} \bullet\mathbb{R}$ contains elements $(x_i)_{i \in \mathbb{N}}$ such that the set $\{\omega(x_i)\}_{i \in \mathbb{N}}$ has no finite upper bound ($x_i = t^{1/(i+1)}$ for instance).

Now we discuss function spaces. Let X and Y be diffeological spaces. Since the category \mathbf{Diff} of diffeological spaces is Cartesian closed, $\mathbf{Diff}(X, Y)$ ¹³ is also a diffeological space, with the natural diffeology (called the *functional diffeology*) consisting of all maps $U \rightarrow \mathbf{Diff}(X, Y)$ such that the corresponding adjoint maps $U \times X \rightarrow Y$ are smooth. So $\bullet(\mathbf{Diff}(X, Y)) = \mathbf{colim}_{U \in \mathcal{S}/\mathbf{Diff}(X, Y)} \bullet U$. On the other hand, we can apply the adding infinitesimal functor to the adjoint maps $U \times X \rightarrow Y$ to get $\bullet U \times \bullet X \cong \bullet(U \times X) \rightarrow \bullet Y$. Since the category $\bullet C^\infty$ of Fermat spaces is Cartesian closed, we can take the adjoint back and get Fermat maps $\bullet U \rightarrow \bullet C^\infty(\bullet X, \bullet Y)$. It is easy to check that we get a Fermat map $i : \bullet(\mathbf{Diff}(X, Y)) \rightarrow \bullet C^\infty(\bullet X, \bullet Y)$. Moreover, the composite

$$|\mathbf{Diff}(X, Y)| \xrightarrow{i_{\mathbf{Diff}(X, Y)}} |\bullet(\mathbf{Diff}(X, Y))| \xrightarrow{|i|} |\bullet C^\infty(\bullet X, \bullet Y)|$$

exactly sends a smooth map f to its Fermat extension $\bullet f$.

In general, one cannot expect the Fermat map $i : \bullet(\mathbf{Diff}(X, Y)) \rightarrow \bullet C^\infty(\bullet X, \bullet Y)$ to be an isomorphism in $\bullet C^\infty$. For example, when $X = Y = \mathbb{R}$, $\bullet(\mathbf{Diff}(\mathbb{R}, \mathbb{R}))$ consists of $\bullet f(u, -) : \bullet\mathbb{R} \rightarrow \bullet\mathbb{R}$, where $f : U \times \mathbb{R} \rightarrow \mathbb{R}$ is a smooth map with U some open subset of a Euclidean space, and $u \in \bullet U$ is some fixed point; $\bullet C^\infty(\bullet\mathbb{R}, \bullet\mathbb{R})$ is the set of all Fermat maps $\bullet\mathbb{R} \rightarrow \bullet\mathbb{R}$; the map $i : \bullet(\mathbf{Diff}(\mathbb{R}, \mathbb{R})) \rightarrow \bullet C^\infty(\bullet\mathbb{R}, \bullet\mathbb{R})$ is the inclusion map, which is hence not an isomorphism in $\bullet C^\infty$.

On the other hand, we will show in next subsection that both $\bullet(\mathbf{Diff}(X, Y))$ and $\bullet C^\infty(\bullet X, \bullet Y)$ have the same “underlying diffeological space”.

4.14. THE DELETING INFINITESIMAL FUNCTOR $\bullet(-)$. In this subsection, we introduce a functor $\bullet C^\infty \rightarrow \mathbf{Diff}$ which deletes all infinitesimal points. This is the left inverse of the adding infinitesimal functor $\bullet(-)$ introduced in the previous subsection.

Recall that we write \mathcal{F}' for the full subsite of the Fermat site \mathcal{F} , consisting of objects of the form $\bullet U$ for U an open subset of \mathbb{R}^n for all $n \in \mathbb{N}$. We first observe that

4.15. PROPOSITION. $\mathcal{F}' \rightarrow \mathcal{S}$ defined by

$$f : \bullet U \rightarrow \bullet V \mapsto \bullet f : U \rightarrow V,$$

with $\bullet f(u) = ev_0 \circ f \circ i_U(u) = \circ(f(u))$ is a functor.

PROOF. Note that f is quasi-standard smooth, i.e., for every $a \in \bullet U$, there exist an open neighborhood U' of $\circ a$ in U , an open subset U'' of a Euclidean space, a fixed point $b \in \bullet U''$, and a smooth map $g : U'' \times U' \rightarrow \mathbb{R}^n$ with $n = \dim(V)$, such that for any $x \in \bullet U'$, $f(x) = \bullet g(b, x)$. Hence, for any $u \in U'$, $\bullet f(u) = \circ(f(u)) = \circ(\bullet g(b, u)) = g(\circ b, u)$. Therefore, $\bullet f$ is a smooth map.

Clearly $\bullet(1_{\bullet U}) = 1_U$.

¹³Here $|\mathbf{Diff}(X, Y)|$ denotes the set of all smooth maps $X \rightarrow Y$.

Let $f : \bullet U \rightarrow \bullet V$ and $g : \bullet V \rightarrow \bullet W$ be quasi-standard smooth maps. Then for any $u \in U$

$$\begin{aligned} \bullet g(\bullet f(u)) &= \bullet g(\circ(f(u))) \\ &= \circ(g(\circ(f(u)))) \\ &= \circ(g(f(u))) \\ &= \bullet(g \circ f)(u), \end{aligned}$$

where the third equality follows from Taylor's expansion of the local expression of g as a Fermat extension of a smooth function. Therefore, $\bullet g \circ \bullet f = \bullet(g \circ f)$. ■

Hence, we get a functor from Fermat spaces to diffeological spaces:

4.16. PROPOSITION. $\bullet(-) : \bullet C^\infty \rightarrow \mathbf{Diff}$ defined by

$$X \mapsto \bullet X = \mathbf{colim}_{U \in \mathcal{F}'/X} U$$

is a functor.

We call this functor the *deleting infinitesimal functor*; see next proposition for explanation.

PROOF. This is clear from Proposition 4.15. Indeed, this is the left Kan extension of the composite of functors $\mathcal{F}' \rightarrow \mathcal{S} \hookrightarrow \mathbf{Diff}$ along the inclusion functor $\mathcal{F}' \hookrightarrow \bullet C^\infty$. ■

It is easy to check that the composite $\mathcal{S} \rightarrow \mathcal{F}' \rightarrow \mathcal{S}$ is identity, where the first functor is introduced in Proposition 4.2, and the second one is given by Proposition 4.15. This property can be extended to the corresponding concrete sheaf categories:

4.17. PROPOSITION. *The composite*

$$\mathbf{Diff} \xrightarrow{\bullet(-)} \bullet C^\infty \xrightarrow{\bullet(-)} \mathbf{Diff}$$

is the identity functor.

In other words, the deleting infinitesimal functor is the left inverse of the adding infinitesimal functor.

PROOF. For any diffeological space X , we prove below that $\bullet(\bullet X) = X$. From the proof, it is clear that the composite of these two functors acts as identity on morphisms.

Recall that

$$X = \mathbf{colim}_{U \in \mathcal{S}/X} U, \bullet X = \mathbf{colim}_{U \in \mathcal{S}/X} \bullet U, \text{ and } \bullet(\bullet X) = \mathbf{colim}_{U \in \mathcal{F}'/\bullet X} U.$$

We define a functor $\mathcal{S}/X \rightarrow \mathcal{F}'/\bullet X$ by

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ & \searrow p & \swarrow q \\ & X & \end{array} \mapsto \begin{array}{ccc} \bullet U & \xrightarrow{\bullet f} & \bullet V \\ & \searrow \bullet p & \swarrow \bullet q \\ & \bullet X & \end{array}$$

It is straightforward to check that $\bullet(\bullet f) = f : U \rightarrow V$, and hence we get a natural smooth map $X \rightarrow \bullet(\bullet X)$.

On the other hand, for any Fermat plot $p : \bullet U \rightarrow \bullet X$, write $\bar{p} : U \rightarrow X$ for the composite

$$U \xrightarrow{i_U} \bullet U \xrightarrow{p} \bullet X \xrightarrow{ev_0} X.$$

By a similar proof as Proposition 4.15, one can check that $\mathcal{F}'/\bullet X \rightarrow \mathcal{S}/X$ defined by

$$\begin{array}{ccc} \bullet U & \xrightarrow{f} & \bullet V \\ & \searrow p & \swarrow q \\ & \bullet X & \end{array} \mapsto \begin{array}{ccc} U & \xrightarrow{\bullet f} & V \\ & \searrow \bar{p} & \swarrow \bar{q} \\ & X & \end{array}$$

is a well-defined functor, and hence we get another natural smooth map $\bullet(\bullet X) \rightarrow X$.

Although p and $\bullet\bar{p}$ can be different, it is straightforward to check that the two composites

$$U \xrightarrow{i_U} \bullet U \xrightarrow[\bullet\bar{p}]{p} \bullet X \xrightarrow{ev_0} X$$

are the same, and hence the two maps $X \rightarrow \bullet(\bullet X)$ and $\bullet(\bullet X) \rightarrow X$ are inverse to each other. ■

By a similar method, one can show that if X and Y are diffeological spaces, and $f : \bullet X \rightarrow \bullet Y$ is a Fermat map, then $\bullet f : \bullet(\bullet X) \rightarrow \bullet(\bullet Y)$ after natural diffeomorphisms as constructed in the proof of Proposition 4.17 corresponds to $\bar{f} : X \rightarrow Y$, i.e., the composite

$$X \xrightarrow{i_X} \bullet X \xrightarrow{f} \bullet Y \xrightarrow{ev_0} Y.$$

In particular, from a commutative triangle in $\bullet C^\infty$:

$$\begin{array}{ccc} \bullet X & \xrightarrow{f} & \bullet Y \\ & \searrow h & \swarrow g \\ & \bullet Z & \end{array}$$

we get a commutative triangle in **Diff**:

$$\begin{array}{ccc} X & \xrightarrow{\bar{f}} & Y \\ & \searrow \bar{h} & \swarrow \bar{g} \\ & Z & \end{array}$$

where X, Y, Z are diffeological spaces, and f, g, h are Fermat maps.

As an easy application, we have:

4.18. COROLLARY. *Let X and Y be diffeological spaces. Then the Fermat map $i : \bullet(\mathbf{Diff}(X, Y)) \rightarrow \bullet C^\infty(\bullet X, \bullet Y)$ introduced at the end of last subsection induces a diffeomorphism $\mathbf{Diff}(X, Y) \cong \bullet(\bullet C^\infty(\bullet X, \bullet Y))$.*

PROOF. A morphism

$$\begin{array}{ccc} \bullet U & \xrightarrow{f} & \bullet V \\ & \searrow \bar{p} & \swarrow \bar{q} \\ & \bullet C^\infty(\bullet X, \bullet Y) & \end{array}$$

in $\mathcal{F}'/\bullet C^\infty(\bullet X, \bullet Y)$ is equivalent to a commutative triangle

$$\begin{array}{ccc} \bullet U \times \bullet X & \xrightarrow{f \times 1_{\bullet X}} & \bullet V \times \bullet X \\ & \searrow p & \swarrow q \\ & \bullet Y & \end{array}$$

in $\bullet C^\infty$ by Cartesian closedness of $\bullet C^\infty$. By the observation above this corollary, we get a commutative triangle

$$\begin{array}{ccc} U \times X & \xrightarrow{\bar{f} \times 1_X} & V \times X \\ & \searrow \bar{p} & \swarrow \bar{q} \\ & Y & \end{array}$$

in \mathbf{Diff} . By Cartesian closedness of \mathbf{Diff} , we get a morphism

$$\begin{array}{ccc} U & \xrightarrow{\bar{f} = \bullet f} & V \\ & \searrow & \swarrow \\ & \mathbf{Diff}(X, Y) & \end{array}$$

in $\mathcal{S}/\mathbf{Diff}(X, Y)$. Hence, we get a smooth map $\bullet(\bullet C^\infty(\bullet X, \bullet Y)) \rightarrow \mathbf{Diff}(X, Y)$, and one can check easily that this is the inverse of $\bullet i$. ■

On the other hand, we will show in next subsection that $\bullet(\bullet Y)$ is not isomorphic to Y in $\bullet C^\infty$ for a general Fermat space Y .

Here is another way to think of the deleting infinitesimal functor. The inclusion of the concrete sites $\mathcal{F}' \hookrightarrow \mathcal{F}$ gives rise to the restriction functor $\bullet C^\infty \rightarrow \mathbf{CSh}(\mathcal{F}')$, and Proposition 4.15 induces a functor $\mathbf{CSh}(\mathcal{F}') \rightarrow \mathbf{Diff}$. One can check that the functor $\mathcal{F}' \rightarrow \mathcal{S}$ naturally preserves finite products, and as a result of Remark 4.12 we have:

4.19. PROPOSITION. *For any Fermat spaces X and Y , $\bullet(X \times Y)$ is naturally diffeomorphic to $\bullet X \times \bullet Y$.*

For application of the adding and the deleting infinitesimal functors to integrals, see the subsection “Standard and infinitesimal parts of an integral” in [9, Section7].

4.20. WHY WE CHOOSE \mathcal{F} TO BE THE FERMAT SITE. In Example 3.22, we defined \mathcal{F} with objects all subsets of $\bullet\mathbb{R}^n$ for all $n \in \mathbb{N}$, morphisms all quasi-standard smooth maps between them, and covers the Fermat open coverings to be the Fermat site. We have explained in Section 2 that instead of taking morphisms to consist of only Fermat extension of smooth maps, we get a concrete site. In order to relate Fermat spaces with diffeological spaces, there is another natural choice – we can take \mathcal{F}' to be the full subcategory of \mathcal{F} consisting of objects of the form $\bullet U$ with U an open subset of a Euclidean space. Then by Example 3.22, \mathcal{F}' is also a concrete site. In this subsection, we explain in what sense the category $\bullet C^\infty$ is better than the category $\mathbf{CSh}(\mathcal{F}')$.

One naive reason is, we want to develop geometry of Fermat spaces and diffeological spaces using general spaces like $D_\infty = \{x \in \bullet\mathbb{R} \mid \circ x = 0\}$, $D = \{x \in \bullet\mathbb{R} \mid x^2 = 0\}$ and $D_{\geq 0} = \{x \in D \mid x \geq 0\}$, but none of them are of the form $\bullet U$ for some open subset U of a Euclidean space. On the other hand, these spaces are in the site \mathcal{F} . However, since $\mathbf{CSh}(\mathcal{F}')$ is the category of concrete sheaves over the concrete site \mathcal{F}' , we can think of D_∞ , D and $D_{\geq 0}$ as subspaces of $\bullet\mathbb{R}$ in $\mathbf{CSh}(\mathcal{F}')$.

We can also define the adding infinitesimal functor $\mathbf{Diff} \rightarrow \mathbf{CSh}(\mathcal{F}')$ as the one in Subsection 4.1, denoted by $X \mapsto \bullet X'$ ¹⁴. The actual reason is, the Fermat space $\bullet X \in \bullet C^\infty$ still keeps record of the diffeological information of X , but $\bullet X' \in \mathbf{CSh}(\mathcal{F}')$ does not, for the following explanations:

4.21. PROPOSITION. *Let X be a diffeological space, and write $(X \leq \bullet X)$ for the Fermat subspace of $\bullet X$ via the inclusion map $i_X : X \rightarrow \bullet X$ (Proposition 4.3). Then for any open subset U of a Euclidean space, $|\bullet C^\infty(U, (X \leq \bullet X))| = |\mathbf{Diff}(U, X)|$.*

In other words, the Fermat space $\bullet X$ still remembers X as a diffeological space.

PROOF. We only need to prove the two inclusions since both $|\bullet C^\infty(U, (X \leq \bullet X))|$ and $|\mathbf{Diff}(U, X)|$ are subsets of $\mathbf{Set}(|U|, |X|)$.

For any plot $p : U \rightarrow X$, we have a commutative square in \mathbf{Set} :

$$\begin{array}{ccc} |U| & \xrightarrow{i_U} & |\bullet U| \\ |p| \downarrow & & \downarrow |\bullet p| \\ |(X \leq \bullet X)| & \xrightarrow{i_X} & |\bullet X|, \end{array}$$

Since i_U is quasi-standard smooth and $\bullet p$ is a Fermat map, $p \in \bullet C^\infty(U, (X \leq \bullet X))$. So we have $|\mathbf{Diff}(U, X)| \subseteq |\bullet C^\infty(U, (X \leq \bullet X))|$.

For any Fermat plot $q : U \rightarrow (X \leq \bullet X)$, since $\bullet X = \mathbf{colim}_{V \in \mathcal{S}/X} \bullet V$, for every $u \in U$, there exist a Fermat open neighborhood U' of u in U , some plot $r : V \rightarrow X$ and a quasi-standard smooth map $f : U' \rightarrow \bullet V$ such that $i_X \circ q|_{U'} = \bullet r \circ f$. Hence, we have the

¹⁴This meaning of the notation $\bullet X'$ is valid only in this subsection.

following commutative diagram in **Set**:

$$\begin{array}{ccccc}
 |U'| & \xrightarrow{|f|} & |\bullet V| & \xrightarrow{ev_0} & |V| \\
 |q|_{U'} \downarrow & & \downarrow |\bullet r| & & \downarrow |r| \\
 |(X \leq \bullet X)| & \xrightarrow{i_X} & |\bullet X| & \xrightarrow{ev_0} & |(X \leq \bullet X)|.
 \end{array}$$

Note that the composite of the bottom horizontal maps is $1_{|(X \leq \bullet X)|}$, and the composite of the upper horizontal maps is smooth. So $q \in \mathbf{Diff}(U, X)$, i.e., $|\bullet C^\infty(U, (X \leq \bullet X))| \subseteq |\mathbf{Diff}(U, X)|$. ■

As a corollary, we have:

4.22. THEOREM. *The assignment $\mathbf{Diff} \rightarrow \bullet C^\infty$ defined by*

$$f : X \rightarrow Y \mapsto f : (X \leq \bullet X) \rightarrow (Y \leq \bullet Y)$$

is a functor, which makes \mathbf{Diff} a full subcategory of $\bullet C^\infty$.

Therefore, every diffeological space is canonically a Fermat space with the same underlying set, and every smooth map between diffeological spaces is canonically a Fermat map between the corresponding Fermat spaces.

Proposition 4.21 also implies the following:

4.23. EXAMPLE. Let U and V be open subsets of Euclidean spaces. Then $|\mathcal{F}(U, V)| = |\mathcal{S}(U, V)|$, i.e., a map $U \rightarrow V$ is quasi-standard smooth if and only if it is smooth.

4.24. REMARK. In fact, every Fermat space is automatically a diffeological space in the following way. Let Y be a Fermat space. For any open subset U of a Euclidean space, we define $U \rightarrow Y$ to be a plot if it is in $\bullet C^\infty(U, Y)$. In this way, Y is a diffeological space. Indeed, this defines a forgetful functor $\bullet C^\infty \rightarrow \mathbf{Diff}$, which has a right inverse given by the functor defined in Theorem 4.22.

By Proposition 4.21, if U is an open subset of some Euclidean space with $\dim(U) > 0$, then the Fermat space $(U \leq \bullet U)$ is not discrete in $\bullet C^\infty$.

Now we show that the object $(U \leq \bullet U')$ in $\mathbf{CSh}(\mathcal{F}')$ is discrete. Let $p : \bullet V \rightarrow U$ be a section, i.e., the composite

$$\bullet V \xrightarrow{p} U \xrightarrow{i_U} \bullet U'$$

is a quasi-standard smooth map. So for every $v \in \bullet V$, there exist an open connected neighborhood \tilde{V} of $\circ v$ in V , an open subset W of some Euclidean space, a fixed point $w \in \bullet W$ and a smooth map $f : W \times \tilde{V} \rightarrow \mathbb{R}^n$ with $n = \dim(U)$, such that $i_U(p(x)) = \bullet f(w, x)$ for every $x \in \bullet \tilde{V}$. So

$$\begin{aligned}
 p(x) &= ev_0(i_U(p(x))) \\
 &= ev_0(\bullet f(w, x)) \\
 &= f(\circ w, \circ x) \in U
 \end{aligned}$$

for every $x \in \bullet\tilde{V}$. In other words, $\bullet f(w, x) = f(\circ w, \circ x) \in U$ for the fixed $w \in \bullet W$ and every $x \in \bullet\tilde{V}$. Therefore, by Taylor's expansion of f , $\frac{\partial f}{\partial x_i}(\circ w, \circ x) = 0$ for every variable x_i , which implies that $f(\circ w, \circ x)$ is a constant independent of $\circ x \in \tilde{V}$ since \tilde{V} is connected. That is, every section of $(U \leq \bullet U')$ is locally constant, so $(U \leq \bullet U')$ is discrete in $\mathbf{CSh}(\mathcal{F}')$.

We summarize the above discussion as the following proposition:

4.25. PROPOSITION. *Let U and V be open subsets of Euclidean spaces. Then the set $|\bullet C^\infty(\bullet V, (U \leq \bullet U))|$ consists of only locally constant maps.*

Here are some easy corollaries. Note that if U is an open subset of a Euclidean space of positive dimension, then as a representing concrete sheaf in $\bullet C^\infty$, it is exactly $(U \leq \bullet U)$. (Indeed, by definition, for any object $A \subseteq \bullet \mathbb{R}^n$ in \mathcal{F} , as a representing concrete sheaf in $\bullet C^\infty$, it is exactly $(A \leq \bullet \mathbb{R}^n)$.) Therefore, by Proposition 4.25, the map $ev_0 : \bullet U \rightarrow U$ is not quasi-standard smooth. We also have:

4.26. COROLLARY. *Let U be an open subset of some Euclidean space. Then $\bullet(U \leq \bullet U)$ is the set U with the discrete diffeology.*

Therefore, $\bullet(\bullet Y)$ is not necessarily isomorphic to Y in $\bullet C^\infty$ for a general Fermat space Y .

4.27. CALCULATIONS. In this subsection, we will do a few calculations for $\bullet X$ and $\bullet Y$ for diffeological space X and Fermat space Y .

Here is the general situation we will meet frequently, in both \mathbf{Diff} and $\bullet C^\infty$:

4.28. THEOREM. *Let \mathcal{A} be a concrete site, let \mathcal{I} be a small category, let \mathcal{J} be a subcategory of \mathcal{I} with the inclusion $\ell : \mathcal{J} \hookrightarrow \mathcal{I}$, and let $F : \mathcal{I} \rightarrow \mathbf{CSh}(\mathcal{A})$ be a functor. Then the natural map $\mathbf{colim}_{\mathcal{J}}(F \circ \ell) \rightarrow \mathbf{colim}_{\mathcal{I}} F$ is an isomorphism in $\mathbf{CSh}(\mathcal{A})$ if the following conditions hold:*

(1) *for any object i in \mathcal{I} and for any section $c : A \rightarrow F(i)$, there exists a cover $\{c_\lambda : A_\lambda \rightarrow A\}_{\lambda \in \Lambda}$ of A such that for each λ , there exist an object j in \mathcal{J} and a section $d_\lambda : A_\lambda \rightarrow F(j)$ making the following diagram commutative:*

$$\begin{array}{ccc} A_\lambda & \xrightarrow{c \circ c_\lambda} & F(i) \longrightarrow \mathbf{colim}_{\mathcal{I}} F \\ d_\lambda \downarrow & & \uparrow \\ F(j) & \longrightarrow & \mathbf{colim}_{\mathcal{J}}(F \circ \ell); \end{array}$$

(2) *it induces an injective set map $|\mathbf{colim}_{\mathcal{J}}(F \circ \ell)| \rightarrow |\mathbf{colim}_{\mathcal{I}} F|$.*

PROOF. Recall that a colimit in a category of concrete sheaves is the colifting of the corresponding colimit in \mathbf{Set} . Condition (1) means that the map $|\mathbf{colim}_{\mathcal{J}}(F \circ \ell)| \rightarrow |\mathbf{colim}_{\mathcal{I}} F|$ is surjective, so together with Condition (2), this map is a bijection. Then use Condition (1) again, it is easy to see that the inverse map $\mathbf{colim}_{\mathcal{I}} F \rightarrow \mathbf{colim}_{\mathcal{J}}(F \circ \ell)$ is also a morphism in $\mathbf{CSh}(\mathcal{A})$. \blacksquare

The hard part of applying this theorem is to check Condition (2). We will make it more explicit in the following cases:

4.28.1. CALCULATIONS OF $\bullet X$. In Subsection 4.1, we defined $\bullet X = \mathbf{colim}_{U \in \mathcal{S}/X} \bullet U$ for every diffeological space X . One can use this definition to show that if X is a discrete diffeological space, then $\bullet X$ is a discrete Fermat space with $|\bullet X| = |X|$. But in general, the plot category \mathcal{S}/X is huge. We need to find a more efficient way to calculate $\bullet X$.

In many examples, the diffeological space is given as a colimit of Euclidean spaces over a small subcategory of its plot category. The following proposition tells us when we can use this colimit to calculate the corresponding Fermat space:

4.29. PROPOSITION. *Let X be a diffeological space, and let \mathcal{B} be a subcategory of the plot category \mathcal{S}/X . Assume $X = \mathbf{colim}_{U \in \mathcal{B}} U$. Then the natural Fermat map $\mathbf{colim}_{U \in \mathcal{B}} \bullet U \rightarrow \bullet X$ is surjective. If it is also injective, then it is an isomorphism in $\bullet C^\infty$.*

PROOF. This is an easy corollary of Theorem 4.28. ■

To apply this proposition to calculate $\bullet X$, the key part is to check the injectivity of the natural map $\mathbf{colim}_{U \in \mathcal{B}} \bullet U \rightarrow \bullet X$. Injectivity is equivalent to the condition that for any plots $p : U \rightarrow X$ and $q : U' \rightarrow X$ in \mathcal{B} , and any points $u \in \bullet U$ and $u' \in \bullet U'$, if there exist plots $V_1 \rightarrow X, \dots, V_n \rightarrow X$, points $v_1 \in \bullet V_1, \dots, v_n \in \bullet V_n$, and zig-zag morphisms among these plots together with p and q in \mathcal{S}/X such that the Fermat extension of the zig-zag connects u and u' via these v_i 's, (by applying ev_0 , this implies that ${}^\circ u$ and ${}^\circ(u')$ represent the same point in X), then there exist plots $U_1 \rightarrow X, \dots, U_m \rightarrow X$ for some $m \in \mathbb{N}$, points $u_1 \in \bullet U_1, \dots, u_m \in \bullet U_m$, and zig-zag morphisms among these plots together with p and q in \mathcal{B} such that the Fermat extension of this zig-zag connects u and u' via these u_j 's. We will use this description to calculate the following examples, and from these examples, we abstract some general results.

4.30. EXAMPLE.

1. Let M be a smooth manifold, and let $\{(U_i, \phi_i)\}_{i \in I}$ be a smooth atlas. Then we can construct a category \mathcal{I} with objects finite subsets of I and morphisms inclusion maps. There is a canonical functor $\mathcal{I}^{op} \rightarrow \mathbf{Diff}$ sending a finite subset $\{i_1, \dots, i_n\} \subseteq I$ to $U_{i_1} \cap \dots \cap U_{i_n}$, and sending the inclusion map to the corresponding inclusion map. It is easy to see that M is the colimit of this functor, so we write $M = \mathbf{colim}_{i \in \mathcal{I}^{op}} U_i$. One can also check that the injectivity of Proposition 4.29 holds by the definition of a smooth atlas on a smooth manifold, and hence $\bullet M = \mathbf{colim}_{i \in \mathcal{I}^{op}} \bullet U_i$.

2. Let X be the pushout of

$$\mathbb{R} \xleftarrow{0} \mathbb{R}^0 \xrightarrow{0} \mathbb{R}$$

in \mathbf{Diff} , i.e., X is two real lines glued at the origin. One can show that the injectivity of Proposition 4.29 holds, and hence $\bullet X$ is the pushout of

$$\bullet \mathbb{R} \xleftarrow{0} \bullet \mathbb{R}^0 \xrightarrow{0} \bullet \mathbb{R}$$

in $\bullet C^\infty$, i.e., $\bullet X$ is two Fermat reals glued at the origin.

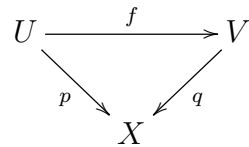
- Let V be a fine diffeological vector space (see [10, Chapter 3]), and let \mathcal{I} be the poset with objects finite-dimensional linear subspaces of V and morphisms inclusions. Then by [10, 3.8], it is easy to see that $V = \mathbf{colim}_{W \in \mathcal{I}} W$ and that the injectivity of Proposition 4.29 holds, which implies that $\bullet V = \mathbf{colim}_{W \in \mathcal{I}} \bullet W$.

Here is a general result from these three examples:

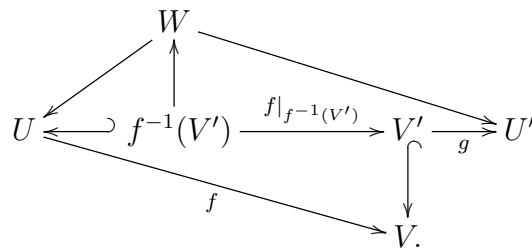
4.31. PROPOSITION. *Let \mathcal{B} be a subcategory of the plot category \mathcal{S}/X over a diffeological space X . Assume that every object $U \rightarrow X$ in \mathcal{B} is an injective map such that the pullback diffeology on U coincides with the standard diffeology, and for any objects $p : U \rightarrow X$ and $q : V \rightarrow X$ with $p(U) \cap q(V) \neq \emptyset$, there exist an object $r : W \rightarrow X$ with $r(W) = p(U) \cap q(V)$ and morphisms $r \rightarrow p$ and $r \rightarrow q$ in \mathcal{B} . Moreover, if $X = \mathbf{colim}_{U \in \mathcal{B}} U$, then $\bullet X = \mathbf{colim}_{U \in \mathcal{B}} \bullet U$.*

PROOF. By Proposition 4.29, we are left to check the injectivity of the natural map $\mathbf{colim}_{U \in \mathcal{B}} \bullet U \rightarrow \bullet X$. We split zig-zag diagrams into two kinds of pieces, and study them separately:

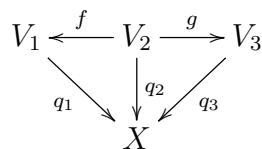
- Assume that we have a commutative triangle in **Diff**:



with p an object in \mathcal{B} , q a plot and f a smooth map, and $u \in \bullet U$ and $v \in \bullet V$ are fixed points such that $\bullet f(u) = v$. Since $X = \mathbf{colim}_{U \in \mathcal{B}} U$, there exist an open neighborhood V' of $\circ v$ in V , an object $r : U' \rightarrow X$ in \mathcal{B} and a smooth map $g : V' \rightarrow U'$ such that $r \circ g = q|_{V'}$. Therefore, $p(U) \cap r(U') \neq \emptyset$. By the assumption of the proposition, there exist an object $s : W \rightarrow X$ with $s(W) = p(U) \cap r(U')$ and morphisms $s \rightarrow p$ and $s \rightarrow r$ in \mathcal{B} . So eventually we have the following commutative diagram in **Diff**:



- Assume that we have a commutative diagram in **Diff**:



with q_1, q_2, q_3 plots and f, g smooth maps, and $v_1 \in \bullet V_1, v_2 \in \bullet V_2, v_3 \in \bullet V_3$ are fixed points such that $\bullet f(v_2) = v_1$ and $\bullet g(v_2) = v_3$. Since $X = \mathbf{colim}_{U \in \mathcal{B}} U$, there exist open neighborhoods V'_1 and V'_3 of $\circ v_1$ and $\circ v_3$ in V_1 and V_3 , respectively, objects $p_1 : U_1 \rightarrow X$ and $p_3 : U_3 \rightarrow X$ in \mathcal{B} and smooth maps $h_1 : V'_1 \rightarrow U_1$ and $h_3 : V'_3 \rightarrow U_3$ such that $p_1 \circ h_1 = q_1|_{V'_1}$ and $p_3 \circ h_3 = q_3|_{V'_3}$. Write $V'_2 := f^{-1}(V'_1) \cap g^{-1}(V'_3)$. It is clear that $p_1(U_1) \cap p_3(U_3) \neq \emptyset$. By the assumption of the proposition, there exist an object $r : W \rightarrow X$ with $r(W) = p_1(U_1) \cap p_3(U_3)$ and morphisms $r \rightarrow p_1$ and $r \rightarrow p_3$ in \mathcal{B} . So eventually we have the following commutative diagram in **Diff**:

$$\begin{array}{ccccc}
 U_1 & \longleftarrow & W & \longrightarrow & U_3 \\
 \uparrow h_1 & & \uparrow & & \uparrow h_3 \\
 V'_1 & \longleftarrow & V'_2 & \longrightarrow & V'_3 \\
 \downarrow & & \downarrow & & \downarrow \\
 V_1 & \xleftarrow{f} & V_2 & \xrightarrow{g} & V_3.
 \end{array}$$

From these diagrams, we know that the natural map $\mathbf{colim}_{U \in \mathcal{B}} \bullet U \rightarrow \bullet X$ is injective. ■

4.32. **EXAMPLE.** Let X be the 1-dimensional irrational torus of slope θ , i.e., X is the quotient group $\mathbb{R}/(\mathbb{Z} + \theta\mathbb{Z})$ with the quotient diffeology, where θ is a fixed irrational number. Let \mathcal{J} be the category associated to the additive group $\mathbb{Z} + \theta\mathbb{Z}$, i.e., \mathcal{J} has one object, the morphisms in \mathcal{J} are indexed by the set $\mathbb{Z} + \theta\mathbb{Z}$, and the composition corresponds to the addition in the additive group $\mathbb{Z} + \theta\mathbb{Z}$. There is a functor $\mathcal{J} \rightarrow \mathbf{Diff}$ sending $a + \theta b : \cdot \rightarrow \cdot$ to $\mathbb{R} \rightarrow \mathbb{R}$ with $x \mapsto x + (a + \theta b)$. It is straightforward to check that the colimit of this functor is X . Since the projection $\pi : \mathbb{R} \rightarrow X$ is a diffeological covering (see [10, Chapter 8]), one can show that the injectivity of Proposition 4.29 holds, and hence $\bullet X = \mathbf{colim}_{\mathcal{J}} \bullet \mathbb{R}$, or more precisely, $\bullet X$ is the quotient group $\bullet \mathbb{R}/(\mathbb{Z} + \theta\mathbb{Z})$.

However, the adding infinitesimal functor does not always preserve colimits. That is why the calculation of $\bullet X$ is not easy. In particular, this implies that the adding infinitesimal functor does not have a right adjoint.

4.33. **EXAMPLE.** Let R be the category associated to the additive group \mathbb{R} , and let $F : R \rightarrow \mathbf{Diff}$ be the functor sending the object in R to \mathbb{R} and sending the morphism $r \in \mathbb{R}$ to the translation $\mathbb{R} \rightarrow \mathbb{R}$ by $x \mapsto x + r$. Then $\mathbf{colim} F = \mathbb{R}^0$. One can easily check that $|\mathbf{colim}(\bullet(-) \circ F)| = |D_\infty|$, since there are only translations by reals, but $|\bullet(\mathbf{colim} F)| = |\bullet \mathbb{R}^0| = |\mathbb{R}^0|$. Therefore, $\mathbf{colim}(\bullet(-) \circ F) \neq \bullet(\mathbf{colim} F)$.

4.33.1. **CALCULATIONS OF $\bullet Y$.** In previous subsections, we have already calculated some examples of $\bullet Y$, where Y is a Fermat space: in Subsection 4.14, we showed that $\bullet(\bullet X) = X$ and $\bullet(\bullet C^\infty(\bullet X, \bullet Z)) = \mathbf{Diff}(X, Z)$ for any diffeological spaces X and Z , and in Subsection 4.20, we showed that $\bullet(U \leq \bullet U)$ is a discrete Fermat space for any open subset U of a Euclidean space. We will calculate one more example below, which will be useful for defining tangent spaces and tangent bundles for Fermat spaces and diffeological spaces (which is different from the approaches presented in [2]) in a future paper.

4.34. EXAMPLE. Let A be an ideal of $\bullet\mathbb{R}$. Then $A \subseteq D_\infty$. (See Section 2 for all possible expressions of A , which are not needed in this example.) Let Y be the quotient ring $\bullet\mathbb{R}/A$, equipped with the quotient Fermat space structure from $\bullet\mathbb{R}$. Then $\bullet Y$ is diffeomorphic to \mathbb{R} . Here is the proof. Note that the quotient map $\pi : \bullet\mathbb{R} \rightarrow Y$ induces a smooth map $\bullet\pi : \mathbb{R} \rightarrow \bullet Y$. By Theorem 4.28, we are left to show that $\bullet\pi$ is injective. Assume that $x, y \in \mathbb{R}$ are mapped to the same point in $\bullet Y$. So there is a zig-zag diagram in \mathcal{F}'/Y connecting two copies of $\pi : \bullet\mathbb{R} \rightarrow Y$, such that after applying the deleting infinitesimal functor on the zig-zag, there is a fixed point on U for each $\bullet U \rightarrow Y$ in the original zig-zag so that x and y gets connected by the new zig-zag via these points. We break the original zig-zag into small pieces as follows and study them to get information of the new zig-zag on the corresponding small pieces:

(1) Assume that we have a commutative triangle

$$\begin{array}{ccc}
 \bullet\mathbb{R} & \xrightarrow{f} & \bullet U \\
 & \searrow \pi & \swarrow p \\
 & & Y
 \end{array}$$

in \mathcal{F}'/Y , and points $x \in \mathbb{R}$ and $u \in U$ such that $\bullet f(x) = u$. Since Y is a quotient Fermat space of $\bullet\mathbb{R}$, there is a Fermat open neighborhood $\bullet V$ of $f(x)$ in $\bullet U$ and a quasi-standard smooth map $g : \bullet V \rightarrow \bullet\mathbb{R}$ so that $p|_{\bullet V} = \pi \circ g$. So we get a commutative square

$$\begin{array}{ccc}
 & \bullet W & \\
 & \swarrow & \searrow \\
 \bullet\mathbb{R} & & \bullet\mathbb{R} \\
 & \searrow \pi & \swarrow \pi \\
 & & Y
 \end{array}$$

where $\bullet W$ is a Fermat open neighborhood of x in $f^{-1}(\bullet V)$. We will deal with this situation in (2).

(2) Assume that we have a commutative square

$$\begin{array}{ccc}
 & \bullet U & \\
 & \swarrow f & \searrow g \\
 \bullet\mathbb{R} & & \bullet V \\
 & \searrow \pi & \swarrow p \\
 & & Y
 \end{array}$$

in \mathcal{F}'/Y , and points $x \in \mathbb{R}$, $u \in U$ and $v \in V$ such that $\bullet f(u) = x$ and $\bullet g(u) = v$. By a similar argument as (1), we may assume that $\bullet V = \bullet\mathbb{R}$ and $p = \pi$. By the commutativity of the square, we have $f(u) - g(u) \in A$, which implies that $x = \bullet f(u) = \bullet g(u) = v$.

Together with these, we can conclude that $x = y \in \mathbb{R}$, i.e., the map $\bullet\pi : \mathbb{R} \rightarrow \bullet Y$ is injective, and hence a diffeomorphism.

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