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HOPF POLYADS, HOPF CATEGORIES AND HOPF GROUP MONOIDS VIEWED AS HOPF MONADS

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ABSTRACT. We associate, in a functorial way, a monoidal bicategory $\mathsf{Span}|\mathcal{V}$ to any monoidal bicategory $\mathcal{V}$. Two examples of this construction are of particular interest: *Hopf polyads* of [Bruguières 2015] can be seen as Hopf monads in $\mathsf{Span}|\mathsf{Cat}$ while *Hopf group monoids* in the spirit of [Zunino 2004, Turaev 2000] in a braided monoidal category V, and *Hopf categories* of [Batista-Caenepeel-Vercruysse 2016] over V both turn out to be Hopf monads in $\mathsf{Span}|V$. Hopf group monoids and Hopf categories are Hopf monads on a distinguished type of monoidales fitting the framework of [Böhm-Lack 2016]. These examples are related by a monoidal pseudofunctor $V \to \mathsf{Cat}$.

1. Introduction

A *Hopf monad* [Chikhladze-Lack-Street 2010] in a monoidal bicategory is an opmonoidal monad on a monoidale (also called a pseudo monoid) such that certain fusion 2-cells are invertible (cf. Section 2.1). In the monoidal 2-category Cat of categories, functors and natural transformations, the Hopf monads of [Bruguières-Lack-Virelizier 2011] on monoidal categories are re-obtained. Opmonoidal monads (in any bicategory) have the characteristic feature that their Eilenberg-Moore object — provided that it exists — is a monoidale too such that the forgetful morphism is a strict morphism of monoidales. If the base monoidale is also closed, then the Hopf property is equivalent to the lifting of the closed structure to the Eilenberg-Moore object, see [Chikhladze-Lack-Street 2010].

A monoidale is said to be a *map* monoidale if its multiplication and unit 1-cells possess right adjoints. We say that it is an *opmap* monoidale if it is a map monoidale in the vertically opposite bicategory (that is, in the original bicategory the multiplication and the unit are right adjoints themselves). Thus passing to the vertically opposite bicategory, opmonoidal monads on opmap monoidales can be seen as monoidal comonads on map monoidales, the central objects of the study in [Böhm-Lack 2016].

An (op)map monoidale is said to be *naturally Frobenius* [López Franco-Street-Wood 2011, López Franco 2009] if two canonical 2-cells (explicitly recalled in [Böhm-Lack 2016, Paragraph 2.4]), relating the multiplication and its adjoint, are invertible. The endohom category of a naturally Frobenius (op)map monoidale in any monoidal bicategory admits a duoidal structure [Street 2012] (what was called a 2-monoidal structure in [Aguiar-

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Mahajan 2010]). The Hopf monads on a naturally Frobenius opmap monoidale can be regarded as Hopf monoids in this duoidal endohom category. In this setting, many equivalent characterizations — including the existence of an antipode — of Hopf monads were obtained in [Böhm-Lack 2016].

Hopf monads in monoidal bicategories unify various structures like groupoids, Hopf algebras, weak Hopf algebras [Böhm-Nill-Szlachányi 1999], Hopf algebroids [Schauenburg 2000], Hopf monads of [Bruguières-Virelizier 2007] and — more generally — of [Bruguières-Lack-Virelizier 2011]. Some of these, namely groupoids, Hopf algebras, weak Hopf algebras [Böhm-Nill-Szlachányi 1999], Hopf algebroids over commutative algebras as in [Ravenel 1986] and the Hopf monads of [Bruguières-Virelizier 2007] live on naturally Frobenius opmap monoidales, see [Böhm-Lack 2016].

The aim of this note is to show that some structures that recently appeared in the literature fit this framework as well: we show that *Hopf group monoids* (thus in particular *Hopf group algebras* in [Turaev 2000, Zunino 2004, Caenepeel-De Lombaerde 2006]), *Hopf categories* in [Batista-Caenepeel-Vercruysse 2016] and *Hopf polyads* in [Bruguières 2015] can be seen as Hopf monads in suitable monoidal bicategories. Hopf group monoids and Hopf categories are even Hopf monads on naturally Frobenius opmap monoidales; explaining e.g. the existence and the properties of their antipodes.

Note that all of Hopf polyads, Hopf group monoids, and Hopf categories can be seen as lax functors from a suitable category (provided by an arbitrary category, a group, and an indiscrete category, respectively) to a monoidal bicategory $\mathcal{V}$ (equal to Cat and a braided monoidal category regarded as a monoidal bicategory with a single object, respectively); so they are objects of a bicategory of lax functors, lax natural transformations and modifications. However, this bicategory does not admit a suitable monoidal structure allowing for a study of Hopf monads.

So in order to achieve our goal, we embed it into a larger bicategory $\mathsf{Span}|\mathcal{V}$. The bicategory $\mathsf{Span}|\mathcal{V}$ is constructed for any bicategory $\mathcal{V}$. Whenever $\mathcal{V}$ is a monoidal bicategory, also $\mathsf{Span}|\mathcal{V}$ is proven to be so. This correspondence is functorial in the sense that any lax functor (respectively, monoidal lax functor) $F : \mathcal{V} \to \mathcal{W}$ induces a lax functor (respectively, monoidal lax functor) $\mathsf{Span}|F : \mathsf{Span}|\mathcal{V} \to \mathsf{Span}|\mathcal{W}$. This construction is applied to two examples:

— A monad in $\mathsf{Span}|\mathsf{Cat}$ is precisely a *polyad* of [Bruguières 2015]. Furthermore, any set of monoidal categories can be regarded as a monoidale in $\mathsf{Span}|\mathsf{Cat}$. The opmonoidal structures of a monad on such a monoidale correspond bijectively to opmonoidal structures of the polyad in the sense of [Bruguières 2015]. Finally, such an opmonoidal monad is a Hopf monad if and only if the corresponding opmonoidal polyad is a Hopf polyad (in the sense of [Bruguières 2015]) over a groupoid.

— Any braided monoidal category V can be regarded as a monoidal bicategory with a single object. Hence there is an associated monoidal bicategory $\mathsf{Span}|V$ in which any object carries the structure of a naturally Frobenius opmap monoidale.

On the one hand, we identify categories enriched in V with certain monads; categories enriched in the category of comonoids in V with certain opmonoidal monads; and Hopf categories over V with certain Hopf monads on these naturally Frobenius opmap monoidales in $\mathsf{Span}|V$.

On the other hand, we also identify monoids in V graded by ordinary monoids with monads; semi-Hopf group monoids in V with opmonoidal monads; and Hopf group monoids in V with Hopf monads on a trivial naturally Frobenius opmap monoidale in $\mathsf{Span}|V$.

The above examples are related by a monoidal pseudofunctor $V \to \mathsf{Cat}$. It induces a monoidal pseudofunctor $\mathsf{Span}|V \to \mathsf{Span}|\mathsf{Cat}$ which takes both Hopf group monoids and Hopf categories to Hopf polyads.

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2. The general construction

Throughout this section $\mathcal{V}$ will denote a bicategory [Borceux 1994, Vol. 1 Section 7.7] whose horizontal composition will be denoted by $\circ$ and whose vertical composition will be denoted by $*$. Although the horizontal composition is required to be neither strictly associative nor strictly unital, we will omit explicitly denoting the associativity and unitality iso 2-cells.

2.1. Hopf monads in monoidal bicategories. We briefly recall some definitions for later reference. For more details we refer e.g. to [Chikhladze-Lack-Street 2010].

A *monad* on a *category* A consists of an endofunctor $f : A \to A$ together with natural transformations μ (the multiplication) from the two-fold iterate $f \circ f$ to f and η (the unit) from the identity functor 1 to f. They are subject to the associativity and unitality axioms.

From the 2-category Cat of categories, functors and natural transformations, this notion can be generalized to *any bicategory*, see [Street 1972]. Then a monad consists of a 1-cell $f : A \to A$ and 2-cells $\mu : f \circ f \to f$ and $\eta : 1 \to f$ such that μ is associative with unit η.

For *monoidal categories* A and A' (with respective monoidal products $\otimes$ and $\otimes'$; monoidal units K and K'), we can ask about the relation of a functor $f : A \to A'$ and the monoidal structures; there are some dual possibilities of their compatibility. An *opmonoidal* (by some authors called *comonoidal*) structure on f consists of natural transformations $f_2 : f(- \otimes -) \to f(-) \otimes' f(-)$ and $f_0 : f(K) \to K'$ which satisfy the evident coassociativity and counitality conditions (see these conditions spelled out explicitly in a more general case below). The functor f is said to be *strict monoidal* if f_2 and f_0 are identity morphisms.

A natural transformation between opmonoidal functors f and f' is said to be *opmonoidal* if compatible with the opmonoidal structures of f and f' (for the precise form of this compatibility see the more general case below).

It is straightforward to see that monoidal categories, opmonoidal functors and opmonoidal natural transformations constitute a 2-category OpMon. The monads therein are termed *opmonoidal monads*. Recall from [Moerdijk 2002] and [McCrudden 2002] that for any monoidal category A and any monad f on the category A, there is a bijective correspondence between

— opmonoidal structures of the functor f making it an opmonoidal monad;

— monoidal structures of the category A^f of Eilenberg–Moore f-algebras such that the forgetful functor $A^f \to A$ is strict monoidal (that is, the *liftings* of the monoidal structure of A to A^f).

To any opmonoidal monad (f, f_2, f_0, μ, η) on a monoidal category A, one associates a natural transformation, the so-called *fusion morphism*,

$$f(f(-)\otimes -) \xrightarrow{f_2} f(f(-))\otimes f(-) \xrightarrow{\mu\otimes 1} f(-)\otimes f(-).$$

The opmonoidal monad f is said to be a *Hopf monad* precisely if the fusion morphism is invertible, see [Bruguières-Lack-Virelizier 2011]. Whenever the monoidal category A is *closed*, the invertibility of the fusion morphism is equivalent to the lifting of the closed structure of A to the Eilenberg–Moore category A^f, see again [Bruguières-Lack-Virelizier 2011].

The above notions can be generalized from the Cartesian monoidal 2-category Cat to any *monoidal bicategory* $\mathcal{V}$ (with monoidal product $\otimes$ and monoidal unit K). Then monoidal category is generalized to what is known as *monoidale* (alternatively called *pseudo monoid*). Such a gadget consists of an object A of $\mathcal{V}$ together with 1-cells m from the monoidal square $A\otimes A$ to A and u from the monoidal unit K to A; as well as invertible 2-cells $m\circ(m\otimes 1)\to m\circ(1\otimes m)$, $m\circ(u\otimes 1)\to 1$ and $m\circ(1\otimes u)\to 1$ which satisfy Mac Lane's coherence axioms.

For monoidales A and A', an *opmonoidal 1-cell* consists of a 1-cell $f: A\to A'$ together with 2-cells $f_2: f\circ m\to m'\circ(f\otimes f)$ and $f_0: f\circ u\to u'$ satisfying the usual coassociativity and counitality conditions

$$
\begin{array}{ccccc}
f\circ m\circ(m\otimes 1) & \xrightarrow{f_2\circ 1} & \begin{array}{c} m'\circ(f\otimes f)\circ(m\otimes 1) \\ \cong m'\circ(f\circ m\otimes f) \end{array} & \xrightarrow{1\circ(f_2\otimes 1)} & \begin{array}{c} m'\circ(m'\otimes 1)\circ(f\otimes f\otimes f) \\ \cong m'\circ(m'\circ(f\otimes f)\otimes f) \end{array} \\
\downarrow{\scriptstyle\cong} & & & & \downarrow{\scriptstyle\cong} \\
f\circ m\circ(1\otimes m) & \xrightarrow[f_2\circ 1]{} & \begin{array}{c} m'\circ(f\otimes f)\circ(1\otimes m) \\ \cong m'\circ(f\otimes f\circ m) \end{array} & \xrightarrow[1\circ(1\otimes f_2)]{} & \begin{array}{c} m'\circ(1\otimes m')\circ(f\otimes f\otimes f) \\ \cong m'\circ(f\otimes m'\circ(f\otimes f)) \end{array}
\end{array}
$$

$$
\begin{array}{ccccc}
f\circ m\circ(u\otimes 1) & \xrightarrow{f_2\circ 1} & \begin{array}{c} m'\circ(f\otimes f)\circ(u\otimes 1) \\ \cong m'\circ(f\circ u\otimes f) \end{array} & \xrightarrow{1\circ(f_0\otimes 1)} & \begin{array}{c} m'\circ(u'\otimes 1)\circ f \\ \cong m'\circ(u'\otimes f) \end{array} \\
\downarrow{\scriptstyle\cong} & & & & \downarrow{\scriptstyle\cong} \\
f & \multicolumn{3}{c}{=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=} & f \\
\uparrow{\scriptstyle\cong} & & & & \uparrow{\scriptstyle\cong} \\
f\circ m\circ(1\otimes u) & \xrightarrow[f_2\circ 1]{} & \begin{array}{c} m'\circ(f\otimes f)\circ(1\otimes u) \\ \cong m'\circ(f\otimes f\circ u) \end{array} & \xrightarrow[1\circ(1\otimes f_0)]{} & \begin{array}{c} m'\circ(1\otimes u')\circ f \\ \cong m'\circ(f\otimes u'). \end{array}
\end{array}
$$

A *strict monoidal* 1-cell is an opmonoidal 1-cell f with f_2 and f_0 the identity 2-cells.

A 2-cell $\varphi : f \to f'$ between opmonoidal 1-cells is *opmonoidal* if the diagrams

$$
\begin{array}{ccc}
f\circ m & \xrightarrow{f_2} & m'\circ(f\otimes f) \\
{\scriptstyle\varphi\circ 1}\downarrow & & \downarrow{\scriptstyle 1\circ(\varphi\otimes\varphi)} \\
f'\circ m & \xrightarrow[f'_2]{} & m'\circ(f'\otimes f')
\end{array}
\qquad\qquad
\begin{array}{ccc}
f\circ u & \xrightarrow{f_0} & u' \\
{\scriptstyle\varphi\circ 1}\downarrow & & \| \\
f'\circ u & \xrightarrow[f'_0]{} & u'
\end{array}
$$

commute.

Once again, monoidales, opmonoidal 1-cells and opmonoidal 2-cells constitute a bicategory $\mathsf{OpMon}(\mathcal{V})$; the monads therein are termed *opmonoidal monads*. Assume that in $\mathcal{V}$ the Eilenberg-Moore object A^f exists for any monad f on some object A. Then for any monad f on A, and for any monoidale with object part A, there is a bijective correspondence between

— 2-cells $f\circ m \to m\circ(f\otimes f)$ and $f\circ u \to u$ yielding an opmonoidal monad f;

— 1-cells $A^f\otimes A^f \to A^f$ and $K \to A^f$ yielding a monoidale A^f such that the forgetful 1-cell $A^f \to A$ is strict monoidal.

The *fusion 2-cell* associated to an opmonoidal monad (f, f_2, f_0, μ, η) takes now the form

$$
f\circ m\circ(f\otimes 1) \xrightarrow{f_2\circ 1} m\circ(f\otimes f)\circ(f\otimes 1) \cong m\circ(f\circ f\otimes f) \xrightarrow{1\circ(\mu\otimes 1)} m\circ(f\otimes f).
$$

Its invertibility defines f to be a *Hopf monad*. As shown in [Chikhladze-Lack-Street 2010], in the case when the base monoidale is *closed*, the invertibility of the fusion 2-cell is again equivalent to the lifting of the closed structure to the Eilenberg-Moore object of f. For some equivalent characterizations of Hopf monads (among opmonoidal monads) in favorable situations, we refer to [Böhm-Lack 2016].

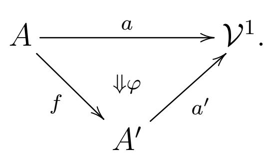
2.2. THE BICATEGORY $\mathsf{Span}|\mathcal{V}$ ASSOCIATED TO A BICATEGORY $\mathcal{V}$. The **0-cells** of $\mathsf{Span}|\mathcal{V}$ are pairs consisting of a set X and a map x from X to the set $\mathcal{V}^0$ of 0-cells in $\mathcal{V}$.

The **1-cells** from $X \xrightarrow{x} \mathcal{V}^0$ to $Y \xrightarrow{y} \mathcal{V}^0$ consist of a span $Y \xleftarrow{l} A \xrightarrow{r} X$ — inducing a span $\mathcal{V}^0 \xleftarrow{y.l} A \xrightarrow{x.r} \mathcal{V}^0$ — and a map a from A to the set $\mathcal{V}^1$ of 1-cells in $\mathcal{V}$, such that with the source and target maps s and t of $\mathcal{V}$ the following compatibility diagram commutes (that is to say, a is a map of spans over the set $\mathcal{V}^0$).

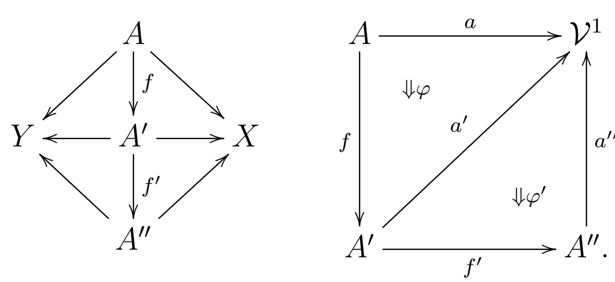
$$\begin{array}{ccccc} Y & \xleftarrow{l} & A & \xrightarrow{r} & X \\ {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle x} \\ \mathcal{V}^0 & \xleftarrow[t]{} & \mathcal{V}^1 & \xrightarrow[s]{} & \mathcal{V}^0 \end{array} \tag{2.1}$$

The **2-cells** from $(Y \leftarrow A \rightarrow X, a)$ to $(Y \leftarrow A' \rightarrow X, a')$ consist of a map of spans $f : A \to A'$ and a set $\varphi = \{\varphi_c : a(c) \Rightarrow a'f(c) | c \in A\}$ of 2-cells in $\mathcal{V}$.

If we regard the maps a and a' as functors from the discrete categories A and A', respectively, to the vertical category of $\mathcal{V}$, then φ is a natural transformation from a to the composite of the functors $f : A \to A'$ and a'. By this motivation we use the diagrammatic notation



The **vertical composite** of the 2-cells $(f, \varphi) : (Y \leftarrow A \rightarrow X, a) \Rightarrow (Y \leftarrow A' \rightarrow X, a')$ and $(f', \varphi') : (Y \leftarrow A' \rightarrow X, a') \Rightarrow (Y \leftarrow A'' \rightarrow X, a'')$ is the pair



In other words, it is the pair $(f'.f, \{\varphi'_{f(c)} * \varphi_c | c \in A\})$.

The identity 2-cell of $(Y \leftarrow A \rightarrow X, a)$ consists of the identity map $1 : A \to A$ and the set $\{1_{a(c)} | c \in A\}$ of identity 2-cells.

The **horizontal composite** of the 1-cells $(\, Y \xleftarrow{l} A \xrightarrow{r} X,\, a)$ and $(\, Z \xleftarrow{l} B \xrightarrow{r} Y,\, b)$ is the pair consisting of the pullback span

$$Z \leftarrow B \circ A := \{(d,c) \in B \times A | r(d) = l(c)\} \to X, \qquad l(d) \leftarrow\!\shortmid (d,c) \mapsto r(c)$$

and the map

$$B \circ A \to \mathcal{V}^1, \qquad (d,c) \mapsto b(d) \circ a(c).$$

The 1-cells $b(d)$ and $a(c)$ are composable indeed thanks to (2.1).

The horizontal composite of 2-cells $(f,\varphi) : (\, Y \leftarrow A \rightarrow X,\, a) \Rightarrow (\, Y \leftarrow A' \rightarrow X,\, a')$ and $(g,\gamma) : (\, Z \leftarrow B \rightarrow Y,\, b) \Rightarrow (\, Z \leftarrow B' \rightarrow Y,\, b')$ consists of the map

$$g \circ f : B \circ A \to B' \circ A', \qquad (d,c) \mapsto (g(d), f(c))$$

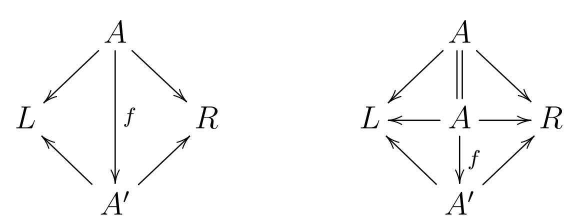
and the following set of 2-cells in $\mathcal{V}$.

$$\{\gamma_d \circ \varphi_c : b(d) \circ a(c) \Rightarrow b' g(d) \circ a' f(c) | (d,c) \in B \circ A\}$$

The identity 1-cell of (X, x) consists of the trivial span $X = X = X$ and the map $1_{x(-)} : X \to \mathcal{V}^1$. The associativity and unitality natural transformations are pairs of the analogous natural transformations in Span and $\mathcal{V}$.

Using that both Span and $\mathcal{V}$ are bicategories, it is straightforward to see that so is $\mathsf{Span}|\mathcal{V}$ above.

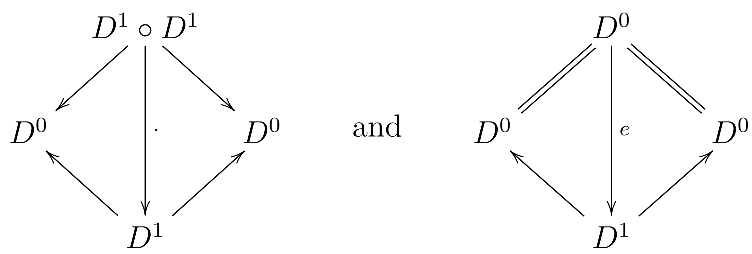
We are not aware of any construction yielding $\mathsf{Span}|\mathcal{V}$ as a comma bicategory. However, regarding it as a tricategory (with only identity 3-cells), it embeds into a comma tricategory obtained by a lax version of the *3-comma category construction* in [Gray 1974, Section I.2.7]: Consider the tricategory $\mathsf{SpanSpan}$ whose 0-cells are sets $X, Y, \ldots$, whose hom-bicategory $\mathsf{SpanSpan}(X,Y)$ is the bicategory of spans in the category $\mathsf{Span}(X,Y)$, and in which the 1-composition is the pullback of spans with the evident coherence 2-and 3-cells. Regarding Span as a tricategory with only identity 3-cells, and interpreting a map of spans in the first diagram of



as a span in the second diagram, we obtain a functor of tricategories $\mathsf{Span} \to \mathsf{SpanSpan}$. On the other hand, any bicategory $\mathcal{V}$ determines an evident (1- and 2-) lax functor of tricategories from the trivial tricategory $\mathbf{1}$ (with a single 0-cell and only identity higher cells) to $\mathsf{SpanSpan}$. The comma tricategory arising from the lax functors $\mathsf{Span} \longrightarrow \mathsf{SpanSpan} \xleftarrow{\mathcal{V}} \mathbf{1}$ contains $\mathsf{Span}|\mathcal{V}$ as a sub-tricategory.

Note for later application that a 1-cell $(\, Y \leftarrow A \rightarrow X,\, a)$ possesses a right adjoint in $\mathsf{Span}|\mathcal{V}$ if and only if $Y \leftarrow A \rightarrow X$ has a right adjoint in Span and for all $c \in A$, $a(c)$ has a right adjoint in $\mathcal{V}$. Equivalently, if and only if it is isomorphic to a 1-cell of the form $(\, Y \leftarrow X = X,\, h)$ such that for all $p \in X$, $h(p)$ has a right adjoint in $\mathcal{V}$.

2.3. Monads in $\mathsf{Span}|\mathcal{V}$. Let us fix an arbitrary 0-cell $(D^0,\ D^0 \xrightarrow{f} \mathcal{V}^0)$ in $\mathsf{Span}|\mathcal{V}$ and describe a monad on it. The underlying 1-cell consists of a span $D^0 \xleftarrow{t} D^1 \xrightarrow{s} D^0$ and a map F associating a 1-cell $F(h) : fs(h) \to ft(h)$ in $\mathcal{V}$ to each element h of D^1. The multiplication and unit 2-cells consist of respective maps of spans



and respective sets of 2-cells $\{\mu_{h,k} : F(h) \circ F(k) \to F(h.k) | (h,k) \in D^1 \circ D^1\}$ and $\{\eta_x : 1_{f(x)} \to F(e_x) | x \in D^0\}$ in $\mathcal{V}$. The associativity and unitality conditions precisely say that there is a category

$$D^0 \underset{t}{\overset{s}{\leftleftarrows}}\!\!\xrightarrow{e} D^1 \xleftarrow{\ \cdot\ } D^1 \circ D^1 \tag{2.2}$$

with object set D^0, morphism set D^1, source and target maps s and t, composition $\cdot$ and identity morphisms $\{e_x | x \in D^0\}$ and — regarding this category as a bicategory with only identity 2-cells — a lax functor $D \to \mathcal{V}$ with object map f, hom functor F (from the discrete hom category D^1), and comparison natural transformations μ and η. Summarizing, for any bicategory $\mathcal{V}$, the following notions coincide.

— A pair consisting of a category D and a lax functor $D \to \mathcal{V}$.

— A monad in $\mathsf{Span}|\mathcal{V}$.

2.4. Bicategories of monads in $\mathsf{Span}|\mathcal{V}$. Consider a category (2.2) and lax functors $((f,F),\mu,\eta)$ and $((f',F'),\mu',\eta')$ from D to $\mathcal{V}$. Regard them as monads in $\mathsf{Span}|\mathcal{V}$ as in Section 2.3.

A 1-cell of the form $(\ D^0 = D^0 = D^0\ ,\ D^0 \xrightarrow{h} \mathcal{V}^1\ )$ from $D^0 \xrightarrow{f} \mathcal{V}^0$ to $D^0 \xrightarrow{f'} \mathcal{V}^0$ and the 2-cell $(D^0 \circ D^1 \cong D^1 \cong D^1 \circ D^0,\ ht(-) \circ F(-) \xrightarrow{\varphi} F'(-) \circ hs(-)\ )$ constitute a monad morphism (in the sense of [Street 1972]) in $\mathsf{Span}|\mathcal{V}$ if and only if (h,φ) is a lax natural transformation.

A 2-cell of the form $(D^0 = D^0,\ h \xrightarrow{\gamma} h'\ )$ is a monad transformation (in the sense of [Street 1972]) in $\mathsf{Span}|\mathcal{V}$ if and only if γ is a modification $(h,\varphi) \to (h',\varphi')$.

These observations amount to the isomorphism of the following bicategories, for any category D and any bicategory $\mathcal{V}$.

— The bicategory $[D,\mathcal{V}]$ of lax functors $D \to \mathcal{V}$, lax natural transformations and modifications.

— The following locally full sub-bicategory in the bicategory of monads in $\mathsf{Span}|\mathcal{V}$. The 0-cells are those monads which live on 0-cells $D^0 \to \mathcal{V}^0$ (for the given object set D^0 of D), whose 1-cells are of the form $(D^0 \xleftarrow{t} D^1 \xrightarrow{s} D^0,\ F)$ (in terms of the given data s,t), and whose multiplication and unit 2-cells have the respective forms $(\cdot,\mu)$ and (e,η) (with the given maps $\cdot$ and e). The 1-cells are those monad morphisms $((H,h),(f,\varphi))$ whose underlying span H is the trivial span $D^0 = D^0 = D^0$ and whose map f is the canonical isomorphism $D^0 \circ D^1 \cong D^1 \cong D^1 \circ D^0$. The 2-cells are all possible monad transformations (g,γ) (g in them is necessarily the identity map $D^0 \to D^0$).

2.5. The monoidal bicategory $\mathsf{Span}|\mathcal{V}$ for a monoidal bicategory $\mathcal{V}$. In this section $\mathcal{V}$ is taken to be a monoidal bicategory — that is, a single object tricategory [Gordon-Power-Street 1995] — with monoidal operation $\otimes$ and monoidal unit K. Then we can equip $\mathsf{Span}|\mathcal{V}$ with a monoidal structure as follows.

The monoidal product of 0-cells $X \xrightarrow{x} \mathcal{V}^0$ and $Z \xrightarrow{z} \mathcal{V}^0$ consists of the Cartesian product set $X \times Z$ and the map

$$X \times Z \to \mathcal{V}^0, \qquad (k,l) \mapsto x(k) \otimes z(l).$$

The monoidal product lax functor on the local hom categories takes a pair of 2-cells (f,φ) : $(Y \leftarrow A \rightarrow X, a) \Rightarrow (Y \leftarrow A' \rightarrow X, a')$ and $(g,\gamma) : (W \leftarrow B \rightarrow Z, b) \Rightarrow (W \leftarrow B' \rightarrow Z, b')$ to the 2-cell consisting of the Cartesian product map $f \times g : A \times B \to A' \times B'$ between the Cartesian product spans and the following set of 2-cells.

$$\{\varphi_c \otimes \gamma_d : a(c) \otimes b(d) \to a'f(c) \otimes b'g(d) | (c,d) \in A \times B\}$$

The natural transformations establishing the compatibility of these functors $\otimes$ with the identity 1-cells and the horizontal composition are inherited from $\mathcal{V}$. The monoidal unit is the singleton set with the map K. The lax natural transformations measuring the non-associativity and non-unitality of $\otimes$, as well as their invertible coherence modifications are induced by those in $\mathcal{V}$.

It requires some patience to check that this is a monoidal bicategory indeed. No conceptual difficulties arise, however, one has to use repeatedly that Span is a monoidal bicategory via the Cartesian product of sets together with the assumed monoidal bicategory structure of $\mathcal{V}$.

Note that the sub-bicategory of $\mathsf{Span}|\mathcal{V}$ occurring in Section 2.4 is not a monoidal sub-bicategory. Hence it is not suitable for our study of Hopf monads.

2.6. The bicategory $\mathsf{Span}|\mathsf{OpMon}(\mathcal{V})$ for a monoidal bicategory $\mathcal{V}$. The 2-full (i.e. both horizontally and vertically full) sub-bicategory in $\mathsf{OpMon}(\mathsf{Span})$ whose objects are the opmap monoidales, is in fact isomorphic to Span via the forgetful functor.

Consider next a monoidale in $\mathsf{Span}|\mathcal{V}$ whose multiplication and unit 1-cells have underlying spans which possess left adjoints in Span. It consists of a 0-cell $X \xrightarrow{C} \mathcal{V}^0$ together with multiplication and unit 1-cells which must be of the form

$$(\ X = X \xrightarrow{\Delta} X \times X,\ X \xrightarrow{m} \mathcal{V}^1\ ) \quad \text{and} \quad (\ X = X \xrightarrow{!} 1,\ X \xrightarrow{u} \mathcal{V}^1\ ) \qquad (2.3)$$

— where Δ is the diagonal map $p \mapsto (p,p)$ and ! denotes the unique map to the singleton set 1 — and associativity and unit 2-cells provided by the identity map of X, and maps sending $p \in X$ to 2-cells in $\mathcal{V}$, $\alpha_p : m_p \circ (m_p \otimes 1_{C_p}) \to m_p \circ (1_{C_p} \otimes m_p)$, $\lambda_p : m_p \circ (u_p \otimes 1_{C_p}) \to 1_{C_p}$ and $\varrho_p : m_p \circ (1_{C_p} \otimes u_p) \to 1_{C_p}$, respectively. The axioms for these data to constitute a monoidale in $\mathsf{Span}|\mathcal{V}$ say precisely that $(C_p, m_p, u_p, \alpha_p, \lambda_p, \varrho_p)$ is a monoidale in $\mathcal{V}$ for all $p \in X$.

If each member $(C_p, m_p, u_p, \alpha_p, \lambda_p, \varrho_p)$ in a monoidale as in (2.3) is a naturally Frobenius opmap monoidale in $\mathcal{V}$, then so is the induced monoidale in $\mathsf{Span}|\mathcal{V}$. The left adjoints of its multiplication and unit are given in terms of the left adjoints $(m_p)_* \dashv m_p$ and $(u_p)_* \dashv u_p$ as

$$(X \times X \xleftarrow{\Delta} X = X \ , X \ni p \mapsto (m_p)_*) \quad \text{and} \quad (1 \xleftarrow{!} X = X \ , X \ni p \mapsto (u_p)_*).$$

The 2-cells of [Böhm-Lack 2016, Paragraph 2.4] are invertible for the induced opmap monoidale since they are so for each member $(C_p, m_p, u_p, \alpha_p, \lambda_p, \varrho_p)$.

In a symmetric manner, a set $\{(C_p, d_p, e_p, \alpha_p, \lambda_p, \varrho_p) | p \in X\}$ of comonoidales in $\mathcal{V}$ induces a comonoidale in $\mathsf{Span}|\mathcal{V}$. The underlying 0-cell is $(X, X \ni p \mapsto C_p)$; the comultiplication and counit 1-cells are

$$(X \times X \xleftarrow{\Delta} X = X \ , X \ni p \mapsto d_p) \quad \text{and} \quad (1 \xleftarrow{!} X = X \ , X \ni p \mapsto e_p),$$

respectively; while the coassociativity and the counit isomorphisms are given by the sets $\{\alpha_p \mid p \in X\}$, $\{\lambda_p \mid p \in X\}$ and $\{\varrho_p \mid p \in X\}$ of the analogous 2-cells for C_p.

An opmonoidal 1-cell between monoidales (X, C) and (Y, H) of the form in (2.3) consists of a span $Y \xleftarrow{l} A \xrightarrow{r} X$ and a map a sending each element h of A to a 1-cell $a(h) : C_{r(h)} \to H_{l(h)}$ in $\mathcal{V}$; together with an opmonoidal structure which consists of opmonoidal structures on each 1-cell $a(h)$ for $h \in A$. A 2-cell (f, φ) between opmonoidal 1-cells $(Y \leftarrow A \rightarrow X, a)$ and $(Y \leftarrow A' \rightarrow X, \ a')$ as above is opmonoidal precisely if each component $\varphi_h : a(h) \to a' f(h)$ is opmonoidal, for $h \in A$.

Putting in other words, from the considerations of the previous paragraph isomorphism of the following bicategories follows.

- — $\mathsf{Span}|\mathsf{OpMon}(\mathcal{V})$.
- — The 2-full sub-bicategory of $\mathsf{OpMon}(\mathsf{Span}|\mathcal{V})$ whose objects are of the kind in (2.3).

2.7. Bicategories of monads in $\mathsf{Span}|\mathsf{OpMon}(\mathcal{V})$. Combining the isomorphisms of Section 2.4 and Section 2.6, we obtain isomorphism of the following bicategories, for any category D and any monoidal bicategory $\mathcal{V}$.

- — $[D, \mathsf{OpMon}(\mathcal{V})]$.
- — The following locally full sub-bicategory in the bicategory of monads (in the sense of [Street 1972]) in $\mathsf{Span}|\mathsf{OpMon}(\mathcal{V})$. The 0-cells are those monads which live on 0-cells

$D^0 \to \mathsf{OpMon}(\mathcal{V})^0$ (for the given object set D^0 of D), whose 1-cells are of the form $(D^0 \xleftarrow{t} D^1 \xrightarrow{s} D^0 , \mathbf{d} : D^1 \to \mathsf{OpMon}(\mathcal{V})^1)$ (in terms of the given data s, t), and whose multiplication and unit 2-cells have the respective forms $(\cdot, \mu)$ and (e, η) (with the given maps $\cdot$ and e). The 1-cells are those monad morphisms $((H, \mathbf{h}), (f, \varphi))$ whose underlying span H is the trivial span $D^0 = D^0 = D^0$ (hence $\mathbf{h}$ is a map $D^0 \to \mathsf{OpMon}(\mathcal{V})^1$) and whose map f is the canonical isomorphism $D^0 \circ D^1 \cong D^1 \cong D^1 \circ D^0$. The 2-cells are all possible monad transformations (g, γ) (g in them is necessarily the identity map $D^0 \to D^0$).

— The following locally full sub-bicategory in the bicategory of monads (in the sense of [Street 1972]) in $\mathsf{OpMon}(\mathsf{Span}|\mathcal{V})$. The 0-cells are those monads which live on monoidales with object part $D^0 \to \mathcal{V}^0$ (for the given object set D^0 of D) and with multiplication and unit of the form in (2.3), whose 1-cells are of the form $(D^0 \xleftarrow{t} D^1 \xrightarrow{s} D^0 , d : D^1 \to \mathcal{V}^1)$ (in terms of the given data s, t), and whose multiplication and unit 2-cells have the respective forms $(\cdot, \mu)$ and (e, η) (with the given maps $\cdot$ and e). (There are no restrictions on the opmonoidal structure of the 1-cell $(D^0 \xleftarrow{t} D^1 \xrightarrow{s} D^0 , d)$ in $\mathsf{Span}|\mathcal{V}$.) The 1-cells are those monad morphisms $((H, h), (f, \varphi))$ whose underlying span H is the trivial span $D^0 = D^0 = D^0$ and whose map f is the canonical isomorphism $D^0 \circ D^1 \cong D^1 \cong D^1 \circ D^0$. (There are no restrictions on the opmonoidal structure of the 1-cell $(D^0 = D^0 = D^0, h : D^0 \to \mathcal{V}^1)$ in $\mathsf{Span}|\mathcal{V}$.) The 2-cells are all possible monad transformations (g, γ) (g in them is necessarily the identity map $D^0 \to D^0$).

2.8. Functoriality. Any lax functor $F : \mathcal{V} \to \mathcal{W}$ between arbitrary bicategories $\mathcal{V}$ and $\mathcal{W}$ induces a lax functor $\mathsf{Span}|F : \mathsf{Span}|\mathcal{V} \to \mathsf{Span}|\mathcal{W}$ as follows. It sends a 0-cell $X \xrightarrow{x} \mathcal{V}^0$ to the 0-cell

$$X \xrightarrow{x} \mathcal{V}^0 \xrightarrow{F^0} \mathcal{W}^0,$$

and it sends a 2-cell $(f, \varphi) : (Y \leftarrow A \to X, a) \Rightarrow (Y \leftarrow A' \to X, a')$ to

$$(f, \{F(\varphi_c)|c \in A\}) : (Y \leftarrow A \to X, F(a(-))) \Rightarrow (Y \leftarrow A' \to X, F(a'(-))).$$

The natural transformations establishing its compatibility with the horizontal composition and the identity 1-cells come from those for F. Hence if F is a pseudofunctor then so is $\mathsf{Span}|F$.

If $\mathcal{V}$ and $\mathcal{W}$ are monoidal bicategories and F is a monoidal lax functor (cf. [Gordon-Power-Street 1995, Definition 3.1]) then a monoidal structure is induced on $\mathsf{Span}|F$ in a natural way. All the needed axioms hold for $\mathsf{Span}|F$ thanks to the fact that they hold for F.

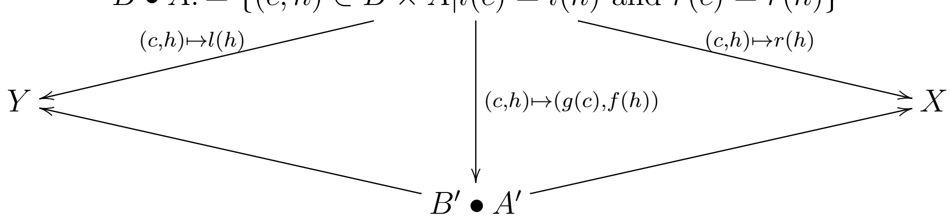
Since any lax functor preserves monads, so does $\mathsf{Span}|F$ for any lax functor F. Since any monoidal lax functor preserves monoidales, so does $\mathsf{Span}|F$ for any monoidal lax functor F. Any monoidal lax functor whose unit- and product-compatibilities are pseudonatural transformations preserves opmonoidal 1- and 2-cells. Hence so does $\mathsf{Span}|F$ whenever the unit- and product-compatibilities of F are invertible.

2.9. Convolution monoidal hom categories and their opmonoidal monads. If M is a monoidale and C is a comonoidale in any monoidal bicategory $\mathcal{M}$ then the hom category $\mathcal{M}(C,M)$ admits a monoidal structure of the convolution type: the monoidal product of 2-cells $\gamma : b \Rightarrow b'$ and $\varphi : a \Rightarrow a'$ between 1-cells $C \to M$ is obtained taking the horizontal composite of the comultiplication of the comonoidale C (which is a 1-cell from C to $C \otimes C$) with the monoidal product of γ and φ in $\mathcal{V}$ (which goes from $C \otimes C$ to $M \otimes M$) and with the multiplication of the monoidale M (which is a 1-cell from $M \otimes M$ to M). The monoidal unit is the horizontal composite of the counit $C \to K$ with the unit $K \to M$.

Via horizontal composition any monad $a : M \to M$ in any bicategory $\mathcal{M}$ induces a monad $\mathcal{M}(C,a)$ in Cat on the hom category $\mathcal{M}(C,M)$, for any 0-cell C of $\mathcal{M}$. If C is a comonoidale, M is a monoidale, and a is an opmonoidal monad in $\mathcal{M}$, then $\mathcal{M}(C,a)$ is canonically an opmonoidal monad in Cat on the above convolution-monoidal category $\mathcal{M}(C,M)$. Moreover, if a is a left or right Hopf monad in $\mathcal{M}$ in the sense of [Chikhladze-Lack-Street 2010], then $\mathcal{M}(C,a)$ is a left or right Hopf monad in Cat in the sense of [Bruguières-Lack-Virelizier 2011].

These considerations apply, in particular, to an induced monoidale $(Y,M) := \{M_p | p \in Y\}$ and an induced comonoidale $(X,C) := \{C_q | q \in X\}$ in $\mathsf{Span}|\mathcal{V}$ (cf. Section 2.6) for any monoidal bicategory $\mathcal{V}$. In the category $\mathsf{Span}|\mathcal{V}((X,C),(Y,M))$ the monoidal product any two morphisms — that is, of 2-cells $(g,\gamma) : (B,b) \Rightarrow (B',b')$ and $(f,\varphi) : (A,a) \Rightarrow (A',a')$ between 1-cells $(X,C) \to (Y,M)$ — is the morphism consisting of the map of spans

$$B \bullet A := \{(c,h) \in B \times A | l(c) = l(h) \text{ and } r(c) = r(h)\} \tag{2.4}$$



and the set

$$\{1 \circ (\gamma_c \otimes \varphi_h) \circ 1 : m_{l(c)} \circ (b(c) \otimes a(h)) \circ d_{r(h)} \to m_{l(c)} \circ (b'g(c) \otimes a'f(h)) \circ d_{r(h)}\}$$

of 2-cells in $\mathcal{V}$ labelled by the elements $(c,h) \in B \bullet A$. The monoidal unit J consists of the complete span $Y \leftarrow Y \times X \rightarrow X$ (whose maps are the first and the second projection, respectively), and the map sending $(i,j) \in Y \times X$ to the 1-cell $u_i \circ e_j : C_j \to M_i$ in $\mathcal{V}$.

Now if (A,a) is an opmonoidal monad on (Y,M), then $\mathsf{Span}|\mathcal{V}((X,C),(A,a))$ is an opmonoidal monad in Cat on the above monoidal category $\mathsf{Span}|\mathcal{V}((X,C),(Y,M))$; which belongs to the realm of the theory of opmonoidal monads in [Bruguières-Lack-Virelizier 2011].

3. Hopf polyads as Hopf monads

In this section we apply the general construction of the previous section to the 2-category Cat of categories, functors and natural transformations; with the monoidal structure provided by the Cartesian product.

3.1. Monads in $\mathsf{Span}|\mathsf{Cat}$ versus polyads. From Section 2.3 we conclude on the coincidence of the following notions.

— A *polyad* in [Bruguières 2015]; that is, a pair consisting of a category and – regarding this category as a bicategory with only identity 2-cells – a lax functor from it to Cat (see [Bruguières 2015, Remark 2.1]).

— A monad in $\mathsf{Span}|\mathsf{Cat}$.

By the application of Section 2.4, the following bicategories are isomorphic, for any given category (2.2).

— The bicategory of polyads over the category (2.2) in [Bruguières 2015, Section 3]. That is, the bicategory of lax functors from (2.2) to Cat, lax natural transformations and modifications.

— The following locally full sub-bicategory in the bicategory of monads in $\mathsf{Span}|\mathsf{Cat}$. The 0-cells are those monads which live on 0-cells $D^0 \to \mathsf{Cat}^0$ (for the given object set D^0), whose 1-cells are of the form ($D^0 \xleftarrow{t} D^1 \xrightarrow{s} D^0$, $d : D^1 \to \mathsf{Cat}^1$) (in terms of the given data s, t), and whose multiplication and unit 2-cells have the respective forms $(\cdot, \mu)$ and (e, η) (with the given maps $\cdot$ and e). The 1-cells are those monad morphisms $((H, h), (f, \varphi))$ whose underlying span H is the trivial span $D^0 = D^0 = D^0$ and whose map f is the canonical isomorphism $D^0 \circ D^1 \cong D^1 \cong D^1 \circ D^0$. The 2-cells are all possible monad transformations (g, γ) (g in them is necessarily the identity map $D^0 \to D^0$).

3.2. The induced monad in Cat. Since a polyad is eventually a monad (D^1, d) in $\mathsf{Span}|\mathsf{Cat}$ on some 0-cell (D^0, C), it induces a monad $\mathsf{Span}|\mathsf{Cat}((Y, H), (D^1, d))$ in Cat on the category $\mathsf{Span}|\mathsf{Cat}((Y, H), (D^0, C))$ for any 0-cell (Y, H) of $\mathsf{Span}|\mathsf{Cat}$, see Section 2.9. An object of the Eilenberg-Moore category of this induced monad is a pair consisting of a 1-cell $(Q, q) : (Y, H) \to (D^0, C)$, and a 2-cell $(r, \varrho) : (D^1, d) \circ (Q, q) \Rightarrow (Q, q)$ in $\mathsf{Span}|\mathsf{Cat}$ which satisfy the associativity and unitality conditions. The morphisms are 2-cells $(Q, q) \Rightarrow (Q', q')$ in $\mathsf{Span}|\mathsf{Cat}$ which are compatible with the actions (r, ϱ) and (r', ϱ').

Let us consider the particular case when the above Y is the singleton set 1 and H takes its single element to the terminal category $\mathbf{1}$; and the corresponding Eilenberg-Moore category of the monad $\mathsf{Span}|\mathsf{Cat}((1, \mathbf{1}), (D^1, d))$. For any monad (D^1, d) on any 0-cell (D^0, C) in $\mathsf{Span}|\mathsf{Cat}$, the following categories are isomorphic (the notation of 2.2) is used).

— The *category of modules* of the polyad (D^1, d) in [Bruguières 2015, Section 2.2]. Recall that an object consists of objects $\{q_x\}$ in C_x for all $x \in D^0$, together with morphisms $\{ d(f)q_{s(f)} \xrightarrow{\varrho_f} q_{t(f)} \}$ in $C_{t(f)}$ for all $f \in D^1$, such that the following diagrams commute for all $x \in D^0$ and all $(f, g) \in D^1 \circ D^1$.

$$\begin{array}{ccc} (d(f) \circ d(g))q_{s(g)} & \xrightarrow{d(f)\varrho_g} & d(f)q_{s(f)} \\ {\scriptstyle (\mu_{f,g})_{q_{s(g)}}}\downarrow & & \downarrow{\scriptstyle \varrho_f} \\ d(f.g)q_{s(g)} & \xrightarrow[\varrho_{f.g}]{} & q_{t(f)} \end{array} \qquad \begin{array}{ccc} q_x & & \\ {\scriptstyle (\eta_x)_{q_x}}\downarrow & \searrow\!\!= & \\ d(e_x)q_x & \xrightarrow[\varrho_{e_x}]{} & q_x \end{array}$$

A morphism $(q, \varrho) \to (q', \varrho')$ consists of morphisms $\{ q_x \xrightarrow{\chi_x} q'_x \}$ in C_x for all $x \in D^0$ such that the following diagram commutes for all $g \in D^1$.

$$\begin{array}{ccc} d(g)q_{s(g)} & \xrightarrow{d(g)\chi_{s(g)}} & d(g)q'_{s(g)} \\ {\scriptstyle \varrho_g}\downarrow & & \downarrow{\scriptstyle \varrho'_g} \\ q_{t(g)} & \xrightarrow[\chi_{t(g)}]{} & q'_{t(g)} \end{array}$$

— The full subcategory of the Eilenberg-Moore category of the monad $\mathsf{Span}|\mathsf{Cat}((1, \mathbf{1}), (D^1, d))$ on $\mathsf{Span}|\mathsf{Cat}((1, \mathbf{1}), (D^0, C))$ whose objects are precisely those Eilenberg–Moore algebras $((Q, q), (r, \varrho))$ whose underlying span Q is $D^0 = D^0 \xrightarrow{!} 1$.

For any monad (D^1, d) on any 0-cell (D^0, C) in $\mathsf{Span}|\mathsf{Cat}$, also the following categories are isomorphic (where the notation of (2.2) is used).

— The *category of representations* of the polyad (D^1, d) in [Bruguières 2015, Section 2.3]. Recall that an object consists of objects $\{W_k\}$ of $C_{t(k)}$ for all $k \in D^1$ together with morphisms $\{ d(g)W_k \xrightarrow{\varrho_{g,k}} W_{g.k} \}$ for $(g, k) \in D^1 \circ D^1$, rendering commutative the following diagrams for all $f, g, k \in D^1 \circ D^1 \circ D^1$.

$$\begin{array}{ccc} (d(f) \circ d(g))W_k & \xrightarrow{d(f)\varrho_{g,k}} & d(f)W_{g.k} \\ {\scriptstyle (\mu_{f,g})_{W_k}}\downarrow & & \downarrow{\scriptstyle \varrho_{f,g.k}} \\ d(f.g)W_k & \xrightarrow[\varrho_{f.g,k}]{} & W_{f.g.k} \end{array} \qquad \begin{array}{ccc} W_k & & \\ {\scriptstyle (\eta_{t(k)})_{W_k}}\downarrow & \searrow\!\!= & \\ d(e_{t(k)})W_k & \xrightarrow[\varrho_{e_{t(k)},k}]{} & W_k \end{array}$$

A morphism $(W, \varrho) \to (W', \varrho')$ consists of morphisms $\{ W_k \xrightarrow{\varphi_k} W'_k \}$ such that the following diagram commutes for all $(g, k) \in D^1 \circ D^1$.

$$\begin{array}{ccc} d(g)W_k & \xrightarrow{d(g)\varphi_k} & d(g)W'_k \\ {\scriptstyle \varrho_{g,k}}\downarrow & & \downarrow{\scriptstyle \varrho'_{g,k}} \\ W_{g.k} & \xrightarrow[\varphi_{g.k}]{} & W'_{g.k} \end{array}$$

— The following non-full subcategory of the Eilenberg-Moore category of the monad $\mathsf{Span|Cat}((1,\mathbf{1}),(D^1,d))$ on $\mathsf{Span|Cat}((1,\mathbf{1}),(D^0,C))$. The objects are precisely those Eilenberg–Moore algebras $((Q,q),(r,\varrho))$ whose underlying span Q is $D^0 \xleftarrow{t} D^1 \xrightarrow{!} 1$ and whose map $r : D^1 \circ D^1 \to D^1$ is the composition in the category D^1. The morphisms are those morphisms of Eilenberg–Moore algebras (f,φ) in which $f : D^1 \to D^1$ is the identity map.

3.3. Opmonoidal monads in $\mathsf{Span|Cat}$ versus opmonoidal polyads. Combining the descriptions in Sections 2.3 and 2.6, we obtain coincidence of the following notions.

— *Opmonoidal polyad* in [Bruguières 2015, Paragraph 2.5]. That is, a pair consisting of a category and – regarding this category as a bicategory with only identity 2-cells – a lax functor from it to OpMon.

— Monad in $\mathsf{Span|OpMon}$.

— Opmonoidal monad in $\mathsf{Span|Cat}$ living on a monoidale of the form in (2.3).

From the isomorphism in Section 2.7, for any given category (2.2) we have isomorphism of the following bicategories.

— The bicategory of opmonoidal polyads over the category (2.2) in [Bruguières 2015, Section 3] (see the top of its page 18). That is, the bicategory of lax functors from (2.2) to OpMon, lax natural transformations and modifications.

— The following locally full sub-bicategory in the bicategory of monads (in the sense of [Street 1972]) in $\mathsf{Span|OpMon}$. The 0-cells are those monads which live on 0-cells $D^0 \to \mathsf{OpMon}^0$ (for the given object set D^0), whose 1-cells are of the form $(D^0 \xleftarrow{t} D^1 \xrightarrow{s} D^0 , \mathbf{d} : D^1 \to \mathsf{OpMon}^1)$ (in terms of the given data s,t), and whose multiplication and unit 2-cells have the respective forms $(\cdot,\mu)$ and (e,η) (with the given maps $\cdot$ and e). The 1-cells are those monad morphisms $((H,\mathbf{h}),(f,\varphi))$ whose underlying span H is the trivial span $D^0 = D^0 = D^0$ and whose map f is the canonical isomorphism $D^0 \circ D^1 \cong D^1 \cong D^1 \circ D^0$. The 2-cells are all possible monad transformations (g,γ) (g in them is necessarily the identity map $D^0 \to D^0$).

— The following locally full sub-bicategory in the bicategory of monads in the bicategory $\mathsf{OpMon}(\mathsf{Span|Cat})$. The 0-cells are those monads which live on monoidales with object part $D^0 \xrightarrow{C} \mathsf{Cat}^0$ (for the given object set D^0) and with multiplication and unit of the form

$$(D^0 = D^0 \xrightarrow{\Delta} D^0 \times D^0 \ , \ D^0 \xrightarrow{\otimes} \mathsf{Cat}^1) \quad \text{and} \quad (D^0 = D^0 \xrightarrow{!} 1 \ , \ D^0 \xrightarrow{K} \mathsf{Cat}^1),$$

whose 1-cells are of the form $(D^0 \xleftarrow{t} D^1 \xrightarrow{s} D^0 , d : D^1 \to \mathsf{Cat}^1)$ (in terms of the given data s,t), and whose multiplication and unit 2-cells have the respective forms $(\cdot,\mu)$ and (e,η) (with the given maps $\cdot$ and e). The 1-cells are those monad morphisms $((H,h),(f,\varphi))$ whose underlying span H is the trivial span $D^0 = D^0 = D^0$ and whose map f is the canonical isomorphism $D^0 \circ D^1 \cong D^1 \cong D^1 \circ D^0$. The 2-cells are all possible monad transformations (g,γ) (g in them is necessarily the identity map $D^0 \to D^0$).

3.4. Hopf monads in $\mathsf{Span}|\mathsf{Cat}$ versus Hopf polyads. Our next task is to compute the fusion 2-cells as in [Chikhladze-Lack-Street 2010] for the opmonoidal monads in $\mathsf{Span}|\mathsf{Cat}$ of Section 3.3. The left fusion 2-cell consists of the map of spans

$$(D^0 \leftarrow \{(p,q) \in D^1 \times D^1 | s(p) = t(q)\} \rightarrow D^0 \times D^0, t(p) \leftarrow\!\shortmid (p,q) \mapsto (s(q), s(p))) \to$$
$$(D^0 \leftarrow \{(p,q) \in D^1 \times D^1 | t(p) = t(q)\} \rightarrow D^0 \times D^0, t(p) \leftarrow\!\shortmid (p,q) \mapsto (s(p), s(q)))$$

sending (p,q) to $(p.q, p)$; and the set of natural transformations

$$d(p)(d(q)(-) \underset{s(p)}{\otimes} (-)) \xrightarrow{d_p^2} (d(p) \circ d(q))(-) \underset{t(p)}{\otimes} d(p)(-) \xrightarrow{\mu_{p,q} \underset{t(p)}{\otimes} 1} d(p.q)(-) \underset{t(p)}{\otimes} d(p)(-) \tag{3.5}$$

between functors $C_{s(q)} \times C_{s(p)} \to C_{t(p)}$, labelled by $(p,q) \in D^1 \circ D^1$ (a label $x \in D^0$ on $\otimes$ refers to the category C_x in which it serves as the monoidal product). This coincides with the left fusion operator of [Bruguières 2015, Definition 2.15].

Clearly, this left fusion 2-cell above is invertible in $\mathsf{Span}|\mathsf{Cat}$ if and only if the underlying category (2.2) is a groupoid and each natural transformation in the set (3.5) is invertible. So we obtained the coincidence of the following notions.

— *Left Hopf polyad* in the sense of [Bruguières 2015, Definition 2.17] whose underlying category is a groupoid. That is, an opmonoidal polyad whose underlying category is a groupoid and for which each of the natural transformations (3.5) is invertible.

— A Hopf monad in $\mathsf{Span}|\mathsf{Cat}$ living on a monoidale of the form in (2.3).

The case of the right fusion 2-cell is symmetric.

3.5. The induced Hopf monad in Cat. Since the monoidal product in Cat is Cartesian, any 0-cell (that is, any category) is a comonoidale in a unique way. Hence the construction in Section 2.6 yields an induced comonoidale (Y, C) in $\mathsf{Span}|\mathsf{Cat}$ for any set of categories $\{C_y | y \in Y\}$.

On the other hand, as described in Section 2.6, any set of *monoidal* categories $\{(M_x, \otimes_x, K_x) | x \in X\}$ induces a monoidale (X, M) in $\mathsf{Span}|\mathsf{Cat}$. So there is a monoidal category $\mathsf{Span}|\mathsf{Cat}((Y,C),(X,M))$ as in Section 2.9.

Let (D^1, d) be an opmonoidal polyad on (D^0, M); that is, an opmonoidal monad in $\mathsf{Span}|\mathsf{Cat}$. It induces an opmonoidal monad in Cat on the category $\mathsf{Span}|\mathsf{Cat}((Y,C),(D^0,M))$, see again Section 2.9. One can define its *Hopf modules* as in [Bruguières-Virelizier 2007] and [Bruguières-Lack-Virelizier 2011, Section 6.5]. Criteria for the equivalence between the category of these Hopf modules and $\mathsf{Span}|\mathsf{Cat}((Y,C),(D^0,M))$ were obtained in [Bruguières-Lack-Virelizier 2011, Theorem 6.11]; known as the *fundamental theorem of Hopf modules*.

The inclusion of the category of representations of a polyad into the Eilenberg-Moore category of the induced monad in Section 3.2 lifts to an inclusion of the category of *Hopf representations* in [Bruguières 2015, Section 6.2] into the above category of Hopf modules in the sense of [Bruguières-Lack-Virelizier 2011], for $(Y, C) = (1, \mathbf{1})$. Hence if the fundamental theorem of Hopf modules in [Bruguières-Lack-Virelizier 2011] holds, then the equivalence therein induces an equivalence between this subcategory in [Bruguières 2015, Section 6.2] and a suitable subcategory of $\mathsf{Span}|\mathsf{Cat}((1, \mathbf{1}), (D^0, M))$. This gives an alternative proof of [Bruguières 2015, Theorem 6.3].

On the other hand, the category of *Hopf modules* in [Bruguières 2015, Section 6.1] does not seem to be a subcategory of the above category of Hopf modules in the sense of [Bruguières-Lack-Virelizier 2011], for $(Y, C) = (1, \mathbf{1})$; and [Bruguières 2015, Theorem 6.1] seems to be of different nature.

4. Hopf group monoids and Hopf categories as Hopf monads on naturally Frobenius opmap monoidales

For an arbitrary object X in any bicategory $\mathcal{M}$, a *monad on* X is exactly the same thing as a *monoid in* the monoidal endohom category $\mathcal{M}(X, X)$ — though one of these equivalent descriptions may turn out to be more convenient in one or another situation.

If X is an *opmap monoidale* (that is, a monoidale or pseudo-monoid whose multiplication and unit 1-cells possess left adjoints) in a *monoidal bicategory* $\mathcal{M}$, then the endohom category $\mathcal{M}(X, X)$ possesses the richer structure of a so-called *duoidal category*; see [Street 2012].

A *duoidal* (or *2-monoidal* in the terminology of [Aguiar-Mahajan 2010]) category is a category with two monoidal structures $(\circ, I)$ and $(\bullet, J)$ which are compatible in the sense that the functors $\circ$ and I, as well as their associativity and unitality natural isomorphisms are opmonoidal for the $\bullet$-product. Equivalently, the functors $\bullet$ and J, as well as their associativity and unitality natural isomorphisms are monoidal for the $\circ$-product. In technical terms it means the existence of four natural transformations (the binary and nullary parts of two (op)monoidal functors) subject to a number of conditions spelled out e.g. in [Aguiar-Mahajan 2010].

For an opmap monoidale X in a monoidal bicategory $\mathcal{M}$, the first monoidal product $\circ$ on $\mathcal{M}(X, X)$ comes from the horizontal composition $\circ$ in $\mathcal{M}$. Since X possesses both structures of a monoidale and a comonoidale (the latter one with the comultiplication and the counit provided by the adjoints of the multiplication and the unit), $\mathcal{M}(X, X)$ has a second monoidal product $\bullet$ of the convolution type, see Section 2.9. Thanks to the (adjunction) relation between the monoidale and the comonoidale X, these monoidal structures $\circ$ and $\bullet$ render $\mathcal{M}(X, X)$ with the structure of duoidal category.

This observation turns out to be very useful: the coincidence of a *monad on* X and a *monoid in* $(\mathcal{M}(X, X), \circ)$ is supplemented with the coincidence of an *opmonoidal endo 1-*

cell on X and a *comonoid in* $(\mathcal{M}(X,X),\bullet)$; see [Böhm-Lack 2016, Section 3.3]. Combining these correspondences, an *opmonoidal monad on* an opmap monoidale X in a monoidal bicategory $\mathcal{M}$ turns out to be exactly the same thing as a *bimonoid in* the duoidal endohom category $\mathcal{M}(X,X)$ (in the sense of [Aguiar-Mahajan 2010, Definition 6.25]), see again [Street 2012] or a review in [Böhm-Lack 2016, Section 3.3].

Although these are *mathematically equivalent* points of view, one of them may turn out to be *more convenient* in one or another situation. Recall for example, that no sensible notion of *antipode* for Hopf monads *on* arbitrary monoidales of monoidal bicategories is known. It is one of the key observations in [Böhm-Lack 2016], however, that for a Hopf monad living on a *naturally Frobenius opmap monoidale*, it can be given a natural meaning. In this situation, the antipode axioms are formulated most easily *in* the duoidal endohom category, see [Böhm-Lack 2016, Theorem 7.2].

Since in this section we shall study Hopf-like structures — *Hopf group monoids* and *Hopf categories* — defined in terms of antipode morphisms, we are to apply this language.

A braided monoidal small category $(V,\otimes,K,c)$ can be regarded as a monoidal bicategory with a single object, in this section we will work with that.

4.1. The bicategory $\mathsf{OpMon}(V)$ for a braided monoidal category V. An object of $\mathsf{OpMon}(V)$ — that is, a monoidale in V — consists of two objects M and U of V (the multiplication and the unit) and three coherence isomorphisms $\alpha : M\otimes M \to M\otimes M$, $\lambda : M\otimes U \to K$ and $\varrho : M\otimes U\to K$ subject to the appropriate pentagon and triangle conditions.

Here we are not interested in arbitrary monoidales in V. The one which plays a relevant role is the *trivial* one which has both the multiplication and the unit equal to the monoidal unit K and all coherence isomorphisms built up from the coherence isomorphisms of V.

A 1-cell of $\mathsf{OpMon}(V)$ — that is, an opmonoidal 1-cell in V — is an object A of V equipped with morphisms $a^2 : A\otimes M \to M'\otimes A\otimes A$ and $a^0 : A\otimes U\to U'$ subject to appropriate coassociativity and counitality conditions.

The endo 1-cells of the trivial monoidale are then the same as the comonoids (A,a^2,a^0) in V.

A 2-cell of $\mathsf{OpMon}(V)$ — that is, an opmonoidal 2-cell in V — is a morphism $A\to A'$ in V which is appropriately compatible with the opmonoidal structures (a^2,a^0) and (a'^2,a'^0).

Between endo 1-cells of the trivial monoidale, the 2-cells are then the same as the comonoid morphisms $(A,a^2,a^0)\to(A',a'^2,a'^0)$.

So for any braided monoidal category V, we obtain isomorphism of the following monoidal categories.

— The endohom category of the trivial monoidale in $\mathsf{OpMon}(V)$.

— The category $\mathsf{Cmd}(V)$ of comonoids in V.

4.2. Sets as naturally Frobenius opmap monoidales in $\mathsf{Span}|V$. Since there is only one 0-cell of the bicategory V, the 0-cells of $\mathsf{Span}|V$ are simply sets. Moreover, the only 0-cell of the bicategory V is the monoidal unit, hence it is a trivial monoidale, so in particular a naturally Frobenius opmap monoidale. Thus for any set X the construction in Section 2.6 yields a naturally Frobenius opmap monoidale in $\mathsf{Span}|V$ with multiplication and unit 1-cells consisting of the respective spans

$$X = X \xrightarrow{\Delta} X \times X \quad \text{and} \quad X = X \xrightarrow{!} 1$$

and in both cases the constant map sending each element of X to the monoidal unit K of V; and trivial (i.e. built up from coherence isomorphisms of V) associativity and unitality coherence 2-cells.

4.3. THE BICATEGORY $\mathsf{Span}|\mathsf{OpMon}(V)$. The isomorphism of Section 2.7 takes an object of $\mathsf{OpMon}(\mathsf{Span}|V)$ of the form in Section 4.2 to the object of $\mathsf{Span}|\mathsf{OpMon}(V)$ which consists of the set X and the constant map sending each element of X to the trivial monoidale in V (see Section 4.1). For brevity we will denote simply by X also this object of $\mathsf{Span}|\mathsf{OpMon}(V)$. We are interested in the 2-full sub-bicategory of $\mathsf{Span}|\mathsf{OpMon}(V)$ defined by these objects.

For any sets X and Y, an object of $\mathsf{Span}|\mathsf{OpMon}(V)(X,Y)$ consists of a span $Y \leftarrow A \rightarrow X$ and a map from A to the object set of the endohom category of the trivial monoidale in $\mathsf{OpMon}(V)$. That is, in view of the isomorphism of Section 4.1, a map a from A to the set of comonoids in V.

A morphism in $\mathsf{Span}|\mathsf{OpMon}(V)(X,Y)$ consists of a map of spans $f: A \to A'$ and morphisms $a(p) \to a'f(p)$ in the endohom category of the trivial monoidale in $\mathsf{OpMon}(V)$, for all $p \in A$. That is, in view of Section 4.1, a set of comonoid morphisms $\{a(p) \to a'f(p) \mid p \in A\}$ in V.

This leads to an isomorphism between the following categories, for any sets X, Y and any braided monoidal category V.

— $\mathsf{OpMon}(\mathsf{Span}|V)(X,Y)$.

— $\mathsf{Span}|\mathsf{OpMon}(V)(X,Y)$.

— $\mathsf{Span}|\mathsf{Cmd}(V)(X,Y)$.

4.4. THE DUOIDAL ENDOHOM CATEGORIES. The structure of an opmap monoidale that we constructed in Section 4.2 on any set X, induces a duoidal structure on the endohom category $\mathsf{Span}|V(X,X)$ which we describe next. It is obtained by a straightforward application of the general construction in [Street 2012], see also [Böhm-Lack 2016, Section 3.3].

The objects of $\mathsf{Span}|V(X,X)$ are pairs consisting of an X-span A and a map a from the set A to the set of objects in V. The morphisms $(A,a) \to (A',a')$ are pairs consisting of a map of X-spans $f: A \to A'$ and a set $\{\varphi_h : a(h) \to a'f(h) | h \in A\}$ of morphisms in V.

The first monoidal product $\circ$ on $\mathsf{Span}|V(X,X)$ comes from the horizontal composition in $\mathsf{Span}|V$; thus in fact from the monoidal product in V: the product of any two morphisms

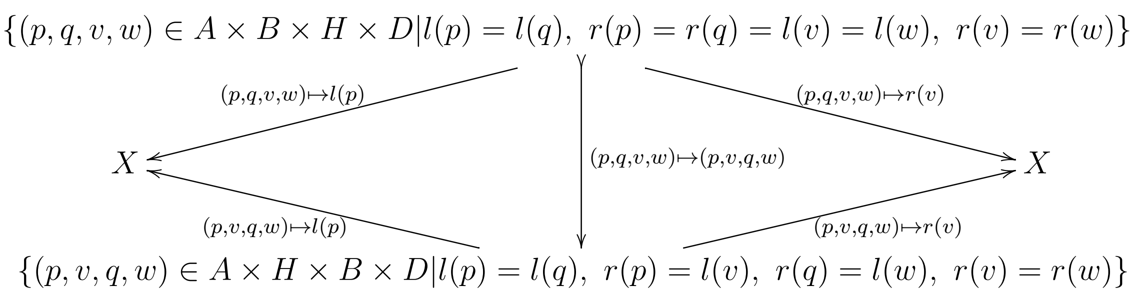
$(g,\gamma):(B,b)\to(B',b')$ and $(f,\varphi):(A,a)\to(A',a')$ is

$$(g\circ f: B\circ A\to B'\circ A', \{\gamma_d\otimes\varphi_h : b(d)\otimes f(h)\to b'g(d)\otimes a'f(h) | (d,h)\in B\circ A\}).$$

The monoidal unit I is the identity 1-cell of X: it consists of the trivial X-span and the map sending each element of X to the monoidal unit K of V.

For any (possibly different) opmap monoidales X and Y of the kind discussed in Section 4.2, the hom category $\mathsf{Span}|V(X,Y)$ admits a monoidal product $\bullet$ which is of the convolution type, see Section 2.9. Now the product of 2-cells $(g,\gamma):(B,b)\Rightarrow(B',b')$ and $(f,\varphi):(A,a)\Rightarrow(A',a')$ between 1-cells $X\to Y$ is the pair consisting of the map of spans in (2.4) and the set $\{\gamma_d\otimes\varphi_h : b(d)\otimes a(h)\to b'g(d)\otimes a'f(h)|(d,h)\in B\bullet A\}$ of morphisms in V. The monoidal unit J consists of the complete span $Y \leftarrow Y\times X \rightarrow X$ and the map sending each element of $Y\times X$ to the monoidal unit K of V.

The above monoidal structures combine into a duoidal structure on $\mathsf{Span}|V(X,X)$. The four structure morphisms take the following forms. The first one is a morphism $((A,a)\bullet(B,b))\circ((H,h)\bullet(D,d))\to((A,a)\circ(H,h))\bullet((B,b)\circ(D,d))$ which is natural in each object $(A,a),(B,b),(H,h),(D,d)$. It consists of the map of spans

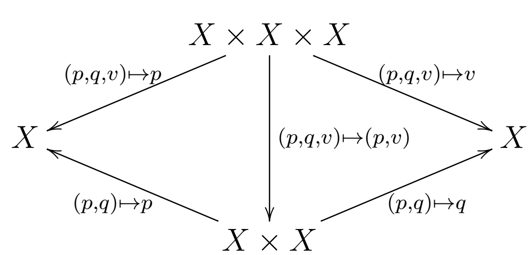


and the set

$$\{1\otimes c\otimes 1 : a(p)\otimes b(q)\otimes h(v)\otimes d(w)\to a(p)\otimes h(v)\otimes b(q)\otimes d(w)\}$$

of morphisms in V, labelled by the elements $(p,q,v,w)\in(A\bullet B)\circ(H\bullet D)$.

Next we need a morphism $J\circ J\to J$; it consists of the map of spans



and the map sending each element of $X\times X\times X$ to the identity morphism of the monoidal unit K of V.

Then we need a morphism $I\to I\bullet I=I$; it is the identity morphism.

Finally we need a morphism $I\to J$. It is given by the diagonal map $\Delta: X\to X\times X$ from the trivial to the complete span and the map sending each element of X to the identity morphism of the monoidal unit K of V.

4.5. The Zunino category. There is a particular duoidal category $\mathsf{Span}|V(1,1)$ of the above form in Section 4.4 for the singleton set 1. Here both monoidal products $\circ$ and $\bullet$ turn out to be equal, and sending any pair of 2-cells $(g,\gamma):(B,b)\Rightarrow(B',b')$ and $(f,\varphi):(A,a)\Rightarrow(A',a')$ between 1-cells $1\to 1$ to

$$(g\times f : B\times A\to B'\times A', \{\gamma_d\otimes\varphi_h : b(d)\otimes f(h)\to b'g(d)\otimes a'f(h)|(d,h)\in B\times A\}).$$

This amounts to saying that the duoidal category $\mathsf{Span}|V(1,1)$ coincides with the braided monoidal *Zunino category*; for its explicit description (in the case when V is the symmetric monoidal category of modules over a commutative ring) see [Caenepeel-De Lombaerde 2006, Section 2.2].

4.6. Hopf group monoids. For an ordinary monoid G (that is, a monoid in the Cartesian monoidal category of sets), a *G-algebra* was defined in [Caenepeel-De Lombaerde 2006, Definition 1.6] as a monoidal functor from G — regarded as a discrete category with object set G and monoidal structure coming from the multiplication $\cdot$ and unit e of G — to the monoidal category of vector spaces (over a given field). Following this idea, we define a *G-monoid* in any monoidal category V as a monoidal functor from G to V. This is the same as a lax functor from the 1-object category G (regarded as a bicategory with only identity 2-cells) to V (regarded as a bicategory with a single 0-cell). Hence from Section 2.3, and from the correspondence between monads on some object and monoids in its composition-monoidal endohom category, we obtain the coincidence of the following notions for any monoidal category V.

— A pair consisting of an ordinary monoid G and a *G-monoid* in V.

— A monad in $\mathsf{Span}|V$ on the singleton set 1.

— A monoid in the Zunino category $\mathsf{Span}|V(1,1)$.

Combining the isomorphism of Section 4.3, and the correspondence of opmonoidal 1-cells on some opmap monoidale and comonoids in its convolution-monoidal endohom category, the following categories are isomorphic for any braided monoidal category V.

— The endohom category of the singleton set 1 in $\mathsf{Span}|\mathsf{Cmd}(V)$.

— The endohom category of the singleton set 1 — regarded as an opmap monoidale in Section 4.2 — in $\mathsf{OpMon}(\mathsf{Span}|V)$.

— The category of comonoids in the Zunino category $\mathsf{Span}|V(1,1)$.

For any monoid G, a *semi Hopf G-algebra* was defined in [Caenepeel-De Lombaerde 2006, Definition 1.7] as a G-monoid (in the above sense) in the monoidal category of coalgebras (over a given field). Following this idea, we define a *semi Hopf G-monoid* in any braided monoidal category V as a G-monoid in $\mathsf{Cmd}(V)$. Hence combining the isomorphism above, and the correspondence between opmonoidal monads on some opmap monoidale and bimonoids in its duoidal endohom category, we obtain the coincidence of the following notions for any monoid monoidal category V.

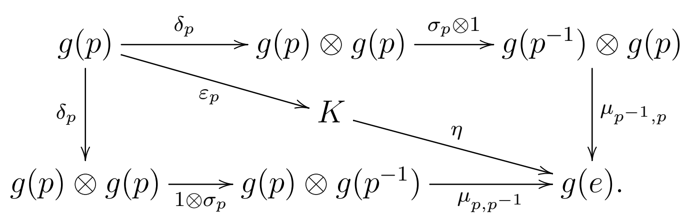
— A pair consisting of a monoid G and a semi Hopf G-monoid in V.

— A monad in $\mathsf{Span}|\mathsf{Cmd}(V)$ on the singleton set 1.

— An opmonoidal monad in $\mathsf{Span}|V$ on the monoidale 1.

— A bimonoid in the Zunino category $\mathsf{Span}|V(1,1)$.

For a group G, a semi Hopf G-algebra — that is, a monoidal functor from the discrete category on the object set G to the monoidal category of coalgebras, sending $p \in G$ to a coalgebra $(g(p), \delta_p, \varepsilon_p)$; with binary part of the monoidal structure denoted by $\{\ g(p) \otimes g(q) \xrightarrow{\mu_{p,q}} g(p.q)\ \}_{p,q \in G}$ and nullary part denoted by $K \xrightarrow{\eta} g(e)$ — was termed a *Hopf G-algebra* in [Caenepeel-De Lombaerde 2006, Definition 1.8] if equipped with linear maps (the so-called *antipode*) $\{\ g(p) \xrightarrow{\sigma_p} g(p^{-1})\ \}_{p \in G}$ rendering commutative the following diagram for all $p \in G$.



By this motivation we define a *Hopf G-monoid* in any braided monoidal category V as a monoidal functor $((g, \delta, \varepsilon), \mu, \eta)$ from the discrete category on the object set G to $\mathsf{Cmd}(V)$ together with morphisms $\{\ g(p) \xrightarrow{\sigma_p} g(p^{-1})\ \}_{p \in G}$ in V rendering commutative the same diagram.

Note that this diagram encodes precisely the antipode axioms of [Böhm-Lack 2016, Theorem 7.2] for the bimonoid g in the duoidal Zunino category $\mathsf{Span}|V(1,1)$; which are in turn the same as the usual antipode axioms for the bimonoid g in the braided monoidal Zunino category $\mathsf{Span}|V(1,1)$. Thus since the singleton set is regarded as a naturally Frobenius opmap monoidale in $\mathsf{Span}|V$ (in the way described in Section 4.2), from [Böhm-Lack 2016, Theorem 7.2] we deduce the coincidence of the following notions for any braided monoidal category V.

— A pair consisting of a group G and a Hopf G-monoid in V.

— A Hopf monoid in the Zunino category $\mathsf{Span}|V(1,1)$.

— A Hopf monad in $\mathsf{Span}|V$ on the monoidale 1.

4.7. Monads in $\mathsf{Span}|V$ versus categories enriched in V. We turn to the interpretation of V-enriched categories in [Batista-Caenepeel-Vercruysse 2016, Section 2] as monads in $\mathsf{Span}|V$, matrices of comonoids in V as in [Batista-Caenepeel-Vercruysse 2016, Section 3] as opmonoidal 1-cells in $\mathsf{Span}|V$, categories enriched in the category of comonoids in V as in [Batista-Caenepeel-Vercruysse 2016, Proposition 3.1] as opmonoidal monads in $\mathsf{Span}|V$, and finally the Hopf categories of [Batista-Caenepeel-Vercruysse 2016, Definition 3.3] as Hopf monads in $\mathsf{Span}|V$.

Recall that a category enriched in V can be described as a pair consisting of a set X (it plays the role of the set of objects) and a lax functor from the indiscrete category on the object set X, regarded as a bicategory with only identity 2-cells, to V, regarded as a bicategory with a single object. An identity-on-objects V-enriched functor is precisely a lax natural transformation whose 1-cell part is trivial.

On the other hand, between monads on the same object in any bicategory, a monad morphism (in the sense of [Street 1972]) with trivial 1-cell part is precisely the same thing as a morphism between the corresponding monoids in the composition-monoidal endohom category.

Using these observations and the fact that the complete span $X \leftarrow X \times X \rightarrow X$ is terminal in $\mathsf{Span}(X, X)$, from Section 2.4 we obtain isomorphism of the following categories, for any braided monoidal category V and any set X.

— The category whose objects are the V-enriched categories with object set X, and whose morphisms are the identity-on-object V-enriched functors. (This category is used in [Batista-Caenepeel-Vercruysse 2016], see its page 1176.)

— The category whose objects are those monads on X in $\mathsf{Span}|V$ which live on such 1-cells of $\mathsf{Span}|V$ whose underlying X-span is the complete span $X \leftarrow X \times X \rightarrow X$; and whose morphisms are those monad morphisms in $\mathsf{Span}|V$ (in the sense of [Street 1972]) whose 1-cell part is the identity 1-cell $X \to X$ in $\mathsf{Span}|V$.

— The full subcategory of the category of monoids in $(\mathsf{Span}|V(X, X), \circ, I)$ whose objects live on such 1-cells of $\mathsf{Span}|V$ in which the underlying X-span is the complete span $X \leftarrow X \times X \rightarrow X$.

4.8. Opmonoidal 1- and 2-cells in $\mathsf{Span}|V$ versus matrices of comonoids, and of comonoid morphisms in V. Again, we are not interested in arbitrary opmonoidal 1- and 2-cells only in those between opmap monoidales X and Y of the kind discussed in Section 4.2.

Let us use again the fact that the complete span $Y \leftarrow Y \times X \rightarrow X$ is terminal in $\mathsf{Span}(X, Y)$. Then from the isomorphism of Section 4.3 on the one hand, and from the correspondence between opmonoidal 1-cells on some opmap monoidale and comonoids in its convolution-monoidal endohom category on the other hand, we obtain the following isomorphism of full subcategories, for any braided monoidal category V and any sets X, Y.

— The category whose objects are matrices of comonoids in V with columns labelled by the elements of X and rows labelled by the elements of Y; and whose morphisms are X by Y matrices of comonoid morphisms in V.

— The full subcategory of opmonoidal 1-cells $X \to Y$ in $\mathsf{Span}|V$ and opmonoidal 2-cells between them, for whose objects the underlying span is the complete span $Y \leftarrow Y \times X \rightarrow X$.

— The full subcategory of comonoids in $(\mathsf{Span}|V(X,Y),\bullet,J)$ for whose objects the underlying span is the complete span $Y \leftarrow Y\times X \rightarrow X$.

4.9. Opmonoidal monads in $\mathsf{Span}|V$ versus categories enriched in $\mathsf{Cmd}(V)$. From the isomorphisms of Section 4.7 and Section 4.3 on the one hand, and the correspondence between opmonoidal monads on an opmap monoidale and the bimonoids in its duoidal endohom category on the other hand, isomorphism of the following categories follows, for any set X and any braided monoidal category V.

— The category whose objects are the $\mathsf{Cmd}(V)$-enriched categories with object set X; and whose morphisms are the identity-on-object $\mathsf{Cmd}(V)$-enriched functors. (This category is used in [Batista-Caenepeel-Vercruysse 2016], see its page 1177.)

— The category in which the objects are those opmonoidal monads in $\mathsf{Span}|V$ on the opmap monoidale X of Section 4.2 in whose 1-cell part $X \to X$ the underlying span is the complete span $X \leftarrow X\times X \rightarrow X$; and whose morphisms are those opmonoidal monad morphisms whose 1-cell part is the identity 1-cell $X \to X$ in $\mathsf{OpMon}(\mathsf{Span}|V)$.

— The full subcategory of the category of bimonoids (in the sense of [Aguiar-Mahajan 2010, Definition 6.25]) in the duoidal category $\mathsf{Span}|V(X,X)$, defined by those objects which live on 1-cells $X \to X$ in $\mathsf{Span}|V$ with underlying span the complete span $X \leftarrow X\times X \rightarrow X$.

4.10. The induced opmonoidal monad in Cat. Regard a V-enriched category with object set X as a monad in $\mathsf{Span}|V$ on the 0-cell X as in Section 4.7. Via horizontal composition it induces a monad in Cat on the category $\mathsf{Span}|V(Y,X)$ for any set Y, see Section 2.9.

If we start with a category enriched in the category of comonoids in V — that is, as a monad in $\mathsf{Span}|V$ it admits an opmonoidal structure with respect to the monoidale of Section 4.2, see Section 4.9 — then so does the induced monad in Cat with respect to the convolution monoidal structure of $\mathsf{Span}|V(Y,X)$, see again Section 2.9. This implies the monoidality (via the product $\bullet$) of the Eilenberg-Moore category of the induced monad.

Consider a $\mathsf{Cmd}(V)$-enriched category with object set X and hom objects $(a(x,y), \delta_{x,y}, \varepsilon_{x,y})$ for $(x,y)\in X\times X$. Denote the composition compatibility morphisms by $\mu_{x,y,z} : a(x,y)\otimes a(y,z) \to a(x,z)$ and denote the unit compatibility morphisms by $\eta_x : K \to a(x,x)$, for all $x,y,z\in X$. For these data, the following monoidal categories are isomorphic.

— The *category of modules* in [Batista-Caenepeel-Vercruysse 2016, Definition 4.1]. Recall that its objects are sets $\{v(p,q)\}_{p,q\in X}$ of objects in V together with morphisms $\{\, a(x,y)\otimes v(y,z) \xrightarrow{\psi_{x,y,z}} v(x,z)\,\}_{x,y,z\in X}$ in V making commutative for all x,y,z,

$u \in X$ the following associativity and unitality diagrams.

$$\begin{array}{ccc} a(x,y)\otimes a(y,z)\otimes v(z,u) & \xrightarrow{\mu_{x,y,z}\otimes 1} & a(x,z)\otimes v(z,u) \\ {\scriptstyle 1\otimes\psi_{y,z,u}}\downarrow & & \downarrow{\scriptstyle \psi_{x,z,u}} \\ a(x,y)\otimes v(y,u) & \xrightarrow[\psi_{x,y,u}]{} & v(x,u) \end{array} \qquad \begin{array}{ccc} v(x,y) & \xrightarrow{\eta_x\otimes 1} & a(x,x)\otimes v(x,y) \\ & \searrow^{=} & \downarrow{\scriptstyle \psi_{x,x,y}} \\ & & v(x,y) \end{array}$$

The morphisms $(v,\psi) \to (v',\psi')$ are sets $\{\, v(x,y) \xrightarrow{\varphi_{x,y}} v'(x,y) \,\}_{x,y\in X}$ of morphisms in V for which the following diagram commutes for all $x,y,z \in X$.

$$\begin{array}{ccc} a(x,y)\otimes v(y,z) & \xrightarrow{1\otimes\varphi_{y,z}} & a(x,y)\otimes v'(y,z) \\ {\scriptstyle \psi_{x,y,z}}\downarrow & & \downarrow{\scriptstyle \psi'_{x,y,z}} \\ v(x,z) & \xrightarrow[\varphi_{x,z}]{} & v'(x,z) \end{array}$$

By [Batista-Caenepeel-Vercruysse 2016, Proposition 4.2] this is a monoidal category with the product $(v\otimes v')(x,y) := v(x,y)\otimes v'(x,y)$ for all $x,y \in X$ and

$$a(x,y)\otimes(v\otimes v')(y,z) \xrightarrow{\delta_{x,y}\otimes 1} a(x,y)\otimes a(x,y)\otimes v(y,z)\otimes v'(y,z) \xrightarrow{1\otimes c\otimes 1}$$
$$a(x,y)\otimes v(y,z)\otimes a(x,y)\otimes v'(y,z) \xrightarrow{\psi_{x,y,z}\otimes\psi'_{x,y,z}} (v\otimes v')(x,z)$$

for $x,y,z \in X$.

— The monoidal full subcategory of the Eilenberg–Moore category of the opmonoidal monad $\mathsf{Span}|V(X,a)$ on $\mathsf{Span}|V(X,X)$, whose objects live on the complete X-span.

4.11. Hopf monads in $\mathsf{Span}|V$ versus Hopf categories. Consider again a $\mathsf{Cmd}(V)$-enriched category with object set X and hom objects $(a(x,y),\delta_{x,y},\varepsilon_{x,y})$ for $(x,y)\in X\times X$. Denote the composition compatibility morphisms by μ and denote the unit compatibility morphisms by η as in the previous section. As we saw in Section 4.9, it can be regarded equivalently as a bimonoid in the duoidal category $\mathsf{Span}|V(X,X)$. In the current situation the *antipode* in the sense of [Böhm-Lack 2016, Theorem 7.2] turns out to be a set of morphisms in V, $\{\, a(v,w) \xrightarrow{\sigma_{v,w}} a(w,v) \,\}_{v,w\in X}$, subject to the axioms in [Böhm-Lack 2016, Theorem 7.2]. The first antipode axiom in [Böhm-Lack 2016, Theorem 7.2] takes now the form in Figure 1. In that figure, for natural numbers $n \geq m$, we denote by p_m the m^{th} projection from the n-fold Cartesian product of X to X, sending $(q_1,\ldots,q_n)$ to q_m.

The second antipode axiom is handled symmetrically. Comparing these diagrams with [Batista-Caenepeel-Vercruysse 2016, Definition 3.3] we conclude by [Böhm-Lack 2016, Theorem 7.2] that for any braided monoidal category V, the following notions coincide.

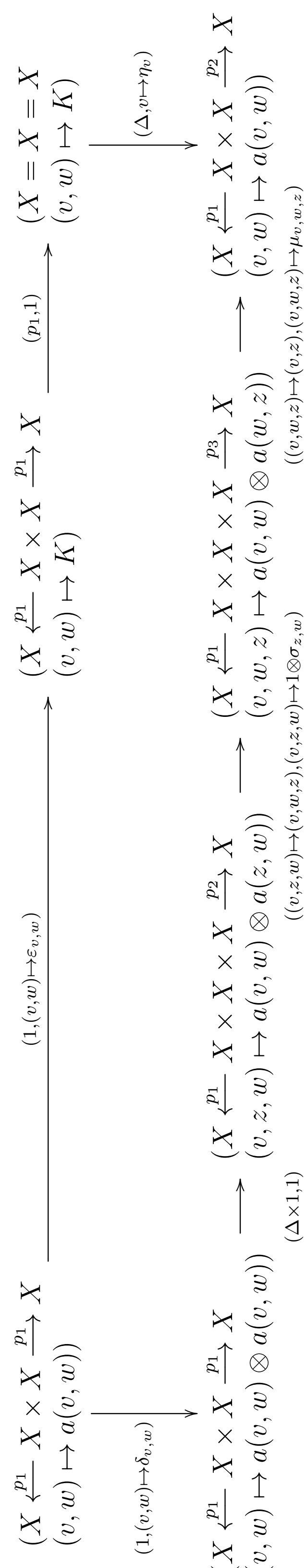


Figure 1: The first antipode axiom

— A *Hopf V-category* in [Batista-Caenepeel-Vercruysse 2016, Definition 3.3]. Explicitly, this means a $\mathsf{Cmd}(V)$-enriched category with some object set X and hom objects $(a(p,q), \delta_{p,q}, \varepsilon_{p,q})$ for $(p,q) \in X \times X$, composition compatibility morphisms $\mu_{p,q,r} : a(p,r) \otimes a(q,r) \to a(p,q)$ and unit compatibility morphisms $\eta_p : K \to a(p,p)$, for all $p,q,r \in X$; equipped with a further set $\{ \, a(p,q) \overset{\sigma_{p,q}}{\dashrightarrow} a(q,p) \, \}_{p,q \in X}$ of morphisms in V rendering commutative the following diagrams for all $p,q \in X$.

$$
\begin{array}{ccc}
a(p,q) & \xrightarrow{\delta_{p,q}} & a(p,q) \otimes a(p,q) \\
 & & \downarrow {\scriptstyle 1 \otimes \sigma_{p,q}} \\
{\scriptstyle \varepsilon_{p,q}} \downarrow & & a(p,q) \otimes a(q,p) \\
 & & \downarrow {\scriptstyle \mu_{p,q,p}} \\
K & \xrightarrow[\eta_p]{} & a(p,p)
\end{array}
\qquad
\begin{array}{ccc}
a(p,q) & \xrightarrow{\delta_{p,q}} & a(p,q) \otimes a(p,q) \\
 & & \downarrow {\scriptstyle \sigma_{p,q} \otimes 1} \\
{\scriptstyle \varepsilon_{p,q}} \downarrow & & a(q,p) \otimes a(p,q) \\
 & & \downarrow {\scriptstyle \mu_{q,p,q}} \\
K & \xrightarrow[\eta_q]{} & a(q,q)
\end{array}
$$

— A Hopf monad in $\mathsf{Span}|V$ on the naturally Frobenius opmap monoidale X of Section 4.2, in whose 1-cell part $X \to X$ the underlying span is the complete span $X \leftarrow X \times X \rightarrow X$.

4.12. The functorial relation of Hopf group monoids and Hopf categories to Hopf polyads. Regarding a braided monoidal category as a monoidal bicategory with a single 0-cell, there is a monoidal pseudofunctor $V \to \mathsf{Cat}$ as follows.

The single 0-cell of the bicategory V is sent to the category V. A 2-cell in the bicategory V — that is, a morphism $f : p \to q$ in the category V — is sent to the natural transformation $f \otimes (-) : p \otimes (-) \to q \otimes (-)$ between endofunctors on V. This is clearly a pseudofunctor. It is monoidal as well via the following ingredients. The unit-compatibility pseudo natural transformation is provided by the 1-cell of Cat (i.e. functor) from the terminal category to V sending the only object to the monoidal unit K; and the isomorphism $K \otimes K \cong K$ in V. The product-compatibility pseudonatural transformation has the object part provided by the monoidal product $\otimes : V \times V \to V$ and the morphism part given by the braiding c of V as $1 \otimes c \otimes 1 : p \otimes (-) \otimes q \otimes (-) \to p \otimes q \otimes (-) \otimes (-)$ for any object (p,q) of $V \times V$. The associativity and unitality modifications are induced by the associativity and unitality natural isomorphisms of V.

This monoidal pseudofunctor $V \to \mathsf{Cat}$ induces a monoidal pseudofunctor from $\mathsf{Span}|V$ to $\mathsf{Span}|\mathsf{Cat}$ whose unit- and product-compatibilities are pseudonatural transformations as well. Since such monoidal pseudofunctors preserve monoidales (but not necessarily opmap monoidales!), monads and opmonoidal morphisms, as well as the invertibility of 2-cells, we conclude that they preserve Hopf monads. In particular, the above monoidal pseudofunctor $\mathsf{Span}|V \to \mathsf{Span}|\mathsf{Cat}$ takes both Hopf group monoids and Hopf categories to Hopf polyads. Hopf polyads in the range of this monoidal pseudofunctor $\mathsf{Span}|V \to \mathsf{Span}|\mathsf{Cat}$ were termed *representable* in [Bruguières 2015, Section 7.2].

References

Marcelo Aguiar and Swapneel Mahajan, *Monoidal Functors, Species and Hopf Algebras.* CRM Monograph Series 29, American Math. Soc. Providence, 2010. Electronically available at: http://www.math.tamu.edu/ maguiar/a.pdf.

Eliezer Batista, Stefaan Caenepeel and Joost Vercruysse, *Hopf Categories,* Algebr. Represent. Theory 19 no. 5 (2016), 1173-1216.

Francis Borceux, *Handbook of Categorical Algebra: Volume 1, Basic Category Theory.* Cambridge University Press 1994.

Gabriella Böhm and Stephen Lack, *Hopf comonads on naturally Frobenius map-monoidales,* J. Pure Appl. Algebra 220 no. 6 (2016), 2177-2213.

Gabriella Böhm, Florian Nill and Kornél Szlachányi, *Weak Hopf algebras. I. Integral theory and C^*-structure,* J. Algebra, 221 no. 2 (1999) 385-438.

Alain Bruguières, *Hopf polyads,* preprint available at http://arxiv.org/abs/1511.04639.

Alain Bruguières, Steve Lack and Alexis Virelizier, *Hopf monads on monoidal categories,* Adv. Math. 227 no. 2 (2011) 745-800.

Alain Bruguières and Alexis Virelizier, *Hopf monads,* Adv. Math. 215 no. 2 (2007) 679-733.

Stefaan Caenepeel and Michel De Lombaerde, *A categorical approach to Turaev's Hopf group-coalgebras,* Comm. Algebra 34 (2006), 2631-2657.

Dimitri Chikhladze, Stephen Lack and Ross Street, *Hopf monoidal comonads,* Theory Appl. Categ. 24 no. 19 (2010) 554-563.

Robert Gordon, Anthony John Power and Ross Street, *Coherence for tricategories.* Mem. Amer. Math. Soc. 117 no. 558, 1995.

John W. Gray, *Formal Category Theory: Adjointness for 2-Categories.* Springer LNM 391, 1974.

Ignacio López Franco, Ross Street and Richard J. Wood, *Duals invert,* Appl. Categ. Structures 19 no. 1 (2011) 321-361.

Ignacio López Franco, *Formal Hopf algebra theory. I. Hopf modules for pseudomonoids,* J. Pure Appl. Algebra 213 no. 6 (2009) 1046-1063.

Paddy McCrudden, *Opmonoidal monads.* Theory Appl. Categ. 10 no 19 (2002) 469-485.

Ieke Moerdijk, *Monads on tensor categories.* J. Pure Appl. Algebra 168 no. 23 (2002) 189-208.

Douglas C. Ravenel, *Complex cobordism and stable homotopy groups of spheres,* Pure Appl. Math., vol. 121, Academic Press, Orlando, FL, 1986.

Peter Schauenburg, *Duals and doubles of quantum groupoids ($\times_R$-Hopf algebras),* AMS Contemp. Math. 267 pp 273-299, AMS, Providence, RI, 2000.

Ross Street, *The formal theory of monads,* J. Pure Appl. Algebra 2 no. 2 (1972) 149-168.

Ross Street, *Monoidal categories in, and linking, geometry and algebra,* Bull. Belg. Math. Soc. Simon Stevin, 19 no. 5 (2012) 769-821.

Vladimir G. Turaev, *Homotopy field theory in dimension 3 and crossed group-categories,* preprint available at http://arxiv.org/abs/0005291.

Marco Zunino, *Double construction for crossed Hopf coalgebras,* J. Algebra 278 (2004), 43-75.

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