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T_0 TOPOLOGICAL SPACES AND T_0 POSETS IN THE TOPOS OF M-SETS

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ABSTRACT. In this paper, we introduce the concept of a topological space in the topos M-**Set** of M-sets, for a monoid M. We do this by replacing the notion of open "subset" by open "subobject" in the definition of a topology. We prove that the resulting category has an open subobject classifier, which is the counterpart of the Sierpinski space in this topos. We also study the relation between the given notion of topology and the notion of a poset in this universe. In fact, the counterpart of the specialization pre-order is given for topological spaces in M-**Set**, and it is shown that, similar to the classic case, for a special kind of topological spaces in M-**Set**, namely T_0 ones, it is a partial order. Furthermore, we obtain the universal T_0 space, and give the adjunction between topological spaces and T_0 posets, in M-**Set**.

1. Introduction

Topoi are categories which behave like the category **Set**, of sets and functions between them. Naturally one tries to reconstruct the classical mathematics in such categories. The difference between classical mathematics and topos base mathematics is that the latter is in some sense constructive. In fact, the internal logic of a topos is intuitionistic, and so topoi provide a convenient set up for constructive mathematics.

One of the most important features of constructive mathematics is in computer science and semantic of programing languages. Since programs are constructive in nature we expect that their semantics would be so.

Dana Scott introduced a topology as a mathematical semantic of a functional programming language (see [A Compendium of Continuous Lattices], [Continuous Lattices and Domains]). The Scott topology is a T_0-space (see [Domain Theory]). It is known that there exists an adjunction between the category of pre-ordered sets and topological spaces (see [Topological Duality in Semantics]). Also, if one changes pre-ordered sets to ordered sets, the adjunction will be between posets and T_0 topological spaces. Dana Scott in a talk in 1980 proposed that intuitionistic set theory might provide a more powerful framework for denotational semantics, and suggested the study of mathematical structures in a topos

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as mathematical semantics.

In this paper, to get through the first step toward studying topological semantics in a topos, we study the notion of a topology, a T_0 topology, and its relation with the notion of a poset in a topos. The topos which we chose as our universe is one of the most famous and useful Grothendieck topoi (elementary topoi), namely the topos of presheaves on a monoid. This topos is isomorphic to the the topos of monoid actions, also called dynamical systems, a kind of automata. The T_0-spaces we introduce, are inspired by Hyland's generalization of T_0 spaces in a topos, called Σ-spaces in [First steps in synthetic domain theory], where Σ is the generalization of the Sierpinski space. Recall that the classic Sierpinski space classifies open subsets of topological spaces. We define the notion of the "Sierpinski space" Σ in the topos of monoid actions in such a way that it becomes the open subobject classifier for topological spaces in our universe. Moreover, we give an adjunction between T_0 topological spaces and a kind of posets in this topos, which we call them T_0 posets.

2. Preliminaries

In this section, we recall the preliminary notions needed in this paper. For more information about the category of M-sets one can see [The category of M-sets], [Topoi: The Categorial Analysis of Logic]; and about topos theory may see [Sketches of an Elephant: A Topos Theory Compendium].

2.1. **M-sets**. Let M be a monoid with e as its identity. An *M-set* is a set X with a function $\mu : M \times X \to X$, called the *action* of M on X, such that, denoting $\mu(s,x) = sx$,

$$ex = x, \qquad (st)x = s(tx).$$

A subset A of an M-set X is called a *sub M-set* of X if it is closed under the action of M on X. That is, for each $x \in A$ and $s \in M$ we have $sx \in A$.

A function $f : X \to Y$ between M-sets is called *equivariant* (or *action-preserving*) if for each $s \in M$ and $x \in X$ we have $f(sx) = sf(x)$.

Notice that, one can consider an M-set X as an algebra such that the action of each $s \in M$ is considered as a unary operation $\lambda_s : X \to X, \lambda_s(x) = sx$. In this way an action-preserving map between two M-sets is just a homomorphism between such algebras. Thus, one has the equational category M-**Set**, of all M-sets with equivariant maps between them. This view, in particular, gives that products in M-**Set** is the cartesian product with the pointwise action. Also, monomorphisms in this category are one-one action-preserving maps, and therefore subobjects (monomorphisms) can be identified by sub M-sets.

2.2. **The topos of M-sets.** If one considers a monoid M as a one object category (M as its only object, elements of M as morphisms, and the binary operation of M as the composition), then the category of M-sets is isomorphic to the functor category $\mathbf{Set}^M$ (see [The category of M-sets]). Thus one can immediately conclude that M-**Set** is a topos.

In what follows we bring some of the ingredients of this topos which we need in this paper. For details one can see [The category of M-sets] and [Topoi: The Categorial Analysis of Logic].

A subset I of a monoid M is said to be a *left ideal* if it is a sub M-set of the M-set M (with its binary operation as the action). That is, for all $s \in M$, $x \in I$, we have $sx \in I$.

The set of all left ideals of M is denoted by L_M. Notice that L_M is an M-set with the following action of M on it, called *the action by division*:

$$s.I = \{t \in M : ts \in I\},$$

for $s \in M$ and $I \in L_M$.

Subobject Classifier. Recall that the subobject classifer in a topos $\mathcal{E}$, with $\mathbf{1}$ as its terminal object, is an object Ω together with an arrow $t : \mathbf{1} \to \Omega$, called *the truth arrow*, such that for every monomorphism $f : A \to X$ there is a unique arrow χ_A or $\chi_f : X \to \Omega$, called *the classifying arrow* making the square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "!"'] & X \arrow[d, "\chi_f"] \\ 1 \arrow[r, "t"'] & \Omega \end{tikzcd}$$

a pullback. The subobject classifier Ω in the topos M-**Set** is L_M, and the truth arrow is $t : \mathbf{1} = \{*\} \to L_M$ with $t(*) = M$. Also, the classifying morphism of a sub M-set A of an M-set X is defined as $\chi_A : X \longrightarrow L_M$,

$$a \mapsto \chi_A(a) = \{s \in M : sa \in A\}.$$

Exponentiation. Also, recall that for objects X and Y in a topos, the exponential object is an object Y^X together with an arrow $ev : Y^X \times X \to Y$, called *the evaluation arrow*, such that for every arrow $g : Z \times X \to Y$ there is a unique arrow $\hat{g} : Z \to Y^X$ with $ev \circ (\hat{g} \times id_X) = g$.

The exponential object Y^X of two M-sets X and Y in the topos M-**Set** is the set

$$\{f : M \times X \to Y : f \text{ is action-preserving}\},$$

equipped with the action

$$(mf) : M \times X \to Y, \qquad (s, x) \mapsto f(sm, x),$$

for $s, m \in M$ and $f \in Y^X$. The evaluation arrow $ev^X_Y : Y^X \times X \to Y$ is defined by

$$ev^X_Y(f, x) = f(e, x).$$

Also, given an equivariant map $g : Z \times X \to Y$, the unique arrow $\hat{g} : Z \to Y^X$ is defined by $\hat{g}(z)(s, x) = g(sz, x)$, for $x \in X,\ s \in M, z \in Z$.

Notice, that we may consider an element $f : M \times X \to Y$ as a family $(f_s)_{s\in M}$ of maps $f_s : X \to Y$ (having f, one defines each f_s as $f_s(x) = f(s, x)$) with the compatibility property that $mf_s(x) = f_{ms}(mx)$ for $s, m \in M$ and $x \in X$. Applying this notation, the action of M on Y^X is then defined as $m(f_s)_{s\in M} = (f_{sm})_{s\in M}$, for $s, m \in M$.

3. Topological spaces in M-**Set**

There is a rich connection between classical domain theory and general topology (see [Domain Theory] and [Continuous Lattices and Domains]). Our aim in this section is to introduce a notion of topology in the topos of M-sets. We then define a counterpart of the Sierpinski space in this topos, and prove that, similar to the classic case, it is the *open subobject classifier* in this topos. Furthermore, we show that the category of topological spaces in the topos of M-sets has arbitrary products.

3.1. Definition. *A topological space in M-**Set** or simply an M-topological space, is an M-set X with a topology on it such that its open subsets are sub M-sets of X.*

We call open sub M-sets of an M-topological space X, an M-open subset, and denote the set of all M-open subsets by $MO(X)$.

3.2. Example. (1) For any M-set X, $\{X, \emptyset\}$ is clearly an M-topology on it.

(2) For any M-set X, the set $Sub(X)$ of all sub M-sets forms the largest M-topology on X. This is because, sub M-sets are closed under arbitrary union and finite intersections (also, $\emptyset$ and M are sub M-sets).

Another example of an M-topology which we use it later on, is the "upper topology" on an M-poset. To introduce it, we need to give the definition of an M-poset (for the definition of a poset in a topos, in general, one can see [Sketches of an Elephant: A Topos Theory Compendium]).

3.3. Definition. *A poset in M-**Set** or simply, an M-poset is a poset $(P; \leq)$ with an action of M on it such that for each $x, y \in P$ and $s \in M$ we have $x \leq y$ implies $sx \leq sy$.*

An upper sub M-set U of an M-poset P is a sub M-set of P which is also upper closed; that is, for $u \in U$ and $x \in P$ with $u \leq x$, we have $x \in U$.

3.4. Example. Since the upper sub M-sets of an M-poset P are clearly closed under arbitrary unions and finite intersections (also, $\emptyset$ and P are upper sub M-sets) we get that upper sub M-sets of an M-poset form an M-topology. We call this M-topological space, the *upper set M-topology* on P.

3.5. Definition. *By a base for an M-topology on an M-set X, we mean a base for its related topology whose members are sub M-sets. That is, a collection $\mathcal{B}$ of sub M-sets of X such that (1) $\bigcup \mathcal{B} = X$; (2) for $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$.*

As in the classic case, we have

3.6. REMARK. If $\mathcal{B}$ is a base for an M-set X, then there exists a smallest M-topology on X containing $\mathcal{B}$. This is because, one can define the M-topology τ generated by $\mathcal{B}$ by taking M-open subsets as the unions of elements of $\mathcal{B}$.

3.7. DEFINITION. *We call an action-preserving continuous map between M-topological spaces, an M-continuous map.*

We denote the category of M-topological spaces together with M-continuous maps by M-**Top**.

3.8. REMARK. (a) The category M-**Top** is a complete category. In fact

(a1) For a family $\{X_i : i \in I\}$ of M-topological spaces, the product M-set $\prod_{i\in I} X_i$ (with the pointwise actions) equipped with the M-topology generated by the base

$$\mathcal{B} = \{\prod_{i\in I} U_i : U_i \in MO(X_i) \text{ and } U_i = X_i \text{ for all but a finite number}\}$$

is an M-topological space. This product with the classic projection maps $p_i : \prod_{i\in I} X_i \to X_i$, $(x_i)_{i\in I} \mapsto x_i$, $i \in I$, which are clearly M-continuous, is actually the product in the category M-**Top**. In fact, for a given family $f_i : Y \to X_i$, $i \in I$, of M-continuous maps, there exists a unique M-continuous map $f \doteq (f_i)_{i\in I} : Y \to \prod_{i\in I} X_i$ such that $p_i f = f_i$, for each $i \in I$. The continuity of f is because for a base member $\prod_{i\in I} U_i$, where $U_i \in MO(X_i)$, and $U_i = X_i$ for all i but a finite number, we have

$$f^{-1}(\prod_{i\in I} U_i) = \bigcap_{i\in I} f_i^{-1}(U_i) = Y \cap \bigcap_{k=1}^{n} f_{i_k}^{-1}(U_i) = \bigcap_{k=1}^{n} f_{i_k}^{-1}(U_i) \in MO(Y)$$

for some $n \in N$.

(a2) For M-topological spaces X, Y, and M-continuous maps $f, g : X \to Y$, the sub M-set $E = \{x \in X : f(x) = g(x)\}$ of X is the equalizer of f and g in M-**Top** with the subspace topology; that is M-open subsets of E are intersections of M-open subsets of X with E.

(b) The category M-**Top** is a cocomplete category. In fact,

(b1) For a family $\{X_i : i \in I\}$ of M-topological spaces, the coproduct M-topological space $\coprod_{i\in I} X_i$ is the disjoint union of X_i's with the same action sx in X_i for $s \in M$ and $x \in X_i$, and with the M-topology generated by the union of $MO(X_i)$, for all $i \in I$.

(b2) The coequalizer of a pair $f, g : X \to Y$ in M-**Top** is computed as in M-**Set** with the quotient topology, that is, if $q : Y \to Z$ is the coequalizer in M-**Set**, then one takes the sub M-sets U of Z for which $q^{-1}(U) \in MO(Y)$ as M-open subsets of Z.

We close this section by introducing the M-topology counterpart of the Sierpinski space Σ which classifies M-open subsets; that is, for each M-open subset U of an M-

topological space X, there exists a unique M-continuous map χ_U such that

$$\begin{tikzcd} U \arrow[r, "i_U"] \arrow[d, "!"'] & X \arrow[d, "\chi_U"] \\ 1 \arrow[r, "t"'] & \Sigma \end{tikzcd}$$

is a pullback square in M-**Top**.

3.9. Definition. *We call the M-topological space $(L_M; \tau = \{L_M, \{M\}, \emptyset\})$, the Sierpinski M- topological space, and denote it by Σ.*

3.10. Lemma. *For each M-open subset U of an M-topological space X, the classifying arrow $\chi_U : X \to \Sigma$, $x \mapsto \chi_U(x) = \{s \in M : sx \in U\}$, is M-continuous.*

Proof. As we remarked in preliminaries, L_M is the subobject classifier in M-**Set** and the above defined χ_U is the classifying arrow in M-**Set**. To show that χ_U is M-continuous, it is enough to notice that $\chi_U^{-1}(\{M\}) = U$ is M-open in X. ■

Also, notice that the truth arrow $t : 1 = \{*\} \to \Sigma$ is clearly M-continuous.

3.11. Note. (1) Notice that, the truth arrow $t : 1 = \{*\} \to \Sigma$, $* \mapsto M$, is clearly M-continuous.

(2) The converse of Lemma 3.10 is also true; that is, the pullback of the truth arrow along any M-continuous map $f : X \to \Sigma$ gives an M-open subset of X, which is $f^{-1}(\{M\})$.

3.12. Theorem. *Σ is the M-open subset classifier in M-**Top**.*

Proof. Let X be an M-topological space and U be an M-open subset of it. Then, since L_M is the subobject classifier in M-**Set**, the classifying arrow χ_U makes the square

$$\begin{tikzcd} U \arrow[r, "i_U"] \arrow[d, "!"'] & X \arrow[d, "\chi_U"] \\ 1 \arrow[r, "t"'] & L_M \end{tikzcd}$$

into a pullback square in M-**Set**. But, by Lemma 3.10, χ_U is continuous, so the above square is a commutative square in M-**Top**, where U is considered with subspace topology. To see that this square is a pullback in M-**Top**, let A be an M-topological space and $f : A \to X$ be an M-continuous function such that $\chi_U f = t \circ !_A$, where $!_A : A \to 1$ is the unique arrow to the terminal object. Then there exists a unique action-preserving map $g : A \to U$ such that $i_U g = f$, $!_U \circ g = !_A$. To show that g is also continuous, take an M-open subset V of U. Since we have $g^{-1}(V) = g^{-1} i_U^{-1}(V) = f^{-1}(V)$, V is an M-open subset of X, and f is continuous, we get that $g^{-1}(V)$ is M-open in A.

Now, we show that χ_U is the only M-continuous map which makes the above diagram a pullback in M**-Top**. Let $\psi : X \to \Sigma$ be an M-continuous map such that

$$\begin{tikzcd} U \arrow[r, "i_U"] \arrow[d, "!"'] & X \arrow[d, "\psi"] \\ 1 \arrow[r, "t"'] & \Sigma \end{tikzcd}$$

is a pullback in M**-Top**. We show that the above diagram is also a pullback in M-**Set** and then it is concluded that $\psi = \chi_U$. Let A be an M-set and $f : A \to X$ be an equivariant map such that $\psi f = t \circ !_A$. Then, by endowing the sub M-sets $f^{-1}(V)$ to A, for M-open subsets V of X, we make A into an M-topological space and f into an M-continuous map. Thus there exists a unique M-continuous map $g : A \to U$ such that $i_U g = f$. Finally, one can see that if there exists another action-preserving map $h : A \to U$ such that $i_U h = f$, then h is also continuous, since f is so, and hence $h = g$. ■

3.13. Corollary. *For each M-topological space X,*

$$MO(X) \cong Hom_{M-\mathbf{Top}}(X, \Sigma).$$

4. T_0 topological spaces in M-Set

Recall that, in the theory of classical topology, a T_0 space is a topological space whose open subsets separate distinct points of the space. Also, recall that the *specialization pre-order* on topological spaces, which relates posets and T_0 spaces and it is proved that specialization pre-order is a partial order if and only if the space is T_0.

In this section, we introduce the notion of T_0 M-topological spaces, and define a "specialization order" on these spaces which gives a relation between M-topological spaces and M-posets in a similar way to the ordinary case.

More precisely, a topological space X is T_0 if and only if for each distinct elements x and y of X, there exists an open subset U such that $x \in U$ and $y \notin U$ or there exists an open subset U such that $y \in U$ and $x \notin U$. Recalling the definition of the characteristic (classifying) map of U in **Set** ($\chi_U(z) = 1$ if and only if $z \in U$), the separation axiom of T_0 states that for each $x \neq y$ in X, there exists an open subset U such that $\chi_U(x) \neq \chi_U(y)$. Now, applying this interpretation, we give the following notion of T_0 for M-topological spaces.

4.1. Definition. *We say that an M-topological space X is T_0 (or M-T_0) whenever for each pair of distinct points $x, y \in X$, there exists an M-open subset U such that $\chi_U(x) \neq \chi_U(y)$.*

In other words, for each $x \neq y$ in X, there exists an M-open subset U such that $sx \in U$ and $sy \notin U$ for some $s \in M$ (or $ty \in U$ and $tx \notin U$, for some $t \in M$).

We denote the category of T_0 M-topological spaces with M-continuous maps between them by M-$\mathbf{Top_0}$.

4.2. Example. (a) The Sierpinski space Σ defined in Definition 3.9 is a T_0 M-topological space. This is because, for every ideal I of M, using the fact that $s \cdot I = M$ if and only $s \in I$, one gets

$$\chi_{\{M\}}(I) = \{s \in M : s \cdot I = M\} = I.$$

(b) Arbitrary products of T_0 M-topological spaces is clearly a T_0 M-topological space. So, any product of Σ with itself is T_0.

The following theorem gives a characterization of T_0 M-topological spaces as subspaces of a product of the Sierpinski M-topological space.

4.3. Theorem. *An M-topological space X is T_0 if and only if X is homeomorphic to a subspace of the product M-topological space $\prod_{MO(X)} \Sigma$.*

Proof. It is enough to notice that X is a T_0 M-topological space if and only if the universal M-continuous map $(\chi_U)_{U \in MO(X)} : X \to \prod_{MO(X)} \Sigma$ to the product (see Remark 3.8) is one-one. Notice that $\ker(\chi_U)_{U \in MO(X)} = \bigcap_{U \in MO(X)} \ker(\chi_U)$. ■

In the following, we give some equivalent conditions to the T_0 axiom.

4.4. Remark. An M-topological space X is T_0 if and only if for each $x, y \in X$ there exist $s \in M$ and an M-open subset U of X such that $\chi_U(sx) \neq \chi_U(sy)$.

To prove the non-clear part, let $x, y \in X$ and there exist $s \in M$ such that $\chi_U(sx) \neq \chi_U(sy)$. Then, taking $m \in M$ with $m \in \chi_U(sx) \setminus \chi_U(sy)$, we have $ms \in \chi_U(x) \setminus \chi_U(y)$.

Recall that in the classic topology, the specialization order is a partial order if and only if the space is T_0. In the following, we define the counterpart of the specialization order for topological spaces in M-**Set**, and show the similar result regarding this order and T_0 M-topological spaces.

4.5. Definition. *Let X be an M-topological space. Define the M- specialization pre-order on X as*

$$x \leq_{Ms} y \Leftrightarrow \forall U \in MO(X),\ \chi_U(x) \subseteq \chi_U(y).$$

As in the classical case, the M-specialization order is a pre-order. Also, similar to the ordinary case we have the following result.

4.6. Lemma. *An M-topological space X is T_0 if and only if the M-specialization pre-order is a partial order on X.*

Proof. Let X be a T_0 M-topological space. To see that $\leq_{Ms}$ is anti-symmetric, assume

$$x \leq_{Ms} y \text{ and } y \leq_{Ms} x.$$

Then, for each M-open subset U of X, we have $\chi_U(x) \subseteq \chi_U(y)$, $\chi_U(y) \subseteq \chi_U(x)$, and hence $\chi_U(x) = \chi_U(y)$. Since X is T_0, we get $x = y$.

On the other hand, let $\leq_{Ms}$ be a partial order on X, $x, y \in X$, and for each M-open subset U of X, we have $\chi_U(x) = \chi_U(y)$. Then $x \leq_{Ms} y$ and $y \leq_{Ms} x$, and hence $x = y$. ■

Now, applying the above lemma and using the fact that for any ideals I and J of M, $I \subseteq J$ implies $s \cdot I \subseteq s \cdot J$, for all $s \in M$, we get the following theorem.

4.7. THEOREM. *An M-topological space X is T_0 if and only if $(X; \leq_{Ms})$ is an M-poset.*

5. Constructing T_0 topological spaces in M-**Set**

In this section, we show how one can construct a T_0 M-topological space from a given M-topological space, and in a universal way.

5.1. DEFINITION. *Let X be a T_0 M-topological space. Define the relation $\equiv$ on X by*

$$x \equiv y \Leftrightarrow \forall\, U \in MO(X),\ \chi_U(x) = \chi_U(y).$$

5.2. REMARK. For any T_0 M-topological space X, $\equiv$ is an M-set congruence on X. It is clearly an equivalence relation, also it is compatible with the action, because each χ_U is equivariant.

5.3. LEMMA. *The M-set $X/{\equiv}$ with the quotient topology is an M-topological space.*

PROOF. Recall that the open subsets of the quotient topology are of the form $V \subseteq X/{\equiv}$, where $\pi^{-1}(V)$ is an open subset of X, and $\pi : X \to X/{\equiv}$ is the natural onto map which takes $x \in X$ to $[x]_{\equiv}$. Now, it is enough to notice that for each open subset V of $X/{\equiv}$, $\pi^{-1}(V)$ is a sub M-set of the M-topological space X, and hence $V = \pi(\pi^{-1}(V))$ is a sub M-set of $X/{\equiv}$, because π is an M-set map. ■

5.4. THEOREM. *For any topological M-topological space X, the M-topology $X/{\equiv}$ is T_0.*

PROOF. Let $[x]$ and $[y]$ be members of $X/{\equiv}$ such that for each M-open subset V of $X/{\equiv}$ we have $\chi_V([x]) = \chi_V([y])$. We show that $[x] = [y]$, that is, for every $U \in MO(X)$, $\chi_U(x) = \chi_U(y)$.

First, we prove that $\pi : X \to X/{\equiv}$ is an open map. Let $U \in MO(X)$. Then to prove that $\pi(U)$ is open in $X/{\equiv}$, we show $\pi^{-1}(\pi(U)) = U$. But, the fact that $U \subseteq \pi^{-1}(\pi(U))$ is true for any function, and for the converse, take $z \in \pi^{-1}(\pi(U))$. Then $[z] \in \pi(U)$, and hence there exists $z' \in U$ such that $[z] = [z']$. Thus, $z \equiv z'$, and so $\chi_U(z) = \chi_U(z') = M$. Therefore, $z \in U$ as required.

Now, let $U \in MO(X)$ and $m \in \chi_U(x)$. Then $mx \in U$, and hence $m[x] = [mx] \in \pi(U)$. This means $m \in \chi_{\pi(U)}([x])$. But, by hypothesis,

$$\chi_{\pi(U)}([x]) = \chi_{\pi(U)}([y]),$$

since $\pi(U)$ is an M-open subset. Therefore, $m \in \chi_{\pi(U)}([y])$, and so $[my] \in \pi(U)$. Thus, we have $[my] = [y']$, for some $y' \in U$, and hence $\chi_U(my) = \chi_U(y') = M$. Therefore, $my \in U$, and so $m \in \chi_U(y)$, as required. So, $\chi_U(x) \subseteq \chi_U(y)$; the reverse inclusion is proved similarly. ■

5.5. Lemma. *The assignment $X \mapsto X/\equiv$ is functorial.*

Proof. Define $F : M\text{-}\mathbf{Top} \to M\text{-}\mathbf{Top_0}$ by $F(X) = X/\equiv$, for an M-topological space X. Also, for an M-continuous map $f : X \to Y$, define $F(f) = \bar{f}$, where $\bar{f} : X/\equiv \to Y/\equiv$ is defined as $\bar{f}([x]) = [f(x)]$. Using the definition of $\equiv$ and the fact that f is equivariant, one gets that $\bar{f}$ is well-defined and equivariant. To show that $\bar{f}$ is continuous, let V be an M-open subset of $Y/\equiv$. Then, $\pi_Y^{-1}(V)$ is M-open in Y. To see that $\pi_X^{-1}(\bar{f}^{-1}(V))$ is M-open in X, we notice that

$$\pi_X^{-1}(\bar{f}^{-1}(V)) = (\bar{f}\pi_X)^{-1}(V) = (\pi_Y f)^{-1}(V)) = f^{-1}(\pi_Y^{-1}(V)),$$

which is M-open in X, because f is continuous. ∎

5.6. Lemma. *For any M-topological space X, $\pi : X \to X/\equiv$ is a universal arrow to T_0 M-topological spaces.*

Proof. Let X be an M-topological space and Y be a T_0 M-topological space, and $f : X \to Y$ be an M-continuous map. Then we show that there exists a unique M-continuous map $g : X/\equiv \to Y$ such that $g\pi_X = f$. Define $g : X/\equiv \to Y$ by $g([x]) = f(x)$. The assignment g is well-defined, for if $[x] = [y]$, then for every $U \in MO(X)$, we have $\chi_U(x) = \chi_U(y)$. But, for every $V \in MO(Y)$, using the fact that f is equivariant, one can easily get that $\chi_V(f(x)) = \chi_{f^{-1}(V)}(x)$. So, for every $V \in MO(Y)$, we have

$$\chi_V(f(x)) = \chi_{f^{-1}(V)}(x) = \chi_{f^{-1}(V)}(y) = \chi_V(f(y)).$$

Now, since Y is a T_0 M-topological space, we get $f(x) = f(y)$.

Obviously, g is action-preserving. Also, g is continuous, because for any M-open subset V of Y, we have $g^{-1}(V) = \pi_X(f^{-1}(V))$, which is M-open in $X/\equiv$, since π_X is M-open and f is continuous. ∎

Now, applying the above lemma, we get the following result.

5.7. Theorem. *The functor F in Lemma 5.5 is a left adjoint to the inclusion functor $i : M\text{-}\mathbf{Top_0} \to \mathbf{M\text{-}Top}$.*

6. Adjunction between T_0 topological spaces and T_0 posets in M-**Set**

In the category **Set**, for each ordinary poset $(P; \leq)$, there exists a topological space X such that $(X; \leq_s) \cong (P; \leq)$. The space X is in fact the poset P with upper topology (upper subsets as open subsets). The following example shows that this is not generally true in M-**Set**.

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6.1. Example. Let M be the monoid $(N;+,0)$ of natural numbers and $P=\{0,1\}$ the M-poset with the equality relation and the action given by

μ	0	1
2k	0	1
2k+1	1	0

Then there is no M-topological space X with $(X;\leq_{Ms})\cong(P;=)$. On the contrary, if there exists such an M-topological space X. Then, taking $\phi : P\to X$ as the assumed isomorphism, we have $X=\{\phi(0),\phi(1)\}$. Now, for any M-open subset U of X, we have $\phi(0)\in U$ if and only if $\phi(1)\in U$. Thus the only possible topology on X is $\{X,\emptyset\}$. But, the specialization pre-order on X is not a partial order, since for each M-open subset U, we have $\chi_U(0)=\chi_U(1)$ while $0\neq 1$.

In this section, to prove the counterpart of the above result for M-topological spaces and M-posets, we define the notion of a T_0 M-poset, and show that there exists an adjunction between the category of T_0 M-topological spaces and T_0 M-posets.

6.2. Definition. *We call an M-poset P a T_0 M-poset if its upper set M-topology is a T_0 M-topological space.*

In other words, for each $p,q\in P$ there exists an upper sub M-set U of P such that $\chi_U(x)\neq\chi_U(y)$.

We denote the category of T_0 M-posets with action-preserving monotone (call them *M-monotone*) *maps* between them by M-$\mathbf{Pos_0}$.

The M-poset P given in Example 6.1 is not T_0. The following theorem shows that one may correspond a T_0 M-poset related to any T_0 M-topological space.

6.3. Lemma. *For any M-topological space X, the M-open subsets of X are upper closed subsets of the pre-ordered set $(X;\leq_{Ms})$.*

Proof. Let U be an M-open subset of an M-topological space X, $a\in U$ and $a\leq_{Ms} b$. Then $\chi_U(a)=M$ and $\chi_U(a)\subseteq\chi_U(b)$. Therefore, $\chi_U(b)=M$ and so $b\in U$. ∎

6.4. Theorem. *For any T_0 M-topological space X, the pre-ordered set $(X;\leq_{Ms})$ is a T_0 M-poset.*

Proof. By Theorem 4.7, $P\doteq(X;\leq_{Ms})$ is an M-poset. Also, by Lemma 6.3, the M-open subsets of X are upper closed subsets of P.

Now, taking $p,q\in P$, since X is T_0, there exists a M-set of X such that $\chi_U(x)\neq\chi_U(y)$. But, by Lemma 6.3, the M-open subsets of X are upper closed subsets of P. So, P is a T_0 M-poset. ∎

6.5. Theorem. *There is an adjunction between the categories M-$\mathbf{Pos_0}$ and M-$\mathbf{Top_0}$.*

Proof. Define $F : M\text{-}\mathbf{Pos_0} \to M\text{-}\mathbf{Top_0}$ by $F(P) = U(P)$, where $U(P)$ is P with the upper set M-topology on it, which is T_0, by Lemma 6.4. Also, for a monotone map $f : P \to Q$ between T_0 M-posets, define $Ff : U(P) \to U(Q)$ as $Ff = f$. Notice that $f : U(P) \to U(Q)$ is M-continuous, because for any upper sub M-set A of Q, $f^{-1}(A)$ is an upper sub M-set of P.

Also, define $G : M\text{-}\mathbf{Top_0} \to M\text{-}\mathbf{Pos_0}$ by $G(X) = (X; \leq_{Ms})$ which is T_0, by Lemma 6.4, and for an M-continuous map $f : X \to Y$, define $Gf = f$. Notice that $f : (X; \leq_{Ms}) \to (Y; \leq_{Ms})$ is M-monotone. This is because, for $x, y \in X$ if $x \leq_{Ms} y$, and V is an M-open subset of Y, then we have $\chi_V(f(x)) \subseteq \chi_V(f(y))$. To see this, let $m \in \chi_V(f(x))$. Then $f(mx) = mf(x) \in V$, and so $mx \in f^{-1}(V)$, where $f^{-1}(V)$ is an M-open subset of X, since f is continuous. This means $m \in \chi_{f^{-1}(V)}(x)$. But, since $x \leq_{Ms} y$, we have $\chi_{f^{-1}(V)}(x) \subseteq \chi_{f^{-1}(V)}(y)$. Thus $m \in \chi_{f^{-1}(V)}(y)$, and hence $my \in f^{-1}(V)$. Therefore, $mf(y) = f(my) \in V$ and so $m \in \chi_V(f(y))$. Thus $f(x) \leq_{Ms} f(y)$.

Now we show that F is the left adjoint to G. To see this, we give the unit $\eta : id_{M-\mathbf{Pos_0}} \to GF$ of the adjunction. Let P be a T_0 M-poset. Then, define $\eta_P(p) = p$, for $p \in P$. It is obvious that η_P is action-preserving. To see that it is monotone, let $p \leq q$ in P and U be an M-open subset of $U(P)$. That is, U is an upper closed sub M-set of P. We have to show that $\chi_U(p) \subseteq \chi_U(q)$. Let $m \in \chi_U(p)$. Then $mp \in U$, but $mp \leq mq$, so $mq \in U$, and hence $m \in \chi_U(q)$.

Now, let $f : P \to G(X)$ be a monotone map. Then $f : F(P) \to X$ is the unique M-continuous map such that $f\eta_P = f$. The fact that f is action-preserving, is obvious. To see that it is continuous, let $U \subseteq X$ be an M-open subset of X. We have to show that $f^{-1}(U)$ is an upper closed sub M-set of P. Let $x \in f^{-1}(U)$ and $x \leq y$ in P. Then $f(x) \in U$ and $f(x) \leq_{Ms} f(y)$. So $\chi_U(f(x)) = M$. But $\chi_U(f(x)) \subseteq \chi_U(f(y))$, and so we have $\chi_U(f(y)) = M$. Therefore $f(y) \in U$, and so $y \in f^{-1}(U)$. ∎

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