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A UNIFYING APPROACH TO THE ACYCLIC MODELS METHOD AND OTHER LIFTING LEMMAS

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Abstract. We prove a fundamental lemma of homological algebra and show how it sets a framework for many different lifting (or comparison) theorems of homological algebra and algebraic topology. Among these are different versions of the acyclic models method.

Introduction

Historical context. In [EM53], Eilenberg and Mac Lane introduced the method of acyclic models. This powerful tool was invented to compare the homology of different chain complex functors while avoiding lots of computations. Since then, several "acyclic models" theorem have been proved, as for example the one from Barr and Beck [BB66]. Even though both approaches can be used to prove some classical results of algebraic topology, it is not clear how they are logically related.

Goal of this note. In this short note, I will show how these different situations (and many others) can be subsumed in a simple unifying approach. Note that this note is self-contained and no prior knowledge of the acyclic models method is required. This paper is more about a fundamental lemma of homological algebra than anything else.

This note doesn't claim to prove any major new result. Actually, the lifting lemma at the heart of this paper can be found in [Bar02, Proposition 5.5, chapter 5]. However, it is used there only to prove a technical side result. The goal of this paper is to show that this lemma is central, as it can be used to recover many different lifting theorems (such as acyclic models theorems), hence the pompous name "general lifting lemma".

Perhaps one idea hidden behind this unifying approach is the following. In Barr and Beck's original form of acyclic models theorem [BB66], they are dealing with functors from a category $\mathcal{C}$ to the category of chain complexes (let's say of abelian groups). Such functors can also be considered as chain complexes in the abelian category $[\mathcal{C}, \mathrm{Ab}]$ of functors from $\mathcal{C}$ to the category of abelian groups. As it turns out, by slightly modifying Barr and Beck's theorem, we can totally forget that this was a category of functors and replace it with an arbitrary abelian category.

Taking this idea seriously, the "general lifting lemma" gives a very general criterion that works in any abelian category and from which all acyclic models theorems (and other

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lifting theorems) can be derived as particular cases.

Organization. The first section is devoted to the "general lifting lemma". The proof is very simple (yet carried out thoroughly) and readers familiar with the definition of chain complexes in an abelian category should be able to prove it themselves.

In section 3, 4 and 5 we show that many lifting theorems (including different acyclic models theorems) can be deduced directly from the "general lifting lemma".

In section 6 and 7 we give some ideas on how to improve the "general lifting lemma" in two opposite directions leading to refinements of some results presented in this paper.

1. General lifting lemma

1.1. Preliminaries. Let $\mathcal{A}$ be an abelian category. By *chain complex* in $\mathcal{A}$ we mean a bounded below $\mathbb{Z}$-graded object of $\mathcal{A}$ with a differential of degree -1. We denote by $\mathrm{C}(\mathcal{A})$ the category of chain complexes in $\mathcal{A}$ and chain complex maps (of degree 0). We denote by $\mathrm{Gr}(\mathcal{A})$ the category of $\mathbb{Z}$-graded objects in $\mathcal{A}$ and $U : \mathrm{C}(\mathcal{A}) \to \mathrm{Gr}(\mathcal{A})$ the obvious forgetful functor.

Note that any additive functor $F : \mathcal{A} \to \mathcal{B}$ induces a functor from $\mathrm{Gr}(\mathcal{A})$ to $\mathrm{Gr}(\mathcal{B})$ and from $\mathrm{C}(\mathcal{A})$ to $\mathrm{C}(\mathcal{B})$. We still write F to denote these functors.

For any $n \in \mathbb{Z}$, we define the *truncation functor* $\sigma_{\leq n} : \mathrm{C}(\mathcal{A}) \to \mathrm{C}(\mathcal{A})$ that sends a complex X to the complex

$$\sigma_{\leq n} X := \cdots \xleftarrow{\partial^X_{n-1}} X_{n-1} \xleftarrow{\partial^X_n} X_n \leftarrow 0 \leftarrow \cdots$$

and whose definition on morphisms is straightforward. Similarly, we define the truncation functor $\sigma_{\geq n}$.

We use the following convention: an *augmented chain complex* means a chain complex bounded below by degree -1:

$$\cdots \leftarrow 0 \leftarrow X_{-1} \xleftarrow{\partial^X_0} X_0 \xleftarrow{\partial^X_1} X_1 \leftarrow \cdots$$

Note that an augmented chain complex X could also be seen as a morphism of chain complex:

$$\partial^X_0 : \sigma_{\geq 0} X \to X_{-1},$$

where X_{-1} is seen as a chain complex concentrated in degree 0. We freely switch between both points of view throughout this paper.

A chain complex X is said to be *exact* or *acyclic* if all its homology groups $H_n(X)$ ($n \in \mathbb{Z}$) are trivial. Note that an augmented chain complex is exact if and only if the map

$$\partial^X_0 : \sigma_{\geq 0} X \to X_{-1},$$

is a quasi-isomorphism (i.e. becomes an isomorphism in homology).

An augmented chain complex X is said to be *contractible* if the map

$$\partial_0^X : \sigma_{\geq 0} X \to X_{-1}$$

is a homotopy equivalence. Note that it is equivalent to ask that the map id_X be homotopic to 0.

Lastly, we do not require that our abelian categories be locally small. The main reason for that is that we will sometimes want to work in the (abelian) category $[\mathcal{C}, \mathrm{Ab}]$ of functors from a (non necessarily small) category $\mathcal{C}$ to the category of abelian groups. This is the case of section 3, with $\mathcal{C} = \mathrm{Top}$.[1]

We now prove the main result of this paper.

1.2. Lemma. [General lifting lemma] *Let X and Y be two augmented chain complexes in an abelian category $\mathcal{A}$. Suppose that for every $n \geq 0$ the augmented chain complex of abelian groups*

$$0 \longleftarrow \mathrm{Hom}(X_n, Y_{-1}) \overset{\partial_0^Y}{\longleftarrow} \mathrm{Hom}(X_n, Y_0) \overset{\partial_1^Y}{\longleftarrow} \mathrm{Hom}(X_n, Y_1) \longleftarrow \cdots \qquad (1)$$

is exact.[2] *Then for every map $f_{-1} : X_{-1} \to Y_{-1}$ there exists a chain complex map $f : \sigma_{\geq 0} X \to \sigma_{\geq 0} Y$ such that*

$$\begin{array}{ccc} X_{-1} & \overset{\partial_0^X}{\longleftarrow} & \sigma_{\geq 0} X \\ \downarrow{\scriptstyle f_{-1}} & & \downarrow{\scriptstyle f} \\ Y_{-1} & \underset{\partial_0^Y}{\longleftarrow} & \sigma_{\geq 0} Y \end{array}$$

commutes.

Moreover, f is unique up to homotopy.

Proof. Existence part: We proceed by induction. $f_{-1} \circ \partial_0^X$ is an element $\mathrm{Hom}(X_0, Y_{-1})$ and since

$$0 \longleftarrow \mathrm{Hom}(X_0, Y_{-1}) \overset{\partial_0^Y}{\longleftarrow} \mathrm{Hom}(X_0, Y_0)$$

is exact, there exists an element $f_0 \in \mathrm{Hom}(X_0, Y_0)$ such that $\partial_0^Y \circ f_0 = f_{-1} \circ \partial_0^X$.

Now suppose f_i is constructed for $i < n$. We have

$$\partial_{n-1}^Y \circ f_{n-1} \circ \partial_n^X = f_{n-2} \circ \partial_{n-1}^Y \circ \partial_n^Y = 0$$

and since the sequence

$$\mathrm{Hom}(X_n, Y_{n-2}) \overset{\partial_{n-1}^Y}{\longleftarrow} \mathrm{Hom}(X_n, Y_{n-1}) \overset{\partial_n^Y}{\longleftarrow} \mathrm{Hom}(X_n, Y_n)$$

[1] If one is really reluctant to use this category, one could also work instead with the category of small functors from $\mathcal{C}$ to Ab, i.e. functors that are small colimits of representable functors in an obvious meaning. This category is then locally small.

[2] I make the small *abus de notation* of denoting ∂_k^Y instead of $\mathrm{Hom}(X_n, \partial_k^Y)$.

is exact, there exists an element $f_n \in \operatorname{Hom}(X_n, Y_n)$ such that

$$\partial^Y_n \circ f_n = f_{n-1} \circ \partial^X_n.$$

By induction the existence is proved.

Uniqueness part: We proceed again by induction. Let f and g be two solutions of the problem. Set $h_{-1} = 0 : X_{-1} \to Y_0$. We have

$$\partial^Y_0 \circ (f_0 - g_0) = (f_{-1} - f_{-1}) \circ \partial^X_0 = 0$$

and since the sequence

$$\operatorname{Hom}(X_0, Y_{-1}) \xleftarrow{\partial^Y_0} \operatorname{Hom}(X_0, Y_0) \xleftarrow{\partial^Y_1} \operatorname{Hom}(X_0, Y_1)$$

is exact, there exists an element $h_0 \in \operatorname{Hom}(X_0, Y_1)$ such that

$$\partial^Y_1 \circ h_0 = \partial^Y_1 \circ h_0 + h_{-1} \circ \partial^X_0 = f_{-1} - g_{-1}.$$

Now suppose h_i is constructed for $i < n$. We have

$$\begin{aligned}\partial^Y_n \circ (h_{n-1} \circ \partial^X_n - (f_n - g_n)) &= (\partial^Y_n \circ h_{n-1} - (f_{n-1} - g_{n-1})) \circ \partial^X_n \\ &= -h_{n-1} \circ \partial^X_{n-1} \circ \partial^X_n \\ &= 0\end{aligned}$$

and since the sequence

$$\operatorname{Hom}(X_n, Y_{n-1}) \xleftarrow{\partial^Y_n} \operatorname{Hom}(X_n, Y_n) \xleftarrow{\partial^Y_{n+1}} \operatorname{Hom}(X_n, Y_{n+1})$$

is exact, there exists and element $h_n \in \operatorname{Hom}(X_n, Y_{n+1})$ such that

$$\partial^Y_{n+1} \circ h_n + h_{n-1} \circ \partial^Y_n = f_n - g_n.$$

By induction we have constructed the homotopy. ■

1.3. Remark. By looking carefully at the proof, we easily see that we didn't need the full exactness of the sequence (1).

More precisely, it is clear that we only need that for each $n \geq 0$ the sequence

$$\operatorname{Hom}(X_n, Y_{n-2}) \xleftarrow{\partial^Y_{n-1}} \operatorname{Hom}(X_n, Y_{n-1}) \xleftarrow{\partial^Y_n} \operatorname{Hom}(X_n, Y_n)$$

be exact in order to prove the existence of the lifting. In the same vein, we only need that for each $n \geq 0$ the sequence

$$\operatorname{Hom}(X_n, Y_{n-2}) \xleftarrow{\partial^Y_{n-1}} \operatorname{Hom}(X_n, Y_{n-1}) \xleftarrow{\partial^Y_n} \operatorname{Hom}(X_n, Y_n) \xleftarrow{\partial^Y_{n+1}} \operatorname{Hom}(X_n, Y_{n+1})$$

be exact to prove existence and uniqueness up to homotopy. What does our extra hypotheses say? One thing that it says is that the homotopy witnessing the "uniqueness up to homotopy" of the lifting is itself unique up to a homotopy of homotopies. That homotopy of homotopies is in turn unique up to a homotopy of homotopies of homotopies and so on.

1.4. Remark. The theorem was stated for chain complexes bounded below by degree -1, but it is clear from the proof that any other bound could be used.

2. Basic homological algebra

As a warm-up, we show how two basic results of homological algebra come for free. Note that the so-called "classical lifting lemma" below is at the heart of the theory of projective resolutions, and constitute as such a cornerstone in the elementary theory of derived functors.

2.1. Proposition. [Classical lifting lemma] *Let X and Y be two augmented chain complexes in an abelian category $\mathcal{A}$. Suppose that*

- ***i.*** *for every $n \geq 0$, X_n is projective,*
- ***ii.*** *Y is acyclic.*

Then for every map $f_{-1} : X_{-1} \to Y_{-1}$ there exists a chain complex map $f : \sigma_{\geq 0}X \to \sigma_{\geq 0}Y$ such that

$$\begin{array}{ccc} X_{-1} & \xleftarrow{\partial_0^X} & \sigma_{\geq 0}X \\ {\scriptstyle f_{-1}}\downarrow & & \downarrow{\scriptstyle f} \\ Y_{-1} & \xleftarrow[\partial_0^Y]{} & \sigma_{\geq 0}Y \end{array}$$

commutes.

Moreover, f is unique up to homotopy.

Proof. Since X_n is projective for each $n \geq 0$, the functor $\mathrm{Hom}(X_n, -)$ preserves (long) exact sequences. ■

There is also a "dummy" version of the above lifting lemma.

2.2. Proposition. [Dummy lifting lemma] *Let X and Y be two augmented chain complexes in an abelian category $\mathcal{A}$. Suppose that Y is contractible.*

Then for every map $f_{-1} : X_{-1} \to Y_{-1}$ there exists a chain complex map $f : \sigma_{\geq 0}X \to \sigma_{\geq 0}Y$ such that

$$\begin{array}{ccc} X_{-1} & \xleftarrow{\partial_0^X} & \sigma_{\geq 0}X \\ {\scriptstyle f_{-1}}\downarrow & & \downarrow{\scriptstyle f} \\ Y_{-1} & \xleftarrow[\partial_0^Y]{} & \sigma_{\geq 0}Y \end{array}$$

commutes.

Moreover, f is unique up to homotopy.

Proof. A homotopy is preserved by any functor, hence the complex:

$$\mathrm{Hom}(X_n, Y) = \cdots \leftarrow 0 \leftarrow \mathrm{Hom}(X_n, Y_{-1}) \leftarrow \mathrm{Hom}(X_n, Y_0) \leftarrow \cdots$$

is contractible for every $n \geq 0$. ■

3. Eilenberg and Mac Lane's Acyclic Models

A category with models[3] is a pair $(\mathcal{C}, \mathcal{M})$ where $\mathcal{C}$ is a category and $\mathcal{M}$ is a (small) subcategory of $\mathcal{C}$. Actually, we should define a category with models as a functor $\nu : \mathcal{M} \to \mathcal{C}$ (that need not even be injective on objects). In practice, $\mathcal{M}$ is usually a discrete category (i.e. it has no morphisms aside from identities) and ν will be understood. The objects of $\mathcal{M}$ are called *models*. The central notion needed for the (classical) acyclic model theorem is the following.

3.1. Definition. *Let* $F : \mathcal{C} \to \mathrm{Ab}$ *be a functor from* $\mathcal{C}$ *to the category of abelian groups.* F *is* free on models *if there is a natural isomorphism:*

$$F \cong \bigoplus_{i \in I} \mathbb{Z}[\mathcal{C}(M_i, -)]$$

where $(M_i)_{i \in I}$ *is a (small) family of models and* $\mathbb{Z}[-]$ *is the free abelian group functor from* Sets *to* Ab*.*

Note that if $\mathcal{C} = \mathcal{M} = \mathbf{1}$ (the terminal category), a functor $F : \mathcal{C} \to \mathrm{Ab}$ can be seen as an abelian group. In that case, it is free on models if and only if it is free as an abelian group.

As the reader might have guessed it, there is an obvious "Yoneda" lemma.

3.2. Lemma. [Yoneda] *Let* X *be an object of* $\mathcal{C}$ *and* $G : \mathcal{C} \to \mathrm{Ab}$ *be a functor. The following morphism of abelian groups*

$$\begin{aligned} \mathrm{Hom}\left(\mathbb{Z}[\mathcal{C}(X, -)], G\right) &\to G(X) \\ \eta &\to \eta_X(\mathrm{id}_X) \end{aligned}$$

is an isomorphism natural in G *(and* X*).*

Proof. Left to the reader. ■

[3] Really bad terminology: it has nothing to do with model categories. We keep using it only for historical reasons.

3.3. Corollary. *Let* $(\mathcal{C}, \mathcal{M})$ *be a category with models and* $F : \mathcal{C} \to \text{Ab}$ *a functor free on models as before. For any functor* $G : \mathcal{C} \to \text{Ab}$ *there is an isomorphism of abelian groups*

$$\text{Hom}\,(F, G) \overset{\cong}{\to} \prod_{i \in I} G(M_i)$$

natural in G.

The (classical) acyclic model theorem is now at hand. From now on $(\mathcal{C}, \mathcal{M})$ will be some fixed category with models and $\mathcal{A}$ will be the abelian category $[\mathcal{C}, \text{Ab}]$ of functors from $\mathcal{C}$ to Ab. Note that the abelian category of chain complexes in $\mathcal{A}$ is (abelian) isomorphic to the abelian category of functors from $\mathcal{C}$ to C(Ab). We will make no distinction between these two categories.

3.4. Theorem. [Classical Acyclic Model Theorem] *Let* F *and* G *be two augmented chain complexes in* $\mathcal{A}$. *Suppose that*

- ***i.*** *for every* $n \geq 0$, F_n *is free on models,*
- ***ii.*** *for every model* M, $G(M)$ *is acyclic.*

Then for every map $f_{-1} : F_{-1} \to G_{-1}$ *there exists a chain complex map* $f : F_{\geq 0} \to G_{\geq 0}$ *that makes the usual diagram commutes. Moreover this chain complex map is unique up to homotopy.*

Proof. Using corollary 3.3, we see that for each $n \geq 0$ the sequence of abelian groups

$$0 \longleftarrow \text{Hom}(F_n, G_{-1}) \overset{\partial_0^G}{\longleftarrow} \text{Hom}(F_n, G_0) \overset{\partial_1^G}{\longleftarrow} \text{Hom}(F_n, G_1) \longleftarrow \cdots$$

is isomorphic to the sequence

$$0 \longleftarrow \prod_{i \in I_n} G_{-1}(M_i) \overset{\Pi\partial_0^G}{\longleftarrow} \prod_{i \in I_n} G_0(M_i) \overset{\Pi\partial_1^G}{\longleftarrow} \cdots$$

(where I_n comes from the freeness of F_n). Note that the naturality in G in corollary 3.3 is essential. Now, an arbitrary product of exact sequences of abelian groups is still an exact sequence[4]. Thus, the previous sequence is exact and we can apply theorem 1.2. ∎

4. Barr and Beck's Acyclic Models

As was already said in the introduction, the authors of [BB66] prove an "acyclic models theorem" (which we prefer to call G-lifting theorem). This section shows how this theorem can be deduced from the general lifting lemma.

[4]A fact which is not true in every abelian category!

4.1. Remark. Actually, the main theorem of this section is slightly more general than the one from [BB66]. As this was also hinted in the introduction, the reason for that is that in [BB66], the authors only considered the case where the abelian category is of the form: $[\mathcal{C}, \mathcal{A}_0]$, with $\mathcal{A}_0$ an arbitrary abelian category. Note, however, that this generalization is not due to the unifying approach of this note and could be proved using only the technique of [BB66].

An *augmented endofunctor* on an abelian category $\mathcal{A}$, is a pair (G, ϵ) where G is an additive endofunctor on $\mathcal{A}$ and $\epsilon : G \Rightarrow \mathrm{Id}$ is a natural transformation.

An object X of $\mathcal{A}$ is *G-projective* if the map

$$G(X) \xrightarrow{\epsilon_X} X$$

admits a section.

Note that since G is an additive functor, then given a chain complex Y, $G(Y)$ is again a chain complex.

4.2. Theorem. [G-lifting theorem] *Let (G, ϵ) an augmented endofunctor on an abelian category $\mathcal{A}$ and let X and Y be two augmented chain complexes in $\mathcal{A}$. Suppose that*

i. *for every $n \geq 0$, X_n is G-projective,*

ii. *the augmented chain complex $G(Y)$ is contractible.*

Then for every map $f_{-1} : X_{-1} \to Y_{-1}$ there exists a chain complex map $f : \sigma_{\geq 0}X \to \sigma_{\geq 0}Y$ that makes the usual diagram commutes. Moreover, f is unique up to homotopy.

Proof. For every $n \geq 0$, let θ_n be a section of $\epsilon_{X_n} : G(X_n) \to X_n$. Let $n \geq 0$ and Z be any object of our abelian category. It's easy to see that the map:

$$\begin{aligned} \mathrm{Hom}(X_n, Z) &\to \mathrm{Hom}(G(X_n), G(Z)) \\ f &\mapsto G(f) \end{aligned}$$

admits the map

$$\begin{aligned} \mathrm{Hom}(G(X_n), G(Z)) &\to \mathrm{Hom}(X_n, Z) \\ f &\mapsto \epsilon_Z \circ f \circ \theta_n \end{aligned}$$

as a section. Moreover, these two maps are natural in Z. Hence, the first map above can be turned into a chain complex map from

$$0 \longleftarrow \mathrm{Hom}(X_n, Y_{-1}) \longleftarrow \mathrm{Hom}(X_n, Y_0) \longleftarrow \cdots \tag{2}$$

to

$$0 \longleftarrow \mathrm{Hom}(G(X_n), G(Y_{-1})) \longleftarrow \mathrm{Hom}(G(X_n), G(Y_0)) \longleftarrow \cdots \tag{3}$$

Similarly, the second map above can be turn into a section of this chain complex map. Now, because $G(Y)$ is contractible, the chain complex (3) is contractible. Since a retract of an acyclic chain complex is an acyclic chain complex, we conclude that (3) is acyclic. ∎

5. Dold, Mac Lane and Oberst's lifting theorem

In [DMO67], the three authors prove a lifting theorem[5] that subsumes many different lifting theorems. In this section, we show how this lifting theorem from [DMO67] can be deduced from the general lifting lemma.

5.1. Definition. *Let $\mathcal{E}$ be any class of morphisms of an abelian category $\mathcal{A}$. An object P of $\mathcal{A}$ is said to be $\mathcal{E}$-projective if for every $e : A \to B \in \mathcal{E}$, the morphism of abelian groups*

$$\mathrm{Hom}(P, e) : \mathrm{Hom}(P, A) \to \mathrm{Hom}(P, B)$$

is surjective.

Note that when $\mathcal{E}$ is the class of epimorphisms, we recover the usual definition of projective object.

Recall that U is the obvious forgetful functor from $\mathrm{C}(\mathcal{A})$ to $\mathrm{Gr}(\mathcal{A})$. The differential of a chain complex X is then a morphism of *degree* -1, $d : U(X) \to U(X)$.

Let $i : Z(X) \to U(X)$ be the kernel of d (it is a map of degree 0). Since $d \circ d = 0$, there exists a unique map (of degree -1), $e : U(X) \to Z(X)$ such that $i \circ e = d$.

5.2. Definition. *Let X be an (augmented) chain complex in $\mathcal{A}$ and let $\mathcal{E}$ a class of morphisms of $\mathcal{A}$. X is said to be $\mathcal{E}$-acyclic, if the map $e : U(X) \to Z(X)$ is degreewise in $\mathcal{E}$. This means that for each n, $e_n : X_n \to Z(X)_{n-1}$ belongs to $\mathcal{E}$.*

Note that when $\mathcal{E}$ is the class of epimorphisms, we recover the usual definition of acyclicity. We can now state and prove theorem 1 of [DMO67].

5.3. Theorem. *Let $\mathcal{E}$ be any class in an abelian category $\mathcal{A}$, and let X and Y be two augmented chain complexes in $\mathcal{A}$. Suppose that*

i. *for every $n \geq 0$, X_n is $\mathcal{E}$-projective,*

ii. *Y is $\mathcal{E}$-acyclic.*

Then for every map $f_{-1} : X_{-1} \to Y_{-1}$ there exists a chain complex map $f : \sigma_{\geq 0}X \to \sigma_{\geq 0}Y$ that makes the usual diagram commutes. Moreover, f is unique up to homotopy.

Proof. This is a generalization of proposition 2.1. The only thing to prove is that the functor $\mathrm{Hom}(P, -)$ sends $\mathcal{E}$-acyclic complexes to acyclic complexes. Let X be a $\mathcal{E}$-acyclic complex, and i and e as before. We then have

$$\mathrm{Hom}(P, i) \circ \mathrm{Hom}(P, e) = \mathrm{Hom}(P, d).$$

Moreover, $\mathrm{Hom}(P, -)$ preserves kernels[6], thus

$$\mathrm{Hom}(P, i) : \mathrm{Hom}(P, Z(X)) \to \mathrm{Hom}(P, U(X))$$

[5] The theorem is called "comparison theorem" in [DMO67], but we chose to call it "lifting theorem" to be consistent with the rest of this note.

[6] A fact that is true for any P.

is the kernel of $\mathrm{Hom}(P, d)$. Because of the hypothesis, $\mathrm{Hom}(P, e)$ is degreewise in $\mathcal{E}$ and in light of what we said following definition 5.2, this proves that $\mathrm{Hom}(P, X)$ is an acyclic complex. ■

6. Future work: weaker hypotheses

In practise, we often use the unifying lemma (or any of its "corollary" proved in this note) in the following way. We show that X_{-1} and Y_{-1} are isomorphic and we deduce that X and Y are homotopic and, thus, have the same homology groups. However, it could happen that the two complexes have the same homology groups without being homotopic. One might wonder if there could be a version of the general lifting lemma where the uniqueness up to homotopy (or even existence) of the lifting would be replaced by something weaker that would still carry information on the level of homology. In [Bar02], the author proves such a lifting theorem in a particular case.

Roughly, it goes this way. Suppose you have an augmented endofunctor (G, ϵ) acting an abelian category $\mathcal{A}$. For each object X of $\mathcal{A}$, the standard construction (see [Bar02] for details[7]) gives an augmented chain complex

$$X \leftarrow G(X) \leftarrow G^2(X) \leftarrow \cdots$$

Say that X is weakly G-projective if this augmented chain complex is exact. It is easy to show that G-projectiveness implies weak G-projectiveness.

Let Y be a augmented chain complex. Say that Y is G-acyclic if for every $k \geq 1$, the augmented chain complex $G^k(Y)$ is acyclic. It is obvious that if $G(Y)$ is contractible then Y is G-acyclic.

Recall that the derived category of an abelian category $\mathcal{A}$ is the localization of $\mathrm{C}(\mathcal{A})$ with respect to quasi-isomorphisms. We often denote this category by $\mathrm{D}(\mathcal{A})$.

6.1. THEOREM. *[Bar02] Let (G, ϵ) be an augmented endofunctor on an abelian category $\mathcal{A}$ and let X and Y be two augmented chain complexes in $\mathcal{A}$. Suppose that*

- ***i.*** *for every $n \geq 0$, X_n if weakly G-projective,*
- ***ii.*** *Y is G-acyclic.*

Then for every map $f_{-1} : X_{-1} \to Y_{-1}$, there exists a unique map in $\mathrm{D}(\mathcal{A})$, $f : \sigma_{\geq 0}X \to \sigma_{\geq 0}Y$, such that

$$\begin{array}{ccc} X_{-1} & \xleftarrow{\partial_0^X} & \sigma_{\geq 0}X \\ {\scriptstyle f_{-1}}\downarrow & & \downarrow{\scriptstyle f} \\ Y_{-1} & \xleftarrow[\partial_0^Y]{} & \sigma_{\geq 0}Y \end{array}$$

commutes.

[7]Although I do not use exactly the same terminology.

6.2. Remark. In [Bar02], the author doesn't necessarily localize $\mathrm{C}(\mathcal{A})$ with respect to quasi-isomorphisms but use instead abstract classes that have some good properties. Of course, the class of quasi-isomorphisms is one of the prototypal example of such a "good class" of morphisms. I didn't want to get into too many details, and that is why I only stated a simplified version of the theorem from [Bar02].

6.3. Remark. The same remark as the one on the G-lifting theorem (theorem 4.2 of this note) applies: in [Bar02] the author only works with an abelian category which is itself a category of functors to an abelian category. As before, this hypothesis is useless.

In the spirit of this note, it would be reasonable to expect a generalization of our general lifting lemma that would admit the previous theorem as a particular case. With some hope, this should be done in a future work.

7. Future work: stronger hypotheses

Consider the following remark. If we replace, in the general lifting lemma, the hypothesis that

$$0 \longleftarrow \mathrm{Hom}(X_n, Y_{-1}) \xleftarrow{\partial_0^Y} \mathrm{Hom}(X_n, Y_0) \xleftarrow{\partial_1^Y} \mathrm{Hom}(X_n, Y_1) \longleftarrow \cdots$$

are exact for all $n \geq 0$ by the hypothesis that

$$\mathrm{Hom}(X_n, Y_{n-2}) \xleftarrow{\partial_{n-1}^Y} \mathrm{Hom}(X_n, Y_{n-1}) \xleftarrow{\partial_n^Y} \mathrm{Hom}(X_n, Y_n) \longleftarrow 0$$

are exact for all $n \geq 0$, then we obtain that the lifting is truly unique (not only up to homotopy). This condition was inspired by [Pro83, Pro84] and some unpublished work of the same author[8] and has non-trivial consequences in algebraic topology. With some hope, a forthcoming article will show how these results fit in a framework similar to this note, where the general lifting lemma is replaced with the one with the stronger hypothesis we have just stated.

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