

FROBENIUS ALGEBRAS AND HOMOTOPY FIXED POINTS OF GROUP ACTIONS ON BICATEGORIES

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ABSTRACT. We explicitly show that symmetric Frobenius structures on a finite-dimensional, semi-simple algebra stand in bijection to homotopy fixed points of the trivial $SO(2)$ -action on the bicategory of finite-dimensional, semi-simple algebras, bimodules and intertwiners. The results are motivated by the 2-dimensional Cobordism Hypothesis for oriented manifolds, and can hence be interpreted in the realm of Topological Quantum Field Theory.

1. Introduction

While fixed points of a group action on a set form an ordinary subset, homotopy fixed points of a group action on a category as considered in [Kir02, EGNO15] provide additional structure.

In this paper, we take one more step on the categorical ladder by considering a topological group G as a 3-group via its fundamental 2-groupoid. We provide a detailed definition of an action of this 3-group on an arbitrary bicategory \mathcal{C} , and construct the bicategory of homotopy fixed points \mathcal{C}^G as a suitable limit of the action. Contrarily from the case of ordinary fixed points of group actions on sets, the bicategory of homotopy fixed points \mathcal{C}^G is strictly “larger” than the bicategory \mathcal{C} . Hence, the usual fixed-point condition is promoted from a property to a structure.

Our paper is motivated by the 2-dimensional Cobordism Hypothesis for oriented manifolds: according to [Lur09b], 2-dimensional oriented fully-extended topological quantum field theories are classified by homotopy fixed points of an $SO(2)$ -action on the core of fully-dualizable objects of the symmetric monoidal target bicategory. In case the target bicategory of a 2-dimensional oriented topological field theory is given by Alg_2 , the bicategory of algebras, bimodules and intertwiners, it is claimed in [FHLT10, Example 2.13] that the additional structure of a homotopy fixed point should be given by the structure of a symmetric Frobenius algebra.

As argued in [Lur09b], the $SO(2)$ -action on Alg_2 should come from rotating the 2-framings in the framed cobordism category. By [Dav11, Proposition 3.2.8], the induced action on the core of fully-dualizable objects of Alg_2 is actually trivializable. Hence,

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instead of considering the action coming from the framing, we may equivalently study the *trivial* $SO(2)$ -action on Alg_2^{fd} .

Our main result, namely Theorem 4.1, computes the bicategory of homotopy fixed points $\mathcal{C}^{SO(2)}$ of the trivial $SO(2)$ -action on an arbitrary bicategory \mathcal{C} . It follows then as a corollary that the bicategory $(\mathcal{K}(\text{Alg}_2^{\text{fd}}))^{SO(2)}$ consisting of homotopy fixed points of the trivial $SO(2)$ -action on the core of fully-dualizable objects of Alg_2 is equivalent to the bicategory Frob of semisimple symmetric Frobenius algebras, compatible Morita contexts, and intertwiners. This bicategory, or rather bigroupoid, classifies 2-dimensional oriented fully-extended topological quantum field theories, as shown in [SP09]. Thus, unlike fixed points of the trivial action on a set, homotopy fixed-points of the trivial $SO(2)$ -action on Alg_2 are actually interesting, and come equipped with the additional structure of a symmetric Frobenius algebra.

If Vect_2 is the bicategory of linear abelian categories, linear functors and natural transformations, we show in corollary 4.8 that the bicategory $(\mathcal{K}(\text{Vect}_2^{\text{fd}}))^{SO(2)}$ given by homotopy fixed points of the trivial $SO(2)$ -action on the core of the fully dualizable objects of Vect_2 is equivalent to the bicategory of Calabi-Yau categories, which we introduce in Definition 4.6.

The two results above are actually intimately related to each other via natural considerations from representation theory. Indeed, by assigning to a finite-dimensional, semi-simple algebra its category of finitely-generated modules, we obtain a functor $\text{Rep} : \mathcal{K}(\text{Alg}_2^{\text{fd}}) \rightarrow \mathcal{K}(\text{Vect}_2^{\text{fd}})$. This 2-functor turns out to be $SO(2)$ -equivariant, and thus induces a morphism on homotopy fixed point bicategories, which is moreover an equivalence. More precisely, one can show that a symmetric Frobenius algebra is sent by the induced functor to its category of representations equipped with the Calabi-Yau structure given by the composite of the Frobenius form and the Hattori-Stallings trace. These results have appeared in [Hes16].

The present paper is organized as follows: we recall the concept of Morita contexts between symmetric Frobenius algebras in section 2. Although most of the material has already appeared in [SP09], we give full definitions to mainly fix the notation. We give a very explicit description of compatible Morita contexts between finite-dimensional semi-simple Frobenius algebras not present in [SP09], which will be needed to relate the bicategory of symmetric Frobenius algebras and compatible Morita contexts to the bicategory of homotopy fixed points of the trivial $SO(2)$ -action. The expert reader might wish to at least take notice of the notion of a compatible Morita context between symmetric Frobenius algebras in definition 2.4 and the resulting bicategory Frob in definition 2.9.

In section 3, we recall the notion of a group action on a category and of its homotopy fixed points, which has been named “equivariantization” in [EGNO15, Chapter 2.7]. By categorifying this notion, we arrive at the definition of a group action on a bicategory and its homotopy fixed points. This definition is formulated in the language of tricategories. Since we prefer to work with bicategories, we explicitly spell out the definition in Remark 3.13.

In section 4, we compute the bicategory of homotopy fixed points of the trivial $SO(2)$ -

action on an arbitrary bicategory. Corollaries 4.3 and 4.8 then show equivalences of bicategories

$$\begin{aligned} (\mathcal{K}(\text{Alg}_2^{\text{fd}}))^{SO(2)} &\cong \text{Frob} \\ (\mathcal{K}(\text{Vect}_2^{\text{fd}}))^{SO(2)} &\cong \text{CY} \end{aligned} \tag{1.1}$$

where CY is the bicategory of Calabi-Yau categories. We note that the bicategory Frob has been proven to be equivalent [Dav11, Proposition 3.3.2] to a certain bicategory of 2-functors. We clarify the relationship between this functor bicategory and the bicategory of homotopy fixed points $(\mathcal{K}(\text{Alg}_2^{\text{fd}}))^{SO(2)}$ in Remark 4.4.

Throughout the paper, we use the following conventions: all algebras considered will be over an algebraically closed field \mathbb{K} . All Frobenius algebras appearing will be symmetric.

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2. Frobenius algebras and Morita contexts

In this section we will recall some basic notions regarding Morita contexts, mostly with the aim of setting up notations. We will mainly follow [SP09], though we point the reader to Remark 2.5 for a slight difference in the statement of the compatibility condition between Morita context and Frobenius forms.

2.1. DEFINITION. *Let A and B be two algebras. A Morita context \mathcal{M} consists of a quadruple $\mathcal{M} := ({}_B M_A, {}_A N_B, \varepsilon, \eta)$, where ${}_B M_A$ is a (B, A) -bimodule, ${}_A N_B$ is an (A, B) -bimodule, and*

$$\begin{aligned} \varepsilon : {}_A N \otimes_B M_A &\rightarrow {}_A A_A \\ \eta : {}_B B_B &\rightarrow {}_B M \otimes_A N_B \end{aligned} \tag{2.1}$$

are isomorphisms of bimodules, so that the two diagrams

$$\begin{array}{ccc} {}_B M \otimes_A N_B \otimes_B M_A & \xrightarrow{\text{id}_M \otimes \varepsilon} & {}_B M \otimes_A A_A \\ \eta \otimes \text{id}_M \uparrow & & \downarrow \\ {}_B B \otimes_B M_A & \longrightarrow & {}_B M_A \end{array} \tag{2.2}$$

$$\begin{array}{ccc}
 {}_A N \otimes_B M \otimes_A N_B & \xleftarrow{\text{id}_N \otimes \eta} & {}_A N \otimes_B B_B \\
 \downarrow \varepsilon \otimes \text{id}_N & & \downarrow \\
 {}_A A \otimes_A N_B & \longrightarrow & {}_A N_B
 \end{array} \tag{2.3}$$

commute.

Note that Morita contexts are the adjoint 1-equivalences in the bicategory Alg_2 of algebras, bimodules and intertwiners. These form a category, where the morphisms are given by the following:

2.2. DEFINITION. Let $\mathcal{M} := ({}_B M_A, {}_A N_B, \varepsilon, \eta)$ and $\mathcal{M}' := ({}_B M'_A, {}_A N'_B, \varepsilon', \eta')$ be two Morita contexts between two algebras A and B . A morphism of Morita contexts consists of a morphism of (B, A) -bimodules $f : M \rightarrow M'$ and a morphism of (A, B) -bimodules $g : N \rightarrow N'$, so that the two diagrams

$$\begin{array}{ccc}
 {}_B M \otimes_A N_B & \xrightarrow{f \otimes g} & {}_B M' \otimes_A N'_B \\
 \uparrow \eta & \nearrow \eta' & \\
 B & &
 \end{array}
 \qquad
 \begin{array}{ccc}
 {}_A N \otimes_B M_A & \xrightarrow{g \otimes f} & {}_A N' \otimes_B M'_A \\
 \downarrow \varepsilon & \nwarrow \varepsilon' & \\
 A & &
 \end{array} \tag{2.4}$$

commute.

If the algebras in question have the additional structure of a symmetric Frobenius form $\lambda : A \rightarrow \mathbb{K}$, we would like to formulate a compatibility condition between the Morita context and the Frobenius forms. We begin with the following two observations: if A is an algebra, the map

$$\begin{aligned}
 A/[A, A] &\rightarrow A \otimes_{A \otimes A^{\text{op}}} A \\
 [a] &\mapsto a \otimes 1
 \end{aligned} \tag{2.5}$$

is an isomorphism of vector spaces, with inverse given by $a \otimes b \mapsto [ab]$. Furthermore, if B is another algebra, and $({}_B M_A, {}_A N_B, \varepsilon, \eta)$ is a Morita context between A and B , there is a canonical isomorphism of vector spaces

$$\begin{aligned}
 \tau : (N \otimes_B M) \otimes_{A \otimes A^{\text{op}}} (N \otimes_B M) &\rightarrow (M \otimes_A N) \otimes_{B \otimes B^{\text{op}}} (M \otimes_A N) \\
 n \otimes m \otimes n' \otimes m' &\mapsto m \otimes n' \otimes m' \otimes n.
 \end{aligned} \tag{2.6}$$

Using the results above, we can formulate a compatibility condition between Morita context and Frobenius forms, as in the following lemma.

2.3. LEMMA. Let A and B be two algebras, and let $({}_B M_A, {}_A N_B, \varepsilon, \eta)$ be a Morita context between A and B . Then, there is a canonical isomorphism of vector spaces

$$\begin{aligned}
 f : A/[A, A] &\rightarrow B/[B, B] \\
 [a] &\mapsto \sum_{i,j} [\eta^{-1}(m_j \cdot a \otimes n_i)]
 \end{aligned} \tag{2.7}$$

where n_i and m_j are defined by

$$\varepsilon^{-1}(1_A) = \sum_{i,j} n_i \otimes m_j \in N \otimes_B M. \tag{2.8}$$

PROOF. Consider the following chain of isomorphisms:

$$\begin{aligned} f : A/[A, A] &\cong A \otimes_{A \otimes A^{\text{op}}} A && \text{(by equation 2.5)} \\ &\cong (N \otimes_B M) \otimes_{A \otimes A^{\text{op}}} (N \otimes_B M) && \text{(using } \varepsilon \otimes \varepsilon) \\ &\cong (M \otimes_A N) \otimes_{B \otimes B^{\text{op}}} (M \otimes_A N) && \text{(by equation 2.6)} \\ &\cong B \otimes_{B \otimes B^{\text{op}}} B && \text{(using } \eta \otimes \eta) \\ &\cong B/[B, B] && \text{(by equation 2.5)} \end{aligned} \tag{2.9}$$

Chasing through those isomorphisms, we can see that the map f is given by

$$f([a]) = \sum_{i,j} [\eta^{-1}(m_j \cdot a \otimes n_i)] \tag{2.10}$$

as claimed. ■

The isomorphism f described in Lemma 2.3 allows to introduce the following relevant definition.

2.4. DEFINITION. *Let (A, λ^A) and (B, λ^B) be two symmetric Frobenius algebras, and let $({}_B M_A, {}_A N_B, \varepsilon, \eta)$ be a Morita context between A and B . Since the Frobenius algebras are symmetric, the Frobenius forms necessarily factor through $A/[A, A]$ and $B/[B, B]$. We call the Morita context compatible with the Frobenius forms, if the diagram*

$$\begin{array}{ccc} A/[A, A] & \xrightarrow{f} & B/[B, B] \\ & \searrow \lambda^A & \swarrow \lambda^B \\ & \mathbb{K} & \end{array} \tag{2.11}$$

commutes.

2.5. REMARK. The definition of compatible Morita context of [SP09, Definition 3.72] requires another compatibility condition on the coproduct of the unit of the Frobenius algebras. However, a calculation using proposition 2.8 shows that the condition of [SP09] is already implied by our condition on Frobenius form of definition 2.4; thus the two definitions of compatible Morita context do coincide.

For later use, we give a very explicit way of expressing the compatibility condition between Morita context and Frobenius forms: if (A, λ^A) and (B, λ^B) are two finite-dimensional semi-simple symmetric Frobenius algebras over an algebraically closed field

\mathbb{K} , and $({}_B M_A, {}_B N_A, \varepsilon, \eta)$ is a Morita context between them, the algebras A and B are isomorphic to direct sums of matrix algebras by Artin-Wedderburn:

$$A \cong \bigoplus_{i=1}^r M_{d_i}(\mathbb{K}), \quad \text{and} \quad B \cong \bigoplus_{j=1}^r M_{n_j}(\mathbb{K}). \tag{2.12}$$

By Theorem 3.3.1 of [EGH⁺11], the simple modules (S_1, \dots, S_r) of A and the simple modules (T_1, \dots, T_r) of B are given by $S_i := \mathbb{K}^{d_i}$ and $T_i := \mathbb{K}^{n_i}$, and every module is a direct sum of copies of those. Since simple finite-dimensional representations of $A \otimes_{\mathbb{K}} B^{\text{op}}$ are given by tensor products of simple representations of A and B^{op} by Theorem 3.10.2 of [EGH⁺11], the most general form of ${}_B M_A$ and ${}_A N_B$ is given by

$$\begin{aligned} {}_B M_A &:= \bigoplus_{i,j=1}^r \alpha_{ij} T_i \otimes_{\mathbb{K}} S_j \\ {}_A N_B &:= \bigoplus_{k,l=1}^r \beta_{kl} S_k \otimes_{\mathbb{K}} T_l \end{aligned} \tag{2.13}$$

where α_{ij} and β_{kl} are multiplicities. First, we show that the multiplicities are trivial:

2.6. LEMMA. *In the situation as above, the multiplicities are trivial after a possible re-ordering of the simple modules: $\alpha_{ij} = \delta_{ij} = \beta_{ij}$ and the two bimodules M and N are actually given by*

$$\begin{aligned} {}_B M_A &= \bigoplus_{i=1}^r T_i \otimes_{\mathbb{K}} S_i \\ {}_A N_B &= \bigoplus_{j=1}^r S_j \otimes_{\mathbb{K}} T_j. \end{aligned} \tag{2.14}$$

PROOF. Suppose for a contradiction that there is a term of the form $(T_i \oplus T_j) \otimes S_k$ in the direct sum decomposition of M . Let $f : T_i \rightarrow T_j$ be a non-trivial linear map, and define $\varphi \in \text{End}_A((T_i \oplus T_j) \otimes S_k)$ by setting $\varphi((t_i + t_j) \otimes s_k) := f(t_i) \otimes s_k$. The A -module map φ induces an A -module endomorphism on all of ${}_A M_B$ by extending φ with zero on the rest of the direct summands. Since $\text{End}_A({}_B M_A) \cong B$ as algebras by Theorem 3.5 of [Bas68], the endomorphism φ must come from left multiplication, which cannot be true for an arbitrary linear map f . This shows that the bimodule M is given as claimed in equation (2.14). The statement for the other bimodule N follows analogously. ■

Lemma 2.6 shows how the bimodules underlying a Morita context of semi-simple algebras look like. Next, we consider the Frobenius structure.

2.7. LEMMA. [Koc03, Lemma 2.2.11] *Let (A, λ) be a symmetric Frobenius algebra. Then, every other symmetric Frobenius form on A is given by multiplying the Frobenius form with a central invertible element of A .*

By Lemma 2.7, we conclude that the Frobenius forms on the two semi-simple algebras A and B are given by

$$\lambda^A = \bigoplus_{i=1}^r \lambda_i^A \operatorname{tr}_{M_{d_i}(\mathbb{K})} \quad \text{and} \quad \lambda^B = \bigoplus_{i=1}^r \lambda_i^B \operatorname{tr}_{M_{n_i}(\mathbb{K})} \tag{2.15}$$

where λ_i^A and λ_i^B are non-zero scalars. We can now state the following proposition, which will be used in the proof of corollary 4.3.

2.8. PROPOSITION. *Let (A, λ^A) and (B, λ^B) be two finite-dimensional, semi-simple symmetric Frobenius algebras and suppose that $\mathcal{M} := (M, N, \varepsilon, \eta)$ is a Morita context between them. Let λ_i^A and λ_j^B be as in equation (2.15), and define two invertible central elements*

$$\begin{aligned} a &:= (\lambda_1^A, \dots, \lambda_r^A) \in \mathbb{K}^r \cong Z(A) \\ b &:= (\lambda_1^B, \dots, \lambda_r^B) \in \mathbb{K}^r \cong Z(B) \end{aligned} \tag{2.16}$$

Then, the following are equivalent:

1. The Morita context \mathcal{M} is compatible with the Frobenius forms in the sense of definition 2.4.
2. We have $m.a = b.m$ for all $m \in {}_B M_A$ and $n.b^{-1} = a^{-1}.n$ for all $n \in {}_A N_B$.
3. For every $i = 1, \dots, r$, we have that $\lambda_i^A = \lambda_i^B$.

PROOF. With the form of M and N determined by equation (2.14), we see that the only isomorphisms of bimodules $\varepsilon : N \otimes_B M \rightarrow A$ and $\eta : B \rightarrow M \otimes_A N$ must be given by multiples of the identity matrix on each direct summand:

$$\begin{aligned} \varepsilon : N \otimes_B M &\cong \bigoplus_{i=1}^r M(d_i \times d_i, \mathbb{K}) \rightarrow \bigoplus_{i=1}^r M(d_i \times d_i, \mathbb{K}) = A \\ &\sum_{i=1}^r M_i \mapsto \sum_{i=1}^r \varepsilon_i M_i \end{aligned} \tag{2.17}$$

Similarly, η is given by

$$\begin{aligned} \eta : B &= \bigoplus_{i=1}^r M(n_i \times n_i, \mathbb{K}) \mapsto M \otimes_A B \cong \bigoplus_{i=1}^r M(n_i \times n_i, \mathbb{K}) \\ &\sum_{i=1}^r M_i \mapsto \sum_{i=1}^r \eta_i M_i \end{aligned} \tag{2.18}$$

Here, ε_i and η_i are non-zero scalars. The condition that this data should be a Morita context then demands that $\varepsilon_i = \eta_i$, as a short calculation in a basis confirms. By calculating

the action of the elements a and b defined above in a basis, we see that conditions (2) and (3) of the above proposition are equivalent.

Next, we show that (1) and (3) are equivalent. In order to see when the Morita context is compatible with the Frobenius forms, we calculate the map $f : A/[A, A] \rightarrow B/[B, B]$ from equation (2.11). One way to do this is to notice that $[A, A]$ consists precisely of trace-zero matrices (cf. [AM57]); thus

$$\begin{aligned} A/[A, A] &\rightarrow \mathbb{K}^r \\ [A_1 \oplus A_2 \oplus \cdots \oplus A_r] &\mapsto (\text{tr}(A_1), \dots, \text{tr}(A_r)) \end{aligned} \tag{2.19}$$

is an isomorphism of vector spaces. Using this identification, we see that the map f is given by

$$\begin{aligned} f : A/[A, A] &\rightarrow B/[B, B] \\ [A_1 \oplus A_2 \oplus \cdots \oplus A_r] &\mapsto \bigoplus_{i=1}^r \text{tr}_{M_{d_i}}(A_i) \begin{bmatrix} E_{11}^{(n_i \times n_i)} \end{bmatrix} \end{aligned} \tag{2.20}$$

Note that this map is independent of the scalars ε_i and η_i coming from the Morita context. Now, the two Frobenius algebras A and B are Morita equivalent via a compatible Morita context if and only if the diagram in equation (2.11) commutes. This is the case if and only if $\lambda_i^A = \lambda_i^B$ for all i , as a straightforward calculation in a basis shows. ■

Having established how compatible Morita contexts between semi-simple algebras over an algebraic closed field look like, we arrive at following definition.

2.9. DEFINITION. *Let \mathbb{K} be an algebraically closed field. Let Frob be the bicategory where*

- *objects are given by finite-dimensional, semisimple, symmetric Frobenius \mathbb{K} -algebras,*
- *1-morphisms are given by compatible Morita contexts, as in definition 2.4,*
- *2-morphisms are given by isomorphisms of Morita contexts.*

Note that Frob has got the structure of a symmetric monoidal bigroupoid, where the monoidal product is given by the tensor product over the ground field, which is the monoidal unit.

The bicategory Frob will be relevant for the remainder of the paper, due to the following theorem.

2.10. THEOREM. [Oriented version of the Cobordism Hypothesis, [SP09]] *The weak 2-functor*

$$\begin{aligned} \text{Fun}_{\otimes}(\text{Cob}_{2,1,0}^{or}, \text{Alg}_2) &\rightarrow \text{Frob} \\ Z &\mapsto Z(+) \end{aligned} \tag{2.21}$$

is an equivalence of bicategories.

3. Group actions on bicategories and their homotopy fixed points

For a group G , we denote with BG the category with one object and G as morphisms. Similarly, if \mathcal{C} is a monoidal category, $B\mathcal{C}$ will denote the bicategory with one object and \mathcal{C} as endomorphism category of this object. Furthermore, we denote by \underline{G} the discrete monoidal category associated to G , i.e. the category with the elements of G as objects, only identity morphisms, and monoidal product given by group multiplication.

Recall that an action of a group G on a set X is a group homomorphism $\rho : G \rightarrow \text{Aut}(X)$. The set of fixed points X^G is then defined as the set of all elements of X which are invariant under the action. In equivalent, but more categorical terms, a G -action on a set X can be defined to be a functor $\rho : BG \rightarrow \text{Set}$ which sends the one object of the category BG to the set X .

If $\Delta : BG \rightarrow \text{Set}$ is the constant functor sending the one object of BG to the set with one element, one can check that the set of fixed points X^G stands in bijection to the set of natural transformations from the constant functor Δ to ρ , which is exactly the limit of the functor ρ . Thus, we have bijections of sets

$$X^G \cong \lim_{*/G} \rho \cong \text{Nat}(\Delta, \rho). \tag{3.1}$$

3.1. REMARK. A further equivalent way of providing a G -action on a set X is by giving a monoidal functor $\rho : \underline{G} \rightarrow \underline{\text{Aut}}(X)$, where we regard both G and $\text{Aut}(X)$ as categories with only identity morphisms. This definition however does not allow us to express the set of homotopy fixed points in a nice categorical way as in equation (3.1), and thus turns out to be less useful for our purposes.

Categorifying the notion of a G -action on a set yields the definition of a discrete group acting on a category:

3.2. DEFINITION. *Let G be a discrete group and let \mathcal{C} be a category. Let $B\underline{G}$ be the 2-category with one object and \underline{G} as the category of endomorphisms of the single object. A G -action on \mathcal{C} is defined to be a weak 2-functor $\rho : B\underline{G} \rightarrow \text{Cat}$ with $\rho(*) = \mathcal{C}$.*

Note that just as in remark 3.1, we could have avoided the language of 2-categories and have defined a G -action on a category \mathcal{C} to be a monoidal functor $\rho : \underline{G} \rightarrow \underline{\text{Aut}}(\mathcal{C})$.

Next, we would like to define the homotopy fixed point category of this action to be a suitable limit of the action, just as in equation (3.1). The appropriate notion of a limit of a weak 2-functor with values in a bicategory appears in the literature as a *pseudo-limit* or *indexed limit*, which we will simply denote by lim . We will only consider limits indexed by the constant functor. For background, we refer the reader to [Lac10], [Kel89], [Str80] and [Str87].

We are now in the position to introduce the following definition:

3.3. DEFINITION. *Let G be a discrete group, let \mathcal{C} be a category, and let $\rho : B\underline{G} \rightarrow \text{Cat}$ be a G -action on \mathcal{C} . Then, the category of homotopy fixed points \mathcal{C}^G is defined to be the pseudo-limit of ρ .*

Just as in the 1-categorical case in equation (3.1), it is shown in [Kel89] that the limit of any weak 2-functor with values in Cat is equivalent to the category of pseudo-natural transformations and modifications $\text{Nat}(\Delta, \rho)$. Hence, we have an equivalence of categories

$$\mathcal{C}^G \cong \lim \rho \cong \text{Nat}(\Delta, \rho). \tag{3.2}$$

Here, $\Delta : B\underline{G} \rightarrow \text{Cat}$ is the constant functor sending the one object of $B\underline{G}$ to the terminal category with one object and only the identity morphism. By spelling out definitions, one sees:

3.4. REMARK. Let $\rho : B\underline{G} \rightarrow \text{Cat}$ be a G -action on a category \mathcal{C} , and suppose that $\rho(e) = \text{id}_{\mathcal{C}}$, i.e. the action respects the unit strictly. Then, the homotopy fixed point category \mathcal{C}^G is equivalent to the “equivariantization” introduced in [EGNO15, Definition 2.7.2].

3.5. G-ACTIONS ON BICATEGORIES. Next, we would like to step up the categorical ladder once more, and define an action of a group G on a bicategory. Moreover, we would also like to account for the case where our group is equipped with a topology. This will be done by considering the fundamental 2-groupoid of G , referring the reader to [HKK01] for additional details.

3.6. DEFINITION. *Let G be a topological group. The fundamental 2-groupoid of G is the monoidal bicategory $\Pi_2(G)$ where*

- *objects are given by points of G ,*
- *1-morphisms are given by paths between points,*
- *2-morphisms are given by homotopy classes of homotopies between paths, called 2-tracks.*

The monoidal product of $\Pi_2(G)$ is given by the group multiplication on objects, by pointwise multiplication of paths on 1-morphisms, and by pointwise multiplication of 2-tracks on 2-morphisms. Notice that this monoidal product is associative on the nose, and all other monoidal structure like associators and unitors can be chosen to be trivial.

We are now ready to give a definition of a G -action on a bicategory. Although the definition we give uses the language of tricategories as defined in [GPS95] or [Gur07], we provide a bicategorical description in Remark 3.9.

3.7. DEFINITION. *Let G be a topological group, and let \mathcal{C} be a bicategory. A G -action on \mathcal{C} is defined to be a trifunctor*

$$\rho : B\Pi_2(G) \rightarrow \text{Bicat} \tag{3.3}$$

with $\rho() = \mathcal{C}$. Here, $B\Pi_2(G)$ is the tricategory with one object and with $\Pi_2(G)$ as endomorphism-bicategory, and Bicat is the tricategory of bicategories.*

3.8. REMARK. If \mathcal{C} is a bicategory, let $\text{Aut}(\mathcal{C})$ be the bicategory consisting of auto-equivalences of bicategories of \mathcal{C} , pseudo-natural isomorphisms and invertible modifications. Observe that $\text{Aut}(\mathcal{C})$ has the structure of a monoidal bicategory, where the monoidal product is given by composition. Since there are two ways to define the horizontal composition of pseudo-natural transformation, which are *not* equal to each other, there are actually two monoidal structures on $\text{Aut}(\mathcal{C})$. It turns out that these two monoidal structures are equivalent; see [GPS95, Section 5] for a discussion in the language of tricategories.

With either monoidal structure of $\text{Aut}(\mathcal{C})$ chosen, note that as in Remark 3.1 we could equivalently have defined a G -action on a bicategory \mathcal{C} to be a weak monoidal 2-functor $\rho : \Pi_2(G) \rightarrow \text{Aut}(\mathcal{C})$.

Since we will only consider trivial actions in this paper, the hasty reader may wish to skip the next remark, in which the definition of a G -action on a bicategory is unpacked. We will, however use the notation introduced here in our explicit description of homotopy fixed points in remark 3.13.

3.9. REMARK. [Unpacking Definition 3.7] Unpacking the definition of a weak monoidal 2-functor $\rho : \Pi_2(G) \rightarrow \text{Aut}(\mathcal{C})$, as for instance in [SP09, Definition 2.5], or equivalently of a trifunctor $\rho : B\Pi_2(G) \rightarrow \text{Bicat}$, as in [GPS95, Definition 3.1], shows that a G -action on a bicategory \mathcal{C} consists of the following data and conditions:

- For each group element $g \in G$, an equivalence of bicategories $F_g := \rho(g) : \mathcal{C} \rightarrow \mathcal{C}$,
- For each path $\gamma : g \rightarrow h$ between two group elements, the action assigns a pseudo-natural isomorphism $\rho(\gamma) : F_g \rightarrow F_h$,
- For each 2-track $m : \gamma \rightarrow \gamma'$, the action assigns an invertible modification $\rho(m) : \rho(\gamma) \rightarrow \rho(\gamma')$.
- There is additional data making ρ into a weak 2-functor, namely: if $\gamma_1 : g \rightarrow h$ and $\gamma_2 : h \rightarrow k$ are paths in G , we obtain invertible modifications

$$\phi_{\gamma_2\gamma_1} : \rho(\gamma_2) \circ \rho(\gamma_1) \rightarrow \rho(\gamma_2 \circ \gamma_1) \quad (3.4)$$

- Furthermore, for every $g \in G$ there is an invertible modification $\phi_g : \text{id}_{F_g} \rightarrow \rho(\text{id}_g)$ between the identity endotransformation on F_g and the value of ρ on the constant path id_g .

There are three compatibility conditions for this data: one condition making ϕ_{γ_2,γ_1} compatible with the associators of $\Pi_2(G)$ and $\text{Aut}(\mathcal{C})$, and two conditions with respect to the left and right unitors of $\Pi_2(G)$ and $\text{Aut}(\mathcal{C})$.

- Finally, there are data and conditions for ρ to be monoidal. These are:
 - A pseudo-natural isomorphism

$$\chi : \rho(g) \otimes \rho(h) \rightarrow \rho(g \otimes h) \quad (3.5)$$

- A pseudo-natural isomorphism

$$\iota : \text{id}_{\mathcal{C}} \rightarrow F_e \tag{3.6}$$

- For each triple (g, h, k) of group elements, an invertible modification ω_{ghk} in the diagram

$$\begin{array}{ccc}
 F_g \otimes F_h \otimes F_k & \xrightarrow{\chi_{gh} \otimes \text{id}} & F_{gh} \otimes F_k \\
 \text{id} \otimes \chi_{hk} \downarrow & \swarrow \omega_{ghk} & \downarrow \chi_{gh,k} \\
 F_g \otimes F_{hk} & \xrightarrow{\chi_{g,hk}} & F_{ghk}
 \end{array} \tag{3.7}$$

- An invertible modification γ in the triangle below

$$\begin{array}{ccc}
 & F_e \otimes F_g & \\
 \iota \otimes \text{id} \nearrow & \Downarrow \gamma & \searrow \chi_{e,g} \\
 \text{id}_{\mathcal{C}} \otimes F_g & \xrightarrow{\text{id}_{F_g}} & F_g
 \end{array} \tag{3.8}$$

- Another invertible modification δ in the triangle

$$\begin{array}{ccc}
 & F_g \otimes F_e & \\
 \text{id} \otimes \iota \nearrow & \Uparrow \delta & \searrow \chi_{g,e} \\
 F_g \otimes \text{id}_{\mathcal{C}} & \xrightarrow{\text{id}_{F_g}} & F_g
 \end{array} \tag{3.9}$$

The data $(\rho, \chi, \iota, \omega, \gamma, \delta)$ then has to obey equations (HTA1) and (HTA2) in [GPS95, p. 17].

Just as in the case of a group action on a set and a group action on a category, we would like to define the bicategory of homotopy fixed points of a group action on a bicategory as a suitable limit. However, the theory of trilimits is not very well established. Therefore we will take the description of homotopy fixed points as natural transformations as in equation (3.1) as a definition, and define homotopy fixed points of a group action on a bicategory as the bicategory of pseudo-natural transformations between the constant functor and the action.

3.10. DEFINITION. *Let G be a topological group and \mathcal{C} a bicategory. Let*

$$\rho : B\Pi_2(G) \rightarrow \text{Bicat} \tag{3.10}$$

be a G -action on \mathcal{C} . The bicategory of homotopy fixed points \mathcal{C}^G is defined to be

$$\mathcal{C}^G := \text{Nat}(\Delta, \rho) \tag{3.11}$$

Here, Δ is the constant functor which sends the one object of $B\Pi_2(G)$ to the terminal bicategory with one object, only the identity 1-morphism and only identity 2-morphism. The bicategory $\text{Nat}(\Delta, \rho)$ then has objects given by tritransformations $\Delta \rightarrow \rho$, 1-morphisms are given by modifications, and 2-morphisms are given by perturbations.

3.11. REMARK. The notion of the “equivariantization” of a strict 2-monad on a 2-category has already appeared in [MN14, Section 6.1]. Note that definition 3.10 is more general than the definition of [MN14], in which some modifications have been assumed to be trivial.

3.12. REMARK. In principle, even higher-categorical definitions are possible: for instance in [FV15] a homotopy fixed point of a higher character ρ of an ∞ -group is defined to be a (lax) morphism of ∞ -functors $\Delta \rightarrow \rho$.

3.13. REMARK. [Unpacking objects of \mathcal{C}^G] Since unpacking the definition of homotopy fixed points is not entirely trivial, we spell it out explicitly in the subsequent remarks, following [GPS95, Definition 3.3]. In the language of bicategories, a homotopy fixed point consists of:

- an object c of \mathcal{C} ,
- a pseudo-natural equivalence

$$\begin{array}{ccc}
 & \Delta_c & \\
 \text{\scriptsize $\Pi_2(G)$} \curvearrowright & \Downarrow \Theta & \curvearrowright \mathcal{C} \\
 & \text{\scriptsize $ev_c \circ \rho$} &
 \end{array} \tag{3.12}$$

where Δ_c is the constant functor which sends every object to $c \in \mathcal{C}$, and ev_c is the evaluation at the object c . In components, the pseudo-natural transformation Θ consists of the following:

- for every group element $g \in G$, a 1-equivalence in \mathcal{C}

$$\Theta_g : c \rightarrow F_g(c) \tag{3.13}$$

- and for each path $\gamma : g \rightarrow h$, an invertible 2-morphism Θ_γ in the diagram

$$\begin{array}{ccc}
 c & \xrightarrow{\Theta_g} & F_g(c) \\
 \text{id}_c \downarrow & \swarrow \Theta_\gamma & \downarrow \rho(\gamma)_c \\
 c & \xrightarrow{\Theta_h} & F_h(c)
 \end{array} \tag{3.14}$$

which is natural with respect to 2-tracks.

- an invertible modification Π in the diagram

(3.15)

$\Downarrow \Pi$

which in components means that for every tuple of group elements (g, h) we have an invertible 2-morphism Π_{gh} in the diagram

(3.16)

- for the unital structure, another invertible modification M , which only has the component given in the diagram

$$\begin{array}{ccc}
 & \Theta_e & \\
 c & \begin{array}{c} \curvearrowright \\ \downarrow M \\ \curvearrowleft \end{array} & F_e(c) \\
 & \iota_c &
 \end{array} \tag{3.17}$$

with ι as in equation (3.6). The data (c, Θ, Π, M) of a homotopy fixed point then has to obey the following three conditions. Using the equation in [GPS95, p.21-22] we find the condition

$$\begin{array}{ccccc}
 & & F_x F_y c & \xrightarrow{F_x F_y(\Theta_z)} & F_x F_y F_z c \\
 & \nearrow^{F_x(\Theta_y)} & & \searrow^{\chi_{xy}^c} & \\
 F_x c & & & & \\
 & \parallel^{\Pi_{xy}} & & & \\
 & & F_{xy} c & \xrightarrow{F_{xy}(\Theta_z)} & F_{xy} F_z c \\
 & \nwarrow_{\Theta_x} & \nearrow_{\Theta_{xy}} & \parallel_{\Pi_{xy,z}} & \nwarrow_{\chi_{xy,z}} \\
 & & c & \xrightarrow{\Theta_{xyz}} & F_{xyz} c
 \end{array} \cong$$

$$\begin{array}{ccccc}
 & & F_x F_y c & \xrightarrow{F_x F_y(\Theta_z)} & F_x F_y F_z c \\
 & \nearrow^{F_x(\Theta_y)} & & \searrow^{F_x(\Pi_{yz})} & \\
 F_x c & \xrightarrow{F_x(\Theta_{yz})} & F_x F_{yz} c & \xleftarrow{F_x(\chi_{yz}^c)} & F_x F_y F_z c \\
 & \nwarrow_{\Theta_x} & \nearrow_{\Pi_{x,yz}} & \xleftarrow{\omega_{xyz}} & \\
 & & c & \xrightarrow{\Theta_{xyz}} & F_{xyz} c
 \end{array} \tag{3.18}$$

whereas the equation on p.23 of [GPS95] demands that we have

$$\begin{array}{ccccc}
 & F_e c & \xrightarrow{F_e(\Theta_x)} & F_e F_x c & \\
 & \uparrow \Theta_e & & \searrow \chi_{ex} & \\
 c & \xrightarrow{\Theta_x} & F_x c & \xrightarrow{\text{id}_{F_x(c)}} & F_x c \\
 & & \downarrow \Pi_{ex} & & \\
 & & \Theta_x & & \\
 & & \cong & &
 \end{array}$$

||

(3.19)

$$\begin{array}{ccccc}
 & F_e c & \xrightarrow{F_e(\Theta_x)} & F_e F_x c & \\
 & \uparrow \Theta_e & & \searrow \chi_{ex} & \\
 c & \xrightarrow{\Theta_x} & F_x c & \xrightarrow{\text{id}_{F_x(c)}} & F_x c \\
 & & \downarrow \gamma & & \\
 & & \iota_{F_x(c)} & & \\
 & & \cong & &
 \end{array}$$

||

and finally the equation on p.25 of [GPS95] demands that

$$\begin{array}{ccc}
 \begin{array}{ccc}
 F_x c & \xrightarrow{F_x(\Theta_e)} & F_x F_e c \\
 \uparrow \Theta_x & & \downarrow \chi_{xe} \\
 c & \xrightarrow{\Theta_x} & F_x c \\
 & & \downarrow \Pi_{xe}
 \end{array} & = & \begin{array}{ccc}
 & \xrightarrow{F_x(\Theta_e)} & F_x F_e c \\
 & \downarrow F_x(M) & \\
 F_x c & \xrightarrow{F_x(\iota_c)} & F_x F_e c \\
 \uparrow \Theta_x & \searrow \text{id}_{F_x(c)} & \downarrow \chi_{xe} \\
 c & \xrightarrow{\Theta_x} & F_x c \\
 & & \downarrow \delta^{-1}
 \end{array}
 \end{array}$$

(3.20)

3.14. REMARK. Suppose that (c, Θ, Π, M) and (c', Θ', Π', M') are homotopy fixed points. A 1-morphism between these homotopy fixed points consists of a trimodification. In detail, this means:

- A 1-morphism $f : c \rightarrow c'$,
- An invertible modification m in the diagram

$$\begin{array}{ccc}
 \begin{array}{c}
 \Delta_c \\
 \curvearrowright \\
 \Pi_2(G) \xrightarrow{\text{ev}_c \circ \rho} \mathcal{C} \\
 \curvearrowleft \\
 \text{ev}_f * \text{id} \Downarrow \\
 \text{ev}_{c'} \circ \rho
 \end{array}
 & \xRightarrow{m} &
 \begin{array}{c}
 \Delta_c \\
 \curvearrowright \\
 \Pi_2(G) \xrightarrow{\Delta_{c'}} \mathcal{C} \\
 \curvearrowleft \\
 \Theta' \Downarrow \\
 \text{ev}_{c'} \circ \rho
 \end{array}
 \end{array} \tag{3.21}$$

In components, m_g is given by

$$\begin{array}{ccc}
 c & \xrightarrow{\Theta_g} & F_g(c) \\
 f \downarrow & \swarrow m_g & \downarrow F_g(f) \\
 c' & \xrightarrow{\Theta'_g} & F_g(c')
 \end{array} \tag{3.22}$$

The data (f, m) of a 1-morphism of homotopy fixed points has to satisfy the following two equations as on p.25 and p. 26 of [GPS95]:

$$\begin{array}{ccccccc}
 c & \xrightarrow{\Theta_g} & F_g(c) & \xrightarrow{F_g(\Theta_h)} & F_g(F_h(c)) & \xrightarrow{\chi_{gh}^c} & F_{gh}(c) \\
 \downarrow f & \searrow & & \downarrow \Pi_{gh} & & \nearrow & \downarrow F_{gh}(f) \\
 & & & \Theta_{gh} & & & \\
 & & & \downarrow m_{gh} & & & \\
 c' & \xrightarrow{\Theta'_g} & F_g(c') & \xrightarrow{F_g(\Theta'_h)} & F_g(F_h(c')) & \xrightarrow{\chi_{gh}^{c'}} & F_{gh}(c') \\
 & & & \downarrow \Pi'_{gh} & & & \\
 & & & \Theta'_{gh} & & & \\
 & & & \downarrow m'_g & & & \\
 & & & c' & & &
 \end{array}$$

||

(3.23)

$$\begin{array}{ccccccc}
 c & \xrightarrow{\Theta_g} & F_g(c) & \xrightarrow{F_g(\Theta_h)} & F_g(F_h(c)) & \xrightarrow{\chi_{gh}^c} & F_{gh}(c) \\
 \downarrow f & & \downarrow F_g(f) & \swarrow F_g(m_h) & \downarrow F_g(F_h(f)) & & \downarrow F_{gh}(f) \\
 & & F_g(c') & \xrightarrow{F_g(\Theta'_h)} & F_g(F_h(c')) & \cong & \\
 & \swarrow \Theta'_g & \leftarrow m_g & & \downarrow \chi_{gh}^{c'} & & \\
 c' & \xrightarrow{\Theta'_g} & F_g(c') & \xrightarrow{F_g(\Theta'_h)} & F_g(F_h(c')) & \xrightarrow{\chi_{gh}^{c'}} & F_{gh}(c') \\
 & & & \downarrow \Pi'_{gh} & & & \\
 & & & \Theta'_{gh} & & &
 \end{array}$$

whereas the second equation reads

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & \Theta_e & \\
 & \downarrow M & \\
 c & \xrightarrow{\iota_c} & F_e(c) \\
 \downarrow f & \swarrow \iota_f & \downarrow F_e(f) \\
 c' & \xrightarrow{\iota_{c'}} & F_e(c')
 \end{array} & = & \begin{array}{ccc}
 c & \xrightarrow{\Theta_e} & F_e(c) \\
 \downarrow f & \swarrow m_e & \downarrow F_e(f) \\
 c' & \xrightarrow{\Theta'_e} & F_e(c') \\
 & \downarrow M' & \\
 & \iota_{c'} &
 \end{array}
 \end{array}$$

(3.24)

3.15. REMARK. The condition saying that m , as introduced in equation (3.21), is a modification will be vital for the proof of Theorem 4.1 and states that for every path $\gamma : g \rightarrow h$ in G , we must have the following equality of 2-morphisms in the two diagrams:

$$\begin{array}{c}
 \begin{array}{ccccccc}
 c & \xrightarrow{\Theta_g} & F_g(c) & \xrightarrow{F_g(f)} & F_g(c') & \xrightarrow{\rho(\gamma)^{c'}} & F_h(c') \\
 & \searrow f & \Downarrow m_g & \nearrow \Theta'_g & \Downarrow \Theta'_\gamma & \nearrow \Theta'_h & \\
 & & c' & \xrightarrow{\text{id}_{c'}} & c' & & \\
 \text{id}_c & \searrow & & \cong & & \nearrow & \Theta'_h \\
 & & c & \xrightarrow{f} & c' & &
 \end{array} \\
 \parallel & & & & & & (3.25)
 \end{array}$$

$$\begin{array}{ccccccc}
 c & \xrightarrow{\Theta_g} & F_g(c) & \xrightarrow{F_g(f)} & F_g(c') & \xrightarrow{\rho(\gamma)^{c'}} & F_h(c') \\
 & & \downarrow \rho(\gamma)^c & & \Downarrow \rho(\gamma)_f^{-1} & \nearrow & \\
 & & F_h(c) & \xrightarrow{F_h(f)} & & & \\
 \text{id}_c & \searrow & \leftarrow \Theta_\gamma & & \downarrow m_h & \nearrow & \Theta'_h \\
 & & c & \xrightarrow{f} & c' & &
 \end{array}$$

Next, we come to 2-morphisms of the bicategory \mathcal{C}^G of homotopy fixed points:

3.16. REMARK. Let $(f, m), (\xi, n) : (c, \Theta, \Pi, M) \rightarrow (c', \Theta', \Pi', M')$ be two 1-morphisms of homotopy fixed points. A 2-morphism of homotopy fixed points consists of a perturbation between those trimodifications. In detail, a 2-morphism of homotopy fixed points consists of a 2-morphism $\alpha : f \rightarrow \xi$ in \mathcal{C} , so that

$$\begin{array}{ccc}
 \begin{array}{ccc}
 c & \xrightarrow{\Theta_g} & F_g(c) \\
 \downarrow f & \nearrow m_g & \downarrow F_g(f) \\
 c' & \xrightarrow{\Theta'_g} & F_g(c')
 \end{array} & = & \begin{array}{ccc}
 c & \xrightarrow{\Theta_g} & F_g(c) \\
 \downarrow \xi & \nearrow n_g & \downarrow F_g(\xi) \\
 c' & \xrightarrow{\Theta'_g} & F_g(c')
 \end{array} \\
 \leftarrow \alpha & & \leftarrow F_g(\alpha)
 \end{array} \tag{3.26}$$

Let us give an example of a group action on bicategories and its homotopy fixed points:

3.17. EXAMPLE. Let G be a discrete group, and let \mathcal{C} be any bicategory. Suppose $\rho : \Pi_2(G) \rightarrow \text{Aut}(\mathcal{C})$ is the trivial G -action. Then, by remark 3.13 a homotopy fixed point, i.e. an object of \mathcal{C}^G consists of

- an object c of \mathcal{C} ,
- a 1-equivalence $\Theta_g : c \rightarrow c$ for every $g \in G$,
- a 2-isomorphism $\Pi_{gh} : \Theta_h \circ \Theta_g \rightarrow \Theta_{gh}$,
- a 2-isomorphism $M : \Theta_e \rightarrow \text{id}_c$.

This is exactly the same data as a functor $B\underline{G} \rightarrow \mathcal{C}$, where $B\underline{G}$ is the bicategory with one object, G as morphisms, and only identity 2-morphisms. Extending this analysis to 1- and 2-morphisms of homotopy fixed points shows that we have an equivalence of bicategories

$$\mathcal{C}^G \cong \text{Fun}(B\underline{G}, \mathcal{C}). \tag{3.27}$$

When one specializes to $\mathcal{C} = \text{Vect}_2$, the functor bicategory $\text{Fun}(B\underline{G}, \mathcal{C})$ is also known as $\text{Rep}_2(G)$, the bicategory of 2-representations of G . Thus, we have an equivalence of bicategories $\text{Vect}_2^G \cong \text{Rep}_2(G)$. This result generalizes the 1-categorical statement that the homotopy fixed point 1-category of the trivial G -action on Vect is equivalent to $\text{Rep}(G)$, cf. [EGNO15, Example 4.15.2].

4. Homotopy fixed points of the trivial $SO(2)$ -action

We are now in the position to state and prove the main result of the present paper. Applying the description of homotopy fixed points in Remark 3.13 to the trivial action of the topological group $SO(2)$ on an arbitrary bicategory yields Theorem 4.1. Specifying the bicategory in question to be the core of the fully-dualizable objects of the Morita-bicategory Alg_2 then shows in corollary 4.3 that homotopy fixed points of the trivial $SO(2)$ -action on $\mathcal{K}(\text{Alg}_2^{\text{fd}})$ are given by symmetric, semi-simple Frobenius algebras.

4.1. THEOREM. *Let \mathcal{C} be a bicategory, and let $\rho : \Pi_2(SO(2)) \rightarrow \text{Aut}(\mathcal{C})$ be the trivial $SO(2)$ -action on \mathcal{C} . Then, the bicategory of homotopy fixed points $\mathcal{C}^{SO(2)}$ is equivalent to the bicategory where*

- *objects are given by pairs (c, λ) where c is an object of \mathcal{C} , and $\lambda : \text{id}_c \rightarrow \text{id}_c$ is a 2-isomorphism,*
- *1-morphisms $(c, \lambda) \rightarrow (c', \lambda')$ are given by 1-morphisms $f : c \rightarrow c'$ in \mathcal{C} , so that the*

diagram of 2-morphisms

$$\begin{array}{ccccc}
 f & \xrightarrow{\sim} & f \circ \text{id}_c & \xrightarrow{\text{id}_f * \lambda} & f \circ \text{id}_c \\
 \downarrow \wr & & & & \downarrow \wr \\
 \text{id}_{c'} \circ f & \xrightarrow{\lambda' * \text{id}_f} & \text{id}_{c'} \circ f & \xrightarrow{\sim} & f
 \end{array} \tag{4.1}$$

commutes, where $*$ denotes horizontal composition of 2-morphisms. The unlabeled arrows are induced by the canonical coherence isomorphisms of \mathcal{C} .

- 2-morphisms of \mathcal{C}^G are given by 2-morphisms $\alpha : f \rightarrow f'$ in \mathcal{C} .

PROOF. First, notice that we do not require any conditions on the 2-morphisms of $\mathcal{C}^{SO(2)}$. This is due to the fact that the action is trivial, and that $\pi_2(SO(2)) = 0$. Hence, all naturality conditions with respect to 2-morphisms in $\Pi_2(SO(2))$ are automatically fulfilled.

To start, we observe that the fundamental 2-groupoid $\Pi_2(SO(2))$ is equivalent to the bicategory consisting of only one object, \mathbb{Z} worth of morphisms, and only identity 2-morphisms which we denote by $B\mathbb{Z}$. Thus, it suffices to consider the homotopy fixed point bicategory of the trivial action $B\mathbb{Z} \rightarrow \text{Aut}(\mathcal{C})$. In this case, the definition of a homotopy fixed point as in 3.10 reduces to

- An object c of \mathcal{C} ,
- A 1-equivalence $\Theta := \Theta_* : c \rightarrow c$,
- For every $n \in \mathbb{Z}$, an invertible 2-morphism $\Theta_n : \text{id}_c \circ \Theta \rightarrow \Theta \circ \text{id}_c$. Since Θ is a pseudo-natural transformation, it is compatible with respect to composition of 1-morphisms in $B\mathbb{Z}$. Therefore, Θ_{n+m} is fully determined by Θ_n and Θ_m , cf. [SP09, Figure A.1] for the relevant commuting diagram. Thus, it suffices to specify Θ_1 .

By using the canonical coherence isomorphisms of \mathcal{C} , we see that instead of giving Θ_1 , we can equivalently specify an invertible 2-morphism

$$\tilde{\lambda} : \Theta \rightarrow \Theta. \tag{4.2}$$

which will be used below.

- A 2-isomorphism

$$\text{id}_c \circ \Theta \circ \Theta \rightarrow \Theta \tag{4.3}$$

which is equivalent to giving a 2-isomorphism

$$\Pi : \Theta \circ \Theta \rightarrow \Theta. \tag{4.4}$$

- A 2-isomorphism

$$M : \Theta \rightarrow \text{id}_c. \tag{4.5}$$

Note that equivalently to the 2-isomorphism $\tilde{\lambda}$, one can specify an invertible 2-isomorphism

$$\lambda : \text{id}_c \rightarrow \text{id}_c \tag{4.6}$$

where

$$\lambda := M \circ \tilde{\lambda} \circ M^{-1}. \tag{4.7}$$

with M as in equation (4.5). This data has to satisfy the following three equations: Equation (3.18) says that we must have

$$\Pi \circ (\text{id}_\Theta * \Pi) = \Pi \circ (\Pi * \text{id}_\Theta) \tag{4.8}$$

whereas equation (3.19) demands that Π equals the composition

$$\Theta \circ \Theta \xrightarrow{\text{id}_\Theta * M} \Theta \circ \text{id}_c \cong \Theta \tag{4.9}$$

and finally equation (3.20) tells us that Π must also be equal to the composition

$$\Theta \circ \Theta \xrightarrow{M * \text{id}_\Theta} \text{id}_c \circ \Theta \cong \Theta. \tag{4.10}$$

Hence Π is fully specified by M . An explicit calculation using the two equations above then confirms that equation (4.8) is automatically fulfilled. Indeed, by composing with Π^{-1} from the right, it suffices to show that $\text{id}_\Theta * \Pi = \Pi * \text{id}_\Theta$. Suppose for simplicity that \mathcal{C} is a strict 2-category. Then,

$$\begin{aligned} \text{id}_\Theta * \Pi &= \text{id}_\Theta * (M * \text{id}_\Theta) && \text{by equation (4.10)} \\ &= (\text{id}_\Theta * M) * \text{id}_\Theta && \\ &= \Pi * \text{id}_\Theta && \text{by equation (4.9)} \end{aligned} \tag{4.11}$$

Adding appropriate associators shows that this is true in a general bicategory.

If $(c, \Theta, \lambda, \Pi, M)$ and $(c', \Theta', \lambda', \Pi', M')$ are two homotopy fixed points, the definition of a 1-morphism of homotopy fixed points reduces to

- A 1-morphism $f : c \rightarrow c'$ in \mathcal{C} ,
- A 2-isomorphism $m : f \circ \Theta \rightarrow \Theta' \circ f$ in \mathcal{C}

satisfying two equations. The condition due to equation (3.24) demands that the following isomorphism

$$f \circ \Theta \xrightarrow{\text{id}_f * M} f \circ \text{id}_c \cong f \tag{4.12}$$

is equal to the isomorphism

$$f \circ \Theta \xrightarrow{m} \Theta' \circ f \xrightarrow{M' * \text{id}_f} \text{id}_{c'} \circ f \cong f \tag{4.13}$$

and thus is equivalent to the equation

$$m = \left(f \circ \Theta \xrightarrow{\text{id}_f * M} f \circ \text{id}_c \cong f \cong \text{id}_{c'} \circ f \xrightarrow{M'^{-1} * \text{id}_f} \Theta' \circ f \right). \tag{4.14}$$

Thus, m is fully determined by M and M' . The condition due to equation (3.23) reads

$$m \circ (\text{id}_f * \Pi) = (\Pi' * \text{id}_f) \circ (\text{id}_{\Theta'} * m) \circ (m * \text{id}_{\Theta}) \tag{4.15}$$

and is automatically satisfied, as an explicit calculation confirms. Indeed, if \mathcal{C} is a strict 2-category we have that

$$\begin{aligned} & (\Pi' * \text{id}_f) \circ (\text{id}_{\Theta'} * m) \circ (m * \text{id}_{\Theta}) \\ &= (\Pi' * \text{id}_f) \circ \left[\text{id}_{\Theta'} * (M'^{-1} * \text{id}_f \circ \text{id}_f * M) \right] \circ \left[(M'^{-1} * \text{id}_f \circ \text{id}_f * M) * \text{id}_{\Theta} \right] \\ &= (\Pi' * \text{id}_f) \circ (\text{id}_{\Theta'} * M'^{-1} * \text{id}_f) \circ (\text{id}_{\Theta'} * \text{id}_f * M) \\ &\quad \circ (M'^{-1} * \text{id}_f * \text{id}_{\Theta}) \circ (\text{id}_f * M * \text{id}_{\Theta}) \\ &= (\Pi' * \text{id}_f) \circ (\Pi'^{-1} * \text{id}_f) \circ (\text{id}_{\Theta'} * \text{id}_f * M) \circ (M'^{-1} * \text{id}_f * \text{id}_{\Theta}) \circ (\text{id}_f * \Pi) \\ &= (\text{id}_{\Theta'} * \text{id}_f * M) \circ (M'^{-1} * \text{id}_f * \text{id}_{\Theta}) \circ (\text{id}_f * \Pi) \\ &= (M^{-1} * \text{id}_f) \circ (\text{id}_f * M) \circ (\text{id}_f * \Pi) \\ &= m \circ (\text{id}_f * \Pi) \end{aligned}$$

as desired. Here, we have used equation (4.14) in the first and last line, and equations (4.9) and (4.10) in the third line. Adding associators shows this for an arbitrary bicategory.

The condition that m is a modification as spelled out in equation (3.25) demands that

$$(\tilde{\lambda}' * \text{id}_f) \circ m = m \circ (\text{id}_f * \tilde{\lambda}) \tag{4.16}$$

as equality of 2-morphisms between the two 1-morphisms

$$f \circ \Theta \rightarrow \Theta' \circ f. \tag{4.17}$$

Using equation (4.14) and replacing $\tilde{\lambda}$ by λ as in equation (4.7), we see that this requirement is equivalent to the commutativity of diagram (4.1).

If (f, m) and (g, n) are two 1-morphisms of homotopy fixed points, a 2-morphism of homotopy fixed points consists of a 2-morphisms $\alpha : f \rightarrow g$. The condition coming from equation (3.26) then demands that the diagram

$$\begin{array}{ccc} f \circ \Theta & \xrightarrow{m} & \Theta' \circ f \\ \alpha * \text{id}_{\Theta} \downarrow & & \downarrow \text{id}_{\Theta'} * \alpha \\ g \circ \Theta & \xrightarrow{n} & \Theta' \circ g \end{array} \tag{4.18}$$

commutes. Using the fact that both m and n are uniquely specified by M and M' , one quickly confirms that the diagram commutes automatically.

Our analysis shows that the forgetful functor U which forgets the data M , Θ and Π on objects, which forgets the data m on 1-morphisms, and which is the identity on 2-morphisms is an equivalence of bicategories. Indeed, let (c, λ) be an object in the strictified homotopy fixed point bicategory. Choose $\Theta := \text{id}_c$, $M := \text{id}_\Theta$ and Π as in equation (4.9). Then, $U(c, \Theta, M, \Pi, \lambda) = (c, \lambda)$. This shows that the forgetful functor is essentially surjective on objects. Since m is fully determined by M and M' , it is clear that the forgetful functor is essentially surjective on 1-morphisms. Since (4.18) commutes automatically, the forgetful functor is bijective on 2-morphisms and thus an equivalence of bicategories. ■

In the following, we specialise Theorem 4.1 to the case of symmetric Frobenius algebras and Calabi-Yau categories.

4.2. SYMMETRIC FROBENIUS ALGEBRAS AS HOMOTOPY FIXED POINTS. In order to state the next corollary, recall that the fully-dualizable objects of the Morita bicategory Alg_2 consisting of algebras, bimodules and intertwiners are precisely given by the finite-dimensional, semi-simple algebras [SP09]. Furthermore, recall that the core $\mathcal{K}(\mathcal{C})$ of a bicategory \mathcal{C} consists of all objects of \mathcal{C} , the 1-morphisms are given by 1-equivalences of \mathcal{C} , and the 2-morphisms are restricted to be isomorphisms.

4.3. COROLLARY. *Suppose $\mathcal{C} = \mathcal{K}(\text{Alg}_2^{\text{fd}})$, and consider the trivial $SO(2)$ -action on \mathcal{C} . Then $\mathcal{C}^{SO(2)}$ is equivalent to the bicategory of finite-dimensional, semi-simple symmetric Frobenius algebras Frob , as defined in definition 2.9. This implies a bijection of isomorphism-classes of symmetric, semi-simple Frobenius algebras and homotopy fixed points of the trivial $SO(2)$ -action on $\mathcal{K}(\text{Alg}_2^{\text{fd}})$.*

PROOF. Indeed, by Theorem 4.1, an object of $\mathcal{C}^{SO(2)}$ is given by a finite-dimensional semisimple algebra A , together with an isomorphism of Morita contexts $\text{id}_A \rightarrow \text{id}_A$. By definition, a morphism of Morita contexts consists of two intertwiners of (A, A) -bimodules $\lambda_1, \lambda_2 : A \rightarrow A$. The diagrams in definition 2.2 then require that $\lambda_1 = \lambda_2^{-1}$. Thus, λ_2 is fully determined by λ_1 . Let $\lambda := \lambda_1$. Since λ is an automorphism of (A, A) -bimodules, it is fully determined by $\lambda(1_A) \in Z(A)$. This gives A , by Lemma 2.7, the structure of a symmetric Frobenius algebra.

We analyze the 1-morphisms of $\mathcal{C}^{SO(2)}$ in a similar way: if (A, λ) and (A', λ') are finite-dimensional semi-simple symmetric Frobenius algebras, a 1-morphism in $\mathcal{C}^{SO(2)}$ consists of a Morita context $\mathcal{M} : A \rightarrow A'$ so that (4.1) commutes.

Suppose that $\mathcal{M} = ({}_A M_{A'}, {}_{A'} N_A, \varepsilon, \eta)$ is a Morita context, and let $a := \lambda(1_A)$ and $a' := \lambda'(1_{A'})$. Then, the condition that (4.1) commutes demands that

$$\begin{aligned} m.a &= a'.m \\ a^{-1}.n &= n.a'^{-1} \end{aligned} \tag{4.19}$$

for every $m \in M$ and every $n \in N$. By proposition 2.8 this condition is equivalent to the fact that the Morita context is compatible with the Frobenius forms as in definition 2.4.

It follows that the 2-morphisms of $\mathcal{C}^{SO(2)}$ and Frob are equal to each other, proving the result. ■

4.4. REMARK. In [Dav11, Proposition 3.3.2], the bigroupoid Frob of corollary 4.3 is shown to be equivalent to the bicategory of 2-functors $\text{Fun}(B^2\mathbb{Z}, \mathcal{K}(\text{Alg}_2^{\text{fd}}))$. Assuming a homotopy hypothesis for bigroupoids, as well as an equivariant homotopy hypothesis in a bicategorical framework, this bicategory of functors should agree with the bicategory of homotopy fixed points of the trivial $SO(2)$ -action on $\mathcal{K}(\text{Alg}_2^{\text{fd}})$ in corollary 4.3. Concretely, one might envision the following strategy for an alternative proof of corollary 4.3, which should roughly go as follows:

1. By [Dav11, Proposition 3.3.2], there is an equivalence of bigroupoids

$$\text{Frob} \cong \text{Fun}(B^2\mathbb{Z}, \mathcal{K}(\text{Alg}_2^{\text{fd}})).$$

2. Then, use the homotopy hypothesis for bigroupoids. By this, we mean that the fundamental 2-groupoid should induce an equivalence of tricategories

$$\Pi_2 : \text{Top}_{\leq 2} \rightarrow \text{BiGrp}. \tag{4.20}$$

Here, the right hand-side is the tricategory of bigroupoids, whereas the left hand side is a suitable tricategory of 2-types. Such an equivalence of tricategories induces an equivalence of bicategories

$$\text{Fun}(B^2\mathbb{Z}, \mathcal{K}(\text{Alg}_2^{\text{fd}})) \cong \Pi_2(\text{Hom}(BSO(2), X)), \tag{4.21}$$

where X is a 2-type representing the bigroupoid $\mathcal{K}(\text{Alg}_2^{\text{fd}})$.

3. Now, consider the trivial homotopy $SO(2)$ -action on the 2-type X . Using the fact that we work with the trivial $SO(2)$ -action, we obtain a homotopy equivalence $\text{Hom}(BSO(2), X) \cong X^{hSO(2)}$, cf. [Dav11, Page 50].
4. In order to identify the 2-type $X^{hSO(2)}$ with our definition of homotopy fixed points, we additionally need an equivariant homotopy hypothesis: namely, we need to use that a homotopy action of a topological group G on a 2-type Y is equivalent to a G -action on the bicategory $\Pi_2(Y)$ as in definition 3.7 of the present paper. Furthermore, we also need to assume that the fundamental 2-groupoid is G -equivariant, namely that there is an equivalence of bicategories $\Pi_2(Y^{hG}) \cong \Pi_2(Y)^G$. Using this equivariant homotopy hypothesis for the trivial $SO(2)$ -action on the 2-type X then should give an equivalence of bicategories

$$\Pi_2(X^{hSO(2)}) \cong \Pi_2(X)^{SO(2)} \cong (\mathcal{K}(\text{Alg}_2^{\text{fd}}))^{SO(2)}. \tag{4.22}$$

Combining all four steps gives an equivalence of bicategories between the bigroupoid of Frobenius algebras and homotopy fixed points:

$$\text{Frob} \underset{(1)}{\cong} \text{Fun}(B^2\mathbb{Z}, \mathcal{K}(\text{Alg}_2^{\text{fd}})) \underset{(2)}{\cong} \Pi_2(\text{Hom}(BSO(2), X)) \underset{(3)}{\cong} \Pi_2(X^{hSO(2)}) \underset{(4)}{\cong} (\mathcal{K}(\text{Alg}_2^{\text{fd}}))^{SO(2)}.$$

In order to turn this argument into a full proof, we would need to provide a proof of the homotopy hypothesis for bigroupoids in equation (4.20), as well as a proof for the equivariant homotopy hypothesis in equation (4.22). While the homotopy hypothesis as formulated in equation (4.20) is widely believed to be true, we are not aware of a proof of this statement in the literature. A step in this direction is [MS93], which proves that the homotopy categories of 2-types and 2-groupoids are equivalent. We however really need the full tricategorical version of this statement as in equation (4.20), since we need to identify the (higher) morphisms in BiGrp with (higher) homotopies. Notice that statements of this type are rather subtle, see [KV91, Sim98].

While certainly interesting and conceptually illuminating, a proof of the equivariant homotopy hypothesis in a bicategorical language in equation (4.22) is beyond the scope of the present paper, which aims to give an *algebraic* description of homotopy fixed points on bicategories. Although an equivariant homotopy hypothesis for ∞ -groupoids follows from [Lur09a, Theorem 4.2.4.1], we are not aware of a proof of the bicategorical statement in equation (4.22).

Next, we compute homotopy fixed points of the trivial $SO(2)$ -action on $\text{Vect}_2^{\text{fd}}$ and show that they are given by Calabi-Yau categories. This result is new and has not yet appeared in the literature.

4.5. CALABI-YAU CATEGORIES AS HOMOTOPY FIXED POINTS. We now apply Theorem 4.1 to Calabi-Yau categories, as considered in [MS06]. Let Vect_2 be the bicategory consisting of linear, abelian categories, linear functors, and natural transformations.

Recall that a \mathbb{K} -linear, abelian category \mathcal{C} is called finite, if it has finite-dimensional Hom-spaces, every object has got finite length, the category \mathcal{C} has got enough projectives, and there are only finitely many isomorphism classes of simple objects.

The fully-dualizable objects of Vect_2 are then precisely the finite, semi-simple linear categories, cf. [BDSV15, Appendix A]. For convenience, we recall the definition of a finite Calabi-Yau category.

4.6. DEFINITION. *Let \mathbb{K} be an algebraically closed field. A Calabi-Yau category $(\mathcal{C}, \text{tr}^{\mathcal{C}})$ is a \mathbb{K} -linear, finite, semi-simple category \mathcal{C} , together with a family of \mathbb{K} -linear maps*

$$\text{tr}_c^{\mathcal{C}} : \text{End}_{\mathcal{C}}(c) \rightarrow \mathbb{K} \tag{4.23}$$

for each object c of \mathcal{C} , so that:

1. for each $f \in \text{Hom}_{\mathcal{C}}(c, d)$ and for each $g \in \text{Hom}_{\mathcal{C}}(d, c)$, we have that

$$\text{tr}_c^{\mathcal{C}}(g \circ f) = \text{tr}_d^{\mathcal{C}}(f \circ g), \tag{4.24}$$

2. for each $f \in \text{End}_{\mathcal{C}}(x)$ and each $g \in \text{End}_{\mathcal{C}}(d)$, we have that

$$\text{tr}_{c \oplus d}^{\mathcal{C}}(f \oplus g) = \text{tr}_c^{\mathcal{C}}(f) + \text{tr}_d^{\mathcal{C}}(g), \tag{4.25}$$

3. for all objects c of \mathcal{C} , the induced pairing

$$\begin{aligned} \langle -, - \rangle_c : \mathrm{Hom}_{\mathcal{C}}(c, d) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathcal{C}}(d, c) &\rightarrow \mathbb{K} \\ f \otimes g &\mapsto \mathrm{tr}_c^{\mathcal{C}}(g \circ f) \end{aligned} \quad (4.26)$$

is a non-degenerate pairing of \mathbb{K} -vector spaces.

We will call the collection of morphisms $\mathrm{tr}_c^{\mathcal{C}}$ a trace on \mathcal{C} .

An equivalent way of defining a Calabi-Yau structure on a linear category \mathcal{C} is by specifying a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(c, d) \rightarrow \mathrm{Hom}_{\mathcal{C}}(d, c)^*, \quad (4.27)$$

cf. [Sch13, Proposition 4.1].

4.7. DEFINITION. Let $(\mathcal{C}, \mathrm{tr}^{\mathcal{C}})$ and $(\mathcal{D}, \mathrm{tr}^{\mathcal{D}})$ be two Calabi-Yau categories. A linear functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called a Calabi-Yau functor, if

$$\mathrm{tr}_c^{\mathcal{C}}(f) = \mathrm{tr}_{F(c)}^{\mathcal{D}}(F(f)) \quad (4.28)$$

for each $f \in \mathrm{End}_{\mathcal{C}}(c)$ and for each $c \in \mathrm{Ob}(\mathcal{C})$. Equivalently, one may require that

$$\langle Ff, Fg \rangle_{\mathcal{D}} = \langle f, g \rangle_c \quad (4.29)$$

for every pair of morphisms $f : c \rightarrow d$ and $g : d \rightarrow c$ in \mathcal{C} .

If $F, G : \mathcal{C} \rightarrow \mathcal{D}$ are two Calabi-Yau functors between Calabi-Yau categories, a Calabi-Yau natural transformation is just an ordinary natural transformation.

This allows us to define the symmetric monoidal bicategory CY consisting of Calabi-Yau categories, Calabi-Yau functors and natural transformations. The monoidal structure is given by the Deligne tensor product of abelian categories.

4.8. COROLLARY. Suppose $\mathcal{C} = \mathcal{K}(\mathrm{Vect}_2^{\mathrm{fd}})$, and consider the trivial $SO(2)$ -action on \mathcal{C} . Then $\mathcal{C}^{SO(2)}$ is equivalent to the bicategory of Calabi-Yau categories.

PROOF. Indeed, by Theorem 4.1 a homotopy fixed point consists of a category \mathcal{C} , together with a natural transformation $\lambda : \mathrm{id}_{\mathcal{C}} \rightarrow \mathrm{id}_{\mathcal{C}}$. Let X_1, \dots, X_n be the simple objects of \mathcal{C} . Then, the natural transformation $\lambda : \mathrm{id}_{\mathcal{C}} \rightarrow \mathrm{id}_{\mathcal{C}}$ is fully determined by giving an endomorphism $\lambda_X : X \rightarrow X$ for every simple object X . Since λ is an invertible natural transformation, the λ_X must be central invertible elements in $\mathrm{End}_{\mathcal{C}}(X)$. Since we work over an algebraically closed field, Schur's Lemma shows that $\mathrm{End}_{\mathcal{C}}(X) \cong \mathbb{K}$ as vector spaces. Hence, the structure of a natural transformation of the identity functor of \mathcal{C} boils down to choosing a non-zero scalar for each simple object of \mathcal{C} . This structure is equivalent to giving \mathcal{C} the structure of a Calabi-Yau category.

Now note that by equation (4.1) in Theorem 4.1, 1-morphisms of homotopy fixed points consist of equivalences of categories $F : \mathcal{C} \rightarrow \mathcal{C}'$ so that $F(\lambda_X) = \lambda'_{F(X)}$ for every object X of \mathcal{C} . This is exactly the condition saying that F must be a Calabi-Yau functor.

Finally, one can see that 2-morphisms of homotopy fixed points are given by natural isomorphisms of Calabi-Yau functors. \blacksquare

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