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SPHERES AS FROBENIUS OBJECTS

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Abstract.

Following the pattern of the Frobenius structure usually assigned to the 1-dimensional sphere, we investigate the Frobenius structures of spheres in all other dimensions. Starting from dimension $d = 1$, all the spheres are commutative Frobenius objects in categories whose arrows are $(d+1)$-dimensional cobordisms. With respect to the language of Frobenius objects, there is no distinction between these spheres—they are all free of additional equations formulated in this language. The corresponding structure makes out of the 0-dimensional sphere not a commutative but a symmetric Frobenius object. This sphere is mapped to a matrix Frobenius algebra by a 1-dimensional topological quantum field theory, which corresponds to the representation of a class of diagrammatic algebras given by Richard Brauer.

1. Introduction

A Frobenius structure of one dimensional sphere S^1 is thoroughly investigated in a series of papers and books (see [6], [1], [14] and references therein). It is not the case that S^1 as a commutative Frobenius object of the category of 2-cobordisms is dealt with separately, but always in the context of two dimensional topological quantum field theories and in connection with Frobenius algebras. A Frobenius structure of spheres of other dimensions is investigated in [6] and [25].

It is straightforward to conclude that for every $d \geq 1$, the sphere S^{d-1} is a symmetric Frobenius object in the category $dCob$ of d-cobordisms. Also, it is straightforward to conclude that for every $d \geq 2$, the sphere S^{d-1} is a commutative Frobenius object in this category. (The author of [25] claims in Proposition 1 that every sphere is a commutative Frobenius object, which is not true for the case of S^0.) This means that increasing the dimension of a sphere from 0 to 1 produces a narrowing of the class of symmetric to the class of commutative Frobenius objects. Hence, it is natural to ask the following question:

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how many such steps are there, which produce new classes of Frobenius objects, induced by increasing the dimension of spheres?

The notion of commutative Frobenius object is not Post complete, i.e. adding a new equality between the canonical arrows (those relevant for the Frobenius structure) does not produce a collapse—some canonical arrows with the same source and target remain different. Hence, there are different classes of commutative Frobenius objects. If for a pair of different closed 2-manifolds, one forms the corresponding equality of canonical arrows, then all the commutative Frobenius objects satisfying this new equality form a proper subclass of commutative Frobenius objects. There are infinitely many such classes and [21, Proposition 2.4] provides a way to classify all the commutative Frobenius objects into classes corresponding to pairs of closed 2-manifolds.

For example, the class of commutative Frobenius objects satisfying the equality: comultiplication followed by multiplication equals identity (a *special* Frobenius algebra is such an object) is a proper subclass of the commutative Frobenius objects satisfying the equality: unit followed by comultiplication followed by multiplication followed by counit equals unit followed by counit. In terms of 2-manifolds, the latter class corresponds to the pair consisting of the torus $S^1 \times S^1$ and the sphere S^2.

The purpose of this paper is to show that no proper subclass of commutative Frobenius objects includes S^{d-1}, for $d \geq 2$. In order to do this, we construct a symmetric monoidal category K with a universal commutative Frobenius object, and show that for every $d \geq 2$, every symmetric monoidal functor from K to $dCob$ that maps this object to S^{d-1} is faithful.

The paper is organized so that some basic notions from category theory, which are necessary for understanding the results, are given in this introductory section. The category $dCobS$, whose objects are finite collections of $(d-1)$-dimensional spheres and arrows are equivalence classes of topological d-cobordisms, is introduced in Sections 2 and 9. This category is an ambient for a Frobenius object S^{d-1}. The category $dCobS$ is a full subcategory of the category $dCob$ whose objects are the $(d-1)$-dimensional closed topological manifolds.

In Section 3, we justify our restriction of objects of the category of d-cobordisms to collections of spheres. The results of this section heavily depend on some topological facts that are listed in Section 10. In Section 4, the pattern followed by us is explained in order to define a Frobenius structure of a sphere.

Section 5 is devoted to the case of S^0 and a classical result of Richard Brauer concerning a matrix representation of a class of diagrammatic algebras. This matrix representation is generalized by Došen and the second author (see [10] and [11]) to cover a category and not just a monoid of diagrams. This generalization is a one dimensional topological quantum field theory that maps S^0 to a matrix Frobenius algebra, which is usually the first example of a Frobenius algebra one finds in the literature.

Section 6 serves to define a symmetric strict monoidal category K with a universal commutative Frobenius object in it. This category is built out of a syntax material. Technical details of this construction are given in Section 11. A normal form for arrows of this category is given in Section 7.

The main result of Section 8 is that, for every $d \geq 2$, the category K is embeddable into $dCobS$. The image of the universal Frobenius object through this embedding is the sphere S^{d-1}. Such a result is a completeness result from the point of view of a logician and a coherence result from the point of view of a category theorist. It says that with respect to the language of Frobenius objects there is no distinction between spheres starting from dimension $d = 1$, i.e. they are all free of additional equations formulated in this language. This provides the answer to the question from the second paragraph.

Almost all the categories we deal with in this paper are *skeletal* in the sense that there are no two different isomorphic objects in them. Hence, all the monoidal categories mentioned below will be strict monoidal. In this way we lose some interesting combinatorics tied to associativity, but this enables us to emphasize the combinatorial structure we investigate.

A *strict monoidal* category is a triple $(\mathcal{M}, \otimes, e)$ consisting of a category $\mathcal{M}$, a bifunctor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, which is associative, and an object e, which is a left and right unit for $\otimes$. It is *symmetric* when there is a natural transformation τ with components

$$\tau_{A,B} : A \otimes B \to B \otimes A,$$

which means that for every pair of arrows $f : A \to A'$ and $g : B \to B'$ the diagram

$$\begin{tikzcd} A \otimes B \arrow[r, "\tau_{A,B}"] \arrow[d, "f \otimes g"'] & B \otimes A \arrow[d, "g \otimes f"] \\ A' \otimes B' \arrow[r, "\tau_{A',B'}"'] & B' \otimes A' \end{tikzcd}$$

commutes, this transformation is self-inverse, i.e. $\tau_{B,A} \circ \tau_{A,B} = \mathbf{1}_{A \otimes B}$, and it satisfies $\tau_{A \otimes B, C} = (\tau_{A,C} \otimes \mathbf{1}_B) \circ (\mathbf{1}_A \otimes \tau_{B,C})$ (cf. the equations (*str*), (*cat*), (*fun*), (*nat*), (*inv*) and (*hex*) of Section 11). The main example of symmetric strict monoidal categories in this paper are the categories $dCobS$ and $dCob$ introduced in Section 2.

A *monoid* $(M, \mu^\Diamond : M \otimes M \to M, \eta^\Diamond : e \to M)$ in a strict monoidal category $\mathcal{M}$ is a triple consisting of an object M of $\mathcal{M}$, and two arrows $\mu^\Diamond$ and $\eta^\Diamond$ of $\mathcal{M}$, such that the following diagrams commute

$$\begin{tikzcd} M \otimes M \otimes M \arrow[r, "\mu^\Diamond \otimes \mathbf{1}_M"] \arrow[d, "\mathbf{1}_M \otimes \mu^\Diamond"'] & M \otimes M \arrow[d, "\mu^\Diamond"] \\ M \otimes M \arrow[r, "\mu^\Diamond"'] & M \end{tikzcd} \qquad \begin{tikzcd} M \otimes M \arrow[dr, "\mu^\Diamond"'] & M \arrow[l, "\eta^\Diamond \otimes \mathbf{1}_M"'] \arrow[r, "\mathbf{1}_M \otimes \eta^\Diamond"] \arrow[d, "\mathbf{1}_M"] & M \otimes M \arrow[dl, "\mu^\Diamond"] \\ & M & \end{tikzcd}$$

(cf. the equations (*assoc*) and (*unit*) of Section 11).

A *comonoid* $(M, \mu^\Box : M \to M \otimes M, \eta^\Box : M \to e)$ in $\mathcal{M}$ is defined in a dual manner (cf. the equations (*coass*) and (*counit*) of Section 11). A *Frobenius object* in $\mathcal{M}$ is a quintuple

$$(M, \mu^\Diamond : M \otimes M \to M, \eta^\Diamond : e \to M, \mu^\Box : M \to M \otimes M, \eta^\Box : M \to e)$$

such that $(M, \mu^\Diamond, \eta^\Diamond)$ is a monoid, $(M, \mu^\Box, \eta^\Box)$ is a comonoid, and the following *Frobenius equations* (cf. the equations (*Frob*) of Section 11) hold

$$(\mathbf{1}_M \otimes \mu^\Diamond) \circ (\mu^\Box \otimes \mathbf{1}_M) = \mu^\Box \circ \mu^\Diamond = (\mu^\Diamond \otimes \mathbf{1}_M) \circ (\mathbf{1}_M \otimes \mu^\Box).$$

If $\mathcal{M}$ is symmetric, then a Frobenius object $(M, \mu^\Diamond, \eta^\Diamond, \mu^\Box, \eta^\Box)$ is *commutative* when

$$\mu^\Diamond \circ \tau_{M,M} = \mu^\Diamond \quad \text{and} \quad \tau_{M,M} \circ \mu^\Box = \mu^\Box$$

(cf. the equations (*com*) and (*cocom*) of Section 11, which are interderivable in the presence of other equations), and it is *symmetric* when

$$\eta^\Box \circ \mu^\Diamond \circ \tau_{M,M} = \eta^\Box \circ \mu^\Diamond \quad \text{and} \quad \tau_{M,M} \circ \mu^\Box \circ \eta^\Diamond = \mu^\Box \circ \eta^\Diamond.$$

A functor between two symmetric strict monoidal categories is *symmetric monoidal* when it preserves the symmetric monoidal structure on the nose, i.e. it maps tensor to tensor, unit to unit and symmetry to symmetry. According to our intention to work with strict monoidal structures, by a d-dimensional *topological quantum field theory* (dTQFT) we mean here a symmetric monoidal functor between the category $dCob$ and a strict monoidal category equivalent to the category of finite dimensional vector spaces over a chosen field. This strictification is supported by [19, Section XI.3, Theorem 1].

In some parts of the text, a natural number (finite ordinal) n is considered to be the set $\{0, \ldots, n-1\}$. It will be clear from the context when this is assumed.

2. The category $dCobS$

By a *d-manifold* we mean here a compact, oriented d-dimensional ∂-manifold (see Section 9). It is *closed* when its boundary is empty.

For $d \geq 1$ and $i \in \mathbf{N}$, let S_i be the $(d-1)$-dimensional sphere in $\mathbf{R}^d$ with the center $(3i, 0, \ldots, 0)$ and the radius 1. Assume that an orientation of S_0 is chosen, and that S_i is oriented so that the translation by the vector $(3i, 0, \ldots, 0)$ is an orientation preserving homeomorphism from S_0 to S_i. Let $\underline{0}$ denote the empty set, and for $n > 0$, let $\underline{n}$ denote the closed $(d-1)$-manifold $S_0 \cup \ldots \cup S_{n-1}$.

Let M be a d-manifold such that its boundary ∂M is a disjoint union of Σ_0 homeomorphic to $\underline{n}$ and Σ_1 homeomorphic to $\underline{m}$. We assume that the orientations of Σ_0 and Σ_1 are induced from the orientation of M (see Section 9).

Let $f_0 : \underline{n} \to M$ and $f_1 : \underline{m} \to M$ be two embeddings whose images are respectively Σ_0 and Σ_1. Assume that f_0 preserves, while f_1 reverses the orientation. The triple (M, f_0, f_1) is a *d-cobordism*, or simply a cobordism, from $\underline{n}$ to $\underline{m}$. We call Σ_0 and Σ_1, respectively, the *ingoing* and *outgoing* boundary of M in this cobordism.

Two d-cobordisms $K = (M, f_0, f_1)$ and $K' = (M', f'_0, f'_1)$ are *equivalent*, which we denote by $K \sim K'$, when there is an orientation preserving homeomorphism $F : M \to M'$ such that the following diagram commutes.

$$\begin{tikzcd}
& M \arrow[dd, "F"] & \\
\underline{n} \arrow[ur, "f_0"] \arrow[dr, "f_0'"'] & & \underline{m} \arrow[ul, "f_1"'] \arrow[dl, "f_1'"] \\
& M' &
\end{tikzcd}$$

The category $dCobS$ has $\underline{0}, \underline{1}, \underline{2}, \dots$ as objects and equivalence classes of d-cobordisms as arrows. The *identity arrow* from $\underline{n}$ to $\underline{n}$ in $dCobS$ is the equivalence class of the d-cobordism

$$\underline{n} \xrightarrow{\langle \mathbf{1}, c_0 \rangle} \underline{n} \times I \xleftarrow{\langle \mathbf{1}, c_1 \rangle} \underline{n}$$

where I is the unit interval $[0,1]$, $\mathbf{1}$ is the identity map on $\underline{n}$, $c_0, c_1 : \underline{n} \to I$ are the constant maps $c_0(x) = 0$ and $c_1(x) = 1$, and for $f : C \to A$, and $g : C \to B$, the pairing $\langle f, g \rangle : C \to A \times B$ is defined by $\langle f, g \rangle(c) = (f(c), g(c))$.

Composition of cobordisms $(M, f_0, f_1) : \underline{n} \to \underline{m}$ and $(N, g_0, g_1) : \underline{m} \to \underline{k}$ consists of the d-manifold $N +_{g_0, f_1} M$ obtained by gluing (see Section 9) and two maps $j \circ f_0$ and $i \circ g_1$, where $i : N \to N +_{g_0, f_1} M$ and $j : M \to N +_{g_0, f_1} M$ are the embeddings in the corresponding pushout diagram (see Section 9). Equivalence of cobordisms is a congruence with respect to the composition.

When $d = 2$, the category $dCobS$ is isomorphic to the category **2-Cobord** of [1, Section 4]. The category $dCobS$ is strict monoidal with respect to the sum on objects ($\underline{n} + \underline{m} = \underline{n+m}$) and the following operation of "putting side by side" on arrows. First, for two d-manifolds N and M, we denote by $N + M$ the disjoint union $(N \times \{0\}) \cup (M \times \{1\})$, and for two functions $f : \underline{n} \to N$ and $g : \underline{m} \to M$, we denote by $f + g : \underline{n+m} \to N + M$ the following function

$$(f+g)(x) = \begin{cases} (f(x), 0), & x \in \underline{n} \\ (g(x - (3n, 0, \dots, 0)), 1), & x \notin \underline{n}. \end{cases}$$

Then, the "putting side by side" of (N, f_0, f_1) and (M, g_0, g_1) is the d-cobordism

$$(N + M, f_0 + g_0, f_1 + g_1).$$

The category $dCobS$ is also symmetric monoidal with respect to the family of d-cobordisms $\underline{\tau_{n,m}}$, defined as

$$\underline{n} + \underline{m} \xrightarrow{\langle \mathbf{1}, c_0 \rangle} (\underline{n+m}) \times I \xleftarrow{\langle f, c_1 \rangle} \underline{m} + \underline{n},$$

where $f : \underline{n+m} \to \underline{n+m}$ translates the spheres S_i, $0 \leq i \leq n-1$, by the vector $(3m, 0, \dots, 0)$, and the spheres S_j, $n \leq j \leq n+m-1$, by the vector $(-3n, 0, \dots, 0)$.

The category $dCobS$ is skeletal, i.e. there are no two different isomorphic objects in $dCobS$. This is shown below (see Section 5 and Corollary 8.4). It is a full subcategory of the category $dCob$, whose objects are all closed $(d-1)$-manifolds, and whose arrows are based on arbitrary d-manifolds, and not only on those with boundaries homeomorphic to collections of spheres. The symmetric monoidal structure of the category $dCob$ is defined as for $dCobS$.

3. Why spheres?

In this section we explain why we work in *dCobS* and not in *dCob*, and why we deal with topological and not with smooth manifolds. The main reason is that dealing with arrows of *dCobS* is simplified to a certain extent by "irrelevance" of gluing. Section 10 serves to prepare the ground for the results of this section. The ambient consisting of collections of spheres is sufficient for our purposes, since we investigate spheres as Frobenius objects.

3.1. LEMMA. *If $f : \underline{1} \to \underline{1}$ is an orientation preserving homeomorphism, then the cobordisms $(\underline{1} \times I, \langle \mathbf{1}, c_0\rangle, \langle \mathbf{1}, c_1\rangle)$ and $(\underline{1} \times I, \langle \mathbf{1}, c_0\rangle, \langle f, c_1\rangle)$ are equivalent.*

PROOF. Let $F : \underline{1} \times I \to \underline{1} \times I$ be the homeomorphism from Proposition 10.9 such that $F(x, 0) = (x, 0)$ and $F(x, 1) = (f(x), 1)$. Then F makes the following diagram commutative.

$$\begin{tikzcd}
& \underline{1}\times I \arrow[dd, "F"] & \\
\underline{1} \arrow[ur, "{\langle \mathbf{1}, c_0\rangle}"] \arrow[dr, "{\langle \mathbf{1}, c_0\rangle}"'] & & \underline{1} \arrow[ul, "{\langle \mathbf{1}, c_1\rangle}"'] \arrow[dl, "{\langle f, c_1\rangle}"] \\
& \underline{1}\times I &
\end{tikzcd}$$

■

3.2. LEMMA. *If $u, v : \underline{1} \to \Sigma$ are two orientation preserving homeomorphisms, then the cobordisms $K_1 = (\Sigma\times I, \langle v, c_0\rangle, \langle v, c_1\rangle)$, $K_2 = (\Sigma\times I, \langle v, c_0\rangle, \langle u, c_1\rangle)$ and $(\underline{1}\times I, \langle \mathbf{1}, c_0\rangle, \langle \mathbf{1}, c_1\rangle)$ are equivalent.*

PROOF. The homeomorphism F in the center of the following diagram is the one from Lemma 3.1 obtained for $f = v^{-1} \circ u$.

$$\begin{tikzcd}
& \Sigma\times I \arrow[d, "v^{-1}\times\mathbf{1}"] & \\
& \underline{1}\times I \arrow[dd, "F"] & \\
\underline{1} \arrow[uur, bend left, "{\langle v, c_0\rangle}"] \arrow[ur, "{\langle \mathbf{1}, c_0\rangle}"] \arrow[dr, "{\langle \mathbf{1}, c_0\rangle}"'] \arrow[ddr, bend right, "{\langle v, c_0\rangle}"'] & & \underline{1} \arrow[uul, bend right, "{\langle v, c_1\rangle}"'] \arrow[ul, "{\langle \mathbf{1}, c_1\rangle}"'] \arrow[dl, "{\langle f, c_1\rangle}"] \arrow[ddl, bend left, "{\langle u, c_1\rangle}"] \\
& \underline{1}\times I \arrow[d, "v\times\mathbf{1}"] & \\
& \Sigma\times I &
\end{tikzcd}$$

■

3.3. LEMMA. *If $u, v : \underline{1} \to \Sigma$ are two orientation preserving homeomorphisms, where Σ is a part of the boundary of a d-manifold M, then the cobordisms $(M, f + u + g, h)$ and $(M, f + v + g, h)$ are equivalent.*

PROOF. Let K_1 and K_2 be the cobordisms from Lemma 3.2. For $\underline{n}$ and $\underline{m}$ being the sources of f and g respectively, we have

$$\begin{aligned}(M, f + u + g, h) &\sim (M, f + u + g, h) \circ \mathbf{1}_{\underline{n+1+m}} \\ &\sim (M, f + u + g, h) \circ (\mathbf{1}_{\underline{n}} + K_2 + \mathbf{1}_{\underline{m}}) \\ &= (M, f + v + g, h) \circ (\mathbf{1}_{\underline{n}} + K_1 + \mathbf{1}_{\underline{m}}) \\ &\sim (M, f + v + g, h) \circ \mathbf{1}_{\underline{n+1+m}} \\ &\sim (M, f + v + g, h).\end{aligned}$$

∎

By iterating Lemma 3.3 and an analogous result concerning the outgoing boundary of M, we obtain the following result in which "connected components" should be replaced by "pairs of points", when $d = 1$.

3.4. COROLLARY. *Every arrow of $dCobS$ is completely determined by a d-manifold and two sequences—one of connected components of the ingoing boundary and the other of connected components of the outgoing boundary.*

Hence, we may denote an arrow from $\underline{n}$ to $\underline{m}$ by (M, Σ_0, Σ_1), where $\Sigma_0 = (\Sigma_0^0, \dots, \Sigma_0^{n-1})$ is a sequence of all the connected components (or pairs of points, when $d = 1$) of the ingoing boundary and $\Sigma_1 = (\Sigma_1^0, \dots, \Sigma_1^{m-1})$ is a sequence of all the connected components of the outgoing boundary of M.

3.5. PROPOSITION. *Two cobordisms (M, Σ_0, Σ_1) and (N, Δ_0, Δ_1) are equivalent iff the corresponding sequences are of the same length and there is a homeomorphism $F : M \to N$ such that for every $i \in \{0, 1\}$ and every j, the image of F restricted to Σ_i^j is Δ_i^j.*

PROOF. The direction from left to right follows from the definition of equivalence. For the other direction, for every j, let $h_0^j : \underline{1} \to \Sigma_0^j$ be an orientation preserving homeomorphism and let $h_1^j : \underline{1} \to \Sigma_1^j$ be an orientation reversing homeomorphism. Define $g_i^j : \underline{1} \to \Delta_i^j$ to be $F \circ h_i^j$. Then F underlies the equivalence of $(M, \sum_{j=0}^{n-1} h_0^j, \sum_{j=0}^{m-1} h_1^j)$ and $(N, \sum_{j=0}^{n-1} g_0^j, \sum_{j=0}^{m-1} g_1^j)$. ∎

However, if for $d \geq 3$ we allow closed $(d{-}1)$-manifolds other than collections of spheres to be objects of the category of d-cobordisms, then it would not be the case that the arrows of such a category are determined just by manifolds and sequences of ingoing and outgoing boundaries. For example, a solid torus with the torus as the ingoing boundary and the empty set as the outgoing boundary does not determine a 3-cobordism. The identity map and an orientation preserving homeomorphism of the torus that interchanges parallels and meridians define two different 3-cobordisms. By the result of Lickorish, [16], every closed, connected, 3-manifold is obtainable from S^3 by removing a finite collection of solid tori, and then sewing them back. For example, if one removes an unknotted solid torus from S^3 and sew it back according to a homeomorphism of torus that interchanges parallels and meridians, then the resulting 3-manifold is $S^2 \times S^1$.

In case of the category of smooth d-cobordisms as arrows and collections of spheres as objects, the analogues of Corollary 3.4 and Proposition 3.5 do not hold for every d. For example, the manifold $S^{d-1} \times I$ with $S^{d-1} \times \{0\}$ as the ingoing and $S^{d-1} \times \{1\}$ as the outgoing boundary does not determine a d-cobordism. This is shown as follows.

A *pseudo-isotopy* of a smooth closed manifold M is a diffeomorphism F of $M \times I$ that restricts to the identity on $M \times \{0\}$. The restriction of F to $M \times \{1\}$ is, up to the identification of $M \times \{1\}$ with M, a diffeomorphism $f : M \to M$. One says that f is *pseudo-isotopic* to the identity.

By a definition analogous to the one given in Section 2 (cf. [14, 1.2.17]), two smooth d-cobordisms $(S^{d-1} \times I, \langle \mathbf{1}, c_0\rangle, \langle \mathbf{1}, c_1\rangle)$ and $(S^{d-1} \times I, \langle \mathbf{1}, c_0\rangle, \langle f, c_1\rangle)$ are equivalent when there is an orientation preserving diffeomorphism $F : S^{d-1} \times I \to S^{d-1} \times I$ such that the following diagram commutes.

$$\begin{tikzcd}
 & S^{d-1} \times I \arrow[dd, "F"] & \\
S^{d-1} \arrow[ur, "{\langle \mathbf{1}, c_0\rangle}"] \arrow[dr, "{\langle \mathbf{1}, c_0\rangle}"'] & & S^{d-1} \arrow[ul, "{\langle \mathbf{1}, c_1\rangle}"'] \arrow[dl, "{\langle f, c_1\rangle}"] \\
 & S^{d-1} \times I &
\end{tikzcd}$$

This is equivalent to the fact that f is pseudo-isotopic to the identity on S^{d-1}. Since it is not the case that for every d every orientation preserving diffeomorphism of S^{d-1} is pseudo-isotopic to the identity (see [12], [3] and [5]), we have that there is not always a unique d-cobordism corresponding to $S^{d-1} \times I$, with chosen ingoing and outgoing boundaries.

However, for $d \leq 6$ (and not only for these dimensions), every orientation preserving diffeomorphism of S^{d-1} is pseudo-isotopic to the identity. This fact, for $d = 2$, is implicitly used by Kock, [14], in order to pass from smooth 2-cobordisms to the pictures representing the underlying manifolds. A result analogous to our Corollary 3.4 holds for 2-cobordisms of [14].

4. A Frobenius structure of spheres

In this section we follow the pattern given for S^1 in [1] and [14] in order to define a Frobenius structure for a sphere of any finite dimension.

For an oriented d-disc D and its boundary ∂D, let $\underline{\eta^{\lozenge}}$ be the d-cobordism $(D, \emptyset, (\partial D))$ and let $\underline{\eta^{\Box}}$ be the d-cobordism $(D, (\partial D), \emptyset)$.

On the other hand, for D_1 and D_2 being two nonintersecting d-discs in the interior of D, let M be a d-manifold obtained from D by removing the interiors of D_1 and D_2. We define $\underline{\mu^{\lozenge}}$ to be the d-cobordism $(M, (\partial D_1, \partial D_2), (\partial D))$ and $\underline{\mu^{\Box}}$ to be the d-cobordism $(M, (\partial D), (\partial D_1, \partial D_2))$.

Figure 1: unit, counit

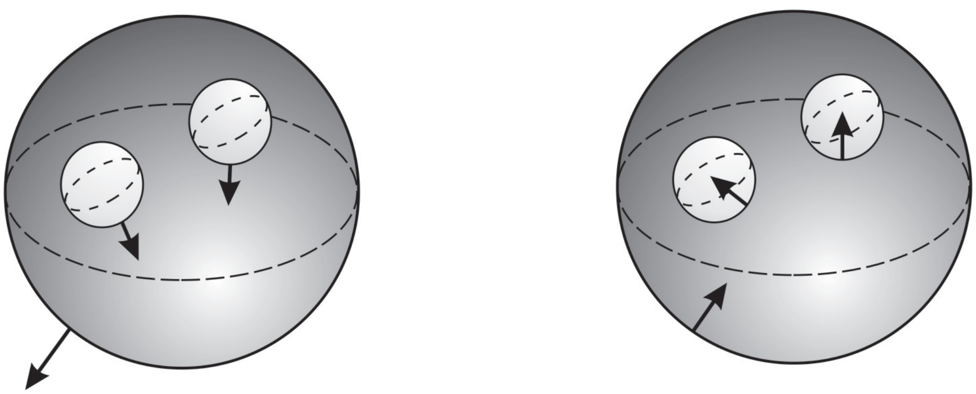


Figure 2: multiplication, comultiplication

It is not difficult to see that the above cobordisms, together with the symmetric monoidal structure of $dCobS$, satisfy the conditions necessary for S^0 to be a symmetric Frobenius object of $1CobS$, and S^{d-1}, for $d \geq 2$ to be a commutative Frobenius object of $dCobS$. For example, the equation (*assoc*), for $d = 3$, is illustrated by the following picture.

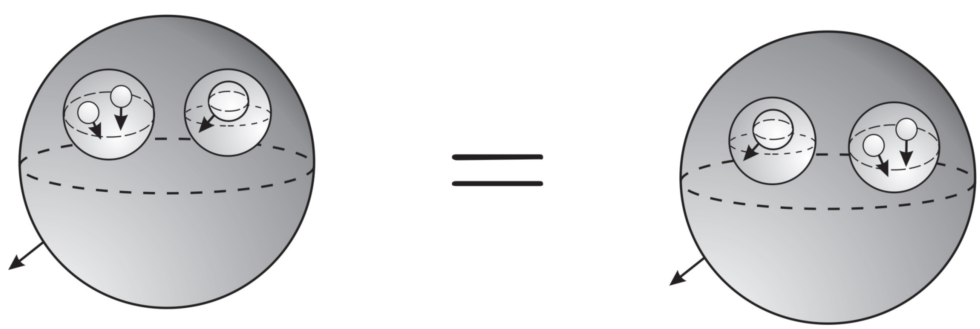


Figure 3: associativity

The defined Frobenius structure of S^{d-1} guarantees that every dTQFT maps this sphere to a Frobenius algebra. The image of S^{d-1} by a dTQFT is a commutative Frobenius algebra when $d \geq 2$. This is a part of [25, Proposition 1], which is essentially due to

Dijkgraaf, [6].

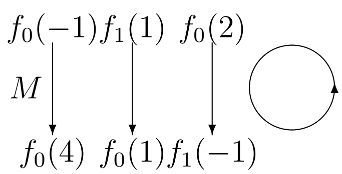
5. Brauerian representation as a 1TQFT

In this section, we pay attention to $1CobS$ in particular. We show that Brauer, [2], anticipated 1TQFT by his matrix representation of a class of diagrammatic algebras. When restricted to $1CobS$, such a representation determines a matrix Frobenius algebra as the image of the Frobenius object S^0.

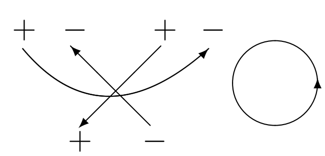
Following the definition given in Section 2, the category $1CobS$ has objects $\underline{0}, \underline{1}, \underline{2}, \ldots$, where $\underline{0}$ is the empty set and $\underline{n}$ is the 0-dimensional manifold $\{-1, 1, \ldots, 3n-4, 3n-2\}$ for which we fix the orientation

$$\varepsilon(x) = \begin{cases} \;\;\, 1, & x = 3i-1, \\ -1, & x = 3i+1. \end{cases}$$

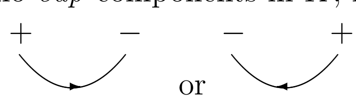
Hence, we may envisage an object of $1CobS$ as a finite sequence built out of the pair $+-$. The arrows of $1CobS$ are the equivalence classes of 1-cobordisms. For example, the cobordism $(M, f_0, f_1) : \underline{2} \to \underline{1}$



is illustrated by the following picture



The category $1CobS$ is skeletal. If there is an isomorphism K between $\underline{n}$ and $\underline{m}$, then it is easy to see that there are no *cup* components in K, i.e. components presented by



Otherwise, there would be such components in $K^{-1} \circ K : \underline{n} \to \underline{n}$, which is impossible. Analogously, there are no *cap* components in K, hence $n = m$, which implies $\underline{n} = \underline{m}$ (cf. Proposition 8.3).

For the infinite sequence $-1, 1, 2, 4, \ldots, 3i-1, 3i+1, \ldots$ let $\underline{\underline{n}}$ denote the set of its first n members. In order to obtain a symmetric strict monoidal category containing $1CobS$ as a full subcategory, let $1Cob$ be the category whose set of objects is

$$\{(\underline{\underline{n}}, \varepsilon) \mid n \in N, \varepsilon : \underline{\underline{n}} \to \{-1, 1\}\}$$

and whose arrows are the equivalence classes of cobordisms of the form

$$(M, f_0 : (\underline{\underline{n}}, \varepsilon_0) \to M, f_1 : (\underline{\underline{m}}, \varepsilon_1) \to M),$$

where M is a 1-manifold such that its boundary ∂M is a disjoint union of Σ_0 and Σ_1, and f_0 is an orientation preserving embedding whose image is Σ_0, while f_1 is an orientation reversing embedding whose image is Σ_1. The symmetric monoidal structure of $1Cob$ is defined as for $1CobS$ by "putting side by side" and by using the symmetry defined in an analogous way as $\underline{\tau_{n,m}}$ defined in Section 2. A connected component of M homeomorphic to S^1 is called *circular* component of the cobordism. Again, as in Corollary 3.4, every arrow of $1Cob$ is completely determined by a 1-manifold M and two sequences $\Sigma_0 = (\Sigma_0^0, \ldots, \Sigma_0^{n-1})$ and $\Sigma_1 = (\Sigma_1^0, \ldots, \Sigma_1^{m-1})$ of points, one of the ingoing boundary and the other of the outgoing boundary. The category $1Cob$ is not skeletal since we have two different objects $(\underline{\underline{2}}, \varepsilon_0)$ and $(\underline{\underline{2}}, \varepsilon_1)$ with $\varepsilon_0(-1) = 1, \varepsilon_0(1) = -1, \varepsilon_1(-1) = -1, \varepsilon_1(1) = 1$, which are isomorphic via symmetry.

Brauer, [2], introduced a class of diagrammatic algebras and found their matrix representation. In [9, Section 6] a generalization of this representation to a category of diagrams is given (see also [8] and [11, Section 14]). This generalization leads to the following assignment of matrices to the arrows of $1Cob$.

Let $\mathcal{F}$ be a field of characteristic 0 and let p be a natural number greater than or equal to 2. For an arrow $K = (M, \Sigma_0, \Sigma_1) : (\underline{\underline{n}}, \varepsilon_0) \to (\underline{\underline{m}}, \varepsilon_1)$ of $1Cob$, let ρ_K be the following equivalence relation on the disjoint union $(n \times \{0\}) \cup (m \times \{1\})$ of finite ordinals $n = \{0, \ldots, n-1\}$ and $m = \{0, \ldots, m-1\}$. For (i, k) and (j, l) elements of $(n \times \{0\}) \cup (m \times \{1\})$, we have that $(i, k)\rho_K(j, l)$ when the points Σ_k^i and Σ_l^j belong to the same connected component of M.

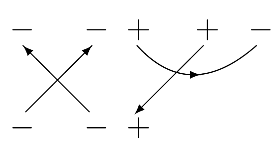
For every $K : (\underline{\underline{n}}, \varepsilon_0) \to (\underline{\underline{m}}, \varepsilon_1)$ we define a matrix $A(K) \in \mathcal{M}_{p^m \times p^n}$ in the following way. For a_0 such that $0 \le a_0 < p^n$, which denotes a column of $A(K)$, and a_1 such that $0 \le a_1 < p^m$, which denotes a row of $A(K)$, write a_0 in the base p system with n digits $a_0^0 \ldots a_0^{n-1}$, and a_1 in the base p system with m digits $a_1^0 \ldots a_1^{m-1}$. For example, if $p = 2$, $n = 5$, $m = 3$, $a_0 = 10$, $a_1 = 5$, we have $a_0 = 01010$ and $a_1 = 101$.

We define the (a_1, a_0) element of $A(K)$ to be 1 when for every (i, k) and (j, l) from $(n \times \{0\}) \cup (m \times \{1\})$ we have that

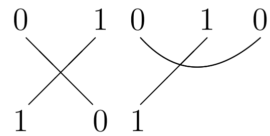
$$(i, k)\rho_K(j, l) \;\Rightarrow a_k^i = a_l^j;$$

otherwise it is 0.

If we take K to be given by the following picture,



and we take $p = 2$ as above, then the $(5, 10)$ element of $A(K)$ is 1 since the sequences 01010 and 101 "match" into the picture of ρ_K.



Let $\mathbf{Mat}_{\mathcal{F}}$ be the category whose objects are vector spaces $\mathcal{F}^n$, $n \geq 1$, and whose arrows from $\mathcal{F}^n$ to $\mathcal{F}^m$ are $m \times n$ matrices over the field $\mathcal{F}$. The identity matrix of order n is the identity arrow on $\mathcal{F}^n$ and matrix multiplication is the composition of arrows. One can identify the objects of $\mathbf{Mat}_{\mathcal{F}}$ with natural numbers (the dimensions of vector spaces) as it was done in [11]. The category $\mathbf{Mat}_{\mathcal{F}}$ may be considered as a skeleton of the category $\mathbf{Vect}_{\mathcal{F}}$ of finite-dimensional vector spaces over $\mathcal{F}$. Hence, $\mathbf{Mat}_{\mathcal{F}}$ and $\mathbf{Vect}_{\mathcal{F}}$ are equivalent.

The category $\mathbf{Mat}_{\mathcal{F}}$ is symmetric strict monoidal with respect to the multiplication on objects considered as natural numbers, and the Kronecker product on arrows (matrices). The symmetry is brought by the family of $nm \times mn$ permutation matrices $S_{n,m}$. The matrix $S_{n,m}$ is the matrix representation of the linear map $\sigma : \mathcal{F}^n \otimes \mathcal{F}^m \to \mathcal{F}^m \otimes \mathcal{F}^n$ with respect to the standard ordered bases, defined on the basis vectors by $\sigma(e_i \otimes f_j) = f_j \otimes e_i$. For example, $S_{3,2}$ is the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Consider the following functor B from $1Cob$ to $\mathbf{Mat}_{\mathcal{F}}$. On objects it is defined by $B(\underline{\underline{n}}, \varepsilon) = p^n$ and on arrows we define it as

$$B(K) = p^c \cdot A(K),$$

where c is the number of circular components of K, and $A(K)$ is the $0-1$ matrix defined above. That this is indeed a functor stems from [9, Section 5, Proposition 4] and that it is faithful stems from [11, Section 14]. We shall not go here into any more detail about this matter.

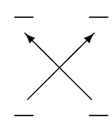
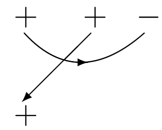
In order to conclude that this functor is monoidal, note that for matrices $X \in \mathcal{M}_{m \times n}$ and $Y \in \mathcal{M}_{k \times l}$ we have $Z = X \otimes Y \in \mathcal{M}_{(m \cdot k) \times (n \cdot l)}$ and

$$x_{i,j} \cdot y_{q,r} = z_{i \cdot k+q, j \cdot l+r}.$$

If K is obtained from K_1 and K_2 by "putting side by side" and Z is the matrix $A(K)$, while X and Y are $A(K_1)$ and $A(K_2)$ respectively, then

$$z_{i \cdot k+q, j \cdot l+r} = 1 \ \text{ iff } \ x_{i,j} = y_{q,r} = 1.$$

In our example for K_1 and K_2, respectively being

 and 

we have $z_{5,10} = x_{2,1} \cdot y_{1,2}$.

It is easy to check that B maps symmetry to symmetry. Consequently, the functor B may be said to be a 1TQFT.

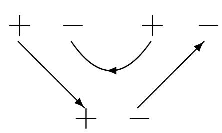
Let us now restrict the functor B to the category $1CobS$. Since S^0, i.e. the object $\underline{1}$ is equipped with a Frobenius structure in $1CobS$, consequently in $1Cob$, the image of $\underline{1}$ by the monoidal functor B is a Frobenius algebra. It is interesting that B brings to $B(\underline{1})$ the structure of a matrix Frobenius algebra (for the notion of matrix Frobenius algebra see [14, 2.2.16]).

Note that $B(\underline{1})$ is p^2, i.e. the vector space $\mathcal{F}^{p^2}$. Every vector

$$\vec{v} = \begin{bmatrix} v_0 \\ \vdots \\ v_{p^2-1} \end{bmatrix} \in \mathcal{F}^{p^2}$$

corresponds to the matrix $H(\vec{v}) \in \mathcal{M}_{p\times p}$ whose (i,j) member is $v_{i\cdot p+j}$. This is the standard isomorphism $H : \mathcal{F}^{p^2} \to \mathcal{M}_{p\times p}$. In order to show that B brings the structure of a matrix Frobenius algebra to $\mathcal{M}_{p\times p} = B(\underline{1})$, it suffices to show that $B(\underline{\mu^\Diamond})$ represents the multiplication of matrices and that $B(\underline{\eta^\Box})$ represents the trace form.

The arrow $\underline{\mu^\Diamond} : \underline{2} \to \underline{1}$ of $1Cob$ is presented by the following picture



and the corresponding matrix $B(\underline{\mu^\Diamond})$ is in $\mathcal{M}_{p^2\times p^4}$. Our goal is to show that for the standard isomorphism

$$H_2 : \mathcal{F}^{p^4} \to \mathcal{M}_{p^2\times p^2}$$

defined as H above (i.e. (i,j) member of $H_2(\vec{v})$ is $v_{i\cdot p^2+j}$) and arbitrary matrices $X, Y \in \mathcal{M}_{p\times p}$ we have that

$$H(B(\underline{\mu^\Diamond})\, H_2^{-1}(X \otimes Y)) = XY.$$

When $p = 2$, the matrix $B(\underline{\mu^\Diamond})$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and $H_2^{-1}(X \otimes Y)$ is the vector

$$\vec{v} = \begin{bmatrix} v_0 \\ \vdots \\ v_{15} \end{bmatrix} \in \mathcal{F}^{16}$$

where $v_0 = x_{00} \cdot y_{00}$, $v_1 = x_{00} \cdot y_{01}$, $v_6 = x_{01} \cdot y_{10}$, $v_7 = x_{01} \cdot y_{11}$, $v_8 = x_{10} \cdot y_{00}$, $v_9 = x_{10} \cdot y_{01}$, $v_{14} = x_{11} \cdot y_{10}$ and $v_{15} = x_{11} \cdot y_{11}$. Hence, $B(\underline{\mu}^{\Diamond})\, H_2^{-1}(X \otimes Y)$ is

$$\begin{bmatrix} x_{00} \cdot y_{00} + x_{01} \cdot y_{10} \\ x_{00} \cdot y_{01} + x_{01} \cdot y_{11} \\ x_{10} \cdot y_{00} + x_{11} \cdot y_{10} \\ x_{10} \cdot y_{01} + x_{11} \cdot y_{11} \end{bmatrix}$$

which is mapped to XY by H.

For the general case, let $\vec{u} = B(\underline{\mu}^{\Diamond})\, H_2^{-1}(X \otimes Y)$ and $A = H(\vec{u})$. We want to show that for $0 \leq i, j \leq p-1$,

$$a_{i,j} = \sum_{k=0}^{p-1} x_{i,k} \cdot y_{k,j}.$$

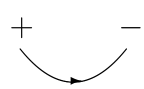
Since the element $a_{i,j}$ is equal to $u_{i \cdot p + j}$, we are interested in the $(i \cdot p + j)$-th row of the matrix $B(\underline{\mu}^{\Diamond})$. In this row, which in the base p system is presented by the sequence ij, the entry 1 occurs p times in the columns presented in the base p system by the sequences

$$i00j, \quad i11j, \quad \ldots \quad ikkj, \quad \ldots \quad i(p-1)(p-1)j,$$

and all the other elements are 0. The column presented by $ikkj$ is actually the $(i \cdot p^3 + k \cdot p^2 + k \cdot p + j)$-th column of the matrix $B(\underline{\mu}^{\Diamond})$. Since the corresponding row of $H_2^{-1}(X \otimes Y)$ is equal to $x_{i,k} \cdot y_{k,j}$, we have that

$$a_{i,j} = u_{i \cdot p + j} = \sum_{k=0}^{p-1} x_{i,k} \cdot y_{k,j}.$$

The arrow $\underline{\eta}^{\Box} : \underline{1} \to \underline{0}$ of $1Cob$ is presented by the following picture



and the corresponding matrix $B(\underline{\eta}^{\Box})$ is in $\mathcal{M}_{1 \times p^2}$. Our goal is to show that for an arbitrary matrix $X \in \mathcal{M}_{p \times p}$ we have that

$$B(\underline{\eta}^{\Box})\, H^{-1}(X) = \mathrm{tr}(X).$$

When $p = 2$, this equality reads

$$\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{00} \\ x_{01} \\ x_{10} \\ x_{11} \end{bmatrix} = x_{00} + x_{11}.$$

For the general case, in the row of the matrix $B(\underline{\eta^\square})$ the entry 1 occurs p times in the columns presented in the base p system by the sequences

$$00, \quad 11, \quad \dots \quad kk, \quad \dots \quad (p-1)(p-1),$$

and all the other elements are 0. The column presented by kk is actually the $(k \cdot p + k)$-th column of the matrix $B(\underline{\eta^\square})$. Since the corresponding row of $H^{-1}(X)$ is equal to $x_{k,k}$, we have that

$$B(\underline{\eta^\square})\, H^{-1}(X) = \sum_{k=0}^{p-1} x_{k,k}.$$

6. The category $\mathbf{K}$

Our intention is to define the category $\mathbf{K}$ as a PROP, in the sense of [18, Chapter V], having 1 as the universal commutative Frobenius object, in the same sense as 1, as an object of the simplicial category Δ is the universal monoid. The category Δ is introduced in [19, Section VII.5] as the concrete category of monotone functions between finite ordinals. Alternatively, this category could be introduced in a pure syntactical manner by generators and relations, via [19, Proposition 2, Section VII.5].

We choose this alternative approach and present the category $\mathbf{K}$ by generators and relations. In this way we stipulate the intended universal property in its definition.

More formally, consider the category $\mathcal{F}$ whose objects are symmetric strict monoidal categories with one distinguished commutative Frobenius object and whose arrows are symmetric monoidal functors preserving distinguished objects and their Frobenius structures. Since the notions of symmetric strict monoidal category and commutative Frobenius object are purely equational, the forgetful functor G from $\mathcal{F}$ to the category *Set* of sets and functions, which maps an object of $\mathcal{F}$, i.e. a symmetric monoidal category, to the set of its objects, has a left adjoint F. As in universal algebra, FX, for a set X is built out of a term model. Our category $\mathbf{K}$ is $F\emptyset$. What follows is a brief description of our construction of $\mathbf{K}$ and we refer to Section 11 for details.

The category $\mathbf{K}$ has the set of finite ordinals ω as the set of objects. The ordinal n is interpreted as the n-th tensor power of the distinguished commutative Frobenius object. Hence, the monoidal structure on objects is given by addition. In order to define the arrows of this category, an equational system is introduced in Section 11.

Briefly, as in every syntactical construction of a free object, words built out of $\mathbf{1}$, $\circ$, $\otimes$, τ, $\mu^\lozenge$, $\eta^\lozenge$, $\mu^\square$ and $\eta^\square$ denoting the arrows of $\mathbf{K}$ are introduced. We call these words *terms*. Every such a term has its source and target. The terms are quotient by the smallest equivalence relation guaranteeing that 1 is a commutative Frobenius object in $\mathbf{K}$. (See Section 11 for details.) The equivalence class of a term f is denoted by $[f]$ and $\{[f] \mid f \text{ is a term}\}$ is the set of arrows of $\mathbf{K}$. The source of $[f]$ is the source of f and the same holds for targets. The identity arrow on n is $[\mathbf{1}_n]$ and $[g] \circ [f]$ is $[g \circ f]$.

The category $\mathbf{K}$ is strict monoidal with respect to the monoidal structure given by $\otimes$ and 0. Its symmetry is given by the family of τ arrows. It is *skeletal* by Corollary 8.4.

The category $\mathbf{K}$, since it is the image of the initial object in *Set* under the functor F, has the following universal property: for every commutative Frobenius object M in a symmetric strict monoidal category $\mathcal{M}$, there is a unique symmetric monoidal functor $U : \mathbf{K} \to \mathcal{M}$ such that $U(1) = M$, and U preserves the Frobenius structure. Hence, for $d \geq 2$, there is a unique symmetric monoidal functor from $\mathbf{K}$ to $dCobS$ that maps 1 to $\underline{1}$. We call this functor the *interpretation* of $\mathbf{K}$ in $dCobS$. That the interpretation is faithful is shown in Section 8.

The equations (*cat*) (see Section 11) are usually not mentioned in the calculations that follow. Hence, we omit parenthesis tied to nested compositions, and erase or add compositions with identities, when necessary.

6.1. REMARK. We could start with the category $2CobS$ instead of $\mathbf{K}$ (cf. Corollary 8.8), which would be more in the style of the definition of the simplicial category given in [19, Section VII.5]. However, for the proof of our main result, if we relied on 2CobS instead on K, then we would miss the syntax necessary for our approach. This would lead to a certain amount of imprecision.

7. Normal form for arrows of $\mathbf{K}$

In this section, we define a normal form for terms and show that every arrow of $\mathbf{K}$ is representable by a term in normal form. This normal form is essentially the same as the one given in [14, 1.4.16]. The normal form is then used in Section 8 for the proof of faithfulness of the interpretation. Some proofs are illustrated by pictures corresponding to the interpretation of $\mathbf{K}$ in $2CobS$.

We start with some auxiliary notions. Let $V_{-1} = \eta^\Diamond$, $\Lambda_{-1} = \eta^\Box$, $V_0 = H_0 = \Lambda_0 = \mathbf{1}_1$, and for $n \geq 1$, let

$$V_n = \mu^\Diamond \circ (\mu^\Diamond \otimes \mathbf{1}_1) \circ \ldots \circ (\mu^\Diamond \otimes \mathbf{1}_{n-1}) : n+1 \to 1,$$

$$H_n = \underbrace{(\mu^\Diamond \circ \mu^\Box) \circ \ldots \circ (\mu^\Diamond \circ \mu^\Box)}_{n} : 1 \to 1,$$

$$\Lambda_n = (\mu^\Box \otimes \mathbf{1}_{n-1}) \circ \ldots \circ (\mu^\Box \otimes \mathbf{1}_1) \circ \mu^\Box : 1 \to n+1.$$

With the help of these terms, for $n, m, p \geq 0$, we define $E_{p,m,n}$ as

$$\Lambda_{p-1} \circ H_m \circ V_{n-1} : n \to p$$

A term is a τ-*term* when $\mu^\Diamond$, $\eta^\Diamond$, $\mu^\Box$ and $\eta^\Box$ do not occur in it. For every τ-term $f : n \to n$ there exists a unique permutation on n that corresponds to f.

A term is *special* when it is a τ-term, or for $k \geq 1$, it is of the form

$$\pi \circ \bigotimes_{i=1}^{k} E_{p_i,m_i,n_i} \circ \chi,$$

where π and χ are τ-terms. We call χ, the *head*, $\bigotimes_{i=1}^{k} E_{p_i,m_i,n_i}$, the *center*, and π, the *tail* of this term.

7.1. Proposition. *Every term is equal to a special term.*

We use the following lemmata in the proof of Proposition 7.1.

7.2. Lemma. *Every term is equal to a term of the form* $f_n \circ \ldots \circ f_0$, $n \geq 0$, *where every* f_i *is of the form* $\mathbf{1}_l \otimes \beta \otimes \mathbf{1}_r$, *for* $l, r \geq 0$ *and* $\beta \in \{\tau, \mu^{\square}, \eta^{\square}, \eta^{\Diamond}, \mu^{\Diamond}\}$.

Proof. By relying on the equations

$$f_1 \otimes f_2 = (f_1 \otimes \mathbf{1}_{m_2}) \circ (\mathbf{1}_{n_1} \otimes f_2) \quad \text{and} \quad (g \circ f) \otimes \mathbf{1}_m = (g \otimes \mathbf{1}_m) \circ (f \otimes \mathbf{1}_m),$$

derived from (*cat*) and (*fun*). ■

7.3. Lemma. *For every permutation on* n, *there is a* τ*-term* $\pi : n \to n$ *such that this permutation corresponds to* π. *If the permutations corresponding to two* τ*-terms are equal, then these terms are equal in* $\mathcal{K}$.

Proof. By symmetric monoidal coherence (see [17]). ■

From now on, we identify a τ-term with the corresponding permutation.

7.4. Lemma. *For every* τ*-term* $\pi : p \to p$ *and every* $l \in p$, *there is a* τ*-term* $\pi' : p-1 \to p-1$ *such that for* $j = \pi^{-1}(l)$, π *is equal to*

$$(\tau_{1,l} \otimes \mathbf{1}_{p-l-1}) \circ (\mathbf{1}_1 \otimes \pi') \circ (\tau_{j,1} \otimes \mathbf{1}_{p-j-1}).$$

Proof. The permutation corresponding to

$$(\tau_{l,1} \otimes \mathbf{1}_{p-l-1}) \circ \pi \circ (\tau_{1,j} \otimes \mathbf{1}_{p-j-1})$$

has 0 as a fix point. Hence, by Lemma 7.3, there is a τ-term π' such that this permutation corresponds to $\mathbf{1}_1 \otimes \pi'$. By Lemma 7.3 and (*inv*) this concludes the proof. ■

By relying on (*fun*), (*coass*) and (*counit*), we obtain the following two lemmata.

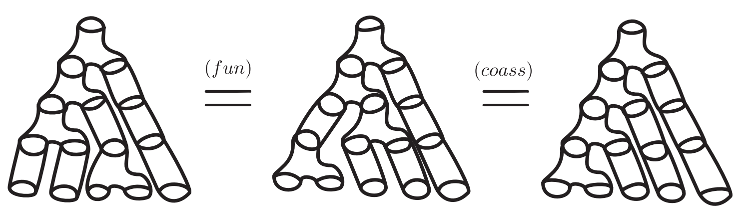


Figure 4: Lemma 7.5

7.5. Lemma. *For* $l + r = n \geq 0$, *we have* $(\mathbf{1}_l \otimes \mu^{\square} \otimes \mathbf{1}_r) \circ \Lambda_n = \Lambda_{n+1}$.

7.6. Lemma. *For* $l + r = n \geq 0$, *we have* $(\mathbf{1}_l \otimes \eta^{\square} \otimes \mathbf{1}_r) \circ \Lambda_n = \Lambda_{n-1}$.

7.7. Lemma. *For every* τ*-term* $\pi : l + r \to l + r$, *we have*

$$(\mathbf{1}_l \otimes \eta^{\Diamond} \otimes \mathbf{1}_r) \circ \pi = (\tau_{1,l} \otimes \mathbf{1}_r) \circ (\mathbf{1}_1 \otimes \pi) \circ (\eta^{\Diamond} \otimes \mathbf{1}_{l+r}).$$

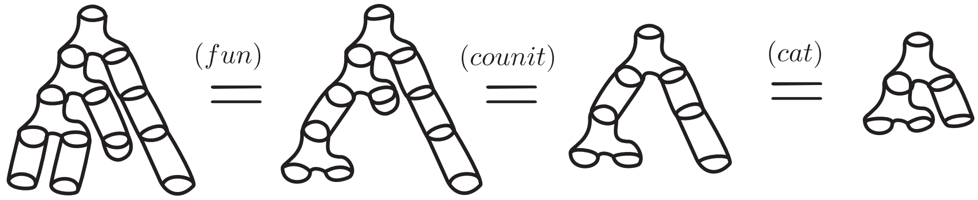


Figure 5: Lemma 7.6

Proof. We prove this from right to left, by relying on (*fun*), (*nat*) and the equation $\tau_{0,l} = \mathbf{1}_l$, which is derivable by (*inv*), (*hex*) and (*str*) as follows

$$\begin{aligned}\mathbf{1}_l &= \tau_{l,0} \circ \tau_{0,l} = \tau_{l,0} \circ \tau_{0+0,l} = \tau_{l,0} \circ (\tau_{0,l} \otimes \mathbf{1}_0) \circ (\mathbf{1}_0 \otimes \tau_{0,l}) \\ &= \tau_{l,0} \circ \tau_{0,l} \circ \tau_{0,l} = \tau_{0,l}.\end{aligned}$$

■

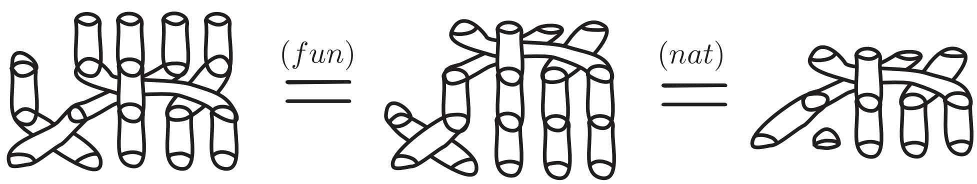


Figure 6: Lemma 7.7

By relying on (*coass*), (*assoc*), (*cocom*), (*com*), and the fact that every permutation is equal to a composition of transpositions, we can prove the following.

7.8. Lemma. *For every τ-term $\pi : n+1 \to n+1$, we have $\pi \circ \Lambda_n = \Lambda_n$, and $V_n \circ \pi = V_n$.*

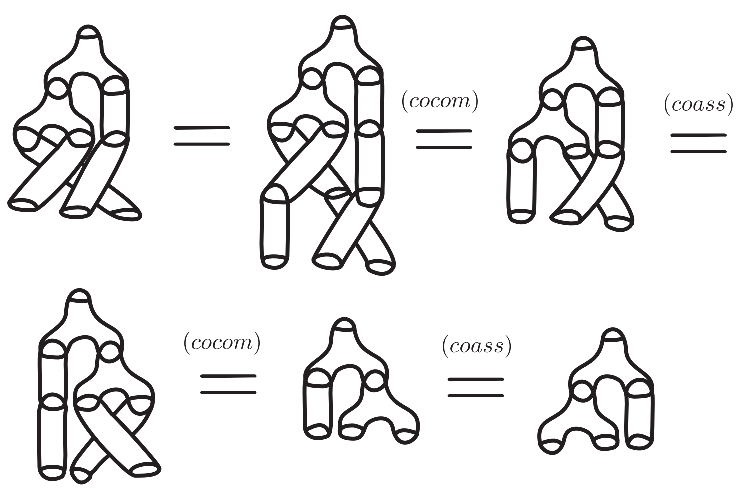


Figure 7: Lemma 7.8

For a τ-term $\pi : p \to p$ with $p \geq 2$, we say that $l, l+1 \in p$ are *parallel* in π when $\pi^{-1}(l+1) = \pi^{-1}(l) + 1$, i.e. for some $j \in p$, $\pi(j) = l$ and $\pi(j+1) = l+1$.

7.9. Lemma. *For a special term f, which is not a τ-term, with the target $p \geq 2$, and every $l \in p-1$, there is a special term equal to f, such that $l, l+1$ are parallel in its tail.*

Proof. Let f be $\pi \circ \bigotimes_{i=1}^{k} E_{p_i,m_i,n_i} \circ \chi$. If l and $l+1$ are tied by π to the target of one E in the center of f, i.e. there is $j \in \{1, \dots, k\}$ such that

$$\sum_{i=1}^{j-1} p_i \leq \pi^{-1}(l), \pi^{-1}(l+1) < \sum_{i=1}^{j} p_i,$$

then, if necessary, by Lemma 7.8 a τ-term could be added in between the tail and the center of f in order to obtain a new tail such that $l, l+1$ are parallel in it.

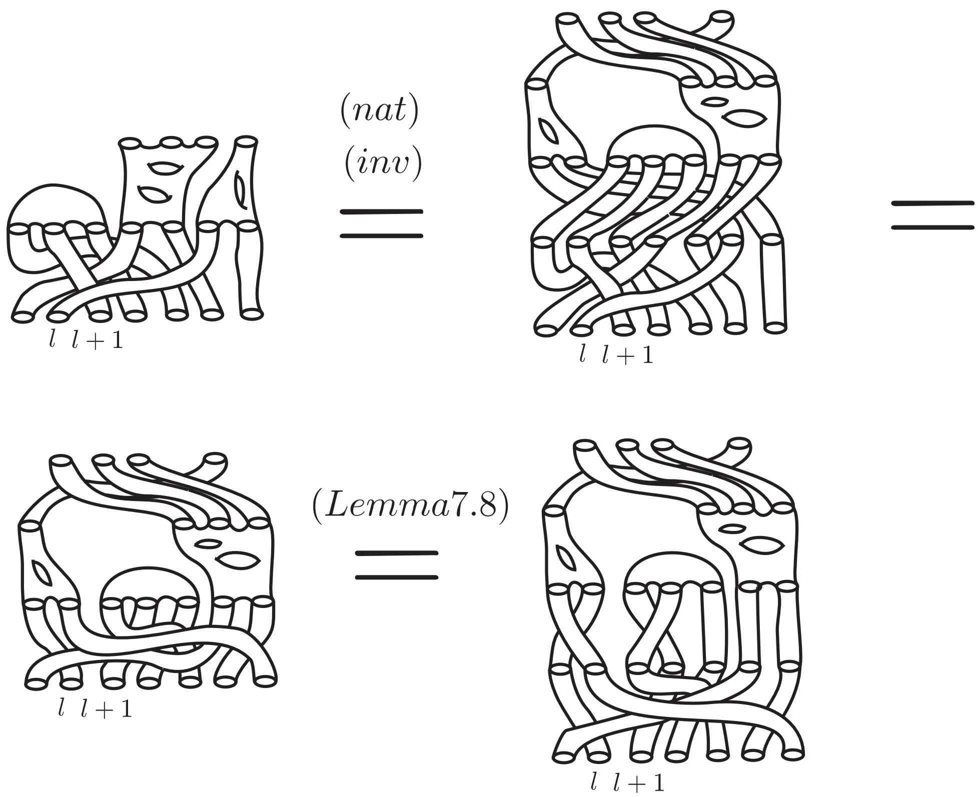


Figure 8: Lemma 7.9

If this is not the case, then by the following corollary of (*nat*) and (*inv*)

$$f_1 \otimes f_2 = \tau_{m_2,m_1} \circ (f_2 \otimes f_1) \circ \tau_{n_1,n_2},$$

we may assume, without loss of generality, that there is $j \in \{1, \dots, k\}$ such that

$$\sum_{i=1}^{j-1} p_i \leq \pi^{-1}(l) < \sum_{i=1}^{j} p_i \leq \pi^{-1}(l+1) < \sum_{i=1}^{j+1} p_i.$$

If necessary, by Lemma 7.8 a new τ-term could be added in between the tail and the center of f in order to obtain a new tail such that $l, l+1$ are parallel in it. ∎

The proof of the following lemma is akin to the proof of Lemma 7.4.

7.10. Lemma. *For every τ-term $\pi : p \to p$ and every $l \in p-1$ such that $l, l+1$ are parallel in π, there is a τ-term $\pi' : p-2 \to p-2$ such that for $j = \pi^{-1}(l)$, π is equal to*

$$(\tau_{2,l} \otimes \mathbf{1}_{p-l-2}) \circ (\mathbf{1}_2 \otimes \pi') \circ (\tau_{j,2} \otimes \mathbf{1}_{p-j-2}).$$

By Lemma 7.8 and the equations (*nat*), (*fun*) and (*Frob*), we have the following.

7.11. Lemma. *For $n \geq 1$, $(\mathbf{1}_l \otimes \mu^\Diamond \otimes \mathbf{1}_{n-l-1}) \circ \Lambda_n = \Lambda_{n-1} \circ H_1$.*

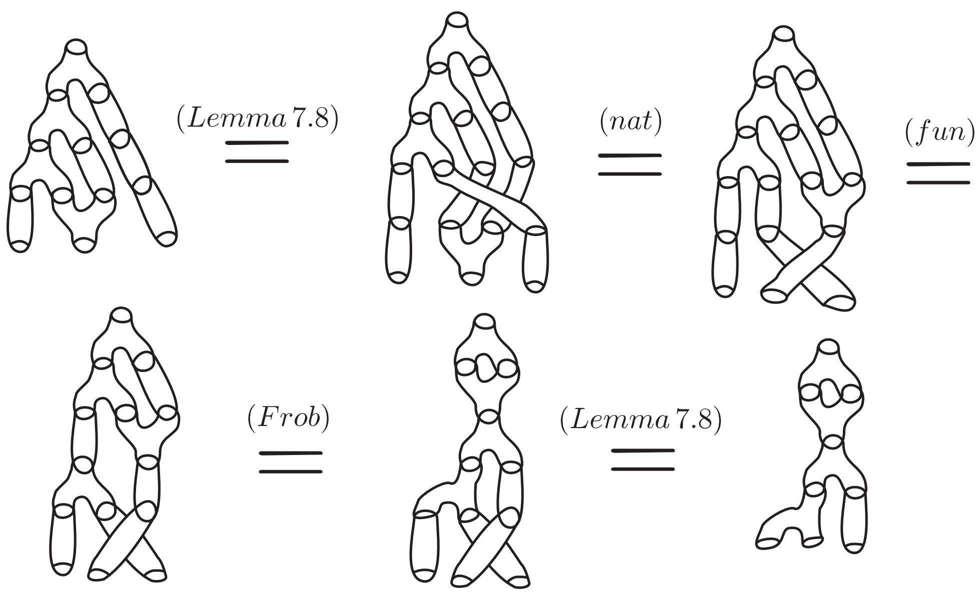


Figure 9: Lemma 7.11

By the equations (*fun*) and (*Frob*), we have the following.

7.12. Lemma. *For $n, m \geq 0$, $(\mathbf{1}_n \otimes \mu^\Diamond \otimes \mathbf{1}_m) \circ (\Lambda_n \otimes \Lambda_m) = \Lambda_{n+m} \circ \mu^\Diamond$.*

By the equations (*fun*), (*assoc*) and (*Frob*), we have the following.

7.13. Lemma. *For $n, m \geq 0$, $\mu^\Diamond \circ (H_n \otimes H_m) = H_{n+m} \circ \mu^\Diamond$.*

By the equations (*fun*), (*assoc*) or (*unit*), we have the following.

7.14. Lemma. *For $n, m \geq -1$, $\mu^\Diamond \circ (V_n \otimes V_m) = V_{n+m+1}$.*

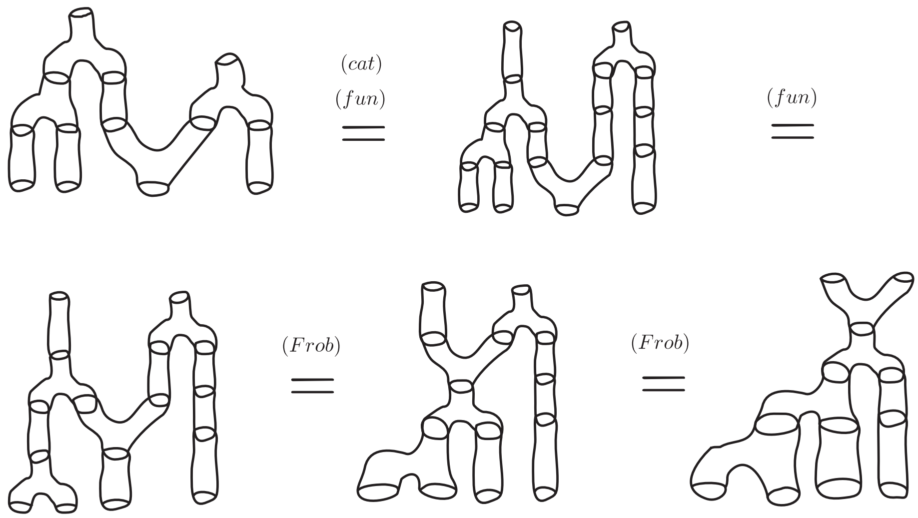


Figure 10: Lemma 7.12

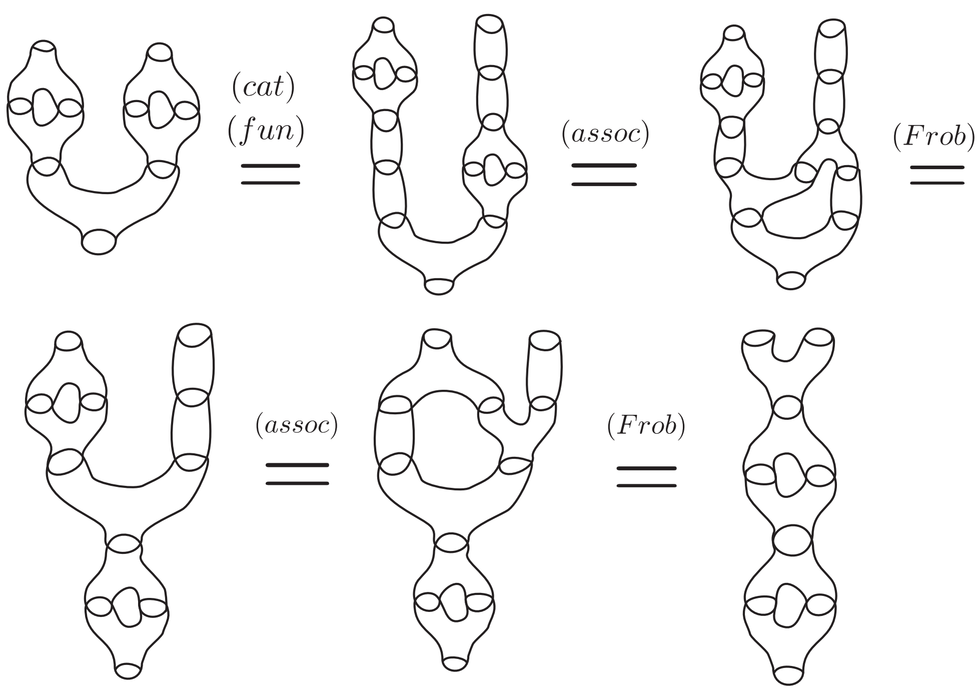


Figure 11: Lemma 7.13

Proof of Proposition 7.1. Let f be a term. By Lemma 7.2, f is equal to a term of the form $f_n \circ \ldots \circ f_0$, where every f_i is of the form $\mathbf{1}_l \otimes \beta \otimes \mathbf{1}_r$, for $l, r \geq 0$ and $\beta \in \{\tau, \mu^{\Box}, \eta^{\Box}, \eta^{\Diamond}, \mu^{\Diamond}\}$. We proceed by induction on $n \geq 0$. (The indices of identities not important for our calculations are usually omitted.)

If $n = 0$, then since $\mathbf{1}_l \otimes \beta \otimes \mathbf{1}_r$ is special, we are done.

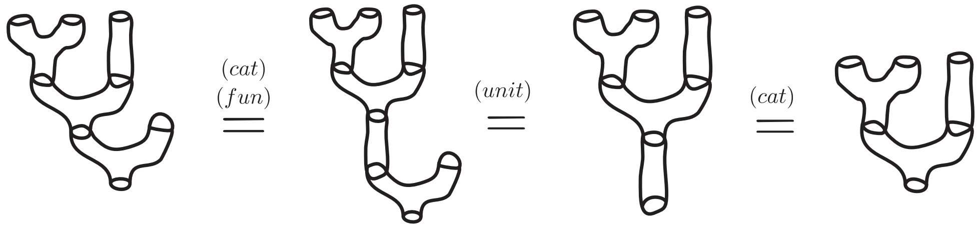


Figure 12: Lemma 7.14, $m = -1$

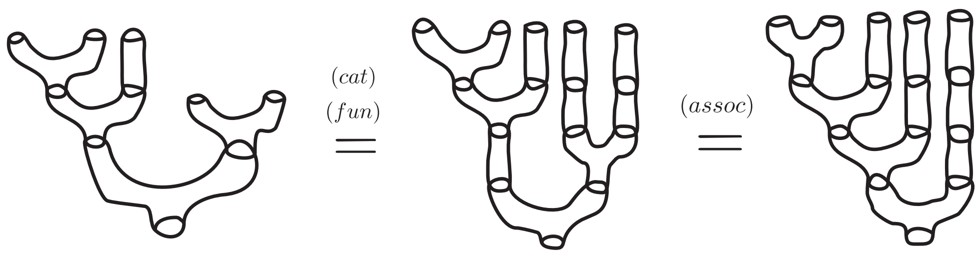


Figure 13: Lemma 7.14

If $n > 0$, then by the induction hypothesis, $f_{n-1} \circ \ldots \circ f_0$ is equal to a term of the form $\pi \circ \bigotimes_{i=1}^{k} E_{p_i,m_i,n_i} \circ \chi$. We have the following cases concerning f_n.

If f_n is $\mathbf{1}_l \otimes \tau \otimes \mathbf{1}_r$, then $f_n \circ \pi$ is a τ-term and we are done.

If f_n is $\mathbf{1}_l \otimes \mu^{\Box} \otimes \mathbf{1}_r$, then by Lemma 7.4, we have a τ-term π' such that

$$\begin{aligned}
f_n \circ \pi &= (\mathbf{1}_l \otimes \mu^{\Box} \otimes \mathbf{1}) \circ (\tau_{1,l} \otimes \mathbf{1}) \circ (\mathbf{1}_1 \otimes \pi') \circ (\tau_{j,1} \otimes \mathbf{1}) \\
&= (\tau_{2,l} \otimes \mathbf{1}) \circ (\mu^{\Box} \otimes \mathbf{1}) \circ (\mathbf{1}_1 \otimes \pi') \circ (\tau_{j,1} \otimes \mathbf{1}) && (nat) \\
&= (\tau_{2,l} \otimes \mathbf{1}) \circ (\mathbf{1}_2 \otimes \pi') \circ (\mu^{\Box} \otimes \mathbf{1}) \circ (\tau_{j,1} \otimes \mathbf{1}) && (cat), (fun) \\
&= (\tau_{2,l} \otimes \mathbf{1}) \circ (\mathbf{1}_2 \otimes \pi') \circ (\tau_{j,2} \otimes \mathbf{1}) \circ (\mathbf{1}_j \otimes \mu^{\Box} \otimes \mathbf{1}) && (nat)
\end{aligned}$$

Then for some $u \in k$, by (fun), $(\mathbf{1}_j \otimes \mu^{\Box} \otimes \mathbf{1}) \circ \bigotimes_{i=1}^{k} E_{p_i,m_i,n_i}$ is equal to

$$(\mathbf{1} \otimes ((\mathbf{1} \otimes \mu^{\Box} \otimes \mathbf{1}) \circ E_{p_u,m_u,n_u}) \otimes \mathbf{1}) \circ (\bigotimes_{i=1}^{u-1} E_{p_i,m_i,n_i} \otimes \mathbf{1}_{n_u} \otimes \bigotimes_{i=u+1}^{k} E_{p_i,m_i,n_i}),$$

which is, with the help of Lemma 7.5 and (fun) again, equal to the new center

$$\bigotimes_{i=1}^{u-1} E_{p_i,m_i,n_i} \otimes E_{p_u+1,m_u,n_u} \otimes \bigotimes_{i=u+1}^{k} E_{p_i,m_i,n_i}.$$

If f_n is $\mathbf{1}_l \otimes \eta^{\Box} \otimes \mathbf{1}_r$, then we proceed as in the preceding case, just by relying on Lemma 7.6 instead of Lemma 7.5 in order to obtain the new center

$$\bigotimes_{i=1}^{u-1} E_{p_i,m_i,n_i} \otimes E_{p_u-1,m_u,n_u} \otimes \bigotimes_{i=u+1}^{k} E_{p_i,m_i,n_i}.$$

If f_n is $\mathbf{1}_l \otimes \eta^\Diamond \otimes \mathbf{1}_r$, then by relying on Lemma 7.7 we have the following

$$\begin{aligned} f_n \circ \pi \circ \textstyle\bigotimes_{i=1}^k E_{p_i,m_i,n_i} &= (\tau_{1,l} \otimes \mathbf{1}) \circ (\mathbf{1}_1 \otimes \pi) \circ (\eta^\Diamond \otimes \mathbf{1}) \circ \textstyle\bigotimes_{i=1}^k E_{p_i,m_i,n_i} \\ &= (\tau_{1,l} \otimes \mathbf{1}) \circ (\mathbf{1}_1 \otimes \pi) \circ \textstyle\bigotimes_{i=0}^k E_{p_i,m_i,n_i} \qquad (fun), \end{aligned}$$

where $p_0 = 1$ and $m_0 = n_0 = 0$.

If f_n is $\mathbf{1}_l \otimes \mu^\Diamond \otimes \mathbf{1}_r$, then by Lemmata 7.9 and 7.10, we may assume that the tail π of a special term equal to $f_{n-1} \circ \ldots \circ f_0$ is of the form

$$(\tau_{2,l} \otimes \mathbf{1}_{p-l-2}) \circ (\mathbf{1}_2 \otimes \pi') \circ (\tau_{j,2} \otimes \mathbf{1}_{p-j-2}).$$

As above, we obtain

$$f_n \circ \pi = (\tau_{1,l} \otimes \mathbf{1}) \circ (\mathbf{1}_1 \otimes \pi') \circ (\tau_{j,1} \otimes \mathbf{1}) \circ (\mathbf{1}_j \otimes \mu^\Diamond \otimes \mathbf{1}).$$

There are two possibilities concerning the term

$$(\mathbf{1}_j \otimes \mu^\Diamond \otimes \mathbf{1}) \circ \bigotimes_{i=1}^k E_{p_i,m_i,n_i}.$$

Either it is equal to

$$(\mathbf{1} \otimes ((\mathbf{1} \otimes \mu^\Diamond \otimes \mathbf{1}) \circ E_{p_u,m_u,n_u}) \otimes \mathbf{1}) \circ (\bigotimes_{i=1}^{u-1} E_{p_i,m_i,n_i} \otimes \mathbf{1}_{n_u} \otimes \bigotimes_{i=u+1}^{k} E_{p_i,m_i,n_i})$$

when we apply Lemma 7.11, with the help of (fun), in order to obtain

$$\bigotimes_{i=1}^{u-1} E_{p_i,m_i,n_i} \otimes E_{p_u-1,m_u+1,n_u} \otimes \bigotimes_{i=u+1}^{k} E_{p_i,m_i,n_i},$$

or it is equal to

$$\begin{aligned} &(\mathbf{1} \otimes ((\mathbf{1} \otimes \mu^\Diamond \otimes \mathbf{1}) \circ (E_{p_u,m_u,n_u} \otimes E_{p_{u+1},m_{u+1},n_{u+1}})) \otimes \mathbf{1}) \circ \\ &\qquad (\textstyle\bigotimes_{i=1}^{u-1} E_{p_i,m_i,n_i} \otimes \mathbf{1}_{n_u+n_{u+1}} \otimes \textstyle\bigotimes_{i=u+2}^{k} E_{p_i,m_i,n_i}) \end{aligned}$$

when we apply Lemmata 7.12, 7.13 and 7.14 in order to obtain

$$\bigotimes_{i=1}^{u-1} E_{p_i,m_i,n_i} \otimes E_{p_u+p_{u+1}-1,m_u+m_{u+1},n_u+n_{u+1}} \otimes \bigotimes_{i=u+2}^{k} E_{p_i,m_i,n_i}$$

∎

For $a, b, c, d \geq 0$, and $n_i, q_i, s_i, u_i \geq 1$ consider a special term of the form

$$\pi \circ (\bigotimes_{i=1}^{a} E_{0,m_i,0} \otimes \bigotimes_{i=1}^{b} E_{0,p_i,n_i} \otimes \bigotimes_{i=1}^{c} E_{q_i,r_i,0} \otimes \bigotimes_{i=1}^{d} E_{s_i,t_i,u_i}) \circ \chi.$$

If $b \geq 1$, let $\beta^1 = 0$, and let $\beta^i = n_1 + \ldots + n_{i-1}$, for $i \in \{2, \ldots, b\}$.

If $d \geq 1$, let $\delta^1 = n_1 + \ldots + n_b$, $\delta_1 = q_1 + \ldots + q_c$, and for $i \in \{2, \ldots, d\}$, let $\delta^i = n_1 + \ldots + n_b + \ldots + u_1 + \ldots + u_{i-1}$ and $\delta_i = q_1 + \ldots + q_c + \ldots + s_1 + \ldots + s_{i-1}$.

If $c \geq 1$, let $\gamma_1 = 0$, and for $i \in \{2, \ldots, c\}$, let $\gamma_i = q_1 + \ldots + q_{i-1}$. Such a special term is in *normal form* when

$$m_1 \leq m_2 \leq \ldots \leq m_a,$$

$$\chi^{-1}(\beta^1) < \chi^{-1}(\beta^2) < \ldots < \chi^{-1}(\beta^b),$$

$$\pi(\gamma_1) < \pi(\gamma_2) < \ldots < \pi(\gamma_c),$$

$$\pi(\delta_1) < \pi(\delta_2) < \ldots < \pi(\delta_d),$$

for every $i \in \{1, \ldots, b\}$

$$\chi^{-1}(\beta^i) < \chi^{-1}(\beta^i + 1) < \ldots < \chi^{-1}(\beta^i + n_i - 1),$$

for every $i \in \{1, \ldots, d\}$

$$\chi^{-1}(\delta^i) < \chi^{-1}(\delta^i + 1) < \ldots < \chi^{-1}(\delta^i + u_i - 1),$$

$$\pi(\delta_i) < \pi(\delta_i + 1) < \ldots < \pi(\delta_i + s_i - 1),$$

and finally, for every $i \in \{1, \ldots, c\}$

$$\pi(\gamma_i) < \pi(\gamma_i + 1) < \ldots < \pi(\gamma_i + q_i - 1).$$

By Proposition 7.1, Lemma 7.8 and the equation

$$f_1 \otimes f_2 = \tau_{m_2,m_1} \circ (f_2 \otimes f_1) \circ \tau_{n_1,n_2},$$

which follows from (*nat*) and (*inv*), we can prove the following.

7.15. THEOREM. *Every term is equal to a term in normal form.*

8. Faithfulness of the interpretation

The aim of this section is to prove the following result.

8.1. Theorem. *For every $d \geq 2$, the interpretation of* $\mathbf{K}$ *in dCobS is faithful.*

For the proof of this theorem, we need some auxiliary notions and results. Every d-cobordism $K = (M, \Sigma_0, \Sigma_1) : \underline{n} \to \underline{m}$, $\Sigma_0 = (\Sigma_0^0, \ldots, \Sigma_0^{n-1})$ and $\Sigma_1 = (\Sigma_1^0, \ldots, \Sigma_1^{m-1})$, induces the following equivalence relation ρ_K on the set $(n \times \{0\}) \cup (m \times \{1\})$ (cf. the relation with the same name defined in Section 5). For (i, k) and (j, l) elements of $(n \times \{0\}) \cup (m \times \{1\})$, we have that $(i, k)\rho_K(j, l)$ when Σ_k^i and Σ_l^j belong to the same connected component of M.

From Proposition 3.5, and since homeomorphisms preserve connected components, we have the following lemma.

8.2. Lemma. *If two d-cobordisms $K = (M, \Sigma_0, \Sigma_1)$ and $L = (N, \Delta_0, \Delta_1)$ are equivalent, then $\rho_K = \rho_L$.*

The following proposition serves to prove that our categories are skeletal.

8.3. Proposition. *If $K : \underline{n} \to \underline{m}$ is an isomorphism, then $n = m$.*

Proof. We prove that every equivalence class of ρ_K has two elements, one with the second component 0 and the other with the second component 1, from which the proposition follows. Let $L : \underline{m} \to \underline{n}$ be the inverse of K.

Suppose that an equivalence class of ρ_K is a singleton $\{(i, 0)\}$. Then $\{(i, 0)\}$ is an equivalence class of $\rho_{L\circ K}$, which is impossible by Lemma 8.2, since $L \circ K$ is equivalent to the identity d-cobordism.

Suppose that for $i \neq j$, an equivalence class of ρ_K contains both $(i, 0)$ and $(j, 0)$. Then an equivalence class of $\rho_{L\circ K}$ contains both $(i, 0)$ and $(j, 0)$, which is again impossible by Lemma 8.2.

We proceed analogously, by relying on $\rho_{K\circ L}$, in cases when an equivalence class of ρ_K is a singleton $\{(i, 1)\}$ or when for $i \neq j$, an equivalence class of ρ_K contains both $(i, 1)$ and $(j, 1)$. ∎

8.4. Corollary. *The categories dCobS, for $d \geq 2$, and* $\mathbf{K}$ *are skeletal.*

That $1CobS$ is also skeletal is proved in Section 5. The following implication has a trivial converse.

8.5. Lemma. *If $\underline{E_{0,n,0}} \sim \underline{E_{0,m,0}}$, then $n = m$.*

Proof. The d-manifolds underlying the cobordisms $\underline{E_{0,n,0}}$ and $\underline{E_{0,m,0}}$ are closed, and these d-cobordisms can be identified with the underlying manifolds. Moreover, $\underline{E_{0,n,0}} \sim \underline{E_{0,m,0}}$ means that these manifolds are homeomorphic. In the case when $d = 2$, we have that n is the genus of $\underline{E_{0,n,0}}$, and when $d \geq 3$, we have that $\underline{E_{0,1,0}}$ is homeomorphic to $S^{d-1} \times S^1$, which with the help of Van Kampen's Theorem asserts that the fundamental group of $\underline{E_{0,n,0}}$ is the free group with n generators. ∎

In the sequel, we assume that f and f' are two normal forms

$$\pi \circ (\bigotimes_{i=1}^{a} E_{0,m_i,0} \otimes \bigotimes_{i=1}^{b} E_{0,p_i,n_i} \otimes \bigotimes_{i=1}^{c} E_{q_i,r_i,0} \otimes \bigotimes_{i=1}^{d} E_{s_i,t_i,u_i}) \circ \chi$$

and

$$\pi' \circ (\bigotimes_{i=1}^{a'} E_{0,m'_i,0} \otimes \bigotimes_{i=1}^{b'} E_{0,p'_i,n'_i} \otimes \bigotimes_{i=1}^{c'} E_{q'_i,r'_i,0} \otimes \bigotimes_{i=1}^{d'} E_{s'_i,t'_i,u'_i}) \circ \chi'.$$

8.6. Proposition. *If $\underline{f} \sim \underline{f'}$, then $a = a'$ and $m_i = m'_i$ for every $1 \leq i \leq a$.*

Proof. Since every homeomorphism justifying $\underline{f} \sim \underline{f'}$ maps the closed components of $\underline{f}$ to the closed components of $\underline{f'}$, there must be a bijection from $\{1, \dots, a\}$ to $\{1, \dots, a'\}$ such that $\underline{E_{0,m_i,0}} \sim \underline{E_{0,m'_j,0}}$, for j corresponding to i by this bijection. Hence, we have $a = a'$, and by Lemma 8.5, since the sequences (m_i) and (m'_i) are increasing, we conclude that $m_i = m'_i$ for every $1 \leq i \leq a$. ∎

The following proposition has Theorem 8.1 as an immediate corollary.

8.7. Proposition. *If $\underline{f} \sim \underline{f'}$, then f and f' are identical.*

Proof. By Proposition 3.5 we have that f and f' are of the same type $n \to m$. We proceed by induction on $n + m$. If $n + m = 0$, then we apply Proposition 8.6.

If $n + m > 0$, let ρ be the equivalence relation corresponding, by Lemma 8.2, both to $\underline{f}$ and $\underline{f'}$. Suppose that $b > 0$, hence E_{0,p_1,n_1} appears in f. The relation ρ guaranties that $b' > 0$. Let $X = \chi^{-1}[\{0, \dots, n_1 - 1\}]$. The set $X \times \{0\}$ is an equivalence class of ρ, namely the equivalence class of $(\chi^{-1}(0), 0)$. Our normal form and the relation ρ guarantee that $n'_1 = n_1$, and that χ and χ' coincide on X.

Let g be the term $g_0 \otimes \dots \otimes g_{n-1}$, where

$$g_i = \begin{cases} \mathbf{1}_1, & i \notin X, \\ \eta^\Diamond, & i \in X. \end{cases}$$

By relying on the equation (*nat*), $f \circ g$ is equal to the normal form f_1

$$\pi \circ (A \otimes \bigotimes_{i=2}^{b} E_{0,p_i,n_i} \otimes C \otimes D) \circ \chi_1,$$

where A is of the form

$$\bigotimes_{i=1}^{k} E_{0,m_i,0} \otimes E_{0,p_1,0} \otimes \bigotimes_{i=k+1}^{a} E_{0,m_i,0},$$

while C is $\bigotimes_{i=1}^{c} E_{q_i,r_i,0}$, and D is $\bigotimes_{i=1}^{d} E_{s_i,t_i,u_i}$. Analogously, we conclude that $f' \circ g$ is equal to the normal form f'_1

$$\pi' \circ (A' \otimes \bigotimes_{i=2}^{b'} E_{0,p'_i,n'_i} \otimes C' \otimes D') \circ \chi'_1,$$

with the abbreviations A', C' and D' as above.

From $\underline{f \circ g} \sim \underline{f' \circ g}$, since the interpretation is a functor, it follows that $\underline{f_1} \sim \underline{f'_1}$. By the induction hypothesis f_1 and f'_1 are identical. We have that $\bigotimes_{i=1}^{a} E_{0,m_i,0}$ and $\bigotimes_{i=1}^{a'} E_{0,m'_i,0}$ are identical, by Proposition 8.6, which together with the fact that A and A' are identical delivers that $p_1 = p'_1$. It remains only to prove that the permutations χ and χ' are equal, which follows from the fact that χ_1 and χ'_1 are equal and that χ and χ' coincide on X.

We proceed analogously in all the other situations ($b = 0$ and $c > 0$, or $b = c = 0$ and $d > 0$). ∎

Since the interpretation of $\mathbf{K}$ in $dCobS$ is one-one on objects, it is an embedding. The following corollary asserting that $2CobS$ is a PROP having $\underline{1}$ as the universal commutative Frobenius object is already given in [14, Theorem 3.6.19].

8.8. Corollary. *The category* $\mathbf{K}$ *is isomorphic to* $2CobS$.

Proof. From the classification theorem for 2-manifolds (see for example [26, VI.40]) it follows that, in the case $d = 2$, the interpretation is full. ∎

However, for $d > 2$, the interpretation is not full, and hence not an isomorphism.

Appendix

9. Topological manifolds, orientation and gluing

For $n \geq 0$, an *n-dimensional manifold* M is a second countable Hausdorff space that is locally Euclidean of dimension n. This means that the topology of M admits a countable basis, that there are disjoint neighborhoods of every pair of distinct points in M, and that every point in M has a neighborhood homeomorphic to an open subset of $\mathbf{R}^n$. A *chart* of M is a homeomorphism $\varphi : U \to U'$, where $U \subseteq M$ and $U' \subseteq \mathbf{R}^n$ are open. An *atlas* of M is a collection of its charts $\{\varphi_i : U_i \to U'_i \mid i \in I\}$ such that $\bigcup\{U_i \mid i \in I\} = M$.

For $n \geq 1$, an n-dimensional *manifold with boundary*, shortly *∂-manifold*, is a second countable Hausdorff space in which every point has a neighborhood homeomorphic to an open subset of the halfspace

$$\pi_n^+ = \{(x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_n \geq 0\}.$$

A *chart* of an n-dimensional ∂-manifold M is a homeomorphism $\varphi : U \to U'$, where $U \subseteq M$ and $U' \subseteq \pi_n^+$ are open. An *atlas* of M is again a collection of its charts whose domains cover M.

A *boundary point* of M is a point mapped to a point in the hyperplane

$$\pi_n = \{(x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_n = 0\}$$

by some chart, otherwise, it is an *interior point* of M. The set of boundary points of M is its *boundary* ∂M, which is an $(n-1)$-dimensional manifold, and the set of interior points of M is its *interior* $\mathrm{Int}M$, which is an n-dimensional manifold. The *interior* $\mathrm{Int}U$ of an open subset U of M is $U - \partial M$. Every n-dimensional manifold, for $n \geq 1$, is an n-dimensional ∂-manifold, with the empty boundary.

A homeomorphism $f : U \to V$ for open $U, V \subseteq \mathbf{R}^n$, $n \geq 1$, is *orientation preserving* when for every $x \in U$ the following isomorphism of homology groups with coefficients in $\mathbf{Z}$ is the identity

$$H_n(\mathbf{R}^n, \mathbf{R}^n - \{0\}) \xrightarrow{\cong} H_n(U, U - \{x\}) \xrightarrow{f_*} H_n(V, V - \{f(x)\}) \xrightarrow{\cong} H_n(\mathbf{R}^n, \mathbf{R}^n - \{0\}).$$

Here, the first isomorphism is the composition

$$H_n(\mathbf{R}^n, \mathbf{R}^n - \{0\}) \xrightarrow{(t_x)_*} H_n(\mathbf{R}^n, \mathbf{R}^n - \{x\}) \xrightarrow{\text{excision}} H_n(U, U - \{x\}),$$

where $t_x : \mathbf{R}^n \to \mathbf{R}^n$ is the translation by x, and the last isomorphism is defined analogously.

9.1. Lemma. *Let $\{W_i \mid i \in I\}$ be an open cover of an open subset U of $\mathbf{R}^n$. A homeomorphism $f : U \to V$, for V an open subset of $\mathbf{R}^n$, is orientation preserving iff for every $i \in I$, the restriction of f to W_i is orientation preserving.*

An atlas $\{\varphi_i : U_i \to U'_i \mid i \in I\}$ of an n-dimensional manifold, $n \geq 1$, is *oriented* when for every $i, j \in I$, the homeomorphism

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j[U_i \cap U_j] \to \varphi_i[U_i \cap U_j]$$

is orientation preserving. A manifold possessing such an atlas is *orientable*. An oriented atlas is *maximal* when it cannot be enlarged to an oriented atlas of the manifold by adding another chart.

Two oriented atlases $\{\varphi_i : U_i \to U'_i \mid i \in I\}$ and $\{\psi_j : V_j \to V'_j \mid j \in J\}$ of the same manifold are *equivalent* when, for every $i \in I$ and every $j \in J$, the homeomorphism

$$\varphi_i \circ \psi_j^{-1} : \psi_j[U_i \cap V_j] \to \varphi_i[U_i \cap V_j]$$

is orientation preserving (cf. [27, Definition 21.11]).

9.2. Proposition. *If two oriented atlases of a manifold are equivalent, then their union is an oriented atlas of this manifold.*

With the help of Lemma 9.1 for transitivity, we can prove the following.

9.3. Proposition. *The above relation is an equivalence relation on the set of oriented atlases of an orientable manifold.*

If an orientable manifold is connected, then this equivalence relation has exactly two classes. As a corollary of Propositions 9.2 and 9.3, we have the following.

9.4. PROPOSITION. *Every oriented atlas could be enlarged to a unique maximal oriented atlas.*

An *orientation* of a 0-dimensional manifold M is a function $\varepsilon : M \to \{-1, 1\}$. For $n \geq 1$, an *orientation* of an orientable n-dimensional manifold M is a choice of its maximal oriented atlas $\mathcal{O}_M$. The orientation opposite to $\mathcal{O}_M$ is obtained by composing every chart in it by a reflection of $\mathbf{R}^n$, for example with respect to π_n.

The *orientation of the product* of two oriented manifolds M and N is given by the maximal oriented atlas containing the products of charts in $\mathcal{O}_M$ with charts in $\mathcal{O}_N$. A homeomorphism f between two oriented n-dimensional manifolds M and N is *orientation preserving* when for every chart $\varphi : U \to U'$ of M, for g being the restriction of f^{-1} to $f[U]$, we have that

$$\varphi \in \mathcal{O}_M \quad \text{iff} \quad \varphi \circ g \in \mathcal{O}_N.$$

An embedding of an n-dimensional manifold into an n-dimensional manifold is *orientation preserving* when its restriction to the image is such. An *orientation reversing* homeomorphism (embedding) from M to N is an orientation preserving homeomorphism (embedding) from M to N with the opposite orientation.

An n-dimensional ∂-manifold, for $n \geq 1$, is orientable when its interior is orientable and an orientation of the interior is an orientation of the ∂-manifold. We denote the orientation of an oriented ∂-manifold M again by $\mathcal{O}_M$. We say that an oriented n-dimensional ∂-manifold $M \subseteq \mathbf{R}^n$ is *oriented by the identity* when its orientation contains the charts $\mathbf{1}_U : U \to U$ for every open $U \subseteq \mathrm{Int}M$.

The orientation of an oriented ∂-manifold *induces the orientation* of its boundary in the following way. For an oriented 1-dimensional ∂-manifold M and $x \in \partial M$, we orient x by $\varepsilon(x) = 1$, when for a neighborhood U of x in M there is a chart $\varphi : U \to U'$, $U' \subseteq \{y \in \mathbf{R} \mid y \geq 0\}$, such that its restriction to $\mathrm{Int}U$ is in $\mathcal{O}_M$. Otherwise, we orient x by $\varepsilon(x) = -1$. For example, if $I = [0, 1]$ is oriented by the identity, then $\varepsilon(0) = 1$ and $\varepsilon(1) = -1$. (Note that this is opposite to the orientation given in [15] but it is consistent with the orientation given in [14].)

An orientation of the sphere S^0 is taken to be induced from an orientation of the interval $[-1, 1]$. Hence, in every orientation of S^0, one point is positive and the other is negative.

For $n \geq 2$, an oriented n-dimensional ∂-manifold M induces the orientation of ∂M given by the maximal oriented atlas containing the restriction of φ to ∂U for every chart $\varphi : U \to U'$, $U' \subseteq \pi_n^+$, whose restriction to $\mathrm{Int}U$ belongs to $\mathcal{O}_M$. For example, if π_n^+ is oriented by the identity, then its boundary π_n is oriented by the identity. If $\pi_n^- = \{(x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_n \leq 0\}$ is oriented by the identity, then its boundary π_n is oriented by the maximal oriented atlas containing the restriction of the reflection $g : \pi_n \to \pi_n$, given by

$$g(x_1, x_2, \ldots, x_{n-1}, 0) = (-x_1, x_2, \ldots, x_{n-1}, 0),$$

to every open $U \subseteq \pi_n$, i.e. it has the opposite orientation from the one in the previous example.

Let Σ_M be a collection of connected components of the boundary of an n-dimensional ∂-manifold M. An embedding of an oriented $(n-1)$-manifold into M, whose image is Σ_M, is *orientation preserving* (*reversing*) when its restriction to the image, with respect to the induced orientation of Σ_M, is such.

We discuss now pushouts in the category of topological spaces, and in particular the case involving ∂-manifolds and oriented ∂-manifolds. For topological spaces X, Y and Z and continuous functions $f : Z \to X$ and $g : Z \to Y$, let $\asymp$ be the smallest equivalence relation on the disjoint union

$$X + Y = (X \times \{0\}) \cup (Y \times \{1\})$$

such that for every $z \in Z$ we have that $(f(z), 0) \asymp (g(z), 1)$.

For functions $i : X \to (X+Y)/\asymp$ and $j : Y \to (X+Y)/\asymp$ defined by $i(x) = [(x, 0)]_{\asymp}$ and $j(y) = [(y, 1)]_{\asymp}$, let the topological space $X +_{f,g} Y$ be given by the set $(X + Y)/\asymp$ with the topology

$$\mathcal{T} = \{U \subseteq (X+Y)/\asymp \;\mid i^{-1}[U] \text{ is open in } X \text{ and } j^{-1}[U] \text{ is open in } Y\}.$$

This is a *pushout* in the category of topological spaces, i.e. we have the commutative diagram

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "j"] \\ X \arrow[r, "i"'] & X +_{f,g} Y \end{tikzcd}$$

with the following universal property. For every pair of continuous functions $i' : X \to A$ and $j' : Y \to A$ such that $i' \circ f = j' \circ g$, there is a unique continuous function $h : X +_{f,g} Y \to A$ such that $h \circ i = i'$ and $h \circ j = j'$.

Let M and N be two n-dimensional ∂-manifolds and let Σ_M and Σ_N be collections of connected components of ∂M and ∂N respectively, such that Σ_M and Σ_N are both homeomorphic to an $(n-1)$-dimensional manifold Σ. Let $f : \Sigma \to M$ and $g : \Sigma \to N$ be two embeddings whose images are Σ_M and Σ_N respectively.

9.5. PROPOSITION. *The space $M +_{f,g} N$ is an n-dimensional ∂-manifold.*

PROOF. Note that for an n-dimensional ∂-manifold M we have that if K is a connected component of ∂M, then $M - K$ is an n-dimensional ∂-manifold whose boundary is $\partial M - K$. Then we rely on [7, Chapter VIII, Proposition 1.11]. ∎

In the case when M and N are two orientable n-dimensional ∂-manifolds and Σ_M, Σ_N and Σ are as above, let $f : \Sigma \to M$ be an orientation preserving embedding whose image is Σ_M, and let $g : \Sigma \to N$ be an orientation reversing embedding whose image is Σ_N. Then the n-dimensional ∂-manifold $M +_{f,g} N$ is orientable.

For charts $\varphi : U \to U'$ and $\psi : V \to V'$ of M and N respectively, such that there is $\Gamma \subseteq \Sigma$, possibly empty, with $\partial U = f[\Gamma]$ and $\partial V = g[\Gamma]$, let $\varphi +_{f,g} \psi$ be the homeomorphism from $U +_{f,g} V$ to $U' +_{\varphi\circ f,\psi\circ g} V'$, where by f and g we mean their restrictions to Γ. This homeomorphism exists by the universal property of pushout. We define the orientation of $M +_{f,g} N$ to be the maximal oriented atlas containing $\varphi +_{f,g} \psi$ for every pair of charts φ and ψ as above such that the restriction of φ to Int$U$ is in $\mathcal{O}_M$ and the restriction of ψ to Int$V$ is in $\mathcal{O}_N$. In this way the restrictions to the interiors of the embeddings $i : M \to M +_{f,g} N$ and $j : N \to M +_{f,g} N$ are orientation preserving.

Locally, the situation is completely illustrated by the following example. For $n \geq 2$, let π_n^+ and π_n be oriented by the identity. For $f, g : \pi_n \to \pi_n^+$ being orientation preserving, respectively orientation reversing, embeddings, with π_n as the image, consider the n-dimensional manifold $\pi_n^+ +_{f,g} \pi_n^+$. Without loss of generality, we may assume that f is the inclusion and that g is the reflection

$$g(x_1, x_2, \ldots, x_{n-1}, 0) = (-x_1, x_2, \ldots, x_{n-1}, 0).$$

Let $\mathbf{g} : \pi_n^+ \to \pi_n^-$ be the composition of two reflections of $\mathbf{R}^n$—one with respect to the hyperplane $\pi_1 = \{(x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_1 = 0\}$ and the other with respect to the hyperplane π_n. Note that $\mathbf{g}$ is orientation preserving and its restriction to π_n is the reflection $g : \pi_n \to \pi_n$ from above. Hence, $\mathbf{g}$ reverses the orientation of the boundary. However, the composition $\mathbf{g} \circ g : \pi_n \to \pi_n^-$ is the inclusion.

Now we have the following commutative diagram

$$\begin{tikzcd}
\pi_n^+ \arrow[dd, "\mathbf{1}"'] \arrow[dr, "i"'] & \pi_n \arrow[l, "\supseteq"'] \arrow[r, "g"] & \pi_n^+ \arrow[dd, "\mathbf{g}"] \arrow[dl, "j"] \\
& \pi_n^+ +_{f,g} \pi_n^+ \arrow[d, dotted, "h"] & \\
\pi_n^+ \arrow[r, "\subseteq"] & \mathbf{R}^n & \pi_n^- \arrow[l, "\supseteq"']
\end{tikzcd}$$

which by the universal property of pushout leads to the homeomorphism $h : \pi_n^+ +_{f,g} \pi_n^+ \to \mathbf{R}^n$. This homeomorphism is orientation preserving when $\mathbf{R}^n$ is oriented by the identity.

10. Some topological remarks

The classical results formulated in this section are used in Section 3. The following theorem is proved for $n = 2$ by Radó, [24], for $n = 3$ by Moise, [20], for $n = 4$ by Quinn, [22], for $n \geq 5$ by Kirby, [13], and it is trivial for $n = 1$.

10.1. THEOREM. [Annulus conjecture, AC_n] *Let $f, g : S^{n-1} \to \mathbf{R}^n$ be disjoint, locally flat embeddings with $f[S^{n-1}]$ inside the bounded component of $\mathbf{R}^n - g[S^{n-1}]$. Then the closed region bounded by $f[S^{n-1}]$ and $g[S^{n-1}]$ is homeomorphic to $S^{n-1} \times I$.*

A homeomorphism from $\mathbf{R}^n$ to $\mathbf{R}^n$ or from S^n to S^n is called *stable*, when it is equal to a finite composition of homeomorphisms each of which is the identity on some non-empty open set. Brown and Gluck, [4], proved that Annulus conjecture is equivalent to the following statement, which is hence a theorem.

10.2. Theorem. [Stable homeomorphism conjecture] *Any orientation preserving homeomorphism of $\mathbf{R}^n$ is stable.*

Two homeomorphisms $f, g : X \to Y$ are *isotopic* when there is a homotopy $\Phi : X \times I \to Y$ from f to g such that every $\Phi_t : X \to Y$ is a homeomorphism. Such a homotopy is called *isotopy.*

10.3. Theorem. [Alexander] *Every homeomorphism from $\mathbf{R}^n$ to $\mathbf{R}^n$, or from S^n to S^n, whose restriction to some non-empty open set is the identity, is isotopic to the identity.*

10.4. Lemma. *If $\Phi_t : X \to X$ is an isotopy from f to g and $\Gamma_t : X \to X$ is an isotopy from u to v, then $\Gamma_t \circ \Phi_t$ is an isotopy from $u \circ f$ to $v \circ g$.*

10.5. Proposition. *Every orientation preserving homeomorphism $f : S^n \to S^n$ is isotopic to the identity.*

Proof. Let $p \in S^n$ and let $g : S^n \to S^n$ be a homeomorphism whose restriction to some non-empty open set is the identity, and such that $g(f(p)) = p$. (It is not difficult to construct such a g). For $h = g \circ f$ we have that its restriction to $S^n - \{p\}$, which is homeomorphic to $\mathbf{R}^n$, is a homeomorphism from $S^n - \{p\}$ to $S^n - \{p\}$.

By Theorem 10.2, we have that this restriction is equal to a composition of homeomorphisms $h_k \circ \ldots \circ h_1$ such that every h_i restricted to some non-empty open set is the identity. If we define $h_i(p) = p$, then every $h_i : S^n \to S^n$ is a homeomorphism and $f = g^{-1} \circ h_k \circ \ldots \circ h_1$. Hence, f is stable. By Theorem 10.3, and Lemma 10.4, f is isotopic to the identity. ∎

10.6. Theorem. [Invariance of Domain Theorem] *If M and N are topological n-manifolds without boundaries and $f : M \to N$ is a continuous 1-1 map, then f is an open map.*

10.7. Lemma. [Pasting Lemma] *For X, Y both closed or both open subsets of $A = X \cup Y$, if for $f : A \to B$ both its restrictions to X and Y are continuous, then f is continuous too.*

10.8. Proposition. *If $\Phi_t : S^n \to S^n$ is an isotopy from the identity to f, then $F : S^n \times I \to S^n \times I$ defined by $F(x,t) = (\Phi_t(x), t)$ is a homeomorphism.*

Proof. We have that F is continuous and that F^{-1} defined by $F^{-1}(x,t) = (\Phi_t^{-1}(x), t)$ is its inverse. It remains to prove that F^{-1} is continuous.

Let $G : S^n \times \mathbf{R} \to S^n \times \mathbf{R}$ be defined by

$$G(x,t) = \begin{cases} (x,t), & (x,t) \in S^n \times (-\infty, 0], \\ F(x,t), & (x,t) \in S^n \times [0,1], \\ (f(x),t), & (x,t) \in S^n \times [1, +\infty). \end{cases}$$

We have that G is 1-1 and by Lemma 10.7 it is continuous. The $(n+1)$-manifold $S^n \times \mathbf{R}$ is without boundary, and by Theorem 10.6, we have that G is open. Hence, F is open, which means that F^{-1} is continuous. ∎

As a corollary of Propositions 10.5 and 10.8, we have the following.

10.9. Proposition. *If $f : S^n \to S^n$ is an orientation preserving homeomorphism, then there exists a homeomorphism $F : S^n \times I \to S^n \times I$ such that $F(x, 0) = (x, 0)$ and $F(x, 1) = (f(x), 1)$.*

11. The equational system $\mathcal{K}$

To define the arrows of $\mathbf{K}$, we need an equational system, denoted by $\mathcal{K}$. We start with an inductive definition of *terms*.

1. For $n, m \in \omega$, the words $\mathbf{1}_n : n \to n$, $\tau_{n,m} : n+m \to m+n$, $\mu^{\Diamond} : 2 \to 1$, $\eta^{\Diamond} : 0 \to 1$, $\mu^{\Box} : 1 \to 2$, $\eta^{\Box} : 1 \to 0$, are *primitive* terms.

2. If $f : n \to m$ and $g : m \to p$ are terms, then $(g \circ f) : n \to p$ is a term.

3. If $f_1 : n_1 \to m_1$ and $f_2 : n_2 \to m_2$ are terms, then $(f_1 \otimes f_2) : n_1 + n_2 \to m_1 + m_2$ is a term.

4. Nothing else is a term.

A *type* is a word of the form $n \to m$, where $n, m \in \omega$. We say that $n \to m$ is a type of a term $f : n \to m$, and we say that this term has n as the *source* and m as the *target*. Usually, we omit the type in writing a term and by a term we mean just its part before the colon symbol. Also, we omit the outermost parenthesis in terms.

The *language* of $\mathcal{K}$ consists of words of the form $f = g$, where f and g are terms with the same type. Besides $f = f$, the *axiom schemata* of $\mathcal{K}$ are the following.

$$f \otimes \mathbf{1}_0 = f = \mathbf{1}_0 \otimes f, \quad (f_1 \otimes f_2) \otimes f_3 = f_1 \otimes (f_2 \otimes f_3). \tag{\textit{str}}$$

For $f : n \to m$, $g : m \to p$ and $h : p \to q$,

$$f \circ \mathbf{1}_n = f = \mathbf{1}_m \circ f, \quad (h \circ g) \circ f = h \circ (g \circ f). \tag{\textit{cat}}$$

$$\mathbf{1}_n \otimes \mathbf{1}_m = \mathbf{1}_{n+m}, \quad (g_1 \circ f_1) \otimes (g_2 \circ f_2) = (g_1 \otimes g_2) \circ (f_1 \otimes f_2). \tag{\textit{fun}}$$

$$\tau_{m_1,m_2} \circ (f_1 \otimes f_2) = (f_2 \otimes f_1) \circ \tau_{n_1,n_2}. \tag{\textit{nat}}$$

$$\tau_{m,n} \circ \tau_{n,m} = \mathbf{1}_{n+m}. \tag{\textit{inv}}$$

$$\tau_{n+m,p} = (\tau_{n,p} \otimes \mathbf{1}_m) \circ (\mathbf{1}_n \otimes \tau_{m,p}). \tag{hex}$$

$$\mu^\Diamond \circ (\mu^\Diamond \otimes \mathbf{1}_1) = \mu^\Diamond \circ (\mathbf{1}_1 \otimes \mu^\Diamond). \tag{$assoc$}$$

$$\mu^\Diamond \circ (\eta^\Diamond \otimes \mathbf{1}_1) = \mathbf{1}_1 = \mu^\Diamond \circ (\mathbf{1}_1 \otimes \eta^\Diamond). \tag{$unit$}$$

$$(\mu^\Box \otimes \mathbf{1}_1) \circ \mu^\Box = (\mathbf{1}_1 \otimes \mu^\Box) \circ \mu^\Box. \tag{$coass$}$$

$$(\eta^\Box \otimes \mathbf{1}_1) \circ \mu^\Box = \mathbf{1}_1 = (\mathbf{1}_1 \otimes \eta^\Box) \circ \mu^\Box. \tag{$counit$}$$

$$(\mu^\Diamond \otimes \mathbf{1}_1) \circ (\mathbf{1}_1 \otimes \mu^\Box) = \mu^\Box \circ \mu^\Diamond = (\mathbf{1}_1 \otimes \mu^\Diamond) \circ (\mu^\Box \otimes \mathbf{1}_1). \tag{$Frob$}$$

$$\mu^\Diamond \circ \tau_{1,1} = \mu^\Diamond. \tag{com}$$

$$\tau_{1,1} \circ \mu^\Box = \mu^\Box. \tag{$cocom$}$$

The *inference figures* of $\mathcal{K}$ are the following.

$$\frac{f = g}{g = f} \qquad \frac{f = g \quad g = h}{f = h}$$

$$\frac{f_1 : n \to m = f_2 : n \to m \qquad g_1 : m \to p = g_2 : m \to p}{g_1 \circ f_1 = g_2 \circ f_2}$$

$$\frac{f_1 = g_1 \qquad f_2 = g_2}{f_1 \otimes f_2 = g_1 \otimes g_2}$$

We say that terms f and g are *equal*, when $f = g$ is a theorem of $\mathcal{K}$, and we denote this by $f \equiv g$. The relation $\equiv$ is an equivalence relation and $[f]$ is the equivalence class of a term f.

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