Theory and Applications of Categories, Vol. 32, No. 38, 2017, pp. 1258–1362.

FIBERED MULTIDERIVATORS AND (CO)HOMOLOGICAL DESCENT

FRITZ HÖRMANN

Abstract. The theory of derivators enhances and simplifies the theory of triangulated categories. In this article a notion of fibered (multi)derivator is developed, which similarly enhances fibrations of (monoidal) triangulated categories. We present a theory of cohomological as well as homological descent in this language. The main motivation is a descent theory for Grothendieck's six operations.

1. Introduction

This article proposes a general theory of homological and cohomological descent[1]. Our main motivation came from the problem of extending Grothendieck six-functor-formalisms to stacks in a purely formal way. In the present article we deal with six-functor-formalisms only in an appendix. However, for the sake of this introduction, we start by reviewing them *to motivate* the need for notions of (co)homological descent to understand "glueing" properties of the six functors. Our theory of (co)homological descent is build on the notion of **derivator** of Grothendieck. For this purpose it is essential to have a theory of **fibered derivators** because although in the classical, 1-categorical world a fibered category is the same as a pseudo-functor with values in category, for derivators such a statement is not true — pseudo-functors with values in usual derivators in a naive sense do not carry enough information. As a special case of *cohomological* descent we recover the theory of Grothendieck and Deligne developed in [SGA72, Exposé $\mathrm{V}^{\mathrm{bis}}$]. The present theory, however, is more general in that it is not restricted to diagrams of simplicial shape and is completely **self-dual**, leading to a theory of homological descent as well.

Grothendieck's six functors and descent. Let $\mathcal{S}$ be a category, for instance, a suitable category of schemes, topological spaces, analytic manifolds, etc. A six-functor-formalism on $\mathcal{S}$ consists of a collection of (derived) categories $\mathcal{D}_S$, one for each "base space" S in $\mathcal{S}$ with the following six types of operations:

Received by the editors 2016-12-07 and, in final form, 2017-09-25.
Transmitted by Lawrence Breen. Published on 2017-09-27.
2010 Mathematics Subject Classification: 55U35, 14F05, 18D10, 18D30, 18E30, 18G99.
Key words and phrases: Derivators, fibered derivators, multiderivators, fibered multicategories, Grothendieck's six-functor-formalism, cohomological descent, homological descent, fundamental localizers, well-generated triangulated categories, equivariant derived categories.

[1] a homotopic version of decent and codescent.

$$\begin{array}{lll} f^* & f_* & \textit{for each } f \textit{ in } \mathrm{Mor}(\mathcal{S}) \\ f_! & f^! & \textit{for each } f \textit{ in } \mathrm{Mor}(\mathcal{S}) \\ \otimes & \mathcal{HOM} & \textit{in each fiber } \mathcal{D}_S \end{array}$$

The fiber $\mathcal{D}_S$ is, in general, a *derived* category of “sheaves” over S, for example coherent sheaves, l-adic sheaves, abelian sheaves, D-modules, motives, etc. and f^*, resp. f_* are the *derived* pull-back resp. push-forward functors. In each row the functor on the left hand side is the left adjoint of the functors on the right hand side. The functor $f_!$ and its right adjoint $f^!$ are called “push-forward with proper support”, and “exceptional pull-back”, respectively. The six functors come along with a bunch of compatibility isomorphisms between them (cf. A.2.19) and it is not easy to precisely define which commutative diagrams they have to fulfill in order to define a six-functor-formalism. However, another approach, explained in an appendix to this article, gives a quite simple precise definition:

Definition A.2.16. *Let $\mathcal{S}$ be a category with fiber products. A (symmetric) six-functor-formalism on $\mathcal{S}$ is a bifibration*[2] *of (symmetric) 2-multicategories with 1-categorical fibers*

$$p : \mathcal{D} \to \mathcal{S}^{\mathrm{cor}}$$

where $\mathcal{S}^{\mathrm{cor}}$ is the symmetric 2-multicategory of correspondences in $\mathcal{S}$ (cf. Definition A.2.15).

From such a bifibration we obtain the operations f_*, f^* (resp. $f^!$, $f_!$) as pull-back and push-forward along the correspondences

$$\begin{tikzcd}[column sep=small] & X \arrow[dl, "f"'] \arrow[dr, equal] & \\ Y & ; & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=small] & X \arrow[dl, equal] \arrow[dr, "f"] & \\ X & ; & Y, \end{tikzcd}$$

respectively. We get $\mathcal{E} \otimes \mathcal{F}$ for objects $\mathcal{E}, \mathcal{F}$ above X as the target of a coCartesian 2-ary multimorphism from the pair $\mathcal{E}, \mathcal{F}$ over the correspondence

$$\begin{tikzcd}[column sep=small] & & & X \arrow[dlll, equal] \arrow[dll, equal] \arrow[dr, equal] & \\ X & X & & ; & X. \end{tikzcd}$$

Given such a six-functor-formalism and a simplicial resolution $\pi : U_\bullet \to S$ of a space $S \in \mathcal{S}$ (for example arising from a Čech cover w.r.t. a suitable Grothendieck topology)

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} U_2 \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} U_1 \substack{\longrightarrow\\\longrightarrow} U_0,$$

[2] 1-bifibration and 2-bifibration

and given an object $\mathcal{E}$ in $\mathcal{D}_S$, one can construct complexes in the category $\mathcal{D}_S$:

$$\cdots \longrightarrow \pi_{2,!}\pi_2^!\mathcal{E} \longrightarrow \pi_{1,!}\pi_1^!\mathcal{E} \longrightarrow \pi_{0,!}\pi_0^!\mathcal{E} \longrightarrow 0$$

$$\cdots \longleftarrow \pi_{2,*}\pi_2^*\mathcal{E} \longleftarrow \pi_{1,*}\pi_1^*\mathcal{E} \longleftarrow \pi_{0,*}\pi_0^*\mathcal{E} \longleftarrow 0$$

The *first* question of homological (resp. cohomological) descent asks whether the hyper(co)homology of these complexes recovers the homology (resp. cohomology) of $\mathcal{E}$. Without a suitable enhancement of the situation, this question, however, does not make sense because a double complex, once considered as a complex in the derived category, loses the information of the homology of its total complex. There are several remedies for this problem. Classically, if at least the π_i^* are derived functors and $\mathcal{E}$ is acyclic w.r.t. them, one can derive the whole construction to get a coherent double complex. This does not work, however, for the functors $f_!$, $f^!$ which are often only constructed on the derived category. One possibility is to consider enhancements of the triangulated categories in question as dg-categories or ∞-categories. In this article, we have worked out a different approach based on Grothendieck's idea of derivators which is, perhaps, conceptually even simpler. It is sufficiently powerful to glue the six functors and define them for morphisms between stacks, or even higher stacks.

The *second* question of homological (resp. cohomological) descent asks whether the whole category $\mathcal{D}_S$ of objects on $S \in \mathcal{S}$ is equivalent to a category of suitable collections of objects on the U_i (cf. the notion of (co)Cartesian objects, explained below). This is closely related to the question whether the collection $\{\mathcal{D}_S\}$ and the six functors can be extended to $\mathcal{S}$-stacks. If a diagram $U_\bullet$ like above presents such a stack $\mathcal{X}$ then a candidate for the (new) category $\mathcal{D}_\mathcal{X}$ would be this "suitable collection" of objects on the U_i. (Co)homological descent in this form then ensures that this extension is well-defined, i.e. does not depend on the presentation of the stack. Using just the collection of derived categories and trying usual descent of 1-categories runs into the same problems discussed for the first question.

The questions of (co)homological descent do only concern the pairs of adjoint functors f^*, f_*, resp. $f_!$, $f^!$ separately, which can be encoded (classically) as usual bifibered 1-categories

$$\mathcal{D}^* \to \mathcal{S}^{\mathrm{op}} \qquad \mathcal{D}^! \to \mathcal{S}. \tag{1}$$

This is the situation that we want to enhance using the language of derivators in this article. Therefore we will not speak about six-functor-formalisms anymore (except for appendix A.2). We will discuss those in detail in subsequent articles [Hör16, Hör17a, Hör17b]. However, we will already include the monoidal aspect in the definitions — although irrelevant for (co)descent questions — speaking thus about fibered *multi* derivators.

From the point of view of ∞-categories, the two questions of (say) cohomological descent are related as follows. In this world a bifibration $\mathcal{D} \to \mathcal{S}^{\mathrm{op}}$ can be given equivalently as a functor $F : \mathcal{S}^{\mathrm{op}} \to \infty - \mathcal{CAT}$ such that the functors in the image are right adjoints. Given $S \in \mathcal{S}$, a resolution $\pi : U_\bullet \to S$, and an object $\mathcal{E} \in F(S)$, the first question of cohomological descent asks whether the natural map

$$\mathcal{E} \cong \lim_{\Delta} \pi_{i,*}\pi_i^*\mathcal{E},$$

is an isomorphism (or maybe whether it becomes an isomorphism after applying a further push-forward to a suitable base), where lim is the (homotopy) limit of the diagram $\Delta \to F(S)$ given by $\Delta_i \mapsto \pi_{i,*}\pi_i^*\mathcal{E}$.

The second question of cohomological descent asks whether the functor F itself satisfies a similar property. If we, neglecting non-invertible morphisms, consider it as a functor $F : \mathcal{S}^{\mathrm{op}} \to \infty - \mathcal{GRP}$ to ∞-groupoids the question becomes whether

$$F(S) \cong \lim_{\Delta} F(U_i),$$

where lim is the (homotopy) limit of the diagram $F \circ U_\bullet : \Delta \to \infty - \mathcal{GRP}$. From this point of view it is already clear that the second property is stronger and implies the first. Both questions cannot be formulated within the realm of classical derivators. Although those nicely encode the occurring homotopy limit functors, there is no way to obtain the argument diagrams starting from, say, any kind of pseudo-functor $\mathcal{S}^{(\mathrm{op})} \to \mathcal{DER}$ to the 2-category of derivators. However, the language of *fibered* derivators proposed in this article constitutes a nice solution, albeit the similarity between the two questions becomes slightly obscured.

Fibered multiderivators. The notion of triangulated category developed by Grothendieck and Verdier in the 1960's, as successful as it has been, is not sufficient for many purposes, for both practical reasons (certain natural constructions cannot be performed) as well as for theoretical reasons (the axioms are rather involved and lack conceptual clarity). Grothendieck much later [Gro91], and Franke and Heller independently, with the notion of **derivator**, proposed a marvelously simple remedy to both deficiencies. The basic observation is that all problems mentioned above are based on the following fact: Consider a category $\mathcal{C}$ and a class of morphisms $\mathcal{W}$ (quasi-isomorphisms, weak equivalences, etc.) which one would like to become isomorphisms. Then *homotopy limits and colimits* w.r.t. $(\mathcal{C}, \mathcal{W})$ cannot be reconstructed once passed to the homotopy category $\mathcal{C}[\mathcal{W}^{-1}]$ (for example a derived category, or the homotopy category of a model category). Examples of homotopy (co)limits are the cone and, more generally, the total complex of a complex of complexes. Whereas the former is required to exist in a triangulated category in a brute-force way, but not functorially, the notion of total complex is completely lost in the derived category. Furthermore, very basic and intuitive properties of homotopy limits and colimits, and more general Kan extensions, not only determine the additional structure (triangles, shift functors) on a triangulated category but also *imply* all of its rather involved axioms. This idea has been successfully worked out by Cisinski, Franke, Groth, Grothendieck, Heller, Maltsiniotis, and others. We refer to the introductory article [Gro13] for an overview.

The purpose of this article is to propose a notion of fibered (multi)derivator which enhances the notion of a fibration of (monoidal) triangulated categories in the same way as the notion of usual derivator enhances the notion of triangulated category. We emphasize that this new context is very well suited to reformulate (and reprove the theorems of) the classical theory of cohomological descent and to establish a completely dual theory of homological descent which should be satisfied by the $f_!$, $f^!$-functors.

(Co)homological descent with fibered derivators. Pursuing the idea of derivators, there is a neat conceptual solution to the problem of (co)homological descent: Analogously to a derivator $\mathbb{D}$ which associates a (derived) category $\mathbb{D}(I)$ with each diagram shape I, we should consider a (derived) category $\mathbb{D}(I,F)$ for each diagram $F : I \to \mathcal{S}$ (resp. $F : I \to \mathcal{S}^{\mathrm{op}}$). Let a simplicial resolution $\pi : U_\bullet \to S$ as before be given, considered as a morphism $p : (\Delta^{\mathrm{op}}, U_\bullet) \to (\cdot, S)$ of diagrams in $\mathcal{S}$, resp. $i : (\Delta, (U_\bullet)^{\mathrm{op}}) \to (\cdot, S)$ of diagrams in $\mathcal{S}^{\mathrm{op}}$. Assume that the corresponding pull-back i^* has a right adjoint i_*, (respectively that p^* does have a left adjoint $p_!$). Note that this is a straightforward generalization of the question of existence of homotopy (co)limits in usual derivators! Then the first question becomes:

Q1: Is the corresponding unit $\mathrm{id} \to i_* i^*$ (resp. counit $p_! p^* \to \mathrm{id}$) an isomorphism?

However, we do not take the association $(I,F) \mapsto \mathbb{D}(I,F)$ as the fundamental datum, and rather define a **fibered (multi)derivator** to be a morphism of pre-derivators $p : \mathbb{D} \to \mathbb{S}$ (or even pre-multiderivators) satisfying some basic axioms generalizing those of a derivator. If $\mathbb{S}$ is the pre-derivator associated with a category $\mathcal{S}$, the $\mathbb{D}(I,F)$ are reconstructed as the fibers $\mathbb{D}(I)_F$ of the (op)fibration of usual categories $\mathbb{D}(I) \to \mathbb{S}(I)$. This allows for more general situations, where $\mathbb{S}$ is a general derivator, not necessarily associated with an ordinary category. For six-functor-formalism, it will be even necessary to consider $\mathbb{S}$ which are *pre-2-multiderivators*, a notion which will be introduced and investigated in a forthcoming article [Hör17a]. There we will define (and give examples of) a derivator version of a (symmetric) Grothendieck six-functor formalism, that is, a (symmetric) fibered multiderivator

$$p : \mathbb{D} \to \mathbb{S}^{\mathrm{cor}},$$

where $\mathbb{S}^{\mathrm{cor}}$ is the symmetric pre-2-multiderivator of correspondences in $\mathcal{S}$. For the purpose of this article, it suffices to consider two fibered derivators

$$\mathbb{D}^* \to \mathbb{S}^{\mathrm{op}} \qquad \mathbb{D}^! \to \mathbb{S}$$

which are enhancements of the bifibrations (13) encoding the f^*, f_* functors, or the $f_!$, $f^!$-functors, respectively.

We actually define two notions: **left fibered derivators** and **right fibered derivators**. The left case is an enhancement of the concept of an opfibration of categories to derivators and, at the same time, is a generalization of the notion of left derivator, encoding the theory of homotopy left Kan extensions (in particular homotopy colimits). The right case is similarly an enhancement of the concept of a fibration of categories. A classical opfibration with cocomplete fibers, in which the push-forward functors commute with colimits, gives rise to a left fibered derivator and dually.

For an (op)fibration $\mathcal{D} \to \mathcal{S}$ we will always denote by $f_\bullet$ a push-forward functor along a morphism f in $\mathcal{S}$ and by $f^\bullet$ a pull-back functor along the same morphism. Hence, when both exist, $f_\bullet$ is always left adjoint to $f^\bullet$ by the definition of (op)fibration. In the first example in (13), i.e. for $\mathcal{D}^! \to \mathcal{S}$, we have $f^\bullet = f^!$ and $f_\bullet = f_!$. In the second example, i.e. for $\mathcal{D}^* \to \mathcal{S}^{\mathrm{op}}$, we have $f^\bullet = (f^{\mathrm{op}})_*$ and $f_\bullet = (f^{\mathrm{op}})^*$, which might seem confusing at first sight. However, the notation was supposed to resemble the usual notations for pull-back and push-forward functors and, at the same time, should not cause confusion with the left or right Kan extension functors, which we denote by $\alpha_!$, resp. by α_*, for a functor $\alpha : I \to J$ of diagrams. The choice of notation is thus a reasonable compromise.

Coming back to the two main examples associated with a six-functor-formalism, more generally, we may consider *Cartesian (resp. coCartesian) objects* in the fiber over a diagram $(\Delta^{\mathrm{op}}, U_\bullet)$ (resp. $(\Delta, (U_\bullet)^{\mathrm{op}})$), and denote the corresponding subcategories by $\mathbb{D}^!(\Delta^{\mathrm{op}})^{\mathrm{cart}}_{U_\bullet}$ (resp. $\mathbb{D}^*(\Delta)^{\mathrm{cocart}}_{U_\bullet^{\mathrm{op}}}$). These categories are "coherent enhancements" of the following data: collections $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ where $\mathcal{E}_n$ lies in the fiber over U_n, and for each morphism $\epsilon : \Delta_n \to \Delta_m$ *iso* morphisms $U(\epsilon)^*\mathcal{E}_n \to \mathcal{E}_m$ (resp. $\mathcal{E}_m \to U(\epsilon)^!\mathcal{E}_n$).

The second question of (co)homological descent becomes:

Q2: Do the categories of (co)Cartesian objects depend only on $U_\bullet$ up to taking (finite) hypercovers w.r.t. a fixed Grothendieck topology on $\mathcal{S}$? In particular, if an object S in $\mathbb{S}(\cdot)$ is presented by a Čech cover (or hypercover) $U_\bullet$, do we have

$$\mathbb{D}^!(\Delta^{\mathrm{op}})^{\mathrm{cart}}_{U_\bullet} \cong \mathbb{D}^!(\cdot)_S, \quad \text{resp.} \quad \mathbb{D}^*(\Delta)^{\mathrm{cocart}}_{U_\bullet^{\mathrm{op}}} \cong \mathbb{D}^*(\cdot)_S?$$

The categories of coCartesian objects can also be seen as a generalization of the **equivariant derived categories** of Bernstein and Lunts (cf. 3.4.3).

We call a fibered derivator **(co)local** w.r.t. a given Grothendieck pre-topology on the base (cf. section 2.5) if a few simple axioms are satisfied, and prove that they imply that (co)homological descent as described in Q1 and Q2 for all finite hypercovers is satisfied.

These axioms are: For each covering $\{f_i : U_i \to S\}$ in the given Grothendieck pre-topology, the corresponding pull-backs f_i^* (resp. $f_i^!$)

1. are jointly conservative,

2. satisfy base-change,

3. and commute with homotopy limits (resp. homotopy colimits) as well.

In a six-functor formalism most of these properties follow from isomorphisms of the form $f^! \cong f^*[n]$, see Remark 2.5.7. The stronger form of Q2 is only proven under the stronger technical hypothesis that the fibers are stable, hence triangulated, and well-generated (resp. compactly generated).

Ayoub has considered in [Ayo07a, Ayo07b] a notion of *algebraic derivator*, which is either a pseudo-functor

$$\mathrm{Dia}(\mathcal{S})^{1-\mathrm{op}} \to \mathcal{CAT} \tag{2}$$

in which all functors in the image have left adjoints, or a pseudo-functor

$$\mathrm{Dia}^{\mathrm{op}}(\mathcal{S})^{1-\mathrm{op}} \to \mathcal{CAT} \tag{3}$$

in which all functors in the image have right adjoints. While these definitions would work the same way when the category $\mathcal{S}$ is replaced by a pre-derivator $\mathbb{S}$, they are not the precise analogues of the notions of fibration (resp. opfibration) in category theory, for the following reasons: While a right fibered derivator in our sense gives rise to a datum (2) via

$$(\alpha, f) \mapsto f^{\bullet}\alpha^*$$

(cf. also 2.6.3) the functors in the image only have left adjoints, if the opfibrations $\mathbb{D}(I) \to \mathbb{S}(I)$ are bifibrations as well (hence when one of the axioms of a left fibered derivator holds as well). Similarly a left fibered derivator in our sense gives rise to a datum (3) via

$$(\alpha, f) \mapsto f_{\bullet}\alpha^*.$$

It only has right adjoints, if the fibrations $\mathbb{D}(I) \to \mathbb{S}(I)$ are bifibrations as well (hence when one of the axioms of a right fibered derivator holds as well).

To specify a left *and* right fibered derivator one would need to specify both pseudo-functors (2) and (3) to state the axioms neatly, but then it becomes unclear how to specify that one pseudo-functor determines the other (which they do, in this case). It is possible, though, to consolidate both viewpoints. This will be explained in a subsequent article [Hör16], where it is shown that a left (resp. right) fibered derivator — or even multiderivator — is basically the same as an opfibration (resp. a fibration) of 2-(multi)categories with 1-categorical fibers

$$\mathcal{D} \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$$

where $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ is a very natural 2-(multi)category of correspondences of diagrams in $\mathbb{S}$. Therefore, e.g. a left fibered multiderivator can also be specified by a pseudo-functor

$$\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}) \to \mathcal{CAT}$$

(without the explicit requirement that adjoints of the images exist — they exist already in the 2-category $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$). Also most of the other axioms of a fibered multiderivator follow automatically. This consolidates the viewpoints because we have natural embeddings

$$\mathrm{Dia}(\mathbb{S})^{2-\mathrm{op}} \hookrightarrow \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}) \qquad \mathrm{Dia}^{\mathrm{op}}(\mathbb{S})^{1-\mathrm{op}} \hookrightarrow \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}).$$

Another point of view is the following: Much like the value $\mathbb{D}(\Delta_1)$ of Δ_1 under a derivator $\mathbb{D}$ provides a "coherent enhancement" of the category $\mathbb{D}(\cdot)^{\Delta_1}$ of morphisms in $\mathbb{D}(\cdot)$, the underlying homotopy category (e.g. a derived category), a left fibered derivator over the category Δ_1 should be seen as a coherent enhancement of a *left continuous morphism of derivators*. Similarly a right fibered derivator over the category Δ_1 should be seen as a coherent enhancement of a *right continuous morphism of derivators*. In particular, a left and right fibered derivator over Δ_1 is an enhancement of an adjunction

$$\begin{tikzcd} \mathbb{D}_1 \arrow[r, bend left, "L"] & \mathbb{D}_2 \arrow[l, bend left, "R"] \end{tikzcd}$$

between derivators, and, in particular, gives rise to coherent versions of the unit $\mathcal{E} \to RL\mathcal{E}$ and counit $LR\mathcal{E} \to \mathcal{E}$ as objects in $\mathbb{D}_1(\Delta_1)$ and $\mathbb{D}_2(\Delta_1)$. Note that it is not possible to get these in a functorial way from an adjunction between (even strong) derivators. The collection of left (resp. right) fibered derivators on small categories should therefore be seen as some kind of "**derivator of derivators**". However, we did not attempt to investigate any axioms regarding (homotopy) Kan extensions along functors between the bases.

Overview. In section 2 we give the general definition of a left (resp. right) fibered multiderivator $p : \mathbb{D} \to \mathbb{S}$. The axioms are basically a straight-forward generalization of those of a left (resp. right) derivator. To give a priori some conceptual evidence that these axioms are indeed reasonable, we prove that the notion of fibered multiderivator — like the notion of fibration of categories — is transitive (2.4), and that it gives rise to a pseudo-functor from 'diagrams in $\mathbb{S}$' to categories as mentioned in the previous discussion, for which a neat base-change formula holds (2.6).

In section 3, a theory of (co)homological descent for fibered derivators is developed (the monoidal, i.e. multi-, aspect does not play any role here). We propose a definition of **localizer** (resp. of **system of relative localizers**) in the category of diagrams in $\mathbb{S}$ which is a generalization of Grothendieck's notion of fundamental localizer in categories. The latter gives a nice combinatorial description of weak equivalences of categories in terms of the condition of Quillen's theorem A. In our more general setting the notion of fundamental localizer depends on the choice of a Grothendieck (pre-)topology on $\mathbb{S}$. In section 3.3 we show purely abstractly that a finite hypercover, considered as a morphism of simplicial diagrams, lies in any localizer or system of relative localizers. The formulation in terms of localizers thus has the additional advantage that the notions and theorems of (co)homological descent do not involve in any way the explicit choice of the simplex category Δ.

Note that this more general notion of localizer has a similar relation to weak equivalences of simplicial pre-sheaves like the classical notion of localizer has to weak equivalences of simplicial sets (or topological spaces), although we will not yet give any precise statement in this direction.

In sections 3.4 and 3.5 these new notions of localizer are tied to the theory of fibered derivators. We introduce two notions of (co)homological descent for a fibered derivator $p : \mathbb{D} \to \mathbb{S}$. We call a morphism of $\mathbb{S}$-diagrams $D_1 = (I, F) \to D_2 = (J, G)$ over some $S \in \mathbb{S}(\cdot)$

a **weak $\mathbb{D}$-equivalence** relative to the base S if the corresponding map

$$\pi_{1!}\pi_1^* \to \pi_{2!}\pi_2^*$$

is a natural isomorphism, where the π_i are the respective structural morphisms. (Dually there is a cohomological notion as well). This notion of weak $\mathbb{D}$-equivalences (related to Q1 above) is a straight-forward generalization of Cisinski's notion of $\mathbb{D}$-equivalence [Cis03] for usual derivators. A morphism of $\mathbb{S}$-diagrams $(I,F) \to (J,G)$ is called a **strong $\mathbb{D}$-equivalence** (a notion related to Q2 above) if it induces an equivalence

$$\mathbb{D}(J)_G^{\mathrm{cart}} \to \mathbb{D}(I)_F^{\mathrm{cart}}.$$

In our relative context, both notions of $\mathbb{D}$-equivalence come in a cohomological as well as in a homological flavour (for $\mathbb{S} = \{\cdot\}$, i.e. for usual derivators there is no difference between the homological and cohomological version).

Whenever the fibered derivator is (co)local w.r.t. to the Grothendieck pre-topology — as explained above — then the Main Theorem 3.5.4 (resp. 3.5.5) of this article states that weak $\mathbb{D}$-equivalences form a system of relative localizers under very general conditions (the easier case) and that strong $\mathbb{D}$-equivalences form an absolute localizer, for fibered derivators with stable, well-generated (resp. compactly generated) fibers.

The proof uses results from the theory of triangulated categories due to Neeman and Krause (centering around Brown representability type theorems). The link of these results to our theory of fibered (multi)derivators is explained in section 4.

In section 5 we introduce the notion of bifibration of multi-model-categories. Roughly, those are families of closed monoidal model categories in which all pairs of pull-back and push-forward functors form Quillen adjunctions. The language of bifibrations of multicategories, however, has the tremendous advantage that no axioms for the compatibilities between the functors involved have to be specified explicitly (cf. also the introduction to the six functors above). This is the most favorable standard context in which a fibered multiderivator (whose base is *representable*, i.e. associated with a usual multicategory) can be constructed. We will present more general methods of constructing fibered multiderivators in a forthcoming article, in particular those encoding a full six-functor-formalism.

The author thanks Kevin Carlson, Ian Coley, Ioannis Lagkas and John Zhang for helpful comments on a preliminary version of this article, and the anonymous referee for valuable comments which led to simplifications compared to earlier versions of this article.

Notation

We denote by $\mathcal{CAT}$ the 2-"category"[3] of categories, by $(\mathcal{S})\mathcal{MCAT}$ the 2-"category" of (symmetric) multicategories, and by Cat the 2-category of small categories. We consider a partially ordered set (poset) X as a small category by interpreting the relation $x \le y$ to be

[3] where "category" has classes replaced by 2-classes (or, if the reader prefers, is constructed w.r.t. a larger universe).

equivalent to the existence of a unique morphism $x \to y$. We denote the positive integers (resp. non-negative integers) by $\mathbb{N}$ (resp. $\mathbb{N}_0$). The ordered sets $\{0, \dots, n\} \subset \mathbb{N}_0$ considered as a small category are denoted by Δ_n. We denote by $\mathrm{Mor}(\mathcal{D})$ (resp. $\mathrm{Iso}(\mathcal{D})$) the class of morphisms (resp. isomorphisms) in a category $\mathcal{D}$. The final category (which consists of only one object and its identity) is denoted by $\cdot$ or Δ_0. The same notation is also used for the final multicategory, i.e. that with one object and precisely one n-ary morphism for any $n \in \mathbb{N}_0$. Our conventions about multicategories and fibered (multi)categories are summarized in appendix A.

2. Fibered derivators

2.1. Categories of diagrams.

2.1.1. Definition. *A* **diagram category** *is a full sub-2-category* $\mathrm{Dia} \subset \mathrm{Cat}$*, satisfying the following axioms:*

(Dia1) The empty category $\varnothing$*, the final category* $\cdot$ *(or* Δ_0*), and* Δ_1 *are objects of* Dia.

(Dia2) Dia *is stable under taking finite coproducts and fibered products.*

(Dia3) All comma categories $I \times_{/J} K$ *for functors* $I \to J$ *and* $K \to J$ *in* Dia *are in* Dia.

A diagram category Dia *is called* **self-dual***, if it satisfies in addition:*

(Dia4) If $I \in \mathrm{Dia}$ *then* $I^{\mathrm{op}} \in \mathrm{Dia}$.

A diagram category Dia *is called* **infinite***, if it satisfies in addition:*

(Dia5) Dia *is stable under taking arbitrary coproducts.*

In the following we mean by a **diagram** a small category.

2.1.2. Example. We have the following diagram categories:

Cat the category of all **diagrams**. It is self-dual.

Inv the category of **inverse** diagrams C, i.e. small categories C such that there exists a functor $C \to \mathbb{N}_0$ with the property that the preimage of an identity consists of identities[4]. An example is the injective simplex category Δ°:

$$\cdots \substack{\leftarrow\\ \leftarrow\\ \leftarrow\\ \leftarrow} \cdot \substack{\leftarrow\\ \leftarrow\\ \leftarrow} \cdot \substack{\leftarrow\\ \leftarrow} \cdot$$

Dir the category of **directed** diagrams D, i.e. small categories such that D^{op} is inverse. An example is the opposite of the injective simplex category $(\Delta^{\circ})^{\mathrm{op}}$:

$$\cdots \substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow} \cdot \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \cdot \substack{\rightarrow\\ \rightarrow} \cdot$$

[4] In many sources $\mathbb{N}_0$ is replaced by any ordinal.

Catf, Dirf, and Invf are defined as before but consisting of **finite diagrams**. Those are self-dual and Dirf = Invf.

Catlf, Dirlf, and Invlf are defined as before but consisting of **locally finite diagrams**, i.e. those which have the property that a morphism γ factors as $\gamma = \alpha \circ \beta$ only in a finite number of ways.

Pos, Posf, Dirpos, and Invpos: the categories of **posets, finite posets, directed posets,** and **inverse posets**.

2.2. Pre-(multi)derivators.

2.2.1. Definition. *A* **pre-derivator** *of domain* Dia *is a contravariant (strict) 2-functor*

$$\mathbb{D} : \mathrm{Dia}^{1-\mathrm{op}} \to \mathcal{CAT}.$$

A **pre-multiderivator** *of domain* Dia *is a contravariant (strict) 2-functor*

$$\mathbb{D} : \mathrm{Dia}^{1-\mathrm{op}} \to \mathcal{MCAT}$$

into the 2-"category" of multicategories. A morphism of pre-(multi)derivators is a pseudo-natural transformation of 2-functors.

For a morphism $\alpha : I \to J$ in Dia *the corresponding functor*

$$\mathbb{D}(\alpha) : \mathbb{D}(J) \to \mathbb{D}(I)$$

will be denoted by α^*.

We call a pre-multiderivator **symmetric** *(resp.* **braided***), if its images are symmetric (resp. braided), and the morphisms α^* are compatible with the actions of the symmetric (resp. braid) groups.*

2.2.2. The pre-(multi)derivator **represented by a (multi)category**: Let $\mathcal{S}$ be a (multi-) category. We associate with it the pre-(multi)derivator

$$\mathbb{S} : I \mapsto \mathrm{Hom}(I, \mathcal{S}).$$

The pull-back α^* is defined as composition with α. A 2-morphism $\kappa : \alpha \to \beta$ induces a natural 2-morphism $\mathbb{S}(\kappa) : \alpha^* \to \beta^*$.

2.2.3. The pre-derivator associated with a simplicial class (in particular, the one associated with an ∞-category): Let $\mathcal{S}$ be a simplicial class, i.e. a functor

$$\mathcal{S} : \Delta \to \mathcal{CLASS}$$

into the "category" of classes. We associate with it the pre-derivator

$$\mathbb{S} : I \mapsto \mathrm{Ho}(\mathrm{Hom}(N(I), \mathcal{S})),$$

where $N(I)$ is the nerve of I and Ho is the left adjoint of N. In detail this means the objects of the category $\mathbb{S}(I)$ are morphisms $\alpha : N(I) \to \mathcal{S}$, the class of morphisms in $\mathbb{S}(I)$ is freely generated by morphisms $\mu : N(I \times \Delta_1) \to \mathcal{S}$ considered to be a morphism from its restriction to $N(I \times \{0\})$ to its restriction to $N(I \times \{1\})$ modulo the relations given by morphisms $\nu : N(I \times \Delta_2) \to \mathcal{S}$, i.e. if ν_1, ν_2 and ν_3 are the restrictions of ν to the 3 faces of Δ_2 then we have $\mu_3 = \mu_2 \circ \mu_1$. The pull-back α^* is defined as composition with the morphism $N(\alpha) : N(I) \to N(J)$. A 2-morphism $\kappa : \alpha \to \beta$ can be given as a functor $I \times \Delta_1 \to J$ which yields (applying N and composing) a natural transformation which we call $\mathbb{S}(\kappa)$.

2.2.4. The following will not be needed in this article. More generally, consider the full subcategory $m\Delta \subset \mathcal{MCAT}$ of all *finite connected* multicategories M that are freely generated by a finite set of multimorphisms $f_1, \dots, f_n$ such that each object of M occurs at most once as a source and at most once as the target of one of the f_i. Similarly consider the full subcategory $T \subset \mathcal{SMCAT}$ which is obtained from $m\Delta$ adding images under the operations of the symmetric groups. This category is usually called the symmetric tree category. With a functor

$$\mathcal{S} : m\Delta \to \mathcal{CLASS} \quad \text{resp.} \quad \mathcal{S} : T \to \mathcal{CLASS}$$

we associate the pre-multiderivator (resp. symmetric pre-multiderivator):

$$\mathbb{S} : I \mapsto \mathrm{Ho}(\mathrm{Hom}(N(I), \mathcal{S})),$$

where $N : \mathcal{MCAT} \to \mathcal{CLASS}^{m\Delta}$ (resp. $N : \mathcal{SMCAT} \to \mathcal{CLASS}^{T}$) is the nerve, I is considered to be a multicategory without any n-ary morphisms for $n \geq 2$, and Ho is the left adjoint of N. Objects in $\mathcal{SET}^T$ are called dendroidal sets in [MW07].

2.3. Fibered (multi)derivators.

2.3.1. Let $p : \mathbb{D} \to \mathbb{S}$ be a strict morphism of pre-derivators with domain Dia, and let $\alpha : I \to J$ be a functor in Dia. Consider an object $S \in \mathbb{S}(J)$. The functor α^* induces a morphism between fibers (denoted the same way)

$$\alpha^* : \mathbb{D}(J)_S \to \mathbb{D}(I)_{\alpha^* S}.$$

We are interested in the case that the latter has a left adjoint $\alpha_!^S$, resp. a right adjoint α_*^S. These will be called ***relative*** **left/right homotopy Kan extension** functors with **base** S. For better readability we often omit the base from the notation. Though the base is not determined by the argument of $\alpha_!$, it will often be understood from the context, cf. also 2.3.28.

2.3.2. We are interested in the case in which all morphisms

$$p(I) : \mathbb{D}(I) \to \mathbb{S}(I)$$

are fibrations, resp. opfibrations (A.1) or, more generally, (op)fibrations of multicategories (A.2).

Then we will choose an associated pseudo-functor, i.e. for each $f : S \to T$ in $\mathbb{S}(I)$ a pair of adjoints functors

$$f_\bullet : \mathbb{D}(I)_S \to \mathbb{D}(I)_T,$$

resp.

$$f^\bullet : \mathbb{D}(I)_T \to \mathbb{D}(I)_S,$$

characterized by functorial isomorphisms:

$$\mathrm{Hom}_f(\mathcal{E},\mathcal{F}) \cong \mathrm{Hom}_{\mathrm{id}_S}(\mathcal{E}, f^\bullet\mathcal{F}) \cong \mathrm{Hom}_{\mathrm{id}_T}(f_\bullet\mathcal{E},\mathcal{F}).$$

More generally, in the multicategorical setting, for a multimorphism $f \in \mathrm{Hom}(S_1,\dots,S_n;T)$ for some $n \geq 1$, we get an *adjunction of n variables*

$$f_\bullet : \mathbb{D}(I)_{S_1} \times \cdots \times \mathbb{D}(I)_{S_n} \to \mathbb{D}(I)_T,$$

and

$$f^{\bullet,i} : \mathbb{D}(I)^{\mathrm{op}}_{S_1} \times \overset{\widehat{i}}{\cdots} \times \mathbb{D}(I)^{\mathrm{op}}_{S_n} \times \mathbb{D}(I)_T \to \mathbb{D}(I)_{S_i}.$$

2.3.3. For a diagram of categories

$$\begin{tikzcd} & I \arrow[d, "\alpha"] \\ K \arrow[r, "\beta"'] & J \end{tikzcd}$$

the **slice category** $K \times_{/J} I$ is the category of triples (k,i,μ), where $k \in K$, $i \in I$ and $\mu : \beta(k) \to \alpha(i)$ is a morphism in J. It sits in a corresponding 2-commutative square:

$$\begin{tikzcd} K \times_{/J} I \arrow[r, "B"] \arrow[d, "A"'] \arrow[dr, phantom, "\nearrow^{\mu}"] & I \arrow[d, "\alpha"] \\ K \arrow[r, "\beta"'] & J \end{tikzcd}$$

which is universal w.r.t. such squares. This construction is associative, but of course not commutative unless J is a groupoid. The projection $K \times_{/J} I \to K$ is a fibration and the projection $K \times_{/J} I \to I$ is an opfibration (see A.1). There is an adjunction

$$\begin{tikzcd} I \times_{/J} J \arrow[r, shift left] & I. \arrow[l, shift left] \end{tikzcd}$$

2.3.4. Consider an arbitrary 2-commutative square

$$\begin{tikzcd} L \arrow[r, "B"] \arrow[d, "A"'] \arrow[dr, phantom, "\nearrow^{\mu}"] & I \arrow[d, "\alpha"] \\ K \arrow[r, "\beta"'] & J, \end{tikzcd} \tag{4}$$

let $S \in \mathbb{S}(J)$ be an object, and $\mathcal{E}$ a preimage in $\mathbb{D}(J)$ w.r.t. p. The 2-morphism (natural transformation) μ induces a functorial morphism (the value of μ under the strict 2-functor $\mathbb{S}$)

$$\mathbb{S}(\mu) : A^*\beta^*S \to B^*\alpha^*S$$

and therefore a functorial morphism

$$\mathbb{D}(\mu) : A^*\beta^*\mathcal{E} \to B^*\alpha^*\mathcal{E}$$

over $\mathbb{S}(\mu)$, or — if we are in the (op)fibered situation — equivalently

$$A^*\beta^*\mathcal{E} \to (\mathbb{S}(\mu))^\bullet B^*\alpha^*\mathcal{E}$$

respectively

$$(\mathbb{S}(\mu))_\bullet A^*\beta^*\mathcal{E} \to B^*\alpha^*\mathcal{E}$$

in the fiber above $A^*\beta^*S$, resp. $B^*\alpha^*S$,

Let now $\mathcal{F}$ be an object over α^*S. If relative right homotopy Kan extensions exist, we may form the following composition which will be called the right **base-change morphism**:

$$\beta^*\alpha_*\mathcal{F} \to A_*A^*\beta^*\alpha_*\mathcal{F} \to A_*(\mathbb{S}(\mu))^\bullet B^*\alpha^*\alpha_*\mathcal{F} \to A_*(\mathbb{S}(\mu))^\bullet B^*\mathcal{F}. \tag{5}$$

(We again omit the base S from the notation for better readability — it is always determined by the argument.)

Let now $\mathcal{F}$ be an object over β^*S. If relative left homotopy Kan extensions exist, we may form the composition, the left **base-change morphism**:

$$B_!(\mathbb{S}(\mu))_\bullet A^*\mathcal{F} \to B_!(\mathbb{S}(\mu))_\bullet A^*\beta^*\beta_!\mathcal{F} \to B_!B^*\alpha^*\beta_!\mathcal{F} \to \alpha^*\beta_!\mathcal{F}. \tag{6}$$

We will later say that the square (4) is **homotopy exact** if (5) is an isomorphism for all right fibered derivators (see Definition 2.3.6 below) and (6) is an isomorphism for all left fibered derivators. It is obvious a priori that for a left *and* right fibered derivator (5) is an isomorphism if and only if (6) is, one being the adjoint of the other (see [Gro13, §1.2] for analogous reasoning in the case of usual derivators).

2.3.5. Definition. *We consider the following axioms*[5] *on a pre-(multi)derivator* $\mathbb{D}$*:*

(Der1) For I, J *in* Dia*, the natural functor* $\mathbb{D}(I \coprod J) \to \mathbb{D}(I) \times \mathbb{D}(J)$ *is an equivalence of (multi)categories. Moreover* $\mathbb{D}(\varnothing)$ *is not empty.*

(Der2) For I *in* Dia *the 'underlying diagram' functor*

$$\mathrm{dia} : \mathbb{D}(I) \to \mathrm{Hom}(I, \mathbb{D}(\cdot))$$

is conservative.

In addition, we consider the following axioms for a strict morphism of pre-(multi)derivators

$$p : \mathbb{D} \to \mathbb{S} :$$

(FDer0 left) For each I *in* Dia *the morphism* p *specializes to an opfibered (multi)category and any functor* $\alpha : I \to J$ *in* Dia *induces a diagram*

$$\begin{tikzcd} \mathbb{D}(J) \arrow[r, "\alpha^*"] \arrow[d] & \mathbb{D}(I) \arrow[d] \\ \mathbb{S}(J) \arrow[r, "\alpha^*"] & \mathbb{S}(I) \end{tikzcd}$$

of opfibered (multi)categories, i.e. the top horizontal functor maps coCartesian morphisms to coCartesian morphisms.

(FDer3 left) For each functor $\alpha : I \to J$ *in* Dia *and* $S \in \mathbb{S}(J)$ *the functor* α^* *between fibers*

$$\mathbb{D}(J)_S \to \mathbb{D}(I)_{\alpha^* S}$$

has a left-adjoint $\alpha_!^S$*.*

(FDer4 left) For each functor $\alpha : I \to J$ *in* Dia*, and for any object* $j \in J$*, and the 2-cell*

$$\begin{tikzcd} I \times_{/J} j \arrow[r, "\iota"] \arrow[d, "\alpha_j"'] \arrow[dr, phantom, "\swarrow\mu"] & I \arrow[d, "\alpha"] \\ \{j\} \arrow[r, hook, "j"] & J \end{tikzcd}$$

we get that the induced natural transformation of functors $\alpha_{j!}(\mathbb{S}(\mu))_\bullet \iota^* \to j^* \alpha_!$ *is an isomorphism*[6]*.*

[5] The numbering is compatible with that of [Gro13] in the case of non-fibered derivators.

[6] This is meant to hold w.r.t. all bases $S \in \mathbb{S}(J)$.

(FDer5 left) For any Grothendieck opfibration $\alpha : I \to J$ in Dia*, and for any morphism $\xi \in \mathrm{Hom}(S_1, \dots, S_n; T)$ in $\mathbb{S}(\cdot)$ for some $n \geq 1$, the natural transformations of functors*

$$\alpha_!(\alpha^*\xi)_\bullet(\alpha^*-, \cdots, \alpha^*-,\ -\ , \alpha^*-, \cdots, \alpha^*-) \cong \xi_\bullet(-, \cdots, -,\ \alpha_!-\ , -, \cdots, -)$$

are isomorphisms.

and their dual variants:

(FDer0 right) For each I in Dia *the morphism p specializes to a fibered (multi)category and any* Grothendieck opfibration $\alpha : I \to J$ *in* Dia *induces a diagram*

$$\begin{tikzcd} \mathbb{D}(J) \arrow[r, "\alpha^*"] \arrow[d] & \mathbb{D}(I) \arrow[d] \\ \mathbb{S}(J) \arrow[r, "\alpha^*"] & \mathbb{S}(I) \end{tikzcd}$$

of fibered (multi)categories, i.e. the top horizontal functor maps Cartesian morphisms w.r.t. the i-th slot to Cartesian morphisms w.r.t. the i-th slot.

(FDer3 right) For each functor $\alpha : I \to J$ in Dia *and $S \in \mathbb{S}(J)$ the functor α^* between fibers*

$$\mathbb{D}(J)_S \to \mathbb{D}(I)_{\alpha^*S}$$

has a right-adjoint α_*^S.

(FDer4 right) For each morphism $\alpha : I \to J$ in Dia*, and for any object $j \in J$, and the 2-cell*

$$\begin{tikzcd} j \times_{/J} I \arrow[r, "\alpha_j"] \arrow[d, "\iota"'] \arrow[dr, phantom, "\swarrow\mu"] & \{j\} \arrow[d, hook] \\ I \arrow[r, "\alpha"] & J \end{tikzcd}$$

we get that the induced natural transformation of functors $j^*\alpha_* \to \alpha_{j\,*}(\mathbb{S}(\mu))^\bullet\iota^*$ is an isomorphism[7]*.*

(FDer5 right) For any functor $\alpha : I \to J$ in Dia*, and for any morphism $\xi \in \mathrm{Hom}(S_1, \dots, S_n; T)$ in $\mathbb{S}(\cdot)$ for some $n \geq 1$, the natural transformations of functors*

$$\alpha_*(\alpha^*\xi)^{\bullet,i}(\alpha^*-, \cdots, \alpha^*-\ ;\ -) \cong \xi^{\bullet,i}(-, \cdots, -\ ;\ \alpha_*-)$$

are isomorphisms for all $1 \leq i \leq n$.

[7]This is meant to hold w.r.t. all bases $S \in \mathbb{S}(J)$.

2.3.6. Definition. *A strict morphism of pre-(multi)derivators* $p : \mathbb{D} \to \mathbb{S}$ *with domain* Dia *is called a* **left fibered (multi)derivator** *with domain* Dia*, if axioms (Der1–2) hold for* $\mathbb{D}$ *and* $\mathbb{S}$ *and (FDer0–5 left) hold for* p*. Similarly it is called a* **right fibered (multi)derivator** *with domain* Dia*, if instead the corresponding dual axioms (FDer0–5 right) hold. It is called just* **fibered** *if it is both left and right fibered.*

2.3.7. Remark.

1. In the case of pre-derivators (not pre-multiderivators) the axioms (FDer0 left, FDer3–5 left) are dual to the axioms (FDer0 right, FDer3–5 right) in the sense that any of those axioms in the left variant holds for $p : \mathbb{D} \to \mathbb{S}$ if and only if the corresponding axiom in the right variant holds for $p^{\mathrm{op}} : \mathbb{D}^{\mathrm{op}} \to \mathbb{S}^{\mathrm{op}}$. This is not true for the multiderivator case; besides $\mathbb{D}^{\mathrm{op}}$ would be a (pre-)opmultiderivator. However, we do not develop this notion explicitly.

2. The squares in axioms (FDer4 left/right) are in fact homotopy exact and it follows from the axioms (FDer4 left/right) that many more are (see 2.3.23).

3. There is some redundancy in the axioms, cf. 2.3.9 and 2.3.27. For instance, if $\mathbb{S}$ is strong (cf. Definition 2.3.17 below), (FDer5 left) resp. (FDer5 right) are only needed in the multicase.

4. The condition that α be an opfibration in axioms (FDer0 right) and (FDer5 left) is only needed if f is an n-ary morphism for $n \geq 2$ hence, in particular, only for fibered *multi*derivators. For $n = 1$ the condition on α is not needed and, in fact, the general version (for α arbitrary) follows from the weaker version (for α an opfibration) and the other axioms.

5. The axioms (FDer0) and (FDer3–5) are similar to the axioms of a six-functor-formalism (cf. the introduction or the appendix A.2). It is actually possible to make this analogy precise and *define* a fibered multiderivator as a bifibration of 2-multicategories $\mathcal{D} \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ where $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ is a certain category of multicorrespondences of diagrams in $\mathbb{S}$, similar to our definition of a usual six-functor-formalism (cf. Definition A.2.16). This also clarifies the existence and comparison of the internal and external monoidal structure, resp. duality, in a closed monoidal derivator (i.e. fibered multiderivator over $\{\cdot\}$) or more generally for any fibered multiderivator. We will explain this in detail in a subsequent article [Hör16].

2.3.8. Question. It seems natural to allow also (symmetric) multicategories, in particular operads, as *domain* for a fibered (symmetric) multiderivator. However, the author did not succeed in writing down a neat generalization of (FDer3–4) which would encompass (FDer5).

2.3.9. LEMMA. *For a strict morphism of pre-derivators $\mathbb{D} \to \mathbb{S}$ such that both satisfy (Der1) and (Der2) and such that it induces a bifibration of multicategories $\mathbb{D}(I) \to \mathbb{S}(I)$ for all $I \in \mathrm{Dia}$ we have the following implications:*

$$\text{(FDer0 left) for } n\text{-ary morphisms, } n \geq 1 \quad \Leftrightarrow \quad \text{(FDer5 right)} \tag{7}$$

$$\text{(FDer0 right)} \quad \Leftrightarrow \quad \text{(FDer5 left)} \tag{8}$$

PROOF. We will only show the implication (7), the other being similar. Choosing push-forward functors $f_\bullet$, the remaining part of (FDer0 left) says that the natural 2-morphism

$$\begin{tikzcd}
\mathbb{D}(J)_{S_1} \times \cdots \times \mathbb{D}(J)_{S_n} \arrow[r, "f_\bullet"] \arrow[d, "\alpha^*"] \arrow[dr, phantom, "\swarrow"] & \mathbb{D}(J)_T \arrow[d, "\alpha^*"] \\
\mathbb{D}(I)_{\alpha^* S_1} \times \cdots \times \mathbb{D}(I)_{\alpha^* S_n} \arrow[r, "(\alpha^* f)_\bullet"] & \mathbb{D}(I)_{\alpha^* T}
\end{tikzcd}$$

is an isomorphism. Taking the adjoint of this diagram (of $f_\bullet$ and $(\alpha^* f)_\bullet$ w.r.t. the i-th slot) we get the diagram

$$\begin{tikzcd}
\mathbb{D}(I)^{\mathrm{op}}_{\alpha^* S_1} \times \overset{\widehat{i}}{\cdots} \times \mathbb{D}(I)^{\mathrm{op}}_{\alpha^* S_n} & \times & \mathbb{D}(I)_{\alpha^* T} \arrow[r, "(\alpha^* f)^{\bullet, i}"] \arrow[d, "\alpha_*"] \arrow[dr, phantom, "\swarrow"] & \mathbb{D}(I)_{\alpha^* S_i} \arrow[d, "\alpha_*"] \\
\mathbb{D}(J)^{\mathrm{op}}_{S_1} \times \overset{\widehat{i}}{\cdots} \times \mathbb{D}(J)^{\mathrm{op}}_{S_n} \arrow[u, "(\alpha^*)^{\mathrm{op}}"] & \times & \mathbb{D}(J)_T \arrow[r, "f^{\bullet, i}"] & \mathbb{D}(J)_{S_i}
\end{tikzcd}$$

That its 2-morphism is an isomorphism is the content of (FDer5 left). Hence (FDer0 left) and (FDer5 right) are equivalent in this situation.

For (8) note that for both (FDer0 right) and (FDer5 left), the functor α in question is restricted to the class of Grothendieck opfibrations. ■

2.3.10. The pre-derivator associated with an ∞-category $\mathcal{S}$ is actually a left and right derivator (in the usual sense, i.e. fibered over $\{\cdot\}$) if $\mathcal{S}$ is complete and co-complete [GPS14]. This includes the case of pre-derivators associated with categories, which is, of course, classical — axiom (FDer4) expressing nothing else than Kan's formulas.

2.3.11. Let $S \in \mathbb{S}(\cdot)$ be an object and $p : \mathbb{D} \to \mathbb{S}$ be a (left, resp. right) fibered multiderivator. The association

$$I \mapsto \mathbb{D}(I)_{\pi^* S},$$

where $\pi : I \to \cdot$ is the projection, defines a (left, resp. right) derivator in the usual sense which we call its fiber $\mathbb{D}_S$ over S. The axioms (FDer7–8) stated below involve only these fibers.

2.3.12. DEFINITION. *More generally, if $S \in \mathbb{S}(J)$ we may consider the association*

$$I \mapsto \mathbb{D}(I \times J)_{\mathrm{pr}_2^* S},$$

where $\mathrm{pr}_2 : I \times J \to J$ is the second projection. This defines again a (left, resp. right) derivator in the usual sense which we call its **fiber** *$\mathbb{D}_S$ over S.*

2.3.13. Lemma. [left] *Let $\mathbb{D} \to \mathbb{S}$ be a left fibered multiderivator and let $I \in$ Dia be a diagram and $f \in \mathrm{Hom}_{\mathbb{S}(I)}(S_1, \dots, S_n; T)$ for some $n \geq 1$ be a morphism. Then the collection of functors for each $J \in$ Dia*

$$\begin{array}{rcl} f_\bullet : \mathbb{D}(J \times I)_{\mathrm{pr}_2^* S_1} \times \cdots \times \mathbb{D}(J \times I)_{\mathrm{pr}_2^* S_n} & \to & \mathbb{D}(J \times I)_{\mathrm{pr}_2^* T} \\ \mathcal{E}_1, \dots, \mathcal{E}_n & \mapsto & (\mathrm{pr}_2^* f)_\bullet(\mathcal{E}_1, \dots, \mathcal{E}_n) \end{array}$$

defines a morphism of left derivators $\mathbb{D}_{S_1} \times \cdots \times \mathbb{D}_{S_n} \to \mathbb{D}_T$. Furthermore, for a collection $\mathcal{E}_k \in \mathbb{D}(I), k \neq i$ the morphism of derivators:

$$\begin{array}{rcl} \mathbb{D}(J \times I)_{\mathrm{pr}_2^* S_i} & \to & \mathbb{D}(J \times I)_{\mathrm{pr}_2^* T} \\ \mathcal{E}_i & \mapsto & (\mathrm{pr}_2^* f)_\bullet(\mathrm{pr}_2^* \mathcal{E}_1, \dots, \mathcal{E}_i, \dots, \mathrm{pr}_2^* \mathcal{E}_n) \end{array}$$

is left continuous (i.e. commutes with left Kan extensions).

Proof. The only point which might not be clear is the left continuity of the bottom morphism of pre-derivators. Consider the following 2-commutative square, where $I, J, J' \in$ Dia, $\alpha : J \to J'$ is a functor, and $j' \in J'$

$$\begin{tikzcd} I \times (j' \times_{/J'} J) \arrow[r, "(\mathrm{id},\iota)"] \arrow[d, "(\mathrm{id},p)"'] \arrow[dr, phantom, "\Nearrow"] & I \times J \arrow[d, "(\mathrm{id},\alpha)"] \\ I \times j' \arrow[r] & I \times J' \end{tikzcd}$$

It is homotopy exact by 2.3.23, 4. Therefore we have (using FDer3–5 left):

$$\begin{array}{rl} & (\mathrm{id}, j')^* (\mathrm{id}, \alpha)_! (\mathrm{pr}_2^* f)_\bullet (\mathrm{pr}_2^* \mathcal{E}_1, \dots, \mathcal{E}_i, \dots, \mathrm{pr}_2^* \mathcal{E}_n) \\ \cong & (\mathrm{id}, p)_! (\mathrm{id}, \iota)^* (\mathrm{pr}_2^* f)_\bullet (\mathrm{pr}_2^* \mathcal{E}_1, \dots, \mathcal{E}_i, \dots, \mathrm{pr}_2^* \mathcal{E}_n) \\ \cong & (\mathrm{id}, p)_! (\mathrm{pr}_2^* f)_\bullet ((\mathrm{id}, \iota)^* \mathrm{pr}_2^* \mathcal{E}_1, \dots, (\mathrm{id}, \iota)^* \mathcal{E}_i, \dots, (\mathrm{id}, \iota)^* \mathrm{pr}_2^* \mathcal{E}_n) \\ \cong & (\mathrm{id}, p)_! (\mathrm{pr}_2^* f)_\bullet ((\mathrm{id}, p)^* \mathcal{E}_1, \dots, (\mathrm{id}, \iota)^* \mathcal{E}_i, \dots, (\mathrm{id}, p)^* \mathcal{E}_n) \\ \cong & f_\bullet (\mathcal{E}_1, \dots, (\mathrm{id}, p)_! (\mathrm{id}, \iota)^* \mathcal{E}_i, \dots, \mathcal{E}_n) \\ \cong & f_\bullet (\mathcal{E}_1, \dots, (\mathrm{id}, j')^* (\mathrm{id}, \alpha)_! \mathcal{E}_i, \dots, \mathcal{E}_n) \\ \cong & (\mathrm{id}, j')^* (\mathrm{pr}_2^* f)_\bullet (\mathrm{pr}_2^* \mathcal{E}_1, \dots, (\mathrm{id}, \alpha)_! \mathcal{E}_i, \dots, \mathrm{pr}_2^* \mathcal{E}_n) \end{array}$$

(Note that (id, p) is trivially an opfibration). A tedious check shows that the composition of these isomorphisms is $(\mathrm{id}, j')^*$ applied to the exchange morphism

$$(\mathrm{id}, \alpha)_! (\mathrm{pr}_2^* f)_\bullet (\mathrm{pr}_1^* \mathcal{E}_1, \dots, \mathcal{E}_i, \dots, \mathrm{pr}_2^* \mathcal{E}_n) \to (\mathrm{pr}_2^* f)_\bullet (\mathrm{pr}_2^* \mathcal{E}_1, \dots, (\mathrm{id}, \alpha)_! \mathcal{E}_i, \dots, \mathrm{pr}_2^* \mathcal{E}_n)$$

Since the above holds for any $j' \in J'$ the exchange morphism is therefore an isomorphism by (Der2). ∎

In the right fibered situation the analogously defined morphisms $f^{\bullet,i}$ are not expected to be made into a morphism of fibers this way. For a discussion of how this is solved, we refer the reader to the article [Hör16] in preparation, where a fibered multiderivator is redefined as a certain type of six-functor-formalism. This will let appear the discussion and results of this section in a much more clear fashion. However, we have:

2.3.14. Lemma. [right] *Let $\mathbb{D} \to \mathbb{S}$ be a right fibered multiderivator and let $I \in \mathrm{Dia}$ be a diagram and $f \in \mathrm{Hom}_{\mathbb{S}(I)}(S_1, \dots, S_n; T)$, for some $n \geq 1$, be a morphism. For each $J \in \mathrm{Dia}$ and for each collection $\mathcal{E}_k \in \mathbb{D}(I)$, $k \neq i$, the association*

$$\begin{array}{rcl} \mathbb{D}(J \times I)_{\mathrm{pr}_2^* T} & \to & \mathbb{D}(J \times I)_{\mathrm{pr}_2^* S_i} \\ \mathcal{F} & \mapsto & (\mathrm{pr}_2^* f)^{\bullet,i}(\mathrm{pr}_2^* \mathcal{E}_1, \dots, \mathrm{pr}_2^* \mathcal{E}_n; \mathcal{F}) \end{array}$$

defines a morphism of right derivators which is right continuous (i.e. commutes with right Kan extensions). This is the right adjoint in the pre-derivator sense to the morphism of pre-derivators in the previous lemma, as soon as $\mathbb{D} \to \mathbb{S}$ is left and right fibered.

Proof. Consider the following 2-commutative square where $I, J, J' \in \mathrm{Dia}$, $\alpha : J \to J'$ is a functor, and $j' \in J'$

$$\begin{tikzcd} I \times (J \times_{/J'} j') \arrow[r, "(\mathrm{id},\iota)"] \arrow[d, "(\mathrm{id},p)"'] \arrow[dr, phantom, "\swarrow"] & I \times J \arrow[d, "(\mathrm{id},\alpha)"] \\ I \times j' \arrow[r] & I \times J' \end{tikzcd}$$

It is homotopy exact by 2.3.23, 4.

Therefore we have (using FDer3–5 right):

$$\begin{array}{rl} & (\mathrm{id}, j')^*(\mathrm{id},\alpha)_*(\mathrm{pr}_2^* f)^{\bullet,i}(\mathrm{pr}_2^*\mathcal{E}_1, \overset{\widehat{i}}{\dots}, \mathrm{pr}_2^*\mathcal{E}_n; \mathcal{F}) \\ \cong & (\mathrm{id},p)_*(\mathrm{id},\iota)^*(\mathrm{pr}_2^* f)^{\bullet,i}(\mathrm{pr}_2^*\mathcal{E}_1, \overset{\widehat{i}}{\dots}, \mathrm{pr}_2^*\mathcal{E}_n, \mathcal{F}) \\ \cong & (\mathrm{id},p)_*(\mathrm{pr}_2^* f)^{\bullet,i}((\mathrm{id},\iota)^*\mathrm{pr}_2^*\mathcal{E}_1, \overset{\widehat{i}}{\dots}, (\mathrm{id},\iota)^*\mathrm{pr}_2^*\mathcal{E}_n; (\mathrm{id},\iota)^*\mathcal{F}) \\ \cong & (\mathrm{id},p)_*(\mathrm{pr}_2^* f)^{\bullet,i}((\mathrm{id},p)^*\mathcal{E}_1, \overset{\widehat{i}}{\dots}, (\mathrm{id},p)^*\mathcal{E}_n; (\mathrm{id},\iota)^*\mathcal{F}) \\ \cong & f^{\bullet,i}(\mathcal{E}_1, \overset{\widehat{i}}{\dots}, \mathcal{E}_n; (\mathrm{id},p)_*(\mathrm{id},\iota)^*\mathcal{F}) \\ \cong & f^{\bullet,i}(\mathcal{E}_1, \overset{\widehat{i}}{\dots}, \mathcal{E}_n; (\mathrm{id},j')^*(\mathrm{id},\alpha)_*\mathcal{F}) \\ \cong & (\mathrm{id},j')^*(\mathrm{pr}_2^* f)^{\bullet,i}(\mathrm{pr}_2^*\mathcal{E}_1, \overset{\widehat{i}}{\dots}, \mathrm{pr}_2^*\mathcal{E}_n; (\mathrm{id},\alpha)_*\mathcal{F}) \end{array}$$

Note that (id, ι) is an opfibration, but (id, j') is not. Hence the last step has to be justified further. Consider the 2-commutative diagram:

$$\begin{tikzcd} I \times (J \times_{/J'} j') \arrow[r, "(\mathrm{id},\iota')"] \arrow[d, "(\mathrm{id},p)"'] \arrow[dr, phantom, "\swarrow"] & I \times J' \arrow[d, equal] \\ I \times j' \arrow[r] & I \times J' \end{tikzcd}$$

It is again homotopy exact by 2.3.23, 4. Therefore we have

$$\begin{aligned}
&\cong f^{\bullet,i}(\mathcal{E}_1,\overset{\widehat{i}}{\ldots},\mathcal{E}_n;(\mathrm{id},j')^*(\mathrm{id},\alpha)_*\mathcal{F}) \\
&\cong f^{\bullet,i}(\mathcal{E}_1,\overset{\widehat{i}}{\ldots},\mathcal{E}_n;(\mathrm{id},p)_*(\mathrm{id},\iota')^*(\mathrm{id},\alpha)_*\mathcal{F}) \\
&\cong (\mathrm{id},p)_*(\mathrm{pr}_2^*f)^{\bullet,i}((\mathrm{id},p)^*\mathcal{E}_1,\overset{\widehat{i}}{\ldots},(\mathrm{id},p)^*\mathcal{E}_n;(\mathrm{id},\iota')^*(\mathrm{id},\alpha)_*\mathcal{F}) \\
&\cong (\mathrm{id},p)_*(\mathrm{pr}_2^*f)^{\bullet,i}((\mathrm{id},\iota')^*\mathrm{pr}_2^*\mathcal{E}_1,\overset{\widehat{i}}{\ldots},(\mathrm{id},\iota')^*\mathrm{pr}_2^*\mathcal{E}_n;(\mathrm{id},\iota')^*(\mathrm{id},\alpha)_*\mathcal{F}) \\
&\cong (\mathrm{id},p)_*(\mathrm{id},\iota')^*(\mathrm{pr}_2^*f)^{\bullet,i}(\mathrm{pr}_2^*\mathcal{E}_1,\overset{\widehat{i}}{\ldots},\mathrm{pr}_2^*\mathcal{E}_n;(\mathrm{id},\iota')^*(\mathrm{id},\alpha)_*\mathcal{F}) \\
&\cong (\mathrm{id},j')^*(\mathrm{pr}_2^*f)^{\bullet,i}(\mathrm{pr}_2^*\mathcal{E}_1,\overset{\widehat{i}}{\ldots},\mathrm{pr}_2^*\mathcal{E}_n;(\mathrm{id},\alpha)_*\mathcal{F})
\end{aligned}$$

Note that (id,ι') is an opfibration as well. In other words: the reason why $f^{\bullet,i}$ also commutes with $(\mathrm{id},j')^*$ in this particular case is that the other argument are constant in the J direction.

A tedious check shows the composition of the isomorphisms of the previous computations yield $(\mathrm{id},j')^*$ applied to the exchange morphism

$$(\mathrm{id},\alpha)_*(\mathrm{pr}_2^*f)^{\bullet,i}(\mathrm{pr}_2^*\mathcal{E}_1,\overset{\widehat{i}}{\ldots},\mathrm{pr}_2^*\mathcal{E}_n;\mathcal{F}) \to (\mathrm{pr}_2^*f)^{\bullet,i}(\mathcal{E}_1,\overset{\widehat{i}}{\ldots},\mathcal{E}_n;(\mathrm{id},\alpha)_*\mathcal{F}).$$

Since the above holds for any $j' \in J'$ it is therefore an isomorphism by (Der2). ∎

2.3.15. Let $p : \mathbb{D} \to \mathbb{S}$ be a (left, resp. right) fibered *multi*derivator and $S : \{\cdot\} \to \mathbb{S}(\cdot)$ a functor of multicategories. This is equivalent to the choice of an object $S \in \mathbb{S}(\cdot)$ and a collection of morphisms $\alpha_n \in \mathrm{Hom}_{\mathbb{S}(\cdot)}(\underbrace{S,\dots,S}_{n \text{ times}};S)$ for all $n \ge 2$, compatible with composition. Then the fiber

$$I \mapsto \mathbb{D}(I)_{p^*S}$$

defines even a (left, resp. right) multiderivator (i.e. a fibered multiderivator over $\{\cdot\}$). The same holds analogously for a functor of multicategories $S : \{\cdot\} \to \mathbb{S}(I)$.

Axiom (FDer5 left) and Lemma A.2.6 imply the following:

2.3.16. Proposition. *A left fibered multiderivator $\mathbb{D} \to \{\cdot\}$ is the same as a monoidal left derivator in the sense of Groth [Gro12]. It is also, in addition, right fibered if and only if it is a right derivator and closed monoidal in the sense of [loc. cit.].*

2.3.17. Definition. *We call a pre-derivator $\mathbb{D}$* **strong***, if the following axiom holds:*

(Der6) For any diagram K in Dia *the 'partial underlying diagram' functor*

$$\mathrm{dia} : \mathbb{D}(K \times \Delta_1) \to \mathrm{Hom}(\Delta_1, \mathbb{D}(K))$$

is full and essentially surjective.

2.3.18. Definition. *Let $p : \mathbb{D} \to \mathbb{S}$ be a fibered (left and right) derivator. We say that $p : \mathbb{D} \to \mathbb{S}$ has* **pointed fibers** *if the following axiom holds:*

(FDer7) For any $S \in \mathbb{S}(\cdot)$, the category $\mathbb{D}(\cdot)_S$ has a zero object.

2.3.19. Definition. *Let $p : \mathbb{D} \to \mathbb{S}$ be a fibered (left and right) derivator. We say that $p : \mathbb{D} \to \mathbb{S}$ has* **stable fibers** *if its fibers are* strong *and the following axiom holds:*

(FDer8) For any $S \in \mathbb{S}(\cdot)$, in the category $\mathbb{D}(\Box)_{p^*S}$ an object is homotopy Cartesian if and only if it is homotopy coCartesian.

This condition can be weakened (cf. [GS12, Corollary 8.13]).

2.3.20. If the fibers of a fibered derivator are stable then they are triangulated categories in a natural way (this follows from [Gro13]). Actually the proof shows that it suffices that the fibers are derivators of domain Posf (finite posets).

Since, by Lemma 2.3.13 and Lemma 2.3.14 push-forward, resp. pull-back w.r.t. any slot commute with homotopy colimits, resp. homotopy limits, they induce triangulated functors between the fibers.

2.3.21. [left] The following is a consequence of (FDer0 left): For a functor $\alpha : I \to J$ and a morphism in $f : S \to T \in \mathbb{S}(J)$, we get a natural isomorphism

$$\mathbb{S}(\alpha^* f)_\bullet \alpha^* \to \alpha^* \mathbb{S}(f)_\bullet.$$

W.r.t. this natural isomorphism we have the following:

2.3.22. Lemma. [left] *Given a "pasting" diagram*

$$\begin{tikzcd} N \arrow[r, "G"] \arrow[d, "A"'] \arrow[dr, phantom, "\swarrow^{\nu}"] & L \arrow[r, "B"] \arrow[d, "a"] \arrow[dr, phantom, "\swarrow^{\mu}"] & I \arrow[d, "\alpha"] \\ M \arrow[r, "\gamma"] & K \arrow[r, "\beta"] & J \end{tikzcd}$$

we get for the pasted natural transformation $\nu \odot \mu := (\beta * \nu) \circ (\mu * G)$ that the following diagram is commutative:

$$\begin{tikzcd} A_! \mathbb{S}(\beta * \nu)_\bullet G^* \mathbb{S}(\mu)_\bullet B^* \arrow[r] \arrow[d, "\sim"] & \gamma^* a_! \mathbb{S}(\mu)_\bullet B^* \arrow[r] & \gamma^* \beta^* \alpha_! \\ A_! \mathbb{S}(\beta * \nu)_\bullet \mathbb{S}(G * \mu)_\bullet G^* B^* \arrow[d, "\sim"] & & \\ A_! \mathbb{S}(\nu \odot \mu)_\bullet G^* B^* \arrow[uurr] & & \end{tikzcd}$$

Here the morphisms going to the right are (induced by) the various base-change morphisms. In particular, the pasted square is homotopy exact if the individual two squares are.

Proof. This is an analogue of [Gro13, Lemma 1.17] and proven similarly. ∎

2.3.23. Proposition.

1. *Any square of the form*

$$\begin{tikzcd} I \times_{/J} K \arrow[r, "B"] \arrow[d, "A"'] \arrow[dr, phantom, "\swarrow^{\mu}"] & I \arrow[d, "\alpha"] \\ K \arrow[r, "\beta"] & J \end{tikzcd}$$

(where $I \times_{/J} K$ is the slice category) is homotopy exact (in particular the ones from axiom (FDer4 left) and (FDer4 right) are).

2. *A Cartesian square*

$$\begin{tikzcd} I \times_J K \arrow[r, "B"] \arrow[d, "A"'] & I \arrow[d, "\alpha"] \\ K \arrow[r, "\beta"] & J \end{tikzcd}$$

(where $I \times_J K$ is the fiber product) is homotopy exact, if α is an opfibration or if β is a fibration.

3. *If $\alpha : I \to J$ is a morphism of opfibrations (resp. fibrations) over a diagram E, then*

$$\begin{tikzcd} I_e \arrow[r, hook, "w_I"] \arrow[d, "\alpha_e"'] & I \arrow[d, "\alpha"] \\ J_e \arrow[r, hook, "w_J"] & J \end{tikzcd} \quad \textit{resp.} \quad \begin{tikzcd} I_e \arrow[r, "\alpha_e"] \arrow[d, hook, "w_I"'] & J_e \arrow[d, hook, "w_J"] \\ I \arrow[r, "\alpha"] & J \end{tikzcd}$$

is homotopy exact for all objects $e \in E$, where w_I, w_J are the inclusions of the respective fibers.

4. *If a square*

$$\begin{tikzcd} L \arrow[r, "B"] \arrow[d, "A"'] \arrow[dr, phantom, "\swarrow^{\mu}"] & I \arrow[d, "\alpha"] \\ K \arrow[r, "\beta"'] & J \end{tikzcd}$$

is homotopy exact then so is the square

$$\begin{tikzcd} L \times X \arrow[r, "B \times \mathrm{id}_X"] \arrow[d, "A \times \mathrm{id}_X"'] \arrow[dr, phantom, "\swarrow^{\mu \times X}"] & I \times X \arrow[d, "\alpha \times \mathrm{id}_X"] \\ K \times X \arrow[r, "\beta \times \mathrm{id}_X"'] & J \times X \end{tikzcd}$$

for any diagram X.

Proof. This proof is completely analogous to the non-fibered case. We sketch the arguments here (for the left-case only, the other case follows by logical duality):

3. We only show the case of opfibrations, the other is analogous. Let j be an object in J_e and consider the cube:

$$\begin{tikzcd}
 & I_e \times_{/J_e} j \arrow[rr, "\iota_e"] \arrow[dl, "w"'] \arrow[dd, "p_e"] & & I_e \arrow[dl, "w_I"'] \arrow[dd, "\alpha_e"] \\
I \times_{/J} j \arrow[rr, "\iota"] \arrow[dd, "p"] & & I \arrow[dd, "\alpha"] \arrow[r, phantom, "\swarrow\mu"] & \\
 & \cdot \arrow[rr, "j"] \arrow[dl, equal] \arrow[d, phantom, "\swarrow\mu"] & & J_e \arrow[dl, "w_J"] \\
\cdot \arrow[rr, "j"] & {} & J &
\end{tikzcd} \tag{9}$$

where w is given by the inclusions $\iota_{I,e}$ resp. $\iota_{J,e}$. By standard arguments on homotopy exact squares it suffices to show that the left square is homotopy exact on constant diagrams, i.e. that

$$p_{e,!}w^* \cong p_!$$

holds true for all usual derivators. By [Gro13, Proposition 1.23] it suffices to show that w has a left adjoint.

Denote $\pi_I : I \to E$ and $\pi_J : J \to E$ the given opfibrations. Consider the two functors

$$\begin{tikzcd} I_e \times_{/J,e} j \arrow[r, shift left, "w"] & I \times_{/J} j \arrow[l, shift left, "c"] \end{tikzcd}$$

where c is given by mapping $(i, \mu : \alpha(i) \to j)$ to $(i', \mu' : \alpha(i') \to j)$ where we chose, for any i, a coCartesian morphism $\xi_{i,\mu} : i \to i'$ over $\pi_I(\mu) : \pi_I(i) \to e$. Since α maps coCartesian morphisms to coCartesian morphisms by assumption, $\alpha(\xi_{i,\mu}) : \alpha(i) \to \alpha(i')$ is coCartesian, and therefore there is a unique factorization

$$\alpha(i) \xrightarrow{\alpha(\xi_i)} \alpha(i') \xrightarrow{\mu'} j$$

of μ. A morphism $\alpha : (i_1, \mu_1 : \alpha(i_1) \to j) \to (i_2, \mu_2 : \alpha(i_2) \to j)$, by definition of coCartesian, gives rise to a unique morphism $\alpha' : i_1' \to i_2'$ over $\pi_I(i_1) \to \pi_I(i_2)$ such that $\alpha'\xi_{i_1,\mu_1} = \xi_{i_2,\mu_2}\alpha'$ holds, and we set $c(\alpha) := \alpha'$. We have $c \circ w = \mathrm{id}$, and a morphism $\mathrm{id}_{I \times_{/J} j} \to w \circ c$ given by $(i, \mu) \mapsto \xi_{i,\mu}$. This makes w right adjoint to c.

2. By axiom (Der2) it suffices to show that for any object k of K, the induced morphism

$$k^* A_! B^* \to k^* \beta^* \alpha_!$$

is an isomorphism. Consider the following pasting diagram

$$\begin{tikzcd}
I \times_J k \arrow[r, "j"] \arrow[d, "\pi"] & I \times_J K \times_{/K} k \arrow[r, "\iota"] \arrow[d, "p"] \arrow[dr, phantom, "\swarrow\mu"] & I \times_J K \arrow[r, "B"] \arrow[d, "A"] & I \arrow[d, "\alpha"] \\
\cdot \arrow[r, equal] & \cdot \arrow[r, "k"] & K \arrow[r, "\beta"] & J
\end{tikzcd}$$

Lemma 2.3.22 shows that the following composition

$$\pi_! \mathbb{S}(\beta * \mu * j)_\bullet j^* \iota^* B^* \to \pi_! j^* \mathbb{S}(\beta * \mu)_\bullet \iota^* B^* \to p_! \mathbb{S}(\beta * \mu)_\bullet \iota^* B^* \to k^* A_! B^* \to k^* \beta^* \alpha_!$$

is the base-change associated with the pasting of the 3 squares in the diagram. All morphisms in this sequence are isomorphisms except possibly for the rightmost one. The second from the left is an isomorphism because j is a right adjoint [Gro13, Proposition 1.23]. The base-change morphism of the pasting is an isomorphism because of 3.

1. By axiom (Der2) it suffices to show that for any object k of K the induced morphism

$$k^* A_! \mathbb{S}(\mu)_\bullet B^* \to k^* \beta^* \alpha_!$$

is an isomorphism. Consider the following pasting diagram

$$\begin{tikzcd}
I \times_{/J} k \arrow[r, "\iota"] \arrow[d, "p"] & I \times_{/J} K \arrow[r, "B"] \arrow[d, "A"] \arrow[dr, phantom, "\swarrow \mu"] & I \arrow[d, "\alpha"] \\
\cdot \arrow[r, "k"'] & K \arrow[r, "\beta"'] & J
\end{tikzcd}$$

Lemma 2.3.22 shows that the following diagram is commutative

$$\begin{tikzcd}
p_! \mathbb{S}(\mu\iota)_\bullet \iota^* B^* \arrow[r, "\sim"] \arrow[d, "\mathrm{can}"', "\sim"] & k^* \beta^* \alpha_! \\
p_! \iota^* \mathbb{S}(\mu)_\bullet B^* \arrow[r, "\sim"] & k^* A_! \mathbb{S}(\mu)_\bullet B^* \arrow[u]
\end{tikzcd}$$

where the bottom horizontal morphism is an isomorphism by 2., and the top horizontal morphism is an isomorphism by (FDer4 left). Therefore the right vertical morphism is also an isomorphism.

4. (cf. also [Gro13, Theorem 1.30]). For any $x \in X$ consider the cube

$$\begin{tikzcd}
& L \arrow[rr, "B"] \arrow[dl, "(\mathrm{id},x)"'] \arrow[dd, "A"' near start] & & I \arrow[dl, "(\mathrm{id},x)"'] \arrow[dd, "\alpha"] \\
L \times X \arrow[rr, "B \times \mathrm{id}_X"' near start, crossing over] \arrow[dd, "A \times \mathrm{id}_X"'] & & I \times X \arrow[dr, phantom, "\swarrow \mu"] & \\
& K \arrow[rr, "\beta" near start] \arrow[dl, "(\mathrm{id},x)"'] \arrow[dr, phantom, "\swarrow \mu \times X"] & & J \arrow[dl, "(\mathrm{id},x)"] \\
K \times X \arrow[rr, "\beta \times \mathrm{id}_X"'] & & J \times X \arrow[from=uu, "\alpha \times \mathrm{id}_X" near end, crossing over] &
\end{tikzcd} \tag{10}$$

The left and right hand side squares are homotopy exact because of 3., whereas the rear one is homotopy exact by assumption. Therefore the pasting

$$\begin{tikzcd}
L \arrow[r] \arrow[d, "A"'] & I \times X \arrow[d, "\alpha \times \mathrm{id}_X"] \\
K \arrow[r] & J \times X
\end{tikzcd}$$

is homotopy exact. Therefore we have an isomorphism

$$(\mathrm{id}, x)^* A_! \mathbb{S}(\mu)_\bullet B^* \to (\mathrm{id}, x)^* \beta^* (\alpha \times \mathrm{id}_X)_!$$

where the morphism is induced by the base change of the given 2-commutative square. We may then conclude by axiom (Der2). ∎

2.3.24. [left] If $\mathbb{S}$ is strong the pull-backs and push-forwards along a morphism in $\mathbb{S}(\cdot)$, or more generally along a morphism in $\mathbb{S}(I)$, can be expressed using only the relative Kan-extension functors:

Let $p : \mathbb{D} \to \mathbb{S}$ be a left fibered derivator such that $\mathbb{S}$ is strong. Consider the 2-commutative square

$$\begin{tikzcd} I \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\swarrow\mu"] & I \arrow[d, "\iota_0"] \\ I \arrow[r, "\iota_1"'] & I \times \Delta_1 \end{tikzcd}$$

and consider a morphism $f : S \to T$ in $\mathbb{S}(I)$. By the strongness of $\mathbb{S}$, the morphism f may be lifted to an object $F \in \mathbb{S}(I \times \Delta_1)$, and this means that the morphism

$$\mathbb{S}(\mu) : \iota_0^* F \to \iota_1^* F$$

is isomorphic to f. Since the square is homotopy exact by Proposition 2.3.23 1., we get that the natural transformation

$$f_\bullet \to \iota_1^* \iota_{0,!}$$

is an isomorphism.

2.3.25. [left] Let $\alpha : I \to J$ a functor in Dia and let $f : S \to T$ be a morphism in $\mathbb{S}(J)$. Axiom (FDer0) of a left fibered derivator implies that we have a canonical isomorphism

$$(\alpha^*(f))_\bullet \alpha^* \cong \alpha^* f_\bullet$$

which is determined by the choice of the push-forward functors. We get an associated exchange morphism

$$\alpha_! (\alpha^*(f))_\bullet \to f_\bullet \alpha_!. \tag{11}$$

2.3.26. Proposition. *If $p : \mathbb{D} \to \mathbb{S}$ is a left fibered derivator, and $\mathbb{S}$ is strong, then the natural transformation (11) is an isomorphism. The corresponding dual statement holds for a right fibered derivator.*

Proof. Consider the following 2-commutative squares (the third and fourth are even commutative on the nose):

$$\begin{tikzcd} I \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\swarrow\mu_I"] & I \arrow[d, "p_I"] \\ I \arrow[r, "\iota_I"'] & I \times \Delta_1 \end{tikzcd} \quad \begin{tikzcd} J \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\swarrow\mu_J"] & J \arrow[d, "p_J"] \\ J \arrow[r, "\iota_J"'] & J \times \Delta_1 \end{tikzcd} \quad \begin{tikzcd} I \arrow[r, "\alpha"] \arrow[d, "\iota_I"'] & J \arrow[d, "\iota_J"] \\ I \times \Delta_1 \arrow[r, "\alpha"'] & J \times \Delta_1 \end{tikzcd} \quad \begin{tikzcd} I \arrow[r, "p_I"] \arrow[d, "\alpha"'] & I \times \Delta_1 \arrow[d, "\alpha"] \\ J \arrow[r, "p_J"'] & J \times \Delta_1 \end{tikzcd} \tag{12}$$

They are all homotopy exact. Consider the diagram

$$\begin{tikzcd}
\alpha_!(\alpha^*(f))_\bullet \arrow[rr] \arrow[d] & & f_\bullet \alpha_! \arrow[d] \\
\alpha_! \iota_I^* p_{I,!} \arrow[r] & \iota_J^* \alpha_! p_{I,!} \arrow[r] & \iota_J^* p_{J,!} \alpha_!
\end{tikzcd}$$

where the vertical morphisms come from (2.3.24) — these are the base change morphism for the first and second square in (12) — and the lower horizontal morphisms are respectively the base change for the third diagram in (12), and the natural morphism associated with the commutativity of the fourth diagram in (12). Repeatedly applying Lemma 2.3.22 shows that this diagram is commutative. Therefore the upper horizontal morphism is an isomorphism because all the others in the diagram are. ∎

2.3.27. The last proposition states that push-forward commutes with homotopy colimits (left case) and pull-back commutes with homotopy limits (right case). This is also the content of (FDer5 left/right) for fibered derivators (not multiderivators), and hence this axiom is implied by the other axioms of left fibered derivators if $\mathbb{S}$ is strong. Even in the multi-case, by Lemma 2.3.9, axiom (FDer5 left/right) also follow from both (FDer0 left) *and* (FDer0 right).

2.3.28. [left] Let $\alpha : I \to J$ be a functor in Dia. Proposition 2.3.26 (or FDer5 left) allows us to extend the functor $\alpha_!$ to a functor

$$\alpha_! : \mathbb{D}(I) \times_{\mathbb{S}(I)} \mathbb{S}(J) \to \mathbb{D}(J)$$

which is still left adjoint to α^*, more precisely: to $(\alpha^*, p(J))$. Here the fiber product is formed w.r.t. $p(I)$ and α^* respectively. We sketch its construction: $\alpha_!(\mathcal{E}, S)$ is given by $\alpha_!^S \mathcal{E}$, where $\alpha_!^S$ is the functor from axiom (FDer3 left) with base S. Let a pair of a morphism $f : S \to T$ in $\mathbb{S}(J)$ and $F : \mathcal{E} \to \mathcal{F}$ in $\mathbb{D}(I)$ over $\alpha^*(f)$ be given. We define $\alpha_!(F, f)$ as follows: F corresponds to a morphism

$$(\alpha^* f)_\bullet \mathcal{E} \to \mathcal{F}.$$

Applying $\alpha_!^T$ we get a morphism

$$\alpha_!^T (\alpha^* f)_\bullet \mathcal{E} \to \alpha_!^T \mathcal{F}$$

and composition with the inverse of the morphism (11) yields

$$f_\bullet \alpha_!^S \mathcal{E} \to \alpha_!^T \mathcal{F}$$

or, equivalently, a morphism which we define to be $\alpha_!(F, f)$

$$\alpha_!^S \mathcal{E} \to \alpha_!^T \mathcal{F}$$

over f.

For the adjunction, we have to give a functorial isomorphism

$$\mathrm{Hom}_{\alpha^* f}(\mathcal{E}, \alpha^*\mathcal{F}) \cong \mathrm{Hom}_f(\alpha_!(\mathcal{E}, S), \mathcal{F}),$$

where $\mathcal{E} \in \mathbb{D}(I)_{\alpha^* S}$ and $\mathcal{F} \in \mathbb{D}(J)_T$. We define it to be the following composition of isomorphisms:

$$\begin{aligned} & \mathrm{Hom}_{\alpha^* f}(\mathcal{E}, \alpha^*\mathcal{F}) \\ \cong\; & \mathrm{Hom}_{\mathrm{id}_{\alpha^* T}}((\alpha^* f)_\bullet \mathcal{E}, \alpha^*\mathcal{F}) \\ \cong\; & \mathrm{Hom}_{\mathrm{id}_T}(\alpha_!(\alpha^* f)_\bullet \mathcal{E}, \mathcal{F}) \\ \cong\; & \mathrm{Hom}_{\mathrm{id}_T}(f_\bullet \alpha_! \mathcal{E}, \mathcal{F}) \\ \cong\; & \mathrm{Hom}_f(\alpha_! \mathcal{E}, \mathcal{F}). \end{aligned}$$

A dual statement holds for a right fibered derivator and the functor α_*.

From Proposition 2.3.26 we also get a vertical version of Lemma 2.3.22:

2.3.29. LEMMA. [left] *Given a "pasting" diagram*

$$\begin{tikzcd} N \arrow[r, "B"] \arrow[d, "\Gamma"'] \arrow[dr, phantom, "\swArrow \nu"] & M \arrow[d, "\gamma"] \\ L \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "\swArrow \mu"] & I \arrow[d, "\alpha"] \\ K \arrow[r, "\beta"] & J \end{tikzcd}$$

we get for the pasted natural transformation $\mu \odot \nu := (\mu * \Gamma) \circ (\alpha * \nu)$ that the following diagram is commutative:

$$\begin{tikzcd} a_! \mathbb{S}(\mu)_\bullet \Gamma_! \mathbb{S}(\alpha * \nu)_\bullet B^* \arrow[r] \arrow[d, "\sim"] & a_! \mathbb{S}(\mu)_\bullet b^* \gamma_! \arrow[r] & \beta^* \alpha_! \gamma_! \\ a_! \Gamma_! \mathbb{S}(\mu * \Gamma)_\bullet \mathbb{S}(\alpha * \nu)_\bullet B^* \arrow[d, "\sim"] & & \\ a_! \Gamma_! \mathbb{S}(\mu \odot \nu)_\bullet B^* \arrow[uurr] & & \end{tikzcd}$$

Here the morphisms going to the right are (induced by) the various base-change morphisms and the upper horizontal morphism is the isomorphism from Proposition 2.3.26. In particular, the pasted square is homotopy exact if the two individual squares are.

2.4. TRANSITIVITY.

2.4.1. PROPOSITION. *Let*

$$\mathbb{E} \xrightarrow{p_1} \mathbb{D} \xrightarrow{p_2} \mathbb{S}$$

be two left (resp. right) fibered multiderivators. Then also the composition $p_3 = p_2 \circ p_1 : \mathbb{E} \to \mathbb{S}$ is a left (resp. right) fibered multiderivator.

PROOF. We will show the statement for left fibered multiderivators. The other statement is shown similarly.

Axiom (FDer0): For any $I \in \mathrm{Dia}$, we have a sequence

$$\mathbb{E}(I) \to \mathbb{D}(I) \to \mathbb{S}(I)$$

of fibered multicategories. It is well-known that then also the composition $\mathbb{E}(I) \to \mathbb{S}(I)$ is a fibered multicategory (see A.2). The other statement of (FDer0) is immediate as well. Let $\alpha : I \to J$ be a functor as in axioms (FDer3 left) and (FDer4 left). We denote the relative homotopy Kan-extension functors w.r.t. the two fibered derivators by $\alpha_!^1$, and $\alpha_!^2$, respectively. As always, the base will be understood from the context or explicitly given as extra argument as in (2.3.28).

Axiom (FDer3 left): Let $S \in \mathbb{S}(J)$ be given. We define a functor

$$\alpha_!^3 : \mathbb{E}(I)_{\alpha^* S} \to \mathbb{E}(J)_S$$

in the fiber (under p_2) of $\mathcal{E} \in \mathbb{D}(I)_{\alpha^* S}$ as the composition

$$\mathbb{E}(I)_{\alpha^* S} \xrightarrow{(\nu_\bullet, \alpha_!^2 p_1)} \mathbb{E}(I)_{\alpha^* S} \times_{\mathbb{D}(I)_{\alpha^* S}} \mathbb{D}(J)_S \xrightarrow{\alpha_!^1} \mathbb{E}(J)_S$$

where ν is the unit

$$\nu : \mathcal{E} \to \alpha^* \alpha_!^2 \mathcal{E}$$

and $\alpha_!^1$ with two arguments is the extension given in (2.3.28).

Let $\mathcal{F}_1 \in \mathbb{E}(I)_{\alpha^* S}$ and $\mathcal{F}_2 \in \mathbb{E}(J)_S$ be given with images $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively under p_1. The adjunction is given by the following composition of isomorphisms:

$$\begin{array}{lll}
 & \mathrm{Hom}_S(\alpha_!^3 \mathcal{F}_1, \mathcal{F}_2) & \\
= & \mathrm{Hom}_S(\alpha_!^1(\nu_\bullet \mathcal{F}_1, \alpha_!^2 \mathcal{E}_1), \mathcal{F}_2) & \text{Definition} \\
= & \{ f \in \mathrm{Hom}_S(\alpha_!^2 \mathcal{E}_1, \mathcal{E}_2); \xi \in \mathrm{Hom}_f(\alpha_!^1(\nu_\bullet \mathcal{F}_1, \alpha_!^2 \mathcal{E}_1), \mathcal{F}_2) \} & \text{Definition} \\
\cong & \{ f \in \mathrm{Hom}_S(\alpha_!^2 \mathcal{E}_1, \mathcal{E}_2); \xi \in \mathrm{Hom}_{\alpha^* f}(\nu_\bullet \mathcal{F}_1, \alpha^* \mathcal{F}_2) \} & \text{Adjunction (2.3.28)} \\
\cong & \{ \widetilde{f} \in \mathrm{Hom}_{\alpha^* S}(\mathcal{E}_1, \alpha^* \mathcal{E}_2); \xi \in \mathrm{Hom}_{\widetilde{f}}(\mathcal{F}_1, \alpha^* \mathcal{F}_2) \} & \text{Note below} \\
= & \mathrm{Hom}_{\alpha^* S}(\mathcal{F}_1, \alpha^* \mathcal{F}_2) \} & \text{Definition}
\end{array}$$

Note that the composition

$$\widetilde{f} : \ \mathcal{E}_1 \xrightarrow{\nu} \alpha^* \alpha_!^2 \mathcal{E}_1 \xrightarrow{\alpha^* f} \alpha^* \mathcal{E}_2$$

is determined by f via the adjunction of (FDer3 left) for base S and $p_2 : \mathbb{D} \to \mathbb{S}$.

Axiom (FDer4 left): Let $\mathcal{E}$ be in $\mathbb{E}(I)_{\alpha^* S}$ and let $\mathcal{F}$ be its image under p_1. We have to show that the natural morphism

$$\alpha_{j!}^3 \mathbb{S}(\mu)_\bullet^3 \iota^* \mathcal{E} \to j^* \alpha_!^3$$

is an isomorphism. Inserting the definition of the push-forwards, resp. of the Kan extensions for p_3, we get

$$\alpha^1_{j!}(\nu_j)^1_\bullet \mathrm{cart}^1_\bullet \iota^*\mathcal{E} \to j^*\alpha^1_! \nu^1_\bullet \mathcal{E}.$$

Here $\nu_j : \mathbb{S}(\mu)^2_\bullet \iota^*\mathcal{F} \to \alpha_j^*\alpha^2_{j!}\mathbb{S}(\mu)^2_\bullet\iota^*\mathcal{F}$ is the unit and $\nu : \mathcal{F} \to \alpha^*\alpha^2_!\mathcal{F}$ is the unit. 'cart^1' is the Cartesian morphism $\iota^*\mathcal{F} \to \mathbb{S}(\mu)^2_\bullet\iota^*\mathcal{F}$. Consider the base-change isomorphism (FDer4 for p_2)

$$\mathrm{bc} : \alpha^2_{j!}\mathbb{S}(\mu)^2_\bullet\iota^*\mathcal{F} \to j^*\alpha^2_!\mathcal{F},$$

and the morphism

$$\mathbb{D}(\mu) : \iota^*\alpha^*\alpha^2_!\mathcal{F} \to \alpha_j^* j^*\alpha^2_!\mathcal{F}.$$

Claim: We have the equality

$$(\alpha_j^*\mathrm{bc}) \circ \nu_j \circ \mathrm{cart} = \mathbb{D}(\mu) \circ \iota^*(\nu).$$

Proof of the claim: Consider the diagram (which affects only the fibered derivator $p_2 : \mathbb{D} \to \mathbb{S}$, hence we omit superscripts):

$$\begin{tikzcd}
\alpha_j^*\alpha_{j!}\mathbb{S}(\mu)_\bullet\iota^*\mathcal{F} \arrow[r, "\alpha_j^*\alpha_{j!}\mathbb{S}(\mu)_\bullet\iota^*\nu"'] \arrow[rrr, bend left, "\alpha_j^*\mathrm{bc}"] & \alpha_j^*\alpha_{j!}\mathbb{S}(\mu)_\bullet\iota^*\alpha^*\alpha_!\mathcal{F} \arrow[r] & \alpha_j^*\alpha_{j!}\alpha_j^*j^*\alpha_!\mathcal{F} \arrow[r] & \alpha_j^*j^*\alpha_!\mathcal{F} \\
\mathbb{S}(\mu)_\bullet\iota^*\mathcal{F} \arrow[u, "\nu_j"] \arrow[r, "\mathbb{S}(\mu)_\bullet\iota^*\nu"'] & \mathbb{S}(\mu)_\bullet\iota^*\alpha^*\alpha_!\mathcal{F} \arrow[u] \arrow[r, "\text{induced}"] & \alpha_j^*j^*\alpha_!\mathcal{F} \arrow[u] \arrow[ur, equal] & \\
\iota^*\mathcal{F} \arrow[u, "\mathrm{cart}"] \arrow[r, "\iota^*\nu"'] & \iota^*\alpha^*\alpha_!\mathcal{F} \arrow[u, "\mathrm{cart}"] \arrow[ur, "\mathbb{D}(\mu)"'] & &
\end{tikzcd}$$

Clearly all squares and triangles in this diagram are commutative. The two given morphisms are the compositions of the extremal paths hence they are equal.

We have a natural isomorphism induced by bc:

$$\alpha^1_{j!}(\cdots, \alpha^2_{j!}\mathbb{S}(\mu)^2_\bullet\iota^*\mathcal{F}) \cong \alpha^1_{j!}((\alpha_j^*\mathrm{bc})_\bullet(\cdots), j^*\alpha^2_!\mathcal{F})$$

(this is true for any isomorphism).

We therefore have

$$\begin{aligned}
& \alpha^1_{j!}(\nu_j)^1_\bullet\mathrm{cart}^1_\bullet\iota^*\mathcal{E} \\
\cong\ & \alpha^1_{j!}(\alpha_j^*\mathrm{bc})^1_\bullet(\nu_j)^1_\bullet\mathrm{cart}^1_\bullet\iota^*\mathcal{E} \\
\cong\ & \alpha^1_{j!}\mathbb{D}(\mu)^1_\bullet(\iota^*\nu)^1_\bullet\iota^*\mathcal{E} \\
\cong\ & \alpha^1_{j!}\mathbb{D}(\mu)^1_\bullet\iota^*\nu^1_\bullet\mathcal{E}
\end{aligned}$$

Thus we are left to show that

$$\alpha^1_{j!}\mathbb{D}(\mu)^1_\bullet\iota^*\nu^1_\bullet\mathcal{E} \to j^*\alpha^1_!\nu^1_\bullet\mathcal{E}$$

is an isomorphism. A tedious check shows that this *is* the base change morphism associated with p_1. It is an isomorphism by (FDer4 left) for p_1. ∎

2.5. (Co)Local morphisms.

2.5.1. Let Dia be a diagram category and let $\mathbb{S}$ be a *strong* right derivator with domain Dia. Strongness implies that for each diagram

$$\begin{tikzcd} & U \arrow[d] \\ S \arrow[r] & T \end{tikzcd}$$

in $\mathbb{S}(\cdot)$ there exists a homotopy pull-back "$U \times_T S$" which is well-defined up to (non-unique!) isomorphism. The existence and (weak) uniqueness of these pull-backs is the only property of $\mathbb{S}$ needed in this section. It is hence not necessary to assume that it is a right derivator on the whole of Dia. For instance, it is certainly enough to have this for the restriction of the pre-derivators $\mathbb{S}_I$ to Posf. A Grothendieck pre-topology on $\mathbb{S}$ is basically a Grothendieck pre-topology in the usual sense on $\mathbb{S}(\cdot)$ except that pull-backs are replaced by homotopy pull-backs. We state the precise definition:

2.5.2. Definition. *A* **Grothendieck pre-topology** *on $\mathbb{S}$ is the datum consisting of, for any $S \in \mathbb{S}(\cdot)$, a collection of families $\{U_i \to S\}_{i \in \mathcal{I}}$ of morphisms in $\mathbb{S}(\cdot)$ called* **covers**, *such that*

1. *Every family consisting of one isomorphism is a cover,*
2. *If $\{U_i \to S\}_{i\in\mathcal{I}}$ is a cover and $T \to S$ is any morphism then the family $\{$"$U_i \times_S T$" $\to T\}_{i\in\mathcal{I}}$ is a cover for any choice of particular members of the family $\{$"$U_i \times_S T$"$\}$.*
3. *If $\{U_i \to S\}_{i\in\mathcal{I}}$ is a cover and for each i, the family $\{U_{i,j} \to U_i\}_{j\in\mathcal{J}_i}$ is a cover then the family of compositions $\{U_{i,j} \to U_i \to S\}_{i\in\mathcal{I}, j\in\mathcal{J}_i}$ is a cover.*

2.5.3. Definition. [left] *Let $p : \mathbb{D} \to \mathbb{S}$ be a left fibered derivator satisfying also (FDer0 right). Assume that pull-backs exist in $\mathbb{S}$. We call a morphism $f : U \to S$ in $\mathbb{S}(\cdot)$* $\mathbb{D}$**-local** *if*

(Dloc1 left) The morphism f satisfies **base change***: for any diagram $Q \in \mathbb{D}(\square)$ with underlying diagram*

$$\begin{tikzcd} A \arrow[r, "\widetilde{F}"] \arrow[d, "\widetilde{G}"'] & B \arrow[d, "\widetilde{g}"] \\ C \arrow[r, "\widetilde{f}"'] & D \end{tikzcd}$$

such that $p(Q)$ in $\mathbb{S}(\square)$ is a pull-back-diagram, i.e. is (homotopy) Cartesian, with $p(\widetilde{f}) = f$ the following holds true: If $\widetilde{F}$ and $\widetilde{f}$ are Cartesian, and $\widetilde{g}$ is coCartesian *then also $\widetilde{G}$ is coCartesian.*[8]

(Dloc2 left) The morphism of derivators (cf. Lemma 2.3.14)

$$f^\bullet : \mathbb{D}_S \to \mathbb{D}_U$$

commutes with homotopy colimits.

A morphism $f : U \to S$ in $\mathbb{S}(\cdot)$ is called **universally $\mathbb{D}$-local** *if any homotopy pull-back of f is $\mathbb{D}$-local.*

2.5.4. Definition. [left] *Assume that $\mathbb{S}$ is equipped with a Grothendieck pre-topology (cf. 2.5.2). A left fibered derivator $p : \mathbb{D} \to \mathbb{S}$ as in Definition 2.5.3 is called* **local** *w.r.t. the pre-topology on $\mathbb{S}$, if the following conditions hold:*

1. *Every morphism $U_i \to S$ which is part of a cover is $\mathbb{D}$-local.*
2. *For a cover $\{f_i : U_i \to S\}$ the family*

$$(f_i)^\bullet : \mathbb{D}(S) \to \mathbb{D}(U_i)$$

is jointly conservative.

2.5.5. Definition. [right] *Let $p : \mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ be a right fibered derivator satisfying also (FDer0 left). Assume that pull-backs exist in $\mathbb{S}$. We call a morphism $f : U \to S$ in $\mathbb{S}(\cdot)$* **$\mathbb{D}$-colocal** *if*

(Dloc1 right) The morphism f satisfies **base change***: for any diagram $Q \in \mathbb{D}(\square)$ with underlying diagram:*

$$\begin{tikzcd} A & B \arrow[l, "\widetilde{F}"'] \\ C \arrow[u, "\widetilde{G}"] & D \arrow[l, "\widetilde{f}"] \arrow[u, "\widetilde{g}"'] \end{tikzcd}$$

such that $p(Q)^{\mathrm{op}}$ in $\mathbb{S}(\square)$ is a pull-back-diagram, i.e. is (homotopy) Cartesian, with $p(\widetilde{f}) = f^{\mathrm{op}}$ the following holds true: If $\widetilde{F}$ and $\widetilde{f}$ are coCartesian, and $\widetilde{g}$ is Cartesian then also $\widetilde{G}$ is Cartesian.

[8] In other words, if

$$\begin{tikzcd} \text{“}U \times_S V\text{”} \arrow[r, "F"] \arrow[d, "G"'] & V \arrow[d, "g"] \\ U \arrow[r, "f"'] & S \end{tikzcd}$$

is the underlying diagram of $p(Q)$ then the exchange morphism

$$G_\bullet F^\bullet \to f^\bullet g_\bullet$$

is an isomorphism.

(Dloc2 right) The morphism of derivators (cf. Lemma 2.3.13)

$$(f^{\mathrm{op}})_\bullet : \mathbb{D}_S \to \mathbb{D}_U$$

commutes with homotopy limits.

A morphism $f : U \to S$ in $\mathbb{S}(\cdot)$ is called **universally $\mathbb{D}$-colocal** *if any homotopy pull-back of f is $\mathbb{D}$-colocal.*

2.5.6. Definition. [right] *Assume that $\mathbb{S}$ is equipped with a Grothendieck pre-topology (cf. 2.5.2). A right fibered derivator $p : \mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ as in Definition 2.5.5 is called* **colocal** *w.r.t. the pre-topology on $\mathbb{S}$, if*

1. *Every morphism $f : U_i \to S$ which is part of a cover is $\mathbb{D}$-colocal.*
2. *For a cover $\{f_i : U_i \to S\}$ the family*

$$(f_i^{\mathrm{op}})_\bullet : \mathbb{D}(\cdot)_S \to \mathbb{D}(\cdot)_{U_i}$$

is jointly conservative.

2.5.7. Remark. The reader should keep in mind the two basic examples given in the introduction extracted from a six-functor-formalism:

$$\mathbb{D}^* \to \mathbb{S}^{\mathrm{op}} \qquad \mathbb{D}^! \to \mathbb{S}. \tag{13}$$

In many six-functor-formalisms occurring in nature there will be a Grothendieck pre-topology on $\mathbb{S}$ such that $\mathbb{D}^!$ is local w.r.t. it and such that $\mathbb{D}^*$ is colocal w.r.t. it. Except for the conservativity axioms this follows, for instance, from isomorphisms of the form[9]

$$f^! \cong f^*[n]$$

for any $f : U \to S$ being part of a cover, compatible with base-change in a suitable way. *Proof:* (Dloc1 left/right) follows from base-change for the pair $f^!$, f_* (or f^*, $f_!$), which is part of the six-functor-formalism, replacing $f^!$ by $f^*[n]$ (resp. f^* by $f^![-n]$). (Dloc2 left/right) follows directly from $f^! \cong f^*[n]$ and the fact that $f^!$ has a left-adjoint (resp. f^* has a right-adjoint). The conservativity axioms Definition 2.5.4, 2. and Definition 2.5.6, 2. do not follow automatically, but become equivalent to each other.

2.6. The associated pseudo-functor. Let $p : \mathbb{D} \to \mathbb{S}$ be a morphism of pre-derivators with domain Dia.

2.6.1. [left] Let $\mathrm{Dia}(\mathbb{S})$ be the 2-category of diagrams over $\mathbb{S}$, where the objects are pairs (I, F) such that $I \in \mathrm{Dia}$ and $F \in \mathbb{S}(I)$, the morphisms $(I, F) \to (J, G)$ are pairs (α, f) such that $\alpha : I \to J, f : F \to \alpha^* G$ and the 2-morphisms $(\alpha, f) \to (\beta, g)$ are the natural transformations $\mu : \alpha \Rightarrow \beta$ satisfying $\mathbb{S}(\mu)(G) \circ f = g$.

We call a morphism (α, f) of **fixed shape** if $\alpha = \mathrm{id}$, and of **diagram type** if f consists of identities. Every morphism is obviously a composition of one of diagram type by one of fixed shape.

[9] assuming the stable case — otherwise ignore the shift $[n]$.

2.6.2. [right] There is a dual notion of a 2-category $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S})$. Explicitly, the objects are pairs (I,F) such that $I \in \mathrm{Dia}$ and $F \in \mathbb{S}(I)$, the morphisms $(I,F) \to (J,G)$ are pairs (α, f) such that $\alpha : I \to J, f : \alpha^* G \to F$ and the 2-morphisms $(\alpha, f) \to (\beta, g)$ are the natural transformations $\mu : \alpha \Rightarrow \beta$ satisfying $f \circ \mathbb{S}(\mu)(G) = g$.

The association $(I,F) \mapsto (I^{\mathrm{op}}, F^{\mathrm{op}})$ induces an isomorphism $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S}) \to \mathrm{Dia}(\mathbb{S}^{\mathrm{op}})^{2-\mathrm{op}}$.

We are interested in associating to a fibered derivator a pseudo-functor like for classical fibered categories.

2.6.3. [left] We associate to a morphism of pre-derivators $p : \mathbb{D} \to \mathbb{S}$ which satisfies (FDer0 right) a (contravariant) 2-pseudo-functor

$$\mathbb{D} : \mathrm{Dia}(\mathbb{S})^{1-\mathrm{op}} \to \mathcal{CAT}$$

mapping a pair (I,F) to $\mathbb{D}(I)_F$, and a morphism $(\alpha, f) : (I,F) \to (J,G)$ to $f^\bullet \circ \alpha^* : \mathbb{D}(J)_G \to \mathbb{D}(I)_F$. A natural transformation $\mu : \alpha \Rightarrow \beta$ is mapped to the natural transformation pasted from the following two 2-commutative triangles:

$$\begin{tikzcd}
& \mathbb{D}(I)_{G\circ\alpha} \arrow[dr, "f^\bullet"] & \\
\mathbb{D}(J)_G \arrow[ur, "\alpha^*"] \arrow[dr, "\beta^*"'] \arrow[r, phantom, "\Downarrow^{\mu}"] & {} & \mathbb{D}(I)_F \arrow[l, phantom, "\Downarrow"'] \\
& \mathbb{D}(I)_{G\circ\beta} \arrow[uu, "\mathbb{S}(\mu)(G)^\bullet"] \arrow[ur, "g^\bullet"'] &
\end{tikzcd}$$

Proof of the pseudo-functor property. For a composition $(\beta, g) \circ (\alpha, f) = (\beta \circ \alpha, \alpha^*(g) \circ f)$ we have: $f^\bullet \circ \alpha^* \circ g^\bullet \circ \beta^* \cong f^\bullet \circ (\alpha^* g)^\bullet \circ \alpha^* \circ \beta^*$. This follows from the isomorphism $\alpha^* \circ g^\bullet \cong (\alpha^* g)^\bullet \circ \alpha^*$ (FDer0). One checks that this indeed yields a pseudo-functor. ∎

2.6.4. [right] We associate to a morphism of pre-derivators $p : \mathbb{D} \to \mathbb{S}$ which satisfies (FDer0 left) a (contravariant) 2-pseudo-functor

$$\mathbb{D} : \mathrm{Dia}^{\mathrm{op}}(\mathbb{S})^{1-\mathrm{op}} \to \mathcal{CAT}$$

mapping a pair (I,F) to $\mathbb{D}(I)_{F(I)}$, and a morphism $(\alpha, f) : (I,F) \to (J,G)$ to $f_\bullet \circ \alpha^*$ from $\mathbb{D}(J)_G \to \mathbb{D}(I)_F$. This defines a functor by the same reason as in 2.6.3.

2.6.5. [left] We assume that $\mathbb{S}$ is a strong right derivator. There is a notion of "comma object" in $\mathrm{Dia}(\mathbb{S})$ which we describe here for the case that $\mathbb{S}$ is the pre-derivator associated with a category $\mathcal{S}$ and leave it to the reader to formulate the derivator version. In that case the corresponding object will be determined up to (non-unique!) isomorphism only.

Given diagrams $D_1 = (I_1, F_1), D_2 = (I_2, F_2), D_3 = (I_3, F_3)$ in $\mathrm{Dia}(\mathbb{S})$ and morphisms $\beta_1 : D_1 \to D_3$, $\beta_2 : D_2 \to D_3$, we form the comma diagram $D_1 \times_{/D_3} D_2$ as follows: the underlying diagram $I_1 \times_{/I_3} I_2$ has objects being triples (i_1, i_2, μ) such that $i_1 \in I_1$, $i_2 \in I_2$, and $\mu : \alpha_1(i_1) \to \alpha_2(i_2)$ in I_3. A morphism is a pair $\beta_j : i_j \to i'_j$ for $j = 1, 2$ such that

$$\begin{tikzcd} \alpha_1(i_1) \arrow[r, "\alpha_1(\beta_1)"] \arrow[d, "\mu"] & \alpha_1(i'_1) \arrow[d, "\mu'"] \\ \alpha_2(i_2) \arrow[r, "\alpha_2(\beta_2)"] & \alpha_2(i'_2) \end{tikzcd}$$

commutes in I_3. The corresponding functor $\widetilde{F} \in \mathbb{S}(I_1 \times_{/I_3} I_2)$ maps a triple (i_1, i_2, μ) to

$$F_1(i_1) \times_{F_3(\alpha_2(i_2))} F_2(i_2).$$

We define P_j to be (ι_j, p_j) for $j = 1, 2$, where ι_j maps a triple (i_1, i_2, μ) to i_j, and p_j is the corresponding projection of the fiber product. We then get a 2-commutative diagram

$$\begin{tikzcd} D_1 \times_{/D_3} D_2 \arrow[r, "P_1"] \arrow[d, "P_2"] \arrow[dr, phantom, "\swarrow\mu"] & D_1 \arrow[d, "\beta_1"] \\ D_2 \arrow[r, "\beta_2"'] & D_3 \end{tikzcd}$$

If we are given I_2, I_3 only and two maps $I_1 \to I_3$ and $I_2 \to I_3$ we also form $D_1 \times_{/I_3} I_2$ by the same underlying category, with functor $F_1 \circ \iota_1$.

2.6.6. [right] We assume that $\mathbb{S}$ is a strong left derivator. There is a dual notion of "comma object" in $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S})$ which we describe here again for the case that $\mathbb{S}$ is the pre-derivator associated with a category $\mathcal{S}$ and leave it to the reader to formulate the derivator version. In that case the corresponding object will be determined up to (non-unique!) isomorphism only.

Given three diagrams $D_1^o = (I_1, F_1), D_2^o = (I_2, F_2)$ in $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S})$ mapping to $D_3^o = (I_3, F_3)$, we form the comma diagram $D_1^o \times_{/D_3^o} D_2^o$ as follows: the underlying diagram is $I_1 \times_{/D_3} I_2$ which has object being triples (i_1, i_2, μ) such that $i_1 \in I_1, i_2 \in I_2$ and $\mu : \alpha_1(i_1) \to \alpha_2(i_2)$ in I_3. A morphism is a pair $\beta_j : i_j \to i'_j$ for $j = 1, 2$ such that

$$\begin{tikzcd} \alpha_1(i_1) \arrow[r, "\alpha_1(\beta_1)"] \arrow[d, "\mu"] & \alpha_1(i'_1) \arrow[d, "\mu'"] \\ \alpha_2(i_2) \arrow[r, "\alpha_2(\beta_2)"] & \alpha_2(i'_2) \end{tikzcd}$$

commutes in I_3. The corresponding functor $\widetilde{F}$ maps a triple (i_1, i_2, μ) to

$$F_1(i_1) \sqcup_{F_3(\alpha_1(i_1))} F_2(i_2).$$

We then get a 2-commutative diagram

$$\begin{tikzcd} D_2^o \times_{/D_3^o} D_1^o \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow\mu"] & D_1^o \arrow[d] \\ D_2^o \arrow[r] & D_3^o \end{tikzcd}$$

2.6.7. Definition. *If $\mathbb{S}$ is equipped with a Grothendieck pre-topology (cf. 2.5.2) then we call $(\alpha, f) : (I, F) \to (J, G)$* **$\mathbb{D}$-local** *if $f_i : F(i) \to G \circ \alpha(i)$ is $\mathbb{D}$-local (cf. 2.5.3) for all $i \in I$. Likewise for the notions of universally $\mathbb{D}$-local, $\mathbb{D}$-colocal, and universally $\mathbb{D}$-colocal.*

2.6.8. Proposition. [left] *Let $\mathbb{D} \to \mathbb{S}$ be a left fibered derivator satisfying also (FDer0 right) and such that $\mathbb{S}$ is a strong right derivator. Then the associated pseudo-functor satisfies the following properties:*

1. *For a morphism of diagrams $(\alpha, f) : D_1 \to D_2$ the corresponding pull-back*
$$(\alpha, f)^* : \mathbb{D}(D_2) \to \mathbb{D}(D_1)$$
has a left-adjoint $(\alpha, f)_!$.

2. *For a diagram like in 2.6.5*
$$\begin{tikzcd} D_1 \times_{/D_3} D_2 \arrow[r, "P_1"] \arrow[d, "P_2"'] \arrow[dr, phantom, "\swarrow\alpha"] & D_1 \arrow[d, "\beta_1"] \\ D_2 \arrow[r, "\beta_2"'] & D_3 \end{tikzcd}$$
the corresponding exchange morphism
$$P_{2!} P_1^* \to \beta_2^* \beta_{1!}$$
is an isomorphism in $\mathbb{D}(D_2)$ provided that β_2 is $\mathbb{D}$-local.

Proof. 1. By (FDer0 left) and (FDer3 left) we can form $(\alpha, f)_! := \alpha_! \circ f_\bullet$ which is clearly left adjoint to $(\alpha, f)^*$.

2. We first reduce to the case where I_2 is the trivial category. Indeed consider the diagram

$$\begin{tikzcd} D_1 \times_{/D_3} (\{i_2\}, F_2(i_2)) \arrow[r, "can."] \arrow[d] \arrow[dr, phantom, "\swarrow"] & D_1 \times_{/D_3} D_2 \times_{/D_2} (\{i_2\}, F_2(i_2)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & D_1 \times_{/D_3} D_2 \arrow[r, "P_1"] \arrow[d, "P_2"'] \arrow[dr, phantom, "\swarrow"] & D_1 \arrow[d, "\beta_1"] \\ (\{i_2\}, F_2(i_2)) \arrow[r, equal] & (\{i_2\}, F_2(i_2)) \arrow[r] & D_2 \arrow[r, "\beta_2"'] & D_3 \end{tikzcd}$$

The exchange morphism of the middle square and outmost rectangle are isomorphisms by the reduced case. The morphism *can.* of the left hand square is of diagram type and its underlying diagram functor has an adjoint. The exchange morphism is therefore an isomorphism by [Gro13, 1.23]. Using Lemma 2.6.9 therefore, applying this for all $i_2 \in I_2$, also the exchange morphism of the right square has to be an isomorphism (this uses axiom Der2).

Now we may assume $D_2 = (\{i_2\}, F_2(i_2))$. Consider the following diagram, in which we denote $\beta_1 = (\alpha_1, f_1)$, $\beta_2 = (\alpha_2, f_2)$.

$$\begin{tikzcd}
(I_1 \times_{/I_3} \{i_2\}, \widetilde{F}) \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[dr, phantom, "\swarrow \; 1"] & (I_1 \times_{/I_3} \{i_2\}, F_1 \circ \iota_1) \arrow[r, "\iota_1"] \arrow[d, "\iota_1^* f_1"] \arrow[dr, phantom, "\swarrow \; 4"] & (I_1, F_1) \arrow[d, "f_1"] \\
(I_1 \times_{/I_3} \{i_2\}, \widetilde{F}') \arrow[r, "p_1'"] \arrow[d, "p_2'"] \arrow[dr, phantom, "\swarrow \; 2"] & (I_1 \times_{/I_3} \{i_2\}, F_3 \circ \alpha_1 \circ \iota_1) \arrow[r, "\iota_1"] \arrow[d, "F_3(\mu)"] \arrow[ddr, phantom, "\swarrow \; 5"] & (I_1, F_3 \circ \alpha_1) \arrow[dd, "\alpha_1"] \\
(I_1 \times_{/I_3} \{i_2\}, F_2(i_2)) \arrow[r, "\iota_2^* f_2"] \arrow[d, "\iota_2"] \arrow[dr, phantom, "\swarrow \; 3"] & (I_1 \times_{/I_3} \{i_2\}, F_3(\alpha_2(i_2))) \arrow[d, "\iota_2"] & \\
(\{i_2\}, F_2(i_2)) \arrow[r, "f_2"] & (\{i_2\}, F_3(\alpha_2(i_2))) \arrow[r, "\alpha_2"] & (I_3, F_3)
\end{tikzcd}$$

where $\widetilde{F}$ is the functor defined in 2.6.5 mapping a triple $(i_1, i_2, \mu : \alpha_1(i_1) \to \alpha_2(i_2))$ to

$$F_1(i_1) \times_{F_3(\alpha_2(i_2))} F_2(i_2)$$

and $\widetilde{F}'$ is the functor mapping a triple $(i_1, i_2, \mu : \alpha_1(i_1) \to \alpha_2(i_2))$ to

$$F_3(\alpha_1(i_1)) \times_{F_3(\alpha_2(i_2))} F_2(i_2).$$

We have to show that the exchange morphism for the outer square is an isomorphism. Using Lemma 2.6.9 below it suffices to show this for the squares 1–5. That the exchange morphism for the squares 1 and 2, where the morphisms are of fixed shape, is an isomorphism can be checked point-wise by (Der2). Then it boils down to the base change condition (Dloc1 left). Note that the squares are pull-back squares in $\mathcal{S}$ by construction of $\widetilde{F}'$ resp $\widetilde{F}$. The exchange morphism for 4 is an isomorphism by (FDer0 left). The exchange morphism for 3 is an isomorphism because of (Dloc2 left). The exchange morphism for 5 is an isomorphism because of (FDer4 left). ∎

Dualizing, there is a right-variant of the theorem, which uses $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S})$ instead. We leave its formulation to the reader.

The language of this section allows to restate Lemma 2.3.22 and Lemma 2.3.29 in a more convenient way:

2.6.9. Lemma. [left]

1. *Given a "pasting" diagram in* Dia($\mathbb{S}$)

$$\begin{tikzcd}
D_1 \arrow[r, "\Gamma"] \arrow[d, "A"'] \arrow[dr, phantom, "\swarrow^{\nu}"] & D_3 \arrow[r, "B"] \arrow[d, "a"] \arrow[dr, phantom, "\swarrow^{\mu}"] & D_5 \arrow[d, "\alpha"] \\
D_2 \arrow[r, "\gamma"] & D_4 \arrow[r, "\beta"] & D_6
\end{tikzcd}$$

the pasted natural transformation $\nu \odot \mu := \beta\nu \circ \mu\Gamma$ *satisfies*

$$\nu_! \odot \mu_! = (\nu \odot \mu)_!.$$

2. Given a "pasting" diagram in $\mathrm{Dia}(\mathbb{S})$

$$\begin{tikzcd} D_1 \arrow[r, "B"] \arrow[d, "\Gamma"'] \arrow[dr, phantom, "\swarrow^{\nu}"] & D_2 \arrow[d, "\gamma"] \\ D_3 \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "\swarrow^{\mu}"] & D_4 \arrow[d, "\alpha"] \\ D_5 \arrow[r, "\beta"] & D_6 \end{tikzcd}$$

the pasted natural transformation $\nu \odot \mu := \alpha\nu \circ \mu\Gamma$ *satisfies*

$$\mu_! \odot \nu_! = (\mu \odot \nu)_!.$$

3. (Co)homological descent

3.1. Categories of $\mathbb{S}$-diagrams.

3.1.1. Definition. *Let* $\mathbb{S}$ *be a strong right derivator with Grothendieck pre-topology.*

A **category of $\mathbb{S}$-diagrams in** $\mathrm{Cat}(\mathbb{S})$ *is a full sub-2-category* $\mathcal{DIA} \subset \mathrm{Cat}(\mathbb{S})$*, satisfying the following axioms:*

(SDia1) The empty diagram $(\varnothing, -)$*, the diagrams* $(\cdot, S)$ *for any* $S \in \mathbb{S}(\cdot)$*, and* (Δ_1, f) *for any* $f \in \mathbb{S}(\Delta_1)$ *are objects of* $\mathcal{DIA}$*.*

(SDia2) $\mathcal{DIA}$ *is stable under taking finite coproducts and such fibered products, where one of the morphisms is of pure diagram type.*

(SDia3) For each morphism $\alpha : D_1 \to D_2$ *with* $D_i = (I_i, F_i)$ *in* $\mathcal{DIA}$ *and for each object* $i \in I_2$ *and morphism* $U \to F_2(i)$ *being part of a cover in the chosen pre-topology, the slice diagram* $D_1 \times_{/D_2} (i, U)$ *is in* $\mathcal{DIA}$*, and if* α *is of pure diagram type then also* $(i, F_2(i)) \times_{/D_2} D_1$ *is in* $\mathcal{DIA}$*.*

A **category of $\mathbb{S}$-diagrams** $\mathcal{DIA}$ *is called* **infinite***, if it satisfies in addition:*

(SDia5) $\mathcal{DIA}$ *is stable under taking arbitrary coproducts.*

There is an obvious dual notion of a category of $\mathbb{S}$-diagrams in $\mathrm{Cat}^{\mathrm{op}}(\mathbb{S})$. If $\mathbb{S}$ is the trivial derivator both definitions boil down to the previous definition of a diagram category 2.1.1.

3.2. Fundamental (co)localizers.

3.2.1. Definition. *A class of morphisms $\mathcal{W}$ in a category is called* **weakly saturated**, *if it satisfies the following properties:*

(WS1) *Identities are in $\mathcal{W}$.*

(WS2) *$\mathcal{W}$ has the 2-out-of-3 property.*

(WS3) *If $p : Y \to X$ and $s : X \to Y$ are morphisms such that $p \circ s = \mathrm{id}_X$ and $s \circ p \in \mathcal{W}$ then $p \in \mathcal{W}$ (and hence $s \in \mathcal{W}$ by (WS2)).*

3.2.2. Definition. *Let $\mathbb{S}$ be a strong right derivator with Grothendieck pre-topology (2.5.2). Let $\mathcal{DIA} \subset \mathrm{Cat}(\mathbb{S})$ be a category of $\mathbb{S}$-diagrams (cf. 3.1.1).*

Consider a family of subclasses $\mathcal{W}_S$ of 1-morphisms in $\mathcal{DIA}/S$ [10] *parametrized by all objects $S \in \mathbb{S}(\cdot)$. Such a family $\{\mathcal{W}_S\}_S$ is called a* **system of relative localizers** *if the following properties are satisfied:*

(L0) *For any morphism $S_1 \to S_2$ the induced functor $\mathcal{DIA}/S_1 \to \mathcal{DIA}/S_2$ maps $\mathcal{W}_{S_1}$ to $\mathcal{W}_{S_2}$.*

(L1) *Each $\mathcal{W}_S$ is weakly saturated.*

(L2 left) *If $D = (I, F) \in \mathcal{DIA}$, and I has a final object e, then the projection $D \to (e, F(e))$ is in $\mathcal{W}_{F(e)}$.*

(L3 left) *For any commutative diagram in $\mathcal{DIA}$ over $(\cdot, S)$*

$$\begin{tikzcd} D_1 \arrow[rr, "w"] \arrow[dr] & & D_2 \arrow[dl] \\ & D_3 = (E, F) & \end{tikzcd}$$

and for any chosen covers $\{U_{e,i} \to F(e)\}$ for all $e \in E$, the following implication holds true:

$$\forall e \in E \ \forall i \quad w \times_{/D_3} (e, U_{e,i}) \in \mathcal{W}_{U_{e,i}} \quad \Rightarrow \quad w \in \mathcal{W}_S.$$

(L4 left) *For any morphism $w : D_1 \to D_2 = (E, F)$ of pure diagram type over $(\cdot, S)$ the following implication holds true:*

$$\forall e \in E \quad (e, F(e)) \times_{/D_2} D_1 \to (e, F(e)) \in \mathcal{W}_{F(e)} \quad \Rightarrow \quad w \in \mathcal{W}_S.$$

There is an obvious dual notion of a **system of colocalizers** in $\mathcal{DIA} \subset \mathrm{Cat}^{\mathrm{op}}(\mathbb{S}) = \mathrm{Cat}(\mathbb{S}^{\mathrm{op}})^{2-\mathrm{op}}$ where $\mathbb{S}^{\mathrm{op}}$ is supposed to be a strong right derivator with Grothendieck pre-topology.

[10] where $\mathcal{DIA}/S$ denotes the comma (slice) category $\mathcal{DIA} \times_{/\mathcal{DIA}} (\cdot, S)$.

3.2.3. Definition. *Let $\mathbb{S}$ be a strong right derivator. Assume we are given a Grothendieck pre-topology on $\mathbb{S}$ (cf. 2.5.2). Let $\mathcal{DIA} \subset \mathrm{Cat}(\mathbb{S})$ be a category of $\mathbb{S}$-diagrams (cf. 3.1.1).*

A subclass $\mathcal{W}$ of 1-morphisms in $\mathcal{DIA}$ is called an **absolute localizer** *(or just* **localizer***) if the following properties are satisfied:*

(L1) $\mathcal{W}$ is weakly saturated.

(L2 left) If $D = (I, F) \in \mathcal{DIA}$, and I has a final object e, then the projection $D \to (e, F(e))$ is in $\mathcal{W}$.

(L3 left) For any commutative diagram in $\mathcal{DIA}$

$$\begin{tikzcd} D_1 \arrow[rr, "\alpha"] \arrow[dr] & & D_2 \arrow[dl] \\ & D_3 = (E, F) & \end{tikzcd}$$

and chosen covering $\{U_{i,e} \to F_3(e)\}$ for all $e \in E$, the following implication holds true:

$$\forall e \in E\ \forall i \quad w \times_{/D_3} (e, U_i) \in \mathcal{W} \quad \Rightarrow \quad w \in \mathcal{W}.$$

(L4 left) For any morphism $w : D_1 \to D_2 = (E, F)$ of pure diagram type, the following implication holds true:

$$\forall e \in E \quad (e, F(e)) \times_{/D_2} D_1 \to (e, F(e)) \in \mathcal{W} \quad \Rightarrow \quad w \in \mathcal{W}.$$

There is an obvious dual notion of absolute **colocalizer** in $\mathcal{DIA} \subset \mathrm{Cat}^{\mathrm{op}}(\mathbb{S}) = \mathrm{Cat}(\mathbb{S}^{\mathrm{op}})^{2-\mathrm{op}}$ where $\mathbb{S}^{\mathrm{op}}$ is supposed to be a strong right derivator with Grothendieck pre-topology.

Recall the isomorphism of 2-categories

$$\begin{array}{rcl} \mathrm{Cat}(\mathbb{S}) & \to & \mathrm{Cat}^{\mathrm{op}}(\mathbb{S}^{\mathrm{op}})^{2-\mathrm{op}} \\ (I, F) & \mapsto & (I^{\mathrm{op}}, F^{\mathrm{op}}). \end{array}$$

By abuse of notation, we denote the image of $\mathcal{DIA}$ under this identification by $\mathcal{DIA}^{\mathrm{op}}$.

3.2.4. Remark.

1. If $\mathcal{W}$ is a localizer in $\mathcal{DIA}$, then $\mathcal{W}^{\mathrm{op}}$ is a colocalizer in $\mathcal{DIA}^{\mathrm{op}}$ and vice versa. The same holds true for systems of relative localizers.
2. If $\mathbb{S}$ is the trivial derivator, then a system of relative localizers or a localizer are the same notion, and (L1–L3 left) are precisely the definition of fundamental localizer of Grothendieck.

3.2.5. Proposition. [Grothendieck] *If $\mathbb{S} = \{\cdot\}$ is the trivial derivator, then* $\mathrm{Cat}(\cdot) = \mathrm{Cat}^{\mathrm{op}}(\cdot)$ *as 2-categories. If $\mathcal{DIA}$ is self-dual, i.e. if $\mathcal{DIA}^{\mathrm{op}} = \mathcal{DIA}$ under this identification, then the notions of localizer, localizer without (L4 left), colocalizer, and colocalizer without (L4 right) are all equivalent.*

Proof. [Cis04, Proposition 1.2.6] ∎

3.2.6. Remark. The class of localizers is obviously closed under intersection, hence there is a smallest localizer $\mathcal{W}^{\min}_{\mathcal{DIA}}$. Furthermore the smallest localizer in $\mathcal{DIA}$ and the smallest colocalizer in $\mathcal{DIA}^{\mathrm{op}}$ correspond. If $\mathbb{S}$ is the trivial derivator and $\mathcal{DIA} = \mathrm{Cat}$, Cisinski [Cis04, Théorème 2.2.11] has shown that $\mathcal{W}^{\min}_{\mathrm{Cat}}$ is precisely the class $\mathcal{W}_\infty$ of functors $\alpha : I \to J$ such that $N(\alpha)$ is a weak equivalence in the classical sense (of simplicial sets, resp. topological spaces). For a localizer in the sense of Definition 3.2.3 this implies the following:

3.2.7. Theorem. *If $\mathcal{DIA} = \mathrm{Cat}(\mathbb{S})$ and $\mathcal{W}$ is an absolute localizer in $\mathcal{DIA}$ and $\alpha \in \mathcal{W}_\infty$, i.e. $\alpha : I \to J$ is a functor such that $N(\alpha)$ is a weak equivalence of topological spaces, the morphism $(\alpha, \mathrm{id}) : (I, p_I^* S) \to (J, p_J^* S)$ is in $\mathcal{W}$ for all $S \in \mathbb{S}(\cdot)$. The same holds analogously for a system of relative localizers.*

Proof. The class of functors $\alpha : I \to J$ in Cat such that $(\alpha, \mathrm{id}) : (I, p_I^* S) \to (J, p_J^* S)$ is in $\mathcal{W}$ obviously form a fundamental localizer in the classical sense. ∎

3.2.8. We will for (notational) simplicity assume that the following properties hold:

1. $\mathbb{S}$ has all **relative finite coproducts** (i.e. for each opfibration with finite *discrete* fibers $p : O \to I$ the functor p^* has a left adjoint $p_!$ and Kan's formula holds true for it).

2. For all finite families $(S_i)_{i \in I}$ of objects in $\mathbb{S}(\cdot)$ the collection $\{S_i \to \coprod_{j \in I} S_j\}_{i \in I}$ is a cover.

Let $\varnothing$ be the initial object of $\mathcal{S}$ (which exists by 1.). Then the map

$$\varnothing \to (\cdot, \varnothing),$$

where $\varnothing$ on the left denotes the empty diagram, is in $\mathcal{W}$ (resp. in $\mathcal{W}_\varnothing$, and hence in all $\mathcal{W}_S$) by (L3 left) applied to the empty cover.

From this and (L3 left) again it follows that for a finite collection $(S_i)_{i \in I}$ of objects of $\mathbb{S}(\cdot)$ the map

$$(I, (S_i)_{i \in I}) \to (\cdot, \coprod_{i \in I} S_i)$$

is in $\mathcal{W}$ (resp. in $\mathcal{W}_{\coprod_{i \in I} S_i}$). More generally, if we have an opfibration with finite *discrete* fibers $p : O \to I$ and a diagram $F \in \mathbb{S}(O)$ (over $S \in \mathbb{S}(\cdot)$), then the morphism

$$(O, F) \to (I, p_! F)$$

is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$).

3.2.9. Example. [Mayer-Vietoris] For the simplest non-trivial example of a non-constant map in $\mathcal{W}$ consider a cover $\{U_1 \to S, U_2 \to S\}$ in $\mathbb{S}(\cdot)$ consisting of two *mono*morphisms[11]. Then the projection

$$p : \left(\begin{tikzcd} \text{“}U_1 \times_S U_2\text{”} \arrow[r] \arrow[d] & U_1 \\ U_2 & \end{tikzcd} \right) \to S$$

is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$) as is easily derived from the axioms (L1–L4). See 3.5.13 for how the Mayer-Vietoris long exact sequence is related to this.

3.2.10. Let $\alpha, \beta : D_1 \to D_2$ be two morphisms in $\mathcal{DIA}$. Recall that it is the same to give a 2-morphism $\alpha \Rightarrow \beta$ or a morphism $D_1 \times \Delta_1 \to D_2$ such that for $i = 1, 2$ the compositions $D_1 \xrightarrow{e_i} D_1 \times \Delta_1 \longrightarrow D_2$ are α and β respectively. We call α and β **homotopic** if they are equivalent for the smallest equivalence relation containing by the following relation: $\alpha \sim \beta$, if there exists a 2-morphism $\alpha \Rightarrow \beta$. In other words α and β are homotopic if there is a finite set of 1-morphisms $\gamma_0, \gamma_1, \dots, \gamma_n : D_1 \to D_2$ such that $\gamma_0 = \alpha$ and $\gamma_n = \beta$ and a zig-zag of 2-morphisms:

$$\gamma_0 \Leftarrow \gamma_1 \Rightarrow \gamma_2 \Leftarrow \cdots \Rightarrow \gamma_n.$$

3.2.11. Proposition. *Let $\mathcal{DIA}$ be a category of $\mathbb{S}$-diagrams (cf. 3.1.1) and let $\mathcal{W}$ be localizer in $\mathcal{DIA}$ (resp. let $\{\mathcal{W}_S\}_S$ be a system of relative localizers). Then $\mathcal{W}$ (resp. $\{\mathcal{W}_S\}_S$) satisfies the following properties:*

1. *The localizer $\mathcal{W}$ (resp. each $\mathcal{W}_S$) is closed under coproducts.*
2. *Let $\widetilde{s} = (s, \mathrm{id}) : D_2 = (I_2, s^*F) \to D_1 = (I_1, F)$ be a morphism in $\mathcal{DIA}$ (resp. over $(\cdot, S)$) of pure diagram type such that s has a left adjoint $p : I_1 \to I_2$. Then the obvious morphisms $\widetilde{p} : D_1 \to D_2$ and $\widetilde{s}$ are in $\mathcal{W}$ (resp. in $\mathcal{W}_S$).*
3. *Given a commutative diagram in $\mathcal{DIA}$ (resp. one over S)*

$$\begin{tikzcd} D_1 \arrow[rr, "w"] \arrow[dr] & & D_2 \arrow[dl] \\ & D_3 & \end{tikzcd}$$

where the underlying functors of the morphisms to D_3 are opfibrations and the underlying functor of w is a morphism of opfibrations, and coverings $\{U_{e,i} \to F_3(e)\}$ for all $e \in I_3$, then (in the relative case)

$$\forall e \in I_3 \ \forall i \quad w \times_{D_3} (e, U_{e,i}) \in \mathcal{W}_{U_{e,i}} \quad \Rightarrow \quad w \in \mathcal{W}_S$$

or (in the absolute case)

$$\forall e \in I_3 \ \forall i \quad w \times_{D_3} (e, U_{e,i}) \in \mathcal{W} \quad \Rightarrow \quad w \in \mathcal{W}$$

[11] For an arbitrary $\mathbb{S}$ this means that the projections “$U_i \times_S U_i$” $\to U_i$ are isomorphisms.

4. *If $f : D_1 \to D_2$ is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$) then also $f \times E : D_1 \times E \to D_2 \times E$ is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$) for any $E \in$ Cat such that the morphism $f \times E$ is a morphism in $\mathcal{DIA}$.*

5. *Any morphism which is homotopic (in the sense of 3.2.10) to a morphism in $\mathcal{W}$ (resp. in $\mathcal{W}_S$) is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$).*

Proof. 1. This property follows immediately from (L3 left) applied to a diagram

$$\begin{tikzcd}
\coprod_{i \in I} D_{1,i} \arrow[rr] \arrow[dr] & & \coprod_{i \in I} D_{2,i} \arrow[dl] \\
& (I, p_I^* S) &
\end{tikzcd}$$

where I is considered to be a discrete category. (In the absolute case let S be the final object of $\mathbb{S}(\cdot)$.)

2. We first show that $\widetilde{p} \in \mathcal{W}$. Using (L3 left), it suffices to show that $\widetilde{p}_i : D_1 \times_{/I_2} i \to D_2 \times_{/I_2} i$ is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$) for all $i \in I_2$, however by the adjunction we have $I_1 \times_{/I_2} i = I_1 \times_{/I_1} s(i)$ and therefore $I_1 \times_{/I_2} i$ has a final object. In the diagram

$$\begin{tikzcd}
D_1 \times_{/I_2} i \arrow[r, "\widetilde{p}_i"] \arrow[d] & D_2 \times_{/I_2} i \arrow[d] \\
(\cdot, s(i)^* F) \arrow[r, equal] & (\cdot, s(i)^* F)
\end{tikzcd}$$

the vertical morphisms are thus in $\mathcal{W}$ (resp. $\mathcal{W}_S$) and so is the upper horizontal morphism. That $\widetilde{s}$ is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$) will follow from 5. because this implies that $\widetilde{s} \circ \widetilde{p}$ and $\widetilde{p} \circ \widetilde{s}$ are in $\mathcal{W}$ (resp. in $\mathcal{W}_S$) therefore by (L1) also $\widetilde{s}$ is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$). For note that the unit and the counit extend to 2-morphisms of diagrams.

3. Using (L3 left), we have to show that $D_1 \times_{/D_3} (e, U_{e,i}) \to D_2 \times_{/D_3} (e, U_{e,i})$ is in $\mathcal{W}$ (resp. in $\mathcal{W}_{U_{e,i}}$). Since the underlying functors of $D_1 \to D_3$ and $D_2 \to D_3$ are opfibrations, we have a diagram over $(e, U_{e,i})$:

$$\begin{tikzcd}
D_1 \times_{D_3} (e, U_{e,i}) \arrow[r] \arrow[d, shift left, "\iota_e"] & D_2 \times_{D_3} (e, U_{e,i}) \arrow[d, shift left, "\iota_e"] \\
D_1 \times_{/D_3} (e, U_{e,i}) \arrow[r] \arrow[u, shift left, "s_e"] & D_2 \times_{/D_3} (e, U_{e,i}) \arrow[u, shift left, "s_e"]
\end{tikzcd}$$

where the underlying functor of ι_e is of diagram type and is right adjoint to s_e. Therefore s_e is in $\mathcal{W}$ (resp. in $\mathcal{W}_{U_{e,i}}$) by 2. and hence the same holds for ι_e because $s_e \iota_e = \mathrm{id}$ (using L1). Note: we are not using the not yet proven part of 2. Since the top arrow is in $\mathcal{W}$ (resp. in $\mathcal{W}_{U_{e,i}}$) the same holds for the bottom arrow.

4. This is a special case of 2.

5. A natural transformation $\mu : f \Rightarrow g$ for $f, g : D_1 \to D_2$ can be seen as a morphism of diagrams $\mu : \Delta_1 \times D_1 \to D_2$ such that $\mu \circ e_0 = f$ and $\mu \circ e_1 = g$. Since the projection $p : \Delta_1 \times D_1 \to D_1$ is in $\mathcal{W}$ by 3. also the morphisms $e_{0,1} : D_1 \to \Delta_1 \times D_1$ are in $\mathcal{W}$. Since $\mu \circ e_0 = f$ and $\mu \circ e_1 = g$, the morphism f is in $\mathcal{W}$ if and only if $g \in \mathcal{W}$. ∎

3.2.12. PROPOSITION. *Axiom (L4 left) is, in the presence of (L1–L3 left), equivalent to the following, apparently weaker axiom:*

(L4' left) Let $w : D_1 \to D_2$ be a morphism (resp. a morphism over S) of pure diagram type such that the underlying functor is a fibration. Then (in the relative case)

$$\forall e \in I_2 \quad (e, F_2(e)) \times_{D_2} D_1 \to (e, F_2(e)) \in \mathcal{W}_{F_2(e)} \quad \Rightarrow \quad w \in \mathcal{W}_S$$

or (in the absolute case)

$$\forall e \in I_2 \quad (e, F_2(e)) \times_{D_2} D_1 \to (e, F_2(e)) \in \mathcal{W} \quad \Rightarrow \quad w \in \mathcal{W}.$$

PROOF. (L4' left) implies (L4 left): Consider the following 2-commutative diagram

$$\begin{tikzcd}
(E,F) \times_{/(E,F)} (I, p^*F) \arrow[rr] \arrow[d] \arrow[drr, phantom, "\Nearrow"] & & (I, p^*F) = D_1 \arrow[d] \\
(E,F) \arrow[rr, equal] & & (E,F) = D_2
\end{tikzcd}$$

The underlying diagram functor of the top horizontal map (which is not purely of diagram type) is an opfibration and hence by Proposition 3.2.11, 3. it is in $\mathcal{W}$ (resp. $\mathcal{W}_S$), provided that the morphisms of the fibers $(E \times_{/E} e, \mathrm{pr}_1^* F) \to (\cdot, F(e))$ are in $\mathcal{W}$ (resp. in $\mathcal{W}_{F(e)}$). However $E \times_{/E} e$ has the final object id_e whose value under $\mathrm{pr}_1^* F$ is $F(e)$. The morphisms of the fibers are therefore in $\mathcal{W}$ (resp. in $\mathcal{W}_{F(e)}$) by (L2 left). The underlying diagram functor of the left vertical map is a fibration and $\mathrm{pr}_1^* F$ is constant along the fibers. Therefore the fact that all $(e, F(e)) \times_{/D_2} D_1 \to (e, F(e))$ are in $\mathcal{W}$ (resp. in $\mathcal{W}_{F(e)}$) implies that the left vertical map is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$) by (L4' left). Thus also the right vertical map is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$). (This uses Proposition 3.2.11, 5. and the fact that the two compositions in the diagram are homotopic).

(L4 left) implies (L4' left): If $D_1 \to D_2 = (E, F)$ is a morphism whose underlying functor is a fibration as in Axiom (L4' left), the morphism of *constant* diagrams $(e, F(e)) \times_{D_2} D_1 \to (e, F(e)) \times_{/D_2} D_1$ is in $\mathcal{W}$ (resp. $\mathcal{W}_{F(e)}$) (their underlying functors being part of an adjunction), therefore (L4 left) applies. ∎

3.3. SIMPLICIAL OBJECTS IN A LOCALIZER.

3.3.1. In this section, we fix a strong right derivator $\mathbb{S}$ equipped with a Grothendieck pre-topology and satisfying the assumptions of 3.2.8 and a category of $\mathbb{S}$-diagrams $\mathcal{DIA}$ (cf. 3.1.1). Assume that for all $S_\bullet \in \mathbb{S}(\Delta^{\mathrm{op}})$ the diagrams $(\Delta^{\mathrm{op}}, S_\bullet)$ and also all truncations $((\Delta^{\le n})^{\mathrm{op}}, S_\bullet)$ are in $\mathcal{DIA}$. Later we will assume that also $((\Delta^\circ)^{\mathrm{op}}, S_\bullet)$ for all $S_\bullet \in \mathbb{S}((\Delta^\circ)^{\mathrm{op}})$, and all truncations $((\Delta^{\circ, \le n})^{\mathrm{op}}, S_\bullet)$ are in $\mathcal{DIA}$, where Δ° denotes the injective simplex diagram. The reasoning in this section uses little of the explicit definition of Δ^{op}. For comparison with classical texts on cohomological descent we stick to the particular diagram Δ^{op}.

Consider the category $\mathbb{S}(\Delta^{\mathrm{op}})$. Since $\mathbb{S}$ has all (relative) finite coproducts, $\mathbb{S}(\cdot)$ is actually tensored over $\mathcal{SETF}$, hence $\mathbb{S}(\Delta^{\mathrm{op}})$ will be tensored over $\mathcal{SETF}^{\Delta^{\mathrm{op}}}$. We sketch this construction. A finite simplicial set, i.e. a functor $\xi : \Delta^{\mathrm{op}} \to \mathcal{SETF}$, can be seen as a functor with values in finite discrete categories. The corresponding Grothendieck construction yields an opfibration $\pi_\xi : \int \xi \to \Delta^{\mathrm{op}}$. We define for $X_\bullet \in \mathbb{S}(\Delta^{\mathrm{op}})$:

$$\xi \otimes X_\bullet := (\pi_\xi)_! (\pi_\xi)^* X_\bullet.$$

Recall that the notion '$\mathbb{S}$ has *relative finite coproducts*' means that all functors $(\pi_\xi)_!$ arising this way exist and can be computed fiber-wise.

3.3.2. Consider the full subcategory $\Delta^{\leq n}$ of Δ consisting of $\Delta_0, \dots, \Delta_n$. Since $\mathbb{S}$ is assumed to be a right derivator, the restriction functor

$$\iota^* : \mathbb{S}(\Delta^{\mathrm{op}}) \to \mathbb{S}((\Delta^{\leq n})^{\mathrm{op}})$$

has a right adjoint ι_*, which is usually called the **coskeleton** and denoted cosk^n.

Let some simplicial object $Y_\bullet \in \mathbb{S}(\Delta^{\mathrm{op}})$ and a morphism $\alpha : X_{\leq n} \to \iota^* Y_\bullet$ be given. Consider the full subcategory $(\Delta^{\mathrm{op}} \times \Delta_1)^{0-\leq n}$ of all objects $\Delta_i \times \{1\}$ for all $i \in \mathbb{N}_0$, and $\Delta_i \times \{0\}$ for $i \leq n$. The restriction

$$\iota^* : \mathbb{S}(\Delta^{\mathrm{op}} \times \Delta_1) \to \mathbb{S}((\Delta^{\mathrm{op}} \times \Delta_1)^{0-\leq n})$$

has again an adjoint ι_*. Since $\mathbb{S}$ is assumed to be strong we can consider α as an object over $(\Delta \times \Delta_1)^{0-\leq n}$. The first row of $\iota_* \alpha$ is called the **relative coskeleton** $\mathrm{cosk}^n(X_{\leq n} | Y_\bullet)$ of $X_{\leq n}$. For $n = -1$ we understand $\mathrm{cosk}^{-1}(-|Y_\bullet) = Y_\bullet$.

These constructions work the same way with Δ replaced by Δ°. The functor 'coskeleton' and 'relative coskeleton' is in both cases even *the same functor*, i.e. these functors commute with the restriction of a simplicial to a semi-simplicial object[12]. This would not at all be true for the corresponding left adjoint, the functor 'skeleton'.

In the following, Δ_n denotes the diagram, $\{\Delta_n\}$ denotes the object of Δ, as usual also considered as the corresponding subdiagram with one object, whereas $\Delta_{n,\bullet}$ denotes the represented simplicial set $N(\Delta_m) : \Delta_m \mapsto \mathrm{Hom}_\Delta(\{\Delta_m\}, \{\Delta_n\})$ and $\Delta^\circ_{n,\bullet}$ the semi-simplicial set $\Delta_m \mapsto \mathrm{Hom}_{\Delta^\circ}(\{\Delta_m\}, \{\Delta_n\})$.

We call a diagram I in a diagram category Dia **contractible**, if $I \to \cdot$ lies in every fundamental localizer on Dia.

[12] To see this, e.g., for the case of the 'coskeleton', observe that there is an adjunction:

$$\{\Delta_m\} \times_{/(\Delta^\circ)^{\mathrm{op}}} (\Delta^\circ_{\leq n})^{\mathrm{op}} \rightleftarrows \{\Delta_m\} \times_{/\Delta^{\mathrm{op}}} \Delta^{\mathrm{op}}_{\leq n}.$$

3.3.3. Lemma.

1. *Let I be a category admitting a final object i. Let $N(I)$ be the nerve of I. Then the category*
$$\int_{(\Delta^\circ)^{\mathrm{op}}} N(I)$$
is contractible.

2. *Let I be a category admitting a final object i. Let $N(I)$ be the nerve of I. Then the category*
$$\int_{\Delta^{\mathrm{op}}} N(I)$$
is contractible.

3. *Let I be a* directed *category admitting a final object i. Let $N^\circ(I)$ be the semi-simplicial nerve of I, defined by letting $N^\circ(I)_m$ be the set of functors $\Delta_m \to I$ such that no non-identity morphism is mapped to an identity. Then the category*
$$\int_{(\Delta^\circ)^{\mathrm{op}}} N^\circ(I)$$
is contractible.

Proof. 1. is shown in [Cis04, Proposition 2.2.3]. 2. is the same but considering $N(I)$ as a functor from Δ^{op} to $\mathcal{SET}$. The same proof works when $(\Delta^\circ)^{\mathrm{op}}$ is replaced by Δ^{op}. 3. is also just a small modification of [loc. cit.]. Define a functor $\xi : \int_{(\Delta^\circ)^{\mathrm{op}}} N^\circ(I) \to \int_{(\Delta^\circ)^{\mathrm{op}}} N^\circ(I)$ as follows: an object (n,x), where $x \in N^\circ(I)_n$ is mapped to (n,x) if $x(n) = i$ and to $(n+1, x')$ with
$$x'(k) = \begin{cases} x(k) & k \le n \\ i & k = n+1. \end{cases}$$
otherwise. There are natural transformations
$$\mathrm{id}_{\int_{(\Delta^\circ)^{\mathrm{op}}} N^\circ(I)} \Rightarrow \xi \qquad i \Rightarrow \xi$$
where i denotes here the constant functor with value $(0,i)$, showing that $\int_{(\Delta^\circ)^{\mathrm{op}}} N^\circ(I)$ is contractible. ∎

3.3.4. Corollary. *The diagrams Δ, Δ°, $\int_{\Delta^{\mathrm{op}}} \Delta_{n,\bullet}$, $\int_{(\Delta^\circ)^{\mathrm{op}}} \Delta^\circ_{n,\bullet}$, $\int_{\Delta^{\mathrm{op}}} \Delta_{n,\bullet} \times \Delta_{m,\bullet}$ and $\{\Delta_n\} \times_{/\Delta^{\mathrm{op}}} (\Delta^\circ)^{\mathrm{op}} = \int_{(\Delta^\circ)^{\mathrm{op}}} \Delta_{n,\bullet}$ are contractible.*

Proof. The simplicial set $\Delta_{n,\bullet}$ is just the nerve $N(\Delta_n)$. Likewise the semi-simplicial set $\Delta^\circ_{n,\bullet}$ is the semi-simplicial nerve N° of Δ_n. ∎

Note that the diagram $\int_{(\Delta^\circ)^{\mathrm{op}}} \Delta^\circ_{n,\bullet}$ is even *finite*.

3.3.5. Lemma. *Let $\mathcal{W}$ be a localizer (resp. let $\{\mathcal{W}_S\}_S$ be a system of relative localizers) in $\mathcal{DIA}$.*

Let $((\Delta^{\mathrm{op}})^2, F_{\bullet,\bullet}) \in \mathcal{DIA}$ be a bisimplicial diagram (resp. a bisimplicial diagram over S) and let $\delta : \Delta^{\mathrm{op}} \to (\Delta^{\mathrm{op}})^2$ be the diagonal. Then the morphism

$$(\Delta^{\mathrm{op}}, \delta^* F_{\bullet,\bullet}) \to ((\Delta^{\mathrm{op}})^2, F_{\bullet,\bullet})$$

is in $\mathcal{W}$ (resp. $\mathcal{W}_S$).

Proof. We focus on the absolute case. For the relative case the proof is identical. Since the morphism in the statement is of pure diagram type, we may check the condition of (L4 left): we have to show that the category

$$(\{\Delta_m\} \times \{\Delta_n\}) \times_{/(\Delta^{\mathrm{op}})^2} \Delta^{\mathrm{op}}$$

is contractible, say, on the diagram category of diagrams I such that $(I, F_{m,n}) \in \mathcal{DIA}$. This is the category $\int_{\Delta^{\mathrm{op}}} \Delta_{n,\bullet} \times \Delta_{m,\bullet}$ which is contractible by Corollary 3.3.4. Note that this is the only feature of Δ^{op} used in the proof of this Lemma. ∎

3.3.6. Remark. The previous lemma should be seen in the following context: the Grothendieck construction gives a way of embedding the category of simplicial sets into the category of small categories. This construction maps weak equivalences to weak equivalences and induces an equivalence between the corresponding homotopy categories. A bisimplicial set can be seen as a simplicial object in the category of simplicial sets. Its homotopy colimit is given by the diagonal simplicial set. On the other hand the homotopy colimit in the category of small categories is just given by the Grothendieck construction. From this perspective, the lemma is clear if $\mathbb{S}$ is the derivator associated with the category of sets (equipped with the discrete topology).

3.3.7. Lemma. *Let $\mathcal{W}$ be a localizer (resp. let $\{\mathcal{W}_S\}_S$ be a system of relative localizers) in $\mathcal{DIA}$.*

Consider a simplicial diagram $(\Delta^{\mathrm{op}}, F_\bullet) \in \mathcal{DIA}$ (resp. a simplicial diagram over S). The morphism

$$(\Delta^{\mathrm{op}}, F_\bullet \otimes \Delta_{n,\bullet}) \to (\Delta^{\mathrm{op}}, F_\bullet)$$

is in $\mathcal{W}$ (resp. $\mathcal{W}_S$).

Proof. We focus on the absolute case. For the relative case the proof is identical. The diagram $(\Delta^{\mathrm{op}}, F_\bullet \otimes \Delta_{n,\bullet})$ is equivalent to $(\int_{\Delta^{\mathrm{op}}} \Delta_{n,\bullet}, \pi^* F_\bullet)$ by definition (see 3.3.1) and the conventions 3.2.8. We apply the criterion of (L4 left) to the resulting map

$$(\int_{\Delta^{\mathrm{op}}} \Delta_{n,\bullet}, \pi^* F_\bullet) \to (\Delta^{\mathrm{op}}, F_\bullet)$$

and have to show that

$$\{\Delta_m\} \times_{/\Delta^{\mathrm{op}}} \int_{\Delta^{\mathrm{op}}} \Delta_{n,\bullet}$$

is contractible. This category is again isomorphic to $\int_{\Delta^{\mathrm{op}}} \Delta_{n,\bullet} \times \Delta_{m,\bullet}$ which is contractible by Corollary 3.3.4. ∎

3.3.8. Corollary. *Let $\mathcal{W}$ be a localizer (resp. let $\{\mathcal{W}_S\}_S$ be a system of relative localizers) in $\mathcal{DIA}$.*

Let $f, g : (\Delta^{\mathrm{op}}, F_\bullet) \to (\Delta^{\mathrm{op}}, G_\bullet)$ be two homotopic morphisms of simplicial objects (resp. morphisms over S). Then $f \in \mathcal{W}$ (resp. in $\mathcal{W}_S$) if and only if $g \in \mathcal{W}$ (resp. in $\mathcal{W}_S$).

Proof. The statement follows by the standard argument because the projection $(\Delta^{\mathrm{op}}, F_\bullet \otimes \Delta_{1,\bullet}) \to (\Delta^{\mathrm{op}}, F_\bullet)$ is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$) by Lemma 3.3.7. ∎

3.3.9. Proposition. [Čech resolutions are in $\mathcal{W}$] *Let $\mathcal{W}$ be a localizer (resp. let $\{\mathcal{W}_S\}_S$ be a system of relative localizers) in $\mathcal{DIA}$.*

Let $U \to S$ be a local epimorphism in $\mathbb{S}(\cdot)$. Then the morphism

$$p : (\Delta^{\mathrm{op}}, \mathrm{cosk}^0(U|S)) \to (\cdot, S)$$

is in $\mathcal{W}$ (resp. $\mathcal{W}_S$).

Proof. To simplify the exposition we focus on the case in which $\mathbb{S}$ is associated with a category $\mathcal{S}$. The reader may check however that everything goes through in the general case because the only constructions involved can be expressed as right Kan extensions. The assumption means that there is a cover $\mathcal{U} = \{U_i \to S\}$ in the given pre-topology, such that for all indices i, the induced map

$$p_i : U \times_S U_i \to U_i$$

has a section s_i. By axiom (L3 left) it suffices to show that for all i the map

$$\widetilde{p}_i : (\Delta^{\mathrm{op}}, \mathrm{cosk}^0(U \times_S U_i \,|\, U_i)) \to (\cdot, U_i)$$

is in $\mathcal{W}$ (resp. in $\mathcal{W}_{U_i}$). Explicitly the simplicial object $\mathrm{cosk}^0(U \times_S U_i \,|\, U_i)$ is given by

$$\cdots \begin{smallmatrix}\to\\\to\\\to\\\to\end{smallmatrix} U \times_S U \times_S U \times_S U_i \begin{smallmatrix}\to\\\to\\\to\end{smallmatrix} U \times_S U \times_S U_i \rightrightarrows U \times_S U_i$$

Since Δ^{op} is contractible (in particular the morphism $(\Delta^{\mathrm{op}}, p^*T) \to (\cdot, T)$ is in $\mathcal{W}$, resp. in $\mathcal{W}_T$, for any $T \in \mathbb{S}(\cdot)$), it suffices to show that the map

$$\widetilde{p}_i : (\Delta^{\mathrm{op}}, \mathrm{cosk}^0(U \times_S U_i \,|\, U_i)) \to (\Delta^{\mathrm{op}}, p^* U_i)$$

is in $\mathcal{W}$ (resp. in $\mathcal{W}_{U_i}$). There is a section

$$\widetilde{s}_i : (\Delta^{\mathrm{op}}, p^* U_i) \to (\Delta^{\mathrm{op}}, \mathrm{cosk}^0(U \times_S U_i \,|\, U_i))$$

induced by s_i such that $\widetilde{p}_i \circ \widetilde{s}_i = \mathrm{id}$. By (L1) it then suffices to check that $\widetilde{s}_i \circ \widetilde{p}_i \in \mathcal{W}$ (resp. in $\mathcal{W}_{U_i}$). We will construct a homotopy between id and $\widetilde{s}_i \circ \widetilde{p}_i$

$$(\Delta^{\mathrm{op}}, \mathrm{cosk}^0(U \times_S U_i \,|\, U_i) \otimes \Delta_{1,\bullet}) \to (\Delta^{\mathrm{op}}, \mathrm{cosk}^0(U \times_S U_i \,|\, U_i))$$

in the sense of simplicial objects. This will suffice by Corollary 3.3.8. Since id and $s_i \circ p_i$ become equal after projection to U_i we get a morphism

$$(\mathrm{id}, s_i \circ p_i) : \mathrm{Hom}(\Delta_0, \Delta_1) \times U \times_S U_i \to U \times_S U_i$$

over U_i. Therefore by definition of cosk^0 it extends to a morphism

$$(\Delta^{\mathrm{op}}, \mathrm{cosk}^0(U \times_S U_i \,|\, U_i) \otimes \Delta_{1,\bullet}) \to (\Delta^{\mathrm{op}}, \mathrm{cosk}^0(U \times_S U_i \,|\, U_i)).$$

∎

3.3.10. DEFINITION. *A morphism $X_\bullet \to Y_\bullet$ of simplicial objects is called a* **hypercover** *if the following two equivalent conditions hold:*

1. *In any diagram of simplicial objects*

$$\begin{tikzcd} \partial\Delta_{n,\bullet} \otimes U \arrow[r] \arrow[d] & X_\bullet \arrow[d] \\ \Delta_{n,\bullet} \otimes U \arrow[r] & Y_\bullet \end{tikzcd}$$

   there is a cover $\mathcal{U} = \{U_i \to U\}$ such that for all i there is a lift (indicated by a dotted arrow) in the diagram

$$\begin{tikzcd} \partial\Delta_{n,\bullet} \otimes U_i \arrow[r] \arrow[d] & \partial\Delta_{n,\bullet} \otimes U \arrow[r] & X_\bullet \arrow[d] \\ \Delta_{n,\bullet} \otimes U_i \arrow[r] \arrow[urr, dotted] & \Delta_{n,\bullet} \otimes U \arrow[r] & Y_\bullet \end{tikzcd}$$

2. *For any $n \geq 0$ the morphism*

$$X_n \to \operatorname{cosk}^{n-1}(\iota^*_{\leq n-1} X_\bullet | Y_\bullet)_n$$

   admits local sections in the pre-topology on $\mathbb{S}$ (i.e. it is a local epimorphism).

3.3.11. REMARK.

1. In particular the notion of hypercover depends only on the Grothendieck topology generated by the pre-topology because a morphism is a local epimorphism precisely if the sieve generated by it is a covering sieve.

2. The equivalent condition 1. of the definition of hypercover shows that, if $\mathbb{S}$ is the derivator associated with the category $\mathcal{SET}$ equipped with the discrete topology, then a hypercover is precisely a trivial Kan fibration.

3.3.12. DEFINITION. *If in condition 2. of Definition 3.3.10 the morphism is even an isomorphism for all sufficiently large n, then α is called a* **finite (or bounded) hypercover**. *Equivalently we have $X_\bullet \cong \operatorname{cosk}^n(\iota^*_{\leq n} X_\bullet | Y_\bullet)$ for some n.*

3.3.13. LEMMA. *Let $\mathcal{W}$ be a localizer (resp. $\{\mathcal{W}_S\}_S$ be a system of relative localizers) in $\mathcal{DIA}$.*

For a finite hypercover $X_\bullet \to Y_\bullet$ (resp. one over $(\cdot, S)$) such that $X_\bullet \cong \operatorname{cosk}^i(X_\bullet | Y_\bullet)$ and $\iota^*_{\leq i-1} X_\bullet \cong \iota^*_{\leq i-1} Y_\bullet$ the morphism $(\Delta^{\mathrm{op}}, X_\bullet) \to (\Delta^{\mathrm{op}}, Y_\bullet)$ is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$).

Proof. Again, to simplify the exposition we focus on the case in which $\mathbb{S}$ is associated with a category $\mathcal{S}$. We may assume $i \geq 1$ because otherwise we are in the situation of Lemma 3.3.9. The assumptions imply that the map $X_i \to Y_i$ is a local epimorphism. Indeed, this is the map $X_i \to Y_i = \operatorname{cosk}^{i-1}(\iota_{\leq i-1}^* X_\bullet \,|\, Y_\bullet)_i$ in this case. Therefore the morphism $X_j \to Y_j$ is actually a local epimorphism for all j.

Consider the following commutative diagram in $\mathcal{DIA}$:

$$\begin{tikzcd}
(\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, (X_\bullet \times_{Y_\bullet} X_\bullet \,|\, X_\bullet)) \arrow[r] \arrow[d] & (\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, (X_\bullet \,|\, Y_\bullet)) \arrow[d] \\
(\Delta^{\mathrm{op}}, X_\bullet) \arrow[r] & (\Delta^{\mathrm{op}}, Y_\bullet)
\end{tikzcd}$$

where

$$(X_\bullet \,|\, Y_\bullet)_{m,n} := \operatorname{cosk}^0(X_n \,|\, Y_n)_m = \underbrace{X_n \times_{Y_n} \cdots \times_{Y_n} X_n}_{m+1 \text{ factors}}.$$

$$(X_\bullet \times_{Y_\bullet} X_\bullet \,|\, X_\bullet)_{m,n} := \operatorname{cosk}^0(X_n \times_{Y_n} X_n \,|\, X_n)_m = \underbrace{X_n \times_{Y_n} \cdots \times_{Y_n} X_n}_{m+2 \text{ factors}}.$$

The vertical morphisms are in $\mathcal{W}$ by Proposition 3.2.11, 3. because its columns are in $\mathcal{W}$ by Lemma 3.3.9. Again by Proposition 3.2.11, 3. it then suffices to show that the rows

$$p : (\Delta^{\mathrm{op}}, (X_\bullet \times Y_\bullet \,|\, X_\bullet)_{m,\bullet}) \longrightarrow (\Delta^{\mathrm{op}}, (X_\bullet \,|\, Y_\bullet)_{m,\bullet})$$

of the top horizontal morphism are in $\mathcal{W}$. These are again hypercovers of the form considered in this Lemma, in particular i-coskeletal relative to $Y_\bullet$, where the i-truncation is given by

$$\begin{tikzcd}
\underbrace{X_i \times_{Y_i} \cdots \times_{Y_i} X_i}_{m+2} \arrow[r, shift left] \arrow[r, phantom, "\vdots"] \arrow[r, shift right] \arrow[d] & X_{i-1} = Y_{i-1} \cdots \arrow[d, equal] & \cdots \arrow[r, shift left] \arrow[r, shift right] & X_0 = Y_0 \arrow[d, equal] \\
\underbrace{X_i \times_{Y_i} \cdots \times_{Y_i} X_i}_{m+1} \arrow[r, shift left] \arrow[r, phantom, "\vdots"] \arrow[r, shift right] & Y_{i-1} \cdots & \cdots \arrow[r, shift left] \arrow[r, shift right] & Y_0
\end{tikzcd}$$

where the left-most vertical arrow is induced by the map $\Delta_{m+1} \to \Delta_{m+2}$, $i \mapsto i$. There is a section s, with s_i induced by the map

$$\Delta_{m+2} \to \Delta_{m+1}, \quad i \mapsto \begin{cases} i & i < m+2, \\ m+1 & i = m+2. \end{cases}$$

We will construct a homotopy $\mu : \mathrm{id} \Rightarrow s \circ p$ of truncated simplicial objects:

$$\begin{tikzcd}
\operatorname{Hom}(\Delta_i, \Delta_1) \times \underbrace{X_i \times_{Y_i} \cdots \times_{Y_i} X_i}_{m+2} \arrow[r, shift left] \arrow[r, phantom, "\vdots"] \arrow[r, shift right] \arrow[d, "\mu_i"] & \operatorname{Hom}(\Delta_{i-1}, \Delta_1) \times Y_{i-1} \cdots \arrow[d, "\mu_{i-1}"] & \cdots \arrow[r, shift left] \arrow[r, shift right] & \operatorname{Hom}(\Delta_0, \Delta_1) \times Y_0 \arrow[d, "\mu_0"] \\
\underbrace{X_i \times_{Y_i} \cdots \times_{Y_i} X_i}_{m+2} \arrow[r, shift left] \arrow[r, phantom, "\vdots"] \arrow[r, shift right] & Y_{i-1} \cdots & \cdots \arrow[r, shift left] \arrow[r, shift right] & Y_0
\end{tikzcd}$$

The morphism μ_i at the constant morphism $0 : \Delta_i \to \Delta_1$ is given by the identity, at the constant morphism $1 : \Delta_i \to \Delta_1$ given by $s_i \circ p_i$, and at the other morphisms $\Delta_i \to \Delta_1$ arbitrarily. The existence of this homotopy allows by Lemma 3.3.8 and by (L1) to conclude. ∎

3.3.14. THEOREM. *Let $\mathcal{W}$ be a localizer (resp. $\{\mathcal{W}_S\}_S$ be a system of relative localizers) in $\mathcal{DIA}$.*

Any finite hypercover (resp. one over S) considered as a morphism of diagrams in $\mathcal{DIA}$

$$(\Delta^{\mathrm{op}}, X_\bullet) \to (\Delta^{\mathrm{op}}, Y_\bullet) \tag{14}$$

is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$).

Let $\iota : (\Delta^\circ)^{\mathrm{op}} \to \Delta^{\mathrm{op}}$ be the inclusion. If the morphism (14) exists in $\mathcal{DIA}$ then also

$$((\Delta^\circ)^{\mathrm{op}}, \iota^* X_\bullet) \to ((\Delta^\circ)^{\mathrm{op}}, \iota^* Y_\bullet)$$

is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$).

PROOF. Any finite hypercover is a finite succession of hypercovers of the form considered in Lemma 3.3.13. The additional statement is a consequence of the following Lemma. ∎

3.3.15. LEMMA. *Let $\mathcal{W}$ be a localizer (resp. let $\{\mathcal{W}_S\}_S$ be a system of relative localizers) in $\mathcal{DIA}$.*

Let $\iota : (\Delta^\circ)^{\mathrm{op}} \to \Delta^{\mathrm{op}}$ be the inclusion and let $(\Delta^{\mathrm{op}}, X_\bullet)$ be a simplicial diagram in $\mathcal{DIA}$ (resp. a simplicial diagram over $(\cdot, S)$). Then the morphism

$$((\Delta^\circ)^{\mathrm{op}}, \iota^* X_\bullet) \to (\Delta^{\mathrm{op}}, X_\bullet)$$

(if in $\mathcal{DIA}$) is in $\mathcal{W}$ (resp. in $\mathcal{W}_S$).

PROOF. We focus on the absolute case. For the relative case the proof is identical. Since the morphism in the statement is of pure diagram type, we may check the condition of (L4 left): we have to show that the category

$$\{\Delta_m\} \times_{/\Delta^{\mathrm{op}}} (\Delta^\circ)^{\mathrm{op}}$$

is contractible, say, on the diagram category of diagrams I such that $(I, X_m) \in \mathcal{DIA}$. This is true by Corollary 3.3.4. ∎

3.4. CARTESIAN AND COCARTESIAN OBJECTS.

3.4.1. DEFINITION. *Let $\mathbb{D} \to \mathbb{S}$ be a fibered derivator of domain* Dia*. Let $I, E \in$* Dia *be diagrams and let $\alpha : I \to E$ be a functor in* Dia*. We say that an object*

$$X \in \mathbb{D}(I)$$

*is E-***(co)Cartesian***, if for any morphism $\mu : i \to j$ in I mapping to an identity in E, the corresponding morphism $\mathbb{D}(\mu) : i^* X \to j^* X$ is (co)Cartesian.*

If E is the trivial category, we omit it from the notation, and talk about (co)Cartesian objects.

These notions define full subcategories $\mathbb{D}(I)^{E-\mathrm{cart}}$ (resp. $\mathbb{D}(I)^{E-\mathrm{cocart}}$) of $\mathbb{D}(I)$, and $\mathbb{D}(I)_S^{E-\mathrm{cart}}$ (resp. $\mathbb{D}(I)_S^{E-\mathrm{cocart}}$) of $\mathbb{D}(I)_S$ for any $S \in \mathbb{S}(I)$.

3.4.2. Lemma. *The functor α^* w.r.t. a morphism $\alpha : D_1 \to D_2$ in* $\mathrm{Dia}(\mathbb{S})$ *maps Cartesian objects to Cartesian objects. The functor α^* for a morphism $\alpha : D_1 \to D_2$ in* $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S})$ *maps coCartesian objects to coCartesian objects.*

3.4.3. Remark. The categories of coCartesian objects are a generalization of the **equivariant derived categories** of Bernstein and Lunts [BL94]. For this let $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ be the stable fibered derivator of sheaves of abelian groups on (nice) topological spaces, where $\mathbb{S}$ is the pre-derivator associated with the category of (nice) topological spaces. Let G be a topological group acting on a space X. Then we may form the following simplicial space which is an object of $\mathbb{S}(\Delta^{\mathrm{op}})$:

$$[G\backslash X]_\bullet : \cdots \begin{smallmatrix}\to\\ \to\\ \to\\ \to\end{smallmatrix} G \times G \times X \begin{smallmatrix}\to\\ \to\\ \to\end{smallmatrix} G \times X \rightrightarrows X,$$

cf. [BL94, B1]. Then the category

$$\mathbb{D}(\Delta)^{\mathrm{cocart}}_{[G\backslash X]_\bullet}$$

is equivalent to the (unbounded) equivariant derived category, cf. [BL94, Proposition B4]. Note that all pull-back functors are exact in this context.

3.4.4. Definition. *Let $\mathbb{D} \to \mathbb{S}$ be a fibered derivator of domain* Dia. *We say that $\mathbb{D} \to \mathbb{S}$* **admits left Cartesian projectors** *if for all functors $\alpha : I \to E$ in* Dia *and $S \in \mathbb{S}(I)$, the fully-faithful inclusion*

$$\mathbb{D}(I)^{E-\mathrm{cart}}_S \to \mathbb{D}(I)_S$$

has a left adjoint $\Box^E_!$. More generally, we have four notions with the following notations:

$\Box^E_!$	*left adjoint*	*left Cartesian projector*
$\blacksquare^E_!$	*right adjoint*	*right Cartesian projector*
$\blacksquare^E_*$	*left adjoint*	*left coCartesian projector*
$\Box^E_*$	*right adjoint*	*right coCartesian projector*

We will, in general, only use left Cartesian and right coCartesian projectors, the others being somewhat unnatural. In 4.3.3 we will show (using Brown representability) that for an infinite fibered derivator whose fibers are stable and well-generated (cf. Definitions 4.1.1, 4.1.7) right coCartesian projectors and left Cartesian projectors exists. Note that for a usual (non fibered) derivator, the notions 'Cartesian' and 'coCartesian' are equivalent. If for a fibered derivator with stable fibers both left and right Cartesian projectors exist, then there is actually a *recollement* [Kra10, Proposition 4.13.1]:

$$\begin{tikzcd}[column sep=large]
\mathbb{D}(I)^{E-\mathrm{cart}}_S \arrow[r, "\mathrm{incl.}" description] & \mathbb{D}(I)_S \arrow[l, bend right, "\blacksquare_*"'] \arrow[l, bend left, "\Box_!"] \arrow[r] & \mathbb{D}(I)_S / \mathbb{D}(I)^{E-\mathrm{cart}}_S \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

3.4.5. Example. The projectors are difficult to describe explicitly, except in very special situations. Here a rather trivial example where this is possible. Let $\mathbb{D}$ be a stable derivator and consider $I = \Delta_1$, the projection $p : \Delta_1 \to \cdot$ and the inclusions $e_0, e_1 : \cdot \to \Delta_1$. Then a left and a right Cartesian projector exist and the recollement above is explicitly given by:

$$\begin{tikzcd} \mathbb{D}(\Delta_1)^{\mathrm{cart}} \cong \mathbb{D}(\cdot) \arrow[r, "p^*"] & \mathbb{D}(\Delta_1) \arrow[l, shift right=3ex, "e_0^*"'] \arrow[l, shift left=3ex, "e_1^*"] \arrow[r, "\mathrm{C}"] & \mathbb{D}(\cdot) \arrow[l, shift right=3ex, "e_{1!}"'] \arrow[l, shift left=3ex, "{[-1]} \circ e_{0,*}"] \end{tikzcd}$$

Note that the functor C (Cone) may be described as either $[1] \circ e_0^!$ or $e_1^?$ (cf. [Gro13, §3]) and that the essential image of p^* is precisely the kernel of C, which also coincides with the full subcategory of Cartesian=coCartesian objects.

3.5. Weak and strong $\mathbb{D}$-equivalences.

3.5.1. Definition. [left] *Let* Dia *be a diagram category and let* $\mathbb{S}$ *be a strong right derivator with domain* Dia *equipped with a Grothendieck pre-topology. Let* $\mathbb{D} \to \mathbb{S}$ *be a left fibered derivator satisfying (FDer0 right) and let* $S \in \mathbb{S}(\cdot)$. *A morphism* $f : D_1 \to D_2$ *in* $\mathrm{Dia}(\mathbb{S})/S$ *is called a* **weak $\mathbb{D}$-equivalence** *relative to* S *if the natural transformation*

$$p_{1!}p_1^* \to p_{2!}p_2^*$$

is an isomorphism of functors, where the $p_i : D_i \to (\cdot, S)$ *are the structural morphisms. A morphism* $f \in \mathrm{Dia}(\mathbb{S})$ *is called a* **strong $\mathbb{D}$-equivalence** *if the functor* f^* *induces an equivalence of categories*

$$f^* : \mathbb{D}(D_2)^{\mathrm{cart}} \to \mathbb{D}(D_1)^{\mathrm{cart}}.$$

Note that *weak* is a relative notion whereas *strong* is absolute.

3.5.2. Definition. [right] *Let* Dia *be a diagram category. Let* $\mathbb{D} \to \mathbb{S}$ *be a right fibered derivator satisfying (FDer0 left), where* $\mathbb{S}^{\mathrm{op}}$ *is a strong right derivator with domain* $\mathrm{Dia}^{\mathrm{op}}$ *equipped with a Grothendieck pre-topology. Let* $S \in \mathbb{S}(\cdot)$. *A morphism* $f : D_1 \to D_2$ *in* $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S})/S$ *is called a* **weak $\mathbb{D}$-equivalence** *relative to* S, *if the natural transformation*

$$p_{2*}p_2^* \to p_{1*}p_1^*$$

is an isomorphism of functors, where the $p_i : D_i \to (\cdot, S)$ *are the structural morphisms. A morphism* $f \in \mathrm{Dia}^{\mathrm{op}}(\mathbb{S})$ *is called a* **strong $\mathbb{D}$-equivalence** *if the functor* f^* *induces an equivalence of categories*

$$f^* : \mathbb{D}(D_2)^{\mathrm{cocart}} \to \mathbb{D}(D_1)^{\mathrm{cocart}}.$$

For a (left and right) derivator, i.e. for $\mathbb{S} = \cdot$, there is no difference between $\mathrm{Dia}(\mathbb{S})$ and $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S})$ and then also the two different definitions of weak, resp. strong $\mathbb{D}$-equivalence coincide (for the case of weak $\mathbb{D}$-equivalences, note that the two conditions become adjoint to each other). These notions of $\mathbb{D}$-*equivalence* (right version) should be compared to the classical notions of *cohomological descent*, see [SGA72, Exposé $\mathrm{V}^{\mathrm{bis}}$, Définition 2.2.2., 2.2.4., 2.2.6.].

3.5.3. Lemma. [left] *Let* $f : D_1 \to D_2$ *be a morphism in* $\mathrm{Dia}(\mathbb{S})/S$*. Then the following implication holds:*

$$f \text{ strong } \mathbb{D}\text{-equivalence} \quad \Rightarrow \quad f \text{ weak } \mathbb{D}\text{-equivalence relative to } S.$$

Proof. If f is a strong $\mathbb{D}$-equivalence then f^* is fully-faithful on Cartesian objects. The condition of f being a weak $\mathbb{D}$-equivalence relative to S is in turn equivalent to f^* being fully-faithful on objects of the form $p_2^*\mathcal{E}$ for $\mathcal{E}$ in $\mathbb{D}(\cdot)_S$ (which are, in particular, Cartesian). ■

Of course there is an analogous right version of this lemma. The goal of this section is to prove the following two theorems:

3.5.4. Main Theorem. [right] *Let* Dia *be a diagram category and let* $\mathbb{S}$ *be a strong right derivator with domain* $\mathrm{Dia}^{\mathrm{op}}$ *equipped with a Grothendieck pre-topology.*

1. *Let* $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ *be a fibered derivator with domain* Dia *which is colocal in the sense of Definition 2.5.6 for the Grothendieck pre-topology on* $\mathbb{S}$*. The collection of classes* $\{\mathcal{W}_{\mathbb{D},S}\}_S$*, where* $\mathcal{W}_{\mathbb{D},S}$ *for* $S \in \mathbb{S}(\cdot)$ *is the class of* weak $\mathbb{D}$-equivalences *relative to* S *in* $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S}^{\mathrm{op}})/S$*, forms a system of relative colocalizers.*
2. *Let* $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ *be an infinite fibered derivator with domain* Dia *which is colocal in the sense of Definition 2.5.6 for the Grothendieck pre-topology on* $\mathbb{S}$*, with stable, compactly generated fibers. The class* $\mathcal{W}_{\mathbb{D}}$ *of* strong $\mathbb{D}$-equivalences *in* $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S}^{\mathrm{op}})$ *forms an absolute colocalizer.*

3.5.5. Main Theorem. [left] *Let* Dia *be a diagram category and let* $\mathbb{S}$ *be a strong right derivator with domain* Dia *equipped with a Grothendieck pre-topology.*

1. *Let* $\mathbb{D} \to \mathbb{S}$ *be a fibered derivator with domain* Dia*, which is local in the sense of Definition 2.5.4 for the Grothendieck pre-topology on* $\mathbb{S}$*. The collection of classes* $\{\mathcal{W}_{\mathbb{D},S}\}_S$*, where* $\mathcal{W}_{\mathbb{D},S}$ *for* $S \in \mathbb{S}(\cdot)$ *is the class of* weak $\mathbb{D}$-equivalences *relative to* S *in* $\mathrm{Dia}(\mathbb{S})/S$ *forms a system of relative localizers.*
2. *Let* $\mathbb{D} \to \mathbb{S}$ *be an infinite fibered derivator with domain* Dia*, which is local in the sense of Definition 2.5.4 for the Grothendieck pre-topology on* $\mathbb{S}$*, with stable, well-generated fibers. The class* $\mathcal{W}_{\mathbb{D}}$ strong $\mathbb{D}$-equivalences *in* $\mathrm{Dia}(\mathbb{S})$ *forms an absolute localizer.*

The weak $\mathbb{D}$-equivalences for the case of usual derivators (i.e. for $\mathbb{S} = \{\cdot\}$) were called just '$\mathbb{D}$-equivalences' by Cisinski [Cis08] and it is rather straight-forward to see from the definition of derivator that they from a fundamental localizer in the classical sense (= absolute localizer for $\mathbb{S} = \{\cdot\}$, = system of relative localizers for $\mathbb{S} = \{\cdot\}$).

We will only prove the left-variant of the theorem. The other follows by logical duality. In the right version compactly generated fibers are needed because of the corresponding assumption in Lemma 3.5.10. Before proving the theorem we need a couple of lemmas. We assume for the rest of this section that Dia is a diagram category and that $\mathbb{S}$ is a strong right derivator with domain Dia equipped with a Grothendieck pre-topology.

3.5.6. Definition. *Two morphisms (in* Dia($\mathbb{S}$) *or in* Dia$^{\mathrm{op}}$($\mathbb{S}$)*)*

$$\begin{tikzcd} D_1 \arrow[r, shift left, "p"] & D_2 \arrow[l, shift left, "s"] \end{tikzcd}$$

such that zig-zags of 2-morphisms

$$p \circ s \Rightarrow \cdots \Leftarrow \cdots \Rightarrow \mathrm{id}_{D_1} \qquad s \circ p \Rightarrow \cdots \Leftarrow \cdots \Rightarrow \mathrm{id}_{D_2}$$

exist are called a **homotopy equivalence** *(or p is called as such if an s with this property exists).*

3.5.7. Lemma. [left] *Let $\mathbb{D}$ be a left fibered derivator satisfying (FDer0 right) and let $D_1, D_2 \in$ Dia($\mathbb{S}$). Given any homotopy equivalence (p, s), then the functors p^* and s^* induce an equivalence*

$$\begin{tikzcd} \mathbb{D}(D_2)^{\mathrm{cart}} \arrow[r, shift left, "p^*"] & \mathbb{D}(D_1)^{\mathrm{cart}} \arrow[l, shift left, "s^*"] \end{tikzcd}$$

Proof. The 2-morphisms $\mu : (\alpha, f) \Rightarrow (\beta, g)$ in Definition 3.5.6 induce morphisms between the pull-back functors

$$(\alpha, f)^* \mathcal{E} \to (\beta, g)^* \mathcal{E}$$

which are isomorphisms on Cartesian objects. ∎

3.5.8. Example. [cf. also Proposition 3.2.11, 2.] Let I_1, I_2 be diagrams in Dia. If

$$\begin{tikzcd} I_1 \arrow[r, shift left, "p"] & I_2 \arrow[l, shift left, "s"] \end{tikzcd}$$

is an adjunction where p is left adjoint to s, and if $F \in \mathbb{S}(I_1)$ then we get an equivalence

$$\begin{tikzcd} \mathbb{D}(D_2)^{\mathrm{cart}} \arrow[r, shift left, "p^*"] & \mathbb{D}(D_1)^{\mathrm{cart}} \arrow[l, shift left, "s^*"] \end{tikzcd}$$

where $D_1 = (I_1, F)$ and $D_2 = (I_2, s^* F)$.

3.5.9. Lemma. [left] *Let* Dia *be an infinite diagram category and let $\mathbb{D} \to \mathbb{S}$ be an infinite fibered derivator with domain* Dia *with stable, well-generated fibers. Consider a morphism $D = (I, F) \to (\cdot, S)$ in* Dia($\mathbb{S}$)*. Let $U \to S$ be a universally $\mathbb{D}$-local morphism. Write $D_U := D \times_{(\cdot, S)} (\cdot, U)$. Then the following diagram is 2-commutative (i.e. the exchange natural transformation is an isomorphism):*

$$\begin{tikzcd} \mathbb{D}(D) \arrow[r, "\Box_!"] \arrow[d, "\mathrm{pr}_1^*"'] \arrow[dr, phantom, "\nearrow\sim"] & \mathbb{D}(D)^{\mathrm{cart}} \arrow[d, "\mathrm{pr}_1^*"] \\ \mathbb{D}(D_U) \arrow[r, "\Box_!"] & \mathbb{D}(D_U)^{\mathrm{cart}} \end{tikzcd}$$

Note that left Cartesian projectors exist for D and D_U by Theorem 4.3.4.

PROOF. The functor pr_1^* has a right adjoint pr_{1*} by (Dloc2 left) and by the Brown representability theorem. (Dloc1 left) says that pr_1^* preserves coCartesian morphisms, hence pr_{1*} preserves Cartesian morphisms. Therefore the right adjoint of the given diagram is the following *commutative* diagram:

$$\begin{tikzcd} \mathbb{D}(D) & \mathbb{D}(D)^{\mathrm{cart}} \arrow[l] \\ \mathbb{D}(D_U) \arrow[u, "\mathrm{pr}_{1*}"] & \mathbb{D}(D_U)^{\mathrm{cart}} \arrow[l] \arrow[u, "\mathrm{pr}_{1*}"'] \end{tikzcd}$$

Consequently the exchange morphism of the diagram in the statement is also a natural isomorphism. ∎

3.5.10. LEMMA. [right] *Let* Dia *be an infinite diagram category and let* $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ *be an infinite fibered derivator with domain* Dia *with stable, compactly generated fibers. Consider a morphism* $D = (I, F) \to (\cdot, S)$ *in* $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S}^{\mathrm{op}})$*. Let* $U \to S$ *be a universally* $\mathbb{D}$*-colocal morphism. Write* $D_U := D \times_{(\cdot,S)} (\cdot, U)$*. Then the following diagram is 2-commutative (i.e. the exchange natural transformation is an isomorphism):*

$$\begin{tikzcd} \mathbb{D}(D) \arrow[r, "\square_*"] \arrow[d, "\mathrm{pr}_1^*"'] \arrow[dr, phantom, "\swarrow\sim"] & \mathbb{D}(D)^{\mathrm{cocart}} \arrow[d, "\mathrm{pr}_1^*"] \\ \mathbb{D}(D_U) \arrow[r, "\square_*"] & \mathbb{D}(D_U)^{\mathrm{cocart}} \end{tikzcd}$$

Note that right coCartesian projectors exist for D and D_U by Theorem 4.3.3.

PROOF. The functor pr_1^* has a left adjoint $\mathrm{pr}_{1!}$ by (Dloc2 right) and by the Brown representability theorem for the dual. (Dloc1 right) says that pr_1^* preserves Cartesian morphisms, hence $\mathrm{pr}_{1!}$ preserves coCartesian morphisms. Therefore the right adjoint of the given diagram is the following *commutative* diagram:

$$\begin{tikzcd} \mathbb{D}(D) & \mathbb{D}(D)^{\mathrm{cocart}} \arrow[l] \\ \mathbb{D}(D_U) \arrow[u, "\mathrm{pr}_{1!}"] & \mathbb{D}(D_U)^{\mathrm{cocart}} \arrow[l] \arrow[u, "\mathrm{pr}_{1!}"'] \end{tikzcd}$$

Consequently the exchange morphism of the diagram in the statement is also a natural isomorphism. ∎

3.5.11. LEMMA. [left] *Let* $\mathbb{D} \to \mathbb{S}$ *be a fibered derivator with domain* Dia *admitting left Cartesian projectors (cf. 3.4.4). For any opfibration*

$$\begin{tikzcd} I \arrow[d, "\pi"] \\ E \end{tikzcd}$$

in Dia*, for any diagram in* $F \in \mathbb{S}(I)$*, and for each element* $e \in E$*, the following diagram is 2-commutative (i.e. the exchange natural transformation is an isomorphism):*

$$\begin{tikzcd} \mathbb{D}(I)_F \arrow[r, "\Box_!^E"] \arrow[d, "\iota^*"'] & \mathbb{D}(I)_F^{E-\mathrm{cart}} \arrow[d, "\iota^*"] \\ \mathbb{D}(I_e)_{F_e} \arrow[r, "\Box_!"] \arrow[ur, phantom, "\nearrow^\sim"] & \mathbb{D}(I_e)_{F_e}^{\mathrm{cart}} \end{tikzcd}$$

where $\iota : I_e \to I$ *is the inclusion of the fiber.*

3.5.12. Lemma. [right] *Let* $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ *be a fibered derivator with domain* Dia *admitting right coCartesian projectors (cf. 3.4.4). For a fibration*

$$\begin{tikzcd} I \arrow[d, "\pi"] \\ E \end{tikzcd}$$

in Dia*, for any diagram in* $F \in \mathbb{S}^{\mathrm{op}}(I)$*, and for each element* $e \in E$*, the following diagram is 2-commutative (i.e. the exchange natural transformation is an isomorphism):*

$$\begin{tikzcd} \mathbb{D}(I)_F \arrow[r, "\Box_*^E"] \arrow[d, "\iota^*"'] & \mathbb{D}(I)_F^{E-\mathrm{cocart}} \arrow[d, "\iota^*"] \\ \mathbb{D}(I_e)_{F_e} \arrow[r, "\Box_*"] \arrow[ur, phantom, "\swarrow^\sim"] & \mathbb{D}(I_e)_{F_e}^{\mathrm{cocart}} \end{tikzcd}$$

where $\iota : I_e \to I$ *is the inclusion of the fiber.*

Proof. We restrict to the right-variant, the other being dual. We will show that the functor $\iota_!$ maps coCartesian objects to E-coCartesian ones. Then the left adjoint of the given diagram is the diagram

$$\begin{tikzcd} \mathbb{D}(I)_F & \mathbb{D}(I)_F^{E-\mathrm{cocart}} \arrow[l] \\ \mathbb{D}(I_e)_{F_e} \arrow[u, "\iota_!"] & \mathbb{D}(I_e)_{F_e}^{\mathrm{cocart}} \arrow[l] \arrow[u, "\iota_!"'] \end{tikzcd}$$

which is commutative. Consequently also the diagram of the statement is 2-commutative via the natural exchange morphism.

Let f in E be an object and let $\nu : i_1 \to i_2$ be a morphism in I mapping to id_f. Let α_k be the inclusions of $\cdot$ into I with image i_k. The morphism ν yields a natural transformation

$$\nu : \alpha_1 \Rightarrow \alpha_2.$$

Consider the diagram

$$\begin{tikzcd}
e \times_{/E} f \arrow[r, shift left, "c_k"] \arrow[d, "p'"] & I_e \times_{/I} i_k \arrow[l, shift left, "\pi"] \arrow[r, "A_k"] \arrow[d, "p_k"'] \arrow[dr, phantom, "\swarrow \mu_k"] & I_e \arrow[d, "\iota"] \\
\cdot \arrow[r, equal] & \cdot \arrow[r, "\alpha_k"] & I
\end{tikzcd}$$

where c_k is given on a morphism $\beta : e \to f$ in E by the choice of a Cartesian arrow $i_k' \to i_k$. It is right adjoint to π by the definition of Cartesian arrow.

There is a functor (composition with ν):

$$\widetilde{\nu} : I_e \times_{/I} i_1 \to I_e \times_{/I} i_2$$

such that $A_2\widetilde{\nu} = A_1$ and $p_2\widetilde{\nu} = p_1$. We have therefore a natural (point-wise) coCartesian morphism $\mathbb{S}^{\mathrm{op}}(\mu_1)_\bullet \widetilde{\nu}^* \to \widetilde{\nu}^* \mathbb{S}^{\mathrm{op}}(\mu_2)_\bullet$ of functors $\mathbb{D}(I_e \times_{/I} i_2)_{A_2^* F_e} \to \mathbb{D}(I_e \times_{/I} i_1)$.

We have also a natural transformation $\rho : \widetilde{\nu} c_1 \to c_2$ defined for a morphism $\beta : e \to f$ in E as the unique arrow $\rho(\beta)$ over id_e making the following diagram commutative:

$$\begin{tikzcd}
i_1' \arrow[r, "c_1(\beta)"] \arrow[d, "\rho(\beta)"'] & i_1 \arrow[d, "\nu"] \\
i_2' \arrow[r, "c_2(\beta)"] & i_2
\end{tikzcd}$$

The resulting morphism $\mathbb{D}(\rho) : c_1^* \widetilde{\nu}^* \to c_2^*$ is point-wise coCartesian on coCartesian objects.

We get a commutative diagram of natural transformations

$$\begin{tikzcd}
\mathbb{S}^{\mathrm{op}}(\mu_1)_\bullet A_1^* \arrow[r] \arrow[d, "\sim"] & \mathbb{S}^{\mathrm{op}}(\mu_1)_\bullet A_1^* \iota^* \iota_! \arrow[r, "\mathbb{D}(\mu_1)'"] \arrow[d, "\sim"] & p_1^* \alpha_1^* \iota_! \arrow[d, "\sim"] \\
\mathbb{S}^{\mathrm{op}}(\mu_1)_\bullet \widetilde{\nu}^* A_2^* \arrow[r] \arrow[d] & \mathbb{S}^{\mathrm{op}}(\mu_1)_\bullet \widetilde{\nu}^* A_2^* \iota^* \iota_! \arrow[d] & \widetilde{\nu}^* p_2^* \alpha_1^* \iota_! \arrow[d, "\widetilde{\nu}^* p_2^*(\mathbb{D}(\nu))"] \\
\widetilde{\nu}^* \mathbb{S}^{\mathrm{op}}(\mu_2)_\bullet A_2^* \arrow[r] & \widetilde{\nu}^* \mathbb{S}^{\mathrm{op}}(\mu_2)_\bullet A_2^* \iota^* \iota_! \arrow[r, "\widetilde{\nu}^*(\mathbb{D}(\mu_2)')"] & \widetilde{\nu}^* p_2^* \alpha_2^* \iota_!
\end{tikzcd}$$

where the first two top vertical morphisms are the natural isomorphisms induced by $A_2\widetilde{\nu} = A_1$, the third top vertical morphism is the natural isomorphism induced by $p_2\widetilde{\nu} = p_1$, and the first two lower vertical morphisms are point-wise coCartesian. Here we use the notation $\mathbb{D}(\mu_1)'$ for the morphism $\mathbb{S}^{\mathrm{op}}(\mu_1)_\bullet X \to Y$ induced by a morphism $\mathbb{D}(\mu_1) : X \to Y$.

Now we apply $p_{1!}$ to the outer square:

$$\begin{tikzcd}
p_{1!} \mathbb{S}^{\mathrm{op}}(\mu_1)_\bullet A_1^* \arrow[r] \arrow[d] & p_{1!} p_1^* \alpha_1^* \iota_! \arrow[d] \\
p_{1!} \widetilde{\nu}^* \mathbb{S}^{\mathrm{op}}(\mu_2)_\bullet A_2^* \arrow[r] & p_{1!} \widetilde{\nu}^* p_2^* \alpha_2^* \iota_!
\end{tikzcd}$$

The left vertical map is still coCartesian (homotopy colimits preserve coCartesian morphisms).

There is a canonical isomorphism $p'_! c_i^* \to p_{i!}$ [Gro13, Prop. 1.23] and the natural transformation $\mathbb{D}(\rho) : c_1^* \widetilde{\nu}^* \to c_2^*$ is an isomorphism on coCartesian objects over constant diagrams. Consider the commutative diagram:

$$\begin{tikzcd}
p'_! c_1^* \widetilde{\nu}^* \arrow[r, "\sim"] \arrow[d, equal] & p_{2!} \widetilde{\nu}_! c_{1!} c_1^* \widetilde{\nu}^* \arrow[r] & p_{2!} \arrow[dd, equal] \\
p'_! c_1^* \widetilde{\nu}^* \arrow[r, "\sim"] & p_{2!} c_{2!} c_1^* \widetilde{\nu}^* \arrow[u, "\mathbb{D}(\rho)^{ad}"'] \arrow[d, "\mathbb{D}(\rho)"] & \\
 & p_{2!} c_{2!} c_2^* \arrow[r, "\sim"] & p_{2!}
\end{tikzcd}$$

where the rightmost horizontal morphisms are the respective counits. Since $\mathbb{D}(\rho)$ is an isomorphism on coCartesian objects over constant diagrams, so is the morphism $p'_! c_1^* \widetilde{\nu}^* \to p_{2!}$. Now we have the commutative diagram

$$\begin{tikzcd}
p'_! c_1^* \widetilde{\nu}^* \arrow[r] \arrow[d, "\sim"] & p_{2!} \\
p_{1!} \widetilde{\nu}^* \arrow[r, "\sim"] & p_{2!} \widetilde{\nu}_! \widetilde{\nu}^* \arrow[u]
\end{tikzcd}$$

which shows that also the natural map $p_{1!} \widetilde{\nu}^* \to p_{2!}$ is an isomorphism on coCartesian objects over constant diagrams.

We get a commutative diagram

$$\begin{tikzcd}
p_{1!} \mathbb{S}^{\mathrm{op}}(\mu_1)_\bullet A_1^* \arrow[r] \arrow[d] & p_{1!} p_1^* \alpha_1^* \iota_! \arrow[r] \arrow[d] & \alpha_1^* \iota_! \arrow[dd, "\mathbb{D}(\nu)"] \\
p_{1!} \widetilde{\nu}^* \mathbb{S}^{\mathrm{op}}(\mu_2)_\bullet A_2^* \arrow[r] \arrow[d] & p_{1!} \widetilde{\nu}^* p_2^* \alpha_2^* \iota_! \arrow[d] & \\
p_{2!} \mathbb{S}^{\mathrm{op}}(\mu_2)_\bullet A_2^* \arrow[r] & p_{2!} p_2^* \alpha_2^* \iota_! \arrow[r] & \alpha_2^* \iota_!
\end{tikzcd}$$

where the composition of the left vertical morphisms is coCartesian on coCartesian objects because the functor $\mathbb{S}^{\mathrm{op}}(\mu_2)_\bullet A_2^*$ maps coCartesian objects to coCartesian objects over constant diagrams. The composition of the horizontal morphisms in the top and bottom rows are isomorphisms by (FDer4 left). Hence the rightmost vertical map is coCartesian as well. ∎

Proof of Main Theorem 3.5.5, 1. This is the case of weak $\mathbb{D}$-equivalences.

(L0) and (L1) are clear.

For (L2 left), let $D_1 = (I, F)$ and $D_2 = (\{e\}, F(e))$. The projection p and the inclusion i of the final object induce morphisms:

$$D_1 \underset{i}{\overset{p}{\rightleftarrows}} D_2$$

We have $p \circ i = \mathrm{id}$ and there is a 2-morphism $\beta : \mathrm{id} \Rightarrow i \circ p$. Therefore the statement is clear for weak $\mathbb{D}$-equivalences over any base S.

(L3 left): Let

$$\begin{tikzcd} D_1 \arrow[rr, "w"] \arrow[dr, "p'_1"] \arrow[ddr, bend right, "p_1"'] & & D_2 \arrow[dl, "p'_2"'] \arrow[ddl, bend left, "p_2"] \\ & D_3 = (E,F) \arrow[d, "p"] & \\ & (\cdot, S) & \end{tikzcd}$$

be a morphism as in (L3 left) over a base $S \in \mathbb{S}(\cdot)$. We have to show that

$$p_{1!}\, p_1^* \to p_{2!}\, p_2^*$$

is an isomorphism and it suffices to show that the morphism

$$p'_{1!}\,(p_1)^* \to p'_{2!}\,(p_2)^*$$

is an isomorphism. This may be checked point-wise by (Der2) and after pull-back to an open cover by condition 2. of 'local' for a fibered derivator (see Definition 2.5.4), so fix $e \in E$ and consider the 2-commutative diagrams

$$\begin{tikzcd} D_i \times_{/D_3} (e, U_j) \arrow[r, "\iota_{i,e,j}"] \arrow[d, "p'_{i,e,j}"'] & D_i \arrow[d, "p'_i"] \\ (e, U_j) \arrow[r, "\epsilon_{e,j}"] & D_3 \end{tikzcd}$$

and let $p_{i,e} : D_i \times_{/D_3} (e, U_i) \to D_i$ be the projection. Applying the functor $\epsilon_{e,j}^*$, we get

$$\epsilon_{e,j}^* p'_{1!}(p_1)^* \to \epsilon_{e,j}^* p'_{2!}(p_2)^*$$

which is, using Proposition 2.6.8 (note that $\iota_{e,j}$ is $\mathbb{D}$-local by assumption), the same as

$$(p'_{1,e,j})_!(\iota_{i,e,j})^*(p_1)^* \to (p'_{2,e,j})_!(\iota_{i,e,j})^*(p_2)^*.$$

Now $p_i \circ \iota_{i,e,j} = \pi_j \circ p'_{i,e,j}$, where $\pi : (\cdot, U_j) \to (\cdot, S)$ is the structural morphism. Therefore we get:

$$(p'_{1,e,j})_!(p'_{1,e,j})^* \pi_j^* \to (p'_{2,e,j})_!(p'_{2,e,j})^* \pi_j^*.$$

By Lemma 2.6.9 this is induced by the canonical natural transformation which is an isomorphism by assumption.

(L4 left): By Lemma 3.2.12 we may prove axiom (L4' left) instead. Consider a morphism $p : D_1 \to (E,F) = D_2$ in $\mathrm{Dia}(\mathbb{S})$ of pure diagram type, where the underlying functor of p is a fibration. It suffices to show that the counit

$$p_!\, p^* \to \mathrm{id}$$

is an isomorphism. This is the same as showing that the unit

$$\mathrm{id} \to p_* p^*$$

is an isomorphism. Note that p_* exists because this is a morphism of diagram type and $\mathbb{D} \to \mathbb{S}$ is assumed to be a right fibered derivator as well (this is the only place, where this assumption is used for the case of weak $\mathbb{D}$-equivalences). Now, since p is a fibration, p_* can be computed fiber-wise. So we have to show that

$$\mathrm{id} \to p_{e,*} p_e^*$$

is an isomorphism or, equivalently, that

$$p_{e,!} p_e^* \to \mathrm{id}$$

is an isomorphism. This holds true because by assumption the map of fibers $I_e \to e$ is in $\mathcal{W}_{F(e)}$. ∎

We proceed to state some consequences of the fact that *weak* $\mathbb{D}$-equivalences form a fundamental localizer.

3.5.13. Example. [Mayer-Vietoris] Let $\mathbb{S}$ be a strong right derivator (e.g. represented by a category with limits) with a Grothendieck pre-topology. We saw in Example 3.2.9 that for a cover $\{U_1 \to S, U_2 \to S\}$ consisting of 2 *mono*morphisms, the projection

$$p : \left(\begin{tikzcd} \text{“}U_1 \times_S U_2\text{”} \arrow[r] \arrow[d] & U_1 \\ U_2 & \end{tikzcd} \right) \to S$$

belongs to any fundamental localizer. Let $\mathbb{D}^! \to \mathbb{S}$ be a fibered derivator (for example coming from a six-functor-formalism, as in the introduction) which is local w.r.t. the pre-topology on $\mathbb{S}$. Theorem 3.5.5 implies that p is a weak $\mathbb{D}$-equivalence in $\mathrm{Dia}(\mathbb{S})/S$, i.e. for $\mathcal{E} \in \mathbb{D}(\cdot)_S$ we have

$$p_! p^* \mathcal{E} \cong \mathcal{E},$$

i.e. the homotopy colimit of

$$\begin{tikzcd} i_{1,2,!}\, i_{1,2}^! \mathcal{E} \arrow[r] \arrow[d] & i_{1,!}\, i_1^! \mathcal{E} \\ i_{2,!}\, i_2^! \mathcal{E} & \end{tikzcd}$$

is isomorphic to $\mathcal{E}$, where we now wrote $i_{1,!}$ for $i_{1,\bullet}$, etc. If $\mathbb{D}$ has stable fibers, this translates to the usual distinguished triangle

$$i_{1,2,!}\, i_{1,2}^! \mathcal{E} \to i_{1,!}\, i_1^! \mathcal{E} \oplus i_{2,!}\, i_2^! \mathcal{E} \to \mathcal{E} \to i_{1,2,!}\, i_{1,2}^! \mathcal{E}[1]$$

in the language of triangulated categories.

Dually, if $\mathbb{D}^* \to \mathbb{S}^{\mathrm{op}}$ is a fibered derivator (for example coming from a six-functor-formalism, as in the introduction) which is colocal w.r.t. the pre-topology on $\mathbb{S}$, Theorem 3.5.4 implies that p^{op} is a weak $\mathbb{D}^*$-equivalence in $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S}^{\mathrm{op}})/S$, i.e. for $\mathcal{E} \in \mathbb{D}(\cdot)_S$ we have

$$\mathcal{E} \cong p_* p^* \mathcal{E}.$$

This means that the homotopy limit of

$$\begin{tikzcd} & i_{1,*}\, i_1^* \mathcal{E} \arrow[d] \\ i_{2,*}\, i_2^* \mathcal{E} \arrow[r] & i_{1,2,*}\, i_{1,2}^* \mathcal{E} \end{tikzcd}$$

is isomorphic to $\mathcal{E}$, where we now wrote $i_{1,*}$ for $(i_1^{\mathrm{op}})^\bullet$, etc. If $\mathbb{D}^* \to \mathbb{S}^{\mathrm{op}}$ has stable fibers, this translates to the usual distinguished triangle

$$\mathcal{E} \to i_{1,*}\, i_1^* \mathcal{E} \oplus i_{2,*}\, i_2^* \mathcal{E} \to i_{1,2,*}\, i_{1,2}^* \mathcal{E} \to \mathcal{E}[1]$$

in the language of triangulated categories.

3.5.14. Example. [(Co)homological descent] Let $\mathbb{S}$ be a strong right derivator with a Grothendieck pre-topology and let $X_\bullet \in \mathbb{S}(\Delta^{\mathrm{op}})$ be a simplicial diagram over $S \in \mathbb{S}(\cdot)$ with underlying diagram

$$\cdots \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \; X_2 \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \; X_1 \; \substack{\longrightarrow\\ \longrightarrow} \; X_0$$

such that $(\mathrm{id}, p) : (\Delta^{\mathrm{op}}, X_\bullet) \to (\Delta^{\mathrm{op}}, \pi^* S)$ is a finite hypercover. Here $\pi : \Delta^{\mathrm{op}} \to \cdot$ denotes the projection. If $\mathbb{D}^! \to \mathbb{S}$ is a fibered derivator (for example coming from a six-functor-formalism, as in the introduction) which is local w.r.t. the pre-topology on $\mathbb{S}$, Theorem 3.5.5 implies that (π, p) is a weak $\mathbb{D}^!$-equivalence in $\mathrm{Dia}(\mathbb{S})/S$, i.e. for $\mathcal{E} \in \mathbb{D}^!(\cdot)_S$ we have

$$\mathcal{E} \cong \pi_!\, p_\bullet\, p^\bullet\, \pi^* \mathcal{E}.$$

This means that the homotopy colimit of $p_\bullet\, p^\bullet\, \pi^* \mathcal{E}$ is equal to $\mathcal{E}$. If the fibers of $\mathbb{D}^! \to \mathbb{S}$ are in fact *derived categories*, this yields a spectral sequence of homological descent because the homotopy colimit over a simplicial complex is the total complex of the associated double complex (a well-known fact). This double complex looks like

$$\cdots \longrightarrow p_{2,!}\, p_2^! \mathcal{E} \longrightarrow p_{1,!}\, p_1^! \mathcal{E} \longrightarrow p_{0,!}\, p_0^! \mathcal{E}$$

where we now wrote $p_{0,!}$ for $p_{0,\bullet}$, etc. The point is that we get a *coherent* double complex. Knowing the individual morphisms $p_{i,!}\, p_i^! \mathcal{E} \to p_{i-1,!}\, p_{i-1}^! \mathcal{E}$ as morphisms in the derived category $\mathbb{D}^!(\cdot)_S$ would not be sufficient!

Dually (applying everything to a fibered derivator $\mathbb{D}^* \to \mathbb{S}^{\mathrm{op}}$, and working in $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S}^{\mathrm{op}})$), one obtains the more classical spectral sequence of *co*homological descent.

Proof of Main Theorem 3.5.5, 2. This is the case of strong $\mathbb{D}$-equivalences.

(L1) is clear.

For (L2 left), let $D_1 = (I, F)$ and $D_2 = (e, F(e))$. The projection p and the inclusion i of the final object induce morphisms:

$$D_1 \underset{i}{\overset{p}{\rightleftarrows}} D_2$$

We have $p \circ i = \mathrm{id}$ and there is a 2-morphism $\beta : \mathrm{id} \Rightarrow i \circ p$. Therefore the statement follows from Lemma 3.5.7. (Actually $(i \circ p)^*$ is left adjoint to the inclusion $\mathbb{D}(D_1)^{\mathrm{cart}} \to \mathbb{D}(D_1)$.)

(L3 left): It suffices to prove the following two statements:

1. Consider a morphism of diagrams $w = (\alpha, f) : D_1 = (I_1, F_1) \to D_2 = (I_2, F_2)$ such that we have a commutative diagram

$$\begin{tikzcd} I_1 \arrow[rr, "\alpha"] \arrow[dr, "p_1"'] & & I_2 \arrow[dl, "p_2"] \\ & E & \end{tikzcd}$$

   and such that $w \times_{/E} e$ is a strong $\mathbb{D}$-equivalence for all objects e in E. Then w is a strong $\mathbb{D}$-equivalence.

2. Consider a morphism of diagrams $w : D_1 = (I_1, F_1) \to D_2 = (I_2, F_2)$ over $(\cdot, S)$ and let $\{U_i \to S\}$ be a covering. If $w \times_{(\cdot,S)} (\cdot, U_i)$ is a strong $\mathbb{D}$-equivalence for all i then w is a strong $\mathbb{D}$-equivalence.

We proceed by showing statement 1. Consider the following diagram over E

$$\begin{tikzcd} D_1 \arrow[r, "w"] \arrow[d] & D_2 \arrow[d] \\ D_1 \times_{/E} E \arrow[r, "w'"] & D_2 \times_{/E} E \end{tikzcd}$$

where the vertical morphisms are of pure diagram type. We have an adjunction

$$I_i \underset{\iota_i}{\overset{\kappa_i}{\rightleftarrows}} I_i \times_{/E} E$$

where κ_i maps an object i to $(i, \mathrm{id}_{p(i)})$. We have a natural transformation $\kappa_i \circ \iota_i \Rightarrow \mathrm{id}_{I_i \times_{/E} E}$ and moreover $\iota_i \circ \kappa_i = \mathrm{id}_E$ holds. Actually this defines an adjunction with κ_i left-adjoint to ι_i. Furthermore, we get lifts to diagrams

$$D_i \underset{\widetilde{\iota}_i}{\overset{\widetilde{\kappa}_i}{\rightleftarrows}} (I_i \times_{/E} E, \iota_i \circ F) = D_1 \times_{/E} E,$$

and a 2-morphism $\widetilde{\kappa}_i \circ \widetilde{\iota}_i \Rightarrow \mathrm{id}_{D_1 \times_{/E} E}$, and we have $\widetilde{\iota}_i \circ \widetilde{\kappa}_i = \mathrm{id}_{D_1}$.

Hence, by Lemma 3.5.7, the pull-backs along $\widetilde{\iota}_1$ and $\widetilde{\iota}_2$ induce equivalences on Cartesian objects, so we are reduced to showing that the pull-back along w' induces an equivalence on Cartesian objects. The underlying diagrams $I_k \times_{/E} E$ are opfibratons over E and the functor underlying w' is a map of opfibrations (the push-forward along a map $\mu : e \to f$ in E being given by mapping $(i, \nu : p(i) \to e)$ to $(i, \nu \circ \mu)$). Hence w.l.o.g. we may assume that $I_i \to E$ are opfibrations and the morphism $I_1 \to I_2$ underlying f is a morphism of opfibrations.

We keep the notation $w : D_1 \to D_2$, however, and the assumption translates to the statement that the pull-back

$$\mathbb{D}(D_{2,e})^{\mathrm{cart}} \xrightarrow{w_e^*} \mathbb{D}(D_{1,e})^{\mathrm{cart}}$$

for the fibers is an equivalence with inverse $\Box_! w_{e,!}$.

Consider the two functors:

$$\mathbb{D}(D_2)^{E-\mathrm{cart}} \xrightarrow{\text{incl.}} \mathbb{D}(D_2) \xrightarrow{w^*} \mathbb{D}(D_1).$$

We first show that the counit

$$\Box_!^E w_! w^* \mathcal{E} \to \mathcal{E}$$

is an isomorphism for every E-Cartesian $\mathcal{E}$.

This can be checked after pulling back to the fibers. Let $\iota_k : I_{k,e} \to I_k$ be the inclusion of the fiber over some $e \in E$.

We have the isomorphisms

$$\iota_2^* \Box_!^E w_! w^* \mathcal{E} \cong \Box_! w_{e,!} \iota_1^* w^* \mathcal{E} \cong \Box_! w_{e,!} w^{e,*} \iota_2^* \mathcal{E} \cong \iota_2^* \mathcal{E},$$

where we used the isomorphism $\iota_2^* \Box_!^E \cong \Box_! \iota_2^*$ (Lemma 3.5.11) and the isomorphism $\iota_2^* w_! \cong w_{e,!} \iota_1^*$ (exists for morphisms of pure diagram type because we have a morphism of opfibrations, see Proposition 2.3.23, 3. and for morphisms of fixed shape by axiom (FDer0 left)). The morphism $\Box_! w_{e,!} w^{e,*} \mathcal{E} \to \mathcal{E}$ is an isomorphism for Cartesian $\mathcal{E}$ by assumption.

We now show that the unit

$$\mathcal{E} \to w^* \Box_!^E w_! \mathcal{E}$$

is an isomorphism for every E-Cartesian $\mathcal{E}$. This can be checked again on the fibers:

$$\iota_1^* w^* \Box_!^E w_! \mathcal{E} \cong w_e^* \iota_2^* \Box_!^E w_! \mathcal{E} \cong w_e^* \Box_! w_{e,!} \iota_1^* \mathcal{E} \cong \iota_1^* \mathcal{E}.$$

Therefore we have already proven that the functors

$$\begin{tikzcd} \mathbb{D}(D_2)^{E-\mathrm{cart}} \arrow[r, shift left, "w^*"] & \mathbb{D}(D_1)^{E-\mathrm{cart}} \arrow[l, shift left, "\Box_!^E w_!"] \end{tikzcd}$$

form an equivalence.

We conclude by showing that $\Box_!^E w_!$ maps Cartesian objects to Cartesian objects: Let $\nu : e \to f$ be a morphisms of E. It induces a morphism (choice of push-forward for $I_k \to E$)

$$\widetilde{\nu}_k : D_{k,e} \to D_{k,f}$$

(not of diagram type!) and a 2-morphism: $\iota_{k,e} \to \iota_{k,f} \circ \widetilde{\nu}_k$.

Claim: It suffices to show that for all $\nu : e \to f$ the induced morphism

$$\iota_{2,e}^* \Box_!^E w_! \mathcal{E} \to \widetilde{\nu}_2^* \iota_{2,f}^* \Box_!^E w_! \mathcal{E}$$

is an isomorphism for every Cartesian $\mathcal{E}$.

Proof of the claim: Every morphism $\mu : i \to i''$ in I with $p(\mu) = \nu$, say, is the composition of a coCartesian μ' and some morphism μ'' with $p(\mu'') = \mathrm{id}_f$. Since $\mathcal{E}$ is E-Cartesian, the morphism $\mathcal{E}(\mu'')$ is Cartesian. Hence to show that $\mathcal{E}(\mu)$ is Cartesian it suffices to see that $\mathcal{E}(\mu')$ is Cartesian. A reformulation is, however, that the morphism of the claim be an isomorphism.

Using the same argument as in the first part of the proof, we have to show that

$$\Box_! w_{e,!} \iota_{1,e}^* \mathcal{E} \to \widetilde{\nu}_2^* \Box_! w_{f,!} \iota_{1,f}^* \mathcal{E}$$

is an isomorphism for every Cartesian $\mathcal{E}$. Since both sides are Cartesian objects, this can be checked after applying w_e^* which is an equivalence on Cartesian objects:

$$w_e^* \Box_! w_{e,!} \iota_{1,e}^* \mathcal{E} \to w_e^* \widetilde{\nu}_2^* \Box_! w_{f,!} \iota_{1,f}^* \mathcal{E}.$$

We have $w_e^* \widetilde{\nu}_2^* = \widetilde{\nu}_1^* w_f^*$ because the map of diagrams underlying w is a morphism of opfibrations. Hence, after applying w_e^*, we get

$$w_{e^*} \Box_! w_{e,!} \iota_{1,e}^* \mathcal{E} \to \widetilde{\nu}_1^* w_{f^*} \Box_! w_{f,!} \iota_{1,f}^* \mathcal{E}.$$

Since $w_{e^*} \Box_! w_{e,!}$ and $w_{f^*} \Box_! w_{f,!}$ are equivalences on Cartesian objects, we get

$$\iota_{1,e}^* \mathcal{E} \to \widetilde{\nu}_1^* \iota_{1,f}^* \mathcal{E}.$$

A slightly tedious check shows that this is again the morphism induced by the 2-morphism $\iota_{1,e} \to \iota_{1,f} \circ \widetilde{\nu}_1$. It is an isomorphism because $\mathcal{E}$ is Cartesian.

We will now show statement 2. Consider a diagram

$$\begin{tikzcd} D_1 \arrow[rr, "w"] \arrow[dr, "p_1"'] & & D_2 \arrow[dl, "p_2"] \\ & (\cdot, S) & \end{tikzcd}$$

For any i (index of the cover in statement 2.) we have the following commutative diagram of objects in $\mathrm{Dia}(\mathbb{S})$:

$$\begin{tikzcd} D_1 \times_S U_i \arrow[r, "w_i"] \arrow[d, "\mathrm{pr}_1^{(i)}"'] & D_2 \times_S U_i \arrow[d, "\mathrm{pr}_1^{(i)}"] \\ D_1 \arrow[r, "w"] & D_2 \end{tikzcd}$$

The morphisms $\mathrm{pr}_1^{(i)}$ are of fixed shape. We first show that the unit is an isomorphism

$$\mathcal{E} \to w^* \,\Box_!\, w_! \mathcal{E}$$

for any Cartesian $\mathcal{E}$. Note that by the stability axiom of a Grothendieck pre-topology also the collections $(D_1 \times_S U_i)_j \to D_{1,j}$ are covers for any $j \in I_1$, where I_1 is the underlying diagram of D_1. Since $\mathbb{D}$ is local w.r.t. the Grothendieck pre-topology (and by axiom Der2), the family $(\mathrm{pr}_1^{(i)})^*$ is jointly conservative. Therefore it suffices to show that the unit is an isomorphism after applying $(\mathrm{pr}_1^{(i)})^*$. We get

$$(\mathrm{pr}_1^{(i)})^* \mathcal{E} \to (\mathrm{pr}_1^{(i)})^* \, w^* \,\Box_!\, w_! \mathcal{E}$$

which is the same as

$$(\mathrm{pr}_1^{(i)})^* \mathcal{E} \to w_i^* \,(\mathrm{pr}_1^{(i)})^* \,\Box_!\, w_! \mathcal{E}.$$

Since $(\mathrm{pr}_1^{(i)})^*$ commutes with $\Box_!$ (Lemma 3.5.9) and with $w_!$ (Proposition 2.6.8, 2.), we get

$$(\mathrm{pr}_1^{(i)})^* \mathcal{E} \to w_i^* \,\Box_!\, w_{i,!} \,(\mathrm{pr}_1^{(i)})^* \mathcal{E}.$$

This morphism is an isomorphism by assumption. In the same way one shows that the counit is an isomorphism.

(L4 left): By Lemma 3.2.12 we may prove axiom (L4' left) instead. We have shown during the proof for (L4' left) for the case of weak $\mathbb{D}$-equivalences that

$$p_! \, p^* \to \mathrm{id}$$

is an isomorphism, hence on Cartesian objects the same holds for the natural transformation

$$\Box_! \, p_! \, p^* \to \mathrm{id}.$$

We have to show that also the counit

$$\mathrm{id} \to p^* \,\Box_!\, p_! \tag{15}$$

is an isomorphism on Cartesian objects. First note that p_* also is a right adjoint of p^* when restricted to the full subcategories of Cartesian objects because p_* preserves Cartesian objects. Indeed, p_* can be computed fiber-wise because p is a fibration. The fibers being contractible in the sense of any localizer on Dia implies that the functors

$p_e^*, p_{e,*}$ induce an equivalence $\mathbb{D}(D_e)^{\mathrm{cart}} \cong \mathbb{D}(\cdot)_{F(e)}$. Note: This uses that (L1–L3 left) hold for the class of strong $\mathbb{D}$-equivalences on the fiber $\mathbb{D}_{F(e)}$, a fact which has been proven already. Therefore we pass to the right adjoints of the functors in (15) and have to show that the counit

$$p^*\, p_* \to \mathrm{id}$$

is an isomorphism on Cartesian objects. Again this can be checked fiber-wise, i.e. we have to show that the counit

$$p_e^*\, p_{e,*} \to \mathrm{id}$$

is an isomorphism on Cartesian objects. But the pair of functors is an equivalence as we have seen, and the claim follows. ∎

We proceed to state some consequences of the fact that *strong* $\mathbb{D}$-equivalences form a fundamental localizer.

3.5.15. Corollary. [left] *Let $\mathbb{S}$ be a strong right derivator. If $\mathbb{D} \to \mathbb{S}$ is an infinite fibered derivator which is local w.r.t. the pre-topology on $\mathbb{S}$ (cf. 2.5.2) with stable, well-generated fibers then for any finite hypercover $f : X_\bullet \to Y_\bullet$ considered as 1-morphism in* $\mathrm{Dia}(\mathbb{S})$ *the functor f^* induces an equivalence*

$$\mathbb{D}(Y_\bullet)^{\mathrm{cart}} \to \mathbb{D}(X_\bullet)^{\mathrm{cart}}.$$

3.5.16. Corollary. [right] *Let $\mathbb{S}$ be a strong right derivator. If $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ is an infinite fibered derivator which is colocal w.r.t. the pre-topology on $\mathbb{S}$ (cf. 2.5.2) with stable, compactly generated fibers then for any finite hypercover $f : X_\bullet \to Y_\bullet$ considered as 1-morphism in* $\mathrm{Dia}^{\mathrm{op}}(\mathbb{S}^{\mathrm{op}})$ *the functor f^* induces an equivalence*

$$\mathbb{D}(Y_\bullet)^{\mathrm{cocart}} \to \mathbb{D}(X_\bullet)^{\mathrm{cocart}}.$$

3.5.17. Corollary. *If $\mathbb{D}$ is an infinite derivator (not fibered) with domain* Cat *which is stable and well-generated, then for each homotopy type I, we get a category $\mathbb{D}(I)^{\mathrm{cart}}$ well-defined up to equivalence of categories. Moreover each morphism $I \to J$ of homotopy types gives rise to a corresponding functor $\alpha^* : \mathbb{D}(J)^{\mathrm{cart}} \to \mathbb{D}(I)^{\mathrm{cart}}$. It is, however, not possible to arrange those as a pseudo-functor $\mathcal{HOT} \to \mathcal{CAT}$, but it is possible to arrange them as a pseudo-functor $\mathcal{HOT}^{(2)} \to \mathcal{CAT}$ where $\mathcal{HOT}^{(2)}$ is the homotopy 2-category (2-truncation) of any model for the homotopy theory of spaces (cf. also A.2).*

4. Representability

In this section we exploit the consequences that Brown representability type results have for fibered derivators. In particular these results are useful to see that under certain circumstances a left fibered (multi)derivator is already a right fibered (multi)derivator, provided that its fibers are nice (i.e. stable and well-generated derivators). Furthermore they provide us with (co)Cartesian projectors that are needed for the strong form of (co)homological descent. In contrast to the rest of the article the results are stated in a rather unsymmetric form. This is due to the fact that in applications the stable derivators will rather be well-generated, or compactly generated, whereas their duals will rather not satisfy this condition. All the auxiliary results are taken from [Kra10] and [Nee01].

4.1. Well-generated triangulated categories and Brown representability theorems.

4.1.1. Definition. [cf. [Kra10, 5.1, 6.3]] *Let $\mathcal{D}$ be a category with zero object and small coproducts. We call $\mathcal{D}$* **perfectly generated** *if there is a set of objects $\mathcal{T}$ in $\mathcal{D}$ such that the following conditions hold:*

1. *An object $X \in \mathcal{D}$ is zero if and only if $\mathrm{Hom}(T, X) = 0$ for all $T \in \mathcal{T}$.*
2. *If $\{X_o \to Y_o\}_{o \in O}$ is any set of maps, and $\mathrm{Hom}(T, X_o) \to \mathrm{Hom}(T, Y_o)$ is surjective for all i and $T \in \mathcal{T}$, then $\mathrm{Hom}(T, \coprod_o X_o) \to \mathrm{Hom}(T, \coprod_o Y_o)$ is also surjective for all $T \in \mathcal{T}$.*

The category $\mathcal{D}$ is called **well-generated** *if there is a set of objects $\mathcal{T}$ in $\mathcal{D}$ such that in addition to 1., 2. there is a regular cardinal α such that the following condition holds:*

3. *All objects $T \in \mathcal{T}$ are α-small, cf. [Kra10, 6.3].*

The category $\mathcal{D}$ is called **compactly generated** *if there is a set of objects $\mathcal{T}$ in $\mathcal{D}$ such that in addition to 1., 2. the following two equivalent conditions hold:*

4. *All $T \in \mathcal{T}$ are $\aleph_0$-small.*

4'. *All $T \in \mathcal{T}$ are compact, i.e. for each morphism $\gamma : T \to \coprod_{o \in O} X_o$ there is a finite subset $O' \subseteq O$ such that γ factors through $\coprod_{o \in O'} X_o$.*

Recall (cf. [Kra10, 4.4]) that a functor from a triangulated category $\mathcal{D}$ to an abelian category is called **cohomological** if it sends distinguished triangles to exact sequences.

We recall the following theorem:

4.1.2. Theorem. [right Brown representability] *Let $\mathcal{D}$ be a perfectly generated triangulated category with small coproducts. Then a functor $F : \mathcal{D}^{\mathrm{op}} \to \mathcal{AB}$ is cohomological and sends coproducts to products if and only if it is representable. An exact functor $\mathcal{D} \to \mathcal{E}$ between triangulated categories commutes with coproducts if and only if it has a right adjoint.*

Proof. See [Kra10, Theorem 5.1.1]. ∎

It can be shown that for a compactly generated triangulated category $\mathcal{D}$ with small coproducts, $\mathcal{D}^{\mathrm{op}}$ is perfectly generated and has small coproducts [Kra10, 5.1.2.(2), 5.3]. Therefore the dual version of the previous theorem holds in this case:

4.1.3. THEOREM. [left Brown representability] *Let $\mathcal{D}$ be a compactly generated triangulated category with small coproducts. Then a functor $F : \mathcal{D} \to \mathcal{AB}$ is homological and sends products to products if and only if F is representable. An exact functor $\mathcal{D} \to \mathcal{E}$ between triangulated categories commutes with products if and only if it has a left adjoint.*

4.1.4. THEOREM. *Let $\mathcal{D}$ be a well-generated triangulated category with small coproducts. Consider a functor $F : \mathcal{D} \to \mathcal{AB}$ which is cohomological and commutes with coproducts. Then there exists a right adjoint to the inclusion of the full subcategory of objects X such that $F(X[n]) = 0$ for all $n \in \mathbb{Z}$ (i.e. this subcategory is coreflective).*

PROOF. See [Kra10, Theorem 7.1.1]. ∎

Recall from 2.1.1 that we say that a diagram category Dia is *infinite* if it is closed under infinite coproducts as well.

4.1.5. DEFINITION. *A pre-derivator $\mathbb{D}$ whose domain* Dia *is infinite is called* **infinite** *if the restriction-to-I_o functors induce an equivalence*

$$\mathbb{D}(\coprod_{o \in O} I_o) \cong \prod_{o \in O} \mathbb{D}(I_o)$$

for all sets O.

4.1.6. LEMMA. *Let* Dia *be an infinite diagram category. Let $\mathbb{D} \to \mathbb{S}$ be an infinite left fibered derivator with domain* Dia*. If $\mathbb{D}(\cdot)_S$ for all $S \in \mathbb{S}(\cdot)$ is perfectly generated (resp. well-generated, resp. compactly generated), then the same holds for $\mathbb{D}(I)_{S'}$ for all $I \in$* Dia *and for all $S' \in \mathbb{S}(I)$. Furthermore the categories $\mathbb{D}(I)_{S'}$ all have small coproducts.*

PROOF. A set of generators as requested is given by the set $\mathcal{T}_I := \{i_! T\}_{i \in I, T \in \mathcal{T}}$. Indeed, an object $X \in \mathbb{D}(I)$ is zero if $i^* X$ is zero for all $i \in X$ by (Der2). Therefore X is zero if $\mathrm{Hom}(i_! T, X) = \mathrm{Hom}(T, i^* X) = 0$ for all $i \in I$ and for every $T \in \mathcal{T}$. We have to show that $\mathrm{Hom}(i_! T, \coprod_o X_o) \to \mathrm{Hom}(i_! T, \coprod_o Y_o)$ is an surjective for a family $\{X_o \to Y_o\}_{o \in O}$ of morphisms as in Definition 4.1.1, 2. We have $\mathrm{Hom}(i_! T, \coprod_o X_o) = \mathrm{Hom}(T, i^* \coprod_o X_o) = \mathrm{Hom}(T, \coprod_o i^* X_o)$, where we used that i^* commutes with coproducts. This follows because the Cartesian diagram

$$\begin{tikzcd} O \arrow[r, "\mathrm{id} \times i"] \arrow[d] & O \times I \arrow[d] \\ \cdot \arrow[r, "i"'] & I \end{tikzcd}$$

is homotopy exact. Note that, since $\mathbb{D}$ is infinite, coproducts exist and are equal to the corresponding homotopy coproducts. The map $\mathrm{Hom}(T, \coprod_o i^* X_o) \to \mathrm{Hom}(T, \coprod_o i^* Y_o)$ is surjective by assumption.

For the assumption on well-generatedness, we have to show that a morphism

$$i_!T \to \coprod_{o \in O} Y_o$$

in $\mathbb{D}(I)_{S'}$ factors through $\coprod_{o \in O'} Y_o$ for some subset $O' \subset O$ of cardinality less than α. By the same reasoning as above, we get a morphism

$$T \to \coprod_{o \in O} i^* Y_o$$

Hence, there is some subset $O' \subset O$, as required, such that this morphism factors through it. The same then holds for the original morphism. Since there is no need to enlarge O', the same statement holds for finite subsets.

The categories $\mathbb{D}(I)_{S'}$ have small coproducts because $\mathbb{D} \to \mathbb{S}$ is infinite and left fibered. ∎

4.1.7. Definition. *Let $\mathbb{D} \to \mathbb{S}$ be an infinite left fibered derivator with domain* Dia. *We will say that $\mathbb{D} \to \mathbb{S}$ has* **perfectly-generated** *(resp.* **well-generated**, *resp.* **compactly-generated**) **fibers**, *if all categories $\mathbb{D}(\cdot)_S$ are perfectly-generated (resp. well-generated, resp. compactly-generated) for all $S \in \mathbb{S}(\cdot)$. It follows from the previous Lemma that, in this case, for all $I \in$* Dia *and for all $S' \in \mathbb{S}(I)$ the category $\mathbb{D}(I)_{S'}$ is also perfectly-generated (resp. well-generated, resp. compactly-generated).*

4.2. Left and Right.

4.2.1. Theorem. [left] *Let* Dia *be an infinite diagram category (cf. 2.1.1). Let $\mathbb{D}$ and $\mathbb{E}$ be infinite left derivators with domain* Dia *such that for all $I \in$* Dia *the pre-derivators $\mathbb{D}_I$ and $\mathbb{E}_I$ are stable (left and right) derivators with domain* Posf. *Assume that $\mathbb{D}$ is perfectly generated. Then a morphism of derivators $F : \mathbb{D} \to \mathbb{E}$ commutes with all homotopy colimits w.r.t.* Dia *if and only if it has a right adjoint.*

Proof. Let I be in Dia. Since $\mathbb{D}_I$ and $\mathbb{E}_I$ are stable, $\mathbb{D}(I)$ is canonically triangulated, and we may use Theorem 4.1.2 of right Brown representability. It follows that the functor $F(I) : \mathbb{D}(I) \to \mathbb{E}(I)$ has a right adjoint $G(I)$, because it is triangulated, commutes with small coproducts and $\mathbb{D}(I)$ is perfectly generated. To construct a morphism of derivators out of this collection, for any $\alpha : I \to J$, we have to give an isomorphism: $G(J)\alpha^* \to \alpha^* G(I)$. We may take the adjoint of the isomorphism $\alpha_! F(J) \to F(I)\alpha_!$ expressing that F commutes with all homotopy colimits (see [Gro13, Lemma 2.1] for details). ∎

Analogously, using Theorem 4.1.3 of left Brown representability, we obtain:

4.2.2. Theorem. [right] *Let* Dia *be an infinite diagram category (cf. 2.1.1). Let $\mathbb{D}$ and $\mathbb{E}$ be infinite right derivators with domain* Dia *such that for all $I \in$* Dia*, the pre-derivators $\mathbb{D}_I$ and $\mathbb{E}_I$ are stable (left and right) derivators with domain* Posf. *Assume that $\mathbb{D}(\cdot)$ is compactly generated. Then a morphism of derivators $F : \mathbb{D} \to \mathbb{E}$ commutes with all homotopy limits w.r.t.* Dia *if and only if it has a left adjoint.*

4.2.3. THEOREM. [left] *Let* Dia *be an infinite diagram category (cf. 2.1.1). Let* $\mathbb{D} \to \mathbb{S}$ *be an infinite left fibered (multi)derivator with domain* Dia *whose* fibers $\mathbb{D}_S$ *for every* $I \in$ Dia *and all* $S \in \mathbb{S}(I)$ *are stable (left and right) derivators with domain* Posf*. Assume that* $\mathbb{D}$ *has perfectly generated fibers. Then* $\mathbb{D}$ *is a* right *fibered (multi)derivator as well.*

PROOF. Let $I \in$ Dia and let $f \in \mathrm{Hom}_{\mathbb{S}(I)}(S_1, \dots, S_n; T)$ be a multimorphism. By Lemma 2.3.13, fixing $\mathcal{E}_1, \overset{\widehat{i}}{\dots}, \mathcal{E}_n$, the association

$$\begin{array}{rcl} \mathbb{D}(I \times J)_{p^* S_i} & \to & \mathbb{D}(I)_{p^* T} \\ \mathcal{E}_i & \mapsto & (p^* f)_\bullet(p^* \mathcal{E}_1, \dots, \ \mathcal{E}_i, \ \dots, p^* \mathcal{E}_n) \end{array}$$

defines a morphism of derivators

$$\mathbb{D}_{S_i} \to \mathbb{D}_T$$

which is left continuous. Hence by Theorem 4.2.1 it has a right adjoint. This shows the first part of (FDer0 right), i.e. the functor $\mathbb{D}(I) \to \mathbb{S}(I)$ is an opfibration as well, for every $I \in$ Dia. Then axiom (FDer5 left) implies the remaining assertion of (FDer0 right) while (FDer0 left) implies (FDer5 right), see Lemma 2.3.9.

Similarly a morphism $\alpha : I \to J$ in Dia induces a morphism of derivators

$$\alpha^* : \mathbb{D}_S \to \mathbb{D}_{\alpha^* S}.$$

It commutes with homotopy colimits by Proposition 2.3.23, 2. Therefore α^* has a right adjoint α_* by the previous theorem, i.e. (FDer3 right) holds. (FDer4 right) is then a consequence of Lemma 2.3.23, 1. ∎

Analogously, using Theorem 4.1.3 of left Brown representability, we obtain:

4.2.4. THEOREM. [right] *Let* Dia *be an infinite diagram category (cf. 2.1.1). Let* $\mathbb{D} \to \mathbb{S}$ *be an infinite right fibered (multi)derivator with domain* Dia*, satisfying (FDer0 left) for* 0*-ary morphisms, whose* fibers $\mathbb{D}_S$ *for every* $I \in$ Dia *and for all* $S \in \mathbb{S}(I)$ *are stable (left and right) derivators with domain* Posf*. Assume that* $\mathbb{D}$ *has compactly generated fibers with small coproducts. Then* $\mathbb{D}$ *is a* left *fibered (multi)derivator as well.*

4.3. (CO)CARTESIAN PROJECTORS. In this section the (co)Cartesian projectors that are needed for the strong form of (co)homological descent will be constructed. The following Proposition subsumes some of the results of [Kra10].

4.3.1. PROPOSITION. *Let*

$$\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "F"] & \mathcal{E} \arrow[l, bend left, "G"] \end{tikzcd}$$

be an adjunction between well-generated triangulated categories with small coproducts in which F *and* G *are exact functors, and with* F *right adjoint (resp. left adjoint). Then there is exists a left adjoint (resp. right adjoint) to the inclusion*

$$\ker(F) \hookrightarrow \mathcal{D}.$$

4.3.2. REMARK. Instead of assuming the existence of a left (resp. right) adjoint to F, for the existence of the right adjoint to the inclusion $\ker(F) \hookrightarrow \mathcal{D}$ it suffices also that F commutes with coproducts (then G automatically exists by Brown representability). Similarly, if $\mathcal{D}$ is, in addition, compactly generated, it suffices for the existence of the left adjoint to the inclusion $\ker(F) \hookrightarrow \mathcal{D}$ that F commutes with products (then G automatically exists by Brown representability for the dual).

PROOF. Let F be the right adjoint and let $\mathcal{E}_0$ be the generating set of $\mathcal{E}$ (w.l.o.g. stable under the shift functors). We have

$$\begin{array}{ll} & X \in \ker(F) \\ \Leftrightarrow & \mathrm{Hom}(Y, F(X)) = 0 \ \forall Y \in \mathcal{E}_0 \\ \Leftrightarrow & \mathrm{Hom}(G(Y), X) = 0 \ \forall Y \in \mathcal{E}_0 \\ \Leftrightarrow & X \in G(\mathcal{E}_0)^\perp \\ \Leftrightarrow & X \in \langle G(\mathcal{E}_0)\rangle^\perp \end{array}$$

where $\langle G(\mathcal{E}_0)\rangle$ is the smallest localizing subcategory [Kra10, 5.1.] containing $G(\mathcal{E}_0)$. This category is well-generated by [Kra10, Theorem 7.2.1.]. By [Kra10, Proposition 5.2.1.] there exists therefore a right adjoint to the inclusion $\langle G(\mathcal{E}_0)\rangle \hookrightarrow \mathcal{D}$. By [Kra10, Proposition 4.9.1.] this is equivalent to the existence of a left adjoint to the inclusion $\ker(F) = \langle G(\mathcal{E}_0)\rangle^\perp \hookrightarrow \mathcal{D}$.

If F is the left adjoint then it commutes with coproducts and therefore by [Kra10, Theorem 7.4.1] the triangulated subcategory $\ker(F)$ is well-generated and hence by [Kra10, Proposition 5.2.1.] the inclusion $\ker(F) \hookrightarrow \mathcal{D}$ has a right adjoint. ∎

4.3.3. THEOREM. [left] *Let $\mathbb{D} \to \mathbb{S}$ be an infinite left fibered derivator w.r.t.* Dia *satisfying also (FDer0 right) whose fibers are stable derivators w.r.t.* Posf. *Assume that $\mathbb{D} \to \mathbb{S}$ has well-generated fibers. Then for all $I \in$* Dia*, for all $F \in \mathbb{S}(I)$, and for all functors $I \to E$ in* Dia *the fully-faithful inclusion*

$$\mathbb{D}(I)_F^{E-\mathrm{cocart}} \hookrightarrow \mathbb{D}(I)_F \tag{16}$$

has a right adjoint $\Box_*^E$.

If F is such that $F(\mu)$ satisfies (Dloc2 left) for every μ mapping to an identity in E, then the fully-faithful inclusion

$$\mathbb{D}(I)_F^{E-\mathrm{cart}} \to \mathbb{D}(I)_F$$

has a right adjoint $\blacksquare_*^E$.

PROOF. Consider the set O of morphisms $\mu : i \to j$ which map to an identity in E, and for each morphism $\mu \in O$ the composition D_μ:

$$\mathbb{D}(I)_F \xrightarrow{\mu^*} \mathbb{D}(\to)_{\mu^* F} \xrightarrow{F(\mu)_\bullet} \mathbb{D}(\to)_{j^* F} \xrightarrow{\mathrm{Cone}} \mathbb{D}(\cdot)_{j^* F}$$

We define a functor D as the following composition

$$\mathbb{D}(I)_F \xrightarrow{\mathrm{pr}_1^*} \mathbb{D}(I \times O)_{\mathrm{pr}_1^* F} \xrightarrow{\prod_{\mu \in O} D_\mu} \mathbb{D}(O)_{\iota^* F}$$

where $\iota : O \to I$ is the functor "target".

The functor D commutes with coproducts as all functors that it is composed of do, and it is exact. Therefore by Proposition 4.3.1 and the Remark the inclusion (16) has a right adjoint. In the Cartesian case define D_μ as

$$\mathbb{D}(I)_F \xrightarrow{\mu^*} \mathbb{D}(\to)_{\mu^* F} \xrightarrow{F(\mu)^\bullet} \mathbb{D}(\to)_{i^* F} \xrightarrow{\mathrm{Cone}} \mathbb{D}(\cdot)_{i^* F}$$

Here $F(\mu)^\bullet$ commutes with coproducts if $F(\mu)$ satisfies (Dloc2 left) and the same conclusion holds. ∎

Note that in the following right version of the Theorem, we need to assume that $\mathbb{D} \to \mathbb{S}$ is a left and right fibered derivator. (This holds automatically by Theorem 4.2.4 for a right fibered derivator whose fibers are derivators with domain Posf, compactly generated, and stable).

4.3.4. THEOREM. [right] *Let $\mathbb{D} \to \mathbb{S}$ be an infinite fibered derivator w.r.t.* Dia *with stable fibers. Assume that $\mathbb{D} \to \mathbb{S}$ has well-generated fibers. Then for all $I \in$* Dia*, for all $F \in \mathbb{S}(I)$, and for all functors $I \to E$ in* Dia *the fully-faithful inclusion*

$$\mathbb{D}(I)_F^{E-\mathrm{cart}} \hookrightarrow \mathbb{D}(I)_F \tag{17}$$

has a left adjoint $\square_!^E$.

If $\mathbb{D}(\cdot)_S$ is compactly generated for every $S \in \mathbb{S}(\cdot)$, and if F is such that $F(\mu)$ satisfies (Dloc2 left) for every μ mapping to an identity in E, then the fully-faithful inclusion

$$\mathbb{D}(I)_F^{E-\mathrm{cocart}} \to \mathbb{D}(I)_F$$

has a left adjoint $\blacksquare_!^E$.

PROOF. Consider the set O of morphisms $\mu : i \to j$ which map to an identity in E, and for each morphism $\mu \in O$ the composition D_μ:

$$\mathbb{D}(I)_F \xrightarrow{\mu^*} \mathbb{D}(\to)_{\mu^* F} \xrightarrow{F(\mu)^\bullet} \mathbb{D}(\to)_{i^* F} \xrightarrow{\mathrm{Cone}} \mathbb{D}(\cdot)_{i^* F}$$

We define a functor D as the following composition

$$\mathbb{D}(I)_F \xrightarrow{\mathrm{pr}_1^*} \mathbb{D}(I \times O)_{\mathrm{pr}_1^* F} \xrightarrow{\prod_{\mu \in O} D_\mu} \mathbb{D}(O)_{\iota^* F}$$

where $\iota : O \to I$ is the functor "source".

By assumption D has a left adjoint as all functors that it is composed of do, and it is exact. Therefore, by Proposition 4.3.1, the inclusion (17) has a left adjoint. In the coCartesian case define D_μ as

$$\mathbb{D}(I)_F \xrightarrow{\mu^*} \mathbb{D}(\to)_{\mu^* F} \xrightarrow{F(\mu)_\bullet} \mathbb{D}(\to)_{j^* F} \xrightarrow{\text{Cone}} \mathbb{D}(\cdot)_{j^* F}$$

Here $F(\mu)_\bullet$ commutes with products if $F(\mu)$ satisfies (Dloc2 right) and therefore (by Brown representability for the dual) has a left adjoint as well, and the same conclusion holds. ∎

5. Constructions

5.1. The fibered multiderivator associated with a fibered multicategory.

5.1.1. The most basic situation in which a (non-trivial) fibered multiderivator can be constructed arises from a bifibration of (locally small) multicategories

$$p : \mathcal{D} \to \mathcal{S}$$

equipped with a class of weak equivalences $\mathcal{W}_S \subset \mathrm{Mor}(\mathcal{D}_S)$ for each object S of $\mathcal{S}$. In the examples we have in mind, the objects of $\mathcal{S}$ are spaces (or schemes), the objects of $\mathcal{D}$ are chain complexes of sheaves (coherent, etale Abelian, etc.) on them, and the morphisms in $\mathcal{W}_S$ are the quasi-isomorphisms. In these examples the multicategory-structure arises from the tensor product and it is even, in most cases, the more natural structure because no particular tensor-product is chosen a priori.

5.1.2. Definition. *In the situation above, let $\mathbb{S}$ be the pre-multiderivator associated with the multicategory $\mathcal{S}$. We define a pre-multiderivator $\mathbb{D}$ as follows (cf. A.3.1 for localizations of multicategories):*

$$\mathbb{D}(I) = \mathrm{Hom}(I, \mathcal{D})[\mathcal{W}_I^{-1}]$$

where $\mathcal{W}_I$ is the class of natural transformations which are element-wise in the union $\bigcup_S \mathcal{W}_S$. The functor p obviously induces a morphism of pre-multiderivators

$$\widetilde{p} : \mathbb{D} \to \mathbb{S}$$

Observe that morphisms in $\mathcal{W}_I$, by definition, necessarily map to identities in $\mathrm{Hom}(I, \mathcal{S})$.

In this section we prove that the above morphism of pre-(multi)derivators is a left (resp. right) fibered (multi)derivator on inverse (resp. directed) diagrams, provided that the fibers are model categories whose structure is compatible with the structure of bifibration. We use the definition of a model category from [Hov99]. We denote the cofibrant replacement functor by Q and the fibrant replacement functor by R.

5.1.3. DEFINITION. *A* **bifibration of (multi-)model-categories** *is a bifibration of (multi)categories* $p : \mathcal{D} \to \mathcal{S}$ *together with the collection of a closed model structure on the fiber*

$$(\mathcal{D}_S, \mathrm{Cof}_S, \mathrm{Fib}_S, \mathcal{W}_S)$$

for any object S *in* $\mathcal{S}$ *such that the following two properties hold:*

1. *For any* $n \geq 1$ *and for every multimorphism*

$$\begin{tikzcd} S_1 \arrow[dr, dash] & & \\ \vdots & f \arrow[r] & T \\ S_n \arrow[ur, dash] & & \end{tikzcd}$$

   the push-forward $f_\bullet$ *and the various pull-backs* $f^{\bullet,j}$ *define a Quillen adjunction in* n*-variables*

$$\prod_i (\mathcal{D}_{S_i}, \mathrm{Cof}_{S_i}, \mathrm{Fib}_{S_i}, \mathcal{W}_{S_i}) \xrightarrow{f_\bullet} (\mathcal{D}_T, \mathrm{Cof}_T, \mathrm{Fib}_T, \mathcal{W}_T)$$

$$(\mathcal{D}_T, \mathrm{Cof}_T, \mathrm{Fib}_T, \mathcal{W}_T) \times \prod_{i \neq j} (\mathcal{D}_{S_i}, \mathrm{Cof}_{S_i}, \mathrm{Fib}_{S_i}, \mathcal{W}_{S_i}) \xrightarrow{f^{\bullet,j}} (\mathcal{D}_{S_j}, \mathrm{Cof}_{S_j}, \mathrm{Fib}_{S_j}, \mathcal{W}_{S_j})$$

2. *For any 0-ary morphism* f *in* $\mathcal{S}$*, let* $f_\bullet()$ *be the corresponding unit object (i.e. the object representing the 0-ary morphism functor* $\mathrm{Hom}_f(;-)$*) and consider the cofibrant replacement* $Qf_\bullet() \to f_\bullet()$*. Then the natural morphism*

$$F_\bullet(\mathcal{E}_1, \dots, \mathcal{E}_{i-1}, Qf_\bullet(), \mathcal{E}_i, \dots, \mathcal{E}_n) \to F_\bullet(\mathcal{E}_1, \dots, \mathcal{E}_{i-1}, f_\bullet(), \mathcal{E}_i, \dots, \mathcal{E}_n) \cong (F \circ_i f)_\bullet(\mathcal{E}_1, \dots, \mathcal{E}_n)$$

   is a weak equivalence if all $\mathcal{E}_i$ *are cofibrant. Here* F *is any morphism which is composable with* f*.*

5.1.4. REMARK. If $\mathcal{S} = \{\cdot\}$ is the final multicategory, the above notion coincides with the notion of monoidal model-category in the sense of [Hov99, Definition 4.2.6]. In this case it is enough to claim property 1. for $n = 1, 2$.

5.1.5. THEOREM. *Under the conditions of Definition 5.1.3 the morphism of pre-derivators*

$$\widetilde{p} : \mathbb{D} \to \mathbb{S}$$

(defined in 5.1.2) is a left fibered multiderivator (satisfying also FDer0 right) with domain Dir *and a right fibered multiderivator (satisfying also FDer0 left) with domain* Inv*. Furthermore, for all* $S \in \mathbb{S}(\cdot)$ *its fiber* $\mathbb{D}_S$ *(cf. 2.3.11) is just the pre-derivator associated with the pair* $(\mathcal{D}_S, \mathcal{W}_S)$*.*

There are techniques by Cisinski [Cis03] which allow to extend a derivator to more general diagram categories. We will explain in a forthcoming article [Hör17c] how these can be applied to fibered (multi)derivators.

The proof of the theorem will occupy the rest of this section. First we have:

5.1.6. Proposition. *Let $\mathcal{D} \to \mathcal{S}$ be a bifibration of multicategories with complete fibers. For any diagram category I, the functors*

$$p_I : \mathrm{Hom}(I,\mathcal{D}) \to \mathrm{Hom}(I,\mathcal{S}) = \mathbb{S}(I)$$

are bifibrations of multicategories.

Morphisms in $\mathrm{Hom}(I,\mathcal{D})$ *are Cartesian, if and only if they are point-wise Cartesian. The 1-ary morphisms in* $\mathrm{Hom}(I,\mathcal{D})$ *are coCartesian, if and only if they are point-wise coCartesian.*

Proof (Sketch). We choose push-forward functors $f_\bullet$ and pull-back functors $f^{i,\bullet}$ for $\mathcal{D} \to \mathcal{S}$ as usual. Let $f \in \mathrm{Hom}(S_1,\dots,S_n;T)$ be a morphism in $\mathrm{Hom}(I,\mathcal{S})$. We define a functor

$$f_\bullet : \mathrm{Hom}(I,\mathcal{D})_{S_1} \times \cdots \times \mathrm{Hom}(I,\mathcal{D})_{S_n} \to \mathrm{Hom}(I,\mathcal{D})_T$$

by

$$\mathcal{E}_1,\dots,\mathcal{E}_n \mapsto \{i \mapsto (f_i)_\bullet(\mathcal{E}_1(i),\dots,\mathcal{E}_n(i))\}.$$

Note that a morphism $\alpha : i \to i'$ in I induces a *well-defined* morphism

$$(f_i)_\bullet(\mathcal{E}_1(i),\dots,\mathcal{E}_n(i)) \to (f_{i'})_\bullet(\mathcal{E}_1(i'),\dots,\mathcal{E}_n(i'))$$

lying over $T(\alpha)$. The functor $f_\bullet$ comes equipped with a morphism in

$$\mathrm{Hom}(\mathcal{E}_1,\dots,\mathcal{E}_n; f_\bullet(\mathcal{E}_1,\dots,\mathcal{E}_n))$$

which is checked to be Cartesian in the strong form of Definition A.2.5.

For 1-ary morphisms we can perform the same construction to produce coCartesian morphisms. For $n \geq 2$ the construction is more complicated. Let $f \in \mathrm{Hom}(S_1,\dots,S_n;T)$ be a morphism with $n \geq 2$. To ease notation, we construct a pull-back functor w.r.t. the first slot. The other constructions are completely symmetric.

For any $i_1 \in I$ consider the category (a variant of the twisted arrow category)

$$X_{i_1}(I) \coloneqq \{\ (i_2,\dots,i_n,j,\{\alpha_k\}_{k=1..n}) \mid \alpha_k : i_k \to j\}$$

which is covariant in j and contravariant in $i_2,\dots,i_n$. For any $\beta : i_1 \to i_1'$ we have an induced functor $\widetilde{\beta} : X_{i_1'}(I) \to X_{i_1}(I)$. Any object $\alpha \in X_{i_1}(I)$ defines by pre-composition with $S_k(\alpha_k)$ for all $1 \leq k \leq n$ a morphism $f_\alpha \in \mathrm{Hom}(S_1(i_1),\dots,S_n(i_n);T(j))$.

We define a functor

$$f^{1,\bullet} : (\mathrm{Hom}(I,\mathcal{D})_{S_2})^{\mathrm{op}} \times \cdots \times (\mathrm{Hom}(I,\mathcal{D})_{S_n})^{\mathrm{op}} \times \mathrm{Hom}(I,\mathcal{D})_T \to \mathrm{Hom}(I,\mathcal{D})_{S_1}$$

assigning to $\mathcal{E}_2,\dots,\mathcal{E}_n;\mathcal{F}$ the following functor $X_{i_1}(I) \to \mathcal{D}_{S_1(i_1)}$:

$$\alpha \mapsto (f_\alpha)^{1,\bullet}(\mathcal{E}_2(i_2),\dots,\mathcal{E}_n(i_n);\mathcal{F}(j))$$

and then taking $\lim_{X_{i_1}(I)}$ which exists because the fibers are required to be complete. For the functoriality note that for $\beta : i_1 \to i_1'$ we have a natural morphism

$$\lim_{X_{i_1}(I)} \cdots \to \lim_{X_{i_1'}(I)} \cdots$$

induced by $\widetilde{\beta}$.

We define a morphism

$$\Xi \in \mathrm{Hom}_f(f^{1,\bullet}(\mathcal{E}_2, \dots, \mathcal{E}_n; \mathcal{F}), \mathcal{E}_2, \dots, \mathcal{E}_n; \mathcal{F})$$

and we will show that it is coCartesian w.r.t. the first slot in a weak sense. At some object $i \in I$, the morphism Ξ is given by composing the projections from

$$\lim_{X_{i=i_1}(I)} f_\alpha^{1,\bullet}(\mathcal{E}_2(i_2), \dots, \mathcal{E}_n(i_n); \mathcal{F}(j))$$

to $f_i^{1,\bullet}(\mathcal{E}_2(i), \dots, \mathcal{E}_n(i); \mathcal{F}(i))$ (note that $f_i = f_\alpha$ for $\alpha = \{\mathrm{id}_i\}_k$) and then composing with the coCartesian morphism (in $\mathcal{D}$) in

$$\mathrm{Hom}(f_i^{1,\bullet}(\mathcal{E}_2(i), \dots, \mathcal{E}_n(i); \mathcal{F}(i)), \mathcal{E}_2(i), \dots, \mathcal{E}_n(i); \mathcal{F}(i)).$$

One checks that the so defined Ξ is functorial in i. It remains to be shown that the composition with Ξ induces an isomorphism

$$\mathrm{Hom}_{\mathrm{id}_{S_1}}(\mathcal{E}_1; f^{1,\bullet}(\mathcal{E}_2, \dots, \mathcal{E}_n; \mathcal{F})) \to \mathrm{Hom}_f(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F}). \tag{18}$$

We will give a map in the other direction which is inverse to composition with Ξ. Let

$$a \in \mathrm{Hom}_f(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$$

be a morphism. To give a morphism on the left hand side of (18), for any i_1 and $\alpha \in X_{i_1}(I)$ we have to give a morphism (functorial in i_1)

$$\mathcal{E}_1(i_1) \to f_\alpha^{1,\bullet}(\mathcal{E}_2(i_2); \dots, \mathcal{E}_n(i_n); \mathcal{F}(j))$$

or, which is the same, a morphism

$$\mathrm{Hom}_{f_\alpha}(\mathcal{E}_1(i_1), \mathcal{E}_2(i_2), \dots, \mathcal{E}_n(i_n); \mathcal{F}(j)).$$

But we have such a morphism, namely the pre-composition of a_j with the n-tuple $\{\mathcal{E}_k(\alpha_k)\}_k$. (Because we know already that $\mathrm{Hom}(I, \mathcal{D}) \to \mathrm{Hom}(I, \mathcal{S})$ is an *op*fibration of multicategories, it suffices to establish that Ξ is coCartesian in this weak form.) ∎

5.1.7. REMARK. The construction in the proof of the above Proposition will become much clearer, when we define a fibered multiderivator *itself* as a six-functor-formalism, similar to the definition mentioned in the introduction. For example, for $\mathcal{S} = \{\cdot\}$ we will get an external and internal monoidal product, resp. right adjoints which a clear relation. We have in that case

$$\boxtimes : \mathrm{Hom}(I, \mathcal{D}) \times \mathrm{Hom}(J, \mathcal{D}) \to \mathrm{Hom}(I \times J, \mathcal{D})$$

by applying $\otimes$ *point-wise* and

$$\mathcal{HOM}_{l/r} : \mathrm{Hom}(I, \mathcal{D}) \times \mathrm{Hom}(J, \mathcal{D}) \to \mathrm{Hom}(I^{\mathrm{op}} \times J, \mathcal{D})$$

by applying $\mathrm{Hom}_{l/r}$ *point-wise*. The formula for the internal hom obtained in the proof of the proposition boils down to the formula

$$\mathrm{Hom}_{l/r}(\mathcal{E}, \mathcal{F})(i_1) = \int_i \mathcal{HOM}_{l/r}(\mathcal{E}(i), \mathcal{F}(i))^{\mathrm{Hom}(i_1, i)}$$

where $\int_i$ is the categorical end. We refer to a subsequent article [Hör16] for an explanation of this in the language of six-functor-formalisms.

We will need later the following

5.1.8. LEMMA. *Let $f \in \mathrm{Hom}(S_1, \dots, S_n; T)$ be a morphism in $\mathrm{Hom}(I, \mathcal{S})$ for some $n \geq 2$. Consider the pull-back functor $f^{j,\bullet}$ constructed in the proof of Proposition 5.1.6. Let $p : I \times J \to I$ be the projection and fix objects $\mathcal{E}_1, \overset{\widehat{j}}{\dots}, \mathcal{E}_n, \mathcal{F}$ in $\mathcal{D}$ lying over $S_1, \overset{\widehat{j}}{\dots}, S_n, T$. Then the natural morphism*

$$p^* f^{j,\bullet}(\mathcal{E}_1, \overset{\widehat{j}}{\dots}, \mathcal{E}_n; \mathcal{F}) \to (p^* f)^{j,\bullet}(p^*\mathcal{E}_1, \overset{\widehat{j}}{\dots}, p^*\mathcal{E}_n; p^*\mathcal{F})$$

is an isomorphism, or, in other words, the functor $p^* : \mathrm{Hom}(I, \mathcal{D}) \to \mathrm{Hom}(I \times J, \mathcal{S})$ maps Cartesian morphisms to Cartesian morphisms.

PROOF. Again, we assume $j = 1$ to ease the notation. The statement concerning the other pull-backs is completely symmetric. We have by definition

$$(f^{1,\bullet}(\mathcal{E}_2, \dots, \mathcal{E}_n; \mathcal{F}))(i') = \lim_{\alpha \in X_{i_1}(I)} f_\alpha^{1,\bullet}(\mathcal{E}_2(i_1), \dots, \mathcal{E}_n(i_n); \mathcal{F}(i'))$$

and

$$((p^*f)^{1,\bullet}(\mathcal{E}_2, \dots, \mathcal{E}_n; \mathcal{F}))(i', j')$$
$$= \lim_{\alpha \in X_{i_1, j_1}(I \times J)} (p^* f)_\alpha^{1,\bullet}((p^*\mathcal{E}_2)(i_1, j_1), \dots, (p^*\mathcal{E}_n)(i_n, j_n); (p^*\mathcal{F})(i', j'))$$

The natural map in question is induced by the functor $\widetilde{p} : X_{i_1, j_1}(I \times J) \to X_{i_1}(I)$ which forgets all data involving the J direction. Now there is also a functor $\widetilde{s} : X_{i_1}(I) \to X_{i_1, j_1}(I \times J)$ which is constant on the J-component with value $\{\mathrm{id}_{j_1}\}_{k=1..n}$. We have $\widetilde{p} \circ \widetilde{s} = \mathrm{id}$ and a zig-zag of natural transformations $\widetilde{s} \circ \widetilde{p} \Leftarrow \cdots \Rightarrow \mathrm{id}$ involving only data in the J-direction. However, all the natural transformations are mapped to identities by the functor

$$\alpha \mapsto \lim_{\alpha \in X_{i_1,j_1}(I\times J)} (p^*f)^{1,\bullet}_{\alpha}((p^*\mathcal{E}_2)(i_1,j_1),\dots,(p^*\mathcal{E}_n)(i_n,j_n);(p^*\mathcal{F})(i',j'))$$

because everything is constant along the J-direction. This shows that the natural morphism in the statement is an isomorphism. ∎

If I is directed (resp. inverse) we want to show that also p_I is a bifibration of multi-model-categories in the sense of Definition 5.1.3.

Afterwards we will apply the following variant and generalization to multicategories of the results in [SGA73, Exposé XVII, §2.4].

5.1.9. Proposition. *Let $p : \mathcal{D} \to \mathcal{S}$ be a bifibration of (multi-)model-categories in the sense of 5.1.3. Let $\mathcal{W}$ be the union of the $\mathcal{W}_S$ over all objects $S \in \mathcal{S}$. Then the fibers of $\widetilde{p} : \mathcal{D}[\mathcal{W}^{-1}] \to \mathcal{S}$ (as ordinary categories) are the homotopy categories $\mathcal{D}_S[\mathcal{W}_S^{-1}]$ and $\widetilde{p}$ is again a bifibration of multicategories such that the push-forward $F_\bullet$ along any $F \in \mathrm{Hom}_{\mathcal{S}}(S_1,\dots,S_n;T)$ (for $n \geq 1$) is the left derived functor of the corresponding push-forward w.r.t. p. Similarly the pull-back w.r.t. some slot ist the right derived functor of the corresponding pull-back w.r.t. p.*

5.1.10. The above proposition and its proof have several well-known consequences which we mention, despite being all elementary, because the proof below gives a uniform treatment of all the cases.

1. The homotopy category of a model category is locally small and can be described as the category of cofibrant/fibrant objects modulo homotopy of morphisms. *Apply the proof of the proposition to the (trivial) bifibration of ordinary categories $\mathcal{D} \to \{\cdot\}$.*
2. Quillen adjunctions lead to an adjunction of the derived functors on the homotopy categories. *Apply the proposition to a bifibration of ordinary categories $\mathcal{D} \to \Delta_1$.*
3. The homotopy category of a closed monoidal model category is a closed monoidal category. *Apply the proposition to a bifibration of multicategories $\mathcal{D} \to \{\cdot\}$.*
4. Quillen adjunctions in n variables lead to an adjunction in n variables on the homotopy categories. *Apply the proposition to a bifibration of multicategories $\mathcal{D} \to \Delta_{1,n}$, where the multicategory $\Delta_{1,n}$ consists of $n+1$ objects and one n-ary morphism connecting them.*

Before proving Proposition 5.1.9, we define homotopy relations on $\mathrm{Hom}_F(\mathcal{E}_1,\dots,\mathcal{E}_n;\mathcal{F})$ where $F \in \mathrm{Hom}(X_1,\dots,X_n;Y)$ is a multimorphism in $\mathcal{S}$.

5.1.11. Definition.

1. *Two morphisms f and g in* $\mathrm{Hom}_F(\mathcal{E}_1,\dots,\mathcal{E}_n;\mathcal{F})$ *are called* **right homotopic** *if there is a path object of $\mathcal{F}$*
$$\mathcal{F} \longrightarrow \mathcal{F}' \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} \mathcal{F}$$
and a morphism $\mathrm{Hom}(\mathcal{E}_1,\dots,\mathcal{E}_n;\mathcal{F}')$ *over the same multimorphism F such that the compositions with* pr_1 *and* pr_2 *are f and g, respectively.*

2. *For $n \geq 1$, two morphisms f and g in* $\mathrm{Hom}_F(\mathcal{E}_1,\dots,\mathcal{E}_n;\mathcal{F})$ *are called* **i-left homotopic** *if there is a cylinder object $\mathcal{E}_i'$ of $\mathcal{E}_i$*
$$\mathcal{E}_i \underset{\iota_2}{\overset{\iota_1}{\rightrightarrows}} \mathcal{E}_i' \longrightarrow \mathcal{E}_i$$
and a morphism $\mathrm{Hom}(\mathcal{E}_1,\dots,\mathcal{E}_i',\dots,\mathcal{E}_n;\mathcal{F})$ *over F such that the compositions with ι_1 and ι_2 are f and g, respectively.*

5.1.12. Lemma.

1. *The condition 'right homotopic' is preserved under pre-composition, while the condition 'i-left homotopic' is preserved under post-composition.*

2. *Let $n \geq 1$. If $f,g \in \mathrm{Hom}(\mathcal{E}_1,\dots,\mathcal{E}_n;\mathcal{F})$ are i-left homotopic and all $\mathcal{E}_i$ are cofibrant then f and g are right homotopic. If $f,g \in \mathrm{Hom}(\mathcal{E}_1,\dots,\mathcal{E}_n;\mathcal{F})$ are right homotopic, $\mathcal{F}$ is fibrant, and all $\mathcal{E}_j$ for $j \neq i$ are cofibrant then f and g are i-left homotopic.*

3. *Let $n \geq 1$. In $\mathrm{Hom}(\mathcal{E}_1,\dots,\mathcal{E}_n;\mathcal{F})$ right homotopy is an equivalence relation if all $\mathcal{E}_i$ are cofibrant. In $\mathrm{Hom}(\mathcal{E}_1,\dots,\mathcal{E}_n;\mathcal{F})$ i-left homotopy is an equivalence relation if $\mathcal{F}$ is fibrant, and all $\mathcal{E}_j$, $j \neq i$ are cofibrant*

In particular on the category $\mathcal{D}^{\mathrm{Cof,Fib}}$ of fibrant/cofibrant objects, i-left homotopy=right homotopy is an equivalence relation, which is compatible with composition.

Proof. 1. is obvious.

2. If all $\mathcal{E}_i$ are cofibrant then also $F_\bullet(\mathcal{E}_1,\dots,\mathcal{E}_n)$ is cofibrant and f and g correspond uniquely to morphisms $f',g' : F_\bullet(\mathcal{E}_1,\dots,\mathcal{E}_n) \to \mathcal{F}$. Since f and g are i-left homotopic, there is a cylinder object
$$\mathcal{E}_i \rightrightarrows \mathcal{E}_i' \longrightarrow \mathcal{E}_i$$
realizing the i-left homotopy. Since $\mathcal{E}_i$ is cofibrant so is $\mathcal{E}_i'$. Hence also
$$F_\bullet(\mathcal{E}_1,\dots,\mathcal{E}_n) \rightrightarrows F_\bullet(\mathcal{E}_1,\dots,\mathcal{E}_i',\dots,\mathcal{E}_n) \longrightarrow F_\bullet(\mathcal{E}_1,\dots,\mathcal{E}_n)$$
is a cylinder object because all $\mathcal{E}_j$ are cofibrant, and hence also f' and g' are left homotopic. These are therefore also right homotopic and hence so are f and g. Dually we obtain the second statement.

3. follows from [Hov99, Proposition 1.2.5, (iii)]. ∎

5.1.13. Lemma. *Two i-left homotopic morphisms become equal in $\mathcal{D}^{\mathrm{Cof}}[(\mathcal{W}^{\mathrm{Cof}})^{-1}]$.*

Proof. This follows from the fact that a cylinder object

$$\mathcal{E}_i \overset{\iota_1}{\underset{\iota_2}{\rightrightarrows}} \mathcal{E}_i' \xrightarrow{p} \mathcal{E}_i$$

automatically lies in $\mathcal{D}^{\mathrm{Cof}}$ if $\mathcal{E}_i$ does, and the two maps ι_1 and ι_2 become equal because p becomes invertible. ∎

We have to distinguish the easier case, in which all objects $F_\bullet()$ for 0-ary morphisms F are cofibrant. Otherwise we define a category $\mathcal{D}^{\mathrm{Cof}}[\widetilde{(\mathcal{W}^{\mathrm{Cof}}}{)^{-1}}]$ in which we set $\mathrm{Hom}_F(;\mathcal{F}) \coloneqq \mathrm{Hom}_{\mathcal{D}_S[\mathcal{W}_S^{-1}]}(QF_\bullet();\mathcal{F})$ for all $\mathcal{F}$, where F is a 0-ary morphism with domain S. Composition is given as follows: For a morphism $f \in \mathrm{Hom}_G(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$ with cofibrant $\mathcal{E}_i$ and $\mathcal{F}$ and $\xi : QF_\bullet() \to \mathcal{E}_i$, we define the composition $\xi \circ f$ as the following composition

$$\begin{tikzcd}
\mathcal{E}_1 \arrow[dr, dash] & & & & \\
\vdots^{\widehat{i}} & \text{cocart} \arrow[r] & (F \circ G)_\bullet(\mathcal{E}_2, .\overset{\widehat{i}}{.}., \mathcal{E}_n) \arrow[r, "\sim"] & G_\bullet(\mathcal{E}_1, \dots, F_\bullet(), \dots, \mathcal{E}_n) & \arrow[l] \\
\mathcal{E}_n \arrow[ur, dash] & & & &
\end{tikzcd}$$

$$\longleftarrow G_\bullet(\mathcal{E}_1, \dots, QF_\bullet(), \dots, \mathcal{E}_n) \longrightarrow G_\bullet(\mathcal{E}_1, \dots, \mathcal{E}_n) \longrightarrow \mathcal{F}.$$

One checks that the so-defined composition is associative and independent of the choice of the push-forwards.

5.1.14. Lemma. *If the object $F_\bullet()$ is cofibrant for every 0-ary morphism F then the natural functor*

$$\mathcal{D}^{\mathrm{Cof}}[(\mathcal{W}^{\mathrm{Cof}})^{-1}] \to \mathcal{D}[\mathcal{W}^{-1}]$$

is an equivalence of categories.

Otherwise it is, if we replace $\mathcal{D}^{\mathrm{Cof}}[(\mathcal{W}^{\mathrm{Cof}})^{-1}]$ by $\mathcal{D}^{\mathrm{Cof}}[\widetilde{(\mathcal{W}^{\mathrm{Cof}}}{)^{-1}}]$.

Proof. The inclusion $\mathcal{D}^{\mathrm{Cof}} \to \mathcal{D}$ induces a functor $\Xi : \mathcal{D}^{\mathrm{Cof}}[(\mathcal{W}^{\mathrm{Cof}})^{-1}] \to \mathcal{D}[\mathcal{W}^{-1}]$. If the objects $F_\bullet()$ are not cofibrant then Ξ may be modified to a functor

$$\mathcal{D}^{\mathrm{Cof}}[\widetilde{(\mathcal{W}^{\mathrm{Cof}}}{)^{-1}}] \to \mathcal{D}[\mathcal{W}^{-1}]$$

as follows: a 0-ary morphism $QF_\bullet() \to \mathcal{F}$ is mapped to the composition

$$\circ \xrightarrow{\text{cocart}} F_\bullet() \longleftarrow QF_\bullet() \longrightarrow \mathcal{F}$$

in $\mathcal{D}[\mathcal{W}^{-1}]$.

We now specify a functor Φ in the other direction. Φ maps an object X to a cofibrant replacement QX. For $n \geq 1$, a morphism $f \in \mathrm{Hom}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$ over F is mapped to the following morphism. Composing with the morphisms $Q\mathcal{E}_i \to \mathcal{E}_i$, we get a morphism $f' \in \mathrm{Hom}(Q\mathcal{E}_1, \dots, Q\mathcal{E}_n; \mathcal{F})$ or equivalently a morphism $X_i \to F^{\bullet,i}(Q\mathcal{E}_1, \overset{\widehat{i}}{\dots}, Q\mathcal{E}_n; \mathcal{F})$. Now choose a lift (dotted arrow in the diagram)

$$\begin{tikzcd} & F^{\bullet,i}(Q\mathcal{E}_1, \overset{\widehat{i}}{\dots}, Q\mathcal{E}_n; Q\mathcal{F}) \arrow[d] \\ Q\mathcal{E}_i \arrow[r] \arrow[ur, dotted] & F^{\bullet,i}(Q\mathcal{E}_1, \overset{\widehat{i}}{\dots}, Q\mathcal{E}_n; \mathcal{F}) \end{tikzcd}$$

which exists because the vertical map is again a trivial fibration (because all the $Q\mathcal{E}_i$ are cofibrant). The resulting map in $\mathrm{Hom}(Q\mathcal{E}_1, \dots, Q\mathcal{E}_n; P\mathcal{F})$ is actually well-defined in $\mathcal{D}^{\mathrm{Cof}}[(\mathcal{W}^{\mathrm{Cof}})^{-1}]$. Note that, two different lifts are indeed left homotopic because $Q\mathcal{E}_i$ is cofibrant [Hov99, Proposition 1.2.5. (iv)], and therefore also the two morphisms in $\mathrm{Hom}(Q\mathcal{E}_1, \dots, Q\mathcal{E}_n; Q\mathcal{F})$ become equal in $\mathcal{D}^{\mathrm{Cof}}[(\mathcal{W}^{\mathrm{Cof}})^{-1}]$ by Lemma 5.1.13. From this it follows that Φ is indeed a functor on n-ary morphisms for $n \geq 1$.

For $n = 0$, a morphism $f \in \mathrm{Hom}(;\mathcal{F})$ over F corresponds to a morphism $F_\bullet() \to \mathcal{F}$. If $F_\bullet()$ is cofibrant, this morphism lifts (again uniquely up to right homotopy) to a morphism $F_\bullet() \to Q\mathcal{F}$, i.e. to a morphism in $\mathrm{Hom}_F(;Q\mathcal{F})$. If $F_\bullet()$ is not cofibrant then the composition lifts to a morphism: $QF_\bullet() \to Q\mathcal{F}$ which is defined to be the image of Φ. Furthermore Φ is inverse to Ξ up to isomorphism. ∎

5.1.15. Lemma. *Right homotopic morphisms become equal in* $\mathcal{D}^{\mathrm{Cof,Fib}}[(\mathcal{W}^{\mathrm{Cof,Fib}})^{-1}]$.

Proof. The assertion follows from the fact that there exists a path object

$$\begin{tikzcd} \mathcal{F} & \mathcal{F}' \arrow[l, shift right, "\mathrm{pr}_1"'] \arrow[l, shift left, "\mathrm{pr}_2"] & \mathcal{F} \arrow[l, "i"'] \end{tikzcd}$$

where $\mathcal{F}'$ is cofibrant and fibrant which realizes the right homotopy [Hov99, Proposition 1.2.6.]. This uses that all sources are cofibrant and the domain is fibrant. The two morphisms pr_1 and pr_2 become equal because i becomes invertible. ∎

5.1.16. Lemma. *The functor* $\mathcal{D}^{\mathrm{Fib,Cof}}[(\mathcal{W}^{\mathrm{Fib,Cof}})^{-1}] \to \mathcal{D}^{\mathrm{Cof}}[(\mathcal{W}^{\mathrm{Cof}})^{-1}]$ *and the functor* $\mathcal{D}^{\mathrm{Fib,Cof}}\widetilde{[(\mathcal{W}^{\mathrm{Fib,Cof}})^{-1}]} \to \mathcal{D}^{\mathrm{Cof}}\widetilde{[(\mathcal{W}^{\mathrm{Cof}})^{-1}]}$*, respectively, are equivalences of multicategories.*

Proof. The proof is analogous to that of Lemma 5.1.14 but with some minor changes which require, in particular, the chosen order of restriction to cofibrant and fibrant objects. We specify again a functor Φ in the other direction. On objects, Φ maps $\mathcal{E}$ to a fibrant replacement $R\mathcal{E}$. Note that $R\mathcal{E}$ is still cofibrant. A morphism $f \in \mathrm{Hom}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$ over F corresponds to a morphism $F_\bullet(\mathcal{E}_1, \dots, \mathcal{E}_n) \to \mathcal{F}$. Now choose a lift (dotted arrow in the diagram)

$$\begin{tikzcd}
F_\bullet(\mathcal{E}_1,\dots,\mathcal{E}_n) \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] & R\mathcal{F} \\
F_\bullet(R\mathcal{E}_1,\mathcal{E}_2,\dots,\mathcal{E}_n) \arrow[d] & & \\
\vdots \arrow[d] & & \\
F_\bullet(R\mathcal{E}_1,R\mathcal{E}_2,\dots,R\mathcal{E}_n) \arrow[uuurr, dotted] & &
\end{tikzcd}$$

It exists because the vertical maps are again trivial cofibrations (because all the $\mathcal{E}_i$ and $R\mathcal{E}_i$ are cofibrant). The lift is well-defined in $\mathcal{D}^{\mathrm{Cof,Fib}}[(\mathcal{W}^{\mathrm{Cof,Fib}})^{-1}]$, because two lifts in the triangle above become right homotopic (because $R\mathcal{F}$ is fibrant by [Hov99, Proposition 1.2.5. (iv)]). Therefore also the corresponding morphisms in $\mathrm{Hom}(R\mathcal{E}_1,\dots,R\mathcal{E}_n;R\mathcal{F})$ become equal in $\mathcal{D}^{\mathrm{Cof,Fib}}[(\mathcal{W}^{\mathrm{Cof,Fib}})^{-1}]$ by the previous lemma. It follows that Φ is indeed a functor which is inverse to the inclusion up to isomorphism. ∎

5.1.17. LEMMA. *If the objects $F_\bullet()$ for all 0-ary morphisms in $\mathcal{S}$ are cofibrant then the natural functor*

$$\mathcal{D}^{\mathrm{Fib,Cof}}[(\mathcal{W}^{\mathrm{Fib,Cof}})^{-1}] \to \mathcal{D}^{\mathrm{Fib,Cof}}/\sim$$

is an isomorphism of categories. Otherwise it is, if we modify the 0-ary morphisms as before.

PROOF. The natural functor $\mathcal{D}^{\mathrm{Fib,Cof}} \to \mathcal{D}^{\mathrm{Fib,Cof}}/\sim$ takes weak equivalences to isomorphisms [Hov99, Proposition 1.2.8] and has the universal property of $\mathcal{D}^{\mathrm{Fib,Cof}}[(\mathcal{W}^{\mathrm{Fib,Cof}})^{-1}]$ by the same argument as in [Hov99, Proposition 1.2.9]. ∎

PROOF OF PROPOSITION 5.1.9. The previous lemmas showed that $\mathcal{D}[\mathcal{W}^{-1}]$ is equivalent to $\mathcal{D}^{\mathrm{Fib,Cof}}/\sim$ if all objects of the form $F_\bullet()$ are cofibrant, or if we replace the second multicategory by $\widetilde{\mathcal{D}^{\mathrm{Fib,Cof}}}/\sim$, where we set $\mathrm{Hom}_{F,\widetilde{\mathcal{D}^{\mathrm{Fib,Cof}}}/\sim}(;\mathcal{F}) \coloneqq \mathrm{Hom}_{\mathcal{D}_S[\mathcal{W}_S^{-1}]}(F_\bullet(),\mathcal{F})$ for all 0-ary morphism F in $\mathcal{S}$ with domain S and for every $\mathcal{F}\in\mathcal{D}_S$.

It remains to show that the functor

$$p/\sim\colon\ \mathcal{D}^{\mathrm{Fib,Cof}}/\sim\ \to \mathcal{S}$$

is bifibered if all $F_\bullet()$ are cofibrant or otherwise bifibered for $n\geq 1$ (i.e. (co)Cartesian n-ary morphisms exist for $n\geq 1$). (The modification $\widetilde{\mathcal{D}^{\mathrm{Fib,Cof}}}/\sim$ has been constructed in such a way that it has 0-ary coCartesian morphisms.)

We show that $p/\sim$ is opfibered, the other case being similar. Let F be a multimorphism in $\mathcal{S}$ with codomain S. The set $\mathrm{Hom}_F(\mathcal{E}_1,\dots,\mathcal{E}_n;\mathcal{F})$ modulo right homotopy is in bijection with the set $\mathrm{Hom}_{\mathcal{D}_Y}(F_\bullet(\mathcal{E}_1,\dots,\mathcal{E}_n),\mathcal{F})$ modulo right homotopy. Since $\mathcal{F}$ is fibrant, the latter set is the same as $\mathrm{Hom}_{\mathcal{D}_S}(R(F_\bullet(\mathcal{E}_1,\dots,\mathcal{E}_n)),\mathcal{F})$ modulo right homotopy. Hence morphisms in $\mathrm{Hom}_F(\mathcal{E}_1,\dots,\mathcal{E}_n;\mathcal{F})$ uniquely decompose as the composition

$$\begin{tikzcd}
\mathcal{E}_1 \arrow[dr, dash] & & & \\
\vdots & \mathrm{cocart} \arrow[r] & F_\bullet(\mathcal{E}_1,\dots,\mathcal{E}_n) \arrow[r] & R(F_\bullet(\mathcal{E}_1,\dots,\mathcal{E}_n)) \\
\mathcal{E}_n \arrow[ur, dash] & & &
\end{tikzcd}$$

followed by a morphism in $\mathrm{Hom}_{\mathcal{D}_S}(R(F_\bullet(\mathcal{E}_1,\dots,\mathcal{E}_n)),\mathcal{F})$ modulo right homotopy. More generally, by the same argument, a morphism in some $\mathrm{Hom}_{GF}(\mathcal{F}_1,\dots,\mathcal{E}_1,\dots,\mathcal{E}_n,\dots,\mathcal{F}_m;\mathcal{G})$, where G is another multimorphism in $\mathcal{S}$, modulo right homotopy factorizes uniquely into the above composition followed by a morphism in

$$\mathrm{Hom}_G(\mathcal{F}_1,\dots,R(F_\bullet(\mathcal{E}_1,\dots,\mathcal{E}_n)),\dots,\mathcal{F}_m;\mathcal{G})$$

modulo right homotopy.

It remains to see that the push-forward in $\mathcal{D}[\mathcal{W}^{-1}]$ corresponds to the left derived functor of $F_\bullet$. For any objects $\mathcal{E}_1,\dots,\mathcal{E}_n$ the composition

$$\begin{tikzcd}
RQ\mathcal{E}_1 \arrow[dr, dash] & & & \\
\vdots & \mathrm{cocart} \arrow[r] & F_\bullet(RQ\mathcal{E}_1,\dots,RQ\mathcal{E}_n) \arrow[r] & R(F_\bullet(RQ\mathcal{E}_1,\dots,RQ\mathcal{E}_n)) \\
RQ\mathcal{E}_n \arrow[ur, dash] & & &
\end{tikzcd}$$

is a coCartesian morphism lying over F, with domains isomorphic to the $\mathcal{E}_i$.

However, the object $R(F_\bullet(RQ\mathcal{E}_1,\dots,RQ\mathcal{E}_n))$ is isomorphic to the value of the left derived functor of $F_\bullet$ at $\mathcal{E}_1,\dots,\mathcal{E}_n$. ∎

5.1.18. We now focus on the left case. If I is a inverse diagram, we proceed to construct a model structure on the fibers of the bifibration of multicategories (cf. 5.1.6):

$$\mathrm{Hom}(I,\mathcal{D})\to\mathrm{Hom}(I,\mathcal{S})=\mathbb{S}(I).$$

This model structure is an analogue of the classical Reedy model structure and it has the property that pull-backs w.r.t. diagrams and the corresponding relative left Kan extension functors form a Quillen adjunction.

Let $I\in\mathrm{Dir}$ and let $F:I\to\mathcal{S}$ be a functor. We will define a model-category structure

$$(\mathcal{D}_F,\mathrm{Cof}_F,\mathrm{Fib}_F,\mathcal{W}_F)$$

where $\mathcal{D}_F$ is the fiber of $\mathrm{Hom}(I, \mathcal{D})$ over F and where $\mathcal{W}_F$ is the class of morphisms which are element-wise in the corresponding $\mathcal{W}_{F(i)}$.

For any $G \in \mathcal{D}_F$, and for any $i \in I$, we define a **latching object**

$$L_i G := \mathrm{colim}_{I_i} \{F(\alpha)_\bullet G(j)\}_{\alpha : j \to i},$$

Here I_i is the full subcategory of $I \times_{/I} i$ consisting of all objects except id_i. We have a canonical morphism

$$L_i G \to G(i)$$

in $\mathcal{D}_{F(i)}$. We define Fib_F to be the class of morphisms which are element-wise in the corresponding $\mathrm{Fib}_{F(i)}$. We define Cof_F to be the class of morphisms $G \to H$ such that for any $i \in I$ the induced morphism δ in the diagram

$$\begin{tikzcd} L_i G \arrow[r] \arrow[d] & L_i H \arrow[d] & \\ G(i) \arrow[r] & \text{push-out} \arrow[r, dotted, "\delta"] & H(i) \end{tikzcd}$$

belongs to $\mathrm{Cof}_{F(i)}$. We call a morphism $G \to H$ in Cof_F temporarily an **acyclic cofibration** if δ is, in addition, a weak equivalence. The proof that this yields a model-category structure is completely analogous to the classical case [Hov99, §5.1] (if $\mathcal{S}$ is trivial). We need a couple of lemmas:

5.1.19. LEMMA. *The class of cofibrations (resp. acyclic cofibrations) in $\mathcal{D}_F$ consists precisely of the morphisms which have the left lifting property w.r.t. trivial fibrations (resp. fibrations). These are stable under retracts.*

PROOF. This is shown as in the classical case: we first prove that acyclic cofibrations have the lifting property w.r.t. fibrations. Consider a diagram

$$\begin{tikzcd} G_1 \arrow[r] \arrow[d, "\alpha"] & H_1 \arrow[d, "\beta"] \\ G_2 \arrow[r] & H_2 \end{tikzcd}$$

where α is an acyclic cofibration and β is a fibration. We proceed by induction on n and assume that for all $i \in I$ with $\nu(i) < n$ a map $G_2(i) \to H_1(i)$ has been constructed such that it is a lift in the above diagram, evaluated at i. For each i of degree n consider the following diagram (where the morphism $L_i G_2 \to L_i H_1 \to H_1(i)$ is formed using the already constructed lifts):

$$\begin{tikzcd} G_1(i) \coprod_{L_i G_1} L_i G_2 \arrow[r] \arrow[d, "\alpha'(i)"] & H_1(i) \arrow[d, "\beta(i)"] \\ G_1(i) \arrow[r] & H_2(i) \end{tikzcd}$$

Here $\alpha'(i)$ is a trivial $\mathrm{Cof}_{F(i)}$-cofibration by definition, and $\beta(i)$ is a $\mathrm{Fib}_{F(i)}$-fibration by definition. Hence a lift exists. In the same way the statement for cofibrations and for trivial fibrations is shown. Closure under retracts is left as an exercise for the reader. The assertion that the class of acyclic cofibrations (resp. cofibrations) is *precisely* the class of morphisms that have the left lifting property w.r.t. fibrations (resp. trivial fibrations) follows from the retract argument as for model categories. ■

5.1.20. LEMMA. *There exists a functorial factorization of morphisms in $\mathcal{D}_F$ into a fibration followed by an acyclic cofibration and into a trivial fibration followed by a cofibration.*

PROOF. We show this again by induction on n. We do the first case, the other being similar. Let $G \to K$ a morphism in $\mathcal{D}_F$. We have the following diagram:

$$\begin{tikzcd} L_iG \arrow[r] \arrow[d] & L_iH \arrow[rr] \arrow[d] & & L_iK \arrow[d] \\ G(i) \arrow[r] & G(i) \coprod_{L_iG} L_iH \arrow[r, dotted] & H(i) \arrow[r, dotted] & K(i) \end{tikzcd}$$

Here the top row is constructed using the already defined factorizations. The object $H(i)$ and the dotted maps are constructed as the factorization in the model category $\mathcal{D}_{F(i)}$ into a trivial $\mathrm{Cof}_{F(i)}$-cofibration followed by $\mathrm{Fib}_{F(i)}$-fibration. ■

5.1.21. LEMMA. *The classes of cofibrations, acyclic cofibrations, fibrations and weak equivalences are stable under composition.*

PROOF. This follows from the characterization by a lifting property (resp. by definition for the case of the weak equivalences). ■

5.1.22. LEMMA. *Acyclic cofibrations are precisely the trivial cofibrations.*

PROOF. We begin by showing that an acyclic cofibration is a weak equivalence. It suffices to show that in the diagram

$$\begin{tikzcd} L_iG \arrow[r] \arrow[d] & L_iH \arrow[d] \\ G(i) \arrow[r] & H(i) \end{tikzcd}$$

the top horizontal morphism is a trivial cofibration. Then the lower horizontal morphism is a composition of two trivial cofibrations and hence is a weak equivalence. The top morphism is indeed a trivial cofibration because the morphism of I_i-diagrams (cf. 5.1.18)

$$\{F(\alpha)_\bullet G(j)\}_{\alpha:j\to i} \to \{F(\alpha)_\bullet G(j)\}_{\alpha:j\to i}$$

is a trivial cofibration in the classical sense (i.e. over the constant diagram over I_i with value $F(i)$) because of Lemmas 5.1.23 and 5.1.24.

In the other direction, let f be a trivial cofibration and factor it as $f = p \circ g$, where g is an acyclic cofibration and p is a fibration. It follows that p is a weak-equivalence. Now construct a lift in the diagram

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, "f"'] & H \arrow[d, "p"] \\ G \arrow[r, equal] & G \end{tikzcd}$$

This shows that f is a retract of g, and hence is an acyclic cofibration as well. ∎

5.1.23. Lemma. *For each (1-ary) morphism of diagrams $f \in \mathrm{Hom}_{\mathcal{S}}(X_1; Y)$ there is an associated push-forward and an associated pull-back, defined by taking the point-wise push-forward $f_\bullet$, and point-wise pull-back $f^\bullet$ (cf. 5.1.6), respectively. The push-forward $f_\bullet$ respects the classes of cofibrations and acyclic cofibrations. The pull-back $f^\bullet$ respects the classes of fibrations and trivial fibrations.*

Proof. It suffices (by the lifting property) to show that $f^\bullet$ respects fibrations and trivial fibrations. This is clear because they are defined point-wise. ∎

A posteriori this will say that the pair of functors $f^\bullet, f_\bullet$ form a Quillen adjunction between the corresponding model categories (cf. 5.1.28).

5.1.24. Lemma. *Let $i \in I$ be an object, let $\iota : I_i \to I$ be the corresponding latching category with its natural functor to I, and let $F_i := \iota^* F : I_i \to \mathcal{S}$ be the restriction of F to I_i. The pull-back $\iota^* : \mathcal{D}_F \to \mathcal{D}_{F_i}$ respects cofibrations and acyclic cofibrations.*

Proof. It is easy to see that the pull-back induces an isomorphism of the corresponding latching objects as in the classical case. ∎

5.1.25. Corollary. *The structure constructed in 5.1.18 defines a model category.*

Proof. This follows from the previous Lemmas. ∎

5.1.26. Proposition. *For any morphism of inverse diagrams $\alpha : I \to J$, and for any functor $F : J \to \mathcal{S}$, the functor*

$$\alpha^* : \mathcal{D}_F \to \mathcal{D}_{\alpha^* F}$$

has a left adjoint $\alpha_!^F$. The pair $\alpha^*, \alpha_!^F$ define a Quillen adjunction.

Proof. That the two functors define a Quillen adjunction is clear once we have shown that $\alpha_!$ exists because α^* preserves fibrations and weak equivalences. Let G be an object of $\mathcal{D}_F$. We define

$$(\alpha_! G)(j) := \mathrm{colim}_{I \times_{/J} j} \mathbb{S}(\mu)_\bullet \iota_j^* G.$$

For each morphism $\mu : j \to j'$ we get a functor

$$\widetilde{\mu} : I \times_{/J} j \to I \times_{/J} j'$$

and hence an induced morphism

$$F(\mu)_\bullet \mathbb{S}(\mu)_\bullet \iota_j^* G \to \widetilde{\mu}^* \mathbb{S}(\mu')_\bullet \iota_{j'}^*.$$

Since $F(\mu)_\bullet$ commutes with colimits we get a morphism

$$F(\mu)_\bullet \mathrm{colim}_{I\times_{/J}j}\mathbb{S}(\mu)_\bullet \iota_j^* G \to \mathrm{colim}_{I\times_{/J}j'}\mathbb{S}(\mu')_\bullet \iota_{j'}^*$$

which we define to be $(\alpha_! G)(\mu)$. We now proceed to show that the functor we have constructed is indeed adjoint to α^*. A morphism $\mu : G \to \alpha^* H$ is given by a collection of maps $a(i) : G(i) \to H(\alpha(i))$ for all objects $i \in I$, subject to the condition that the diagram

$$\begin{tikzcd}
F(\alpha(\lambda))_\bullet G(i) \arrow[r, "F(\alpha(\lambda))_\bullet a(i)"] \arrow[d, "\overline{G(\lambda)}"] & F(\alpha(\lambda))_\bullet H(\alpha(i)) \arrow[d, "\overline{H(\alpha(\lambda))}"] \\
G(i') \arrow[r, "a(i')"] & H(\alpha(i'))
\end{tikzcd}$$

commutes for each morphism $\lambda : i \to i'$ in I. For each $j \in J$ and morphism $\mu : \alpha(i) \to j$ we get a morphism

$$\overline{H(\mu)} \circ (F(\mu)_\bullet a(i)) : F(\mu)_\bullet G(i) \to H(j)$$

and therefore for fixed j a morphism

$$\mathrm{colim}_{I\times_{/J}j}\mathbb{S}(\mu)_\bullet \iota_j^* G \to H(j).$$

One checks that this yields a morphism $\alpha_! G \to H$. On the other hand, let $b : \alpha_! G \to H$ be a morphism given by

$$b(j) : \mathrm{colim}_{I\times_{/J}j}\mathbb{S}(\mu)_\bullet \iota_j^* G \to H(j)$$

or equivalently for all $\mu : \alpha(i) \to j$ by morphisms

$$F(\mu)_\bullet G(i) \to H(j).$$

In particular, if μ is the identity of $\alpha(i)$, we get morphisms

$$G(i) \to H(\alpha(i))$$

which constitute a morphism of diagrams $G \to \alpha^* H$. One checks that these associations are inverse to each other. ∎

5.1.27. LEMMA. *Let $\alpha : I \to J$ be a morphism of inverse diagrams and let j be an object of J. The functor $\iota_j^* : \mathcal{D}_I \to \mathcal{D}_{I\times_{/J}j}$ respects cofibrations and trivial cofibrations.*

PROOF. This follows easily from the fact that ι_j induces a canonical identification

$$I_i = (I \times_{/J} j)_\mu$$

for any $\mu = (i, \alpha(i) \to j)$. For this implies that we have a canonical isomorphism $L_i G \cong L_\mu \iota_j^* G$. ∎

5.1.28. LEMMA. *The bifibration of multicategories, defined in 5.1.6*

$$\mathrm{Hom}(I,\mathcal{D}) \to \mathrm{Hom}(I,\mathcal{S}) = \mathbb{S}(I)$$

equipped with the model-category structures constructed in 5.1.18 is a bifibration of multi-model-categories in the sense of 5.1.3.

PROOF. First for each multi-morphism of diagrams $f \in \mathrm{Hom}_{\mathcal{S}}(X_1,\dots,X_n;Y)$ we have to see that the push-forward and the various pull-backs form a Quillen adjunction in n variables. The case $n = 1$ has been treated above. We only work out the case $n = 2$, the proof for higher n being similar. It suffices to check the following: for any cofibration $\mathcal{E}_1 \to \mathcal{E}_1'$ and for any fibration $\mathcal{F} \to \mathcal{F}'$ the dotted induced morphism in the following diagram

$$\begin{tikzcd} f^{\bullet,2}(\mathcal{E}_1';\mathcal{F}) \arrow[r, dotted] & \text{pull-back} \arrow[r] \arrow[d] & f^{\bullet,2}(\mathcal{E}_1';\mathcal{F}') \arrow[d] \\ & f^{\bullet,2}(\mathcal{E}_1;\mathcal{F}) \arrow[r] & f^{\bullet,2}(\mathcal{E}_1;\mathcal{F}') \end{tikzcd}$$

is a fibration. Since fibrations are defined point-wise and fibered products are computed point-wise, we have only to see that the assertion holds point-wise. Now $\mathcal{F} \to \mathcal{F}'$ is a point-wise fibration and $\mathcal{E}_1 \to \mathcal{E}_1'$ is a Reedy cofibration, so by the reasoning in the proof of Lemma 5.1.22 it is in particular a point-wise cofibration. Hence the assertion holds because of the assumption that $\mathcal{D} \to \mathcal{S}$ is a bifibration of multi-model-categories (5.1.3). The requested property for the 0-ary push-forward is easier and is left to the reader. ∎

5.1.29. PROPOSITION. *The functor $\mathbb{D}(I) \to \mathbb{S}(I)$ defined in 5.1.2 is a bifibration of multicategories whose fiber over $S \in \mathbb{S}(I)$ is equivalent to $\mathcal{D}_S[\mathcal{W}_S^{-1}]$. The pull-back and push-forward functors are given by the left derived functors of $f_\bullet$, and by the right derived functors of $f^{\bullet,j}$, respectively.*

PROOF. We have seen in 5.1.28 that the fibers of $\mathrm{Hom}(I,\mathcal{D}) \to \mathbb{S}(I)$ are a bifibration of multi-model-categories in the sense of 5.1.3. Therefore by Proposition 5.1.9 we get that $\mathbb{D}(I) \to \mathbb{S}(I)$ are bifibered multicategories with the requested properties. ∎

PROOF OF THEOREM 5.1.5. (Der1) and (Der2) for $\mathbb{D}$ and $\mathbb{S}$ are obvious.

(FDer0 left) and the first part of (FDer0 right) follow from Theorem 5.1.29.

(FDer3 left) follows from 5.1.26.

(FDer4 left): By construction of $\alpha_!$ the natural base-change

$$\mathrm{colim}\ \mathbb{S}(\mu)_\bullet \iota_j^* G \to j^* \alpha_! G \tag{19}$$

is an isomorphism for the non-derived functors. For the derived functors the same follows because all functors in the equation respect cofibrations and trivial cofibrations and all functors which have to be derived in (19) are left Quillen functors and hence can be derived by composing them with cofibrant replacement.

(FDer3 right) and (FDer4 right) are shown precisely the same way.

(FDer5 left): Fixing a morphism $f \in \mathrm{Hom}(S_1, \dots, S_n; T)$ in $\mathcal{S}$ and objects $\mathcal{E}_2, \dots, \mathcal{E}_n$ over $S_2, \dots, S_n$ we have by Theorem 5.1.29 a push-forward functor

$$\begin{array}{rcl} \mathbb{D}(I \times J)_{p^*S_1} & \to & \mathbb{D}(I \times J)_{p^*T} \\ \mathcal{E}_1 & \mapsto & (p^*f)_\bullet(\mathcal{E}_1, p^*\mathcal{E}_2, \dots, p^*\mathcal{E}_n) \end{array}$$

(we denote it with the same letter as the underived version) which, by (FDer0 left), defines a morphism of pre-derivators

$$\mathbb{D}_{S_1} \to \mathbb{D}_T.$$

We first show that it preserves colimits, i.e. that for $p : J \to \cdot$ we have that for all $\mathcal{E}_1 \in \mathcal{D}_{p^*S_1}(I \times J)$ the natural morphism

$$f_\bullet(p_*\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_n) \to p_*(p^*f)_\bullet(\mathcal{E}_1, p^*\mathcal{E}_2, \cdots, p^*\mathcal{E}_n)$$

(where we wrote p also for the projection $p : I \times J \to I$) is an isomorphism. This is the same as showing that

$$p^*f^{1,\bullet}(\mathcal{E}_1, \dots, \mathcal{E}_n) \to (p^*f)^{1,\bullet}(p^*\mathcal{E}_1, \dots, p^*\mathcal{E}_n)$$

is an isomorphism. This follows from Lemma 5.1.8 because it suffices to check this for the underived functors. Now let $\alpha : I \to J$ be an opfibration. To show that

$$f_\bullet(\alpha_*\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_n) \to \alpha_*(\alpha^*f)_\bullet(\mathcal{E}_1, \alpha^*\mathcal{E}_2, \dots, \alpha^*\mathcal{E}_n)$$

is an isomorphism we may show this point-wise. Indeed, after applying j^* we get

$$(j^*f)_\bullet(j^*\alpha_*\mathcal{E}_1, j^*\mathcal{E}_2, \dots, j^*\mathcal{E}_n) \to j^*\alpha_*(\alpha^*f)_\bullet(\mathcal{E}_1, \alpha^*\mathcal{E}_2, \dots, \alpha^*\mathcal{E}_n)$$

$$(j^*f)_\bullet(p_*\iota_j^*\mathcal{E}_1, j^*\mathcal{E}_2, \dots, j^*\mathcal{E}_n) \to p_*\iota_j^*(\alpha^*f)_\bullet(\mathcal{E}_1, \alpha^*\mathcal{E}_2, \dots, \alpha^*\mathcal{E}_n)$$

where $\iota_j : I_j \to I$ is the inclusion of the fiber. Note that the commutative diagram

$$\begin{tikzcd} I_j \arrow[r, "\iota_j"] \arrow[d, "p"'] & I \arrow[d, "\alpha"] \\ j \arrow[r] & J \end{tikzcd}$$

is homotopy exact by Lemma 2.3.23, 2. because α is an opfibration. Finally we get the morphism

$$(j^*f)_\bullet(p_*\iota_j^*\mathcal{E}_1, j^*\mathcal{E}_2, \dots, j^*\mathcal{E}_n) \to p_*(j^*f)_\bullet(\iota_j^*\mathcal{E}_1, p^*j^*\mathcal{E}_2, \dots, p^*j^*\mathcal{E}_n)$$

which is an isomorphism by the above reasoning.

Since we have bifibrations, by Lemma 2.3.9 the full content of (FDer0 right) follows from (FDer5 left) while (FDer5 right) follows from (FDer0 left). ∎

A. Fibrations of categories

A.1. Grothendieck (op)fibrations.

A.1.1. [right] Let $p : \mathcal{D} \to \mathcal{S}$ be a functor, and let $f : S \to T$ be a morphism in $\mathcal{S}$. A morphism $\xi : \mathcal{E}' \to \mathcal{E}$ over f is called **Cartesian** if the composition with ξ induces an isomorphism

$$\mathrm{Hom}_g(\mathcal{F}, \mathcal{E}') \cong \mathrm{Hom}_{f \circ g}(\mathcal{F}, \mathcal{E})$$

for any morphism $g : R \to S$ in $\mathcal{S}$ and for every $\mathcal{F} \in \mathcal{D}_R$.

The functor p is called a **fibration** if for any $f : S \to T$ and for every object $\mathcal{E}$ in $\mathcal{D}_T$ (i.e. such that $p(\mathcal{E}) = T$) there exists a Cartesian morphism $\mathcal{E}' \to \mathcal{E}$.

A.1.2. [left] Let $p : \mathcal{D} \to \mathcal{S}$ be a functor, and let $f : S \to T$ be a morphism in $\mathcal{S}$. A morphism $\xi : \mathcal{E} \to \mathcal{E}'$ over f is called **coCartesian** if the composition with ξ induces an isomorphism

$$\mathrm{Hom}_g(\mathcal{E}', \mathcal{F}) \cong \mathrm{Hom}_{g \circ f}(\mathcal{E}, \mathcal{F})$$

for any morphism $g : T \to U$ in $\mathcal{S}$ and for every $\mathcal{F} \in \mathcal{D}_U$.

The functor p is called an **opfibration** if for any $f : S \to T$ and for every object $\mathcal{E}$ in $\mathcal{D}_S$ there exists a coCartesian morphism $\mathcal{E} \to \mathcal{E}'$.

A.1.3. The functor p is an opfibration if and only if $p^{\mathrm{op}} : \mathcal{D}^{\mathrm{op}} \to \mathcal{S}^{\mathrm{op}}$ is a fibration. We say that p is a **bifibration** if is a fibration and an opfibration at the same time. If $p : \mathcal{D} \to \mathcal{S}$ is a fibration we may choose an associated pseudo-functor, i.e. to each $S \in \mathcal{S}$ we associate the category $\mathcal{D}_S$, and to each $f : S \to T$ we associate a push-forward functor

$$f_\bullet : \mathcal{D}_S \to \mathcal{D}_T$$

characterized by the fact that for each $\mathcal{E}$ in $\mathcal{D}_S$ there is a coCartesian morphism $\mathcal{E} \to f_\bullet \mathcal{E}$. The same holds similarly for an opfibration with the pull-back $f^\bullet$ instead of the push-forward. If the functor p is a bifibration, $f_\bullet$ is left adjoint to $f^\bullet$. Situations where this is the opposite can be modeled by considering bifibrations $\mathcal{D} \to \mathcal{S}^{\mathrm{op}}$.

A.2. Fibered multicategories and the six functors.

A.2.1. We give a definition of a (op)fibered multicategory. This is a straightforward generalization of the notion of (op)fibered category given in Section A.1. It is very useful to encode the formalism of the six functors. Details about (op)fibered multicategories can be found, for instance, in [Her00, Her04].

The reader should keep in mind that a multicategory abstracts the properties of multilinear maps, and indeed every monoidal category gives rise to a multicategory setting

$$\mathrm{Hom}(A_1, \dots, A_n; B) := \mathrm{Hom}((A_1 \otimes (A_2 \otimes (\cdots))), B). \tag{20}$$

A.2.2. Definition. *A* **multicategory** $\mathcal{D}$ *consists of*

a class of objects $\mathrm{Ob}(\mathcal{D})$*;*

for every $n \in \mathbb{Z}_{\geq 0}$*, and objects* $X_1, \dots, X_n, Y$ *a class*

$$\mathrm{Hom}(X_1, \dots, X_n; Y);$$

a composition law, i.e. for objects $X_1, \dots, X_n$, $Y_1, \dots, Y_m$, Z *and for each integer* $1 \leq i \leq m$ *a map:*

$$\mathrm{Hom}(X_1, \dots, X_n; Y_i) \times \mathrm{Hom}(Y_1, \dots, Y_m; Z) \to \mathrm{Hom}(Y_1, \dots, Y_{i-1}, X_1, \dots, X_n, Y_{i+1}, \dots, Y_m; Z);$$

for each object $X \in \mathrm{Ob}(\mathcal{D})$ *an identity* $\mathrm{id}_X \in \mathrm{Hom}(X; X)$*;*

satisfying associativity and identity laws. The composition w.r.t. independent slots is commutative, i.e. for $1 \leq i < j \leq m$ *if* $f \in \mathrm{Hom}(X_1, \dots, X_n; Y_i)$ *and* $f' \in \mathrm{Hom}(X_1', \dots, X_k'; Y_j)$ *and* $g \in \mathrm{Hom}(Y_1, \dots, Y_m; Z)$ *we have*

$$(g \circ_i f) \circ_{j+n-1} f' = (g \circ_j f') \circ_i f. \tag{21}$$

A symmetric (braided) multicategory is given by an action of the symmetric (braid) groups, i.e. isomorphisms

$$\alpha : \mathrm{Hom}(X_1, \dots, X_n; Y) \to \mathrm{Hom}(X_{\alpha(1)}, \dots, X_{\alpha(n)}; Y)$$

for $\alpha \in S_n$ *(resp.* $\alpha \in B_n$*) forming an action which is compatible with composition in the obvious way (substitution of strings in the braid group).*

In some references the composition is defined in a seemingly more general way; in the presence of identities these descriptions are, however, equivalent. We denote a multimorphism in $f \in \mathrm{Hom}(X_1, \dots, X_n; Y)$ also by

$$\begin{tikzcd} X_1 \arrow[dr, no head] & & \\ \vdots & f \arrow[r] & Y \\ X_n \arrow[ur, no head] & & \end{tikzcd}$$

for $n \geq 1$, or by

$$\circ \xrightarrow{f} Y$$

for $n = 0$.

We will also need the definition of a strict 2-multicategory which is a multicategory enriched in (usual) categories:

A.2.3. DEFINITION. *A (strict)* **2-multicategory** $\mathcal{D}$ *consists of*

- *a class of objects* $\mathrm{Ob}(\mathcal{D})$*;*
- *for every* $n \in \mathbb{Z}_{\geq 0}$*, and objects* $X_1, \dots, X_n, Y$ *a category*

$$\mathrm{Hom}(X_1, \dots, X_n; Y);$$

- *a composition, i.e. for objects* $X_1, \dots, X_n$*,* $Y_1, \dots, Y_m$*,* Z *and for each integer* $1 \leq i \leq m$ *a functor:*

$$\mathrm{Hom}(X_1, \dots, X_n; Y_i) \times \mathrm{Hom}(Y_1, \dots, Y_m; Z) \to \mathrm{Hom}(Y_1, \dots, Y_{i-1}, X_1, \dots, X_n, Y_{i+1}, \dots, Y_m; Z);$$

- *for each object* $X \in \mathrm{Ob}(\mathcal{D})$ *an identity object* id_X *in the category* $\mathrm{Hom}(X; X)$*;*

satisfying strict associativity and identity laws. The composition w.r.t. independent slots is commutative, i.e. for $1 \leq i < j \leq m$ *if* $f \in \mathrm{Hom}(X_1, \dots, X_n; Y_i)$ *and* $f' \in \mathrm{Hom}(X'_1, \dots, X'_k; Y_j)$ *and* $g \in \mathrm{Hom}(Y_1, \dots, Y_m; Z)$ *we have*

$$(g \circ_i f) \circ_{j+n-1} f' = (g \circ_j f') \circ_i f. \tag{22}$$

A symmetric (braided) 2-multicategory is given by an action of the symmetric (braid) groups, i.e. isomorphisms of categories

$$\alpha : \mathrm{Hom}(X_1, \dots, X_n; Y) \to \mathrm{Hom}(X_{\alpha(1)}, \dots, X_{\alpha(n)}; Y)$$

for $\alpha \in S_n$ *(resp.* $\alpha \in B_n$*) forming an action which is strictly compatible with composition in the obvious way (substitution of strings in the braid group).*

The 1-composition of 2-morphisms is (as for usual 2-categories) determined by the following whiskering operations: Let $f, g \in \mathrm{Hom}(X_1, \dots, X_n; Y_i)$ be 1-morphisms and let $h \in \mathrm{Hom}(Y_1, \dots, Y_m; Z)$ be a 1-morphism and let $\mu : f \Rightarrow g$ be a 2-morphism in $\mathrm{Mor}(\mathrm{Hom}(X_1, \dots, X_n; Y_i))$. Then we define

$$h * \mu := \mathrm{id}_h \cdot \mu$$

where the right hand side is the image of the morphism $\mathrm{id}_h \times \mu$ under the composition functor. Similarly we define $\mu * h$ for $\mu : f \Rightarrow g$ with $f, g \in \mathrm{Hom}(Y_1, \dots, Y_m; Z)$ and $h \in \mathrm{Hom}(X_1, \dots, X_n; Y_i)$.

A.2.4. We leave it to the reader to state the obvious definition of a functor between multicategories. Similarly there is a definition of a **opmulticategory**, in which we have classes

$$\mathrm{Hom}(X; Y_1, \dots, Y_n)$$

and similar data. For a multicategory $\mathcal{D}$ we get a natural opmulticategory $\mathcal{D}^{\mathrm{op}}$ by reversing the arrows.

The trivial category $\{\cdot\}$ is considered as a multicategory setting all $\mathrm{Hom}(\cdot,\dots,\cdot;\cdot)$ to the 1-element set. It is the final object in the "category" of multicategories.

To clarify the precise relation between multicategories and monoidal categories we have to define Cartesian and coCartesian morphisms. It turns out that we can actually give a definition which is a common generalization of coCartesian morphisms in opfibered categories and the morphisms expressing the existence of a tensor product:

A.2.5. Definition. *Consider a functor of multicategories $p:\mathcal{D}\to\mathcal{S}$. We call a morphism*

$$\xi\in\mathrm{Hom}(X_1,\dots,X_n;Y)$$

in $\mathcal{D}$ **coCartesian** *w.r.t. p, if for all $Y_1,\dots,Y_m,Z$ with $Y_i=Y$, and for all*

$$f\in\mathrm{Hom}(p(Y_1),\dots,p(Y_m);p(Z))$$

the map

$$\begin{array}{rcl}\mathrm{Hom}_f(Y_1,\dots,Y_m;Z) & \to & \mathrm{Hom}_{f\circ p(\xi)}(Y_1,\dots,Y_{i-1},X_1,\dots,X_n,Y_{i+1},\dots,Y_m;Z)\\ \alpha & \mapsto & \alpha\circ\xi\end{array}$$

is bijective. We call a morphism

$$\xi\in\mathrm{Hom}(X_1,\dots,X_n;Y)$$

in $\mathcal{D}$ **Cartesian** *w.r.t. p at the i-th slot, if for all $Z_1,\dots,Z_m$, and for all $f\in\mathrm{Hom}(p(Z_1),\dots,p(Z_m);p(X_i))$ the map*

$$\mathrm{Hom}_f(Z_1,\dots,Z_m;X_i)\to\mathrm{Hom}_{p(\xi)\circ f}(X_1,\dots,X_{i-1},Z_1,\dots,Z_m,X_{i+1},\dots,X_n;Y).$$

$$\alpha\mapsto\xi\circ\alpha$$

is bijective.

The functor $p:\mathcal{D}\to\mathcal{S}$ is called an **opfibered multicategory** *if for every $g\in\mathrm{Hom}(S_1,\dots,S_n;T)$ in $\mathcal{S}$, and for every collection of objects X_i with $p(X_i)=S_i$ there is some object Y over T and some coCartesian morphism $\xi\in\mathrm{Hom}(X_1,\dots,X_n;Y)$ such that $p(\xi)=g$.*

The functor $p:\mathcal{D}\to\mathcal{S}$ is called a **fibered multicategory** *if for every $1\le j\le n$, for each $g\in\mathrm{Hom}(S_1,\dots,S_n;T)$ in $\mathcal{S}$, for every collection of objects X_i for $i\neq j$ with $p(X_i)=S_i$, and for every Y over T, there is some object X_j and some Cartesian morphism w.r.t. the j-th slot $\xi\in\mathrm{Hom}(X_1,\dots,X_n;Y)$ such that $p(\xi)=g$.*

The functor $p:\mathcal{D}\to\mathcal{S}$ is called a **bifibered multicategory** *if it is both fibered and opfibered.*

A **morphism of (op)fibered multicategories** *is a commutative diagram of functors*

$$\begin{tikzcd}\mathcal{D}_1 \arrow[r, "F"] \arrow[d] & \mathcal{D}_2 \arrow[d] \\ \mathcal{S}_1 \arrow[r, "G"] & \mathcal{S}_2\end{tikzcd}$$

such that F maps (co)Cartesian morphisms to (co)Cartesian morphisms.

It turns out that the composition of Cartesian morphisms is Cartesian (and similarly for coCartesian morphisms if they are composed w.r.t. the right slot)[13].

A.2.6. Lemma.

1. *An opfibered multicategory $p : \mathcal{D} \to \{\cdot\}$ is a monoidal category defining $X \otimes Y$ to be the target of a coCartesian arrow from the pair X, Y over the unique morphism in $\mathrm{Hom}(\cdot, \cdot; \cdot)$ of the final multicategory $\{\cdot\}$.*

   Conversely any monoidal category gives rise to an opfibered multicategory $p : \mathcal{D} \to \{\cdot\}$ via (20). A multicategory $\mathcal{D}$ is a closed *category if and only if it is fibered over $\{\cdot\}$. In particular, the fibers of an (op)fibered multicategory $p : \mathcal{D} \to \mathcal{S}$ are always closed/monoidal in the following sense: given any functor of multicategories[14] $x : \{\cdot\} \to \mathcal{S}$, the category $\mathcal{D}_x$ of objects over x is closed/monoidal.*

2. *Given (op)fibered multicategories $p : \mathcal{C} \to \mathcal{D}$ and $q : \mathcal{D} \to \mathcal{E}$ also the composition $q \circ p$ is an (op)fibered multicategory. In particular, if we have an opfibered multicategory $p : \mathcal{C} \to \mathcal{S}$ and if $\mathcal{S} \to \{\cdot\}$ is opfibered (i.e. $\mathcal{S}$ is monoidal) then also $\mathcal{C} \to \{\cdot\}$ is opfibered (i.e. $\mathcal{C}$ is monoidal). The same holds dually. A morphism α is (co)Cartesian for $q \circ p$ if and only if α is (co)Cartesian for p and $p(\alpha)$ is (co)Cartesian for q.*

Similarly, the unit 1 is just the target of a coCartesian morphism in $\mathrm{Hom}(;1)$ which exists by definition (the existence is also required for the empty set of objects).

The second part of the lemma encapsulates the distinction between internal and external tensor product in a four (or six) functor context, see A.2.17.

A.2.7. Let $\mathcal{D}, \mathcal{S}$ be (usual) multicategories. More generally any opfibered multicategory $\mathcal{D} \to \mathcal{S}$ gives rise to a pseudo-functor of 2-multicategories

$$\mathcal{S} \to \mathcal{MCAT}^{2-\mathrm{op}}$$

where $\mathcal{MCAT}$ is the 2-multi"category" of categories, whose objects are categories and the morphism categories are defined to be:

$$\mathrm{Hom}_{\mathcal{MCAT}}(\mathcal{C}_1, \dots, \mathcal{C}_n; \mathcal{D}) := \mathrm{Fun}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n, \mathcal{D})$$

Here by a pseudo-functor $\Psi : \mathcal{S} \to \mathcal{T}$, where $\mathcal{T}$ is a 2-multicategory, we understand the obvious generalization of the usual concept of a pseudo-functor. This means that for

[13] As with fibered categories there are weaker notions of Cartesian which still uniquely determine a Cartesian morphism (up to isomorphism) from given objects over a given multimorphism, however, do not imply stability under composition. Similarly for coCartesian morphisms.

[14] This specifies also morphisms in $\mathrm{Hom}(\underbrace{X, \dots, X}_{n}; X)$, for all n, compatible with composition.

each $f \in \mathrm{Hom}_{\mathcal{S}}(S_1,\dots,S_n;T)$ we are given a functor $\Psi(f) \in \mathrm{Hom}(\Psi(S_1),\dots,\Psi(S_n);T)$ and for each composition $g\cdot f$ a natural isomorphism

$$\Psi_{f,g} : \Psi(g)\Psi(f) \Rightarrow \Psi(g\cdot f) \tag{23}$$

satisfying the usual relation for composable morphisms f, g and h:

$$(\Psi(h) * \Psi_{f,g})\Psi_{gf,h} = (\Psi_{g,h} * \Psi(f))\Psi_{f,hg}.$$

This definition generalizes readily to the case in which also $\mathcal{S}$ is a 2-multicategory, the only modification being that, on morphisms, we are given *functors*

$$\mathrm{Hom}_{\mathcal{S}}(S_1,\dots,S_n;T) \to \mathrm{Hom}_{\mathcal{T}}(\Psi(S_1),\dots,\Psi(S_n);\Psi(T))$$

and the 2-morphisms (23) have to be functorial in f and g.

A.2.8. Translated back to the language of fibrations we arrive at the following definition: see A.2.9.

First note that the definition of coCartesian morphism (cf. A.2.5) may be stated in the following way: A morphism

$$\xi \in \mathrm{Hom}(X_1,\dots,X_n;Y)$$

in $\mathcal{D}$ coCartesian w.r.t. p, if for all $Y_1,\dots,Y_m, Z$ with $Y_i = Y$ the diagram of sets

$$\begin{tikzcd}
\mathrm{Hom}(Y_1,\dots,Y_m;Z) \arrow[r, "\circ\xi"] \arrow[d] & \mathrm{Hom}(Y_1,\dots,Y_{i-1},X_1,\dots,X_n,Y_{i+1},\dots,Y_m;Z) \arrow[d] \\
\mathrm{Hom}(p(Y_1),\dots,p(Y_m);p(Z)) \arrow[r, "\circ p(\xi)"] & \mathrm{Hom}(p(Y_1),\dots,p(Y_{i-1}),p(X_1),\dots,p(X_n),p(Y_{i+1}),\dots,p(Y_m);p(Z))
\end{tikzcd}$$

is Cartesian.

A.2.9. Definition. *Let $p : \mathcal{D} \to \mathcal{S}$ be a strict functor of 2-multicategories. A 1-morphism*

$$\xi \in \mathrm{Hom}_f(\mathcal{E}_1,\dots,\mathcal{E}_n;\mathcal{F})$$

in $\mathcal{D}$ over $f \in \mathrm{Hom}(S_1,\dots,S_n;T)$ is called **coCartesian** *w.r.t. p, if for all $\mathcal{F}_1,\dots,\mathcal{F}_m,\mathcal{G}$ with $\mathcal{F}_i = \mathcal{F}$ the diagram of categories*

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{F}_1,\dots,\mathcal{F}_m;\mathcal{G}) \arrow[r, "\circ\xi"] \arrow[d] & \mathrm{Hom}(\mathcal{F}_1,\dots,\mathcal{F}_{i-1},\mathcal{E}_1,\dots,\mathcal{E}_n,\mathcal{F}_{i+1},\dots,\mathcal{F}_m;\mathcal{G}) \arrow[d] \\
\mathrm{Hom}(T_1,\dots,T_m;U) \arrow[r, "\circ p(\xi)"] & \mathrm{Hom}(T_1,\dots,T_{i-1},S_1,\dots,S_n,T_{i+1},\dots,T_m;U)
\end{tikzcd}$$

is Cartesian (where we set $T_k := p(\mathcal{F}_k)$ and $U := p(\mathcal{G})$).

The strict functor p is called a **2-opfibered 1-opfibered multicategory (with 1-categorical fibers)** *if for all $f \in \mathrm{Hom}(S_1, \dots, S_n; T)$ and objects $\mathcal{E}_1, \dots, \mathcal{E}_n$ with $p(\mathcal{E}_i) = S_i$ there is a coCartesian 1-morphism with domains $\mathcal{E}_1, \dots, \mathcal{E}_n$. Furthermore the functors*

$$\mathrm{Hom}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F}) \to \mathrm{Hom}(p(\mathcal{E}_1), \dots, p(\mathcal{E}_n); p(\mathcal{F}))$$

have to be opfibrations (with discrete fibers*) and composition has to be a morphism of opfibrations.*

The functor $p : \mathcal{D} \to \mathcal{S}$ is called a **2-fibered 1-opfibered multicategory (with 1-categorical fibers)** *if for every $1 \le j \le n$ and for each $g \in \mathrm{Hom}(S_1, \dots, S_n; T)$ in $\mathcal{S}$, and for each collection of objects $\mathcal{E}_i$ for $i \neq j$ with $p(\mathcal{E}_i) = S_i$, and for each $\mathcal{F}$ over T, there is some object $\mathcal{E}_j$ and some Cartesian 1-morphism w.r.t. the j-th slot $\xi \in \mathrm{Hom}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$ with $p(\xi) = g$. Furthermore the functors*

$$\mathrm{Hom}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F}) \to \mathrm{Hom}(p(\mathcal{E}_1), \dots, p(\mathcal{E}_n); p(\mathcal{F}))$$

have to be opfibrations (with discrete fibers*) and composition has to be a morphism of opfibrations.*

There are several other, partly more general, definitions of an (op)fibration with 2-categorical fibers which we will not need in this section. We will discuss them in a subsequent article [Hör16].

Note that for (op)fibrations *with 1-categorical fibers* the composition is automatically a morphism of opfibrations.

A.2.10. An opfibration $p : \mathcal{D} \to \mathcal{S}$ of 2-multicategories *with 1-categorical fibers* is in particular (forgetting 2-morphisms) a usual opfibration. The additional datum, which makes it into a 2-opfibration is the following: For each 2-morphism $\mu : f \Rightarrow g$ in $\mathcal{S}$ a map of sets (the 2-push-forward):

$$p^*(\mu) : \mathrm{Hom}_f(X_1, \dots, X_n; Y) \to \mathrm{Hom}_g(X_1, \dots, X_n; Y)$$

such that

$$\begin{aligned} p^*(\mathrm{id}_f)(\beta) &= \beta \\ p^*(\mu) \circ p^*(\nu) &= p^*(\mu \circ \nu) \end{aligned}$$

(composition of 2-coCartesian morphisms are 2-coCartesian) and

$$\begin{aligned} p^*(p(\alpha) * \mu)(\alpha \circ \xi) &= \alpha \circ (p^*(\mu)(\xi)) \\ p^*(\mu * p(\alpha))(\xi \circ \alpha) &= (p^*(\mu)(\xi)) \circ \alpha \end{aligned}$$

(1-composition maps coCartesian morphisms to coCartesian morphisms).

The 2-morphisms between α and β in $\mathcal{D}$ lying over f, resp. g in $\mathcal{S}$ can be reconstructed from the datum p^* as

$$\mathrm{Hom}(\alpha, \beta) = \{\mu \in \mathrm{Hom}(f, g) \mid p^*(\mu)(\alpha) = \beta\}.$$

A.2.11. With a pseudo-functor

$$\Psi : \mathcal{S} \to \mathcal{MCAT}^{2-\mathrm{op}}$$

where $\mathcal{S}$ is any strict 2-multicategory, we associate the opfibration

$$\mathcal{D}_\Psi \to \mathcal{S}.$$

The objects of $\mathcal{D}_\Psi$ are pairs

$$(S, X \in \Psi(S))$$

in which S is an object of $\mathcal{S}$. The 1-morphisms

$$\begin{tikzcd}
(S_1, X_1) \arrow[dr, no head] & & \\
\vdots & (f,\alpha) \arrow[r] & (T, Y) \\
(S_n, X_n) \arrow[ur, no head] & &
\end{tikzcd}$$

are pairs of (multi)morphisms

$$f \in \mathrm{Hom}(S_1, \dots, S_n; T) \qquad \alpha : \Psi(f)(X_1, \dots, X_n) \to Y.$$

The 2-morphisms

$$(f,\alpha) \Rightarrow (f', \alpha')$$

are given by 2-morphisms $\mu : f \Rightarrow f'$ such that $\alpha \circ (\Psi(\mu)(X)) = \alpha'$.

The fiber[15] of $\mathcal{D}_\Psi \to \mathcal{S}$ over S is actually a 1-category, namely precisely the category $\Psi(S)$.

We have the following generalization of Lemma A.2.6, 2.:

A.2.12. LEMMA. *Given (op)fibered 2-multicategories $p : \mathcal{C} \to \mathcal{D}$ and $q : \mathcal{D} \to \mathcal{E}$ then the composition $q \circ p$ is an (op)fibered 2-multicategory as well. A 1-morphism α is (co)Cartesian for $q \circ p$ if and only if α is (co)Cartesian for p and $p(\alpha)$ is (co)Cartesian for q.*

A.2.13. EXAMPLE. Let $\mathcal{S}$ be a usual category. Then $\mathcal{S}$ may be turned into a symmetric multicategory by setting

$$\mathrm{Hom}(X_1, \dots, X_n; Y) := \mathrm{Hom}(X_1; Y) \times \cdots \times \mathrm{Hom}(X_n; Y).$$

If $\mathcal{S}$ has coproducts, then $\mathcal{S}$ (with this multicategory structure) is opfibered over $\{\cdot\}$. Let $p : \mathcal{D} \to \mathcal{S}$ be an opfibered (usual) category. Any object X induces a canonical functor of

[15] i.e. the 2-category of those objects, morphisms, and 2-morphisms which Ψ maps to S, id_S, and $\mathrm{id}_{\mathrm{id}_S}$, respectively

multicategories $x : \{\cdot\} \to \mathcal{S}$ with image X, hence the fibers of an opfibered multicategory $p : \mathcal{D} \to \mathcal{S}$ are monoidal and the datum p is equivalent to giving a monoidal structure on the fibers such that the push-forwards $f_\bullet$ are monoidal functors and such that the compatibility morphisms between them are morphisms of monoidal functors. This is called a covariant monoidal pseudo-functor in [LH09, (3.6.7)].

A.2.14. Example. Let $\mathcal{S}$ be a usual category. Then $\mathcal{S}^{\mathrm{op}}$ may be turned into a symmetric multicategory (or equivalently $\mathcal{S}$ into a symmetric opmulticategory) by setting

$$\mathrm{Hom}(X_1, \dots, X_n; Y) := \mathrm{Hom}(Y; X_1) \times \cdots \times \mathrm{Hom}(Y; X_n).$$

If $\mathcal{S}$ has products then $\mathcal{S}^{\mathrm{op}}$ (with this multicategory structure) is opfibered over $\{\cdot\}$. Let $p : \mathcal{D} \to \mathcal{S}^{\mathrm{op}}$ be an opfibered (usual) category. Then an opfibered multicategory structure on p, w.r.t. this multicategory structure on $\mathcal{S}^{\mathrm{op}}$, is equivalent to a monoidal structure on the fibers such that pull-backs f^* (along morphisms in $\mathcal{S}$) are monoidal functors and such that the compatibility morphisms between them are morphisms of monoidal functors. This is called a contravariant monoidal pseudo-functor in [LH09, (3.6.7)].

A.2.15. Definition. *The point is that the notion of (op)fibered multicategory is* not restricted to *the situation of Examples A.2.13 and A.2.14. Let $\mathcal{S}$ be a category with fiber products and define $\mathcal{S}^{\mathrm{cor}}$, denoted the* **symmetric 2-multicategory of correspondences in** $\mathcal{S}$ *to be the symmetric 2-multicategory having the same objects as $\mathcal{S}$, and where the category of morphisms* $\mathrm{Hom}(S_1, \dots, S_n; T)$ *is the category of objects*

$$\begin{tikzcd}
 & & & A \arrow[llld, "g_1"'] \arrow[ld, "g_n"'] \arrow[rd, "f"] & \\
S_1 & \cdots & S_n & ; & T
\end{tikzcd}$$

and where the 2-morphisms $(A, f, g_1, \dots, g_n) \Rightarrow (A', f', g_1', \dots, g_n')$ are iso*morphisms $A \to A'$ compatible with f, f' and $g_1, g_1', \dots, g_n, g_n'$.*

Composition is given by:

$$\begin{tikzcd}
 & & & & & A \times_{Y_i} B \arrow[lld, "\mathrm{pr}_1"'] \arrow[rrd, "\mathrm{pr}_2"] & & & & & \\
 & & & A \arrow[llld] \arrow[ld] \arrow[rrrd] & & & & B \arrow[llld] \arrow[ld] \arrow[rd] \arrow[rrrd] & & & \\
X_1 & \cdots & X_n & ; & Y_1 & \cdots & Y_i & \cdots & Y_m & ; & Z
\end{tikzcd}$$

where strictly associative fiber products have been chosen in $\mathcal{S}$.

This 2-multicategory is representable (i.e. opfibered over $\{\cdot\}$), closed (i.e. fibered over $\{\cdot\}$) and self-dual, with tensor product and internal hom both *given by $\times$ and having as unit the final object of $\mathcal{S}$.*

A.2.16. Definition. *Let $\mathcal{S}$ be a category with fiber products. A* **(symmetric) six-functor-formalism** *on $\mathcal{S}$ is a 1-bifibered and 2-bifibered (symmetric) 2-multicategory with 1-categorical fibers*

$$p : \mathcal{D} \to \mathcal{S}^{\mathrm{cor}}.$$

A.2.17. We have a morphism of opfibered (over $\{\cdot\}$) symmetric multicategories $\mathcal{S}^{\mathrm{op}} \to \mathcal{S}^{\mathrm{cor}}$ where $\mathcal{S}^{\mathrm{op}}$ is equipped with the symmetric multicategory structure as in A.2.14. However there is no reasonable morphism of opfibered multicategories $\mathcal{S} \to \mathcal{S}^{\mathrm{cor}}$. (There is no compatibility involving only '$\otimes$' and '!'.) From a six-functor-formalism we get operations g_*, g^* as the pull-back and the push-forward along the correspondence

$$\begin{tikzcd} & X \arrow[dl, "g"'] \arrow[dr, equal] & \\ Y & ; & X \end{tikzcd}$$

We get $f^!$ and $f_!$ as the pull-back and the push-forward along the correspondence

$$\begin{tikzcd} & X \arrow[dl, equal] \arrow[dr, "f"] & \\ X & ; & Y \end{tikzcd}$$

We get the monoidal product $A \otimes B$ for objects A, B above X as the target of any Cartesian morphism $\otimes$ over the correspondence

$$\xi_X = \left(\begin{tikzcd} & & X \arrow[dll, equal] \arrow[dl, equal] \arrow[dr, equal] & \\ X & X & ; & X \end{tikzcd} \right)$$

Alternatively, we have

$$A \otimes B = \Delta^*(A \boxtimes B)$$

where Δ^* is the push-forward along the correspondence

$$\left(\begin{tikzcd} & X \arrow[dl, "\Delta"'] \arrow[dr, equal, "f"] & \\ X \times X & ; & X \end{tikzcd} \right)$$

induced by the canonical $\xi_X \in \mathrm{Hom}(X, X; X)$, and where $\boxtimes$ is the absolute monoidal product which exists because by Lemma A.2.12 the composition $\mathcal{D} \to \{\cdot\}$ is opfibered as well, i.e. $\mathcal{D}$ is monoidal.

A.2.18. It is easy to derive from the definition of bifibered multicategory over $\mathcal{S}^{\mathrm{cor}}$ that the absolute monoidal product $A \boxtimes B$ can be reconstructed from the fiber-wise product as $\mathrm{pr}_1^* A \otimes \mathrm{pr}_2^* B$ on $X \times Y$, whereas the absolute $\mathbf{HOM}(A, B)$ is given by $\mathcal{HOM}(\mathrm{pr}_1^* A, \mathrm{pr}_2^! B)$ on $X \times Y$. In particular $DA := \mathbf{HOM}(A, 1)$ is given by $\mathcal{HOM}(A, \pi^! 1)$ for $\pi : X \to \cdot$ being the final morphism.

A.2.19. Lemma. *Given a symmetric six-functor-formalism on $\mathcal{S}$*

$$p : \mathcal{D} \to \mathcal{S}^{\mathrm{cor}}$$

for the six operations as extracted in A.2.17 there exist naturally the following compatibility isomorphisms:

	left adjoints	*right adjoints*
$(*,*)$	$(fg)^* \xrightarrow{\sim} g^* f^*$	$f_* g_* \xrightarrow{\sim} (fg)_*$
$(!,!)$	$(fg)_! \xrightarrow{\sim} f_! g_!$	$g^! f^! \xrightarrow{\sim} (fg)^!$
$(!,*)$	$g^* f_! \xrightarrow{\sim} F_! G^*$	$G_* F^! \xrightarrow{\sim} f^! g_*$
$(\otimes,*)$	$f^*(-\otimes-) \xrightarrow{\sim} f^* - \otimes f^* -$	$f_* \mathcal{HOM}(f^*-,-) \xrightarrow{\sim} \mathcal{HOM}(-, f_*-)$
$(\otimes,!)$	$f_!(-\otimes f^*-) \xrightarrow{\sim} (f_!-)\otimes -$	$\mathcal{HOM}(f_!-,-) \xrightarrow{\sim} f_*\mathcal{HOM}(-, f^!-)$
		$f^!\mathcal{HOM}(-,-) \xrightarrow{\sim} \mathcal{HOM}(f^*-, f^!-)$
$(\otimes,\otimes)$	$(-\otimes-)\otimes - \xrightarrow{\sim} -\otimes(-\otimes-)$	$\mathcal{HOM}(-,\mathcal{HOM}(-,-)) \xrightarrow{\sim} \mathcal{HOM}(-\otimes-,-)$

Here f, g, F, G are morphisms in $\mathcal{S}$ which, in the $(!,*)$-row, are related by the Cartesian *diagram*

$$\begin{tikzcd} \cdot \arrow[r, "G"] \arrow[d, "F"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

A.2.20. Remark. In the right column the corresponding adjoint natural transformations are listed. In each case the left hand side natural isomorphism determines the right hand side one and conversely. (In the $(\otimes,!)$-case there are 2 versions of the commutation between the right adjoints; in this case any of the three isomorphisms determines the other two.) The $(!,*)$-isomorphism (between left adjoints) is called **base change**, the $(\otimes,!)$-isomorphism is called the **projection formula**, and the $(*,\otimes)$-isomorphism is usually part of the definition of a **monoidal functor**. The $(\otimes,\otimes)$-isomorphism is the associativity of the tensor product and part of the definition of a monoidal category. The $(*,*)$-isomorphism, and the $(!,!)$-isomorphism express that the corresponding functors arrange as a pseudo-functor with values in categories.

Proof. The existence of all isomorphisms is a consequences of the fact that the composition of coCartesian morphisms is coCartesian. For example, the projection formula $(\otimes,!)$ is derived from the following composition in $\mathcal{S}^{\mathrm{cor}}$:

$$\left(\begin{tikzcd} & & Y \arrow[dll, equal] \arrow[dl, equal] \arrow[dr, equal] & \\ Y & Y & ; & Y \end{tikzcd}\right) \circ_1 \left(\begin{tikzcd} & X \arrow[dl, equal] \arrow[dr, "f"] & \\ X & ; & Y \end{tikzcd}\right) \cong \left(\begin{tikzcd} & & X \arrow[dll, equal] \arrow[dl, "f"] \arrow[dr, "f"] & \\ X & Y & ; & Y \end{tikzcd}\right),$$

where $\circ_1$ means that we compose w.r.t. the first slot.

The "monoidality of f^*" $(*, \otimes)$ is derived from the following composition in $\mathcal{S}^{\mathrm{cor}}$:

$$\left(\begin{tikzcd}[ampersand replacement=\&] \& X \arrow[dl, "f"'] \arrow[dr, equal] \& \\ Y \& ; \& X \end{tikzcd}\right) \circ \left(\begin{tikzcd}[ampersand replacement=\&] \& \& Y \arrow[dll, equal] \arrow[dl, equal] \arrow[dr, equal] \& \\ Y \& Y \& ; \& Y \end{tikzcd}\right) \cong \left(\begin{tikzcd}[ampersand replacement=\&] \& \& X \arrow[dll, "f"'] \arrow[dl, "f"] \arrow[dr, equal] \& \\ Y \& Y \& ; \& X \end{tikzcd}\right).$$

Base change $(!, *)$ is derived from:

$$\left(\begin{tikzcd}[ampersand replacement=\&] \& X \arrow[dl, "g"'] \arrow[dr, equal] \& \\ A \& ; \& X \end{tikzcd}\right) \circ \left(\begin{tikzcd}[ampersand replacement=\&] \& Y \arrow[dl, equal] \arrow[dr, "f"] \& \\ Y \& ; \& A \end{tikzcd}\right) \cong \left(\begin{tikzcd}[ampersand replacement=\&] \& Y \times_A X \arrow[dl, "F"'] \arrow[dr, "G"] \& \\ Y \& ; \& X \end{tikzcd}\right).$$

■

All compatibilities between these isomorphisms can be derived as well. Each of these compatibilities corresponds to an associativity relation in the 2-multicategory $\mathcal{S}^{\mathrm{cor}}$. One can also axiomatize the properties of the morphism $f_! \to f_*$ that often accompanies a six-functor-formalism. Can one give a finite list of compatibility diagrams from which all the others would follow?

A.2.21. The goal and motivation for this research is, as said in the introduction, to define (and to construct in reasonable contexts) a *derivator version* of a six-functor-formalism, i.e. a fibered multiderivator

$$\mathbb{D} \to \mathbb{S}^{\mathrm{cor}}$$

where $\mathbb{S}^{\mathrm{cor}}$ is the pre-2-multiderivator associated with the 2-category $\mathcal{S}^{\mathrm{cor}}$. We will give the definition of a pre-2-multiderivator and of a fibered derivator over such in subsequent articles [Hör16, Hör17a].

A.3. Localization of multicategories.

A.3.1. Proposition. *Let $\mathcal{D}$ be a (symmetric, braided) multicategory and let $\mathcal{W}$ be a subclass of 1-ary morphisms. Then there exists a (symmetric, braided) multicategory $\mathcal{D}[\mathcal{W}^{-1}]$, which is not necessarily locally small, together with a functor $\iota : \mathcal{D} \to \mathcal{D}[\mathcal{W}^{-1}]$ of (symmetric, braided) multicategories with the property that $\iota(w)$ is an isomorphism for all $w \in \mathcal{W}$ and which is universal w.r.t. this property.*

Proof (sketch). This construction is completely analogous to the construction for usual categories. Morphisms $\mathrm{Hom}(X_1, \dots, X_n; Y)$ are formal compositions of i-ary morphisms in $\mathcal{D}$ and formal inverses of morphisms in $\mathcal{W}$, for example:

$$\begin{tikzcd}
X_1 \arrow[dr, no head] & & & & & & \\
X_2 \arrow[r, no head] & f_1 \arrow[r] & \cdot & \cdot \arrow[l, "w_1"'] \arrow[ddr, no head] & & & \\
X_3 \arrow[ur, no head] & & & & & & \\
X_4 \arrow[rrrr, no head] & & & & f_3 \arrow[r] & \cdot & Y \arrow[l, "w_3"'] \\
X_5 \arrow[r, no head] & f_2 \arrow[r] & \cdot & \cdot \arrow[l, "w_2"'] \arrow[ur, no head] & & & \\
X_6 \arrow[ur, no head] & & & & & &
\end{tikzcd}$$

More precisely: Morphisms are defined to be the class of lists of n_i-ary morphisms $f_i \in \operatorname{Hom}(X_{i,1}, \dots, X_{i,n_i}; Y_i)$, morphisms $w_i : Y_i' \to Y_i$ in $\mathcal{W}$ and integers k_i as follows

$$(f_1, w_1), k_1, (f_2, w_2), k_2, \dots, k_{n-1}, (f_n, w_n)$$

such that $Y_i' = X_{i+1,k_i}$, modulo relations coming from composing at independent slots, commutative squares, and forcing the (id, w_i) to become the left and right inverse of (w_i, id). ■

References

[Ayo07a] Joseph Ayoub. Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. I. *Astérisque*, (314):x+466 pp., 2007.

[Ayo07b] Joseph Ayoub. Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. II. *Astérisque*, (315):vi+364 pp., 2007.

[BL94] Joseph Bernstein and Valery Lunts. *Equivariant sheaves and functors*, volume 1578 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1994.

[Cis03] D. C. Cisinski. Images directes cohomologiques dans les catégories de modèles. *Ann. Math. Blaise Pascal*, 10(2):195–244, 2003.

[Cis04] D. C. Cisinski. Le localisateur fondamental minimal. *Cah. Topol. Géom. Différ. Catég.*, 45(2):109–140, 2004.

[Cis08] D. C. Cisinski. Propriétés universelles et extensions de Kan dérivées. *Theory Appl. Categ.*, 20:No. 17, 605–649, 2008.

[GPS14] Moritz Groth, Kate Ponto, and Michael Shulman. Mayer-Vietoris sequences in stable derivators. *Homology Homotopy Appl.*, 16(1):265–294, 2014.

[Gro91] Alexander Grothendieck. Les dérivateurs. Edited by M. Künzer, J. Malgoire and G. Maltsiniotis, `https://www.imj-prg.fr/~georges.maltsiniotis/groth/Derivateurs.html`, 1991.

[Gro12] Moritz Groth. Monoidal derivators and additive derivators. arXiv: 1203.5071, 2012.

[Gro13] Moritz Groth. Derivators, pointed derivators and stable derivators. *Algebr. Geom. Topol.*, 13(1):313–374, 2013.

[GS12] Moritz Groth and Jan Stovicek. Tilting theory via stable homotopy theory. arXiv: 1401.6451, 2012.

[Hör16] F. Hörmann. Six-Functor-Formalisms and Fibered Multiderivators. arXiv: 1603.02146, 2016.

[Hör17a] F. Hörmann. Derivator Six-Functor-Formalisms — Definition and Construction I. arXiv: 1701.02152, 2017.

[Hör17b] F. Hörmann. Derivator Six-Functor-Formalisms — Construction II. in preparation, 2017.

[Hör17c] F. Hörmann. Enlargement of (fibered) derivators. arXiv: 1706.09692, 2017.

[Her00] Claudio Hermida. Representable multicategories. *Adv. Math.*, 151(2):164–225, 2000.

[Her04] Claudio Hermida. Fibrations for abstract multicategories. In *Galois theory, Hopf algebras, and semiabelian categories*, volume 43 of *Fields Inst. Commun.*, pages 281–293. 2004.

[Hov99] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[Kra10] Henning Krause. Localization theory for triangulated categories. In *Triangulated categories*, volume 375 of *London Math. Soc. Lecture Note Ser.*, pages 161–235. Cambridge Univ. Press, Cambridge, 2010.

[LH09] Joseph Lipman and Mitsuyasu Hashimoto. *Foundations of Grothendieck duality for diagrams of schemes*, volume 1960 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009.

[MW07] Ieke Moerdijk and Ittay Weiss. Dendroidal sets. *Algebr. Geom. Topol.*, 7:1441–1470, 2007.

[Nee01] Amnon Neeman. *Triangulated categories*, volume 148 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001.

[SGA72] *Théorie des topos et cohomologie étale des schémas. Tome 2.* Lecture Notes in Mathematics, Vol. 270. Springer-Verlag, Berlin-New York, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[SGA73] *Théorie des topos et cohomologie étale des schémas. Tome 3.* Lecture Notes in Mathematics, Vol. 305. Springer-Verlag, Berlin-New York, 1973. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat.

Mathematisches Institut, Albert-Ludwigs-Universität Freiburg
Eckerstraße 1, 79104 Freiburg, Germany

Email: `fritz.hoermann@math.uni-freiburg.de`

This article may be accessed at `http://www.tac.mta.ca/tac/`