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CATEGORIFICATION OF PRE-LIE ALGEBRAS AND SOLUTIONS OF 2-GRADED CLASSICAL YANG-BAXTER EQUATIONS

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ABSTRACT. In this paper, we introduce the notion of a pre-Lie 2-algebra, which is the categorification of a pre-Lie algebra. We prove that the category of pre-Lie 2-algebras and the category of 2-term pre-Lie$_\infty$-algebras are equivalent. We classify skeletal pre-Lie 2-algebras by the third cohomology group of a pre-Lie algebra. We prove that crossed modules of pre-Lie algebras are in one-to-one correspondence with strict pre-Lie 2-algebras. $\mathcal{O}$-operators on Lie 2-algebras are introduced, which can be used to construct pre-Lie 2-algebras. As an application, we give solutions of 2-graded classical Yang-Baxter equations in some semidirect product Lie 2-algebras.

1. Introduction

Pre-Lie algebras (or left-symmetric algebras, Vinberg algebras, and etc.) arose from the study of affine manifolds and affine Lie groups, convex homogeneous cones and deformations of associative algebras. They appeared in many fields in mathematics and mathematical physics (see the survey article [9] and the references therein). The beauty of a pre-Lie algebra is that the commutator gives rise to a Lie algebra and the left multiplication gives rise to a representation of the commutator Lie algebra. So pre-Lie algebras naturally play important roles in the study involving the representations of Lie algebras on the underlying spaces of the Lie algebras themselves or their dual spaces. For example, they are the underlying algebraic structures of the non-abelian phase spaces of Lie algebras [6, 18], which lead to a bialgebra theory of pre-Lie algebras [7]. They are also regarded as the algebraic structures "behind" the classical Yang-Baxter equations (CYBE) and they provide a construction of solutions of CYBE in certain semidirect product Lie algebras (that is, over the "double" spaces) induced by pre-Lie algebras [5, 19]. Furthermore, pre-Lie algebras are also regarded as the underlying algebraic structures of symplectic Lie algebras [13], which coincides with Drinfeld's observation of the correspondence between invertible (skew-symmetric) classical r-matrices and symplectic forms on Lie algebras [14]. In [11], the authors studied pre-Lie algebras using the theory of operads, and introduced the notion of a pre-Lie$_\infty$-algebra. The author also proved that the PreLie operad is Koszul. The PreLie operad is further studied in [10] recently. Furthermore, the relation between pre-Lie algebras, trees and cohomology operations are studied in [22].

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$\mathcal{O}$-operators on a Lie algebra $\mathfrak{g}$ associated to a representation $(V;\rho)$ were introduced in [19] inspired by the study of the operator form of the CYBE. See [24] for more details. On one hand, an $\mathcal{O}$-operator could give rise to a pre-Lie algebra structure on V. On the other hand, an $\mathcal{O}$-operator could give rise to a solution of the CYBE in the semidirect product Lie algebra $\mathfrak{g}\ltimes_{\rho^*} V$.

Recently, people have paid more attention to higher categorical structures with motivations from string theory. One way to provide higher categorical structures is by categorifying existing mathematical concepts. One of the simplest higher structure is a 2-vector space, which is a categorified vector space. If we further put Lie algebra structures on 2-vector spaces, then we obtain Lie 2-algebras [1]. The Jacobi identity is replaced by a natural transformation, called the Jacobiator, which also satisfies some coherence laws of its own. It is well-known that the category of Lie 2-algebras is equivalent to the category of 2-term L_∞-algebras [1]. The concept of an L_∞-algebra (sometimes called a strongly homotopy (sh) Lie algebra) was originally introduced in [21, 26] as a model for "Lie algebras that satisfy Jacobi identity up to all higher homotopies". The structure of a Lie 2-algebra appears in many areas such as string theory [3, 4], higher symplectic geometry [2], and Courant algebroids [27].

The first aim of this paper is to categorify the relation between $\mathcal{O}$-operators, pre-Lie algebras and Lie algebras. We introduce the notion of an $\mathcal{O}$-operator on a Lie 2-algebra associated to a representation and the notion of a pre-Lie 2-algebra, and establish the following commutative diagram:

$$
\begin{tikzcd}
\mathcal{O}\text{-operators on Lie 2-algebras} \arrow[r] & \text{pre-Lie 2-algebras} \arrow[r] & \text{Lie 2-algebras} \\
\mathcal{O}\text{-operators on Lie algebras} \arrow[r] \arrow[u, "\text{categorification}"'] & \text{pre-Lie algebras} \arrow[r] \arrow[u, "\text{categorification}"'] & \text{Lie algebras.} \arrow[u, "\text{categorification}"']
\end{tikzcd}
$$

In [8], the authors introduced the notion of an $L_\infty[l,k]$-bialgebra. In particular, an $L_\infty[0,1]$-bialgebra is a Lie 2-bialgebra, which is a certain categorification of the concept of a Lie bialgebra. See [12, 17, 23] for more details along this direction. However, the relation between Lie 2-algebras and Khovanov-Lauda's famous work about categorification of quantum groups [16] is still unclear. 2-graded classical Yang-Baxter equations were established in [8], whose solutions could naturally generate examples of Lie 2-bialgebras.

The second aim of this paper is to construct solutions of the 2-graded CYBE. We categorify the relation between $\mathcal{O}$-operators and solutions of the CYBE, and establish the following commutative diagram:

$$
\begin{tikzcd}
\mathcal{O}\text{-operators on Lie 2-algebras} \arrow[r] & \text{solutions of 2-graded CYBE} \arrow[r] & \text{Lie 2-bialgebras} \\
\mathcal{O}\text{-operators on Lie algebras} \arrow[r] \arrow[u, "\text{categorification}"'] & \text{solutions of CYBE} \arrow[r] \arrow[u, "\text{categorification}"] & \text{Lie bialgebras.} \arrow[u, "\text{categorification}"]
\end{tikzcd}
$$

We also find that there are pre-Lie 2-algebras behind the construction of Lie 2-bialgebras in [8].

The paper is organized as follows. In Section 2, we recall Lie 2-algebras and their representations, pre-Lie algebras and their cohomologies, $\mathcal{O}$-operators and solutions of the CYBE. In Section 3, first we prove that a 2-term pre-Lie$_\infty$-algebra gives rise to a Lie 2-algebra with a natural representation on itself. Then we introduce the notion of a pre-Lie 2-algebra. At last, we prove that the category of 2-term pre-Lie$_\infty$-algebras and the category of pre-Lie 2-algebras are equivalent (Theorem 3.13). In Section 4, we study skeletal pre-Lie 2-algebras and strict pre-Lie 2-algebras in detail. Skeletal pre-Lie 2-algebras are classified by the third cohomology group (Theorem 4.1). We find that there is a natural 3-cocycle associated to a pre-Lie algebra with a skew-symmetric invariant bilinear form. By this fact, we construct a natural example of skeletal pre-Lie 2-algebras associated to a pre-Lie algebra with a skew-symmetric invariant bilinear form. We also introduce the notion of crossed modules of pre-Lie algebras and prove that there is a one-to-one correspondence between crossed modules of pre-Lie algebras and strict pre-Lie 2-algebras (Theorem 4.8). In Section 5, we introduce the notion of an $\mathcal{O}$-operator on a Lie 2-algebra $\mathcal{G}$ associated to a representation $(\mathcal{V};\rho)$, and construct a pre-Lie 2-algebra structure on $\mathcal{V}$. In Section 6, we construct solutions of the 2-graded CYBE in the strict Lie 2-algebra $\mathcal{G}\ltimes_{\rho^*}\mathcal{V}^*$ using $\mathcal{O}$-operators (Theorem 6.3). In particular, if the strict Lie 2-algebra under consideration is given by a strict pre-Lie 2-algebra, there is a natural solution of the 2-graded CYBE in the strict Lie 2-algebra $\mathcal{G}(\mathcal{A})\ltimes_{(L_0^*,L_1^*)}\mathcal{A}^*$ (Theorem 6.4). At last, we give the pre-Lie 2-algebra structure behind the construction of Lie 2-bialgebras in [8].

2. Preliminaries

2.1. Lie 2-algebras and 2-term L_∞-algebras. Vector spaces can be categorified to 2-vector spaces. A good introduction for this subject is [1]. Let Vect be the category of vector spaces. A **2-vector space** is a category in the category Vect. Thus, a 2-vector space C is a category with a vector space of objects C_0 and a vector space of morphisms C_1, such that all the structure maps are linear. Let $s,t:C_1\longrightarrow C_0$ be the source and target maps respectively. Let $\cdot_{\mathrm{v}}$ be the composition of morphisms.

It is well known that the category of 2-vector spaces is equivalent to the category of 2-term complexes of vector spaces. Roughly speaking, given a 2-vector space C, $\mathrm{Ker}(s)\xrightarrow{t}C_0$ is a 2-term complex. Conversely, any 2-term complex of vector spaces $\mathcal{V}:V_1\xrightarrow{\mathrm{d}}V_0$ gives rise to a 2-vector space of which the set of objects is V_0, the set of morphisms is $V_0\oplus V_1$, the source map s and the target map t are given by

$$s(u+m)=u,\quad t(u+m)=u+\mathrm{d}m,\quad \forall u,v\in V_0, m\in V_1.$$

The composition of morphisms is given by

$$(u+m)\cdot_{\mathrm{v}}(v+n)=(u+m+n),\quad \forall u,v\in V_0, m,n\in V_1,\ \text{safisfying } v=u+\mathrm{d}m.$$

We denote the 2-vector space associated to the 2-term complex of vector spaces $\mathcal{V}:V_1\xrightarrow{\mathrm{d}}$

V_0 by $\mathbb{V}$:

$$\mathbb{V} = \begin{array}{c} \mathbb{V}_1 := V_0 \oplus V_1 \\ \mathrm{s} \Big\downarrow \Big\downarrow \mathrm{t} \\ \mathbb{V}_0 := V_0. \end{array} \tag{1}$$

In this paper, we always assume that a 2-vector space is of the above form. The identity-assigning map $1 : \mathbb{V}_0 \longrightarrow \mathbb{V}_1$ is given by $1_u = (u, 0)$, for any $u \in \mathbb{V}_0$.

2.2. DEFINITION. [1] *A* Lie 2-algebra *is a 2-vector space C equipped with*

- *a skew-symmetric bilinear functor, the bracket, $[\![\cdot,\cdot]\!] : C \times C \longrightarrow C$,*
- *a skew-symmetric trilinear natural isomorphism, the Jacobiator,*

$$J_{x,y,z} : [\![[\![x,y]\!],z]\!] \longrightarrow [\![x,[\![y,z]\!]]\!] + [\![[\![x,z]\!],y]\!],$$

such that the following Jacobiator identity is satisfied,

$$\begin{aligned} &J_{[\![w,x]\!],y,z} \cdot_{\mathrm{v}} ([\![J_{w,x,z},y]\!] + 1) \cdot_{\mathrm{v}} (J_{w,[\![x,z]\!],y} + J_{[\![w,z]\!],x,y} + J_{w,x,[\![y,z]\!]}) \\ = &[\![J_{w,x,y},z]\!] \cdot_{\mathrm{v}} (J_{[\![w,y]\!],x,z} + J_{w,[\![x,y]\!],z}) \cdot_{\mathrm{v}} ([\![J_{w,y,z},x]\!] + 1) \cdot_{\mathrm{v}} ([\![w,J_{x,y,z}]\!] + 1). \end{aligned}$$

2.3. DEFINITION. *A* 2-term L_∞-algebra *structure on a graded vector space $\mathcal{G} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ consists of the following data:*

- *a linear map $\mathfrak{d} : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_0$,*
- *a skew-symmetric bilinear map $\mathfrak{l}_2 : \mathfrak{g}_i \times \mathfrak{g}_j \longrightarrow \mathfrak{g}_{i+j}$, $0 \leq i+j \leq 1$,*
- *a skew-symmetric trilinear map $\mathfrak{l}_3 : \wedge^3 \mathfrak{g}_0 \longrightarrow \mathfrak{g}_1$,*

such that for any $x_i, x, y, z \in \mathfrak{g}_0$ and $m, n \in \mathfrak{g}_1$, the following equalities are satisfied:

(i) $\mathfrak{d}\mathfrak{l}_2(x,m) = \mathfrak{l}_2(x,\mathfrak{d}m), \quad \mathfrak{l}_2(\mathfrak{d}m,n) = \mathfrak{l}_2(m,\mathfrak{d}n)$,

(ii) $\mathfrak{d}\mathfrak{l}_3(x,y,z) = \mathfrak{l}_2(x,\mathfrak{l}_2(y,z)) + \mathfrak{l}_2(y,\mathfrak{l}_2(z,x)) + \mathfrak{l}_2(z,\mathfrak{l}_2(x,y))$,

(iii) $\mathfrak{l}_3(x,y,\mathfrak{d}m) = \mathfrak{l}_2(x,\mathfrak{l}_2(y,m)) + \mathfrak{l}_2(y,\mathfrak{l}_2(m,x)) + \mathfrak{l}_2(m,\mathfrak{l}_2(x,y))$,

(iv) *the Jacobiator identity:*

$$\begin{aligned} &\sum_{i=1}^{4}(-1)^{i+1}\mathfrak{l}_2(x_i,\mathfrak{l}_3(x_1,\cdots,\widehat{x_i},\cdots,x_4)) \\ &+\sum_{i<j}(-1)^{i+j}\mathfrak{l}_3(\mathfrak{l}_2(x_i,x_j),x_1,\cdots,\widehat{x_i},\cdots,\widehat{x_j},\cdots,x_4) = 0. \end{aligned}$$

Usually, we denote a 2-term L_∞-algebra by $(\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{d}, \mathfrak{l}_2, \mathfrak{l}_3)$, or simply by $\mathcal{G}$. A 2-term L_∞-algebra is called *strict* if $\mathfrak{l}_3 = 0$. Associated to a 2-term strict L_∞-algebra, there is a semidirect product Lie algebra $\mathfrak{g}_0 \ltimes \mathfrak{g}_1 = (\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}, [\cdot,\cdot]_s)$, where the bracket $[\cdot,\cdot]_s$ is given by

$$[x+m, y+n]_s := \mathfrak{l}_2(x,y) + \mathfrak{l}_2(x,n) + \mathfrak{l}_2(m,y). \tag{2}$$

2.4. Definition. *Let $\mathcal{G} = (\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{d}, \mathfrak{l}_2, \mathfrak{l}_3)$ and $\mathcal{G}' = (\mathfrak{g}'_0, \mathfrak{g}'_1, \mathfrak{d}', \mathfrak{l}'_2, \mathfrak{l}'_3)$ be 2-term L_∞-algebras. A* homomorphism *F from $\mathcal{G}$ to $\mathcal{G}'$ consists of: linear maps $F_0 : \mathfrak{g}_0 \to \mathfrak{g}'_0$, $F_1 : \mathfrak{g}_1 \to \mathfrak{g}'_1$ and $\mathcal{F}_2 : \mathfrak{g}_0 \wedge \mathfrak{g}_0 \to \mathfrak{g}'_1$, such that the following equalities hold for all $x, y, z \in \mathfrak{g}_0, a \in \mathfrak{g}_1$,*

(i) $F_0 \circ \mathfrak{d} = \mathfrak{d}' \circ F_1$,

(ii) $F_0\mathfrak{l}_2(x, y) - \mathfrak{l}'(F_0(x), F_0(y)) = \mathfrak{d}'\mathcal{F}_2(x, y)$,

(iii) $F_1\mathfrak{l}_2(x, a) - \mathfrak{l}'(F_0(x), F_1(a)) = \mathcal{F}_2(x, \mathfrak{d}a)$,

(iv) $\mathcal{F}_2(\mathfrak{l}_2(x, y), z) + c.p. + F_1(\mathfrak{l}_3(x, y, z)) = \mathfrak{l}'_2(F_0(x), \mathcal{F}_2(y, z)) + c.p. + \mathfrak{l}'_3(F_0(x), F_0(y), F_0(z))$.

It is well-known that the category of Lie 2-algebras and the category of 2-term L_∞-algebras are equivalent. Thus, when we say "a Lie 2-algebra", we mean a 2-term L_∞-algebra in the sequel.

Let $\mathcal{V} : V_1 \xrightarrow{\mathrm{d}} V_0$ be a complex of vector spaces. Define $\mathrm{End}^0_{\mathrm{d}}(\mathcal{V})$ by

$$\mathrm{End}^0_{\mathrm{d}}(\mathcal{V}) \triangleq \{(A_0, A_1) \in \mathfrak{gl}(V_0) \oplus \mathfrak{gl}(V_1) | A_0 \circ \mathrm{d} = \mathrm{d} \circ A_1\},$$

and define $\mathrm{End}^1(\mathcal{V}) \triangleq \mathrm{Hom}(V_0, V_1)$. There is a differential $\delta : \mathrm{End}^1(\mathcal{V}) \longrightarrow \mathrm{End}^0_{\mathrm{d}}(\mathcal{V})$ given by

$$\delta(\phi) \triangleq \phi \circ \mathrm{d} + \mathrm{d} \circ \phi, \quad \forall\, \phi \in \mathrm{End}^1(\mathcal{V}),$$

and a bracket operation $[\cdot, \cdot]$ given by the graded commutator. More precisely, for any $A = (A_0, A_1), B = (B_0, B_1) \in \mathrm{End}^0_{\mathrm{d}}(\mathcal{V})$ and $\phi \in \mathrm{End}^1(\mathcal{V})$, $[\cdot, \cdot]$ is given by

$$[A, B] = A \circ B - B \circ A = (A_0 \circ B_0 - B_0 \circ A_0, A_1 \circ B_1 - B_1 \circ A_1),$$

and

$$[A, \phi] = A \circ \phi - \phi \circ A = A_1 \circ \phi - \phi \circ A_0. \tag{3}$$

These two operations make $\mathrm{End}^1(\mathcal{V}) \xrightarrow{\delta} \mathrm{End}^0_{\mathrm{d}}(\mathcal{V})$ into a strict Lie 2-algebra, which we denote by $\mathrm{End}(\mathcal{V})$. It plays the same role as $\mathfrak{gl}(V)$ for a vector space V ([20]).

A **representation** of a Lie 2-algebra $\mathcal{G}$ on $\mathcal{V}$ is a homomorphism (ρ_0, ρ_1, ρ_2) from $\mathcal{G}$ to $\mathrm{End}(\mathcal{V})$. A representation of a strict Lie 2-algebra $\mathcal{G}$ on $\mathcal{V}$ is called **strict** if $\rho_2 = 0$. Given a strict representation of a strict Lie 2-algebra $\mathcal{G}$ on $\mathcal{V}$, there is a semidirect product strict Lie 2-algebra $\mathcal{G} \ltimes \mathcal{V}$, in which the degree 0 part is $\mathfrak{g}_0 \oplus V_0$, the degree 1 part is $\mathfrak{g}_1 \oplus V_1$, the differential is $\mathfrak{d} + \mathrm{d} : \mathfrak{g}_1 \oplus V_1 \longrightarrow \mathfrak{g}_0 \oplus V_0$, and for all $x, y \in \mathfrak{g}_0, a \in \mathfrak{g}_1, u, v \in V_0, m \in V_1$, $\mathfrak{l}^s_2$ is given by

$$\begin{aligned} \mathfrak{l}^s_2(x + u, y + v) &= \mathfrak{l}_2(x, y) + \rho_0(x)v - \rho_0(y)u, \\ \mathfrak{l}^s_2(x + u, a + m) &= \mathfrak{l}_2(x, a) + \rho_0(x)m - \rho_1(a)u. \end{aligned}$$

2.5. Pre-Lie algebras and their representations.

2.6. Definition. *A* pre-Lie algebra $(A,\cdot_A)$ *is a vector space A equipped with a bilinear product $\cdot_A:\otimes^2A\longrightarrow A$ such that for any $x,y,z\in A$, the associator $(x,y,z)=(x\cdot_A y)\cdot_A z-x\cdot_A(y\cdot_A z)$ is symmetric in x,y, i.e.,*

$$(x,y,z)=(y,x,z),\quad\text{or equivalently,}\quad (x\cdot_A y)\cdot_A z-x\cdot_A(y\cdot_A z)=(y\cdot_A x)\cdot_A z-y\cdot_A(x\cdot_A z).$$

Let A be a pre-Lie algebra. The commutator $[x,y]_A=x\cdot_A y-y\cdot_A x$ defines a Lie algebra structure on A, which is called the **sub-adjacent Lie algebra** of A and denoted by $\mathfrak{g}(A)$. Furthermore, $L:A\to\mathfrak{gl}(A)$ with $L_xy=x\cdot_A y$ gives a representation of the Lie algebra $\mathfrak{g}(A)$ on A. See [9] for more details.

2.7. Definition. *Let $(A,\cdot_A)$ be a pre-Lie algebra and V a vector space. A* representation *of A on V consists of a pair (ρ,μ), where $\rho:A\longrightarrow\mathfrak{gl}(V)$ is a representation of the Lie algebra $\mathfrak{g}(A)$ on V and $\mu:A\longrightarrow\mathfrak{gl}(V)$ is a linear map satisfying*

$$\rho(x)\mu(y)u-\mu(y)\rho(x)u=\mu(x\cdot_A y)u-\mu(y)\mu(x)u,\quad\forall x,y\in A,\ u\in V.\tag{4}$$

Usually, we denote a representation by $(V;\rho,\mu)$. In this case, we will also say that (ρ,μ) is an **action** of $(A,\cdot_A)$ on V. Define $R:A\longrightarrow\mathfrak{gl}(A)$ by $R_xy=y\cdot_A x$. Then $(A;L,R)$ is a representation of $(A,\cdot_A)$. Furthermore, $(A^*;\mathrm{ad}^*=L^*-R^*,-R^*)$ is also a representation of $(A,\cdot_A)$, where L^* and R^* are given by

$$\langle L^*_x\xi,y\rangle=\langle\xi,-L_xy\rangle,\quad\langle R^*_x\xi,y\rangle=\langle\xi,-R_xy\rangle,\quad\forall x,y\in A,\xi\in A^*.$$

The cohomology complex for a pre-Lie algebra $(A,\cdot_A)$ with a representation $(V;\rho,\mu)$ is given as follows ([15]). The set of $(n+1)$-cochains is given by

$$C^{n+1}(A,V)=\mathrm{Hom}(\wedge^nA\otimes A,V),\ n\geq 0.$$

For all $\omega\in C^n(A,V)$, the coboundary operator $d:C^n(A,V)\longrightarrow C^{n+1}(A,V)$ is given by

$$\begin{aligned}
&d\omega(x_1,x_2,\cdots,x_{n+1})\\
=\;&\sum_{i=1}^n(-1)^{i+1}\rho(x_i)\omega(x_1,x_2,\cdots,\hat{x}_i,\cdots,x_{n+1})\\
&+\sum_{i=1}^n(-1)^{i+1}\mu(x_{n+1})\omega(x_1,x_2,\cdots,\hat{x}_i,\cdots,x_n,x_i)\\
&-\sum_{i=1}^n(-1)^{i+1}\omega(x_1,x_2,\cdots,\hat{x}_i,\cdots,x_n,x_i\cdot_A x_{n+1})\\
&+\sum_{1\leq i<j\leq n}(-1)^{i+j}\omega([x_i,x_j]_A,x_1,\cdots,\hat{x}_i,\cdots,\hat{x}_j,\cdots,x_{n+1}),
\end{aligned}$$

for all $x_i\in\Gamma(A),i=1,2\cdots,n+1$.

2.8. $\mathcal{O}$-operators and solutions of the classical Yang-Baxter equations. Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Lie algebra and $(V;\rho)$ be a representation. A linear map $T: V \to \mathfrak{g}$ is called an **$\mathcal{O}$-operator** on $\mathfrak{g}$ associated to the representation $(V;\rho)$ if T satisfies

$$[T(u), T(v)]_{\mathfrak{g}} = T\Big(\rho(T(u))v - \rho(T(v))u\Big), \quad \forall\, u, v \in V. \tag{5}$$

Associated to a representation $(V;\rho)$, we have the semidirect product Lie algebra $\mathfrak{g} \ltimes_{\rho^*} V^*$, where $\rho^*: \mathfrak{g} \longrightarrow \mathfrak{gl}(V^*)$ is the dual representation. A linear map $T: V \longrightarrow \mathfrak{g}$ can be view as an element $\overline{T} \in \otimes^2(\mathfrak{g} \oplus V^*)$ via

$$\overline{T}(\xi + u, \eta + v) = \langle T(u), \eta\rangle, \quad \forall\, \xi + u, \eta + v \in \mathfrak{g}^* \oplus V. \tag{6}$$

Let σ be the exchanging operator acting on the tensor space, then $r \triangleq \overline{T} - \sigma(\overline{T})$ is skew-symmetric.

2.9. Theorem. *Let $T: V \to \mathfrak{g}$ be a linear map. Then $r = \overline{T} - \sigma(\overline{T})$ is a solution of the classical Yang-Baxter equation in the Lie algebra $\mathfrak{g} \ltimes_{\rho^*} V^*$ if and only if T is an $\mathcal{O}$-operator.*

2.10. Theorem. [5] *Let A be a pre-Lie algebra. Then*

$$r = \sum_{i=1}^{n} (e_i \otimes e_i^* - e_i^* \otimes e_i) \tag{7}$$

is a solution of the CYBE in $\mathfrak{g}(A) \ltimes_{L^*} A^*$, where $\{e_i\}$ is a basis of A, and $\{e_i^*\}$ is the dual basis.

3. 2-term pre-Lie$_\infty$-algebras and Pre-Lie 2-algebras

In this section, we show that a 2-term pre-Lie$_\infty$-algebra can give rise to a Lie 2-algebra, and the left multiplication gives rise to a representation of the Lie 2-algebra. We introduce the notion of a pre-Lie 2-algebra, which is a categorification of a pre-Lie algebra. We prove that the category of 2-term pre-Lie$_\infty$-algebras and the category of pre-Lie 2-algebras are equivalent.

3.1. 2-term Pre-Lie$_\infty$-algebras. The notion of a pre-Lie$_\infty$-algebra was introduced in [11], which is a right-symmetric algebra[1] up to homotopy. By a slight modification, we can obtain a left-symmetric algebra (the pre-Lie algebra we use in this paper) up to homotopy. By truncation, we obtain a 2-term pre-Lie$_\infty$-algebra.

[1] A right-symmetric algebra $(A, \cdot_A)$ is a vector space A equipped with a bilinear product $\cdot_A: \otimes^2 A \longrightarrow A$ such that the associator satisfies $(x, y, z) = (x, z, y)$, for all $x, y, z \in A$.

3.2. Definition. *A* 2-term pre-Lie$_\infty$-algebra *is a 2-term graded vector space* $\mathcal{A}=A_0\oplus A_1$*, together with linear maps* $\mathrm{d}:A_1\longrightarrow A_0$, $\cdot:A_i\otimes A_j\longrightarrow A_{i+j}$, $0\leq i+j\leq 1$, *and* $l_3:\wedge^2A_0\otimes A_0\longrightarrow A_1$*, such that for all* $v,v_i\in A_0$ *and* $m,n\in A_1$*, we have*

(a_1) $\mathrm{d}(v\cdot m)=v\cdot \mathrm{d}m$,

(a_2) $\mathrm{d}(m\cdot v)=(\mathrm{d}m)\cdot v$,

(a_3) $\mathrm{d}m\cdot n=m\cdot \mathrm{d}n$,

(b_1) $v_0\cdot(v_1\cdot v_2)-(v_0\cdot v_1)\cdot v_2-v_1\cdot(v_0\cdot v_2)+(v_1\cdot v_0)\cdot v_2=\mathrm{d}l_3(v_0,v_1,v_2)$,

(b_2) $v_0\cdot(v_1\cdot m)-(v_0\cdot v_1)\cdot m-v_1\cdot(v_0\cdot m)+(v_1\cdot v_0)\cdot m=l_3(v_0,v_1,\mathrm{d}m)$,

(b_3) $m\cdot(v_1\cdot v_2)-(m\cdot v_1)\cdot v_2-v_1\cdot(m\cdot v_2)+(v_1\cdot m)\cdot v_2=l_3(\mathrm{d}m,v_1,v_2)$,

(c)

$$\begin{aligned}&v_0\cdot l_3(v_1,v_2,v_3)-v_1\cdot l_3(v_0,v_2,v_3)+v_2\cdot l_3(v_0,v_1,v_3)\\&+l_3(v_1,v_2,v_0)\cdot v_3-l_3(v_0,v_2,v_1)\cdot v_3+l_3(v_0,v_1,v_2)\cdot v_3\\&-l_3(v_1,v_2,v_0\cdot v_3)+l_3(v_0,v_2,v_1\cdot v_3)-l_3(v_0,v_1,v_2\cdot v_3)\\&-l_3(v_0\cdot v_1-v_1\cdot v_0,v_2,v_3)+l_3(v_0\cdot v_2-v_2\cdot v_0,v_1,v_3)-l_3(v_1\cdot v_2-v_2\cdot v_1,v_0,v_3)\\&=0.\end{aligned}$$

Usually, we denote a 2-term pre-Lie$_\infty$-algebra by $(A_0,A_1,\mathrm{d},\cdot,l_3)$, or simply by $\mathcal{A}$. A 2-term pre-Lie$_\infty$-algebra $(A_0,A_1,\mathrm{d},\cdot,l_3)$ is said to be **skeletal** (**strict**) if $\mathrm{d}=0$ ($l_3=0$).

Given a 2-term pre-Lie$_\infty$-algebra $(A_0,A_1,\mathrm{d},\cdot,l_3)$, we define $\mathfrak{l}_2:A_i\wedge A_j\longrightarrow A_{i+j}$ and $\mathfrak{l}_3:\wedge^3A_0\longrightarrow A_1$ by

$$\begin{aligned}\mathfrak{l}_2(u,v)&=u\cdot v-v\cdot u, &(8)\\ \mathfrak{l}_2(u,m)&=-\mathfrak{l}_2(m,u)=u\cdot m-m\cdot u, &(9)\\ \mathfrak{l}_3(u,v,w)&=l_3(u,v,w)+l_3(v,w,u)+l_3(w,u,v). &(10)\end{aligned}$$

Furthermore, define $L_0:A_0\longrightarrow \mathrm{End}(A_0)\oplus\mathrm{End}(A_1)$ by

$$L_0(u)v=u\cdot v,\quad L_0(u)m=u\cdot m. \tag{11}$$

Define $L_1:A_1\longrightarrow \mathrm{Hom}(A_0,A_1)$ by

$$L_1(m)u=m\cdot u. \tag{12}$$

Define $L_2:\wedge^2A_0\longrightarrow \mathrm{Hom}(A_0,A_1)$ by

$$L_2(u,v)w=-l_3(u,v,w),\quad \forall u,v,w\in A_0. \tag{13}$$

3.3. Theorem. *Let $\mathcal{A} = (A_0, A_1, \mathrm{d}, \cdot, l_3)$ be a 2-term pre-Lie$_\infty$-algebra. Then, we have $(A_0, A_1, \mathrm{d}, \mathfrak{l}_2, \mathfrak{l}_3)$ is a Lie 2-algebra, which we denote by $\mathcal{G}(\mathcal{A})$, where $\mathfrak{l}_2$ and $\mathfrak{l}_3$ are given by (8)-(10) respectively. Furthermore, (L_0, L_1, L_2) is a representation of the Lie 2-algebra $\mathcal{G}(\mathcal{A})$ on the complex of vector spaces $A_1 \xrightarrow{\mathrm{d}} A_0$, where L_0, L_1, L_2 are given by (11)-(13) respectively.*

Proof. By Conditions (a_1)-(a_3), we have

$$\begin{aligned}
\mathrm{d}\mathfrak{l}_2(v,m) &= \mathrm{d}(v\cdot m - m\cdot v) = v\cdot \mathrm{d}m - (\mathrm{d}m)\cdot v = \mathfrak{l}_2(v,\mathrm{d}m),\\
\mathfrak{l}_2(\mathrm{d}m,n) &= (\mathrm{d}m)\cdot n - n\cdot \mathrm{d}m = m\cdot \mathrm{d}n - (\mathrm{d}n)\cdot m = \mathfrak{l}_2(m,\mathrm{d}n).
\end{aligned}$$

By Condition (b_1), we have

$$\begin{aligned}
\mathfrak{l}_2(v_0,\mathfrak{l}_2(v_1,v_2)) + c.p. &= \mathfrak{l}_2(v_0, v_1\cdot v_2 - v_2\cdot v_1) + c.p.\\
&= v_0\cdot(v_1\cdot v_2) - (v_1\cdot v_2)\cdot v_0 - v_0\cdot(v_2\cdot v_1) + (v_2\cdot v_1)\cdot v_0 + c.p.\\
&= \mathrm{d}(l_3(v_0,v_1,v_2) + l_3(v_1,v_2,v_0) + l_3(v_2,v_0,v_1))\\
&= \mathrm{d}\mathfrak{l}_3(v_0,v_1,v_2).
\end{aligned}$$

Similarly, by Conditions (b_2) and (b_3), we have

$$\begin{aligned}
&\mathfrak{l}_2(v_0,\mathfrak{l}_2(v_1,m)) + \mathfrak{l}_2(v_1,\mathfrak{l}_2(m,v_0)) + \mathfrak{l}_2(m,\mathfrak{l}_2(v_0,v_1))\\
={}& \mathfrak{l}_2(v_0, v_1\cdot m - m\cdot v_1) + \mathfrak{l}_2(v_1, m\cdot v_0 - v_0\cdot m) + \mathfrak{l}_2(m, v_0\cdot v_1 - v_1\cdot v_0)\\
={}& v_0\cdot(v_1\cdot m) - (v_1\cdot m)\cdot v_0 - v_0\cdot(m\cdot v_1) + (m\cdot v_1)\cdot v_0\\
&+ v_1\cdot(m\cdot v_0) - (m\cdot v_0)\cdot v_1 - v_1\cdot(v_0\cdot m) + (v_0\cdot m)\cdot v_1\\
&+ m\cdot(v_0\cdot v_1) - (v_0\cdot v_1)\cdot m - m\cdot(v_1\cdot v_0) + (v_1\cdot v_0)\cdot m\\
={}& l_3(v_0,v_1,\mathrm{d}m) + l_3(v_1,\mathrm{d}m,v_0) + l_3(\mathrm{d}m,v_0,v_1)\\
={}& \mathfrak{l}_3(v_0,v_1,\mathrm{d}m).
\end{aligned}$$

At last, by Condition (c), we can get

$$\begin{aligned}
&\mathfrak{l}_2(v_0,\mathfrak{l}_3(v_1,v_2,v_3)) - \mathfrak{l}_2(v_1,\mathfrak{l}_3(v_0,v_2,v_3)) + \mathfrak{l}_2(v_2,\mathfrak{l}_3(v_0,v_1,v_3)) - \mathfrak{l}_2(v_3,\mathfrak{l}_3(v_0,v_1,v_2))\\
={}& \mathfrak{l}_3(\mathfrak{l}_2(v_0,v_1),v_2,v_3) - \mathfrak{l}_3(\mathfrak{l}_2(v_0,v_2),v_1,v_3) + \mathfrak{l}_3(\mathfrak{l}_2(v_0,v_3),v_1,v_2) + \mathfrak{l}_3(\mathfrak{l}_2(v_1,v_2),v_0,v_3)\\
&- \mathfrak{l}_3(\mathfrak{l}_2(v_1,v_3),v_0,v_2) + \mathfrak{l}_3(\mathfrak{l}_2(v_2,v_3),v_0,v_1).
\end{aligned}$$

Thus, $(A_0, A_1, \mathrm{d}, \mathfrak{l}_2, \mathfrak{l}_3)$ is a Lie 2-algebra.

By Condition (a_1), we deduce that $L_0(u) \in \mathrm{End}^0_{\mathrm{d}}(\mathcal{A})$ for all $u \in A_0$. By Conditions (a_2) and (a_3), we have

$$\delta \circ L_1(m) = L_0(\mathrm{d}m). \tag{14}$$

Furthermore, we have

$$\begin{aligned}
L_0(\mathfrak{l}_2(u,v))w &= (u\cdot v)\cdot w - (v\cdot u)\cdot w = u\cdot(v\cdot w) - v\cdot(u\cdot w) - \mathrm{d}l_3(u,v,w)\\
&= [L_0(u), L_0(v)]w - \mathrm{d}l_3(u,v,w),
\end{aligned}$$

which implies that

$$L_0(\mathfrak{l}_2(u,v)) - [L_0(u), L_0(v)] = \mathrm{d}\circ L_2(u,v). \tag{15}$$

Similarly, we have

$$L_1(\mathfrak{l}_2(u,m)) - [L_0(u), L_1(m)] = L_2(u,\mathrm{d}m). \tag{16}$$

At last, by Condition (c) in Definition 3.2, we get

$$-[L_0(u), L_2(v,w)] + L_2(\mathfrak{l}_2(u,v),w) + c.p. + L_1(\mathfrak{l}_3(u,v,w)) = 0. \tag{17}$$

By (14)-(17), we deduce that (L_0, L_1, L_2) is a homomorphism from the Lie 2-algebra $\mathcal{G}(\mathcal{A})$ to $\mathrm{End}(\mathcal{V})$. The proof is finished. ∎

3.4. Example. Let V be a vector space. Denote by $A_0 = \mathfrak{gl}(V)\oplus V$ and $A_1 = V$. Define $\mathrm{d}: A_1 \longrightarrow A_0$, $\cdot : A_i\otimes A_j \longrightarrow A_{i+j}$, $0\le i+j\le 1$, and $l_3 : \wedge^2 A_0\otimes A_0 \longrightarrow A_1$ by

$$\begin{array}{rcll} \mathrm{d}u &=& u, & \forall\ u\in A_1,\\ (A+u)\cdot(B+v) &=& AB+\frac{1}{2}Av, & \forall\ A+u, B+v\in A_0,\\ (A+u)\cdot v &=& \frac{1}{2}Av, & \forall\ A+u\in A_0, v\in A_1,\\ u\cdot(B+v) &=& 0, & \forall\ u\in A_1, B+v\in A_0,\\ l_3(A+u,B+v,C+w) &=& -\frac{1}{4}[A,B]w, & \forall\ A+u,B+v,C+w\in A_0. \end{array}$$

Then it is straightforward to prove that $\mathcal{A} = (A_0, A_1, \mathrm{d}, \cdot, l_3)$ is a 2-term pre-Lie$_\infty$-algebra.

By Theorem 3.3, we obtain a Lie 2-algebra $\mathcal{G}(\mathcal{A})$. Actually, this Lie 2-algebra is the Lie 2-algebra associated to Weinstein's omni-Lie algebra. See [25] for more details.

3.5. Definition. *Let $\mathcal{A} = (A_0, A_1, \mathrm{d}, \cdot, l_3)$ and $\mathcal{A}' = (A_0', A_1', \mathrm{d}', \cdot', l_3')$ be 2-term pre-Lie$_\infty$-algebras. A* homomorphism (F_0, F_1, F_2) *from $\mathcal{A}$ to $\mathcal{A}'$ consists of linear maps $F_0 : A_0 \longrightarrow A_0'$, $F_1 : A_1 \longrightarrow A_1'$, and $F_2 : A_0\otimes A_0 \longrightarrow A_1'$ such that the following equalities hold:*

(i) $F_0\circ\mathrm{d} = \mathrm{d}'\circ F_1$,

(ii) $F_0(u\cdot v) - F_0(u)\cdot' F_0(v) = \mathrm{d}'F_2(u,v)$,

(iii) $F_1(u\cdot m) - F_0(u)\cdot' F_1(m) = F_2(u,\mathrm{d}m)$, $\quad F_1(m\cdot u) - F_1(m)\cdot' F_0(u) = F_2(\mathrm{d}m,u)$,

(iv) $F_0(u)\cdot' F_2(v,w) - F_0(v)\cdot' F_2(u,w) + F_2(v,u)\cdot' F_0(w) - F_2(u,v)\cdot' F_0(w) - F_2(v,u\cdot w) + F_2(u,v\cdot w) - F_2(u\cdot v,w) + F_2(v\cdot u,w) + l_3'(F_0(u),F_0(v),F_0(v)) - F_1 l_3(u,v,w) = 0.$

By straightforward computations, we have

3.6. Proposition. *Let $\mathcal{A} = (A_0, A_1, \mathrm{d}, \cdot, l_3)$ and $\mathcal{A}' = (A'_0, A'_1, \mathrm{d}', \cdot', l'_3)$ be 2-term pre-Lie$_\infty$-algebras, (F_0, F_1, F_2) a homomorphism from $\mathcal{A}$ to $\mathcal{A}'$. Then $(F_0, F_1, \mathcal{F}_2)$ is a homomorphism from the corresponding Lie 2-algebra $\mathcal{G}(\mathcal{A})$ to $\mathcal{G}(\mathcal{A}')$, where $\mathcal{F}_2 : \wedge^2 A_0 \longrightarrow A_1$ is given by*

$$\mathcal{F}_2(u, v) = F_2(u, v) - F_2(v, u), \quad \forall u, v \in A_0. \tag{18}$$

At the end of this subsection, we introduce composition and identity for 2-term pre-Lie$_\infty$-algebra homomorphisms. Let $F = (F_0, F_1, F_2) : \mathcal{A} \longrightarrow \mathcal{A}'$ and $G = (G_0, G_1, G_2) : \mathcal{A}' \longrightarrow \mathcal{A}''$ be 2-term pre-Lie$_\infty$-algebra homomorphisms. Their composition $GF = ((GF)_0, (GF)_1, (GF)_2)$ is defined by $(GF)_0 = G_0 \circ F_0, (GF)_1 = G_1 \circ F_1$, and $(GF)_2$ is given by

$$(GF)_2(u, v) = G_2(F_0(u), F_0(v)) + G_1(F_2(u, v)). \tag{19}$$

It is straightforward to verify that $GF = ((GF)_0, (GF)_1, (GF)_2) : \mathcal{A} \longrightarrow \mathcal{A}''$ is a 2-term pre-Lie$_\infty$-algebra homomorphism. It is obvious that $(\mathrm{id}_{A_0}, \mathrm{id}_{A_1}, 0)$ is the identity homomorphism. Thus, we obtain

3.7. Proposition. *There is a category, which we denote by* **2preLie**, *with 2-term pre-Lie$_\infty$-algebras as objects, homomorphisms between them as morphisms.*

3.8. Pre-Lie 2-algebras.

3.9. Definition. *A* pre-Lie 2-algebra *is a 2-vector space $\mathbb{V}$ endowed with a bilinear functor $\star : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{V}$ and a natural isomorphism $J_{u,v,w}$ for all $u, v, w \in \mathbb{V}_0$,*

$$J_{u,v,w} : (u \star v) \star w - u \star (v \star w) \longrightarrow (v \star u) \star w - v \star (u \star w), \tag{20}$$

such that the following identity is satisfied:

$$\begin{array}{rl}
& (0J_{1,2,3}) \cdot_{\mathrm{v}} (-J_{0,21,3} + 1_{(0(21))3} + J_{0,2,13} - 1_{(02)(13)}) \\
& \cdot_{\mathrm{v}}(1_{(0(21))3-((21)0)3+(21)(03)} + J_{02,1,3} - 1_{((02)1)3} - J_{20,1,3} + 1_{((20)1)3} + 2J_{0,1,3} - 21_{(01)3}) \\
= & (-J_{0,12,3} + 1_{(0(12))3} + J_{0,1,23} - 1_{(01)(23)}) \\
& \cdot_{\mathrm{v}}(J_{1,2,03} + 1_{1(2(03))} + J_{01,2,3} - 1_{((01)2)3} - J_{10,2,3} + 1_{((10)2)3} + 1J_{0,2,3} - 11_{(02)3} + 1_{-((12)0)3+(0(12))3}) \\
& \cdot_{\mathrm{v}}(-J_{1,2,0}3 - 1_{1(20)+2(10)}3 - J_{2,0,1}3 + 1_{(20)1}3 - J_{0,1,2}3 + 1_{(10)2}3 \\
& +1_{(21)(03)-2(1(03))-2((01)3)+2((10)3)-1((02)3)+1((20)3)}).
\end{array} \tag{21}$$

Here, $0, 1, 2, 3$ denote v_0, v_1, v_2, v_3 respectively, ij denotes $v_i \star v_j$, $iJ_{j,k,l}$ denotes $v_i \star J_{v_j,v_k,v_l}$, and $J_{j,k,l}i$ denotes $J_{v_j,v_k,v_l} \star v_i$. Or, in terms of a commutative diagram,

$$\begin{tikzcd}
0((12)3 - 1(23)) \arrow[r, "0J_{1,2,3}"] \arrow[d, "-J_{0,12,3}+1_{(0(12))3}+J_{0,1,23}-1_{(01)(23)}"'] & 0((21)3 - 2(13)) \arrow[d, "-J_{0,21,3}+1_{(0(21))3}+J_{0,2,13}-1_{(02)(13)}"] \\
P \arrow[d, "\epsilon"'] & Q \arrow[d, "\varepsilon"] \\
M \arrow[r, "\kappa"] & N
\end{tikzcd}$$

where

$$
\begin{array}{rcl}
P &=& -((12)0)3+(12)(03)+(0(12))3-(01)(23)+(10)(23)-1(0(23)),\\
Q &=& (0(21))3-((21)0)3+(21)(03)-(02)(13)+(20)(13)-2(0(13)),\\
\epsilon &=& J_{1,2,03}+1_{1(2(03))}+J_{01,2,3}-1_{((01)2)3}-J_{10,2,3}+1_{((10)2)3}+1J_{0,2,3}-11_{(02)3}+1_{-((12)0)3+(0(12))3},\\
\varepsilon &=& 1_{(0(21))3-((21)0)3+(21)(03)}+J_{02,1,3}-1_{((02)1)3}-J_{20,1,3}+1_{((20)1)3}+2J_{0,1,3}-21_{(01)3},\\
\kappa &=& -J_{1,2,0}3-1_{1(20)+2(10)}3-J_{2,0,1}3+1_{(20)1}3-J_{0,1,2}3+1_{(10)2}3\\
&& +1_{(21)(03)-2(1(03))-2((01)3)+2((10)3)-1((02)3)+1((20)3)},\\
M &=& -((12)0)3+(0(12))3+1(2(03))+(21)(03)-2(1(03))-((01)2)3+(2(01))3-2((01)3)\\
&& +((10)2)3-(2(10))3+2((10)3)-1((02)3)+1((20)3)-1(2(03)),\\
N &=& (0(21))3-((21)0)3+(21)(03)-((02)1)3+(1(02))3-1((02)3)\\
&& +((20)1)3-(1(20))3+1((20)3)-2((01)3)+2((10)3)-2(1(03)).
\end{array}
$$

3.10. Definition. *Let* $(\mathbb{V},\star,J)$ *and* $(\mathbb{V}',\star',J')$ *be pre-Lie* 2*-algebras. A* homomorphism $\Phi:\mathbb{V}\longrightarrow\mathbb{V}'$ *consists of*

- *A linear functor* (Φ_0,Φ_1) *from* $\mathbb{V}$ *to* $\mathbb{V}'$,
- *A bilinear natural transformation* $\Phi_2:\Phi_0(u)\star'\Phi_0(v)\longrightarrow\Phi_0(u\star v)$,

such that the following identity holds:

$$
\begin{array}{rl}
& J'_{\Phi_0(u),\Phi_0(v),\Phi_0(w)}\cdot_{\mathrm{v}}(F_2(v,u)\star 1_{\Phi_0(w)}-1_{\Phi_0(v)}\star F_2(u,w))\cdot_{\mathrm{v}}(F_2(v\star u,w)-F_2(v,u\star w))\\
= & (F_2(u,v)\star 1_{\Phi_0(w)}-1_{\Phi_0(u)}\star F_2(v,w))\cdot_{\mathrm{v}}(F_2(u\star v,w)-F_2(u,v\star w))\cdot_{\mathrm{v}}F_1J_{u,v,w},
\end{array}
$$

or, in terms of a commutative diagram:

$$
\begin{tikzcd}
(\Phi_0(u)\star'\Phi_0(v))\star'\Phi_0(w)-\Phi_0(u)\star'(\Phi_0(v)\star'\Phi_0(w)) \arrow[r, "J'_{\Phi_0(u),\Phi_0(v),\Phi_0(w)}"] \arrow[d, "F_2(u,v)\star 1_{\Phi_0(w)}-1_{\Phi_0(u)}\star F_2(v,w)"'] & (\Phi_0(v)\star'\Phi_0(u))\star'\Phi_0(w)-\Phi_0(v)\star'(\Phi_0(u)\star'\Phi_0(w)) \arrow[d, "F_2(v,u)\star 1_{\Phi_0(w)}-1_{\Phi_0(v)}\star F_2(u,w)"] \\
\Phi_0(u\star v)\star\Phi_0(w)-\Phi_0(u)\star\Phi_0(v\star w) \arrow[d, "F_2(u\star v,w)-F_2(u,v\star w)"'] & \Phi_0(v\star u)\star\Phi_0(w)-\Phi_0(v)\star\Phi_0(u\star w) \arrow[d, "F_2(v\star u,w)-F_2(v,u\star w)"] \\
F_0((u\star v)\star w)-F_0(u\star(v\star w)) \arrow[r, "F_1J_{u,v,w}"] & F_0((v\star u)\star w)-F_0(v\star(u\star w)).
\end{tikzcd}
$$

The composition of two homomorphisms $\Phi:\mathbb{V}\longrightarrow\mathbb{V}'$ and $\Psi:\mathbb{V}'\longrightarrow\mathbb{V}''$, which we denote by $\Psi\Phi:\mathbb{V}\longrightarrow\mathbb{V}''$ is defined as follows:

$$
(\Psi\Phi)_0=\Psi_0\circ\Phi_0,\quad (\Psi\Phi)_1=\Psi_1\circ\Phi_1,\quad (\Psi\Phi)_2(u,v)=\Psi_2(\Phi_0(u),\Phi_0(v))\cdot_{\mathrm{v}}\Psi_1(\Phi_2(u,v)).
$$

The identity homomorphism $1_{\mathbb{V}}$ has the identity functor as its underlying functor, together with an identity natural transformation. It is straightforward to obtain

3.11. Proposition. *There is a category, which we denote by* **preLie2**, *with pre-Lie* 2*-algebras as objects, homomorphisms between them as morphisms.*

3.12. The equivalence.

3.13. Theorem. *The categories* **2preLie** *and* **preLie2***, which are given in Proposition 3.7 and Proposition 3.11 respectively, are equivalent.*

Thus, in the following sections, a 2-term pre-Lie$_\infty$-algebra will be called a pre-Lie 2-algebra.

Proof. We only give a sketch of the proof. First we construct a functor T : **2preLie** $\longrightarrow$ **preLie2**.

Given a 2-term pre-Lie$_\infty$-algebra $\mathcal{A} = (A_0, A_1, \mathrm{d}, \cdot, l_3)$, we have a 2-vector space $\mathbb{A}$ given by (1). More precisely, we have $\mathbb{A}_0 = A_0,\ \mathbb{A}_1 = A_0 \oplus A_1$. Define a bilinear functor $\star : \mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{A}$ by

$$(u+m) \star (v+n) = u \cdot v + u \cdot n + m \cdot v + \mathrm{d}m \cdot n, \quad \forall\ u+m, v+n \in \mathbb{A}_1 = A_0 \oplus A_1.$$

Define the Jacobiator $J : \otimes^3 \mathbb{A}_0 \longrightarrow \mathbb{A}_1$ by

$$J_{u,v,w} = (u \cdot v) \cdot w - u \cdot (v \cdot w) + l_3(x, y, z).$$

By the various conditions of $\mathcal{A}$ being a 2-term pre-Lie$_\infty$-algebra, we deduce that $(\mathbb{A}, \star, J)$ is a pre-Lie 2-algebra. Thus, we have constructed a pre-Lie 2-algebra $\mathbb{A} = T(\mathcal{A})$ from a 2-term pre-Lie$_\infty$-algebra $\mathcal{A}$.

For any homomorphism $F = (F_0, F_1, F_2)$ form $\mathcal{A}$ to $\mathcal{A}'$, next we construct a pre-Lie 2-algebra homomorphism $\Phi = T(F)$ from $\mathbb{A} = T(\mathcal{A})$ to $\mathbb{A}' = T(\mathcal{A}')$. Let $\Phi_0 = F_0,\ \Phi_1 = F_0 \oplus F_1$, and Φ_2 be given by

$$\Phi_2(u, v) = F_0(u) \cdot' F_0(v) + F_2(u, v).$$

Then $\Phi_2(u, v)$ is a natural isomorphism from $\Phi_0(u) \cdot' \Phi_0(v)$ to $\Phi_0(u \cdot v)$, and $\Phi = (\Phi_0, \Phi_1, \Phi_2)$ is a homomorphism from $\mathbb{A}$ to $\mathbb{A}'$.

One can also deduce that T preserves the identity homomorphisms and the composition of homomorphisms. Thus, T constructed above is a functor from **2pre-Lie** to **preLie2**.

Conversely, given a pre-Lie 2-algebra $\mathbb{A}$, we construct the 2-term pre-Lie$_\infty$-algebra $\mathcal{A} = S(\mathbb{A})$ as follows. As a complex of vector spaces, $\mathcal{A}$ is obtained as follows: $A_0 = \mathbb{A}_0,\ A_1 = \mathrm{Ker}(s)$, and $\mathrm{d} = t|_{\mathrm{Ker}(s)}$, where s, t are the sauce map and the target map in the 2-vector space $\mathbb{A}$. Define a multiplication $\cdot : A_i \otimes A_j \longrightarrow A_{i+j},\ 0 \le i+j \le 1$, by

$$u \cdot v = u \star v, \quad u \cdot m = 1_u \star m, \quad m \cdot u = m \star 1_u, \quad \forall u, v \in A_0,\ m, n \in A_1.$$

Define $l_3 : \wedge^2 A_0 \otimes A_0 \longrightarrow A_1$ by

$$l_3(u, v, w) = J_{u,v,w} - 1_{s(J_{u,v,w})}.$$

The various conditions of $\mathbb{A}$ being a pre-Lie 2-algebra imply that $\mathcal{A}$ is 2-term pre-Lie$_\infty$-algebra.

Let $\Phi = (\Phi_0, \Phi_1, \Phi_2) : \mathbb{A} \longrightarrow \mathbb{A}'$ be a pre-Lie 2-algebra homomorphism, and $S(\mathbb{A}) = \mathcal{A}$, $S(\mathbb{A}') = \mathcal{A}'$. Define $S(\Phi) = F = (F_0, F_1, F_2)$ as follows. Let $F_0 = \Phi_0$, $F_1 = \Phi_1|_{A_1=\mathrm{Ker}(s)}$ and define F_2 by

$$F_2(u,v) = \Phi_2(u,v) - 1_{s(\Phi_2(u,v))}.$$

It is not hard to deduce that F is a homomorphism between 2-term pre-Lie$_\infty$-algebras. Furthermore, S also preserves the identity homomorphisms and the composition of homomorphisms. Thus, S is a functor from **preLie2** to **2pre-Lie**.

We are left to show that there are natural isomorphisms $\alpha : T \circ S \Longrightarrow \mathrm{id}_{\mathbf{preLie2}}$ and $\beta : S \circ T \Longrightarrow \mathrm{id}_{\mathbf{2preLie}}$. For a pre-Lie 2-algebra $(\mathbb{A}, \star, J)$, applying the functor S to $\mathbb{A}$, we obtain a 2-term pre-Lie$_\infty$-algebra $\mathcal{A} = (A_0, A_1, \mathrm{d} = t|_{\mathrm{Ker}(s)}, \cdot, l_3)$, where $A_0 = \mathbb{A}_0$, $A_1 = \mathrm{Ker}(s)$. Applying the functor T to $\mathcal{A}$, we obtain a pre-Lie 2-algebra $(\mathbb{A}', \star', J')$, with the space A_0 of objects and the space $A_0 \oplus \mathrm{Ker}(s)$ of morphisms. Define $\alpha_{\mathbb{A}} : \mathbb{A}' \longrightarrow \mathbb{A}$ by setting

$$(\alpha_{\mathbb{A}})_0(u) = u, \quad (\alpha_{\mathbb{A}})_1(u+m) = 1_u + m.$$

It is obvious that $\alpha_{\mathbb{A}}$ is an isomorphism of 2-vector spaces. Furthermore, since $\star$ is a bilinear functor, we have $1_u \star 1_v = 1_{u\star v}$, and

$$m \star n = (m \cdot_{\mathrm{v}} 1_{\mathrm{d}m}) \star (1_0 \cdot_{\mathrm{v}} n) = (m \star 1_0) \cdot_{\mathrm{v}} (1_{\mathrm{d}m} \star n) = 1_{\mathrm{d}m} \star n.$$

Therefore, we have

$$\begin{aligned}
\alpha_{\mathbb{A}}((u+m) \star' (v+n)) &= \alpha_{\mathbb{A}}(u \cdot v + u \cdot n + m \cdot v + \mathrm{d}m \cdot n) \\
&= \alpha_{\mathbb{A}}(u \star v + 1_u \star n + m \star 1_v + 1_{\mathrm{d}m} \star n) \\
&= 1_{u\star v} + 1_u \star n + m \star 1_v + 1_{\mathrm{d}m} \star n \\
&= 1_u \star 1_v + 1_u \star n + m \star 1_v + 1_{\mathrm{d}m} \star n \\
&= \alpha_{\mathbb{A}}(u+m) \star \alpha_{\mathbb{A}}(v+n),
\end{aligned}$$

which implies that $\alpha_{\mathbb{A}}$ is also a pre-Lie 2-algebra homomorphism with $(\alpha_{\mathbb{A}})_2$ the identity isomorphism. Thus, $\alpha_{\mathbb{A}}$ is an isomorphism of pre-Lie 2-algebras. It is also easy to see that it is a natural isomorphism.

For a 2-term pre-Lie$_\infty$-algebra $\mathcal{A} = (A_0, A_1, \mathrm{d}, \cdot, l_3)$, applying the functor S to $\mathcal{A}$, we obtain a pre-Lie 2-algebra $(\mathbb{A}, \star, J)$. Applying the functor T to $\mathbb{A}$, we obtain exactly the same 2-term pre-Lie$_\infty$-algebra $\mathcal{A}$. Thus, $\beta_{\mathcal{A}} = \mathrm{id}_{\mathcal{A}} = (\mathrm{id}_{A_0}, \mathrm{id}_{A_1})$ is the natural isomorphism from $T \circ S$ to $\mathrm{id}_{\mathbf{2preLie}}$. This finishes the proof. ■

3.14. Remark. We can further obtain 2-categories **2preLie** and **preLie2** by introducing 2-morphisms and strengthen Theorem 3.13 to the 2-equivalence of 2-categories. We omit details.

4. Skeletal and strict pre-Lie 2-algebras

In this section, we study skeletal pre-Lie 2-algebras and strict pre-Lie 2-algebras in detail.

Let $(A_0, A_1, \mathrm{d} = 0, \cdot, l_3)$ be a skeletal pre-Lie 2-algebra. Condition (b_1) in Definition 3.2 implies that $(A_0, \cdot)$ is a pre-Lie algebra. Define ρ and μ from A_0 to $\mathfrak{gl}(A_1)$ by

$$\rho(u)m = u \cdot m, \quad \mu(u)m = m \cdot u, \quad \forall\, u \in A_0,\ m \in A_1. \tag{22}$$

Condition (b_2) and (b_3) in Definition 3.2 implies that $(A_1; \rho, \mu)$ is a representation of the pre-Lie algebra $(A_0, \cdot)$. Furthermore, Condition (c) exactly means that l_3 is a 3-cocycle on A_0 with values in A_1. Summarize the discussion above, we have

4.1. Theorem. *There is a one-to-one correspondence between skeletal pre-Lie 2-algebras and triples $((A_0, \cdot), (A_1; \rho, \mu), l_3)$, where $(A_0, \cdot)$ is a pre-Lie algebra, $(A_1; \rho, \mu)$ is a representation of $(A_0, \cdot)$, and l_3 is a 3-cocycle on $(A_0, \cdot)$ with values in A_1.*

Recall that a skew-symmetric bilinear form $\omega : \wedge^2 A \longrightarrow A$ on a pre-Lie algebra $(A, \cdot_A)$ is called **invariant** if

$$\omega(u \cdot_A v - v \cdot_A u, w) + \omega(v, u \cdot_A w) = 0, \quad \forall u, v, w \in A. \tag{23}$$

Equivalently, $\omega([u, v]_A, w) + \omega(v, u \cdot_A w) = 0$, where $[\cdot, \cdot]_A$ is the Lie bracket in the sub-adjacent Lie algebra of A.

4.2. Lemma. *Let ω be a skew-symmetric invariant bilinear form on a pre-Lie algebra $(A, \cdot_A)$. Then we have*

$$\omega(u \cdot_A v, w) = \omega(u, w \cdot_A v). \tag{24}$$

Proof. By (23), we have

$$\omega(u \cdot_A w - w \cdot_A u, v) + \omega(w, u \cdot_A v) = 0. \tag{25}$$

Since ω is skew-symmetric, by (23) and (25), we have

$$-\omega(w \cdot_A u, v) - \omega(v \cdot_A u, w) = 0,$$

which implies that $\omega(u \cdot_A v, w) = \omega(u, w \cdot_A v)$. ∎

Define $\varphi : \wedge^2 A \otimes A \longrightarrow \mathbb{R}$ by

$$\varphi(u, v, w) = \omega(u \cdot_A v - v \cdot_A u, w). \tag{26}$$

4.3. Proposition. *Let ω be a skew-symmetric invariant bilinear form on a pre-Lie algebra $(A, \cdot_A)$. Then φ defined by (26) is a 3-cocycle on A with values in $\mathbb{R}$, i.e. $d\varphi = 0$.*

PROOF. For any $u, v, w, p \in A$, by (23) and (24), we have

$$\begin{aligned} d\varphi(u,v,w,p) &= -\varphi(v,w,u\cdot_A p)+\varphi(u,w,v\cdot_A p)-\varphi(u,v,w\cdot_A p)\\ &\quad -\varphi([u,v]_A,w,p)+\varphi([u,w]_A,v,p)-\varphi([v,w]_A,u,p)\\ &= -\omega([v,w]_A,u\cdot_A p)+\omega([u,w]_A,v\cdot_A p)-\omega([u,v]_A,w\cdot_A p)\\ &\quad -\omega([[u,v]_A,w]_A+c.p.,p)\\ &= \omega(w,v\cdot_A(u\cdot_A p))-\omega(w,u\cdot_A(v\cdot_A p))+\omega(w,[u,v]_A\cdot_A p)\\ &= \omega(w,v\cdot_A(u\cdot_A p)-u\cdot_A(v\cdot_A p)+(u\cdot_A v)\cdot_A p-(v\cdot_A u)\cdot_A p)\\ &= 0, \end{aligned}$$

which finishes the proof. ∎

4.4. EXAMPLE. Let ω be a skew-symmetric invariant bilinear form on a pre-Lie algebra $(A, \cdot_A)$. Consider the graded vector space $\mathcal{A} = A_0 \oplus A_1$ where $A_0 = A, A_1 = \mathbb{R}$. Define $\mathrm{d} : \mathbb{R} \longrightarrow A$, $\cdot : A_i \otimes A_j \longrightarrow A_{i+j}$, $0 \le i + j \le 1$, and $l_3 : \otimes A_0 \longrightarrow A_1$ by

$$\begin{aligned} \mathrm{d} &= 0,\\ u\cdot v &= u\cdot_A v,\\ u\cdot m &= m\cdot u = 0,\\ l_3(u,v,w) &= \varphi(u,v,w), \end{aligned}$$

for any $u, v, w \in A$ and $m \in A_1$. By Proposition 4.3, it is straightforward to verify that $\mathcal{A} = (A, \mathbb{R}, \mathrm{d} = 0, \cdot, l_3 = \varphi)$ is a pre-Lie 2-algebra. Furthermore, ω ia a closed 2-form on the Lie algebra $\mathfrak{g}(A)$, i.e.

$$\omega([u,v]_A,w)+\omega([v,w]_A,u)+\omega([w,u]_A,v)=0,$$

which implies that $l_3(u,v,w)+l_3(v,w,u)+l_3(w,u,v) = 0$. Thus, the skeletal Lie 2-algebra $\mathcal{G}(\mathcal{A})$ is strict.

Now we turn to the study on strict pre-Lie 2-algebras. First we introduce the notion of crossed modules of pre-Lie algebras, which can give rise to crossed modules of Lie algebras.

4.5. DEFINITION. A crossed module of pre-Lie algebras *is defined to be a quadruple* $((A_0, \cdot_0), (A_1, \cdot_1), \mathrm{d}, (\rho, \mu))$ *where* $(A_0, \cdot_0)$ *and* $(A_1, \cdot_1)$ *are pre-Lie algebras,* $\mathrm{d} : A_1 \longrightarrow A_0$ *is a homomorphism of pre-Lie algebras, and* (ρ, μ) *is an action of* $(A_0, \cdot_0)$ *on* A_1 *such that for all* $u \in A_0$ *and* $m, n \in A_1$*, the following equalities are satisfied:*

(C1) $\mathrm{d}(\rho(u)m) = u \cdot_0 \mathrm{d}m, \quad \mathrm{d}(\mu(u)m) = (\mathrm{d}m) \cdot_0 u,$

(C2) $\rho(\mathrm{d}m)n = \mu(\mathrm{d}n)m = m \cdot_1 n.$

4.6. EXAMPLE. Let $(A, \cdot_A)$ be a pre-Lie algebra and $B \subset A$ an ideal. Then it is straightforward to see that $((A, \cdot), (B, \cdot|_B), \mathbb{i}, (\rho, \mu))$ is a crossed module of pre-Lie algebras, where $\mathbb{i}$ is the inclusion, and (ρ, μ) are given by $\rho(u)v = u\cdot_A v,\ \mu(u)v = v\cdot_A u$, for all $u \in A, v \in B$.

4.7. PROPOSITION. *Let $((A_0,\cdot_0),(A_1,\cdot_1),\mathrm{d},(\rho,\mu))$ be a crossed module of pre-Lie algebras. Then we have*

$$\begin{aligned} \rho(u)(m\cdot_1 n) &= (\rho(u)m)\cdot_1 n+m\cdot_1\rho(u)n-(\mu(u)m)\cdot_1 n, && (27)\\ \mu(u)(m\cdot_1 n) &= \mu(u)(n\cdot_1 m)+m\cdot_1\mu(u)n-n\cdot_1\mu(u)m. && (28)\end{aligned}$$

Consequently, there is a pre-Lie algebra structure $\cdot$ on the direct sum $A_0\oplus A_1$ given by

$$(u+m)\cdot(v+n)=u\cdot_0 v+\rho(u)n+\mu(v)m+m\cdot_1 n. \tag{29}$$

PROOF. Since (ρ,μ) is an action of A_0 on A_1, we have

$$\begin{aligned} \rho(u)\rho(\mathrm{d}m)n &= \rho(u\cdot_0\mathrm{d}m)n-\rho(\mathrm{d}m\cdot_0 u)n+\rho(\mathrm{d}m)\rho(u)n\\ &= \rho(\mathrm{d}\rho(u)m)n-\rho(\mathrm{d}\mu(u)m)n+\rho(\mathrm{d}m)\rho(u)n.\end{aligned}$$

The second equality is due to (C1). By (C2), we obtain (27). (28) can be obtained similarly. The other conclusion is obvious. ∎

4.8. THEOREM. *There is a one-to-one correspondence between strict pre-Lie 2-algebras and crossed modules of pre-Lie algebras.*

PROOF. Let $(A_0,A_1,\mathrm{d},\cdot,l_3=0)$ be a strict pre-Lie 2-algebra. We construct a crossed module of pre-Lie algebras as follows. Obviously, $(A_0,\cdot)$ is a pre-Lie algebra. Define a multiplication $\cdot_1$ on A_1 by

$$m\cdot_1 n=(\mathrm{d}m)\cdot n=m\cdot\mathrm{d}n. \tag{30}$$

Then by Conditions (a_1) and (b_2) in Definition 3.2, we have

$$\begin{aligned} & m\cdot_1(n\cdot_1 p)-(m\cdot_1 n)\cdot_1 p-n\cdot_1(m\cdot_1 p)+(n\cdot_1 m)\cdot_1 p\\ =\; & (\mathrm{d}m)\cdot((\mathrm{d}n)\cdot p)-\mathrm{d}((\mathrm{d}m)\cdot n)\cdot p-(\mathrm{d}n)\cdot((\mathrm{d}m)\cdot p)+\mathrm{d}((\mathrm{d}n)\cdot m)\cdot_1 p\\ =\; & (\mathrm{d}m)\cdot((\mathrm{d}n)\cdot p)-((\mathrm{d}m)\cdot\mathrm{d}n)\cdot p-(\mathrm{d}n)\cdot((\mathrm{d}m)\cdot p)+((\mathrm{d}n)\cdot\mathrm{d}m)\cdot_1 p=0,\end{aligned}$$

which implies that $(A_1,\cdot_1)$ is a pre-Lie algebra. Also by Condition (a_1), we deduce that d is a homomorphism between pre-Lie algebras. Define $\rho,\mu:A_0\longrightarrow\mathfrak{gl}(A_1)$ by

$$\rho(u)m=u\cdot m,\quad \mu(u)m=m\cdot u. \tag{31}$$

By Conditions (b_2) and (b_3) in Definition 3.2, it is straightforward to deduce that (ρ,μ) is an action of $(A_0,\cdot)$ on A_1. By Conditions (a_1) and (a_2), we deduce that Condition (C1) hold. Condition (C2) follows from the definition of $\cdot_1$ directly. Thus, the data $((A_0,\cdot),(A_1,\cdot_1),\mathrm{d},(\rho,\mu))$ constructed above is a crossed module of pre-Lie algebras.

Conversely, a crossed module of pre-Lie algebras $((A_0,\cdot),(A_1,\cdot_1),\mathrm{d},(\rho,\mu))$ gives rise to a strict pre-Lie 2-algebra $(A_0,A_1,\mathrm{d},\cdot,l_3=0)$, where $\cdot:A_i\otimes A_j\longrightarrow A_{i+j}$, $0\le i+j\le 1$ is given by

$$u\cdot v=u\cdot_0 v,\quad u\cdot m=\rho(u)m,\quad m\cdot u=\mu(u)m.$$

The crossed module conditions give various conditions for a strict pre-Lie 2-algebra. We omit details. ∎

A pre-Lie algebra has its sub-adjacent Lie algebra. Similarly, a crossed module of pre-Lie algebras has its sub-adjacent crossed module of Lie algebras. Recall that **a crossed module of Lie algebras** is a quadruple $(\mathfrak{h}_1, \mathfrak{h}_0, dt, \phi)$, where $\mathfrak{h}_1$ and $\mathfrak{h}_0$ are Lie algebras, $dt : \mathfrak{h}_1 \longrightarrow \mathfrak{h}_0$ is a Lie algebra homomorphism and $\phi : \mathfrak{h}_0 \longrightarrow \mathrm{Der}(\mathfrak{h}_1)$ is an action of Lie algebra $\mathfrak{h}_0$ on Lie algebra $\mathfrak{h}_1$ as a derivation, such that

$$dt(\phi_X(A)) = [X, dt(A)]_{\mathfrak{h}_0}, \quad \phi_{dt(A)}(B) = [A, B]_{\mathfrak{h}_1}, \quad \forall X \in \mathfrak{h}_0, A, B \in \mathfrak{h}_1.$$

4.9. Proposition. *Let $((A_0, \cdot_0), (A_1, \cdot_1), \mathrm{d}, (\rho, \mu))$ be a crossed module of pre-Lie algebras and $\mathfrak{g}(A_0), \mathfrak{g}(A_1)$ the corresponding sub-adjacent Lie algebras of $(A_0, \cdot_0), (A_1, \cdot_1)$ respectively. Then $(\mathfrak{g}(A_0), \mathfrak{g}(A_1), \mathrm{d}, \rho - \mu)$ is a crossed module of Lie algebras.*

Proof. The fact that d is a homomorphism between pre-Lie algebras implies that d is also a homomorphism between Lie algebras. Since $(A_1; \rho, \mu)$ is a representation of $(A_0, \cdot_0)$, $(A_1; \rho-\mu)$ is a representation of the Lie algebra $\mathfrak{g}(A_0)$. By (C1), we have $\mathrm{d}((\rho-\mu)(u)m) = [u, \mathrm{d}m]_0$. By (C2), we have $(\rho - \mu)(\mathrm{d}m)n = [m, n]_1$. Thus, $(\mathfrak{g}(A_0), \mathfrak{g}(A_1), \mathrm{d}, \rho - \mu)$ is a crossed module of Lie algebras. ∎

5. Categorification of $\mathcal{O}$-operators

Let $\mathcal{G} = (\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{d}, \mathfrak{l}_2, \mathfrak{l}_3)$ be a Lie 2-algebra and (ρ_0, ρ_1, ρ_2) be a representation of $\mathcal{G}$ on a 2-term complex of vector spaces $\mathcal{V} = V_1 \xrightarrow{\mathrm{d}} V_0$.

5.1. Definition. *A triple (T_0, T_1, T_2), where $T_0 : V_0 \longrightarrow \mathfrak{g}_0, T_1 : V_1 \longrightarrow \mathfrak{g}_1$ is a chain map, and $T_2 : \wedge^2 V_0 \longrightarrow \mathfrak{g}_1$ is a linear map, is called an* $\mathcal{O}$**-operator** *on $\mathcal{G}$ associated to the representation (ρ_0, ρ_1, ρ_2), if for all $u, v, v_i \in V_0$ and $m \in V_1$ the following conditions are satisfied:*

(i) $T_0\Big(\rho_0(T_0 u)v - \rho_0(T_0 v)u\Big) - \mathfrak{l}_2(T_0 u, T_0 v) = \mathrm{d}T_2(u, v)$;

(ii) $T_1\Big(\rho_1(T_1 m)v - \rho_0(T_0 v)m\Big) - \mathfrak{l}_2(T_1 m, T_0 v) = T_2(\mathrm{d}m, v)$;

(iii)

$$\begin{aligned}&\mathfrak{l}_2(T_0(v_1), T_2(v_2, v_3)) + T_2\Big(v_3, \rho_0(T_0 v_1)v_2 - \rho_0(T_0 v_2)v_1\Big)\\ &+T_1\Big(\rho_1(T_2(v_2, v_3))v_1 + \rho_2(T_0 v_2, T_0 v_3)v_1\Big) + c.p. + \mathfrak{l}_3(T_0 v_1, T_0 v_2, T_0 v_3) = 0.\end{aligned}$$

5.2. Example. Let $\mathcal{A} = (A_0, A_1, \mathrm{d}, \cdot, l_3)$ be a pre-Lie 2-algebra. Then, $(T_0 = \mathrm{id}_{A_0}, T_1 = \mathrm{id}_{A_1}, T_2 = 0)$ is an $\mathcal{O}$-operator on the Lie 2-algebra $\mathcal{G}(\mathcal{A})$ associated to the representation (L_0, L_1, L_2) given in Theorem 3.3.

Define a degree 0 multiplication $\cdot : V_i \otimes V_j \longrightarrow V_{i+j}$, $0 \le i + j \le 1$, on $\mathcal{V}$ by

$$u \cdot v = \rho_0(T_0 u)v, \quad u \cdot m = \rho_0(T_0 u)m, \quad m \cdot u = \rho_1(T_1 m)u. \tag{32}$$

Define $l_3 : \wedge^2 V_0 \otimes V_0 \longrightarrow V_1$ by

$$l_3(v_1, v_2, v_3) = -\rho_1(T_2(v_1, v_2))v_3 - \rho_2(T_0 v_1, T_0 v_2)v_3. \tag{33}$$

Now, Condition (iii) in Definition 5.1 reads

$$\begin{aligned}&\mathfrak{l}_2(T_0(v_1), T_2(v_2, v_3)) + T_2(v_3, v_1 \cdot v_2 - v_2 \cdot v_1) - T_1(l_3(v_1, v_2, v_3)) + c.p.\\ &+\mathfrak{l}_3(T_0 v_1, T_0 v_2, T_0 v_3) = 0.\end{aligned} \tag{34}$$

5.3. Theorem. *Let (ρ_0, ρ_1, ρ_2) be a representation of $\mathcal{G}$ on $\mathcal{V}$ and (T_0, T_1, T_2) an $\mathcal{O}$-operator on $\mathcal{G}$ associated to the representation (ρ_0, ρ_1, ρ_2). Then, $(V_0, V_1, \mathrm{d}, \cdot, l_3)$ is a pre-Lie 2-algebra, where the multiplication "$\cdot$" and l_3 are given by (32) and (33) respectively.*

Proof. By the fact that $\mathrm{d} \circ \rho(x) = \rho(x) \circ \mathrm{d}$ for all $x \in \mathfrak{g}_0$, we deduce that

$$\mathrm{d}(u \cdot m) = \mathrm{d}\rho_0(T_0 u)m = \rho_0(T_0 u)\mathrm{d}m = u \cdot \mathrm{d}m.$$

By the fact that both (T_0, T_1) and (ρ_0, ρ_1) are chain maps, we have

$$\mathrm{d}(m \cdot u) = \mathrm{d}(\rho_1(T_1 m)u) = \delta(\rho_1(T_1 m))u = \rho_0(\mathfrak{d}T_1 m)u = \rho_0(T_0 \mathrm{d}m)u = (\mathrm{d}m) \cdot u.$$

Similarly, we have

$$(\mathrm{d}m) \cdot n = \rho_0(T_0 \mathrm{d}m)n = \rho_0(\mathfrak{d}T_1 m)n = \delta(\rho_1(T_1 m))n = \rho_1(T_1 m)(\mathrm{d}n) = m \cdot (\mathrm{d}n).$$

Thus, Conditions (a_1)-(a_3) in Definition 3.2 hold. For all $u, v, w \in A_0$, we have

$$\begin{aligned}
& u \cdot (v \cdot w) - (u \cdot v) \cdot w - v \cdot (u \cdot w) + (v \cdot u) \cdot w \\
= \;& \rho_0(T_0 u)\rho_0(T_0 v)w - \rho_0(T_0(\rho_0(T_0 u)v))w - \rho_0(T_0 v)\rho_0(T_0 u)w + \rho_0(T_0(\rho_0(T_0 v)u))w \\
= \;& [\rho_0(T_0 u), \rho_0(T_0 v)]w - \rho_0\Big(T_0(\rho_0(T_0 u)v) - T_0(\rho_0(T_0 v)u)\Big)w \\
= \;& \rho_0(\mathfrak{l}_2(T_0 u, T_0 v))w - \mathrm{d}\rho_2(T_0 u, T_0 v)w - \rho_0\Big(T_0(\rho_0(T_0 u)v) - T_0(\rho_0(T_0 v)u)\Big)w \\
= \;& -\rho_0(\mathrm{d}T_2(u, v))w - \mathrm{d}\rho_2(T_0 u, T_0 v)w \\
= \;& -\mathrm{d}\rho_1(T_2(u, v))w - \mathrm{d}\rho_2(T_0 u, T_0 v)w \\
= \;& \mathrm{d}l_3(u, v, w),
\end{aligned}$$

which implies that Condition (b_1) in Definition 3.2 holds. Similarly, Conditions (b_2) and (b_3) also hold.

The left hand side of Condition (c) is equal to

$$\begin{aligned}
&\rho_0(T_0v_0)l_3(v_1,v_2,v_3)-\rho_0(T_0v_1)l_3(v_0,v_2,v_3)+\rho_0(T_0v_2)l_3(v_0,v_1,v_3)\\
&+\rho_1(T_1l_3(v_1,v_2,v_0))v_3-\rho_1(T_1l_3(v_0,v_2,v_1))v_3+\rho_1(T_1l_3(v_0,v_1,v_2))v_3\\
&+\rho_1(T_2(v_1,v_2))(v_0\cdot v_3)+\rho_2(T_0v_1,T_0v_2)(v_0\cdot v_3)-\rho_1(T_2(v_0,v_2))(v_1\cdot v_3)\\
&-\rho_2(T_0v_0,T_0v_2)(v_1\cdot v_3)+\rho_1(T_2(v_0,v_1))(v_2\cdot v_3)+\rho_2(T_0v_0,T_0v_1)(v_2\cdot v_3)\\
&+\rho_1(T_2(\rho_0(T_0v_0)v_1-\rho_0(T_0v_1)v_0,v_2))v_3+\rho_2(T_0(\rho_0(T_0v_0)v_1-\rho_0(T_0v_1)v_0),T_0v_2)v_3\\
&-\rho_1(T_2(\rho_0(T_0v_0)v_2-\rho_0(T_0v_2)v_0,v_1))v_3-\rho_2(T_0(\rho_0(T_0v_0)v_2-\rho_0(T_0v_2)v_0),T_0v_1)v_3\\
&+\rho_1(T_2(\rho_0(T_0v_1)v_2-\rho_0(T_0v_2)v_1,v_0))v_3+\rho_2(T_0(\rho_0(T_0v_1)v_2-\rho_0(T_0v_2)v_1),T_0v_0)v_3\\
=\;&-\rho_0(T_0v_0)\rho_1(T_2(v_1,v_2))v_3-\rho_0(T_0v_0)\rho_2(T_0v_1,T_0v_2)v_3\\
&+\rho_0(T_0v_1)\rho_1(T_2(v_0,v_2))v_3+\rho_0(T_0v_1)\rho_2(T_0v_0,T_0v_2)v_3\\
&-\rho_0(T_0v_2)\rho_1(T_2(v_0,v_1))v_3-\rho_0(T_0v_2)\rho_2(T_0v_0,T_0v_1)v_3\\
&+\rho_1(T_1l_3(v_1,v_2,v_0))v_3-\rho_1(T_1l_3(v_0,v_2,v_1))v_3+\rho_1(T_1l_3(v_0,v_1,v_2))v_3\\
&+\rho_1(T_2(v_1,v_2))\rho_0(v_0)v_3+\rho_2(T_0v_1,T_0v_2)\rho_0(v_0)v_3-\rho_1(T_2(v_0,v_2))\rho_0(v_1)v_3\\
&-\rho_2(T_0v_0,T_0v_2)\rho_0(v_1)v_3+\rho_1(T_2(v_0,v_1))\rho_0(v_2)v_3+\rho_2(T_0v_0,T_0v_1)\rho_0(v_2)v_3\\
&+\rho_1(T_2(\rho_0(T_0v_0)v_1-\rho_0(T_0v_1)v_0,v_2))v_3+\rho_2(T_0(\rho_0(T_0v_0)v_1-\rho_0(T_0v_1)v_0),T_0v_2)v_3\\
&-\rho_1(T_2(\rho_0(T_0v_0)v_2-\rho_0(T_0v_2)v_0,v_1))v_3-\rho_2(T_0(\rho_0(T_0v_0)v_2-\rho_0(T_0v_2)v_0),T_0v_1)v_3\\
&+\rho_1(T_2(\rho_0(T_0v_1)v_2-\rho_0(T_0v_2)v_1,v_0))v_3+\rho_2(T_0(\rho_0(T_0v_1)v_2-\rho_0(T_0v_2)v_1),T_0v_0)v_3\\
=\;&\Big(-[\rho_0(T_0v_0),\rho_1(T_2(v_1,v_2))]+c.p.\Big)v_3+\Big(-[\rho_0(T_0v_0),\rho_2(T_0v_1,T_0v_2)]+c.p.\Big)v_3\\
&+\Big(\rho_1(T_1l_3(v_0,v_1,v_2))+c.p.\Big)v_3+\Big(\rho_1T_2(v_0\cdot v_1-v_1\cdot v_0,v_2)+c.p.\Big)v_3\\
&+\Big(\rho_2(T_0(v_0\cdot v_1-v_1\cdot v_0),T_0v_2)+c.p.\Big)v_3\\
=\;&\Big(-\rho_1\mathfrak{l}_2(T_0v_0,T_2(v_1,v_2))+\rho_2(T_0v_0,\mathrm{d}T_2(v_1,v_2))+c.p.\Big)v_3\\
&+\Big(-[\rho_0(T_0v_0),\rho_2(T_0v_1,T_0v_2)]+c.p.\Big)v_3+\Big(\rho_1(T_1l_3(v_0,v_1,v_2))+c.p.\Big)v_3\\
&+\Big(\rho_1T_2(v_0\cdot v_1-v_1\cdot v_0,v_2)+c.p.\Big)v_3+\Big(\rho_2(T_0(v_0\cdot v_1-v_1\cdot v_0),T_0v_2)+c.p.\Big)v_3.
\end{aligned}$$

By Condition (ii) in Definition 5.1, we have

$$\rho_2(T_0v_0,\mathrm{d}T_2(v_1,v_2))+c.p.+\rho_2(T_0(v_0\cdot v_1-v_1\cdot v_0),T_0v_2)+c.p.=\rho_2(\mathfrak{l}_2(T_0v_0,T_0v_1),T_0v_2)+c.p..$$

By the fact that (ρ_0,ρ_1,ρ_2) is a representation, we have

$$[\rho_0(T_0v_1),\rho_2(T_0v_2,T_0v_3)]+c.p.-\rho_2(\mathfrak{l}_2(T_0v_1,T_0v_2),T_0v_3)+c.p.=\rho_1\mathfrak{l}_3(T_0v_1,T_0v_2,T_0v_3).$$

By (34), we deduce that Condition (c) in Definition 3.2 holds. Thus, $(V_0,V_1,\mathrm{d},\cdot,l_3)$ is a pre-Lie 2-algebra. This finishes the proof. ∎

5.4. Corollary. *Let (ρ_0, ρ_1, ρ_2) be a representation of the Lie 2-algebra $\mathcal{G}$ on $\mathcal{V}$ and (T_0, T_1, T_2) be an $\mathcal{O}$-operator on $\mathcal{G}$ associated to the representation (ρ_0, ρ_1, ρ_2). Then (T_0, T_1, T_2) is a homomorphism from the Lie 2-algebra $\mathcal{G}(\mathcal{V})$ to $\mathcal{G}$.*

6. Solutions of 2-graded Classical Yang-Baxter Equations

Let $\mathcal{G} = (\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{d}, \mathfrak{l}_2)$ be a strict Lie 2-algebra and $r \in \mathfrak{g}_0 \otimes \mathfrak{g}_1 \oplus \mathfrak{g}_1 \otimes \mathfrak{g}_0$ and $\mathfrak{r} \in \mathfrak{g}_1 \otimes \mathfrak{g}_1$. Denote by $R = r - (\mathrm{d} \otimes 1 + 1 \otimes \mathrm{d})\mathfrak{r}$.

6.1. Definition. ([8]) *The classical Yang-Baxter equation for R in the semidirect product Lie algebra $\mathfrak{g}_0 \ltimes \mathfrak{g}_1 = (\mathfrak{g}_0 \oplus \mathfrak{g}_1, [\cdot,\cdot]_s)$ together with $(\mathfrak{d} \otimes 1 - 1 \otimes \mathfrak{d})R = 0$ are called the* **2-graded classical Yang-Baxter Equations** *(2-graded CYBE) in the strict Lie 2-algebra $\mathcal{G}$, where $[\cdot,\cdot]_s$ is the semidirect product Lie algebra structure given by (2).*

More precisely, the 2-graded CYBE reads:

(a) R is skew-symmetric,

(b) $[R_{12}, R_{13}]_s + [R_{13}, R_{23}]_s + [R_{12}, R_{23}]_s = 0$,

(c) $(\mathfrak{d} \otimes 1 - 1 \otimes \mathfrak{d})r = 0$.

For $R = \sum_i a_i \otimes b_i$,

$$R_{12} = \sum_i a_i \otimes b_i \otimes 1; \quad R_{13} = \sum_i a_i \otimes 1 \otimes b_i; \quad R_{23} = \sum_i 1 \otimes a_i \otimes b_i. \tag{35}$$

Let (ρ_0, ρ_1) be a strict representation of the Lie 2-algebra $\mathcal{G} = (\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{d}, \mathfrak{l}_2)$ on the 2-term complex of vector space $\mathcal{V} : V_1 \xrightarrow{\mathrm{d}} V_0$. We view $\rho_0 \oplus \rho_1$ a linear map from $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ to $\mathfrak{gl}(V_0 \oplus V_1)$ by

$$(\rho_0 \oplus \rho_1)(x + a)(u + m) = \rho_0(x)(u) + \rho_0(x)m + \rho_1(a)u. \tag{36}$$

By straightforward computations, we have

6.2. Lemma. *With the above notations, $\rho_0 \oplus \rho_1 : \mathfrak{g}_0 \oplus \mathfrak{g}_1 \longrightarrow \mathfrak{gl}(V_0 \oplus V_1)$ is a representation of $(\mathfrak{g}_0 \oplus \mathfrak{g}_1, [\cdot,\cdot]_s)$ on $V_0 \oplus V_1$. Furthermore, (T_0, T_1) is an $\mathcal{O}$-operator on $\mathcal{G}$ associated to the representation (ρ_0, ρ_1) if and only if*

(a) *$T_0 + T_1 : V_0 \oplus V_1 \longrightarrow \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is an $\mathcal{O}$-operator on the Lie algebra $(\mathfrak{g}_0 \oplus \mathfrak{g}_1, [\cdot,\cdot]_s)$ associated to the representation $\rho_0 \oplus \rho_1$,*

(b) *$T_0 \circ \mathrm{d} = \mathfrak{d} \circ T_1$.*

Let (ρ_0^*, ρ_1^*) be the dual representation of (ρ_0, ρ_1). Then we have the semidirect product Lie 2-algebra $\bar{\mathcal{G}} = \mathcal{G} \ltimes_{(\rho_0^*, \rho_1^*)} \mathcal{V}^*$, where $\bar{\mathcal{G}}_0 = \mathfrak{g}_0 \oplus V_1^*$, $\bar{\mathcal{G}}_1 = \mathfrak{g}_1 \oplus V_0^*$, and $\bar{\mathfrak{d}} = \mathfrak{d} \oplus \mathrm{d}^*$. It is obvious that

$$\overline{T_0} + \overline{T_1} \in V_0^* \otimes \mathfrak{g}_0 \oplus V_1^* \otimes \mathfrak{g}_1 \in (\bar{\mathcal{G}}_1 \otimes \bar{\mathcal{G}}_0) \oplus (\bar{\mathcal{G}}_0 \otimes \bar{\mathcal{G}}_1),$$

where $\overline{T_0}$ and $\overline{T_1}$ are given by (6).

6.3. THEOREM. *Let (ρ_0,ρ_1) be a strict representation of the Lie 2-algebra $\mathcal{G}=(\mathfrak{g}_0,\mathfrak{g}_1,\mathfrak{d},\mathfrak{l}_2)$ on the 2-term complex of vector space $\mathcal{V}: V_1 \xrightarrow{\mathrm{d}} V_0$, and $T_0: V_0 \longrightarrow \mathfrak{g}_0, T_1: V_1 \longrightarrow \mathfrak{g}_1$ linear maps. Then, (T_0,T_1) is an $\mathcal{O}$-operator on the Lie 2-algebra $\mathcal{G}$ associated to the representation (ρ_0,ρ_1) if and only if $\overline{T_0}+\overline{T_1}-\sigma(\overline{T_0}+\overline{T_1})$ is a solution of the 2-graded CYBE in the semidirect product Lie 2-algebra $\bar{\mathcal{G}}$.*

PROOF. It is obvious that $(\rho_0\oplus\rho_1)^*=\rho_0^*\oplus\rho_1^*:\mathfrak{g}_0\oplus\mathfrak{g}_1\longrightarrow\mathfrak{gl}(V_1^*\oplus V_0^*)$. By Lemma 6.2 and Theorem 2.9, (T_0,T_1) is an $\mathcal{O}$-operator on the Lie 2-algebra $\mathcal{G}$ if and only if $\overline{T_0}+\overline{T_1}-\sigma(\overline{T_0}+\overline{T_1})$ is a solution of the CYBE in the semidirect product Lie algebra $(\mathfrak{g}_0\ltimes\mathfrak{g}_1)\ltimes_{\rho_0^*\oplus\rho_1^*}(V_1^*\oplus V_0^*)$, and $T_0\circ\mathrm{d}=\mathfrak{d}\circ T_1$. Note that the semidirect product Lie algebra $(\mathfrak{g}_0\ltimes\mathfrak{g}_1)\ltimes_{\rho_0^*\oplus\rho_1^*}(V_1^*\oplus V_0^*)$ is exactly the same as the semidirect product Lie algebra $\bar{\mathcal{G}}_0\ltimes\bar{\mathcal{G}}_1$. Furthermore, $T_0\circ\mathrm{d}=\mathfrak{d}\circ T_1$ if and only if $(\bar{\mathfrak{d}}\otimes 1-1\otimes\bar{\mathfrak{d}})(\overline{T_0}+\overline{T_1})=0$. Thus, (T_0,T_1) is an $\mathcal{O}$-operator on the Lie 2-algebra $\mathcal{G}$ associated to the representation (ρ_0,ρ_1) if and only if $\overline{T_0}+\overline{T_1}-\sigma(\overline{T_0}+\overline{T_1})$ is a solution of the 2-graded CYBE in the semidirect product Lie 2-algebra $\bar{\mathcal{G}}$. ■

Let $\mathcal{A}=(A_0,A_1,\mathrm{d},\cdot)$ be a strict pre-Lie 2-algebra. Then, $\mathcal{G}(\mathcal{A})=(A_0,A_1,\mathrm{d},\mathfrak{l}_2)$ is a strict Lie 2-algebra, where $\mathfrak{l}_2$ is given by (8) and (9). Furthermore, (L_0,L_1) is a strict representation of the Lie 2-algebra $\mathcal{G}(\mathcal{A})$ on the complex of vector spaces $A_1\xrightarrow{\mathrm{d}}A_0$, where L_0,L_1 are given by (11) and (12) respectively. Let $\{e_i\}_{1\le i\le k}$ and $\{\mathfrak{e}_j\}_{1\le j\le l}$ be the basis of A_0 and A_1 respectively, and denote by $\{e_i^*\}_{1\le i\le k}$ and $\{\mathfrak{e}_j^*\}_{1\le j\le l}$ the dual basis.

6.4. THEOREM. *With the above notations,*

$$R=\sum_{i=1}^{k}(e_i\otimes e_i^*-e_i^*\otimes e_i)+\sum_{j=1}^{l}(\mathfrak{e}_j\otimes\mathfrak{e}_j^*-\mathfrak{e}_j^*\otimes\mathfrak{e}_j) \tag{37}$$

is a solution of the 2-graded CYBE in the strict Lie 2-algebra $\mathcal{G}(\mathcal{A})\ltimes_{(L_0^*,L_1^*)}\mathcal{A}^*$.

PROOF. It is obvious that $(T_0=\mathrm{id}_{A_0},T_1=\mathrm{id}_{A_1})$ is an $\mathcal{O}$-operator on $\mathcal{G}(\mathcal{A})$ associated to the representation (L_0,L_1). By Theorem 6.3,

$$\overline{T_0}+\overline{T_1}-\sigma(\overline{T_0}+\overline{T_1})=\sum_{i=1}^{k}(e_i\otimes e_i^*-e_i^*\otimes e_i)+\sum_{j=1}^{l}(\mathfrak{e}_j\otimes\mathfrak{e}_j^*-\mathfrak{e}_j^*\otimes\mathfrak{e}_j)$$

is a solution of the 2-graded CYBE in the strict Lie 2-algebra $\mathcal{G}(\mathcal{A})\ltimes_{(L_0^*,L_1^*)}\mathcal{A}^*$. ■

At the end of this section, we consider the construction of strict Lie 2-bialgebras in [8, Proposition 4.4]. In fact, there are pre-Lie 2-algebras behind the construction.

Let $(A,\cdot_A)$ be a pre-Lie algebra. Then $(A;L,R)$ is a representation of $(A,\cdot_A)$. Furthermore, $(A^*;L^*-R^*,-R^*)$ is also a representation of $(A,\cdot_A)$. Let $A_0=A$ and $A_1=A^*$. Define a multiplication $\cdot: A_i\otimes A_j\longrightarrow A_{i+j},\ 0\le i+j\le 1$, by

$$x\cdot y=x\cdot_A y,\quad x\cdot\xi=\mathrm{ad}_x^*\xi,\quad \xi\cdot x=-R_x^*\xi,\quad \forall x,y\in A,\ \xi\in A^*. \tag{38}$$

On the other hand, consider its sub-adjacent Lie algebra $\mathfrak{g}(A)$. Define a skew-symmetric operation $\mathfrak{l}_2 : A_i \wedge A_j \longrightarrow A_{i+j},\ 0 \le i+j \le 1$, by

$$\mathfrak{l}_2(x,y) = [x,y]_A = x \cdot_A y - y \cdot_A x, \quad \mathfrak{l}_2(x,\xi) = -\mathfrak{l}_2(\xi,x) = L_x^*\xi. \tag{39}$$

6.5. Proposition. *Let $(A,\cdot_A)$ be a pre-Lie algebra, and $\mathrm{d} : A^* \longrightarrow A$ a linear map. If $(A, A^*, \mathrm{d}, \cdot)$ is a pre-Lie 2-algebra, then $(\mathfrak{g}(A), A^*, \mathrm{d}, \mathfrak{l}_2)$ is a Lie 2-algebra, where $\cdot$ and $\mathfrak{l}_2$ are given by (38) and (39) respectively.*

Conversely, if $(\mathfrak{g}(A), A^*, \mathrm{d}, \mathfrak{l}_2)$ is a Lie 2-algebra, in which $\mathrm{d} : A^* \longrightarrow A$ is skew-symmetric, then $(A, A^*, \mathrm{d}, \cdot)$ is a pre-Lie 2-algebra.

Proof. If $(A, A^*, \mathrm{d}, \cdot)$ is a pre-Lie 2-algebra, then we have

$$\mathrm{d}(\mathrm{ad}_x^*\eta) = x \cdot \mathrm{d}\eta, \quad \mathrm{d}(-R_y^*\xi) = (\mathrm{d}\xi)\cdot y, \quad \mathrm{ad}_{\mathrm{d}\xi}^*\eta = -R_{\mathrm{d}\eta}^*\xi, \quad \forall x,y \in A,\ \xi,\eta \in A^*.$$

Therefore, we have

$$\begin{aligned}
\mathrm{d}\mathfrak{l}_2(x,\eta) &= \mathrm{d}L_x^*\eta = \mathrm{ad}_x^*\eta + R_x^*\eta = x\cdot \mathrm{d}\eta - (\mathrm{d}\eta)\cdot x = \mathfrak{l}_2(x,\mathrm{d}\eta),\\
\mathfrak{l}_2(\mathrm{d}\xi,\eta) &= L_{\mathrm{d}\xi}^*\eta = \mathrm{ad}_{\mathrm{d}\xi}^*\eta + R_{\mathrm{d}\xi}^*\eta = \mathrm{ad}_{\mathrm{d}\xi}^*\eta - \mathrm{ad}_{\mathrm{d}\eta}^*\xi = \mathfrak{l}_2(\xi,\mathrm{d}\eta).
\end{aligned}$$

Since L^* is a representation of the Lie algebra $\mathfrak{g}(A)$ on A^*, it is obvious that the other conditions in the definition of a Lie 2-algebra are also satisfied. Thus, $(\mathfrak{g}(A), A^*, \mathrm{d}, \mathfrak{l}_2)$ is a Lie 2-algebra.

Conversely, if $(\mathfrak{g}(A), A^*, \mathrm{d}, \mathfrak{l}_2)$ is a Lie 2-algebra, we have

$$\mathrm{d}\mathfrak{l}_2(x,\eta) = \mathfrak{l}_2(x,\mathrm{d}\eta), \quad \mathfrak{l}_2(\mathrm{d}\xi,\eta) = \mathfrak{l}_2(\xi,\mathrm{d}\eta),$$

which implies that

$$\mathrm{d}L_x^*\eta = L_x\mathrm{d}\eta - R_x\mathrm{d}\eta, \quad L_{\mathrm{d}\xi}^*\eta = -L_{\mathrm{d}\eta}^*\xi.$$

If d is skew-symmetric, then we can obtain

$$\begin{aligned}
\langle \mathrm{d}R_x^*\eta, \xi\rangle &= \langle R_x^*\eta, -\mathrm{d}\xi\rangle = \langle \eta, R_x\mathrm{d}\xi\rangle\\
&= \langle \eta, L_x\mathrm{d}\xi - \mathrm{d}L_x^*\xi\rangle = \langle \mathrm{d}L_x^*\eta - L_x\mathrm{d}\eta, \xi\rangle\\
&= \langle -R_x\mathrm{d}\eta, \xi\rangle,
\end{aligned}$$

which implies that

$$\mathrm{d}(\eta\cdot x) = (\mathrm{d}\eta)\cdot x. \tag{40}$$

Furthermore, we have

$$\mathrm{d}(\mathrm{ad}_x^*\eta) = \mathrm{d}(L_x^*\eta - R_x^*\eta) = L_x\mathrm{d}\eta,$$

which implies that

$$\mathrm{d}(x\cdot\eta) = x\cdot\mathrm{d}\eta. \tag{41}$$

Also by the fact that d is skew-symmetric, we have

$$\begin{aligned}\langle \mathrm{ad}^*_{\mathrm{d}\xi}\eta, x\rangle &= \langle \eta, L_x\mathrm{d}\xi - R_x\mathrm{d}\xi\rangle = \langle \mathrm{d}L^*_x\eta - \mathrm{d}R^*_x\eta, \xi\rangle \\ &= \langle L_x\mathrm{d}\eta - R_x\mathrm{d}\eta + R_x\mathrm{d}\eta, \xi\rangle \\ &= \langle R_{\mathrm{d}\eta}x, \xi\rangle = \langle x, -R^*_{\mathrm{d}\eta}\xi\rangle,\end{aligned}$$

which implies that $\mathrm{ad}^*_{\mathrm{d}\xi}\eta = -R^*_{\mathrm{d}\eta}\xi$, i.e.

$$(\mathrm{d}\xi)\cdot\eta = \xi\cdot(\mathrm{d}\eta). \tag{42}$$

By (40)-(42), we deduce that Conditions (a_1)-(a_3) in Definition 3.2 hold. It is obvious that the other conditions also hold. Thus, $(A, A^*, \mathrm{d}, \cdot)$ is a pre-Lie 2-algebra. ■

By Proposition 6.5 and Proposition 4.4 in [8], we have

6.6. Corollary. *Let $(A, A^*, \mathrm{d}, \cdot)$ be a pre-Lie 2-algebra, where $\cdot$ is given by (38) and d is skew-symmetric. Then r given by (7) is a solution of the 2-graded CYBE in the strict Lie 2-algebra $(\mathfrak{g}(A), A^*, \mathrm{d}, \mathfrak{l}_2)$, where $\mathfrak{l}_2$ is given by (39).*

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