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A BRAUER-CLIFFORD-LONG GROUP FOR THE CATEGORY OF DYSLECTIC HOPF YETTER-DRINFEL'D (S, H)-MODULE ALGEBRAS

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ABSTRACT. Brauer-Clifford groups are equivariant Brauer groups for which a Hopf algebra acts or coacts nontrivially on the base ring. Brauer-Clifford groups have been established previously in the category of modules for a skew group ring $S\#G$, the category of modules for the smash product $S\#H$ over a cocommutative Hopf algebra H, and its dual category of (S, H)-Hopf modules ${}_S\mathcal{M}^H$ over a commutative Hopf algebra H. In this article the authors introduce a Brauer-Clifford group for the category of dyslectic Hopf Yetter-Drinfel'd (S, H)-modules for an H-commutative base ring S and quantum group H. This is the first such example in a category of modules for a quantum group, and it gives a new example of an equivariant Brauer group in a braided monoidal category.

Introduction

There are now several examples of Brauer groups of symmetric and braided monoidal categories (see [vOZ]). The different examples originally arose one-by-one, beginning with the original Brauer group of a field, until a unifying general pattern of ideas was developed [Pareigis]. A braided monoidal category is a category $\mathcal{C}$ with a tensor product $\otimes$ that has a unit object R and has a family of isomorphisms $\gamma_{M,N} : M \otimes N \to N \otimes M$, one for each pair of objects in $\mathcal{C}$, satisfying natural coherence conditions. If $\gamma_{M,N} \circ \gamma_{M,N} = id_{M\otimes N}$ always holds, then the braided monoidal category is symmetric.

Van Oystaeyen and Zhang gave a general construction for the Brauer group of a braided monoidal category $(\mathcal{C}, \otimes, R, \gamma)$ in [vOZ]. It is the group of equivalence classes of "Azumaya algebras" in the category modulo taking braided products with "trivial" algebras in the category that arise as $End_{\mathcal{C}}(M)$, where M is a faithfully projective R-module in $\mathcal{C}$. Braided monoidal categories frequently occur in categories related to the actions and coactions of Hopf algebras. Caenepeel, van Oystaeyen, and Zhang were in particular inspired by their formulation of the Brauer group in the category of Yetter-Drinfel'd H-module algebras for a Hopf algebra H with bijective antipode. However, in this category both the action and coaction of H on the base ring R were assumed to be trivial.

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Brauer-Clifford groups are equivariant Brauer groups where the action or coaction on the base ring is not assumed to be trivial. In a previous article, the authors defined the Brauer-Clifford group of equivalence classes of (S, H)-Azumaya algebras. This is the Brauer group for the symmetric monoidal category of $S\#H$-modules, where S is a commutative H-module algebra for a cocommutative Hopf algebra H acting nontrivially on S. A dual situation was given for the Brauer-Clifford group of Azumaya algebras in the category of (S, H)-Hopf modules, a category of S-modules with a compatible H-comodule structure, where S is a commutative H-comodule algebra for a commutative Hopf algebra H.

The natural next step in extending the list of Brauer-Clifford groups is to formulate one in a category of Yetter-Drinfel'd H-modules for a Hopf algebra H with bijective antipode. Yetter-Drinfel'd H-modules do form a braided monoidal category when the action and coaction on the base ring is trivial. The Brauer group for the category of Yetter-Drinfel'd H-modules was introduced by Caenepeel, van Oystaeyen, and Zhang in [CvOZ1] and [CvOZ2]. In order to have the base ring S be a nontrivial Yetter-Drinfel'd H-module algebra, we require the *dyslectic* condition introduced by Pareigis in [Pareigis2]. We will assume S to be H-commutative (aka. quantum commutative) and restrict ourselves to the subcategory of *dyslectic* Hopf Yetter-Drinfel'd (S, H)-module algebras in order to obtain a braided monoidal category. (We do not require the base ring S to be commutative.) The goal of this article is to give a detailed description of the Brauer group of this braided monoidal category, which we call the *Brauer-Clifford-Long* group as it is a generalization of both the Brauer-Clifford group and the Brauer-Long group.

1. Preliminaries and Notation

Let H be a Hopf algebra over a commutative ring R. We denote its comultiplication by $\Delta : H \to H \otimes H$, its antipode by $\mathcal{S} : H \to H$, and its counit by $\epsilon : H \to R$. We will use Sweedler-Heyneman notation, omitting sums, so we write $\Delta(h) = h_1 \otimes h_2$. For a Hopf algebra with comultiplication Δ, Δ^{cop} is defined by $\Delta^{cop}(h) = \sum h_2 \otimes h_1$.

We will require a sequence of definitions, all of which are standard. An R-algebra A is an *H-module algebra* if A is a left H-module such that

$$h.(ab) = (h_1.a)(h_2.b) \text{ and } h.1_A = \epsilon(h)a, \text{ for all } a, b \in A, h \in H. \tag{1}$$

H acts trivially on A when $h.a = \epsilon(h)a$ for all $h \in H$ and $a \in A$. A *homomorphism of H-module algebras* is a homomorphism of H-modules which is also a homomorphism of R-algebras. If A is an H-module algebra, then the *smash product algebra $A\#H$* is the R-module $A \otimes H$ with multiplication

$$(a \otimes h)(a' \otimes h') = a(h_1.a') \otimes h_2h', \text{ for all } a, a' \in A \text{ and } h, h' \in H. \tag{2}$$

An R-module M is a *left $A\#H$-module* if it is a left A-module and a left H-module for which

$$h(am) = (h_1.a)(h_2m) \text{ for all } h \in H, a \in A, m \in M. \tag{3}$$

If A is an H-module algebra and S is a sub-H-module algebra of A, then the algebras A and S are left $S\#H$-modules. We will write ${}_{A\#H}\mathcal{M}$ for the category of left $A\#H$-modules. It was observed in [GH] that if H is cocommutative and A is a commutative H-module algebra, then $({}_{A\#H}\mathcal{M}, \otimes_A, A)$ is a symmetric monoidal category.

If H is a Hopf algebra over R, an R-module M is a *right H-comodule* if there exists an R-linear map $\rho_M : M \to M \otimes H$ satisfying $(\rho_M \otimes id_H) \circ \rho_M = (id_H \otimes \Delta_H) \circ \rho_M$ and $(id_M \otimes \epsilon) \circ \rho_M = id_M$. In Sweedler notation, we write $\rho_M(m) = m_0 \otimes m_1$ for all $m \in M$, and the right H-comodule conditions on M are

$$m_{00} \otimes m_{01} \otimes m_1 = m_0 \otimes m_{11} \otimes m_{12}, \text{ and } m_0\epsilon(m_1) = m, \text{ for all } m \in M. \tag{4}$$

H coacts trivially on M when $m_0 \otimes m_1 = m \otimes 1_H$ for all $m \in M$. Let M and N be right H-comodules. A *homomorphism of right H-comodules* (aka. a right H-colinear map) is an R-linear map $f : M \to N$ such that $\rho_N \circ f = (f \otimes id_H) \circ \rho_M$. In Sweedler notation, this is equivalent to

$$f(m)_0 \otimes f(m)_1 = f(m_0) \otimes m_1, \text{ for all } m \in M. \tag{5}$$

If M and N are right H-comodules then $M \otimes N$ is a right H-comodule under the codiagonal coaction:

$$(m \otimes n)_0 \otimes (m \otimes n)_1 = m_0 \otimes n_0 \otimes (m_1 n_1), \text{ for all } m \in M, n \in N. \tag{6}$$

An R-algebra A is an *H-comodule algebra* if A is a right H-comodule and the multiplication in A satisfies

$$(ab)_0 \otimes (ab)_1 = a_0 b_0 \otimes a_1 b_1 \text{ and } \rho(1_A) = 1_A \otimes 1_H, \text{ for all } a, b \in A. \tag{7}$$

A *homomorphism of H-comodule algebras* is a homomorphism of H-comodules which is also a homomorphism of R-algebras.

Let A be a right H-comodule algebra. A R-module M is an *(A, H)-Hopf module* if M is both a left A-module and a right H-comodule, with the property

$$(am)_0 \otimes (am)_1 = a_0 m_0 \otimes a_1 m_1, \text{ for all } a \in A, m \in M. \tag{8}$$

A *homomorphism of (A, H)-Hopf modules* is a left A-linear map which is also a right H-colinear map. We will write ${}_A\mathcal{M}^H$ for the category of (A, H)-Hopf modules. This category is dual to ${}_{A\#H}\mathcal{M}$, and when H is commutative and A is a commutative H-comodule algebra, $({}_A\mathcal{M}^H, \otimes_A, A)$ is a symmetric monoidal category [GH].

Let H be a Hopf algebra with bijective antipode. A *Hopf Yetter-Drinfel'd H-module* (in the literature also called a crossed H-module or a quantum Yang-Baxter H-module) is an R-module M which is both a left H-module and a right H-comodule satisfying the compatibility condition

$$(hm)_0 \otimes (hm)_1 = h_2 m_0 \otimes h_3 m_1 \mathcal{S}^{-1}(h_1), \text{ for all } h \in H, m \in M. \tag{9}$$

A *Hopf Yetter-Drinfel'd H-module homomorphism* between two Hopf Yetter-Drinfel'd H-modules M and N is an R-linear map $M \to N$ which is simultaneously a left H-module homomorphism and a right H^{op}-comodule homomorphism.

A Hopf Yetter-Drinfel'd H-module algebra is both a left H-module algebra and a right H^{op}-comodule algebra satisfying the relation (9). A *Hopf Yetter-Drinfel'd H-module algebra homomorphism* between two Yetter-Drinfel'd H-module algebras A and B is a R-linear map $A \to B$ which is simultaneously a Yetter-Drinfel'd H-module homomorphism and an R-algebra homomorphism. The category of Hopf Yetter-Drinfel'd H-modules is denoted $\mathcal{Q}^H$ [CvOZ2]. For Hopf Yetter-Drinfel'd H-modules M and N, the tensor product $M \otimes N$ has an H-module structure given by

$$h(m \otimes n) = (h_1 m) \otimes (h_2 n), \text{ for all } h \in H, m \in M, n \in N, \tag{10}$$

and an H-comodule structure given by

$$(m \otimes n)_0 \otimes (m \otimes n)_1 = m_0 \otimes n_0 \otimes n_1 m_1, \text{ for all } m \in M, n \in N. \tag{11}$$

These H-structures satisfy the compatibility condition (9) and make $M \otimes N$ a Hopf Yetter-Drinfel'd H-module, denoted by $M \tilde{\otimes} N$.

For Hopf Yetter-Drinfel'd H-modules M and N, there exists a Yetter-Drinfel'd H-module isomorphism $\gamma_{M,N}$ from $M \tilde{\otimes} N$ to $N \tilde{\otimes} M$ defined by (see [CvOZ1, (1.2.4)])

$$\gamma_{M,N}(m \tilde{\otimes} n) = n_0 \tilde{\otimes} n_1 m, \text{ for all } m \in M, n \in N, \tag{12}$$

with inverse

$$\gamma^{-1}_{M,N}(n \tilde{\otimes} m) = S_H(n_1) m \tilde{\otimes} n_0, \text{ for all } m \in M, n \in N. \tag{13}$$

According to [CvOZ1, (1.2.4)] and [CvOZ2, (1.4)], $(\mathcal{Q}^H, \tilde{\otimes}_k, \gamma_{M,N}, k)$ is a braided monoidal category. A monoidal category $(\mathcal{C}, \otimes)$ is *braided* if there are natural isomorphisms $\gamma_{M,N} : M \otimes N \cong N \otimes M$ in $\mathcal{C}$ for all $M, N \in \mathcal{C}$, such that the following hexagonal coherence conditions are satisfied (see [MacLane, p. 180]):

$$\gamma_{M \otimes N, P} = (\gamma_{M,P} \otimes 1) \circ (1 \otimes \gamma_{N,P}) \text{ and } \gamma_{M, N \otimes P} = (1 \otimes \gamma_{M,P}) \circ (\gamma_{M,N} \otimes 1), \text{ for all } M, N, P \in \mathcal{C}.$$

2. The category of Hopf Yetter-Drinfel'd (S, H)-modules

Let S be a Yetter-Drinfel'd H-module algebra. A *Hopf Yetter-Drinfel'd (S, H)-module M* is a left S-module and a Yetter-Drinfel'd H-module satisfying the compatibility conditions

$$h(s \rightharpoonup m) = (h_1.s) \rightharpoonup (h_2 m) \tag{14}$$

and

$$(s \rightharpoonup m)_0 \otimes (s \rightharpoonup m)_1 = (s_0 \rightharpoonup m_0) \otimes m_1 s_1, \text{ for all } h \in H, s \in S, m \in M. \tag{15}$$

Equivalently, M is a left $S\#H$-module and a right (S, H^{op})-Hopf module for which relation (9) is satisfied. A Hopf Yetter-Drinfel'd (k, H)-module is just a Hopf Yetter-Drinfel'd H-module. Furthermore, note that if S is a Yetter-Drinfel'd H-module algebra, then S is a Hopf Yetter-Drinfel'd (S, H)-module: the left action of S is given by $s \rightharpoonup s' = ss'$ for all $s, s' \in S$.

A *Hopf Yetter-Drinfel'd* (S, H)*-module homomorphism* is a Hopf Yetter-Drinfel'd H-module map which is also left S-linear. We denote by ${}_S\mathcal{Q}^H$ the category consisting of Hopf Yetter-Drinfel'd (S, H)-modules and Hopf Yetter-Drinfel'd (S, H)-module homomorphisms.

Let S be a Yetter-Drinfel'd H-module algebra. We say that S is H*-commutative* (or *quantum commutative*) if

$$ss' = s'_0(s'_1.s), \text{ for all } s, s' \in S. \tag{16}$$

If S is an H-commutative Yetter-Drinfel'd H-module algebra, then for every left S-action on an $M \in {}_S\mathcal{Q}^H$ there is a corresponding right S-action defined by

$$m \leftharpoonup s = s_0 \rightharpoonup s_1.m, \text{ for all } s \in S, m \in M. \tag{17}$$

This allows us to view M as an S-S-bimodule. Note that the left S-action and the right S-action are also related by

$$s \rightharpoonup m = \mathcal{S}(s_1)m \leftharpoonup s_0, \text{ for all } s \in S, m \in M. \tag{18}$$

Note also that we have

$$h(m \leftharpoonup s) = (h_1 m) \leftharpoonup (h_2.s) \tag{19}$$

and

$$(m \leftharpoonup s)_0 \otimes (m \leftharpoonup s)_1 = (m_0 \leftharpoonup s_0) \otimes s_1 m_1, \text{ for all } h \in H, m \in M, s \in S. \tag{20}$$

Let S be an H-commutative Yetter-Drinfel'd H-module algebra. Then for M and N in ${}_S\mathcal{Q}^H$, we can endow the tensor product $M \otimes_S N$ with the following S-action and H-module and comodule structures:

$$s \rightharpoonup (m \,\tilde{\otimes}\, n) = (s \rightharpoonup m) \,\tilde{\otimes}\, n, \tag{21}$$

$$h(m \,\tilde{\otimes}\, n) = h_1 m \,\tilde{\otimes}\, h_2 n, \tag{22}$$

and

$$(m \,\tilde{\otimes}\, n)_0 \otimes (m \otimes n)_1 = m_0 \,\tilde{\otimes}\, n_0 \otimes n_1 m_1, \tag{23}$$

for all $h \in H$, $s \in S$, $m \in M$, and $n \in N$, where $m \,\tilde{\otimes}\, n = m \otimes_S n$. According to [CvOZ2], these structures make $M \otimes_S N$ into a Hopf Yetter-Drinfel'd (S, H)-module, denoted by $M \,\tilde{\otimes}_S\, N$. Note that we have

$$(m \,\tilde{\otimes}\, n) \leftharpoonup s = m \,\tilde{\otimes}\, (n \leftharpoonup s), \text{ for all } m \in M, n \in N, s \in S. \tag{24}$$

It follows from [CvOZ1, Theorem 3.2.3] that if S is H-commutative, then $({}_S\mathcal{Q}^H, \tilde{\otimes}_S, S)$ is a monoidal category.

In the remainder of the paper, if M and N are Hopf Yetter-Drinfel'd (S, H)-modules, $Hom_S(M, N)$ means $Hom_S(M_S, N_S)$ and ${}_SHom(M, N)$ means ${}_SHom({}_SM, {}_SN)$.

2.1. LEMMA. *Let S be an H-commutative Yetter-Drinfel'd H-module algebra, and let M and N be Hopf Yetter-Drinfel'd (S,H)-modules. Then the following hold:*

(i) $Hom_S(M,N)$ is a left $S\#H$-module, where the action of S is defined by

$$(s \rightharpoonup f)(m) = s \rightharpoonup f(m), \text{ for all } s \in S, f \in Hom_S(M,N), m \in M, \tag{25}$$

and the action of H is defined by

$$(hf)(m) = h_1[f(\mathcal{S}(h_2)m)], \text{ for all } f \in Hom_S(M,N), h \in H, m \in M. \tag{26}$$

(ii) If M is finitely generated projective as a right S-module, then $Hom_S(M,N)$ is a Hopf Yetter - Drinfel'd (S,H)- module, where the coaction of H is defined by

$$f_0(m) \otimes f_1 = f(m_0)_0 \otimes \mathcal{S}^{-1}(m_1)f(m_0)_1, \text{ for all } f \in Hom_S(M,N), m \in M. \tag{27}$$

PROOF. (i) Let $f \in Hom_S(M,N), h \in H, s \in S$ and $m \in M$. We have

$$\begin{array}{rcl}
(hf)(m \leftharpoonup s) & = & h_1(f(\mathcal{S}(h_2)(m \leftharpoonup s))) \\
& = & h_1(f((\mathcal{S}(h_2)_1 m) \leftharpoonup (\mathcal{S}(h_2)_2.s))) \\
& = & h_1[f((\mathcal{S}(h_{22})m) \leftharpoonup (\mathcal{S}(h_{21}).s))] \\
& = & h_1[f(\mathcal{S}(h_3)m) \leftharpoonup (\mathcal{S}(h_2).s)] \\
& = & (h_{11}.(f(\mathcal{S}(h_3)m))) \leftharpoonup (h_{12}.(\mathcal{S}(h_2).s)) \\
& = & (h_1.(f(\mathcal{S}(h_4)m))) \leftharpoonup (h_2.(\mathcal{S}(h_3).s)) \\
& = & (h_1.(f(\mathcal{S}(h_4)m))) \leftharpoonup (h_2(\mathcal{S}(h_3)).s) \\
& = & (h_1(f(\mathcal{S}(h_3)m))) \leftharpoonup \epsilon(h_2)s \\
& = & ((hf)(m)) \leftharpoonup s.
\end{array}$$

So $hf \in Hom_S(M,N)$, that is, the H-action is well defined. It is easy to see that $Hom_S(M,N)$ is a left H-module. For s' in S, $m \in M$ and $f \in Hom_S(M,N)$ we have

$$\begin{array}{rcl}
(s \rightharpoonup f)(m \leftharpoonup s') & = & s \rightharpoonup (f(m \leftharpoonup s')) \\
& = & s \rightharpoonup (f(m) \leftharpoonup s') \\
& = & (s \rightharpoonup (f(m)) \leftharpoonup s' \\
& = & ((s \rightharpoonup f)(m)) \leftharpoonup s'.
\end{array}$$

So $(s \rightharpoonup f) \in Hom_S(M,N)$, that is, the left S-action is well defined. It is easy to see that $Hom_S(M,N)$ is a left S-module. For all $f \in Hom_S(M,N)$, $s \in S$, $h \in H$, and $m \in M$, we have

$$\begin{array}{rcl}
[h(s \rightharpoonup f)](m) & = & h_1[(s \rightharpoonup f)(\mathcal{S}(h_2)m)] \\
& = & h_1[s \rightharpoonup (f(\mathcal{S}(h_2)m))] \\
& = & (h_{11}.s) \rightharpoonup (h_{12}(f(\mathcal{S}(h_2)m))) \\
& = & (h_1.s) \rightharpoonup (h_{21}(f(\mathcal{S}(h_{22})m))) \\
& = & (h_1.s) \rightharpoonup ((h_2 f)(m)) \\
& = & [(h_1.s) \rightharpoonup (h_2 f)](m),
\end{array}$$

and (14) is satisfied. Therefore $Hom_S(M,N)$ is a left $S\#H$-module.

(ii) When M is a finitely generated projective right S-module, then $Hom_S(M,N) \otimes H \simeq Hom_S(M, N \otimes H)$, so $Hom_S(M,N)$ becomes an H-comodule with the given action. We have that for all $f \in Hom_S(M,N)$, $s \in S$, and $m \in M$,

$$\begin{aligned} f_0(m \leftharpoondown s) \otimes f_1 &= f((m \leftharpoondown s)_0)_0 \otimes \mathcal{S}^{-1}((m \leftharpoondown s)_1) f((m \leftharpoondown s)_0)_1 \\ &= f(m_0 \leftharpoondown s_0)_0 \otimes \mathcal{S}^{-1}(s_1 m_1) f(m_0 \leftharpoondown s_0)_1 \\ &= (f(m_0) \leftharpoondown s_0)_0 \otimes \mathcal{S}^{-1}(s_1 m_1)(f(m_0) \leftharpoondown s_0)_1 \\ &= (f(m_0)_0 \leftharpoondown s_{00}) \otimes \mathcal{S}^{-1}(m_1)\mathcal{S}^{-1}(s_1) s_{01} f(m_0)_1 \\ &= (f(m_0)_0 \leftharpoondown s) \otimes \mathcal{S}^{-1}((m_1) f(m_0)_1 \\ &= (f_0(m)) \leftharpoondown s) \otimes f_1. \end{aligned}$$

So $f_0 \in Hom_S(M,N)$, that is, the right H-coaction is well defined. It is easy to see that $Hom_S(M,N)$ is a right H^{op}-comodule. We have

$$\begin{aligned} (s \rightharpoonup f)_0(m) \otimes (s \rightharpoonup f)_1 &= ((s \rightharpoonup f)(m_0))_0 \otimes \mathcal{S}^{-1}(m_1)((s \rightharpoonup f)(m_0))_1 \\ &= (s \rightharpoonup (f(m_0)))_0 \otimes \mathcal{S}^{-1}(m_1)(s \rightharpoonup (f(m_0)))_1 \\ &= s_0 \rightharpoonup (f(m_0)_0)) \otimes \mathcal{S}^{-1}(m_1) f(m_0)_1 s_1 \\ &= (s_0 \rightharpoonup (f_0(m)) \otimes f_1 s_1 \\ &= (s_0 \rightharpoonup f_0))(m) \otimes f_1 s_1, \end{aligned}$$

for all $f \in Hom_S(M,N)$, $s \in S$, and $m \in M$, and so equation (15) is satisfied. Therefore $Hom_S(M,N)$ is a right H^{op}-comodule.

We have that for all $f \in Hom_S(M,N)$, $h \in H$, and $m \in M$,

$$\begin{aligned} &(hf)_0(m) \otimes (hf)_1 \\ &= ((hf)(m_0))_0 \otimes \mathcal{S}^{-1}(m_1)((hf)(m_0))_1 \\ &= (h_1(f(\mathcal{S}(h_2)m_0)))_0 \otimes \mathcal{S}^{-1}(m_1)(h_1(f(\mathcal{S}(h_2)m_0)))_1 \\ &= h_{12}(f(\mathcal{S}(h_2)m_0)_0) \otimes \mathcal{S}^{-1}(m_1) h_{13}(f(\mathcal{S}(h_2)m_0)_1)\mathcal{S}^{-1}(h_{11}) \\ &= h_2(f(\mathcal{S}(h_4)m_0)_0) \otimes \mathcal{S}^{-1}(m_1) h_3(f(\mathcal{S}(h_4)m_0)_1)\mathcal{S}^{-1}(h_1) \\ &= h_2(f(\mathcal{S}(h_4)m_0)_0) \otimes h_6 \mathcal{S}^{-1}(h_5)\mathcal{S}^{-1}(m_1) h_3(f(\mathcal{S}(h_4)m_0)_1)\mathcal{S}^{-1}(h_1) \\ &= h_{21}(f(\mathcal{S}(h_{222})m_0)_0) \otimes h_3 \mathcal{S}^{-1}(h_{223})\mathcal{S}^{-1}(m_1) h_{221}(f(\mathcal{S}(h_{222})m_0)_1)\mathcal{S}^{-1}(h_1) \\ &= h_{21}(f(\mathcal{S}(h_{222})m_0)_0) \otimes h_3 \mathcal{S}^{-1}(\mathcal{S}(h_{221}) m_1 h_{223})(f(\mathcal{S}(h_{222})m_0)_1)\mathcal{S}^{-1}(h_1) \\ &= h_{21}(f(\mathcal{S}(h_{22})_2 m_0)_0) \otimes h_3 \mathcal{S}^{-1}(\mathcal{S}(h_{22})_3 m_1 \mathcal{S}^{-1}(\mathcal{S}(h_{22})_1)(f(\mathcal{S}(h_{22})_2 m_0)_1)\mathcal{S}^{-1}(h_1) \\ &= h_{21}(f((\mathcal{S}(h_{22})m)_0)_0) \otimes h_3 \mathcal{S}^{-1}((\mathcal{S}(h_{22})m)_1)(f((\mathcal{S}(h_{22})_2 m)_0)_1)\mathcal{S}^{-1}(h_1) \\ &= (h_{21}(f_0(\mathcal{S}(h_{22})m) \otimes h_3 f_1 \mathcal{S}^{-1}(h_1) \\ &= (h_2 f_0)(m) \otimes h_3 f_1 \mathcal{S}^{-1}(h_1), \end{aligned}$$

so equation (9) is satisfied. ■

Since S is not necessarily commutative, we need to consider the left and right S-module homomorphisms separately.

2.2. LEMMA. *Let S be an H-commutative Yetter-Drinfel'd H-module algebra, and let M and N be Hopf Yetter-Drinfel'd (S,H)-modules.*

(i) Then ${}_SHom(M,N)$ is a left $S\#H$-module, where the action of S is defined by

$$(s \rightharpoonup f)(m) = f(m \leftharpoondown s), \text{ for all } s \in S, f \in {}_SHom(M,N), m \in M, \tag{28}$$

and the action of H is defined by

$$(hf)(m) = h_2[f(\mathcal{S}^{-1}(h_1)m)], \text{ for all } f \in {}_SHom(M,N), h \in H, m \in M. \tag{29}$$

(ii) If M is finitely generated projective as a left S-module, then ${}_SHom(M,N)$ is a Hopf Yetter-Drinfel'd (S,H)- module, where the coaction of H^{op} is defined by

$$f_0(m) \otimes f_1 = f(m_0)_0 \otimes f(m_0)_1 \mathcal{S}(m_1), \text{ for all } f \in {}_SHom(M,N), m \in M. \tag{30}$$

Proof. This proof is dual to that of Lemma 2.1. ∎

2.3. Lemma. *Assume S is an H-commutative Yetter-Drinfel'd H-module algebra. Let M, N and P be Hopf Yetter-Drinfel'd (S,H)-modules with P finitely generated projective as a right S-module. Then we have an R-module isomorphism*

$${}_{S\#H}Hom^{H^{op}}(N \tilde{\otimes}_S P, Q) \simeq {}_{S\#H}Hom^{H^{op}}(N, Hom_S(P,Q)).$$

Proof. We consider the R-linear map

$$\phi : {}_{S\#H}Hom^{H^{op}}(N \tilde{\otimes}_S P, Q) \to {}_{S\#H}Hom^{H^{op}}(N, Hom_S P, Q))$$

given by $\phi(f)(n)(p) = f(n \tilde{\otimes} p)$. Let f be an element of ${}_{S\#H}Hom^{H^{op}}(N \tilde{\otimes}_S P, Q)$. For $n \in N$, $p \in P$, and $s \in S$, we have

$$\begin{aligned}
\phi(f)(n)(p \leftharpoonup s) &= f(n \tilde{\otimes} (p \leftharpoonup s)) \\
&= f((n \tilde{\otimes} p) \leftharpoonup s) \\
&= f(s_0 \rightharpoonup s_1(n \tilde{\otimes} p)) \\
&= s_0[s_1(f(n \tilde{\otimes} p))] \\
&= (f(n \tilde{\otimes} p)) \leftharpoonup s \\
&= [\phi(f)(n)(p)] \leftharpoonup s.
\end{aligned}$$

So $\phi(f)(n)$ is S-linear. We also have

$$\begin{aligned}
(\phi(f)(s \rightharpoonup n)](p) &= f((s \rightharpoonup n) \tilde{\otimes} p) \\
&= f(s \rightharpoonup (n \tilde{\otimes} p)) \\
&= s \rightharpoonup (f(n \tilde{\otimes} p)) \\
&= [s \rightharpoonup \phi(f)(n)](p),
\end{aligned}$$

so $\phi(f)$ is S-linear.

Let $h \in H$. we have

$$\begin{aligned}
\phi(f)(hn)(p) &= f(hn \tilde{\otimes} p) \\
&= f(h_1 n \tilde{\otimes} \epsilon(h_2)p) \\
&= f(h_1 n \tilde{\otimes} h_{21}\mathcal{S}(h_{22})p) \\
&= f(h_{11} n \tilde{\otimes} h_{12}\mathcal{S}(h_2)p) \\
&= f(h_1(n \tilde{\otimes} \mathcal{S}(h_2)p)) \\
&= h_1[f(n \tilde{\otimes} \mathcal{S}(h_2)p)] \\
&= h_1[\phi(f)(n)(\mathcal{S}(h_2)p)] \\
&= [h(\phi(f)(n))](p),
\end{aligned}$$

so $\phi(f)$ is left H-linear, and therefore, $\phi(f)$ is $S\#H$-linear.

Assume f is right H^{op}-colinear. Then

$$\begin{aligned}(\phi(f)(n))_0(p) \otimes (\phi(f)(n))_1 &= [(\phi(f)(n))(p_0)]_0 \otimes \mathcal{S}_H^{-1}(p_1)[(\phi(f)(n))(p_0)]_1\\ &= f(n \tilde{\otimes} p_0)_0 \otimes \mathcal{S}_H^{-1}(p_1) f(n \tilde{\otimes} p_0)_1\\ &= f((n \tilde{\otimes} p_0)_0) \otimes \mathcal{S}_H^{-1}(p_1)(n \tilde{\otimes} p_0)_1\\ &= f(n_0 \tilde{\otimes} p_{00}) \otimes \mathcal{S}_H^{-1}(p_1) p_{01} n_1\\ &= f(n_0 \tilde{\otimes} p_0) \otimes \mathcal{S}_H^{-1}(p_{12}) p_{11} n_1\\ &= f(n_0 \tilde{\otimes} p_0) \otimes \epsilon(p_1) n_1\\ &= f(n_0 \tilde{\otimes} p) \otimes n_1\\ &= \phi(f)(n_0)(p) \otimes n_1.\end{aligned}$$

We deduce that $(\phi(f)(n))_0 \otimes (\phi(f)(n))_1 = \phi(f)(n_0) \otimes n_1$, that is, $\phi(f)$ is right H^{op}-colinear.

It follows that ϕ is well defined. Let us consider the R-linear map

$$\psi : {}_{S\#H}Hom^{H^{op}}(N, Hom_S(P,Q)) \to {}_{S\#H}Hom^{H^{op}}(N \tilde{\otimes}_S P, Q)$$

defined by $\psi(g)(n \tilde{\otimes} p) = g(n)(p)$, for all $g \in {}_{S\#H}Hom^{H^{op}}(N, Hom_S(P,Q))$, $n \in N$, and $p \in P$. For $h \in H$ we have

$$\begin{aligned}\psi(g)(h(n \tilde{\otimes} p)) &= \psi(g)(h_1 n \tilde{\otimes} h_2 p)\\ &= g(h_1 n)(h_2 p)\\ &= [h_1(g(n))](h_2 p)\\ &= h_{11}[g(n)(\mathcal{S}(h_{12}) h_2 p)]\\ &= h_1[g(n)(\mathcal{S}(h_{21}) h_{22} p)]\\ &= h_1[g(n)(\epsilon(h_2) p)]\\ &= h[g(n)(p)] = h(\psi(g)(n \tilde{\otimes} p)),\end{aligned}$$

so $\psi(g)$ is H-linear. Let $s \in S$. We have

$$\begin{aligned}\psi(g)(s \rightharpoonup (n \tilde{\otimes} p)) &= \psi(g)((s \rightharpoonup n) \tilde{\otimes} p)\\ &= g(s \rightharpoonup n)(p)\\ &= [s \rightharpoonup (g(n))](p)\\ &= s \rightharpoonup (\psi(g)(n \tilde{\otimes} p)),\end{aligned}$$

so $\psi(g)$ is S-linear. Therefore $\psi(g)$ is $S\#H$-linear. Let us assume that g is right H^{op}-colinear. We have

$$\begin{aligned}\psi(g)((n \tilde{\otimes} p)_0) \otimes (n \tilde{\otimes} p)_1 &= \psi(g)(n_0 \tilde{\otimes} p_0) \otimes (p_1 n_1)\\ &= g(n_0)(p_0) \otimes p_1 n_1\\ &= (g(n)_0)(p_0) \otimes p_1 (g(n_0))_1\\ &= (g(n)(p_{00}))_0 \otimes p_1 \mathcal{S}^{-1}(p_{01})(g(n)(p_{00}))_1\\ &= (g(n)(p_0))_0 \otimes p_{12} \mathcal{S}^{-1}(p_{11})(g(n)(p_0))_1\\ &= (g(n)(p_0))_0 \otimes \epsilon(p_1)(g(n)(p_0))_1\\ &= (g(n)(p))_0 \otimes (g(n)(p))_1\\ &= (\psi(g)(n \tilde{\otimes} p))_0 \otimes (\psi(g)(n \tilde{\otimes} p))_1.\end{aligned}$$

We deduce that $\psi(g)$ is right H^{op}-colinear. It follows that ψ is well defined. It is easy to see that ϕ and ψ are inverse of each other. ∎

From Lemma 2.3, we deduce that the functor $Hom_S(P,-)$ defined from ${}_S\mathcal{Q}^H$ to ${}_S\mathcal{Q}^H$ with P finitely generated projective as a right S-module is right adjoint to the functor $-\tilde{\otimes}_S P$ defined from ${}_S\mathcal{Q}^H$ to ${}_S\mathcal{Q}^H$. In the notation of [vOZ], $Hom_S(P,Q)=[P,Q]$. It also follows from Lemma 2.3 that if N and P are projective as right S-modules, then $N\tilde{\otimes}_S P$ is projective as a right S-module.

Again, we need to consider the left S-module structures separately.

2.4. Lemma. *Assume S is an H-commutative Yetter-Drinfel'd H-module algebra. Let M, N, and P be Hopf Yetter-Drinfel'd (S,H)-modules with P finitely generated projective as a left S-module. Then we have an R-module isomorphism*

$$ {}_{S\#H}Hom^{H^{op}}(P\tilde{\otimes}_S N,Q)\simeq {}_{S\#H}Hom^{H^{op}}(N,{}_SHom(P,Q)). $$

Proof. Let f be an element of ${}_{S\#H}Hom^{H^{op}}(P\tilde{\otimes}_S N,Q)$. We consider the R-linear map

$$ \phi: {}_{S\#H}Hom^{H^{op}}(P\tilde{\otimes}_S N,Q)\to {}_{S\#H}Hom^{H^{op}}(N,{}_SHom(P,Q)) $$

given by $\phi(f)(n)(p)=f(p\tilde{\otimes}n)$. Then $\phi(f)(n)$ is S-linear, $\phi(f)$ is S-linear, $\phi(f)$ is H-linear and $\phi(f)$ is right H^{op}-colinear, therefore, $\phi(f)$ is left $S\#H$-linear and right H^{op}-colinear. It follows that ϕ is well defined.

Let g be an element of ${}_{S\#H}Hom^{H^{op}}(N,{}_SHom(P,Q))$. Let us consider the R-linear map

$$ \psi: {}_{S\#H}Hom^{H^{op}}(N,{}_SHom(P,Q))\to {}_{S\#H}Hom^{H^{op}}(P\tilde{\otimes}_S N,Q) $$

defined by $\psi(g)(p\tilde{\otimes}n)=g(n)(p)$. Then $\psi(g)$ is S-linear, left H-linear and right H-colinear. It follows that ψ is well defined. It is easy to see that ϕ and ψ are inverse of each other. ∎

From Lemma 2.4, we deduce that the functor ${}_SHom(P,-)$ defined from ${}_S\mathcal{Q}^H$ to ${}_S\mathcal{Q}^H$ with P finitely generated projective as a left S-module is right adjoint to the functor $P\tilde{\otimes}_S-$ defined from ${}_S\mathcal{Q}^H$ to ${}_S\mathcal{Q}^H$. In the notation of [vOZ], ${}_SHom(P,Q)=\{P,Q\}$. It also follows from Lemma 2.4 that if N and P are projective as left S-modules, then $P\tilde{\otimes}_S N$ is projective as a left S-module.

The results of the following lemma are useful for some computations.

2.5. Lemma. *Assume S is an H-commutative Yetter-Drinfel'd H-module algebra. Let M, N be Hopf Yetter-Drinfel'd (S,H)-modules.*

(i) If M is finitely generated projective as a right S-module, then $(f\leftharpoonup s)(m)=f(s\rightharpoonup m)$ for every $f\in Hom_S(M,N)$, $m\in M$ and $s\in S$.

(ii) If M is finitely generated projective as a left S-module, then $(f\leftharpoonup s)(m)=f(m)\leftharpoonup s$ for every $f\in {}_SHom(M,N)$, $m\in M$ and $s\in S$.

Proof. We will prove (ii). The proof of (i) is easier. We have

$$\begin{aligned}
(f \leftharpoonup s)(m) &= [s_0(s_1 f)](m)\\
&= (s_1 f)(m \leftharpoonup s_0)\\
&= s_{12}[f(\mathcal{S}^{-1}(s_{11})(m \leftharpoonup s_0))]\\
&= s_3[f((\mathcal{S}^{-1}(s_2)m) \leftharpoonup (\mathcal{S}^{-1}(s_1).s_0))]\\
&= s_3[f((\mathcal{S}^{-1}(s_1).s_0)_0 \rightharpoonup ((\mathcal{S}^{-1}(s_1).s_0)_1\mathcal{S}^{-1}(s_2)m))]\\
&= s_3[f((\mathcal{S}^{-1}(s_{12}).s_{00}) \rightharpoonup ((\mathcal{S}^{-1}(s_{11})s_{01}\mathcal{S}^{-2}(s_{13})\mathcal{S}^{-1}(s_2)m))]\\
&= s_6[f((\mathcal{S}^{-1}(s_3).s_0) \rightharpoonup ((\mathcal{S}^{-1}(s_2)s_1\mathcal{S}^{-2}(s_4)\mathcal{S}^{-1}(s_5)m))]\\
&= s_2[f((\mathcal{S}^{-1}(s_1).s_0) \rightharpoonup m)]\\
&= s_2[(\mathcal{S}^{-1}(s_1).s_0) \rightharpoonup f(m)]\\
&= (s_{21}\mathcal{S}^{-1}(s_1).s_0) \rightharpoonup (s_{22}(f(m))\\
&= s_0 \rightharpoonup (s_1(f(m)))\\
&= f(m) \leftharpoonup s.
\end{aligned}$$

∎

3. The category of dyslectic Hopf Yetter-Drinfel'd (S, H) modules

In this section, H denotes a Hopf algebra with bijective antipode and S is an H-commutative Yetter-Drinfel'd H-module algebra. Our objective for this section is to define the subcategory of dyslectic Hopf Yetter-Drinfel'd (S, H)-modules.

We do this because the category ${}_SQ^H$ might not be braided. It is not clear that the analogous braiding maps $\gamma_{M,N} : M \tilde{\otimes}_S N \to N \tilde{\otimes}_S M$ defined as in (12) by $m \tilde{\otimes} n \mapsto n_0 \tilde{\otimes} n_1 m$ will be morphisms in the ${}_SQ^H$ category. An object M of ${}_S\mathcal{Q}^H$ is *dyslectic* if $h_M \circ \gamma_{M,S} \circ \gamma_{S,M} = h_M$, where $h_M : S \otimes M \to M$ denotes the left action of S on M [Pareigis2] . It follows that an object M of ${}_S\mathcal{Q}^H$ is dyslectic if and only if

$$s \rightharpoonup m = (m_1.s)_0 \rightharpoonup ((m_1.s)_1 m_0), \text{ for all } s \in S, m \in M. \tag{31}$$

Note that we can also use the inverse braiding to define dyslectic modules as in [Wang].

Clearly, S is a dyslectic Hopf Yetter-Drinfel'd (S, H)-module, and every Hopf Yetter-Drinfel'd H-module can be regarded as a dyslectic Hopf Yetter-Drinfel'd (k, H)-module. A dyslectic Hopf Yetter-Drinfel'd (S, H)-module homomorphism is a Hopf Yetter-Drinfel'd (S, H)-module homomorphism between dyslectic modules.

Let M be a Hopf Yetter-Drinfel'd (S, H)-module and let us consider the condition

$$s \rightharpoonup m = m_0 \leftharpoonup (m_1.s), \text{ for all } s \in S, m \in M, \tag{32}$$

which is equivalent to the equation

$$m \leftharpoonup s = (\mathcal{S}(m_1).s) \rightharpoonup m_0, \text{ for all } s \in S, m \in M. \tag{33}$$

3.1. Lemma. *Let N be a Hopf Yetter-Drinfel'd (S, H)- module. Then the condition (32) is satisfied for N if and only if $\gamma_{M,N}$ is well defined for all M in ${}_S\mathcal{Q}^H$.*

PROOF. Let $m \in M$, $n \in N$ and $s \in S$. If (32) is satisfied for N, then

$$\begin{aligned}
\gamma_{M,N}((m \leftharpoonup s) \tilde{\otimes} n) &= \gamma_{M,N}((s_0 \rightharpoonup (s_1 m)) \tilde{\otimes} n)\\
&= n_0 \tilde{\otimes} n_1(s_0 \rightharpoonup (s_1 m))\\
&= n_0 \tilde{\otimes} ((n_{11}.s_0) \rightharpoonup n_{12}(s_1 m))\\
&= (n_0 \leftharpoonup (n_{11}.s_0)) \tilde{\otimes} n_{12}(s_1 m)\\
&= (n_{00} \leftharpoonup (n_{01}.s_0)) \tilde{\otimes} n_1(s_1 m)\\
&= (s_0 \rightharpoonup n_0) \tilde{\otimes} n_1(s_1 m)\\
&= (s \rightharpoonup n)_0 \tilde{\otimes} (s \rightharpoonup n)_1 m\\
&= \gamma_{M,N}(m \tilde{\otimes} (s \rightharpoonup n)).
\end{aligned}$$

So $\gamma_{M,N}$ is well defined. If $\gamma_{M,N}$ is well-defined for all M in ${}_S\mathcal{Q}^H$, then $\gamma_{S,N}$ is well defined. Let $n \in N$ and $s \in S$. We have

$$\gamma_{S,N}((1_S \leftharpoonup s) \tilde{\otimes} n) = \gamma_{S,N}(1_S \tilde{\otimes} (s \rightharpoonup n)).$$

But we also have

$$\begin{aligned}
\gamma_{S,N}((1_S \leftharpoonup s) \tilde{\otimes} n) &= n_0 \tilde{\otimes} n_1.(1 \leftharpoonup s)\\
&= n_0 \tilde{\otimes} n_1.s\\
&= (n_0 \leftharpoonup n_1.s) \tilde{\otimes} 1_S
\end{aligned}$$

and

$$\begin{aligned}
\gamma_{S,N}(1_S \tilde{\otimes} (s \rightharpoonup n)) &= (s \rightharpoonup n)_0 \tilde{\otimes} (s \rightharpoonup n)_1.1_S\\
&= (s \rightharpoonup n)_0 \leftharpoonup (s \rightharpoonup n)_1.1_S \tilde{\otimes} 1_S\\
&= (s \rightharpoonup n) \tilde{\otimes} 1_S.
\end{aligned}$$

Therefore, condition (32) is satisfied for N. ∎

3.2. LEMMA. *Let M be a Hopf Yetter-Drinfel'd (S, H)- module. Then the condition (33) is satisfied for M if and only if $\gamma^{-1}_{M,N}$ is well defined for all N in ${}_S\mathcal{Q}^H$.*

PROOF. Let $m \in M$, $n \in N$ and $s \in S$. If (33) is well-defined, we have

$$\begin{aligned}
\gamma^{-1}_{M,N}((m \leftharpoonup s) \tilde{\otimes}_S n) &= \mathcal{S}((m \leftharpoonup s)_1) n \tilde{\otimes}_S (m \leftharpoonup s)_0\\
&= \mathcal{S}(s_1 m_1) n \tilde{\otimes} (m_0 \leftharpoonup s_0)\\
&= (\mathcal{S}(m_1)\mathcal{S}(s_1)) n \tilde{\otimes} (m_0 \leftharpoonup s_0)\\
&= (\mathcal{S}(m_1)\mathcal{S}(s_1)) n \tilde{\otimes} ((\mathcal{S}(m_{01}).s_0) \rightharpoonup m_{00})\\
&= (\mathcal{S}(m_{12})\mathcal{S}(s_1)) n \tilde{\otimes} ((\mathcal{S}(m_{11}).s_0) \rightharpoonup m_0)\\
&= (\mathcal{S}(m_1)_1\mathcal{S}(s_1)) n \tilde{\otimes} ((\mathcal{S}(m_1)_2.s_0) \rightharpoonup m_0)\\
&= [((\mathcal{S}(m_1)_1\mathcal{S}(s_1)) n) \leftharpoonup (\mathcal{S}(m_1)_2.s_0)] \tilde{\otimes} m_0\\
&= \mathcal{S}(m_1)((\mathcal{S}(s_1) n) \leftharpoonup s_0) \tilde{\otimes} m_0\\
&= \gamma^{-1}_{M,N}(m \tilde{\otimes} ((\mathcal{S}(s_1) n) \leftharpoonup s_0))\\
&= \gamma^{-1}_{M,N}(m \tilde{\otimes} (s \rightharpoonup n)).
\end{aligned}$$

So $\gamma^{-1}_{M,N}$ is well defined.

If $\gamma^{-1}_{M,N}$ is well defined for all N in ${}_S\mathcal{Q}^H$, then $\gamma^{-1}_{M,S}$ is well defined. Let $m \in M$ and $s \in S$. We have

$$\gamma^{-1}_{M,S}(m \tilde{\otimes} (s \rightharpoonup 1_S)) = \gamma^{-1}_{M,S}((m \leftharpoonup s) \tilde{\otimes}_S 1_S).$$

But

$$\begin{aligned} \gamma^{-1}_{M,S}(m \tilde{\otimes} (s \rightharpoonup 1_S)) &= \gamma^{-1}_{M,S}(m \tilde{\otimes} s) \\ &= \mathcal{S}(m_1).s \tilde{\otimes} m_0 \\ &= 1_S \tilde{\otimes} (\mathcal{S}(m_1).s \rightharpoonup m_0) \end{aligned}$$

and

$$\begin{aligned} \gamma^{-1}_{M,S}((m \leftharpoonup s) \tilde{\otimes} 1_S) &= \mathcal{S}((m \leftharpoonup s)_1).1_S \tilde{\otimes} (m \leftharpoonup s)_0 \\ &= \mathcal{S}((m \leftharpoonup s)_1).1_S \tilde{\otimes} [\mathcal{S}((m \leftharpoonup s)_1).1_S \rightharpoonup (m \leftharpoonup s)_0] \\ &= 1_S \tilde{\otimes} (m \leftharpoonup s). \end{aligned}$$

So condition (33) is satisfied for M. ∎

The following lemma provides an easiest necessary and sufficient condition to show that a Hopf Yetter-Drinfel'd (S, H)- module is dyslectic.

3.3. Lemma. *Let M be a Hopf Yetter-Drinfel'd (S, H)- module. Then M is dyslectic if and only if the condition (32) is satisfied for M.*

Proof. Assume condition (32) is satisfied for M. Then we have

$$\begin{aligned} s \rightharpoonup m &= m_0 \leftharpoonup m_1.s \\ &= (m_1.s)_0 \rightharpoonup ((m_1.s)_1 m_0). \end{aligned}$$

So M is dyslectic.

If M dyslectic, then

$$\begin{aligned} s \rightharpoonup m &= (m_1.s)_0 \rightharpoonup ((m_1.s)_1 m_0) \\ &= [\mathcal{S}((m_1.s)_{01})((m_1.s)_1 m_0)] \leftharpoonup (m_1.s)_{00} \\ &= [\mathcal{S}((m_1.s)_{11})((m_1.s)_{12} m_0)] \leftharpoonup (m_1.s)_0 \\ &= [\epsilon((m_1.s)_1) m_0] \leftharpoonup (m_1.s)_0 \\ &= m_0 \leftharpoonup (m_1.s), \end{aligned}$$

and the condition (32) is satisfied for M. ∎

Since conditions (32) and (33) are equivalent, a Hopf Yetter-Drinfel'd (S, H)- module M is dyslectic if and only if the condition (33) is satisfied for M. However we can prove this result directly as in Lemma 3.3.

We denote by $Dys\text{-}{}_S\mathcal{Q}^H$ the category of dyslectic Hopf Yetter - Drinfel'd (S, H) modules with dyslectic Hopf Yetter-Drinfel'd (S, H)-modules homomorphisms; it is a full subcategory of ${}_S\mathcal{Q}^H$.

3.4. Lemma. *Let M and N be dyslectic Hopf Yetter-Drinfel'd (S, H)-modules. Then $M \tilde{\otimes}_S N$ is a dyslectic Hopf Yetter - Drinfel'd (S, H)- module.*

PROOF. Suppose M and N are dyslectic Hopf Yetter - Drinfel'd (S,H)- modules. Let $m \in M$, $n \in N$ and $s \in S$. We have

$$\begin{array}{rcl}
(\mathcal{S}((m \tilde{\otimes} n)_1).s) \rightharpoonup (m \tilde{\otimes} n)_0 & = & (\mathcal{S}((n_1 m_1).s) \rightharpoonup (m_0 \tilde{\otimes} n_0) \\
& = & ((\mathcal{S}(m_1)\mathcal{S}(n_1)).s) \rightharpoonup (m_0 \tilde{\otimes} n_0) \\
& = & (\mathcal{S}(m_1).(\mathcal{S}(n_1).s)) \rightharpoonup (m_0 \tilde{\otimes} n_0) \\
& = & ((\mathcal{S}(m_1).(\mathcal{S}(n_1).s)) \rightharpoonup m_0) \tilde{\otimes} n_0 \\
& = & (m \leftharpoonup (\mathcal{S}(n_1).s)) \tilde{\otimes} n_0 \\
& = & m \tilde{\otimes} ((\mathcal{S}(n_1).s) \rightharpoonup n_0) \\
& = & m \tilde{\otimes} (n \leftharpoonup s) \\
& = & (m \tilde{\otimes} n) \leftharpoonup s.
\end{array}$$

So the equation (33) is satisfied for $M \tilde{\otimes}_S N$, and $M \tilde{\otimes}_S N$ is dyslectic. ∎

The above Lemmas show the subcategory of dyslectic Hopf Yetter-Drinfel'd(S,H)-modules is monoidal with respect to $\tilde{\otimes}_S$ and S. Wang showed that when H has a bijective antipode, S is H-commutative, and all Hopf Yetter-Drinfel'd(S,H)-modules are dyslectic, then the monoidal category is braided [Wang, Theorem 2.2] . So we deduce from the above Lemmas that the category of dyslectic Hopf Yetter-Drinfel'd(S,H)-modules can be viewed as a braided monoidal category.

3.5. LEMMA. *Let M and N be dyslectic Hopf Yetter-Drinfel'd (S,H)-modules.*

(i) If M is finitely generated projective as a right S-module, then $Hom_S(M,N)$ is a dyslectic Hopf Yetter-Drinfel'd (S,H)- module.

(ii) If M is finitely generated projective as a left S-module, then ${}_SHom(M,N)$ is a dyslectic Hopf Yetter-Drinfel'd (S,H)- module.

PROOF. (i) Suppose M and N are dyslectic Hopf Yetter-Drinfel'd (S,H)- modules with M finitely generated projective as a right S-module. Let $f \in Hom_S(M,N)$, $m \in M$, $n \in N$ and $s \in S$. We have

$$\begin{array}{rcl}
((\mathcal{S}(f_1).s) \rightharpoonup f_0)(m) & = & (\mathcal{S}(f_1).s) \rightharpoonup f_0(m) \\
& = & (\mathcal{S}[\mathcal{S}^{-1}(m_1)f(m_0)_1].s) \rightharpoonup f(m_0)_0 \\
& = & ([\mathcal{S}(f(m_0)_1)m_1].s) \rightharpoonup f(m_0)_0 \\
& = & (\mathcal{S}(f(m_0)_1)(m_1.s)) \rightharpoonup f(m_0)_0 \\
& = & f(m_0) \leftharpoonup (m_1.s) \\
& = & f(m_0 \leftharpoonup (m_1.s)) \\
& = & f(s \rightharpoonup m) \\
& = & f([\mathcal{S}(s_1)m] \leftharpoonup s_0) \\
& = & f(\mathcal{S}(s_1)m) \leftharpoonup s_0 \\
& = & s_{00} \rightharpoonup (s_{01}(f(\mathcal{S}(s_1)m))) \\
& = & s_0 \rightharpoonup (s_{11}(f(\mathcal{S}(s_{12})m))) \\
& = & s_0 \rightharpoonup ((s_1 f)(m)) \\
& = & (s_0 \rightharpoonup (s_1 f))(m) \\
& = & (f \leftharpoonup s)(m).
\end{array}$$

So condition (33) is satisfied, and $Hom_S(M,N)$ is dyslectic.

(ii) Suppose M and N are dyslectic Hopf Yetter-Drinfel'd (S,H)- modules with M finitely generated projective as a left S-module. Let $f \in {}_S Hom(M,N)$, $m \in M$, $n \in N$ and $s \in S$. We have

$$\begin{aligned}
& f_0(m) \leftharpoonup (f_1.s) \\
&= f(m_0)_0 \leftharpoonup ((f(m_0)_1\mathcal{S}(m_1)).s) \\
&= [(f(m_0)_1\mathcal{S}(m_1).s)_0] \rightharpoonup [(f(m_0)_1\mathcal{S}(m_1).s)_1 f(m_0)_0] \\
&= [f(m_0)_{12}\mathcal{S}(m_{12}).s_{00}] \rightharpoonup [f(m_0)_{13}\mathcal{S}(m_{11})s_1\mathcal{S}^{-1}(f(m_0)_{11}\mathcal{S}(m_{13}))f(m_0)_0] \\
&= [f(m_0)_{12}\mathcal{S}(m_1)][s_0 \rightharpoonup (s_1 m_{13}\mathcal{S}^{-1}(f(m_0)_{11})f(m_0)_0] \\
&= [f(m_0)_{12}\mathcal{S}(m_1)][(m_{13}\mathcal{S}^{-1}(f(m_0)_{11})f(m_0)_0) \leftharpoonup s] \\
&= [f(m_0)_{121}\mathcal{S}(m_{12})m_{13}\mathcal{S}^{-1}(f(m_0)_{11})(f(m_0)_0)] \leftharpoonup [f(m_0)_{122}\mathcal{S}(m_1).s)] \\
&= f(m_0)_0)(\leftharpoonup [f(m_0)_1\mathcal{S}(m_1).s)] \\
&= (\mathcal{S}(m_1).s)(f(m_0)) \\
&= f[(\mathcal{S}(m_1).s) \leftharpoonup m_0] \\
&= f(m \leftharpoonup s) \\
&= (sf)(m).
\end{aligned}$$

So condition (32) is satisfied, and ${}_S Hom(M,N)$ is dyslectic. ■

We deduce from Lemmas 2.3, 3.4 and 3.5(i) that if P is finitely generated projective as a right S-module, then the functor $Hom_S(P,-)$ defined from $Dys\text{-}_S\mathcal{Q}^H$ to $Dys\text{-}_S\mathcal{Q}^H$ is right adjoint to the functor $-\tilde{\otimes}_S P$ defined from $Dys\text{-}_S\mathcal{Q}^H$ to $Dys\text{-}_S\mathcal{Q}^H$. Likewise, we deduce from Lemmas 2.4, 3.4 and 3.5(ii) that if P is finitely generated projective as a left S-module, then the functor ${}_S Hom(P,-)$ defined from $Dys\text{-}_S\mathcal{Q}^H$ to $Dys\text{-}_S\mathcal{Q}^H$ is right adjoint to the functor $-\tilde{\otimes}_S P$ defined from $Dys\text{-}_S\mathcal{Q}^H$ to $Dys\text{-}_S\mathcal{Q}^H$. Since $Dys\text{-}_S\mathcal{Q}^H$ is a braided monoidal category, by [Femić], we have an isomorphism of dyslectic Hopf Yetter -Drinfel'd (S,H)-modules $Hom_S(P,Q) = {}_S Hom(P,Q)$ for all objects P,Q in $Dys\text{-}_S\mathcal{Q}^H$ with P finitely generated projective as a left and as a right S-module: more precisely, the isomorphism is the map $\phi : Hom_S(P,Q) \to {}_S Hom(P,Q)$ defined by $\phi(f)(p) = f_0(f_1 p)$. Note that in $Dys\text{-}_S\mathcal{Q}^H$, if N and P are finitely generated projective as right and left S-modules, then $N \tilde{\otimes}_S P$ is finitely generated projective as a right and left S-module.

We know from [vOZ] that there is a Brauer group for the braided monoidal category $Dys\text{-}_S\mathcal{Q}^H$. Most of the remainder of the paper is concerned with developing the details of the ingredients necessary to define this Brauer group precisely.

4. Dyslectic Hopf Yetter -Drinfel'd (S,H)-module algebras

In this section, H is a Hopf algebra with bijective antipode, and S is an H-commutative Yetter-Drinfel'd H-module algebra.

A *dyslectic Hopf Yetter-Drinfel'd* (S,H)*-module algebra* is an algebra in the braided monoidal category $Dys\text{-}_S\mathcal{Q}^H$, that is, an object A of $Dys\text{-}_S\mathcal{Q}^H$ such that there are two dyslectic Hopf Yetter-Drinfel'd (S,H)-module homomorphisms $\pi : A\tilde{\otimes}_S A \to A$ and $\mu : S \to A$ satisfying the associativity and the unitary conditions of usual algebras.

Since S is H-commutative, S is a dyslectic Hopf Yetter-Drinfel'd (S,H)-module algebra. Note that a dyslectic Hopf Yetter-Drinfel'd (S,H)-module algebra is an algebra in the monoidal category ${}_S\mathcal{Q}^H$ which is dyslectic as a Hopf Yetter-Drinfel'd (S,H)-module. Every Yetter-Drinfel'd H-module algebra is a dyslectic Hopf Yetter-Drinfel'd (R,H)-module algebra.

A *dyslectic Hopf Yetter Drinfel'd (S,H)-module algebra homomorphism* is a dyslectic Hopf Yetter-Drinfel'd (S,H)-module homomorphism which is compatible with the product and is a unitary algebra homomorphism.

4.1. LEMMA. *Assume that M is a dyslectic Hopf Yetter-Drinfel'd (S,H)-module that is finitely generated projective as a right S-module. Then $End_S(M)$ is a dyslectic Hopf Yetter-Drinfel'd (S,H)-module algebra: the product map is defined by $\pi(f\tilde{\otimes} g)=f\circ g$ for all $f,g\in End_S(M)$ and the unit map $\mu:S\to End_S(M)$ is defined by $\mu(s)(m)=s\rightharpoonup m$.*

PROOF. By Lemma 3.5(i), $End_S(M)$ is a dyslectic Hopf Yetter-Drinfel'd (S,H)-module. It is easy to see that π and μ are well-defined, left S-linear, and H-linear. Let us show that π is H^{op}-colinear. We have

$$\begin{aligned}
(f_0\circ g_0)(m)\otimes g_1f_1 &= f_0(g_0(m))\otimes g_1f_1\\
&= f_0(g(m_0)_0)\otimes \mathcal{S}^{-1}(m_1)g(m_0)_1f_1\\
&= f(g(m_0)_{00})_0\otimes \mathcal{S}^{-1}(m_1)g(m_0)_1\mathcal{S}^{-1}(g(m_0)_{01})f(g(m_0)_{00})_1\\
&= f(g(m_0)_0)_0\otimes \mathcal{S}^{-1}(m_1)g(m_0)_{12}\mathcal{S}^{-1}(g(m_0)_{11})f(g(m_0)_0)_1\\
&= f(g(m_0)_0)_0\otimes \mathcal{S}^{-1}(m_1)\epsilon(g(m_0)_1)f(g(m_0)_0)_1\\
&= f(g(m_0))_0\otimes \mathcal{S}^{-1}(m_1)f(g(m_0))_1\\
&= ((f\circ g)(m_0))_0\otimes \mathcal{S}^{-1}(m_1)((f\circ g)(m_0))_1\\
&= (f\circ g)_0(m)\otimes (f\circ g)_1.
\end{aligned}$$

So π is H^{op}-colinear. We have

$$\begin{aligned}
\mu(s)_0(m)\otimes\mu(s)_1 &= (\mu(s)(m_0))_0\otimes \mathcal{S}^{-1}(m_1)(\mu(s)(m_0))_1\\
&= (s\rightharpoonup m_0)_0\otimes \mathcal{S}^{-1}(m_1)(s\rightharpoonup m_0)_1\\
&= (s_0\rightharpoonup m_{00})\otimes \mathcal{S}^{-1}(m_1)m_{01}s_1\\
&= (s_0\rightharpoonup m)\otimes s_1\\
&= \mu(s_0)(m)\otimes s_1.
\end{aligned}$$

So μ is H^{op}-colinear. Clearly, $h.id_M=\epsilon(h)id_M$ and $\rho(id_M)=id_M\otimes 1_H$, where id_M is the identity element of $End_S(M)$. It is well-known that the composition law is associative. ∎

4.2. LEMMA. *Assume that M is a dyslectic Hopf Yetter-Drinfel'd (S,H)-module that is finitely generated projective as a left S-module. Then ${}_SEnd(M)$ is a dyslectic Hopf Yetter-Drinfel'd (S,H)-module algebra: the product map is defined by $\pi(f\tilde{\otimes} g)=fg=g\circ f$ for all $f,g\in {}_SEnd(M)$ and the unit map $\mu:S\to {}_SEnd(M)$ is defined by $\mu(s)(m)=s\rightharpoonup m$.*

Proof. It is easy to show μ is well-defined, S-linear, H-linear, and H^{op}-colinear. To show π is well-defined, let $f, g \in {}_S End(M)$, $s \in S$, and $m \in M$. We have

$$\begin{aligned} [(f \leftharpoonup s)g](m) &= g((f \leftharpoonup s)(m)) \\ &= g(f(m) \leftharpoonup s) \\ &= (s \rightharpoonup g)(f(m)) \\ &= [f(s \rightharpoonup g)](m), \end{aligned}$$

so π is well-defined. Also

$$\begin{aligned} [(s \rightharpoonup f)g](m) &= g((s \rightharpoonup f)(m)) \\ &= g(f(m \leftharpoonup s)) \\ &= [fg](m \leftharpoonup s) \\ &= (s \rightharpoonup [fg])(m), \end{aligned}$$

so π is S-linear.

We also have (see the proof Proposition 4.1 in [CvOZ1]) $h(fg) = (h_1 f)(h_2 g)$ and $(fg)_0 \otimes (fg)_1 = (f_0 g_0) \otimes g_1 f_1$, that is, π is H-linear and H^{op}-colinear. ∎

Let A be a dyslectic Hopf Yetter-Drinfel'd (S, H)-module algebra. The H-opposite algebra $\bar{A}$ of A is defined as follows: $\bar{A} = A$ as a dyslectic Hopf Yetter-Drinfel'd (S, H)-module, but with multiplication $m_A \circ \gamma$, where m_A is the multiplication of A (see [vOZ, page 100]). In other words,

$$\bar{a}\bar{a'} = \overline{a'_0(a'_1.a)} \ \forall\, a, a' \in A.$$

The action of S on $\bar{A}$ is defined by $s \rightharpoonup \bar{a} = \overline{s \rightharpoonup a}$, the H-action by $h.\bar{a} = \overline{h.a}$, and the H-coaction by $(\bar{a})_0 \otimes (\bar{a})_1 = \overline{a_0} \otimes a_1$, for all $a \in A$, $h \in H$, and $s \in S$. If the action of H or the coaction of H is trivial, then $\bar{A} = A^{op}$, the ordinary opposite algebra of A. Note that $\bar{S} \simeq S$ when S is H-commutative.

4.3. Lemma. *Suppose that A is a dyslectic Hopf Yetter-Drinfel'd (S, H)-module algebra. Then $\bar{A}$ is a dyslectic Hopf Yetter-Drinfel'd (S, H)-module algebra.*

Proof. We need to show that the action of S is compatible with the product. Let $s \in S$, $a, b \in A$. Then

$$\begin{aligned} \bar{a}(s \rightharpoonup \bar{b}) &= \bar{a}\overline{(s \rightharpoonup b)} \\ &= \overline{(s \rightharpoonup b)_0((s \rightharpoonup b)_1.a)} \\ &= \overline{(s_0 \rightharpoonup b_0)((b_1 s_1).a)} \\ &= \overline{s_0 \rightharpoonup (b_0(b_1.(s_1.a)))} \\ &= s_0 \rightharpoonup \overline{b_0(b_1.(s_1.a))} \\ &= s_0 \rightharpoonup (\overline{s_1.a}\,\overline{b}) \\ &= (s_0 \rightharpoonup (\overline{s_1.a}))\overline{b} \\ &= \overline{s_0 \rightharpoonup (s_1.a)}\,\overline{b} \\ &= \overline{a \leftharpoonup s)}\,\overline{b} \\ &= (\overline{a} \leftharpoonup s)\overline{b}, \end{aligned}$$

so the multiplication in $\bar{A}$ is well-defined. On the other hand, we have

$$\begin{array}{rcl} (s \rightharpoonup \bar{a})\bar{b} & = & (\overline{s \rightharpoonup a})\bar{b} \\ & = & \overline{b_0.(b_1.(s \rightharpoonup a))} \\ & = & \overline{b_0.((b_{11}.s) \rightharpoonup (b_{12}.a))} \\ & = & \overline{b_{00}.((b_{01}.s) \rightharpoonup (b_1.a))} \\ & = & \overline{(b_{00} \leftharpoondown (b_{01}.s)).(b_1.a))} \\ & = & \overline{(s \rightharpoonup b_0).(b_1.a)} \\ & = & \overline{s \rightharpoonup (b_0.(b_1.a))} \\ & = & s \rightharpoonup \overline{b_0.(b_1.a)} \\ & = & s \rightharpoonup (\bar{a}\bar{b}), \end{array}$$

so the multiplication in $\bar{A}$ is S-linear.

Clearly, $h.(\bar{a}\bar{b}) = (h_1.\bar{a})(h_2.\bar{b})$ and

$$\begin{array}{rcl} (\bar{a}\bar{b})_0 \otimes (\bar{a}\bar{b})_1 & = & \overline{b_0(b_1.a)}_0 \otimes \overline{b_0(b_1.a)}_1 \\ & = & \overline{(b_0(b_1.a))_0} \otimes (b_0(b_1.a))_1 \\ & = & \overline{b_{00}(b_1.a)_0} \otimes (b_1.a)_1 b_{01} \\ & = & \overline{b_{00}(b_{12}.a_0)} \otimes (b_{13}a_1\mathcal{S}^{-1}(b_{11})b_{01}) \\ & = & \overline{b_0(b_3.a_0)} \otimes (b_4 a_1 \mathcal{S}^{-1}(b_2)b_1) \\ & = & \overline{b_0(b_1.a_0)} \otimes b_2 a_1 \\ & = & \overline{b_{00}(b_{01}.a_0)} \otimes b_1 a_1 \\ & = & \bar{a}_0\bar{b}_0 \otimes \bar{b}_1\bar{a}_1. \end{array}$$

Furthermore the H-action and the H-coaction preserve the identity element of $\bar{A}$. ■

If A and B are dyslectic Hopf Yetter-Drinfel'd (S, H)-module algebras, we define a new multiplication in $A \tilde{\otimes}_S B$ by

$$(A \tilde{\otimes}_S B) \tilde{\otimes}_S (A \tilde{\otimes}_S B) \overset{1\otimes\gamma\otimes 1}{\to} (A \tilde{\otimes}_S A) \tilde{\otimes}_S (B \tilde{\otimes}_S B) \overset{m_A \otimes m_B}{\to} A \tilde{\otimes}_S B.$$

In other words,

$$(a \# b)(a' \# b') = aa'_0 \# (a'_1.b)b', \text{ for all } a, a' \in A, b, b' \in B.$$

This new multiplication on $A \tilde{\otimes}_S B$ is called the *braided product* and $A \tilde{\otimes}_S B$ with the braided product will be denoted by $A \#_S B$.

4.4. Proposition. *Let A, B and C be dyslectic Hopf Yetter-Drinfel'd (S, H)- module algebras. Then*

(i) $A \#_S B$ is a dyslectic Hopf Yetter-Drinfel'd (S, H)-module algebra whose identity element is $1_A \otimes 1_B$. The action of H is given by

$$h.(a \# b) = (h_1.a) \# (h_2.b), \text{ for all } h \in H, a \in A, b \in B,$$

and the coaction of H is given by

$$(a \# b)_0 \otimes (a \# b)_1 = (a_0 \# b_0) \otimes b_1 a_1, \text{ for all } a \in A, b \in B.$$

(ii) The canonical injections $A \to A \#_S B$ and $B \to A \#_S B$ are homomorphisms of dyslectic Hopf Yetter-Drinfel'd (S,H)-module algebras.

(iii) The canonical S-linear maps $A \to A \#_S S$ and $A \to S \#_S A$ are isomorphisms of dyslectic Hopf Yetter-Drinfel'd (S,H)-module algebras.

(iv) The map $\phi : (A \#_S B) \#_S C \simeq A \#_S (B \#_S C)$ given by $\phi((a \# b) \# c) = a \# (b \# c)$, for all $a \in A$, $b \in B$, and $c \in C$, is an isomorphism of dyslectic Hopf Yetter-Drinfel'd (S,H)-module algebras.

(v) The map $\phi : \bar{B} \#_S \bar{A} \to \overline{A \#_S B}$ given by $\phi(\bar{b} \# \bar{a}) = \overline{a_0 \# a_1.b}$, for all $a \in A, b \in B$, is an isomorphism of dyslectic Hopf Yetter-Drinfel'd (S,H)-module algebras.

Proof. The verifications of (i), (ii), (iii), and (iv) are long, but easy.

(v) Let $a \in A$ and $b \in B$. For all $s \in S$, we have

$$\begin{aligned}
\phi((\bar{b} \leftharpoonup s) \# \bar{a}) &= \phi(\overline{b \leftharpoonup s} \# \bar{a}) \\
&= \overline{a_0 \# a_1.(b \leftharpoonup s)} \\
&= \overline{a_0 \# ((a_1.b) \leftharpoonup (a_2.s))} \\
&= \overline{a_0 \# ((a_2.s)_0 \rightharpoonup ((a_2.s)_1.(a_1.b)))} \\
&= \overline{(a_0 \leftharpoonup (a_2.s)_0) \# ((a_2.s)_1.(a_1.b))} \\
&= \overline{(a_0 \leftharpoonup (a_{22}.s_0)) \# (a_{23} s_1 \mathcal{S}^{-1}(a_{21}) a_1).b} \\
&= \overline{(a_0 \leftharpoonup (a_1.s_0)) \# (a_2 s_1.b)} \\
&= \overline{(a_{00} \leftharpoonup (a_{01}.s_0)) \# (a_1 s_1.b)} \\
&= \overline{(s_0 \rightharpoonup a_0) \# (a_1 s_1.b)} \\
&= \overline{(s \rightharpoonup a)_0 \# (s \rightharpoonup a)_1.b} \\
&= \phi(\bar{b} \# \overline{s \rightharpoonup a}) \\
&= \phi(\bar{b} \# (s \rightharpoonup \bar{a})).
\end{aligned}$$

The map ϕ is left H-linear, right H^{op}-colinear and left S-linear, since the braiding γ is left H-linear, right H^{op}-colinear and left S-linear. We can prove as in Proposition 2.4.4 of [CvOZ1] that ϕ is compatible with the product. Now we have $\phi(1_{\bar{B}} \# 1_{\bar{A}}) = \overline{1_A \# 1_B}$.

Clearly ϕ is a bijection: its inverse is defined by

$$\phi^{-1}(\overline{a \# b}) = \overline{\mathcal{S}(a_1).b} \# \overline{a_0}, \text{ for all } a \in A, b \in B.$$

∎

A Hopf Yetter-Drinfel'd (S,H) module is right faithfully projective if it is finitely generated projective and faithful as a right S-module; or equivalently, if it is finitely generated projective as a right S-module, and the canonical maps $\phi : P \tilde{\otimes}_S Hom_S(P,S) \to End_S(P)$ and $\psi : Hom_S(P,S) \otimes_{End_S(P)} P \to S$ are isomorphisms, where

$$\phi(p \,\tilde{\otimes}_S\, f)(p') = p \leftharpoonup f(p') \text{ and } \psi(f \,\tilde{\otimes}\, p) = f(p), \text{ for all } p, p' \in P, f \in Hom_S(P,S).$$

We define in a similar way a left faithfully projective Hopf Yetter-Drinfel'd (S, H) module.

A Hopf Yetter-Drinfel'd (S, H) module is said to be *faithfully projective* if it is right and left faithfully projective. Since $Dys\text{-}{}_S\mathcal{Q}^H$ is a braided monoidal category, by [Femić], a dyslectic Hopf Yetter-Drinfel'd (S, H) module is right faithfully projective if and only if it is left faithfully projective. So a dyslectic Hopf Yetter-Drinfel'd (S, H) module is faithfully projective if it is right faithfully projective or left faithfully projective.

It follows from Lemmas 4.1 and 4.2 that $End_S(P)$ and ${}_SEnd(P)$ are dyslectic Hopf Yetter-Drinfel'd (S, H) module algebras for any faithfully projective dyslectic Hopf Yetter-Drinfel'd (S, H)-module P.

For a faithfully projective dyslectic Hopf Yetter-Drinfel'd (S, H)-module M, we know that the left dual ${}_SHom(M, S)$ of M and the right dual $Hom_S(M, S)$ of M coincide in $Dys\text{-}{}_S\mathcal{Q}^H$: we will denote these duals by $M^\star$, which we regard as dyslectic Hopf Yetter-Drinfel'd (S, H)-modules using Lemma 3.5. Note that $M^\star$ is faithfully projective. The following proposition is an illustration of [vOZ].

4.5. Proposition. *Let M be a faithfully projective dyslectic Hopf Yetter-Drinfel'd (S, H)-module. Then*

(i) $End_S(M) \cong {}_SEnd(M^\star)$ *as dyslectic Hopf Yetter-Drinfel'd (S, H) module algebras;*

(ii) ${}_SEnd(M) \cong End_S(M^\star)$ *as dyslectic Hopf Yetter-Drinfel'd (S, H) module algebras;*

(iii) $\overline{End_S(M)} \cong {}_SEnd(M)$ *as dyslectic Hopf Yetter-Drinfel'd (S, H) module algebras; and*

(iv) $\overline{{}_SEnd(M)} \cong End_S(M)$ *as dyslectic Hopf Yetter-Drinfel'd (S, H) module algebras.*

Proof. (i) Define $\phi : End_S(M) \to {}_SEnd(M^\star)$ by $\phi(f)(g) = g \circ f$ for all $f \in End_S(M)$ and $g \in M^\star = Hom_S(M, S)$. Then $\phi(f)(g)$ is in $M^\star$ since f and g are right S-linear. Clearly $\phi(f)$ is in ${}_SEnd(M^\star)$. Therefore $\phi(f)$ is well defined. Using Lemma 4.2 and the definition of the left S-action of ${}_SEnd(M^\star)$, we can show that ϕ is left S-linear. Using the fact that $h(g \circ f) = (h_1 g) \circ (h_2 f)$, it is easy to show that ϕ is H-linear. For every $m \in M$, we have

$$
\begin{aligned}
&(\phi(f)_0(g))(m) \otimes \phi(f)_1 \\
&= (\phi(f)(g_0))_0(m) \otimes (\phi(f)(g_0))_1 \mathcal{S}(g_1) \\
&= (g_0 \circ f)_0(m) \otimes (g_0 \circ f)_1 \mathcal{S}(g_1) \\
&= ((g_0 \circ f)(m_0))_0 \otimes \mathcal{S}^{-1}(m_1)((g_0 \circ f)(m_0))_1 \mathcal{S}(g_1) \\
&= (g_0(f(m_0)))_0 \otimes \mathcal{S}^{-1}(m_1)(g_0(f(m_0)))_1 \mathcal{S}(g_1) \\
&= (g(f(m_0)_0))_{00} \otimes \mathcal{S}^{-1}(m_1)(g(f(m_0)_0))_{01} \mathcal{S}(\mathcal{S}^{-1}(f(m_0)_1)(g(f(m_0)_0))_{11}) \\
&= (g(f(m_0)_0))_{00} \otimes \mathcal{S}^{-1}(m_1)(g(f(m_0)_0))_{01} \mathcal{S}((g(f(m_0)_0))_{11}) f(m_0)_1 \\
&= g(f(m_0)_0) \otimes \mathcal{S}^{-1}(m_1) f(m_0)_1 \\
&= g(f_0(m)) \otimes f_1 \\
&= (\phi(f_0)(g))(m) \otimes f_1.
\end{aligned}
$$

This means that ϕ is H^{op}-colinear. For $f, f' \in End_S(M)$ and $g \in M^\star$, we have

$$\begin{aligned}\phi(ff')(g) &= g \circ (ff') = g \circ f \circ f' \\ &= \phi(f')(g \circ f) = \phi(f')(\phi(f)(g)) \\ &= (\phi(f') \circ \phi(f))(g) \\ &= (\phi(f)\phi(f'))(g).\end{aligned}$$

So ϕ is an algebra map.

Let $\{m^{(i)}, f^{(i)}\}$ be dual bases for the S-modules M and $M^\star$, where $m^{(i)} \in M$ and $f^{(i)} \in M^\star = Hom_S(M, S)$. Define $\psi : {}_SEnd(M^\star) \to End_S(M)$ by $\psi(g)(m) = \sum m^{(i)} \leftharpoonup [g(f^{(i)})](m)$. Since $\sum m^{(i)} \leftharpoonup f^{(i)}(m) = m$, we have

$$f'(m) = \sum f'(m^{(i)})f^{(i)}(m) = \sum (f'(m^{(i)}) \rightharpoonup f^{(i)})(m)$$

for every $f' \in M^\star$ and $m \in M$. So $f' = \sum f'(m^{(i)}) \rightharpoonup f^{(i)}$. For every $g \in {}_SEnd(M^\star)$, we have $g(f') = \sum f'(m^{(i)}) \rightharpoonup [g(f^{(i)})]$. This proves that $\phi \circ \psi$ is the identity map of ${}_SEnd(M^\star)$. In a similar, way we show that $\psi \circ \phi$ is the identity map of $End_S(M)$. So the algebra map ϕ is a bijection with inverse ψ.

(ii) Define $\phi : {}_SEnd(M) \to End_S(M^\star)$ by $\phi(f)(g) = g \circ f$ for all $f \in {}_SEnd(M)$ and $g \in M^\star = {}_SHom(M, S)$. Here $M^\star = {}_SHom(M, S)$. We show as in (i) that ϕ is well defined, left S-linear, left H-linear, right H^{op}-colinear, an algebra homomorphism, and is a bijection with inverse defined by $\psi : End_S(M^\star) \to {}_SEnd(M)$ such that $\psi(g)(m) = [g(f^{(i)})(m)] \rightharpoonup m^{(i)}$, where $\{m^{(i)}, f^{(i)}\}$ is a dual basis of the left S-module M.

(iii) Define $\phi : \overline{End_S(M)} \to {}_SEnd(M)$ by $\phi(\bar{f})(m) = f_0(f_1 m)$, for all $m \in M$ and $f \in {}_SEnd(M)$. We already mentioned that ϕ is an isomorphism of dyslectic Hopf Yetter-Drinfel'd (S, H)-module. It is easy to show that ϕ is an algebra map using the fact that $(g \circ f)_0 \otimes (g \circ f)_1 = (g_0 \circ f_0) \otimes f_1 g_1$ for all $f, g \in End_S(M)$. ■

If M and N are faithfully projective dyslectic Hopf Yetter-Drinfel'd (S, H)-modules, then $M \tilde{\otimes}_S N$ is a faithfully projective dyslectic Hopf Yetter-Drinfel'd (S, H)-module.

4.6. Proposition. *Let M and N be faithfully projective dyslectic Hopf Yetter-Drinfel'd (S, H)-modules. Then*

$$End_S(M) \#_S End_S(N) \simeq End_S(M \tilde{\otimes}_S N) \text{ and } {}_SEnd(M) \#_S {}_SEnd(N) \simeq {}_SEnd(M \tilde{\otimes}_S N)$$

as dyslectic Hopf Yetter-Drinfel'd (S, H)-module algebras.

Proof. Define

$$\phi : End_S(M) \#_S End_S(N) \to End_S(M \tilde{\otimes}_S N)$$

by

$$\phi(f \# g)(m \tilde{\otimes} n) = f(m_0) \tilde{\otimes} (m_1 g)(n), \text{ for all } m \in M, n \in N, f \in End_S(M), g \in End_S(N).$$

$\phi(f\,\#\,g)$ is well defined, since

$$
\begin{aligned}
\phi((f\,\#\,g)(m \leftharpoonup s)\,\tilde{\otimes}\, n) &= f((m \leftharpoonup s)_0)\,\tilde{\otimes}\,((m \leftharpoonup s)_1.g)(n) \\
&= f(m_0 \leftharpoonup s_0)\,\tilde{\otimes}\,(s_1 m_1.g)(n) \\
&= f(m_0)\,\tilde{\otimes}\,(s_0 \rightharpoonup ((s_1 m_1.g)(n))) \\
&= f(m_0)\,\tilde{\otimes}\,[s_0 \rightharpoonup (s_1 m_1)_1(g(\mathcal{S}((s_1 m_1)_2)n))] \\
&= f(m_0)\,\tilde{\otimes}\,[s_0 \rightharpoonup (s_{11} m_{11}(g(\mathcal{S}(s_{12} m_{12})n)))] \\
&= f(m_0)\,\tilde{\otimes}\,[s_0 \rightharpoonup (s_{11} m_{11}(g(\mathcal{S}(m_{12})\mathcal{S}(s_{12})n)))] \\
&= f(m_0)\,\tilde{\otimes}\,[s_{00} \rightharpoonup (s_{01} m_{11}(g(\mathcal{S}(m_{12})\mathcal{S}(s_1)n)))] \\
&= f(m_0)\,\tilde{\otimes}\,[m_{11}(g(\mathcal{S}(m_{12})\mathcal{S}(s_1)n))] \leftharpoonup s_0 \\
&= f(m_0)\,\tilde{\otimes}\,[(m_1.g)(\mathcal{S}(s_1)n)] \leftharpoonup s_0 \\
&= f(m_0)\,\tilde{\otimes}\,[(m_1.g)((\mathcal{S}(s_1)n) \leftharpoonup s_0)] \\
&= f(m_0)\,\tilde{\otimes}\,[(m_1.g)(s \rightharpoonup n)] \\
&= \phi(f\,\#\,g)(m\,\tilde{\otimes}\,(s \rightharpoonup n)),
\end{aligned}
$$

for all $m \in M$, $n \in N$, $s \in S$, $f \in End_S(M)$, and $g \in End_S(N)$.

Let us show that ϕ is well defined. We have

$$
\begin{aligned}
\phi((f \leftharpoonup s)\,\#\,g)(m\,\tilde{\otimes}\,n) &= (f \leftharpoonup s)(m_0)\,\tilde{\otimes}\,(m_1.g)(n) \\
&= [s_0 \rightharpoonup (s_1.f)](m_0)\,\tilde{\otimes}\,(m_1.g)(n) \\
&= s_0 \rightharpoonup [(s_1.f)(m_0)]\,\tilde{\otimes}\,(m_1.g)(n) \\
&= s_0 \rightharpoonup [s_{11}(f(\mathcal{S}(s_{12})m_0))]\,\tilde{\otimes}\,m_{11}(g(\mathcal{S}(m_{12})n)) \\
&= (\mathcal{S}(s_{01})[s_{11}(f(\mathcal{S}(s_{12})m_0))]) \leftharpoonup s_{00}\,\tilde{\otimes}\,m_{11}(g(\mathcal{S}(m_{12})n)) \\
&= (\mathcal{S}(s_1)s_2)(f(\mathcal{S}(s_3)m_0)) \leftharpoonup s_0\,\tilde{\otimes}\,m_{11}(g(\mathcal{S}(m_{12})n)) \\
&= f(\mathcal{S}(s_1)m_0) \leftharpoonup s_0\,\tilde{\otimes}\,m_{11}(g(\mathcal{S}(m_{12})n)) \\
&= f((\mathcal{S}(s_1)m_0) \leftharpoonup s_0)\,\tilde{\otimes}\,m_{11}(g(\mathcal{S}(m_{12})n)) \\
&= f(s \rightharpoonup m_0)\,\tilde{\otimes}\,m_{11}(g(\mathcal{S}(m_{12})n)) \\
&= f(m_{00} \leftharpoonup (m_{01}.s))\,\tilde{\otimes}\,m_{11}(g(\mathcal{S}(m_{12})n)) \\
&= (f(m_{00}) \leftharpoonup (m_{01}.s))\,\tilde{\otimes}\,m_{11}(g(\mathcal{S}(m_{12})n)) \\
&= f(m_{00})\,\tilde{\otimes}\,((m_{01}.s) \rightharpoonup ((m_{11}(g(\mathcal{S}(m_{12})n)))) \\
&= f(m_0)\,\tilde{\otimes}\,((m_1.s) \rightharpoonup (m_2(g(\mathcal{S}(m_3)n)))) \\
&= f(m_0)\,\tilde{\otimes}\,((m_1.s) \rightharpoonup (m_2.g)(n)) \\
&= f(m_0)\,\tilde{\otimes}\,((m_1.s) \rightharpoonup (m_2.g)(n)) \\
&= f(m_0)\,\tilde{\otimes}\,(m_1.(s \rightharpoonup g))(n) \\
&= \phi(f\,\#\,(s \rightharpoonup g))(m\,\tilde{\otimes}\,n)
\end{aligned}
$$

for all $m \in M$, $n \in N$, $s \in S$, $f \in End_S(M)$, and $g \in End_S(N)$. Clearly, ϕ and $\phi(f\,\#\,g)$

are S-linear. That ϕ is H-linear follows from:

$$\begin{aligned}
[h.(\phi(f\,\#\,g)](m\,\tilde{\otimes}\,n) &= h_1[\phi(f\,\#\,g)(\mathcal{S}(h_2)(m\,\tilde{\otimes}\,n))]\\
&= h_1[\phi(f\,\#\,g)(\mathcal{S}(h_{22})m\,\tilde{\otimes}\,\mathcal{S}(h_{21})n))]\\
&= h_1[f((\mathcal{S}(h_{22})m)_0)\,\tilde{\otimes}\,((\mathcal{S}(h_{22})m)_1.g)(\mathcal{S}(h_{21})n))]\\
&= h_1[f((\mathcal{S}(h_{222})m_0))\,\tilde{\otimes}\,((\mathcal{S}(h_{221})m_1\mathcal{S}^{-1}(\mathcal{S}(h_{223})).g)(\mathcal{S}(h_{21})n))]\\
&= h_{11}[f(\mathcal{S}(h_{222})m_0)]\,\tilde{\otimes}\,h_{12}[(\mathcal{S}(h_{221})m_1h_{223}.g)(\mathcal{S}(h_{21})n)])\\
&= h_1[f(\mathcal{S}(h_5)m_0)]\,\tilde{\otimes}\,h_2[((\mathcal{S}(h_4)m_1h_6).g)(\mathcal{S}(h_3)n)]\\
&= h_1(f(\mathcal{S}(h_4)m_0))\,\tilde{\otimes}\,[h_2\mathcal{S}(h_3)m_1h_5.g](n)\\
&= h_1(f(\mathcal{S}(h_2)m_0))\,\tilde{\otimes}\,[m_1h_3.g](n)\\
&= h_{11}(f(\mathcal{S}(h_{12})m_0))\,\tilde{\otimes}\,[m_1h_2.g](n)\\
&= (h_1.f)(m_0)\,\tilde{\otimes}\,[m_1h_2.g](n)\\
&= \phi((h_1.f)\,\#\,(h_2.g))(m\otimes n)\\
&= \phi[h.(f\,\#\,g)](m\otimes n),
\end{aligned}$$

for all $m\in M$, $n\in N$, $f\in End_S(M)$, $g\in End_S(N)$, and $h\in H$. And finally, ϕ is H^{op}-colinear, since

$$\begin{aligned}
&\phi(f\,\#\,g)_0(m\,\tilde{\otimes}\,n)\otimes\phi(f\,\#\,g)_1\\
&=[\phi(f\,\#\,g)((m\,\tilde{\otimes}\,n)_0)]_0\otimes\mathcal{S}^{-1}((m\,\tilde{\otimes}\,n)_1)[\phi(f\,\#\,g)((m\,\tilde{\otimes}\,n)_0)]_1\\
&=[\phi(f\,\#\,g)(m_0\,\tilde{\otimes}\,n_0)]_0\otimes\mathcal{S}^{-1}(n_1m_1)[\phi(f\,\#\,g)(m_0\,\tilde{\otimes}\,n_0)]_1\\
&=[f(m_{00})\,\#\,(m_{01}.g)(n_0)]_0\otimes\mathcal{S}^{-1}(m_1)\mathcal{S}^{-1}(n_1)[f(m_{00})\,\#\,(m_{01}.g)(n_0)]_1\\
&=(f(m_{00})_0\,\#\,[(m_{01}.g)(n_0)]_0)\otimes(\mathcal{S}^{-1}(m_1)\mathcal{S}^{-1}(n_1)[(m_{01}.g)(n_0)]_1f(m_{00})_1)\\
&=(f(m_{00})_0\,\#\,[(m_{01}.g)_0(n)])\otimes(\mathcal{S}^{-1}(m_1)[(m_{01}.g)_1]f(m_{00})_1)\\
&=(f(m_{00})_0\,\#\,[(m_{012}.g_0)(n)])\otimes(\mathcal{S}^{-1}(m_1)[m_{013}.g_1\mathcal{S}^{-1}(m_{011})]f(m_{00})_1)\\
&=(f(m_0)_0\,\#\,[(m_2.g_0)(n)])\otimes(\mathcal{S}^{-1}(m_4)[m_3g_1\mathcal{S}^{-1}(m_1)]f(m_0)_1)\\
&=(f(m_0)_0\,\#\,[(m_2.g_0)(n)])\otimes(g_1\mathcal{S}^{-1}(m_1)f(m_0)_1)\\
&=(f(m_{00})_0\,\#\,[(m_1.g_0)(n)])\otimes(g_1\mathcal{S}^{-1}(m_{01})f(m_{00})_1)\\
&=f_0(m_0)\,\#\,[(m_1.g_0)(n)])\otimes g_1f_1\\
&=\phi(f_0\,\#\,g_0)(m\,\tilde{\otimes}\,n)\otimes g_1f_1\\
&=\phi((f\,\#\,g)_0)(m\,\tilde{\otimes}\,n)\otimes(f\,\#\,g)_1,
\end{aligned}$$

for all $m\in M$, $n\in N$, $f,\in End_S(M)$, and $g\in End_S(N)$. That ϕ is compatible with the product can be shown as in [CvOZ1, Proposition 4.3]. Our assumptions imply that every element of $End_S(M\,\tilde{\otimes}_S\,N)$ has the form $f\,\tilde{\otimes}\,g$, for $f\in End_S(M)$ and $g\in End_S(N)$, so it is easy to show ϕ^{-1} is given by

$$\phi^{-1}(f\,\tilde{\otimes}\,g))=f_0\,\#\,\mathcal{S}^{-1}(f_1).g.$$

Therefore, ϕ is an isomorphism of dyslectic Hopf Yetter-Drinfel'd (S,H)-modules.

To show the second isomorphism, use Lemma 4.5 and apply the first isomorphism to the endomorphism algebras of the duals. ∎

4.7. Lemma. *Let M be a faithfully projective dyslectic Hopf Yetter-Drinfel'd (S,H)-module. Then*

(i) $M\,\tilde{\otimes}_S\,M^\star$ *is a faithfully projective dyslectic Hopf Yetter-Drinfel'd (S,H)-module algebra: the multiplication in $M\,\tilde{\otimes}_S\,M^\star$ is defined by*

$$(m\,\tilde{\otimes}\,f)(m'\,\tilde{\otimes}\,f') = (m \leftharpoonup f(m'))\,\tilde{\otimes}\,f', \text{ for all } m, m' \in M, f, f' \in M^\star.$$

(ii) the natural S-linear map $\phi : M\tilde{\otimes}_S M^\star \simeq End_S(M)$ defined by

$$\phi(m\,\tilde{\otimes}\,f)(m') = m \leftharpoonup f(m'), \text{ for all } m, m' \in M, f \in End_S(M),$$

is an isomorphism of dyslectic Hopf Yetter-Drinfel'd (S,H)-module algebras.

Proof. (i). We know that $M\tilde{\otimes}_S M^\star$ is a faithfully projective dyslectic Hopf Yetter-Drinfel'd (S,H)-module. Let m, $m' \in M$, and $f, f' \in M^\star$. Then we have

$$\begin{aligned}
[(m\,\tilde{\otimes}\,f) \leftharpoonup s](n\,\tilde{\otimes}\,g) &= [m\,\tilde{\otimes}\,(f \leftharpoonup s)](n\,\tilde{\otimes}\,g)\\
&= [m \leftharpoonup ((f \leftharpoonup s)(n))]\,\tilde{\otimes}\,g\\
&= [m \leftharpoonup (f(s \rightharpoonup n))]\,\tilde{\otimes}\,g\\
&= (m\,\tilde{\otimes}\,f)[(s \rightharpoonup n)\,\tilde{\otimes}\,g]\\
&= (m\,\tilde{\otimes}\,f)[s \rightharpoonup (n\,\tilde{\otimes}\,g)].
\end{aligned}$$

It follows that $[(m\,\tilde{\otimes}\,f) \leftharpoonup s](n\,\tilde{\otimes}\,g) = (m\,\tilde{\otimes}\,f)[s \rightharpoonup (n\,\tilde{\otimes}\,g)]$.

We also have

$$\begin{aligned}
[s \rightharpoonup (m\,\tilde{\otimes}\,f)](n\,\tilde{\otimes}\,g) &= [(s \rightharpoonup m)\,\tilde{\otimes}\,f](n\,\tilde{\otimes}\,g)\\
&= [(s \rightharpoonup m) \leftharpoonup f(n)]\,\tilde{\otimes}\,g)\\
&= s \rightharpoonup [(m \leftharpoonup f(n))\,\tilde{\otimes}\,g]\\
&= s \rightharpoonup [(m\,\tilde{\otimes}\,f)(n\,\tilde{\otimes}\,g)].
\end{aligned}$$

We deduce that $[s \rightharpoonup (m\,\tilde{\otimes}\,f)](n\,\tilde{\otimes}\,g) = s \rightharpoonup [(m\,\tilde{\otimes}\,f)(n\,\tilde{\otimes}\,g)]$.

The identity element of $M\,\tilde{\otimes}_S\,M^\star$ is $\sum m^{(i)}\,\tilde{\otimes}\,f^{(i)}$, where $\{m^{(i)}\}$ and $\{f^{(i)}\}$ are dual bases for M and $M^\star$. Let us show that the H-coaction is compatible with the product of $M\,\tilde{\otimes}_S\,M^\star$. We have

$$\begin{aligned}
&(m\,\tilde{\otimes}\,f)_0(m'\,\tilde{\otimes}\,f')_0 \otimes (m'\,\tilde{\otimes}\,f')_1(m\,\tilde{\otimes}\,f)_1\\
&= (m_0\,\tilde{\otimes}\,f_0)(m'_0\,\tilde{\otimes}\,f'_0) \otimes f'_1 m'_1 f_1 m_1\\
&= [(m_0 \leftharpoonup f_0(m'_0))\,\tilde{\otimes}\,f'_0] \otimes f'_1 m'_1 f_1 m_1\\
&= [(m_0 \leftharpoonup f(m'_{00})_0)\,\tilde{\otimes}\,f'_0] \otimes f'_1 m'_1 \mathcal{S}^{-1}(m'_{01}) f(m'_{00})_1 m_1\\
&= [(m_0 \leftharpoonup f(m')_0)\,\tilde{\otimes}\,f'_0] \otimes f'_1 f(m')_1 m_1\\
&= [(m \leftharpoonup f(m'))\,\tilde{\otimes}\,f']_0 \otimes [(m \leftharpoonup f(m'))\,\tilde{\otimes}\,f']_1\\
&= [(m\,\tilde{\otimes}\,f)(m'\,\tilde{\otimes}\,f')]_0 \otimes [(m\,\tilde{\otimes}\,f)(m'\,\tilde{\otimes}\,f')]_1.
\end{aligned}$$

It is easy to show that the H-action is compatible with the product of $M\,\tilde{\otimes}_S\,M^\star$. That the product is associative is well-known. So $M\,\tilde{\otimes}_S\,M^\star$ is a Hopf Yetter-Drinfel'd (S,H)-module algebra.

(ii). Since M is faithfully projective, ϕ is an isomorphism of dyslectic Hopf Yetter-Drinfel'd (S,H)-modules. It is easy to show that ϕ preserves the product and the identity element $\sum m^{(i)}\,\tilde{\otimes}\,f^{(i)}$ of $M\,\tilde{\otimes}_S\,M^\star$, where $\{m^{(i)}\}$ and $\{f^{(i)}\}$ are dual bases of M and $M^\star$. ∎

4.8. Lemma. *Let M be a faithfully projective dyslectic Hopf Yetter-Drinfel'd (S,H)-module. Then*

(i) *$M^\star \tilde{\otimes}_S M$ is a faithfully projective dyslectic Hopf Yetter-Drinfel'd (S,H)-module algebra: the multiplication in $M^\star \tilde{\otimes}_S M$ is defined by*

$$(f \tilde{\otimes} m)(f' \tilde{\otimes} m') = f \tilde{\otimes} [f'(m) \rightharpoonup m'], \text{ for all } m, m' \in M, f, f' \in M^\star; \text{ and}$$

(ii) *the natural R-linear map $\phi : M^\star \tilde{\otimes}_S M \simeq {}_S End(M)$ defined by*

$$\phi(f \otimes m)(m') = f(m') \rightharpoonup m, \text{ for all } m, m' \in M, f \in {}_S End(M),$$

is an isomorphism of dyslectic Hopf Yetter-Drinfel'd (S,H)-module algebras.

4.9. Proposition. *Let A be a dyslectic Hopf Yetter-Drinfel'd (S,H)-module algebra. If M is a dyslectic Hopf Yetter-Drinfel'd (S,H)-module that is faithfully projective as an S-module, then $A \#_S End_S(M) \simeq End_S(M) \#_S A$ and ${}_S End(M) \#_S A \simeq A \#_S {}_S End(M)$ as dyslectic Hopf Yetter-Drinfel'd (S,H)-module algebras.*

Proof. This is shown for general braided monoidal categories in [vOZ, Proposition 2.4(i)]. There the algebra isomorphism is given by

$$\eta(a \# m \otimes f) = \mathcal{S}(a_1)m \otimes f_0 \# f_1.a_0, \text{ for all } a \in A, m \in M, f \in M^\star.$$

This is the composition $(id \otimes \gamma_{A,M^\star}) \circ (\gamma^{-1}_{A,M} \otimes id)$, so it is certainly well-defined morphism in the ${}_S\mathcal{Q}^H$ category. That it is an algebra map is shown with a braiding diagram argument in [vOZ]. Here we verify this directly.

Let $a, b \in A$, $m, n \in M$, and $f, g \in M^\star$. Consider

$$\begin{aligned}
\eta((a \# m \tilde{\otimes} f)(b \# n \tilde{\otimes} g)) &= \eta(ab_0 \# (b_1.(m \tilde{\otimes} f))(n \tilde{\otimes} g)) \\
&= \eta(ab_0 \# [b_1 m \leftharpoonup (b_2 f)(n)] \tilde{\otimes} g) \\
&= \mathcal{S}((ab_0)_1).[b_1 m \leftharpoonup (b_2 f)(n)] \tilde{\otimes} g_0 \# g_1.(ab_0)_0 \\
&= \mathcal{S}(a_1)S(b_{01}).(b_1 m \leftharpoonup (b_2 f)(n)) \tilde{\otimes} g_0 \# g_1.(a_0 b_{00}) \\
&= \mathcal{S}(a_2)m \leftharpoonup (S(a_1)S(b_1).(b_2 f)(n)) \tilde{\otimes} g_0 \# g_1.(a_0 b_0) \\
&= \mathcal{S}(a_2)m \tilde{\otimes} ((S(a_1)S(b_1).(b_2 f)(n)) \rightharpoonup g_0) \# g_1.(a_0 b_0) \\
&= \mathcal{S}(a_2)m \tilde{\otimes} (g_{00} \leftharpoonup (g_{01}.(S(a_1)S(b_1).(b_2 f)(n)))) \# g_1.(a_0 b_0) \\
&= \mathcal{S}(a_2)m \tilde{\otimes} g_{00} \# ((g_{01}.(S(a_1)S(b_1).(b_2 f)(n))) \rightharpoonup g_1.(a_0 b_0)) \\
&= \mathcal{S}(a_2)m \tilde{\otimes} g_0 \# g_1.((S(a_1)S(b_1).(b_2 f)(n)) \rightharpoonup a_0 b_0) \\
&= \mathcal{S}(a_2)m \tilde{\otimes} g_0 \# g_1.(((S(a_1)S(b_1).(b_2 f)(n)) \rightharpoonup a_0) b_0) \\
&= \mathcal{S}(a_1)m \tilde{\otimes} g_0 \# g_1.((a_0 \leftharpoonup (S(b_1).(b_2 f)(n))) b_0) \\
&= \mathcal{S}(a_1)m \tilde{\otimes} g_0 \# g_1.(a_0((S(b_1).(b_2 f)(n)) \rightharpoonup b_0)) \\
&= \mathcal{S}(a_1)m \tilde{\otimes} g_0 \# g_1.(a_0(b_0 \leftharpoonup ((b_1 f)(n)))) \\
&= \mathcal{S}(a_1)m \tilde{\otimes} g_0 \# g_1.((a_0 b_0) \leftharpoonup ((b_1 f)(n))).
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
\eta(a\,\#\,m\,\tilde{\otimes}\,f)\eta(b\,\#\,n\,\tilde{\otimes}\,g) &= (\mathcal{S}(a_1)m\,\tilde{\otimes}\,f_0\,\#\,f_1.a_0)(\mathcal{S}(b_1)n\,\tilde{\otimes}\,g_0\,\#\,g_1.b_0)\\
&= (\mathcal{S}(a_1)m\,\tilde{\otimes}\,f_0)(\mathcal{S}(b_1)n\,\tilde{\otimes}\,g_0)_0\,\#\,((\mathcal{S}(b_1)n\,\tilde{\otimes}\,g_0)_1.(f_1.a_0))(g_1.b_0)\\
&= (\mathcal{S}(a_1)m\,\tilde{\otimes}\,f_0)((\mathcal{S}(b_1)n)_0\,\tilde{\otimes}\,g_{00})\,\#\,(g_{01}(\mathcal{S}(b_1)n)_1 f_1.a_0)(g_1.b_0)\\
&= \mathcal{S}(a_1)m\,\tilde{\otimes}\,(f_0((\mathcal{S}(b_1)n)_0) \rightharpoonup g_0)\,\#\,g_1.(((\mathcal{S}(b_1)n)_1 f_1.a_0)b_0)\\
&= \mathcal{S}(a_1)m\,\tilde{\otimes}\,(g_{00} \leftharpoonup (g_{01}.(f_0((\mathcal{S}(b_1)n)_0)))\,\#\,g_1.(((\mathcal{S}(b_1)n)_1 f_1.a_0)b_0)\\
&= \mathcal{S}(a_1)m\,\tilde{\otimes}\,g_0\,\#\,((g_1.(f_0((\mathcal{S}(b_1)n)_0))) \rightharpoonup (g_2.(((\mathcal{S}(b_1)n)_1 f_1.a_0)b_0)))\\
&= \mathcal{S}(a_1)m\,\tilde{\otimes}\,g_0\,\#\,g_1.((f_0((\mathcal{S}(b_1)n)_0) \rightharpoonup (\mathcal{S}(b_1)n)_1 f_1.a_0)b_0)\\
&= \mathcal{S}(a_1)m\,\tilde{\otimes}\,g_0\,\#\,g_1.((f_0((\mathcal{S}(b_1)n)_0) \rightharpoonup (\mathcal{S}(b_1)n)_1 f_1.a_0)b_0)\\
&= \mathcal{S}(a_1)m\,\tilde{\otimes}\,g_0\,\#\,g_1.((a_0 \leftharpoonup f(\mathcal{S}(b_1)n))b_0)\\
&= \mathcal{S}(a_1)m\,\tilde{\otimes}\,g_0\,\#\,g_1.(a_0(f(\mathcal{S}(b_1)n) \rightharpoonup b_0))\\
&= \mathcal{S}(a_1)m\,\tilde{\otimes}\,g_0\,\#\,g_1.(a_0(b_{00} \leftharpoonup b_{01}f(\mathcal{S}(b_1)n)))\\
&= \mathcal{S}(a_1)m\,\tilde{\otimes}\,g_0\,\#\,g_1.(a_0(b_0 \leftharpoonup (b_1 f)(n)))\\
&= \mathcal{S}(a_1)m\,\tilde{\otimes}\,g_0\,\#\,g_1.((a_0 b_0) \leftharpoonup (b_1 f)(n)),
\end{aligned}
$$

as required.

That ${}_S End(M)\,\#_S\,A \simeq A\,\#_S\,{}_S End(M)$ follows, since ${}_S End(M) \simeq End_S(M^\star)$. ∎

5. Dyslectic Hopf Yetter-Drinfel'd (S,H)-module Azumaya algebras

In this section, H a Hopf algebra with a bijective antipode, and S is an H-commutative Hopf Yetter-Drinfel'd H-module algebra. We will introduce the notion of a dyslectic Hopf Yetter-Drinfel'd (S,H)-module Azumaya algebra and work from there toward our eventual goal of defining the Brauer-Clifford-Long group.

5.1. Proposition. *Let A be a dyslectic Hopf Yetter-Drinfel'd (S,H) module algebra which is faithfully projective as an S-module. We define two S-linear maps*

$$F : A\,\#_S\,\bar{A} \to End_S(A) : \quad F(a\,\#\,\bar{b})(c) = ac_0(c_1.b)$$

and

$$G : \bar{A}\,\#_S\,A \to \overline{End_S(A)} : \quad G(\bar{a}\,\#\,b)(c) = a_0(a_1.c)b$$

for all a, b and c in A.

Then F and G are dyslectic Hopf Yetter-Drinfel'd (S,H)-module algebra homomorphisms.

Proof. To see that F is well-defined, let $a, b, c \in A$ and let $s \in S$. Then

$$
\begin{aligned}
F(a\,\#\,\bar{b})(c \leftharpoonup s) &= a(c \leftharpoonup s)_0((c \leftharpoonup s)_1).b)\\
&= a(c_0 \leftharpoonup s_0)(s_1.(c_1.b))\\
&= ac_0(s_0 \rightharpoonup (s_1.(c_1.b)))\\
&= ac_0((c_1.b) \leftharpoonup s)\\
&= (ac_0(c_1.b)) \leftharpoonup s\\
&= F(a\,\#\,\bar{b})(c) \leftharpoonup s,
\end{aligned}
$$

so $F(a \# \bar{b})$ is S-linear. It is clear that F is additive. Furthermore,

$$\begin{array}{rcl} F((a \leftharpoonup s) \# \bar{b})(c) & = & (a \leftharpoonup s)c_0(c_1.b) \\ & = & a(s \rightharpoonup c_0)(c_1.b) \\ & = & a(c_0 \leftharpoonup c_1.s)(c_2.b) \\ & = & ac_0((c_1.s) \rightharpoonup (c_2.b)) \\ & = & ac_0(c_1.(s \rightharpoonup b)) \\ & = & F(a \# \overline{(s \rightharpoonup b)})(c) \\ & = & F(a \# (s \rightharpoonup \bar{b}))(c), \end{array}$$

so F is a well-defined map from $A \#_S \bar{A}$ to $End_S(A)$.

Next we show F is H-linear. Let $a, b, c \in A$ and $h \in H$. Then

$$\begin{array}{rcl} h.(F(a \# \bar{b}))(c) & = & h_1.(F(a \# \bar{b})(\mathcal{S}(h_2).c)) \\ & = & h_1.(a(\mathcal{S}(h_2).c)_0((\mathcal{S}(h_2).c)_1.b)) \\ & = & h_1.(a(\mathcal{S}(h_3).c_0)(\mathcal{S}(h_2)c_1h_4).b) \\ & = & (h_1.a)c_0(c_1.(h_2.b)) \\ & = & F(h_1.a \# \overline{h_2.b})(c) \\ & = & F(h.(a \# \bar{b}))(c), \end{array}$$

as required. To see that F is H-colinear, let $a, b, c \in A$. We have

$$\begin{array}{rcl} (F(a \# \bar{b}))_0(c) \otimes (F(a \# \bar{b}))_1 & = & (F(a \# \bar{b})(c_0))_0 \otimes \mathcal{S}^{-1}(c_1)(F(a \# \bar{b})(c_0))_1 \\ & = & (ac_0(c_1.b))_0 \otimes \mathcal{S}^{-1}(c_2)(a_0c_0(c_1.b))_1 \\ & = & a_0c_0(c_3.b_0) \otimes \mathcal{S}^{-1}(c_5)c_4b_1S(c_2)c_1a_1 \\ & = & a_0c_0(c_1.b_0) \otimes b_1a_1 \\ & = & F((a \# \bar{b})_0)(c) \otimes (a \# \bar{b})_1, \end{array}$$

so F is H-colinear. Finally, F is an algebra map, since for all $a, b, c, d, e \in A$,

$$\begin{array}{rcl} F(a \# \bar{b})F(c \# \bar{d})(e) & = & F(a \# \bar{b})(ce_0(e_1.d)) \\ & = & a(ce_0(e_1.d))_0((ce_0(e_1.d))_1.b) \\ & = & ac_0e_0(e_2.d)_0((e_2.d)_1e_1c_1).b \\ & = & ac_0e_0(e_3.d_0)((e_4d_1\mathcal{S}^{-1}(e_2)e_1c_1).b) \\ & = & ac_0e_0(e_1.d_0)((e_2d_1c_1).b) \\ & = & (ac_0e_0)e_1.(d_0(d_1.(c_1.b))) \\ & = & F(ac_0 \# \overline{d_0(d_1.(c_1.b))})(e) \\ & = & F(ac_0 \# \overline{c_1.b}\bar{d})(e) \\ & = & F((a \# \bar{b})(c \# \bar{d}))(e). \end{array}$$

For the map G, we view $\overline{End_S(A)}$ as ${}_SEnd(A)$ and use the H-action and H-coaction defined as in Lemma 2.2. That G is well-defined and S-linear is similar to the proof for F. That G is H-linear, H-colinear, and an algebra map is proved exactly as in [CvOZ1, Proposition 5.1] (see also [C, Lemma 12.2.3]). ∎

Let A be a faithfully projective dyslectic Hopf Yetter-Drinfel'd (S,H)-module algebra. We say that A is a *dyslectic Hopf Yetter-Drinfel'd* (S,H)*-module Azumaya algebra* (i.e. an Azumaya algebra in the category $Dys\text{-}_S\mathcal{Q}^H$) if A is faithfully projective, and the dyslectic Hopf Yetter-Drinfel'd (S,H)-module algebra homomorphisms $F : A\,\#_S\,\bar{A} \to End_S(A)$ and $G : \bar{A}\,\#_S\,A \to \overline{End_S(A)}$ are isomorphisms.

Let A be a dyslectic Hopf Yetter-Drinfel'd (S,H)-module algebra. If H is cocommutative and the coaction of H is trivial, then S is commutative, $\bar{A} = A^{op}$, and A is just an S-progenerator (S,H)-algebra for which the natural map $A \otimes_S A^{op} \to End_S(A)$ is an isomorphism of (S,H)-algebras. So A is an (S,H)-Azumaya algebra in the sense of [GH]. If H is commutative and the action of H is trivial, then S is commutative, $\bar{A} = A^{op}$, A is just an S-progenerator (S,H)-Hopf algebra such that the natural map $A \otimes_S A^{op} \to End_S(A)$ is an isomorphism of (S,H)-Hopf algebras. So A is an (S,H)-Hopf Azumaya algebra as in [GH].

5.2. THEOREM. *The following hold:*

(i) If M is a faithfully projective dyslectic Hopf Yetter-Drinfel'd (S,H)-module, then $End_S(M)$ is a dyslectic Hopf Yetter-Drinfel'd (S,H)-module Azumaya algebra.

(ii) If A and B are faithfully projective dyslectic Hopf Yetter-Drinfel'd (S,H) module Azumaya algebras, then $A\,\#_S\,B$ is a dyslectic Hopf Yetter-Drinfel'd (S,H)-module Azumaya algebra.

(iii) If A is a faithfully projective dyslectic Hopf Yetter-Drinfel'd (S,H)-module Azumaya algebra, then $\bar{A}$ is a dyslectic Hopf Yetter-Drinfel'd (S,H)-module Azumaya algebra.

PROOF. It is obvious that $End_S(M)$ is faithfully projective. By Proposition 4.5, we have

$$\overline{End_S(M)} \simeq {}_SEnd(M) \simeq End_S(M^\star)$$

as dyslectic Hopf Yetter-Drinfel'd (S,H)-module algebras. Using Propositions 4.5 and 4.6, and Lemmas 4.7 and 4.8, we get the following dyslectic Hopf Yetter-Drinfel'd (S,H)-module algebra isomorphisms:

$$\begin{array}{rcl} End_S(M)\,\#_S\,\overline{End_S(M)} & \simeq & End_S(M)\,\#_S\,End_S(M^\star) \\ & \simeq & End_S(M\,\tilde{\otimes}_S\,M^\star) \\ & \simeq & End_S(End_S(M)) \end{array}$$

and

$$\begin{array}{rcl} \overline{End_S(M)}\,\#_S\,End_S(M) & \simeq & End_S(M^\star)\,\#_S\,End_S(M) \\ & \simeq & End_S(M^\star\,\tilde{\otimes}_S\,M) \\ & \simeq & End_S((M\,\tilde{\otimes}_S\,M^\star)^\star) \\ & \simeq & End_S((End_S(M))^\star) \\ & \simeq & \overline{End_S(End_S(M))}. \end{array}$$

So F and G are isomorphisms.

(ii) Since $A \otimes_S B$ is faithfully projective so is $A \#_S B$. Using Propositions 4.4, 4.6, and 4.9, we have the following dyslectic Hopf Yetter-Drinfel'd (S, H)-module algebra isomorphisms:

$$
\begin{array}{rcl}
(A \#_S B) \#_S \overline{A \#_S B} & \simeq & A \#_S B \#_S \bar{B} \#_S \bar{A} \\
& \simeq & A \#_S End_S(B) \#_S \bar{A} \\
& \simeq & A \#_S \bar{A} \#_S End_S(B) \\
& \simeq & End_S(A) \#_S End_S(B) \\
& \simeq & End_S(A \tilde{\otimes}_S B) \\
& \simeq & End_S(A \#_S B)
\end{array}
$$

and

$$
\begin{array}{rcl}
\overline{A \#_S B} \#_S (A \#_S B) & \simeq & \bar{B} \#_S \bar{A} \#_S A \#_S B \\
& \simeq & \bar{B} \#_S \overline{End_S(A)} \#_S B \\
& \simeq & \bar{B} \#_S B \#_S \overline{End_S(A)} \\
& \simeq & \overline{End_S(B)} \#_S \overline{End_S(A)} \\
& \simeq & End_S(B^\star) \#_S End_S(A^\star) \\
& \simeq & End_S(B^\star \tilde{\otimes}_S A^\star) \\
& \simeq & End_S((A \tilde{\otimes}_S B)^\star) \\
& \simeq & \overline{End_S(A \tilde{\otimes}_S B)} \\
& \simeq & \overline{End_S(A \#_S B)}.
\end{array}
$$

so F and G are isomorphisms.

(iii) Since A is faithfully projective so is $\bar{A}$. Using Propositions 4.5 and 4.4, we have the following dyslectic Hopf Yetter-Drinfel'd (S, H)-module algebra isomorphisms:

$$
\begin{array}{rcl}
\bar{A} \#_S \bar{\bar{A}} & \simeq & \overline{\bar{A} \#_S A} \\
& \simeq & \overline{\overline{End_S(A)}} \\
& \simeq & End_S(A) \\
& \simeq & End_S(\bar{A}),
\end{array}
$$

and

$$
\begin{array}{rcl}
\bar{\bar{A}} \#_S \bar{A} & \simeq & \overline{A \#_S \bar{A}} \\
& \simeq & \overline{End_S(A)} \\
& \simeq & \overline{End_S(\bar{A})}.
\end{array}
$$

So F and G are isomorphisms. ■

We will say that a dyslectic Hopf Yetter-Drinfel'd (S, H)-module Azumaya algebra E is *trivial* if $E \simeq End_S(P)$ as dyslectic Hopf Yetter-Drinfel'd (S, H)-module algebras, for some faithfully projective dyslectic Hopf Yetter-Drinfel'd (S, H)-module P. If a dyslectic Hopf Yetter-Drinfel'd (S, H)-module Azumaya algebra E is trivial, then so are $E^\star$ and $\bar{E}$. If M and N are faithfully projective dyslectic Hopf Yetter-Drinfel'd (S, H)-modules, then so is $M \tilde{\otimes}_S N$. It follows from Proposition 4.6 and Theorem 5.2 that the braided product of two trivial dyslectic Hopf Yetter-Drinfel'd (S, H)-module Azumaya algebras is a trivial dyslectic Hopf Yetter-Drinfel'd (S, H)-module Azumaya algebra. When A

is a dyslectic Hopf Yetter-Drinfel'd (S, H)-module Azumaya algebra, then we have that $A \#_S \bar{A}$ and $End_S(A)$ are isomorphic as dyslectic Hopf Yetter-Drinfel'd (S, H)-module Azumaya algebras, and $\bar{A} \#_S A$ and $\overline{End_S(A)}$ are isomorphic as dyslectic Hopf Yetter-Drinfel'd (S, H)-module Azumaya algebras.

We will say that two dyslectic Hopf Yetter-Drinfel'd (S, H)-module Azumaya algebras A and B are *equivalent* if there exist trivial dyslectic Hopf Yetter-Drinfel'd (S, H)-module Azumaya algebras E_1 and E_2 such that $A \#_S E_1 \simeq B \#_S E_2$ as dyslectic Hopf Yetter-Drinfel'd (S, H)-module Azumaya algebras.

5.3. Lemma. *The above relation is an equivalence relation on the collection of dyslectic Hopf Yetter-Drinfel'd (S, H)-module Azumaya algebras.*

Proof. The only thing we have to show is transitivity. Suppose A, B, and C are dyslectic Hopf Yetter-Drinfel'd (S, H)-module Azumaya algebras for which A is equivalent to B and B is equivalent to C. Then there exist faithfully projective dyslectic Hopf Yetter-Drinfel'd (S, H)-module N_1, N_2, N_3 and N_4 such that $A \#_S End_S(N_1) \simeq B \#_S End_S(N_2)$ and $B \#_S End_S(N_3) \simeq C \#_S End_S(N_4)$ as dyslectic Hopf Yetter-Drinfel'd (S, H)-module Azumaya algebras. We have the following dyslectic Hopf Yetter-Drinfel'd (S, H)-module Azumaya algebras isomorphisms:

$$\begin{aligned} A \#_S End_S(N_1 \otimes_S N_3) &\simeq A \#_S End_S(N_1) \#_S End_S(N_3) \\ &\simeq B \#_S End_S(N_2) \#_S End_S(N_3) \\ &\simeq B \#_S End_S(N_3) \#_S End_S(N_2) \\ &\simeq C \#_S End_S(N_4) \#_S End_S(N_2) \\ &\simeq C \#_S End_S(N_4 \otimes_S N_2). \end{aligned}$$

This proves the relation is transitive, and hence it is an equivalence relation. ∎

We have now collected all of the ingredients necessary to define the Brauer group for the braided monoidal category $Dys\text{-}_S\mathcal{Q}^H$.

5.4. Definition. *The Brauer-Clifford-Long group for the category of dyslectic Hopf Yetter-Drinfel'd (S, H)-modules Azumaya algebras is the set* $\mathrm{BQ}(S, H)$ *of equivalence classes of dyslectic Hopf Yetter-Drinfel'd (S, H)-module Azumaya algebras modulo the relation defined by taking $\#_S$-products with trivial dyslectic Hopf Yetter-Drinfel'd (S, H)-module Azumaya algebras.*

We remind the reader that our Azumaya algebras in $Dys\text{-}_S\mathcal{Q}^H$ are assumed to be left and right faithfully projective. From the viewpoint of [Femić], these algebras constitute a closed braided monoidal category and so the Brauer-Clifford-Long group we have described is the Brauer group of this category.

5.5. Theorem. *Let H be a Hopf algebra with bijective antipode, and suppose S is an H-commutative Yetter-Drinfel'd H-module algebra. Then $BQ(S, H)$ is a group. If* $[[A]], [[B]]$

denote the equivalence classes of a dyslectic Hopf Yetter-Drinfel'd (S, H)-modules Azumaya algebra A and B, then in $BQ(S, H)$ we will have $[[A]] \cdot [[B]] = [[A \#_S B]]$. The identity of $BQ(S, H)$ is the equivalence class $[[S]]$ consisting of all trivial dyslectic Hopf Yetter-Drinfel'd (S, H)-modules Azumaya algebras, and $[[A]]^{-1} = [[\bar{A}]]$ for all $[[A]] \in BQ(S, H)$.

Proof. The product in $\mathrm{BQ}(S, H)$ is well-defined by Propositions 4.4, 4.6, 4.9, and Theorem 5.2(ii). It follows from Proposition 4.4 that this product is associative and has identity $[[S]]$. That the inverse of the class $[[A]] \in \mathrm{BQ}(S, H)$ is represented by A is $[[\bar{A}]]$ follows from Theorem 5.2(iii). ∎

$\mathrm{BQ}(R, H)$ is precisely the Brauer group for the category $\mathcal{Q}^H$ of Yetter-Drinfel'd H-modules defined by Caenepeel, Van Oystaeyen, and Zhang [CvOZ1], [CvOZ2]. Several basic properties of $\mathrm{BQ}(S, H)$ are immediate from the properties of Brauer groups of braided monoidal categories discussed in [vOZ], we leave these to the reader to explore.

6. Examples

In this section we give an overview of cases where Brauer groups of braided and symmetric monoidal categories that have been previously studied admit nontrivial generalizations to the Brauer-Clifford-Long groups that we have presented.

6.1. Example. *H is triangular.* A Hopf algebra H over commutative ring R is said to be *quasitriangular* if there exists an invertible element $\mathcal{R} = \mathcal{R}_1 \otimes \mathcal{R}_2 \in H \otimes H$ satisfying

$$\begin{array}{llcl}
(QT1) & \mathcal{R}_1^{(1)} \otimes \mathcal{R}_1^{(2)} \otimes \mathcal{R}_2 & = & (\mathcal{R}_1 \otimes 1 \otimes \mathcal{R}_2)(1 \otimes \mathcal{R}_1 \otimes \mathcal{R}_2) := \mathcal{R}_{13}\mathcal{R}_{23} \\
(QT2) & \mathcal{R}_1 \otimes \mathcal{R}_2^{(1)} \otimes \mathcal{R}_2^{(2)} & = & (\mathcal{R}_1 \otimes 1 \otimes \mathcal{R}_2)(\mathcal{R}_1 \otimes \mathcal{R}_2 \otimes 1) := \mathcal{R}_{13}\mathcal{R}_{12} \\
(QT3) & \Delta^{cop}(h) = h_2 \otimes h_1 & = & \mathcal{R}(h_1 \otimes h_2)\mathcal{R}^{-1}, \text{ for all } h \in H.
\end{array}$$

The inverse of $\mathcal{R}$ is $\mathcal{R}^{-1} = \mathcal{S}(\mathcal{R}_1) \otimes \mathcal{R}_2$. By [Majid, Theorem 5.7], the antipode of a quasitriangular Hopf algebra is bijective. Clearly, if H is cocommutative, then H is quasitriangular with $\mathcal{R} = 1_H \otimes 1_H$. If H is quasitriangular with respect to $\mathcal{R} \in H \otimes H$, then every left $S\#H$-module M becomes a Hopf Yetter-Drinfel'd (S, H)-module, with the H^{op}-coaction given by

$$\rho_M(m) = (\mathcal{R}_2 m) \otimes \mathcal{R}_1, \text{ for all } m \in M.$$

A Hopf algebra H is *triangular* if it is quasitriangular and $\mathcal{R}^{-1} = \mathcal{R}_2 \otimes \mathcal{R}_1$. The next lemma shows that Hopf Yetter-Drinfel'd (S, H)-modules are dyslectic when H is triangular and S is an H-commutative left H-module algebra. So in this case the ingredients necessary to define $BQ(S, H)$ are present.

6.2. Lemma. *Let H be a triangular Hopf algebra with respect to $\mathcal{R} \in H \otimes H$. If S is an H-commutative left H-module algebra, then every left $S\#H$-module M is a dyslectic Hopf Yetter-Drinfel'd (S, H)-module.*

PROOF. Since H is triangular, we have $\mathcal{R}_1\mathcal{R}_2 \otimes \mathcal{R}_2\mathcal{R}_1 = 1_H \otimes 1_H$.

Let $m \in M$ and $s \in S$. Then

$$\begin{aligned} m_0 \leftharpoonup (m_1.s) &= (\mathcal{R}_2 m) \leftharpoonup (\mathcal{R}_1.s) \\ &= (\mathcal{R}_1.s)_0 \rightharpoonup ((\mathcal{R}_1.s)_1(\mathcal{R}_2 m) \\ &= (\mathcal{R}_2\mathcal{R}_1.s) \rightharpoonup (\mathcal{R}_1\mathcal{R}_2 m) \\ &= s \rightharpoonup m, \end{aligned}$$

so the equation (32) is satisfied for M. ■

6.3. EXAMPLE. *H faithfully projective with bijective antipode.* Suppose H is a faithfully projective Hopf algebra over a commutative ring R with a bijective antipode. In this case there is a category equivalence between $\mathcal{Q}^H$ and ${}_{D(H)}\mathcal{M}$, where $D(H)$ is the *Drinfel'd double* of H; i.e. the bi-crossed product $H \bowtie H^{*op}$, where H^* is the R-dual of H [Majid91]. When S is an H-commutative Yetter-Drinfel'd H-module algebra, the right H^{op}-comodule structure on S induces a left H^{*op}-module algebra structure on S, and in this way S can be viewed as a $D(H)$-module algebra, with

$$(h \bowtie \phi).s = (h.s_0)\phi(s_1), \text{ for all } h \in H, \phi \in H^{*op}, s \in S.$$

It is well-known that $D(H)$ is a quasitriangular Hopf algebra in this case, whose special element $\mathcal{R} \in D(H) \otimes D(H)$ is constructed using dual bases $(h^{(i)}, \phi^{(i)})$ of H and H^* [Majid91]:

$$\mathcal{R} = \mathcal{R}_1 \otimes \mathcal{R}_2 = \sum_i (h^{(i)} \bowtie \epsilon_H) \otimes (1_H \bowtie \phi^{(i)}),$$

where ϵ_H is the counit of H, i.e. the unit of H^{*op}. We claim that the H-commutativity of S is equivalent to $D(H)$-quantum commutativity in the sense of Cohen-Westreich [CohWest] that is used in [Wang, Corollary 2.5]. $D(H)$-quantum commutativity of S means that for all $s, t \in S$, $ts = (\mathcal{R}_2.s)(\mathcal{R}_1.t)$. Using our characterization of $\mathcal{R}_1$ and $\mathcal{R}_2$, this is equivalent to

$$\begin{aligned} ts &= (\mathcal{R}_2.s)(\mathcal{R}_1.t) \\ &= \textstyle\sum_i ((1_H \bowtie \phi^{(i)}).s)((h^{(i)} \bowtie \epsilon_H).t) \\ &= \textstyle\sum_i (s_0\phi^{(i)}(s_1))(h^{(i)}.t_0\epsilon_H(t_1)) \\ &= \textstyle\sum_i s_0[\phi^{(i)}(s_1)h^{(i)}].[\epsilon_H(t_1)t_0] \\ &= s_0(s_1.t) \\ &= ts, \end{aligned}$$

so H-commutativity of S is equivalent to $D(H)$-quantum commutativity of S in the sense of Cohen-Westreich [CohWest]. So we can conclude from Wang's results [Wang, Lemma 2.1, Theorem 2.2, and Corollary 2.5], that ${}_{S\#D(H)}\mathcal{M}$ is a braided monoidal category. Thus the equivalent category ${}_S\mathcal{Q}^H$ is also braided monoidal, every $M \in {}_S\mathcal{Q}^H$ will be dyslectic. So our Brauer-Clifford-Long group $BQ(S, H)$ will be isomorphic to $BM(S, D(H))$.

6.4. Example. *H is cotriangular.* A Hopf algebra H over a commutative ring R is said to be *coquasitriangular* if there exists a convolution invertible R-linear map $\mathcal{R} : H \otimes H \to k$ satisfying the following conditions:

$$\begin{array}{lrcl} (CQT1) & \mathcal{R}(h, gx) & = & \mathcal{R}(h_1, g)\mathcal{R}(h_2, x) \\ (CQT2) & \mathcal{R}(hg, x) & = & \mathcal{R}(h, x_1)\mathcal{R}(g, x_2) \\ (CQT3) & \mathcal{R}(h_1, g_1)h_2g_2 & = & g_1h_1\mathcal{R}(h_2, g_2). \end{array}$$

The convolution inverse of $\mathcal{R}$ is given by $\mathcal{R}^{-1}(h, g) = \mathcal{R}(\mathcal{S}(h), g)$, for all $g, h \in H$. By [Majid, Theorem 8.6], the antipode of a coquasitriangular Hopf algebra $(H, \mathcal{R})$ is bijective. If $(H, \mathcal{R})$ is coquasitriangular, then every right (S, H^{op})-Hopf module M becomes a Hopf Yetter-Drinfel'd (S, H)-module; the H-action is given by

$$hm = m_0\mathcal{R}(h, m_1) \text{ for all } m \in M.$$

If H is coquasitriangular and S is a right H^{op}-comodule algebra, then S is a Yetter-Drinfel'd H-module algebra with H-action given by

$$h.s = s_0\mathcal{R}(h, s_1) \text{ for all } s \in S.$$

Clearly, if H is commutative, then H is coquasitriangular with $\mathcal{R}(h, g) = 1$.

A coquasitriangular Hopf algebra H is *cotriangular* if

$$\mathcal{R}(h_1, g_1)\mathcal{R}(g_2, h_2) = \epsilon(h)\epsilon(g) \text{ for all } h, g \in H.$$

6.5. Lemma. *Suppose H is a cotriangular Hopf algebra. If S is an H-commutative right H^{op}-comodule algebra, then every (S, H^{op})-Hopf module M is a dyslectic Hopf Yetter-Drinfel'd (S, H)-module.*

Proof. Let $m \in M$ and $s \in S$. Then

$$\begin{array}{rcl} m_0 \leftharpoonup (m_1.s) & = & m_0 \leftharpoonup (s_0\mathcal{R}(m_1, s_1)) \\ & = & s_{00} \rightharpoonup (s_{01}m_0)\mathcal{R}(m_1, s_1) \\ & = & s_0 \rightharpoonup (s_{11}m_0)\mathcal{R}(m_1, s_{12}) \\ & = & s_0 \rightharpoonup (m_{00}\mathcal{R}(s_{11}, m_{01})\mathcal{R}(m_1, s_{12}) \\ & = & (s_0 \rightharpoonup m_0)\mathcal{R}(s_{11}, m_{11})\mathcal{R}(m_{12}, s_{12}) \\ & = & (s_0 \rightharpoonup m_0)\epsilon(s_1)\epsilon(m_1) \\ & = & s \rightharpoonup m, \end{array}$$

so equation (32) is satisfied for M. ∎

If H is cotriangular and S is an H-commutative right H^{op}-comodule algebra, then S is a dyslectic Hopf Yetter-Drinfel'd (S, H)-module algebra. So the ingredients necessary to define $BQ(S, H)$ are present in this situation as well.

7. Elementary homomorphisms between Brauer-Clifford-Long groups

We conclude the article by presenting some elementary homomorphisms between Brauer-Clifford-Long groups that are induced by scalar extensions and central twists.

We first consider scalar extensions. Let R' be a commutative ring with trivial H-action and H-coaction. Fix a ring homomorphism from R' to R. Then $H' = R' \otimes H$ equipped with its natural R'-module structure is a Hopf algebra over R'. If M is a Hopf Yetter-Drinfel'd H-module, then $R' \otimes M$ is a Hopf Yetter-Drinfel'd $(R' \otimes S, H')$-module in a natural way. Let S be an H-commutative Hopf Yetter-Drinfel'd H-module algebra. Then $R' \otimes S$ is an H'-commutative Hopf Yetter-Drinfel'd H'-module algebra. Let M be a Hopf Yetter-Drinfel'd (S, H)-module. Then $R' \otimes M$ equipped with its natural $R' \otimes S$-module structure is a Hopf Yetter-Drinfel'd $(R' \otimes S, H')$-module. If M is dyslectic then so is $R' \otimes M$. If M is faithfully projective as an S-module, then $R' \otimes M$ is faithfully projective as an $R' \otimes S$-module. Furthermore, if A is a Hopf Yetter-Drinfel'd (S, H)-module (Azumaya) algebra, then $R' \otimes A$ will be a Hopf Yetter-Drinfel'd $(R' \otimes S, H')$- module (Azumaya) algebra, and $\overline{R' \otimes A} \simeq R' \otimes \bar{A}$. The canonical nature of these identifications allows us to lift this to a homomorphism between the Brauer-Clifford groups.

7.1. Proposition. *Let S be an H-commutative Hopf Yetter-Drinfel'd H-module algebra. Suppose that R' is a commutative ring with trivial H-action and H-coaction and there is a homomorphism ring from R' to R. Then the map* $\mathrm{BQ}(S, H) \to \mathrm{BQ}(R' \otimes S, R' \otimes H)$ *given by* $[[A]] \mapsto [[R' \otimes A]]$*, for all Hopf Yetter-Drinfel'd (S, H)-module Azumaya algebras A, is a group homomorphism.*

Central twists also induce homomorphisms between Brauer-Clifford-Long groups. Let S be an H-commutative Hopf Yetter-Drinfel'd H-module algebra. Let $H\text{-}\mathrm{Aut}_R(S)$ be the group of Hopf Yetter-Drinfel'd H-module algebra automorphisms of S. We claim there is an action of $H\text{-}\mathrm{Aut}_R(S)$ on the Brauer-Clifford-Long group. For $M \in {}_S\mathcal{Q}^H$ and $\tau \in H\text{-}\mathrm{Aut}_R(S)$, let ${}_\tau M$ be equal to M as a Hopf Yetter-Drinfel'd H-module, but has left S-module structure given by $s \blacktriangleright m = \tau^{-1}(s) \rightharpoonup m$ for all $s \in S$, $m \in M$. Using the H-linearity and the colinearity of τ, we can see that ${}_\tau M \in {}_S\mathcal{Q}^H$. The corresponding right S-module structure on ${}_\tau M$ is given by $m \blacktriangleleft s = m \leftharpoonup \tau^{-1}(s)$. Using the H-linearity and the H-colinearity of τ, we can show that if M is an object of $Dys\text{-}{}_S\mathcal{Q}^H$ then so is ${}_\tau M$.

7.2. Lemma. *Let S be an H-commutative Hopf Yetter-Drinfel'd H-module algebra. Let $\tau \in H\text{-}\mathrm{Aut}_R(S)$. Let $M, N \in {}_S\mathcal{Q}^H$. Then the following hold.*

(i) ${}_\tau(M \tilde{\otimes}_S N) = {}_\tau M \tilde{\otimes}_S {}_\tau N$;

(ii) M is finitely generated projective as a right (left) S-module if and only if ${}_\tau M$ is finitely generated projective as a right (left) S-module;

(iii) If M is finitely generated projective as a right S-module, then ${}_\tau Hom_S(M, N)$ and $Hom_S({}_\tau M, {}_\tau N)$ are isomorphic in ${}_S\mathcal{Q}^H$;

(iv) *If M is finitely generated projective as a left S-module, then ${}_{\tau}({}_{S}Hom(M,N))$ and ${}_{S}Hom({}_{\tau}M,{}_{\tau}N)$ are isomorphic in ${}_{S}\mathcal{Q}^{H}$; and*

(v) *M is S-faithfully projective in Dys-${}_{S}\mathcal{Q}^{H}$ if and only if ${}_{\tau}M$ is S-faithfully projective in Dys-${}_{S}\mathcal{Q}^{H}$.*

Proof. (i) The identity map is linear from ${}_{\tau}(M \tilde{\otimes}_S N)$ to ${}_{\tau}M \tilde{\otimes}_S {}_{\tau}N$.

Clearly, M is finitely generated as a right (left) S-module if and only if ${}_{\tau}M$ is finitely generated as a (right) (left) S-module. A map f is right S-linear from M to N if and only if it is right S-linear from ${}_{\tau}M$ to ${}_{\tau}N$ and $s \blacktriangleright f = s \rightharpoonup f$. Likewise, f is H-colinear (H-linear) from M to N if and only if it is H-colinear (H-linear) from ${}_{\tau}M$ to ${}_{\tau}N$. So the identity map is S-linear from ${}_{\tau}Hom_S(M,N)$ to $Hom_S({}_{\tau}M,{}_{\tau}N)$, and from ${}_{\tau}({}_{S}Hom(M,N))$ to ${}_{S}Hom({}_{\tau}M,{}_{\tau}N)$. The functor τ preserves exact sequences and $\tau \circ \tau$ is the identity. Using these facts, we can show our results.

■

7.3. Definition. *Let S be an H-commutative Yetter-Drinfel'd module algebra. Let A be an algebra in Dys-${}_{S}\mathcal{Q}^{H}$. For any $\tau \in H$-$\mathrm{Aut}_R(S)$, we define ${}_{\tau}A$ to be equal to A as a Hopf Yetter-Drinfel'd H-module algebra, but equal to ${}_{\tau}A$ as an S-module.*

7.4. Lemma. *Let S be a fixed H-commutative Hopf Yetter-Drinfel'd H-module algebra. Let $\tau \in H$-$\mathrm{Aut}_R(S)$. Let A be an algebra in Dys-${}_{S}\mathcal{Q}^{H}$. Then ${}_{\tau}A$ is an algebra in Dys-${}_{S}\mathcal{Q}^{H}$.*

7.5. Lemma. *Let S be an H-commutative Yetter-Drinfel'd module algebra. Let $\tau \in H$-$\mathrm{Aut}_R(S)$. Then the following hold.*

(i) *If M is faithfully projective as an S-module in Dys-${}_{S}\mathcal{Q}^{H}$, then ${}_{\tau}End_S(M) \simeq End_S({}_{\tau}M)$ and ${}_{\tau}({}_{S}End(M)) \simeq {}_{S}End({}_{\tau}M)$ as algebras in Dys-${}_{S}\mathcal{Q}^{H}$;*

(ii) *if A is an algebra in Dys-${}_{S}\mathcal{Q}^{H}$, then ${}_{\tau}A$ is an algebra in Dys-${}_{S}\mathcal{Q}^{H}$, and $\overline{{}_{\tau}A} = {}_{\tau}\bar{A}$ as algebras in Dys-${}_{S}\mathcal{Q}^{H}$;*

(iii) *if A and B are algebras in Dys-${}_{S}\mathcal{Q}^{H}$, then ${}_{\tau}(A \#_S B)$ is an algebra in Dys-${}_{S}\mathcal{Q}^{H}$ and ${}_{\tau}(A \#_S B) \simeq {}_{\tau}A \#_S {}_{\tau}B$ as algebras in Dys-${}_{S}\mathcal{Q}^{H}$; and*

(iv) *if A is an Azumaya algebra in Dys-${}_{S}\mathcal{Q}^{H}$, then so is ${}_{\tau}A$.*

7.6. Proposition. *H-$\mathrm{Aut}_R(S)$ acts by automorphisms on $\mathrm{BQ}(S,H)$. The action is given by $\tau.[[A]] = [[{}_{\tau}A]]$, for any Azumaya algebra A in Dys-${}_{S}\mathcal{Q}^{H}$ and $\tau \in H$-$\mathrm{Aut}_R(S)$.*

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