INVOLUTIVE CATEGORIES, COLORED ∗-OPERADS AND QUANTUM FIELD THEORY

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ABSTRACT. Involutive category theory provides a flexible framework to describe involutive structures on algebraic objects, such as anti-linear involutions on complex vector spaces. Motivated by the prominent role of involutions in quantum (field) theory, we develop the involutive analogs of colored operads and their algebras, named colored ∗-operads and ∗-algebras. Central to the definition of colored ∗-operads is the involutive monoidal category of symmetric sequences, which we obtain from a general product-exponential 2-adjunction whose right adjoint forms involutive functor categories. For ∗-algebras over ∗-operads we obtain involutive analogs of the usual change of color and operad adjunctions. As an application, we turn the colored operads for algebraic quantum field theory into colored ∗-operads. The simplest instance is the associative ∗-operad, whose ∗-algebras are unital and associative ∗-algebras.

1. Introduction and summary

In ordinary category theory, an involution on an object $c \in \mathbf{C}$ of a category $\mathbf{C}$ is an endomorphism $i : c \to c$ that squares to the identity, i.e. $i^2 = \mathrm{id}_c$. Unfortunately, this concept is too rigid to describe many examples of interest. For instance, given an associative and unital ∗-algebra A over $\mathbb{C}$, e.g. the algebra of observables of a quantum system, the involution $* : A \to A$ on its underlying vector space is *not* an endomorphism in the category of complex vector spaces, but rather a complex *anti-linear* map.

Involutive categories [BM09, Egg11, Jac12] were developed in order to introduce the flexibility required to resolve this insufficiency. Their definition is a particular instance of the "microcosm principle" of Baez and Dolan [BD98], which states that *certain algebraic structures can be defined in any category equipped with a categorified version of the same structure.* Hence, an involutive category is a category $\mathbf{C}$ equipped with an endofunctor $J : \mathbf{C} \to \mathbf{C}$ that squares to the identity endofunctor $\mathrm{Id}_{\mathbf{C}}$, up to a given natural isomorphism $j : \mathrm{Id}_{\mathbf{C}} \to J^2$ which has to satisfy certain coherence conditions (cf. Definition 2.2). In an involutive category $(\mathbf{C}, J, j)$, one can introduce a more flexible concept of involution on an object $c \in \mathbf{C}$, which is given by a $\mathbf{C}$-morphism $* : c \to Jc$ satisfying $(J*) * = j_c$ as morphisms from c to $J^2 c$ (cf. Definition 2.16). Such objects (homotopy fixed points, as a matter of fact) are called self-conjugates in [Jac12], involutive objects in [Egg11] and

*-objects in [BM09]. We shall follow the latter terminology because it seems the most natural one to us. If a category is equipped with its trivial involutive structure $J = \mathrm{Id}_{\mathbf{C}}$ and $j = \mathrm{id}_{\mathrm{Id}_{\mathbf{C}}}$ (cf. Example 2.3), then *-objects are just endomorphisms squaring to the identity, i.e. the ordinary involutions mentioned above. This framework, however, becomes much richer and flexible by allowing for non-trivial involutive structures: For example, endowing the category of complex vector spaces $\mathbf{Vec}_{\mathbb{C}}$ with the involutive structure given by the endofunctor that assigns to a complex vector space V its complex conjugate vector space $\overline{V}$, the complex anti-linear map underlying a *-algebra may be regarded as a *-object $* : A \to \overline{A}$ in this involutive category (cf. Examples 2.4 and 2.19).

The observables of a quantum system form a unital and associative *-algebra over $\mathbb{C}$. This shows the relevance of involutive categories for general quantum theory, quantum field theory and also noncommutative geometry. Our main motivation for this paper stems precisely from these areas and more specifically from our recent operadic approach to algebraic quantum field theory [BSW17]. There the axioms of algebraic quantum field theory [HK64, BFV03] are encoded in a colored operad and generalized to richer target categories, such as chain complexes and other symmetric monoidal categories, which are central in modern approaches to quantum gauge theories [CG17, BSS15, BS17, BSW17, BSW18, Yau18]. For their physical interpretation, however, it is essential that quantum systems such as quantum field theories come equipped with involutions. These enable us to perform the GNS construction and recover the usual probabilistic interpretation of quantum theory. We refer to [Jac12] for a generalization of the GNS construction to involutive symmetric monoidal categories.

The purpose of this paper is to combine the theory of colored operads and that of involutive categories, resulting in what we shall call colored *-operads. Despite of our quite concrete motivation, we believe that working out the theory of colored *-operads in full generality provides an interesting and valuable addition to the largely unexplored field of involutive category theory. On the one hand, our constructions naturally lead to interesting new structures such as involutive functor categories, which have not been discussed in the literature. On the other hand, our study of involutive structures on the category of symmetric sequences, which is a monoidal category that does not admit a braiding, provides an interesting example of an involutive monoidal category in the sense of [Jac12], but not in the sense of [BM09, Egg11], see Remark 4.8 for details. This shows that Jacobs' definition of involutive monoidal categories is the one suitable to develop the theory of colored *-operads, consequently we shall use this one in our paper.

The outline of the paper is as follows: Sections 2 and 3 contain a brief review of involutive categories and involutive (symmetric) monoidal categories following mostly [Jac12]. We shall in particular emphasize and further develop the 2-categorical aspects of this theory, including the 2-functorial behavior of the assignments of the categories of *-objects and *-monoids. For the sake of concreteness, we also describe the most relevant constructions and definitions arising this way in fully explicit terms. Theorems 2.25 and 3.19 establish simple criteria that are useful to detect whether an involutive ((symmetric) monoidal) category is isomorphic to one with a trivial involutive structure. In Section

4 we show that the category of colored symmetric sequences, which underlies colored operad theory, carries a canonical involutive monoidal structure in the sense of [Jac12], but not in the sense of [BM09, Egg11]. The relevant involutive structure is obtained by employing a general construction, namely exponentiation of involutive categories, which results in involutive structures on functor categories. Colored ∗-operads with values in any cocomplete involutive closed symmetric monoidal category $(\mathbf{M}, J, j)$ are defined in Section 5 as ∗-monoids in our involutive monoidal category of colored symmetric sequences. In Proposition 5.4 we shall prove that the resulting category is isomorphic to the category of ordinary colored operads with values in the category of ∗-objects in $(\mathbf{M}, J, j)$, which provides an alternative point of view on colored ∗-operads. The possibility to switch between these equivalent perspectives is useful for concrete applications and also to import techniques from ordinary operad theory to the involutive setting. In Section 6 we introduce and study the category of ∗-algebras over colored ∗-operads. In particular, we prove that a change of colored ∗-operad induces an adjunction between the associated categories of ∗-algebras, which generalizes the corresponding crucial and widely used result from ordinary to involutive category theory. Finally, in Section 7 we endow the algebraic quantum field theory operads constructed in [BSW17] with a canonical order-reversing structure of colored ∗-operads and provide a characterization of the corresponding categories of ∗-algebras. As a simple example, we obtain a ∗-operad structure on the associative operad and show that its ∗-algebras behave like ∗-algebras over $\mathbb{C}$ in the sense that the involution reverses the order of multiplication $(a\,b)^* = b^*\,a^*$. It is essential to emphasize that this order-reversal is encoded in our ∗-operad structure. This is radically different from the approach of [BM09, Egg11], whose definition of an involutive monoidal category prescribes that the endofunctor J reverses the monoidal structure up to natural isomorphism, thus recovering unital and associative ∗-algebras over $\mathbb{C}$ directly as ∗-monoids in $\mathbf{Vec}_{\mathbb{C}}$.

Notations: We denote categories by boldface letters like $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$. Objects in categories are indicated by $c \in \mathbf{C}$ and we write $\mathbf{C}(c, c')$ for the set of morphisms from c to c' in $\mathbf{C}$. Functors are denoted by capital letters like $F : \mathbf{C} \to \mathbf{C}'$ or $X : \mathbf{D} \to \mathbf{C}$, and so are the identity functors $\mathrm{Id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$. Natural transformations are denoted by Greek letters like $\zeta : F \to G$ or $\alpha : X \to Y$. Given functors $K : \mathbf{D}' \to \mathbf{D}$, $X : \mathbf{D} \to \mathbf{C}$ and $J : \mathbf{C} \to \mathbf{C}'$, we denote their composition simply by juxtaposition $JXK : \mathbf{D}' \to \mathbf{C}'$. Given also a natural transformation $\alpha : X \to Y$ of functors $X, Y : \mathbf{D} \to \mathbf{C}$, we denote by

$$J\alpha K \; : \; JXK \; \longrightarrow \; JYK$$

the *whiskering* of J, α and K. Explicitly, $J\alpha K$ is the natural transformation with components

$$(J\alpha K)_{d'} = J\alpha_{Kd'} \; : \; JXKd' \; \longrightarrow \; JYKd' \quad ,$$

for all $d' \in \mathbf{D}'$. For $\beta : Y \to Z$ another natural transformation, one easily confirms that

$$(J\beta K)\,(J\alpha K) \; = \; J\big(\beta\alpha\big)K \; : \; JXK \; \longrightarrow \; JZK \quad ,$$

where (vertical) composition of natural transformations is also denoted by juxtaposition. We shall need some basic elements of (strict) 2-category theory, for which we refer to [KS74].

2. Involutive categories

This section contains a brief review of involutive categories. We shall mostly follow the definitions and conventions of Jacobs [Jac12] and refer to this paper for more details and some of the proofs. We strongly emphasize and also develop further the 2-categorical aspects of involutive category theory established in [Jac12], which will be relevant for the development of our present paper. When it comes to notations and terminology, we sometimes prefer the work of Beggs and Majid [BM09] and the one of Egger [Egg11].

2.1. Basic definitions and properties.

2.2. Definition. An *involutive category* is a triple $(\mathbf{C}, J, j)$ consisting of a category $\mathbf{C}$, an endofunctor $J : \mathbf{C} \to \mathbf{C}$ and a natural isomorphism $j : \mathrm{Id}_{\mathbf{C}} \to J^2$ satisfying

$$jJ = Jj : J \longrightarrow J^3 \quad .$$

2.3. Example. For any category $\mathbf{C}$, the triple $(\mathbf{C}, \mathrm{Id}_{\mathbf{C}}, \mathrm{id}_{\mathrm{Id}_{\mathbf{C}}})$ defines an involutive category. We call this the *trivial involutive category* over $\mathbf{C}$.

2.4. Example. Let $\mathbf{Vec}_{\mathbb{C}}$ be the category of complex vector spaces. Consider the endofunctor $\overline{(-)} : \mathbf{Vec}_{\mathbb{C}} \to \mathbf{Vec}_{\mathbb{C}}$ that assigns to any $V \in \mathbf{Vec}_{\mathbb{C}}$ its complex conjugate vector space $\overline{V} \in \mathbf{Vec}_{\mathbb{C}}$ and to any $\mathbb{C}$-linear map $f : V \to W$ the canonically induced $\mathbb{C}$-linear map $\overline{f} : \overline{V} \to \overline{W}$. Notice that $\overline{\overline{(-)}} = \mathrm{Id}_{\mathbf{Vec}_{\mathbb{C}}}$, hence $(\mathbf{Vec}_{\mathbb{C}}, \overline{(-)}, \mathrm{id}_{\mathrm{Id}_{\mathbf{Vec}_{\mathbb{C}}}})$ is an involutive category.

2.5. Example. Let $\mathfrak{C}$ be any non-empty set and $\Sigma_{\mathfrak{C}}$ the associated *groupoid of $\mathfrak{C}$-profiles*. The objects of $\Sigma_{\mathfrak{C}}$ are finite sequences $\underline{c} = (c_1, \ldots, c_n)$ of elements in $\mathfrak{C}$, including also the empty sequence $\emptyset \in \Sigma_{\mathfrak{C}}$. We denote by $|\underline{c}| = n$ the length of the sequence. The morphisms of $\Sigma_{\mathfrak{C}}$ are right permutations $\sigma : \underline{c} \to \underline{c}\sigma := (c_{\sigma(1)}, \ldots, c_{\sigma(n)})$, with $\sigma \in \Sigma_{|\underline{c}|}$ in the symmetric group on $|\underline{c}|$ letters. We define an endofunctor $\mathrm{Rev} : \Sigma_{\mathfrak{C}} \to \Sigma_{\mathfrak{C}}$ as follows: To an object $\underline{c} = (c_1, \ldots, c_n) \in \Sigma_{\mathfrak{C}}$ it assigns the reversed sequence

$$\mathrm{Rev}(\underline{c}) := \underline{c}\,\rho_{|\underline{c}|} := (c_n, \ldots, c_1) \quad ,$$

where $\rho_{|\underline{c}|} \in \Sigma_{|\underline{c}|}$ denotes the order-reversal permutation. To a $\Sigma_{\mathfrak{C}}$-morphism $\sigma : \underline{c} \to \underline{c}\sigma$ it assigns the right permutation

$$\mathrm{Rev}(\sigma) := \rho_{|\underline{c}|}\,\sigma\,\rho_{|\underline{c}|} : \mathrm{Rev}(\underline{c}) \longrightarrow \mathrm{Rev}(\underline{c}\sigma) \quad ,$$

where we also used that $|\underline{c}\sigma| = |\underline{c}|$. Notice that $\mathrm{Rev}^2 = \mathrm{Id}_{\Sigma_{\mathfrak{C}}}$, hence $(\Sigma_{\mathfrak{C}}, \mathrm{Rev}, \mathrm{id}_{\mathrm{Id}_{\Sigma_{\mathfrak{C}}}})$ is an involutive category.

The following very useful result appears in [Jac12, Lemma 1].

2.6. LEMMA. *For every involutive category* $(\mathbf{C}, J, j)$, *the endofunctor* $J : \mathbf{C} \to \mathbf{C}$ *is self-adjoint, i.e.* $J \dashv J$. *As a consequence,* J *preserves all limits and colimits that exist in* $\mathbf{C}$.

2.7. DEFINITION. An *involutive functor* $(F, \nu) : (\mathbf{C}, J, j) \to (\mathbf{C}', J', j')$ consists of a functor $F : \mathbf{C} \to \mathbf{C}'$ and a natural transformation $\nu : FJ \to J'F$ satisfying

$$
\begin{array}{ccc}
F & \!\!\!=\!\!\!=\!\!\!=\!\!\!=\!\!\!=\!\!\!=\!\!\! & F \\
{\scriptstyle Fj}\downarrow & & \downarrow{\scriptstyle j'F} \\
FJ^2 & \xrightarrow[\nu J]{} J'FJ \xrightarrow[J'\nu]{} & J'^2 F
\end{array}
\tag{1}
$$

An *involutive natural transformation* $\zeta : (F, \nu) \to (G, \chi)$ between involutive functors $(F, \nu), (G, \chi) : (\mathbf{C}, J, j) \to (\mathbf{C}', J', j')$ is a natural transformation $\zeta : F \to G$ satisfying

$$
\begin{array}{ccc}
FJ & \xrightarrow{\ \zeta J\ } & GJ \\
{\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \chi} \\
J'F & \xrightarrow[\ J'\zeta\ ]{} & J'G
\end{array}
$$

2.8. PROPOSITION. *Involutive categories, involutive functors and involutive natural transformations form a 2-category* $\mathbf{ICat}$.

2.9. REMARK. Let us describe the 2-category structure on $\mathbf{ICat}$ explicitly.

(i) For any involutive category $(\mathbf{C}, J, j)$, the identity involutive functor is given by $\mathrm{Id}_{(\mathbf{C}, J, j)} := (\mathrm{Id}_{\mathbf{C}}, \mathrm{id}_J) : (\mathbf{C}, J, j) \to (\mathbf{C}, J, j)$.

(ii) For two involutive functors $(F, \nu) : (\mathbf{C}, J, j) \to (\mathbf{C}', J', j')$ and $(F', \nu') : (\mathbf{C}', J', j') \to (\mathbf{C}'', J'', j'')$, their composition is given by

$$
(F', \nu')\,(F, \nu) := \big(F'F, (\nu'F)\,(F'\nu)\big) : (\mathbf{C}, J, j) \ \longrightarrow \ (\mathbf{C}'', J'', j'') \quad .
$$

(iii) Vertical/horizontal composition of involutive natural transformations is given by vertical/horizontal composition of their underlying natural transformations. (It is easy to verify that the latter compositions define involutive natural transformations.)

The following technical lemma is proven in [Jac12, Lemma 2].

2.10. LEMMA. *For every involutive functor* $(F, \nu) : (\mathbf{C}, J, j) \to (\mathbf{C}', J', j')$, *the natural transformation* $\nu : FJ \to J'F$ *is a natural isomorphism.*

As in any 2-category, there exists the concept of adjunctions in the 2-category $\mathbf{ICat}$.

2.11. DEFINITION. An *involutive adjunction*

$$(L, \lambda) : (\mathbf{C}, J, j) \rightleftarrows (\mathbf{D}, K, k) : (R, \rho)$$

consists of two involutive functors $(L, \lambda) : (\mathbf{C}, J, j) \to (\mathbf{D}, K, k)$ and $(R, \rho) : (\mathbf{D}, K, k) \to (\mathbf{C}, J, j)$ together with two involutive natural transformations $\eta : \mathrm{Id}_{(\mathbf{C}, J, j)} \to (R, \rho)(L, \lambda)$ (called *unit*) and $\epsilon : (L, \lambda)(R, \rho) \to \mathrm{Id}_{(\mathbf{D}, K, k)}$ (called *counit*) that satisfy the triangle identities

$$(R, \rho) \xrightarrow{\eta (R,\rho)} (R, \rho)(L, \lambda)(R, \rho) \qquad (L, \lambda) \xrightarrow{(L,\lambda) \eta} (L, \lambda)(R, \rho)(L, \lambda)$$

with $\mathrm{id}_{(R,\rho)}$, $(R,\rho)\epsilon$, (R,ρ) and $\mathrm{id}_{(L,\lambda)}$, $\epsilon(L,\lambda)$, (L,λ)

We also denote involutive adjunctions simply by $(L, \lambda) \dashv (R, \rho)$.

2.12. REMARK. Applying the forgetful 2-functor $\mathbf{ICat} \to \mathbf{Cat}$, every involutive adjunction $(L, \lambda) \dashv (R, \rho)$ defines an ordinary adjunction $L \dashv R$ in the 2-category of categories $\mathbf{Cat}$. Notice that an involutive adjunction is the same thing as an ordinary adjunction $L \dashv R$ (between categories equipped with an involutive structure) whose functors L and R are equipped with involutive structures that are compatible with the unit and counit in the sense that the latter become of involutive natural transformations. This alternative point of view will be useful in Corollary 4.9 and Theorem 6.6 below, where we make use of the construction in the following proposition.

2.13. PROPOSITION. *Let $(R, \rho) : (\mathbf{D}, K, k) \to (\mathbf{C}, J, j)$ be an involutive functor and suppose that $L : \mathbf{C} \to \mathbf{D}$ is a left adjoint to the functor $R : \mathbf{D} \to \mathbf{C}$. Define a natural transformation λ by*

$$
\begin{array}{ccc}
LJ & \xrightarrow{\quad \lambda \quad} & KL \\
{\scriptstyle LJ\eta}\downarrow & & \uparrow{\scriptstyle \epsilon KL} \\
LJRL & \xrightarrow{\quad L\rho^{-1}L \quad} & LRKL
\end{array}
$$

where $\eta : \mathrm{Id}_{\mathbf{C}} \to RL$ and $\epsilon : LR \to \mathrm{Id}_{\mathbf{D}}$ are the unit and counit of the adjunction $L \dashv R$. Then $(L, \lambda) \dashv (R, \rho)$ is an involutive adjunction.

PROOF. The above diagram defines a natural transformation λ because ρ is a natural isomorphism, cf. Lemma 2.10. A slightly lengthy diagram chase shows that $(L, \lambda) : (\mathbf{C}, J, j) \to (\mathbf{D}, K, k)$ is an involutive functor. Furthermore, by the definition of λ, the natural transformations η and ϵ are involutive natural transformations. ∎

2.14. REMARK. Even though we will not need it in the following, let us briefly mention that the dual of Proposition 2.13 also holds true: Let $(L, \lambda) : (\mathbf{C}, J, j) \to (\mathbf{D}, K, k)$ be an involutive functor and suppose that $R : \mathbf{D} \to \mathbf{C}$ is a right adjoint to the functor $L : \mathbf{C} \to \mathbf{D}$. Then $(L, \lambda) \dashv (R, \rho)$ is an involutive adjunction for ρ defined by

$$
\begin{array}{ccc}
JR & \xrightarrow{\rho^{-1}} & RK \\
{\scriptstyle \eta JR} \downarrow & & \uparrow {\scriptstyle RK\epsilon} \\
RLJR & \xrightarrow{R\lambda R} & RKLR
\end{array}
$$

where $\eta : \mathrm{Id}_{\mathbf{C}} \to RL$ and $\epsilon : LR \to \mathrm{Id}_{\mathbf{D}}$ are the unit and counit of the adjunction $L \dashv R$.

2.15. ∗-OBJECTS.

2.16. DEFINITION. A ∗-*object* in an involutive category $(\mathbf{C}, J, j)$ is a $\mathbf{C}$-morphism $* : c \to Jc$ satisfying

$$
\begin{array}{ccc}
c & \xrightarrow{\ *\ } & Jc \\
 & {\scriptstyle j_c} \searrow & \downarrow {\scriptstyle J*} \\
 & & J^2c
\end{array}
$$

A ∗-*morphism* $f : (* : c \to Jc) \to (*' : c' \to Jc')$ is a $\mathbf{C}$-morphism $f : c \to c'$ satisfying

$$
\begin{array}{ccc}
c & \xrightarrow{\ f\ } & c' \\
{\scriptstyle *} \downarrow & & \downarrow {\scriptstyle *'} \\
Jc & \xrightarrow{\ Jf\ } & Jc'
\end{array}
$$

We denote the category of ∗-objects in $(\mathbf{C}, J, j)$ by ∗-$\mathbf{Obj}(\mathbf{C}, J, j)$.

2.17. REMARK. For any ∗-object $(* : c \to Jc) \in $ ∗-$\mathbf{Obj}(\mathbf{C}, J, j)$, the $\mathbf{C}$-morphism $* : c \to Jc$ is an isomorphism with inverse given by $j_c^{-1} J* : Jc \to c$.

2.18. EXAMPLE. Consider the trivial involutive category $(\mathbf{C}, \mathrm{Id}_{\mathbf{C}}, \mathrm{id}_{\mathrm{Id}_{\mathbf{C}}})$ from Example 2.3. A ∗-object consists of an object $c \in \mathbf{C}$ equipped with a $\mathbf{C}$-endomorphism $* : c \to c$ satisfying $*^2 = \mathrm{id}_c$, i.e. an object equipped with an involution.

2.19. EXAMPLE. Consider the involutive category $(\mathbf{Vec}_{\mathbb{C}}, \overline{(-)}, \mathrm{id}_{\mathrm{Id}_{\mathbf{Vec}_{\mathbb{C}}}})$ from Example 2.4. A ∗-object consists of a complex vector space V equipped with a complex *anti-linear* map $* : V \to V$ satisfying $*^2 = \mathrm{id}_V$.

2.20. EXAMPLE. Consider the involutive category $(\Sigma_{\mathfrak{C}}, \mathrm{Rev}, \mathrm{id}_{\mathrm{Id}_{\Sigma_{\mathfrak{C}}}})$ from Example 2.5. A $*$-object consists of a $\mathfrak{C}$-profile $\underline{c} = (c_1, \ldots, c_n)$ equipped with a right permutation $* : \underline{c} \to \mathrm{Rev}(\underline{c}) = \underline{c}\,\rho_{|\underline{c}|}$ satisfying $*\rho_{|\underline{c}|}*\rho_{|\underline{c}|} = e \in \Sigma_{|\underline{c}|}$, where e denotes the identity permutation. In particular, any object $\underline{c} \in \Sigma_{\mathfrak{C}}$ carries a canonical $*$-object structure given by $\rho_{|\underline{c}|} : \underline{c} \to \underline{c}\,\rho_{|\underline{c}|}$. The assignment $\underline{c} \mapsto (\rho_{|\underline{c}|} : \underline{c} \to \underline{c}\,\rho_{|\underline{c}|})$ defines a functor $\rho : \Sigma_{\mathfrak{C}} \to *\text{-}\mathbf{Obj}(\Sigma_{\mathfrak{C}}, \mathrm{Rev}, \mathrm{id}_{\mathrm{Id}_{\Sigma_{\mathfrak{C}}}})$ that is a section of the forgetful functor $U : *\text{-}\mathbf{Obj}(\Sigma_{\mathfrak{C}}, \mathrm{Rev}, \mathrm{id}_{\mathrm{Id}_{\Sigma_{\mathfrak{C}}}}) \to \Sigma_{\mathfrak{C}}$.

For any $(\mathbf{C}, J, j)$, there exists a forgetful functor $U : *\text{-}\mathbf{Obj}(\mathbf{C}, J, j) \to \mathbf{C}$ specified by $(* : c \to Jc) \mapsto c$. If the category $\mathbf{C}$ has coproducts, we can define for any object $c \in \mathbf{C}$ a morphism

$$F(c) := \left(c \sqcup Jc \cong Jc \sqcup c \xrightarrow{\mathrm{id} \sqcup j_c} Jc \sqcup J^2 c \cong J(c \sqcup Jc) \right) \tag{2}$$

in $\mathbf{C}$, where in the last step we used that J preserves coproducts because of Lemma 2.6. One can easily check that (2) defines a $*$-object in $(\mathbf{C}, J, j)$, i.e. $F(c) \in *\text{-}\mathbf{Obj}(\mathbf{C}, J, j)$. Another direct computation shows

2.21. PROPOSITION. *Let $(\mathbf{C}, J, j)$ be an involutive category that admits coproducts. The assignment $c \mapsto F(c)$ given by (2) naturally extends to a functor $F : \mathbf{C} \to *\text{-}\mathbf{Obj}(\mathbf{C}, J, j)$, which is a left adjoint of the forgetful functor $U : *\text{-}\mathbf{Obj}(\mathbf{C}, J, j) \to \mathbf{C}$.*

2.22. REMARK. [Jac12, Lemma 5] shows that $*\text{-}\mathbf{Obj}(\mathbf{C}, J, j)$ inherits all limits and colimits that exist in $\mathbf{C}$. These are preserved by the forgetful functor $U : *\text{-}\mathbf{Obj}(\mathbf{C}, J, j) \to \mathbf{C}$.

As noted in [Jac12, Lemma 6], the assignment of the categories of $*$-objects extends to a 2-functor

$$*\text{-}\mathbf{Obj} : \mathbf{ICat} \longrightarrow \mathbf{Cat} \quad . \tag{3}$$

Concretely, this 2-functor is given by the following assignment:

- an involutive category $(\mathbf{C}, J, j)$ is mapped to its category of $*$-objects $*\text{-}\mathbf{Obj}(\mathbf{C}, J, j)$;

- an involutive functor $(F, \nu) : (\mathbf{C}, J, j) \to (\mathbf{C}', J', j')$ is mapped to the functor $*\text{-}\mathbf{Obj}(F, \nu) : *\text{-}\mathbf{Obj}(\mathbf{C}, J, j) \to *\text{-}\mathbf{Obj}(\mathbf{C}', J', j')$ that acts on objects as

$$*\text{-}\mathbf{Obj}(F, \nu)\big(* : c \to Jc\big) := \left(Fc \xrightarrow{F*} FJc \xrightarrow{\nu_c} J'Fc \right)$$

  and on morphisms as F;

- an involutive natural transformation $\zeta : (F, \nu) \to (G, \chi)$ is mapped to the natural transformation $*\text{-}\mathbf{Obj}(\zeta) : *\text{-}\mathbf{Obj}(F, \nu) \to *\text{-}\mathbf{Obj}(G, \chi)$ whose components are $*\text{-}\mathbf{Obj}(\zeta)_{(*:c \to Jc)} := \zeta_c$, for all $(* : c \to Jc) \in *\text{-}\mathbf{Obj}(\mathbf{C}, J, j)$.

Recalling the trivial involutive categories from Example 2.3, we obtain another 2-functor

$$\mathrm{triv} : \mathbf{Cat} \longrightarrow \mathbf{ICat} \quad . \tag{4}$$

This 2-functor assigns to a category $\mathbf{C}$ the trivial involutive category $(\mathbf{C}, \mathrm{Id}_{\mathbf{C}}, \mathrm{id}_{\mathrm{Id}_{\mathbf{C}}})$, to a functor $F : \mathbf{C} \to \mathbf{C}'$ the involutive functor $(F, \mathrm{id}_F) : (\mathbf{C}, \mathrm{Id}_{\mathbf{C}}, \mathrm{id}_{\mathrm{Id}_{\mathbf{C}}}) \to (\mathbf{C}', \mathrm{Id}_{\mathbf{C}'}, \mathrm{id}_{\mathrm{Id}_{\mathbf{C}'}})$, and to a natural transformation $\zeta : F \to G$ the involutive natural transformation $\zeta : (F, \mathrm{id}_F) \to (G, \mathrm{id}_G)$.

2.23. THEOREM. *The 2-functors* (3) *and* (4) *form a 2-adjunction*

$$\mathrm{triv} : \mathbf{Cat} \; \rightleftarrows \; \mathbf{ICat} : *\text{-}\mathbf{Obj} \quad . \tag{5}$$

The unit $\eta : \mathrm{Id}_{\mathbf{Cat}} \to *\text{-}\mathbf{Obj}\,\mathrm{triv}$ *and counit* $\epsilon : \mathrm{triv}\,*\text{-}\mathbf{Obj} \to \mathrm{Id}_{\mathbf{ICat}}$ *2-natural transformations are stated explicitly in the proof below.*

PROOF. The component at $\mathbf{C} \in \mathbf{Cat}$ of the 2-natural transformation η is the functor

$$\eta_{\mathbf{C}} : \mathbf{C} \; \longrightarrow \; *\text{-}\mathbf{Obj}\big(\mathrm{triv}(\mathbf{C})\big)$$

that equips objects with their identity involution (cf. Example 2.18), i.e. $c \mapsto (\mathrm{id}_c : c \to c)$. The component at $(\mathbf{C}, J, j) \in \mathbf{ICat}$ of the 2-natural transformation ϵ is the involutive functor

$$\epsilon_{(\mathbf{C},J,j)} = (U, \nu) : \mathrm{triv}\big(*\text{-}\mathbf{Obj}(\mathbf{C}, J, j)\big) \; \longrightarrow \; (\mathbf{C}, J, j) \quad ,$$

where $U : *\text{-}\mathbf{Obj}(\mathbf{C}, J, j) \to \mathbf{C}$ is the forgetful functor $(* : c \to Jc) \mapsto c$ and its involutive structure $\nu : U \to JU$ is the natural transformation defined by the components $\nu_{(*:c \to Jc)} = * : c \to Jc$, for all $(* : c \to Jc) \in *\text{-}\mathbf{Obj}(\mathbf{C}, J, j)$. An elementary check shows that η and ϵ are indeed 2-natural transformations that satisfy the triangle identities, hence (5) is a 2-adjunction with unit η and counit ϵ. ∎

2.24. REMARK. Notice that both $\mathbf{Cat}$ and $\mathbf{ICat}$ carry a Cartesian monoidal structure, which is concretely given by the product categories $\mathbf{C} \times \mathbf{D}$ in $\mathbf{Cat}$ and the product involutive categories $(\mathbf{C}, J, j) \times (\mathbf{D}, K, k) = (\mathbf{C} \times \mathbf{D}, J \times K, j \times k)$ in $\mathbf{ICat}$. Because $*\text{-}\mathbf{Obj}$ is a right adjoint functor, it follows that there are canonical isomorphisms

$$*\text{-}\mathbf{Obj}\big((\mathbf{C}, J, j) \times (\mathbf{D}, K, k)\big) \; \cong \; *\text{-}\mathbf{Obj}(\mathbf{C}, J, j) \times *\text{-}\mathbf{Obj}(\mathbf{D}, K, k) \quad ,$$

for all involutive categories $(\mathbf{C}, J, j)$ and $(\mathbf{D}, K, k)$.

We conclude this section with a useful result that allows us to detect involutive categories carrying a trivial involutive structure.

2.25. THEOREM. *Let* $(\mathbf{C}, J, j)$ *be an involutive category. A section* $* : \mathbf{C} \to *\text{-}\mathbf{Obj}(\mathbf{C}, J, j)$ *of the forgetful functor* $U : *\text{-}\mathbf{Obj}(\mathbf{C}, J, j) \to \mathbf{C}$ *canonically determines an isomorphism between* $(\mathbf{C}, J, j)$ *and the trivial involutive category* $(\mathbf{C}, \mathrm{Id}_{\mathbf{C}}, \mathrm{id}_{\mathrm{Id}_{\mathbf{C}}})$. *In particular, if a section of* U *exists, then the involutive categories* $(\mathbf{C}, J, j)$ *and* $(\mathbf{C}, \mathrm{Id}_{\mathbf{C}}, \mathrm{id}_{\mathrm{Id}_{\mathbf{C}}})$ *are isomorphic.*

PROOF. A section $* : \mathbf{C} \to *\text{-}\mathbf{Obj}(\mathbf{C}, J, j)$ of U assigns to each $c \in \mathbf{C}$ a $*$-object $*_c : c \to Jc$ and to each $\mathbf{C}$-morphism $f : c \to c'$ a $*$-morphism

$$
\begin{array}{ccc}
c & \xrightarrow{\ f\ } & c' \\
{\scriptstyle *_c}\downarrow & & \downarrow{\scriptstyle *_{c'}} \\
Jc & \xrightarrow{\ Jf\ } & Jc'
\end{array}
$$

Notice that this diagram implies that $*_c$ are the components of a natural transformation $* : \mathrm{Id}_{\mathbf{C}} \to J$. It is straightforward to check that $(\mathrm{Id}_{\mathbf{C}}, *) : (\mathbf{C}, \mathrm{Id}_{\mathbf{C}}, \mathrm{id}_{\mathrm{Id}_{\mathbf{C}}}) \to (\mathbf{C}, J, j)$ is an involutive functor, which is invertible via the involutive functor $(\mathrm{Id}_{\mathbf{C}}, *^{-1}) : (\mathbf{C}, J, j) \to (\mathbf{C}, \mathrm{Id}_{\mathbf{C}}, \mathrm{id}_{\mathrm{Id}_{\mathbf{C}}})$. ∎

2.26. COROLLARY. *The involutive category* $(\Sigma_{\mathfrak{C}}, \mathrm{Rev}, \mathrm{id}_{\mathrm{Id}_{\Sigma_{\mathfrak{C}}}})$ *of* $\mathfrak{C}$-*profiles equipped with reversal as involutive structure (cf. Examples 2.5 and 2.20) is isomorphic to the trivial involutive category* $(\Sigma_{\mathfrak{C}}, \mathrm{Id}_{\Sigma_{\mathfrak{C}}}, \mathrm{id}_{\mathrm{Id}_{\Sigma_{\mathfrak{C}}}})$.

3. Involutive structures on monoidal categories

In this section we review involutive (symmetric) monoidal categories and $*$-monoids therein. We again shall follow mostly the definitions and conventions of Jacobs [Jac12]. Our main goal is to clarify and work out the 2-functorial behavior of the assignment of the categories of $*$-objects and monoids to involutive (symmetric) monoidal categories. To fix our notations, we start with a brief review of some basic aspects of (symmetric) monoidal categories and monoids therein.

3.1. (SYMMETRIC) MONOIDAL CATEGORIES AND MONOIDS. Recall that a *monoidal category* $(\mathbf{C}, \otimes, I, \alpha, \lambda, \rho)$ consists of a category $\mathbf{C}$, a functor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, an object $I \in \mathbf{C}$ and three natural isomorphisms

$$
\begin{aligned}
\alpha \ : \ \otimes (\otimes \times \mathrm{Id}_{\mathbf{C}}) \ &\longrightarrow \ \otimes (\mathrm{Id}_{\mathbf{C}} \times \otimes) \quad , \\
\lambda \ : \ I \otimes (-) \ &\longrightarrow \ \mathrm{Id}_{\mathbf{C}} \quad , \\
\rho \ : \ (-) \otimes I \ &\longrightarrow \ \mathrm{Id}_{\mathbf{C}} \quad ,
\end{aligned}
$$

which satisfy the pentagon and triangle identities. We follow the usual abuse of notation and often denote a monoidal category by its underlying category $\mathbf{C}$. The associator α and the unitors λ and ρ will always be suppressed. Given two monoidal categories $\mathbf{C}$ and $\mathbf{C}'$, a *(lax) monoidal functor* from $\mathbf{C}$ to $\mathbf{C}'$ is a triple (F, F_2, F_0) consisting of a functor $F : \mathbf{C} \to \mathbf{C}'$, a natural transformation

$$
F_2 \ : \ \otimes' (F \times F) \ \longrightarrow \ F \otimes \quad ,
$$

and a $\mathbf{C}'$-morphism

$$
F_0 \ : \ I' \ \longrightarrow \ FI \quad ,
$$

which are required to satisfy the usual coherence conditions involving the associators and unitors. We often denote a monoidal functor by its underlying functor $F : \mathbf{C} \to \mathbf{C}'$. A *monoidal natural transformation* $\zeta : F \to G$ between monoidal functors $F = (F, F_2, F_0)$ and $G = (G, G_2, G_0)$ is a natural transformation $\zeta : F \to G$ satisfying

$$
\begin{array}{ccc}
\otimes'\,(F \times F) & \xrightarrow{\;\otimes'\,(\zeta \times \zeta)\;} & \otimes'\,(G \times G) \\
{\scriptstyle F_2}\downarrow & & \downarrow{\scriptstyle G_2} \\
F \otimes & \xrightarrow[\;\zeta\,\otimes\;]{} & G \otimes
\end{array}
\qquad
\begin{array}{ccc}
 & I' & \\
{\scriptstyle F_0}\swarrow & & \searrow{\scriptstyle G_0} \\
FI & \xrightarrow[\;\zeta_I\;]{} & GI
\end{array}
$$

3.2. PROPOSITION. *Monoidal categories, (lax) monoidal functors and monoidal natural transformations form a 2-category* **MCat**.

A *symmetric monoidal category* is a monoidal category $\mathbf{C}$ together with a natural isomorphism called *braiding*

$$
\tau : \otimes \;\longrightarrow\; \otimes^{\mathrm{op}} := \otimes\,\sigma
$$

from the tensor product to the opposite tensor product, where $\sigma : \mathbf{C} \times \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ is the flip functor $(c_1, c_2) \mapsto (c_2, c_1)$, which satisfies the hexagon identities and the symmetry constraint

$$
\begin{array}{ccc}
\otimes & \xrightarrow{\;\mathrm{id}_\otimes\;} & \otimes = \otimes\,\sigma^2 \\
{\scriptstyle \tau}\searrow & & \nearrow{\scriptstyle \tau\,\sigma} \\
 & \otimes\,\sigma &
\end{array}
$$

We often denote a symmetric monoidal category by its underling category $\mathbf{C}$. A *symmetric monoidal functor* is a monoidal functor $F : \mathbf{C} \to \mathbf{C}'$ that preserves the braidings, i.e.

$$
\begin{array}{ccc}
\otimes'\,(F \times F) & \xrightarrow{\;\tau'\,(F \times F)\;} & \otimes'\,\sigma(F \times F) = \otimes'\,(F \times F)\sigma \\
{\scriptstyle F_2}\downarrow & & \downarrow{\scriptstyle F_2\sigma} \\
F \otimes & \xrightarrow[\;F\tau\;]{} & F \otimes \sigma
\end{array}
$$

commutes. A *symmetric monoidal natural transformation* is just a monoidal natural transformation between symmetric monoidal functors.

3.3. PROPOSITION. *Symmetric monoidal categories, symmetric monoidal functors and symmetric monoidal natural transformations form a 2-category* **SMCat**.

3.4. DEFINITION. A *monoid* in a (symmetric) monoidal category $\mathbf{C}$ is a triple (M, μ, η) consisting of an object $M \in \mathbf{C}$ and two $\mathbf{C}$-morphisms $\mu : M \otimes M \to M$ (called *multiplication*) and $\eta : I \to M$ (called *unit*) satisfying the associativity and unitality axioms. A

monoid morphism $f : (M, \mu, \eta) \to (M', \mu', \eta')$ is a $\mathbf{C}$-morphism $f : M \to M'$ preserving multiplications and units. We denote the category of monoids in $\mathbf{C}$ by $\mathbf{Mon}(\mathbf{C})$.

The assignment of the categories of monoids extends to a 2-functor

$$\mathbf{Mon} : (\mathbf{S})\mathbf{MCat} \longrightarrow \mathbf{Cat} \quad . \tag{6}$$

Concretely, this 2-functor is given by the following assignment:

- a (symmetric) monoidal category $\mathbf{C}$ is mapped to its category of monoids $\mathbf{Mon}(\mathbf{C})$;

- a (symmetric) monoidal functor $F : \mathbf{C} \to \mathbf{C}'$ is mapped to the functor $\mathbf{Mon}(F) : \mathbf{Mon}(\mathbf{C}) \to \mathbf{Mon}(\mathbf{C}')$ that acts on objects as

$$\mathbf{Mon}(F)\big(M, \mu, \eta\big) :=$$
$$\Big(FM, \ FM \otimes' FM \xrightarrow{F_{2M,M}} F(M \otimes M) \xrightarrow{F\mu} FM, \ I' \xrightarrow{F_0} FI \xrightarrow{F\eta} FM\Big)$$

  and on morphisms as F;

- a (symmetric) monoidal natural transformation $\zeta : F \to G$ is mapped to the natural transformation $\mathbf{Mon}(\zeta) : \mathbf{Mon}(F) \to \mathbf{Mon}(G)$ with components $\mathbf{Mon}(\zeta)_{(M,\mu,\eta)} := \zeta_M$, for all $(M, \mu, \eta) \in \mathbf{Mon}(\mathbf{C})$.

3.5. INVOLUTIVE (SYMMETRIC) MONOIDAL CATEGORIES. The following definition of an involutive (symmetric) monoidal category is due to [Jac12]. We prefer this definition over the one in [Egg11, BM09] as it has the advantage that the category of $*$-objects inherits a monoidal structure (cf. [Jac12, Proposition 1] and Proposition 3.17 in the present paper). This has interesting consequences for the theory of involutive monads in [Jac12] and the developments in our present paper.

3.6. DEFINITION. An *involutive (symmetric) monoidal category* is a triple $(\mathbf{C}, J, j)$ consisting of a (symmetric) monoidal category $\mathbf{C}$, a (symmetric) monoidal endofunctor $J = (J, J_2, J_0) : \mathbf{C} \to \mathbf{C}$ and a (symmetric) monoidal natural isomorphism $j : \mathrm{Id}_{\mathbf{C}} \to J^2$ satisfying

$$jJ = Jj : J \longrightarrow J^3 \quad .$$

The following statement is proven in [Jac12, Lemma 7].

3.7. LEMMA. *For any involutive (symmetric) monoidal category* $(\mathbf{C}, J, j)$, *the (symmetric) monoidal endofunctor* $J = (J, J_2, J_0) : \mathbf{C} \to \mathbf{C}$ *is* strong, *i.e.* $J_2 : \otimes (J \times J) \to J \otimes$ *and* $J_0 : I \to JI$ *are isomorphisms.*

3.8. REMARK. Let us emphasize again and more clearly that our Definition 3.6 of involutive (symmetric) monoidal categories agrees with the one of Jacobs [Jac12]. The definitions in [BM09] and [Egg11] are different because their analog of J_2 is order-reversing, i.e. a natural isomorphism $\otimes^{\mathrm{op}} (J \times J) \to J \otimes$. The reason why we consider order-preserving J_2 as in [Jac12] is that this is better suited for our development of involutive operad theory, cf. Remark 4.8 below.

3.9. REMARK. The condition for $j : \mathrm{Id}_{\mathbf{C}} \to J^2$ to be a (symmetric) monoidal natural transformation explicitly means that the diagrams

$$
\begin{array}{ccc}
\otimes & \xrightarrow{\;\otimes\,(j \times j)\;} & \otimes(J^2 \times J^2) \\[4pt]
{\scriptstyle \mathrm{id}_\otimes}\Big\downarrow & & \Big\downarrow{\scriptstyle J_2(J \times J)} \\[8pt]
 & & J\otimes(J \times J) \\[4pt]
 & & \Big\downarrow{\scriptstyle J\,J_2} \\[8pt]
\otimes & \xrightarrow[\;j\,\otimes\;]{} & J^2\otimes
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & I & \\
{\scriptstyle \mathrm{id}_I}\swarrow & & \searrow{\scriptstyle J_0} \\
I & & JI \\
{\scriptstyle \mathrm{id}_I}\Big\downarrow & & \Big\downarrow{\scriptstyle J J_0} \\
I & \xrightarrow[\;j_I\;]{} & J^2 I
\end{array}
$$

commute. One may reinterpret these diagrams as follows: The left diagram states that $(\otimes, J_2) : (\mathbf{C}, J, j) \times (\mathbf{C}, J, j) \to (\mathbf{C}, J, j)$ is an involutive functor on the product involutive category $(\mathbf{C}, J, j) \times (\mathbf{C}, J, j) = (\mathbf{C} \times \mathbf{C}, J \times J, j \times j)$, see also Remark 2.24. The right diagram states that $(J_0 : I \to JI) \in *\text{-}\mathbf{Obj}(\mathbf{C}, J, j)$ is a $*$-object in $(\mathbf{C}, J, j)$. These two structures allow us to endow the functor $I \otimes (-) : \mathbf{C} \to \mathbf{C}$ with an involutive structure $I \otimes J(-) \to J(I \otimes (-))$ defined by the components

$$
I \otimes Jc \xrightarrow{\;J_0 \otimes \mathrm{id}\;} JI \otimes Jc \xrightarrow{\;J_{2\,I,c}\;} J(I \otimes c) \quad ,
$$

for all $c \in \mathbf{C}$. An analogous statement holds true for the functor $(-) \otimes I : \mathbf{C} \to \mathbf{C}$. The axioms for the (symmetric) monoidal structure on J can then be reinterpreted as the equivalent property that the associator and unitors (as well as the braiding in the symmetric case) are involutive natural transformations.

Hence, we obtain an equivalent description of an involutive (symmetric) monoidal category in terms of the following data: An involutive category $(\mathbf{C}, J, j)$, an involutive functor $(\otimes, J_2) : (\mathbf{C}, J, j) \times (\mathbf{C}, J, j) \to (\mathbf{C}, J, j)$, a $*$-object $(J_0 : I \to JI) \in *\text{-}\mathbf{Obj}(\mathbf{C}, J, j)$ and involutive natural transformations for the associator and unitors (as well as the braiding in the symmetric case), which satisfy analogous axioms as those for (symmetric) monoidal categories. This alternative point of view is useful for (9) and (10) below.

3.10. EXAMPLE. For any (symmetric) monoidal category $\mathbf{C}$, the triple $(\mathbf{C}, \mathrm{Id}_{\mathbf{C}}, \mathrm{id}_{\mathrm{Id}_{\mathbf{C}}})$, with $\mathrm{Id}_{\mathbf{C}}$ the identity (symmetric) monoidal functor and $\mathrm{id}_{\mathrm{Id}_{\mathbf{C}}}$ the identity (symmetric) monoidal natural transformation, defines an involutive (symmetric) monoidal category. We call this the *trivial involutive (symmetric) monoidal category* over $\mathbf{C}$.

3.11. EXAMPLE. Let us equip the category of complex vector spaces $\mathbf{Vec}_{\mathbb{C}}$ with its standard symmetric monoidal structure where $\otimes$ is the usual tensor product, $I = \mathbb{C}$ is the ground field and τ is given by the flip maps $\tau_{V,W} : V \otimes W \to W \otimes V$, $v \otimes w \mapsto w \otimes v$. The endofunctor $\overline{(-)} : \mathbf{Vec}_{\mathbb{C}} \to \mathbf{Vec}_{\mathbb{C}}$ from Example 2.4 can be promoted to a symmetric monoidal functor by using the canonical maps $\overline{(-)}_{2\,V,W} : \overline{V} \otimes \overline{W} \to \overline{V \otimes W}$ and complex conjugation $\overline{(-)}_0 : \mathbb{C} \to \overline{\mathbb{C}}$. The resulting triple $(\mathbf{Vec}_{\mathbb{C}}, \overline{(-)}, \mathrm{id}_{\mathrm{Id}_{\mathbf{Vec}_{\mathbb{C}}}})$ is an involutive symmetric monoidal category.

3.12. EXAMPLE. Recall the groupoid of $\mathfrak{C}$-profiles $\Sigma_{\mathfrak{C}}$ from Example 2.5. The category $\Sigma_{\mathfrak{C}}$ may be equipped with the symmetric monoidal structure given by concatenation of $\mathfrak{C}$-profiles, i.e. $\underline{c} \otimes \underline{d} = (c_1, \ldots, c_n, d_1, \ldots, d_m)$, $I = \emptyset$ is the empty $\mathfrak{C}$-profile and $\tau_{\underline{c},\underline{d}} := \tau\langle|\underline{c}|, |\underline{d}|\rangle : \underline{c} \otimes \underline{d} \to \underline{d} \otimes \underline{c}$ is the block transposition. The reversal endofunctor $\mathrm{Rev} : \Sigma_{\mathfrak{C}} \to \Sigma_{\mathfrak{C}}$ can be promoted to a symmetric monoidal functor by using

$$\mathrm{Rev2}_{\underline{c},\underline{d}} := \tau\langle|\underline{c}|, |\underline{d}|\rangle \ : \ \mathrm{Rev}(\underline{c}) \otimes \mathrm{Rev}(\underline{d}) \ \longrightarrow \ \mathrm{Rev}(\underline{c} \otimes \underline{d})$$

and $\mathrm{Rev}_0 := \mathrm{id}_{\emptyset} : \emptyset \to \mathrm{Rev}(\emptyset) = \emptyset$. The resulting triple $(\Sigma_{\mathfrak{C}}, \mathrm{Rev}, \mathrm{id}_{\mathrm{Id}_{\Sigma_{\mathfrak{C}}}})$ is an involutive symmetric monoidal category.

3.13. DEFINITION. An *involutive (symmetric) monoidal functor* $(F, \nu) : (\mathbf{C}, J, j) \to (\mathbf{C}', J', j')$ consists of a (symmetric) monoidal functor $F = (F, F_2, F_0) : \mathbf{C} \to \mathbf{C}'$ and a (symmetric) monoidal natural transformation $\nu : FJ \to J'F$ satisfying the analog of diagram (1) in Definition 2.7.

An *involutive (symmetric) monoidal natural transformation* $\zeta : (F, \nu) \to (G, \chi)$ between involutive (symmetric) monoidal functors $(F, \nu), (G, \chi) : (\mathbf{C}, J, j) \to (\mathbf{C}', J', j')$ is a natural transformation $\zeta : F \to G$ that is both involutive and (symmetric) monoidal.

3.14. PROPOSITION. *Involutive (symmetric) monoidal categories, involutive (symmetric) monoidal functors and involutive (symmetric) monoidal natural transformations form a 2-category* $\mathbf{I(S)MCat}$.

3.15. REMARK. The condition for the natural transformation $\nu : FJ \to J'F$ to be monoidal explicitly means that the diagrams

commute. From the perspective established in Remark 3.9, one may reinterpret these diagrams as follows: The left diagram states that F_2 is an involutive natural transformation

$$F_2 \ : \ (\otimes', J_2')\big((F, \nu) \times (F, \nu)\big) \ \longrightarrow \ (F, \nu)(\otimes, J_2) \tag{7}$$

of involutive functors from $(\mathbf{C}, J, j) \times (\mathbf{C}, J, j)$ to $(\mathbf{C}', J', j')$. The right diagram states that F_0 defines a morphism

$$F_0 \ : \ \big(J_0' : I' \to J'I'\big) \ \longrightarrow \ *\text{-}\mathbf{Obj}(F, \nu)\big(J_0 : I \to JI\big) \tag{8}$$

in the category $*\text{-}\mathbf{Obj}(\mathbf{C}', J', j')$ of $*$-objects in $(\mathbf{C}', J', j')$.

Hence, we obtain an equivalent description of an involutive (symmetric) monoidal functor in terms of the following data: An involutive functor $(F, \nu) : (\mathbf{C}, J, j) \to (\mathbf{C}', J', j)$, an involutive natural transformation F_2 as in (7) and a *-morphism F_0 as in (8), which satisfy axioms analogous to those for a (symmetric) monoidal functor. This alternative point of view is useful for (12) and (13) below.

3.16. REMARK. Let us summarize Remarks 3.9 and 3.15 by one slogan: Involutive (symmetric) monoidal categories are the same things as (symmetric) monoidal involutive categories.

Let $(\mathbf{C}, J, j)$ be an involutive (symmetric) monoidal category and consider its category of *-objects *-$\mathbf{Obj}(\mathbf{C}, J, j)$. Making use of the 2-functor *-$\mathbf{Obj} : \mathbf{ICat} \to \mathbf{Cat}$ given in (3), we may equip the category *-$\mathbf{Obj}(\mathbf{C}, J, j)$ with a (symmetric) monoidal structure. Concretely, the tensor product functor is given by

$$\text{*-}\mathbf{Obj}(\mathbf{C}, J, j) \times \text{*-}\mathbf{Obj}(\mathbf{C}, J, j) \xrightarrow{\quad\otimes\quad} \text{*-}\mathbf{Obj}(\mathbf{C}, J, j) \tag{9}$$

$$\cong \Big\downarrow \qquad \xrightarrow{\quad \text{*-}\mathbf{Obj}(\otimes, J_2)\quad}$$

$$\text{*-}\mathbf{Obj}\big((\mathbf{C}, J, j) \times (\mathbf{C}, J, j)\big)$$

where the vertical isomorphism was explained in Remark 2.24 and the involutive functor $(\otimes, J_2)$ in Remark 3.9. The unit object

$$\big(J_0 : I \to JI\big) \in \text{*-}\mathbf{Obj}(\mathbf{C}, J, j) \tag{10}$$

is the *-object constructed in Remark 3.9. The associator and unitors (as well as the braiding in the symmetric case) are obtained by applying the 2-functor *-$\mathbf{Obj}$ to the associator and unitors (as well as the braiding in the symmetric case) of $(\mathbf{C}, J, j)$, which makes sense because Remark 3.9 shows that these are involutive natural transformations. Let us also mention that the tensor product of two *-objects $(* : c \to Jc), (*' : c' \to Jc') \in$ *-$\mathbf{Obj}(\mathbf{C}, J, j)$ explicitly reads as

$$(* : c \to Jc) \otimes (*' : c' \to Jc') = \Big(c \otimes c' \xrightarrow{* \otimes *'} Jc \otimes Jc' \xrightarrow{J_{2c,c'}} J(c \otimes c') \Big) \quad .$$

Summing up, we have proven

3.17. PROPOSITION. *Let $(\mathbf{C}, J, j)$ be an involutive (symmetric) monoidal category. Then the category of *-objects *-$\mathbf{Obj}(\mathbf{C}, J, j)$ is a (symmetric) monoidal category with tensor product (9) and unit object (10). Moreover, if $(\mathbf{C}, J, j)$ is also closed, i.e. it has internal homs, then *-$\mathbf{Obj}(\mathbf{C}, J, j)$ is closed too (cf. [Jac12, Proposition 1]).*

The assignment of the (symmetric) monoidal categories of *-objects extends to a 2-functor

$$\text{*-}\mathbf{Obj} : \mathbf{I(S)MCat} \longrightarrow \mathbf{(S)MCat} \quad, \tag{11}$$

which we shall denote with an abuse of notation by the same symbol as the 2-functor in (3). Concretely, this 2-functor is given by the following assignment:

- an involutive (symmetric) monoidal category $(\mathbf{C}, J, j)$ is mapped to the (symmetric) monoidal category $*\text{-}\mathbf{Obj}(\mathbf{C}, J, j)$ given in Proposition 3.17;

- an involutive (symmetric) monoidal functor $(F, \nu) : (\mathbf{C}, J, j) \to (\mathbf{C}', J', j')$ is mapped to the (symmetric) monoidal functor

$$*\text{-}\mathbf{Obj}(F, \nu) \;:\; *\text{-}\mathbf{Obj}(\mathbf{C}, J, j) \;\longrightarrow\; *\text{-}\mathbf{Obj}(\mathbf{C}', J', j') \tag{12}$$

  with underlying functor as in (3) and (symmetric) monoidal structure given by

$$*\text{-}\mathbf{Obj}(F)_2 \;:=\; *\text{-}\mathbf{Obj}(F_2) \quad, \qquad *\text{-}\mathbf{Obj}(F)_0 \;:=\; F_0 \quad, \tag{13}$$

  where F_2 and F_0 should be interpreted according to Remark 3.15;

- an involutive (symmetric) monoidal natural transformation $\zeta : (F, \nu) \to (G, \chi)$ is mapped to the (symmetric) monoidal natural transformation determined by (3).

3.18. REMARK. Notice that the 2-functor $*\text{-}\mathbf{Obj} : \mathbf{I(S)MCat} \to \mathbf{(S)MCat}$ given in (11) is a lift of the 2-functor $*\text{-}\mathbf{Obj} : \mathbf{ICat} \to \mathbf{Cat}$ given in (3) along the forgetful 2-functors $\mathrm{forget}_\otimes : \mathbf{I(S)MCat} \to \mathbf{ICat}$ and $\mathrm{forget}_\otimes : \mathbf{(S)MCat} \to \mathbf{Cat}$ that forget the (symmetric) monoidal structures. More precisely, using the explicit descriptions of our 2-functors, one easily confirms that the diagram

$$
\begin{array}{ccc}
\mathbf{I(S)MCat} & \xrightarrow{\;*\text{-}\mathbf{Obj}\;} & \mathbf{(S)MCat} \\
{\scriptstyle \mathrm{forget}_\otimes}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{forget}_\otimes} \\
\mathbf{ICat} & \xrightarrow[\;*\text{-}\mathbf{Obj}\;]{} & \mathbf{Cat}
\end{array}
$$

of 2-categories and 2-functors commutes (on the nose).

We note the following useful result that generalizes Theorem 2.25 to the (symmetric) monoidal setting. Let us first notice that the forgetful functor $U : *\text{-}\mathbf{Obj}(\mathbf{C}, J, j) \to \mathbf{C}$ satisfies $\otimes (U \times U) = U \otimes$ and $U(J_0 : I \to JI) = I$, hence it can be promoted to a (symmetric) monoidal functor via the trivial (symmetric) monoidal structure $U_2 = \mathrm{id}_{U \otimes}$ and $U_0 = \mathrm{id}_I$.

3.19. THEOREM. *Let $(\mathbf{C}, J, j)$ be an involutive (symmetric) monoidal category. A (symmetric) monoidal section $* : \mathbf{C} \to *\text{-}\mathbf{Obj}(\mathbf{C}, J, j)$ of the forgetful (symmetric) monoidal functor $U : *\text{-}\mathbf{Obj}(\mathbf{C}, J, j) \to \mathbf{C}$ canonically determines an isomorphism between $(\mathbf{C}, J, j)$ and the trivial involutive (symmetric) monoidal category $(\mathbf{C}, \mathrm{Id}_\mathbf{C}, \mathrm{id}_{\mathrm{Id}_\mathbf{C}})$. In particular, if such a section of U exists, then the involutive (symmetric) monoidal categories $(\mathbf{C}, J, j)$ and $(\mathbf{C}, \mathrm{Id}_\mathbf{C}, \mathrm{id}_{\mathrm{Id}_\mathbf{C}})$ are isomorphic.*

PROOF. Using that the (symmetric) monoidal structure on U is trivial, i.e. $U_2 = \mathrm{id}_{U \otimes}$ and $U_0 = \mathrm{id}_I$, and also that U is a faithful functor, one observes that the (symmetric) monoidal structure on the (symmetric) monoidal section $* : \mathbf{C} \to *\text{-}\mathbf{Obj}(\mathbf{C}, J, j)$ is necessarily trivial. The proof then proceeds analogously to the one of Theorem 2.25. ∎

3.20. COROLLARY. *The involutive symmetric monoidal category* $(\Sigma_{\mathfrak{C}}, \mathrm{Rev}, \mathrm{id}_{\mathrm{Id}_{\Sigma_{\mathfrak{C}}}})$ *of* $\mathfrak{C}$-
profiles equipped with reversal as involutive structure (cf. Example 3.12) is isomorphic to
the trivial involutive symmetric monoidal category $(\Sigma_{\mathfrak{C}}, \mathrm{Id}_{\Sigma_{\mathfrak{C}}}, \mathrm{id}_{\mathrm{Id}_{\Sigma_{\mathfrak{C}}}})$.

PROOF. By Theorem 3.19, it is sufficient to construct a symmetric monoidal section
$\rho = (\rho, \rho_2, \rho_0) : \Sigma_{\mathfrak{C}} \to *\text{-}\mathbf{Obj}(\Sigma_{\mathfrak{C}}, \mathrm{Rev}, \mathrm{id}_{\mathrm{Id}_{\Sigma_{\mathfrak{C}}}})$ of the forgetful symmetric monoidal functor
U. Taking the underlying functor as in Example 2.20, i.e. $\rho : \underline{c} \mapsto (\rho_{|\underline{c}|} : \underline{c} \to \underline{c}\rho_{|\underline{c}|})$ with
the order-reversal permutations $\rho_{|\underline{c}|} \in \Sigma_{|\underline{c}|}$, one easily checks that $\otimes(\rho \times \rho) = \rho \otimes$ and
$\rho(\emptyset) = (\mathrm{id}_{\emptyset} : \emptyset \to \emptyset) = (\mathrm{Rev}_0 : \emptyset \to \mathrm{Rev}(\emptyset))$. We choose the trivial symmetric monoidal
structure $\rho_2 = \mathrm{id}_{\rho \otimes}$ and $\rho_0 = \mathrm{id}_{\emptyset}$. ∎

3.21. ∗-MONOIDS. Let us recall the 2-functors $\mathbf{Mon} : (\mathbf{S})\mathbf{MCat} \to \mathbf{Cat}$ given in (6),
$*\text{-}\mathbf{Obj} : \mathbf{ICat} \to \mathbf{Cat}$ given in (3) and its lift $*\text{-}\mathbf{Obj} : \mathbf{I}(\mathbf{S})\mathbf{MCat} \to \mathbf{S}(\mathbf{M})\mathbf{Cat}$ given in
(11). The aim of this subsection is to describe a 2-functor $\mathbf{Mon} : \mathbf{I}(\mathbf{S})\mathbf{MCat} \to \mathbf{ICat}$
that lifts $\mathbf{Mon} : (\mathbf{S})\mathbf{MCat} \to \mathbf{Cat}$ to the involutive setting, such that the diagram

$$\begin{array}{ccc}
\mathbf{I}(\mathbf{S})\mathbf{MCat} & \xrightarrow{\ *\text{-}\mathbf{Obj}\ } & (\mathbf{S})\mathbf{MCat} \\
{\scriptstyle \mathbf{Mon}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathbf{Mon}} \\
\mathbf{ICat} & \xrightarrow[\ *\text{-}\mathbf{Obj}\ ]{} & \mathbf{Cat}
\end{array} \qquad (14)$$

of 2-categories and 2-functors commutes (on the nose). We then define ∗-monoids in terms
of the diagonal 2-functor $*\text{-}\mathbf{Mon} : \mathbf{I}(\mathbf{S})\mathbf{MCat} \to \mathbf{Cat}$ in this square.

Let us start with describing the 2-functor

$$\mathbf{Mon} : \mathbf{I}(\mathbf{S})\mathbf{MCat} \longrightarrow \mathbf{ICat} \qquad (15)$$

that lifts (6) to the involutive setting in some detail:

- an involutive (symmetric) monoidal category $(\mathbf{C}, J, j)$ is mapped to the involutive
  category

  $$\mathbf{Mon}(\mathbf{C}, J, j) := \big(\mathbf{Mon}(\mathbf{C}), \mathbf{Mon}(J), \mathbf{Mon}(j)\big) \in \mathbf{ICat}$$

  given by evaluating the 2-functor (6) on the (symmetric) monoidal category $\mathbf{C}$, on
  the (symmetric) monoidal endofunctor $J : \mathbf{C} \to \mathbf{C}$ and on the (symmetric) monoidal
  natural isomorphism $j : \mathrm{Id}_{\mathbf{C}} \to J^2$;

- an involutive (symmetric) monoidal functor $(F, \nu) : (\mathbf{C}, J, j) \to (\mathbf{C}', J', j')$ is mapped
  to the involutive functor

  $$\mathbf{Mon}(F, \nu) := \big(\mathbf{Mon}(F), \mathbf{Mon}(\nu)\big) : \mathbf{Mon}(\mathbf{C}, J, j) \longrightarrow \mathbf{Mon}(\mathbf{C}', J', j')$$

  given by evaluating the 2-functor (6) on the (symmetric) monoidal functor $F : \mathbf{C} \to$
  $\mathbf{C}'$ and on the (symmetric) monoidal natural transformation $\nu : FJ \to J'F$;

- an involutive (symmetric) monoidal natural transformation $\zeta : (F, \nu) \to (G, \chi)$ is mapped to the involutive natural transformation

$$\mathbf{Mon}(\zeta) : \mathbf{Mon}(F, \nu) \longrightarrow \mathbf{Mon}(G, \chi)$$

given by evaluating the 2-functor (6) on ζ.

3.22. LEMMA. *The diagram* (14) *of 2-categories and 2-functors commutes (on the nose).*

PROOF. This is an elementary check using the explicit definitions of the 2-functors given in (6), (3), (11) and (15). ∎

3.23. DEFINITION. The 2-functor $*\text{-}\mathbf{Mon} : \mathbf{I(S)MCat} \to \mathbf{Cat}$ is defined as the diagonal 2-functor in the commutative square (14), i.e.

$$
\begin{array}{ccc}
\mathbf{I(S)MCat} & \xrightarrow{\;*\text{-}\mathbf{Obj}\;} & \mathbf{(S)MCat} \\
{\scriptstyle \mathbf{Mon}}\Big\downarrow & \;\;\searrow{\scriptstyle *\text{-}\mathbf{Mon}}\;\; & \Big\downarrow{\scriptstyle \mathbf{Mon}} \\
\mathbf{ICat} & \xrightarrow[\;*\text{-}\mathbf{Obj}\;]{} & \mathbf{Cat}
\end{array}
\tag{16}
$$

For an involutive (symmetric) monoidal category $(\mathbf{C}, J, j)$, we call $*\text{-}\mathbf{Mon}(\mathbf{C}, J, j)$ the category of $*$-*monoids* in $(\mathbf{C}, J, j)$.

3.24. REMARK. Let $(\mathbf{C}, J, j)$ be an involutive (symmetric) monoidal category. We provide an explicit description of the objects and morphisms in the associated category of $*$-monoids $*\text{-}\mathbf{Mon}(\mathbf{C}, J, j)$, which we shall call $*$-*monoids* and $*$-*monoid morphisms*. Unpacking Definition 3.23, one obtains that a $*$-monoid is a quadruple $(M, \mu, \eta, *) \in *\text{-}\mathbf{Mon}(\mathbf{C}, J, j)$ consisting of an object $M \in \mathbf{C}$ and three $\mathbf{C}$-morphisms $\mu : M \otimes M \to M$, $\eta : I \to M$ and $* : M \to JM$, which satisfy the following conditions:

(1) (M, μ, η) is a monoid in the (symmetric) monoidal category $\mathbf{C}$;

(2) $* : M \to JM$ is a $*$-object in the involutive category $(\mathbf{C}, J, j)$;

(3) these two structures are compatible in the sense that the diagrams

$$
\begin{array}{ccc}
I & \xrightarrow{\;\eta\;} & M \\
{\scriptstyle J_0}\Big\downarrow & & \Big\downarrow{\scriptstyle *} \\
JI & \xrightarrow[\;J\eta\;]{} & JM
\end{array}
\qquad
\begin{array}{ccccc}
M \otimes M & \xrightarrow{\;*\otimes*\;} & JM \otimes JM & \xrightarrow{\;J_{2\,M,M}\;} & J(M \otimes M) \\
{\scriptstyle \mu}\Big\downarrow & & & & \Big\downarrow{\scriptstyle J\mu} \\
M & & \xrightarrow{\hspace{4cm}*\hspace{4cm}} & & JM
\end{array}
\tag{17}
$$

in $\mathbf{C}$ commute.

As a consequence of Lemma 3.22, these conditions have two equivalent interpretations which correspond to the counterclockwise and clockwise paths in the commutative diagram (16): The first option is to regard $* : (M, \mu, \eta) \to \mathbf{Mon}(J)(M, \mu, \eta)$ as a $*$-object in the

involutive category $\mathbf{Mon}(\mathbf{C}, J, j) \in \mathbf{ICat}$. The second option is to regard $\eta : (J_0 : I \to JI) \to (* : M \to JM)$ and $\mu : (* : M \to JM) \otimes (* : M \to JM) \to (* : M \to JM)$ as the structure maps of a monoid in the (symmetric) monoidal category $*\text{-}\mathbf{Obj}(\mathbf{C}, J, j) \in (\mathbf{S})\mathbf{MCat}$.

A ∗-monoid morphism $f : (M, \mu, \eta, *) \to (M', \mu', \eta', *')$ is a $\mathbf{C}$-morphism $f : M \to M'$ that preserves both the monoid structures and ∗-involutions.

3.25. EXAMPLE. Let us consider a ∗-monoid $(A, \mu, \eta, *)$ in the involutive symmetric monoidal category $(\mathbf{Vec}_{\mathbb{C}}, \overline{(-)}, \mathrm{id}_{\mathrm{Id}_{\mathbf{Vec}_{\mathbb{C}}}})$ from Example 3.11. In particular, the triple (A, μ, η) is an associative and unital algebra over $\mathbb{C}$ with multiplication $a\, b = \mu(a \otimes b)$ and unit $\mathbb{1} = \eta(1)$. By Example 2.19, ∗ is a complex anti-linear automorphism of A that squares to the identity, i.e. $a^{**} = a$. The compatibility conditions in (17) state that $\mathbb{1}^* = \mathbb{1}$ and $(a\, b)^* = a^*\, b^*$. We would like to emphasize that the latter condition *is not* the usual axiom for associative and unital ∗-algebras over $\mathbb{C}$, which is given by order-reversal $(a\, b)^* = b^*\, a^*$. As a consequence, our concept of ∗-monoids given in Definition 3.23 *does not* include the usual associative and unital ∗-algebras over $\mathbb{C}$ as examples. We will show later in Example 7.9 that the usual associative and unital ∗-algebras over $\mathbb{C}$ are recovered as ∗-algebras over a suitable ∗-operad, which provides a sufficiently flexible framework to implement order-reversal $(a\, b)^* = b^*\, a^*$.

4. Involutive structures on colored symmetric sequences

Colored operads can be defined as monoids in the monoidal category of colored symmetric sequences, see e.g. [Yau16, WY18, Yau18, GJ17] and below for a brief review. Let $\mathfrak{C} \in \mathbf{Set}$ be any non-empty set and $\mathbf{M}$ any cocomplete closed symmetric monoidal category. (We denote the monoidal structure on $\mathbf{M}$ by $\otimes$ and I, and the internal hom by $[-, -] : \mathbf{M}^{\mathrm{op}} \times \mathbf{M} \to \mathbf{M}$.) The category of $\mathfrak{C}$-*colored symmetric sequences* with values in $\mathbf{M}$ is defined as the functor category

$$\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) := \mathbf{M}^{\Sigma_{\mathfrak{C}} \times \mathfrak{C}} \quad , \tag{18}$$

where $\Sigma_{\mathfrak{C}}$ is the groupoid of $\mathfrak{C}$-profiles defined in Example 2.5 and the set $\mathfrak{C}$ is regarded as a discrete category. Given $X \in \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$, we write

$$X\binom{t}{\underline{c}} \in \mathbf{M}$$

for the evaluation of this functor on objects $(\underline{c}, t) \in \Sigma_{\mathfrak{C}} \times \mathfrak{C}$ and

$$X(\sigma) : X\binom{t}{\underline{c}} \longrightarrow X\binom{t}{\underline{c}\sigma}$$

for its evaluation on morphisms $\sigma : (\underline{c}, t) \to (\underline{c}\sigma, t)$ in $\Sigma_{\mathfrak{C}} \times \mathfrak{C}$.

The category $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ can be equipped with the following monoidal structure: The tensor product is given by the *circle product* $\circ : \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) \times \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) \to$

$\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$. Concretely, the circle product of $X, Y \in \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ is defined by the coend

$$(X \circ Y)\binom{t}{\underline{c}} := \int^{\underline{a}} \int^{(\underline{b}_1,\dots,\underline{b}_m)} \Sigma_{\mathfrak{C}}\left(\underline{b}_1 \otimes \cdots \otimes \underline{b}_m, \underline{c}\right) \otimes X\binom{t}{\underline{a}} \otimes Y\binom{a_1}{\underline{b}_1} \otimes \cdots \otimes Y\binom{a_m}{\underline{b}_m} \quad (19)$$

for all $(\underline{c}, t) \in \Sigma_{\mathfrak{C}} \times \mathfrak{C}$. Two remarks are in order: (1) This expression makes use of the symmetric monoidal structure on $\Sigma_{\mathfrak{C}}$ that we described in Example 3.12. (2) The tensor product between the Hom-set $\Sigma_{\mathfrak{C}}\left(\underline{b}_1 \otimes \cdots \otimes \underline{b}_m, \underline{c}\right) \in \mathbf{Set}$ and the object $X\binom{t}{\underline{a}} \in \mathbf{M}$ is given by the canonical $\mathbf{Set}$-tensoring of $\mathbf{M}$, i.e. $S \otimes m := \coprod_{s \in S} m$ for any $S \in \mathbf{Set}$ and $m \in \mathbf{M}$. The *circle unit* is the object $I_{\circ} \in \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ defined by

$$I_{\circ}\binom{t}{\underline{c}} := \Sigma_{\mathfrak{C}}(t, \underline{c}) \otimes I \quad , \tag{20}$$

for all $(\underline{c}, t) \in \Sigma_{\mathfrak{C}} \times \mathfrak{C}$.

4.1. PROPOSITION. $(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), \circ, I_{\circ})$ *is a right closed monoidal category.*

The aim of this section is to transfer these structures and results to the setting of involutive categories.

4.2. PRODUCT-EXPONENTIAL 2-ADJUNCTION. Because the category of symmetric sequences (18) is defined as a functor category, we shall start with developing a notion of functor categories in the involutive setting. For this we will first recall the relevant structures for ordinary category theory from a perspective that easily generalizes to involutive category theory.

Let us denote by $\mathbf{Cat} \widetilde{\times} \mathbf{Cat}$ the 2-category with objects given by pairs $(\mathbf{C}, \mathbf{D})$ of categories, morphisms given by pairs (F, G) of functors and 2-morphisms given by pairs (ζ, ξ) of natural transformations, and all compositions given component-wise. (We use the symbol $\widetilde{\times}$ to denote the above product 2-category because we reserve the symbol $\times$ for the 2-functors defined below.) Notice that taking products of categories, functors and natural transformations defines a 2-functor

$$\times \ : \ \mathbf{Cat} \widetilde{\times} \mathbf{Cat} \ \longrightarrow \ \mathbf{Cat} \quad . \tag{21}$$

Let us denote by $\mathbf{Cat}^{\mathrm{op}}$ the opposite 2-category, i.e. morphisms $\mathbf{C} \to \mathbf{D}$ are functors $F : \mathbf{D} \to \mathbf{C}$ going in the opposite direction and 2-morphisms are not reversed. We define the exponential 2-functor

$$(-)^{(-)} \ : \ \mathbf{Cat}^{\mathrm{op}} \widetilde{\times} \mathbf{Cat} \ \longrightarrow \ \mathbf{Cat} \tag{22}$$

as follows:

- a pair $(\mathbf{D}, \mathbf{C})$ of categories is mapped to the functor category $\mathbf{C}^{\mathbf{D}}$;

- a pair $(G : \mathbf{D}' \to \mathbf{D}, F : \mathbf{C} \to \mathbf{C}')$ of functors is mapped to the functor $F^G : \mathbf{C}^{\mathbf{D}} \to \mathbf{C}'^{\mathbf{D}'}$ that acts on objects and morphisms as

$$F^G\big(X : \mathbf{D} \to \mathbf{C}\big) := (FXG : \mathbf{D}' \to \mathbf{C}') \quad,$$
$$F^G\big(\alpha : X \to Y\big) := (F\alpha G : FXG \to FYG) \quad;$$

- a pair $(\xi : G \to G', \zeta : F \to F')$ of natural transformations is mapped to the natural transformation $\zeta^\xi : F^G \to F'^{G'}$ with components given by any of the two compositions in the commutative square

$$
\begin{array}{ccc}
FXG & \xrightarrow{\ \zeta XG\ } & F'XG \\
{\scriptstyle FX\xi}\downarrow & {\scriptstyle (\zeta^\xi)_X} & \downarrow{\scriptstyle F'X\xi} \\
FXG' & \xrightarrow[\ \zeta XG'\ ]{} & F'XG'
\end{array}
$$

for all $X \in \mathbf{C}^{\mathbf{D}}$.

The two 2-functors $\times$ and $(-)^{(-)}$ are related by a family of 2-adjunctions.

4.3. PROPOSITION. *For every* $\mathbf{D} \in \mathbf{Cat}$, *there is a 2-adjunction*

$$(-) \times \mathbf{D} : \mathbf{Cat} \rightleftarrows \mathbf{Cat} : (-)^{\mathbf{D}} \quad.$$

PROOF. The component at $\mathbf{C} \in \mathbf{Cat}$ of the unit 2-natural transformation $\eta : \mathrm{Id}_{\mathbf{Cat}} \to ((-) \times \mathbf{D})^{\mathbf{D}}$ is given by the functor

$$\eta_{\mathbf{C}} : \mathbf{C} \longrightarrow (\mathbf{C} \times \mathbf{D})^{\mathbf{D}}$$

that assigns to $c \in \mathbf{C}$ the inclusion functor $\eta_{\mathbf{C}}(c) : \mathbf{D} \to \mathbf{C} \times \mathbf{D}$ specified by $d \mapsto (c, d)$. The component at $\mathbf{C} \in \mathbf{Cat}$ of the counit 2-natural transformations $\epsilon : (-)^{\mathbf{D}} \times \mathbf{D} \to \mathrm{Id}_{\mathbf{Cat}}$ is given by the evaluation functor

$$\epsilon_{\mathbf{C}} : \mathbf{C}^{\mathbf{D}} \times \mathbf{D} \longrightarrow \mathbf{C} \quad,$$

that assigns to $(X, d) \in \mathbf{C}^{\mathbf{D}} \times \mathbf{D}$ the object $Xd \in \mathbf{C}$. The triangle identities are a straightforward check. ∎

Because of their 2-functoriality, our constructions above can be immediately extended to involutive category theory. Concretely, using the 2-functor (21), we define the product 2-functor

$$\times : \mathbf{ICat} \,\widetilde{\times}\, \mathbf{ICat} \longrightarrow \mathbf{ICat} \tag{23}$$

in the involutive setting as follows:

- a pair of involutive categories is mapped to the involutive category

$$(\mathbf{C}, J, j) \times (\mathbf{D}, K, k) := (\mathbf{C} \times \mathbf{D}, J \times K, j \times k) \quad ;$$

- a pair of involutive functors is mapped to the involutive functor

$$(F, \nu) \times (G, \chi) := (F \times G, \nu \times \chi) \quad ;$$

- a pair of involutive natural transformations is mapped to the involutive natural transformation $\zeta \times \xi$.

Similarly, using the 2-functor (22), we define the exponential 2-functor

$$(-)^{(-)} : \mathbf{ICat}^{\mathrm{op}} \widetilde{\times} \mathbf{ICat} \longrightarrow \mathbf{ICat} \tag{24}$$

in the involutive setting as follows:

- a pair of involutive categories is mapped to the involutive category

$$(\mathbf{C}, J, j)^{(\mathbf{D}, K, k)} := \left(\mathbf{C}^{\mathbf{D}}, J^{K}, j^{k}\right) \quad ;$$

- a pair of involutive functors is mapped to the involutive functor

$$(F, \nu)^{(G, \chi)} := \left(F^{G}, \nu^{\chi^{-1}}\right) \quad ;$$

- a pair of involutive natural transformations is mapped to the involutive natural transformation ζ^{ξ}.

Analogously to Proposition 4.3, one can prove

4.4. PROPOSITION. *For every* $(\mathbf{D}, K, k) \in \mathbf{ICat}$, *there is a 2-adjunction*

$$(-) \times (\mathbf{D}, K, k) : \mathbf{ICat} \rightleftarrows \mathbf{ICat} : (-)^{(\mathbf{D}, K, k)} \quad .$$

4.5. INVOLUTIVE COLORED SYMMETRIC SEQUENCES. Let $(\mathbf{M}, J, j)$ be an involutive closed symmetric monoidal category, which we assume to be cocomplete, and $\mathfrak{C} \in \mathbf{Set}$ a non-empty set of colors. In order to define an involutive analog of the category of symmetric sequences (18), one has to endow $\Sigma_{\mathfrak{C}} \times \mathfrak{C}$ with the structure of an involutive category. The simplest possible choice is the trivial involutive structure from Example 2.3, i.e. $\mathrm{triv}(\Sigma_{\mathfrak{C}} \times \mathfrak{C}) = (\Sigma_{\mathfrak{C}} \times \mathfrak{C}, \mathrm{Id}_{\Sigma_{\mathfrak{C}} \times \mathfrak{C}}, \mathrm{id}_{\mathrm{Id}_{\Sigma_{\mathfrak{C}} \times \mathfrak{C}}}) \in \mathbf{ICat}$. In particular, there is no non-trivial interplay between the involution functor and the ordering of $\mathfrak{C}$-profiles $\underline{c} = (c_1, \ldots, c_n)$. An alternative choice that does describe a non-trivial interplay between involution and ordering of $\mathfrak{C}$-profiles is obtained by considering the involutive symmetric monoidal category $(\Sigma_{\mathfrak{C}}, \mathrm{Rev}, \mathrm{id}_{\mathrm{Id}_{\Sigma_{\mathfrak{C}}}})$ from Examples 2.5, 2.20 and 3.12, where the involution functor is given by order-reversal. Endowing the discrete category $\mathfrak{C}$ with the trivial involutive structure and using the product 2-functor (23), we may form the involutive category $(\Sigma_{\mathfrak{C}}, \mathrm{Rev}, \mathrm{id}_{\mathrm{Id}_{\Sigma_{\mathfrak{C}}}}) \times \mathrm{triv}(\mathfrak{C}) \in \mathbf{ICat}$. Both of these natural choices lead to the same theory of involutive colored sequences. Indeed, by Corollary 3.20, there exists an $\mathbf{ISMCat}$-isomorphism $(\Sigma_{\mathfrak{C}}, \mathrm{Rev}, \mathrm{id}_{\mathrm{Id}_{\Sigma_{\mathfrak{C}}}}) \cong \mathrm{triv}(\Sigma_{\mathfrak{C}})$, which implies that $(\Sigma_{\mathfrak{C}}, \mathrm{Rev}, \mathrm{id}_{\mathrm{Id}_{\Sigma_{\mathfrak{C}}}}) \times \mathrm{triv}(\mathfrak{C}) \cong \mathrm{triv}(\Sigma_{\mathfrak{C}}) \times \mathrm{triv}(\mathfrak{C}) \cong \mathrm{triv}(\Sigma_{\mathfrak{C}} \times \mathfrak{C})$ in $\mathbf{ICat}$. This motivates the following

4.6. DEFINITION. Let $\mathfrak{C} \in \mathbf{Set}$ be a non-empty set. The *involutive category of $\mathfrak{C}$-colored symmetric sequences* with values in a cocomplete involutive closed symmetric monoidal category $(\mathbf{M}, J, j)$ is defined via the exponential 2-functor (24) by

$$\big(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*\big) := (\mathbf{M}, J, j)^{\mathrm{triv}(\Sigma_{\mathfrak{C}} \times \mathfrak{C})} \quad .$$

Concretely, the endofunctor

$$J_* := J^{\mathrm{Id}_{\Sigma_{\mathfrak{C}} \times \mathfrak{C}}} : \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) \longrightarrow \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$$

is given by post-composition with $J : \mathbf{M} \to \mathbf{M}$, i.e. $X \mapsto JX$, and the natural isomorphism

$$j_* := j^{\cdot \mathrm{id}_{\mathrm{Id}_{\Sigma_{\mathfrak{C}} \times \mathfrak{C}}}} : \mathrm{Id}_{\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})} \longrightarrow J_*^2$$

has components $j_{*X} := jX$ given by whiskering the natural isomorphism $j : \mathrm{Id}_{\mathbf{M}} \to J^2$ and the functor $X : \Sigma_{\mathfrak{C}} \times \mathfrak{C} \to \mathbf{M}$, for all $X \in \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$.

We now show that the involutive category $(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*)$ given in Definition 4.6 may be promoted to an involutive monoidal category, extending the monoidal structure of Proposition 4.1 to the involutive setting. Recalling Definition 3.6, this amounts to endowing the endofunctor $J_* : \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) \to \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ with the structure of a monoidal functor such that $j_* : \mathrm{Id}_{\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})} \to J_*^2$ becomes a monoidal natural isomorphism. We first define the natural transformation $J_{*2} : \circ (J_* \times J_*) \to J_* \circ$ in terms of the components

$$\big(J_*X \circ J_*Y\big)\big(\tfrac{t}{\underline{c}}\big) = \int^{\underline{a}} \int^{(\underline{b}_1, \ldots, \underline{b}_m)} \Sigma_{\mathfrak{C}}\big(\underline{b}_1 \otimes \cdots \otimes \underline{b}_m, \underline{c}\big) \otimes JX\big(\tfrac{t}{\underline{a}}\big) \otimes \bigotimes_{i=1}^m JY\big(\tfrac{a_i}{\underline{b}_i}\big) \qquad (25)$$

$$\big\downarrow (J_{*2})_{X,Y} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \big\downarrow \int \int \mathrm{id} \otimes J_2^m$$

$$\big(J_*(X \circ Y)\big)\big(\tfrac{t}{\underline{c}}\big) \xrightarrow[\cong]{} \int^{\underline{a}} \int^{(\underline{b}_1, \ldots, \underline{b}_m)} \Sigma_{\mathfrak{C}}\big(\underline{b}_1 \otimes \cdots \otimes \underline{b}_m, \underline{c}\big) \otimes J\Big(X\big(\tfrac{t}{\underline{a}}\big) \otimes \bigotimes_{i=1}^m Y\big(\tfrac{a_i}{\underline{b}_i}\big)\Big)$$

for all $X, Y \in \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ and all $(\underline{c}, t) \in \Sigma_{\mathfrak{C}} \times \mathfrak{C}$. For the horizontal arrows we used the definition of the circle product (19) and the fact that $J : \mathbf{M} \to \mathbf{M}$ is self-adjoint (cf. Lemma 2.6), hence it preserves coends and the $\mathbf{Set}$-tensoring. In the right vertical arrow we denoted by J_2^m the m-fold iteration of the natural transformation $J_2 : \otimes (J \times J) \to J \otimes$ corresponding to the involutive symmetric monoidal category $(\mathbf{M}, J, j)$. We next define the $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$-morphism $J_{*0} : I_\circ \to J_*I_\circ$ for the circle unit (20) by

$$\begin{array}{ccc} I_\circ\big(\tfrac{t}{\underline{c}}\big) & \xrightarrow{\quad J_{*0} \quad} & (J_*I_\circ)\big(\tfrac{t}{\underline{c}}\big) \\[4pt] \big\| & & \big\downarrow \cong \\[4pt] \Sigma_{\mathfrak{C}}(t, \underline{c}) \otimes I & \xrightarrow[\mathrm{id} \otimes J_0]{} & \Sigma_{\mathfrak{C}}(t, \underline{c}) \otimes JI \end{array} \qquad (26)$$

for all $(\underline{c}, t) \in \Sigma_{\mathfrak{C}} \times \mathfrak{C}$. For the right vertical arrow we used again that $J : \mathbf{M} \to \mathbf{M}$ is self-adjoint and hence it preserves the $\mathbf{Set}$-tensoring. In the bottom horizontal arrow $J_0 : I \to JI$ denotes the morphism corresponding to the involutive symmetric monoidal category $(\mathbf{M}, J, j)$.

4.7. THEOREM. *The involutive category* $(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*)$ *becomes an involutive right closed monoidal category when the underlying category* $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ *is equipped with the circle monoidal structure of Proposition 4.1 and the underlying endofunctor* J_* *is equipped with the monoidal functor structure* (J_{*2}, J_{*0}) *of* (25) *and* (26).

PROOF. It is easy to confirm that $(J_*, J_{*2}, J_{*0}) : \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) \to \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$, as defined in Definition 4.6, (25) and (26), is a monoidal endofunctor with respect to the circle monoidal structure and that the natural isomorphism $j_* : \mathrm{Id}_{\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})} \to J_*^2$ is monoidal. ∎

4.8. REMARK. Because $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ in general does not admit a braiding, the *non-reversing* notion of involutive monoidal category due to [Jac12] (see also Definition 3.6) and the *reversing* one considered in [Egg11, BM09] are a priori inequivalent. This is indeed the case: While Theorem 4.7 equips the monoidal category $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ with a *non-reversing* involutive structure, one cannot obtain a *reversing* one as this requires to specify isomorphisms $J_*X \circ J_*Y \cong J_*(Y \circ X)$, which in general do not exist by the following argument: Assume that $I \not\cong \emptyset$ in $\mathbf{M}$ (e.g. $\mathbf{M} = \mathbf{Vec}_{\mathbb{C}}$) and that the set $\mathfrak{C}$ has cardinality ≥ 2. Define $X, Y \in \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ by setting

$$X\binom{t}{\underline{c}} = \left(\Sigma_{\mathfrak{C}}(t, t_0) \times \Sigma_{\mathfrak{C}}(t_0, \underline{c})\right) \otimes I \quad , \qquad Y\binom{t}{\underline{c}} = \Sigma_{\mathfrak{C}}(\emptyset, \underline{c}) \otimes I \quad ,$$

for some fixed $t_0 \in \mathfrak{C}$. Recalling (19) we obtain

$$(X \circ Y)\binom{t}{\underline{c}} \cong \Sigma_{\mathfrak{C}}(t, t_0) \otimes Y\binom{t}{\underline{c}} \quad , \qquad (Y \circ X)\binom{t}{\underline{c}} \cong Y\binom{t}{\underline{c}} \quad .$$

Since $J_*X \cong X$ and $J_*Y \cong Y$, we find for $t \neq t_0$ that $(J_*X \circ J_*Y)\binom{t}{\emptyset} \cong \emptyset \not\cong I \cong J_*(Y \circ X)\binom{t}{\emptyset}$. This counterexample explains why the *non-reversing* involutive structures defined by [Jac12] are better suited for developing the theory of colored $*$-operads than the *reversing* ones of [Egg11, BM09].

Many interesting constructions in colored operad theory arise from changing the underlying set of colors, see e.g. [BSW17] for examples inspired by quantum field theory. We shall now generalize the relevant constructions to the setting of involutive category theory.

Any map $f : \mathfrak{C} \to \mathfrak{D}$ of non-empty sets induces a functor $f : \Sigma_{\mathfrak{C}} \to \Sigma_{\mathfrak{D}}$ between the associated groupoids of profiles. Concretely, we have that $\underline{c} = (c_1, \ldots, c_n) \mapsto f(\underline{c}) = (f(c_1), \ldots, f(c_n))$. This functor may be equipped with the obvious involutive symmetric monoidal structure such that it defines an involutive symmetric monoidal functor

$$(f, \mathrm{id}_f) : (\Sigma_{\mathfrak{C}}, \mathrm{Id}_{\Sigma_{\mathfrak{C}}}, \mathrm{id}_{\mathrm{Id}_{\Sigma_{\mathfrak{C}}}}) \longrightarrow (\Sigma_{\mathfrak{D}}, \mathrm{Id}_{\Sigma_{\mathfrak{D}}}, \mathrm{id}_{\mathrm{Id}_{\Sigma_{\mathfrak{D}}}}) \quad .$$

Moreover, regarding $\mathfrak{C}$ and $\mathfrak{D}$ as discrete categories, we obtain an involutive functor (denoted by the same symbol)

$$(f, \mathrm{id}_f) : (\mathfrak{C}, \mathrm{Id}_{\mathfrak{C}}, \mathrm{id}_{\mathrm{Id}_{\mathfrak{C}}}) \longrightarrow (\mathfrak{D}, \mathrm{Id}_{\mathfrak{D}}, \mathrm{id}_{\mathrm{Id}_{\mathfrak{D}}})$$

between the associated trivial involutive categories. Using the product and exponential 2-functors (cf. (23) and (24)), we may exponentiate the identity $\mathrm{Id}_{(\mathbf{M},J,j)} = (\mathrm{Id}_{\mathbf{M}}, \mathrm{id}_J)$ involutive functor by the product involutive functor $(f, \mathrm{id}_f) \times (f, \mathrm{id}_f)$ to obtain an involutive functor

$$(f^*, \mathrm{id}_{f^*J_*}) : \big(\mathbf{SymSeq}_{\mathfrak{D}}(\mathbf{M}), J_*, j_*\big) \longrightarrow \big(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*\big) \qquad (27)$$

describing the pullback along f of $\mathfrak{D}$-colored symmetric sequences to $\mathfrak{C}$-colored symmetric sequences. (Notice that $f^*J_* = J_*f^*$ as functors from $\mathbf{SymSeq}_{\mathfrak{D}}(\mathbf{M})$ to $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ because J_* is a pushforward and f^* is a pullback.)

4.9. COROLLARY. *For every map $f : \mathfrak{C} \to \mathfrak{D}$ between non-empty sets, there exists an involutive adjunction (cf. Definition 2.11)*

$$(f_!, \lambda_f) : \big(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*\big) \rightleftarrows \big(\mathbf{SymSeq}_{\mathfrak{D}}(\mathbf{M}), J_*, j_*\big) : (f^*, \mathrm{id}_{f^*J_*}) \qquad .$$

PROOF. By left Kan extension, the functor f^* has a left adjoint $f_!$. The involutive structure λ_f on $f_!$ is the one described in Proposition 2.13, which implies that we have an involutive adjunction. ∎

The pullback functor $f^* : \mathbf{SymSeq}_{\mathfrak{D}}(\mathbf{M}) \to \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ may be equipped with the following canonical monoidal structure: The components of the natural transformation $f_2^* : \circ^{\mathfrak{C}}(f^* \times f^*) \to f^*\circ^{\mathfrak{D}}$ are specified by

$$\Sigma_{\mathfrak{C}}\big(\underline{b}_1 \otimes \cdots \otimes \underline{b}_m, \underline{c}\big) \otimes f^*X\big(\begin{smallmatrix}t\\\underline{a}\end{smallmatrix}\big) \otimes \bigotimes_{i=1}^m f^*Y\big(\begin{smallmatrix}a_i\\\underline{b}_i\end{smallmatrix}\big) \longrightarrow \big(f^*X \circ^{\mathfrak{C}} f^*Y\big)\big(\begin{smallmatrix}t\\\underline{c}\end{smallmatrix}\big) \quad (28)$$

$$\left\downarrow{\scriptstyle f\otimes\mathrm{id}}\right. \qquad\qquad\qquad\qquad\qquad \left\downarrow{\scriptstyle (f_2^*)_{X,Y}}\right.$$

$$\Sigma_{\mathfrak{D}}\big(f(\underline{b}_1) \otimes \cdots \otimes f(\underline{b}_m), f(\underline{c})\big) \otimes f^*X\big(\begin{smallmatrix}t\\\underline{a}\end{smallmatrix}\big) \otimes \bigotimes_{i=1}^m f^*Y\big(\begin{smallmatrix}a_i\\\underline{b}_i\end{smallmatrix}\big) \to \big(f^*(X \circ^{\mathfrak{D}} Y)\big)\big(\begin{smallmatrix}t\\\underline{c}\end{smallmatrix}\big)$$

for all $X, Y \in \mathbf{SymSeq}_{\mathfrak{D}}(\mathbf{M})$ and all $(\underline{c}, t) \in \Sigma_{\mathfrak{C}} \times \mathfrak{C}$. The horizontal arrows are the canonical inclusions into the coend and the left vertical arrow denotes the action of the functor $f : \Sigma_{\mathfrak{C}} \to \Sigma_{\mathfrak{D}}$ on Hom-sets. The $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$-morphism $f_0^* : I_\circ^{\mathfrak{C}} \to f^*I_\circ^{\mathfrak{D}}$ is defined similarly by

$$f_0^* : I_\circ^{\mathfrak{C}}\big(\begin{smallmatrix}t\\\underline{c}\end{smallmatrix}\big) = \Sigma_{\mathfrak{C}}(t, \underline{c}) \otimes I \xrightarrow{\;f\otimes\mathrm{id}\;} \Sigma_{\mathfrak{D}}\big(f(t), f(\underline{c})\big) \otimes I = f^*(I_\circ^{\mathfrak{D}})\big(\begin{smallmatrix}t\\\underline{c}\end{smallmatrix}\big) \quad , \qquad (29)$$

for all $(\underline{c}, t) \in \Sigma_{\mathfrak{C}} \times \mathfrak{C}$.

4.10. THEOREM. *For every map $f : \mathfrak{C} \to \mathfrak{D}$ between non-empty sets, the involutive functor $(f^*, \mathrm{id}_{f^*J_*}) : (\mathbf{SymSeq}_{\mathfrak{D}}(\mathbf{M}), J_*, j_*) \to (\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*)$ of (27) becomes an involutive monoidal functor when equipped with the monoidal structure (f_2^*, f_0^*) of (28) and (29).*

PROOF. By Definition 3.13, it remains to prove that $\mathrm{id}_{J_*f^*} : J_*f^* \to f^*J_* = J_*f^*$ is a monoidal natural transformation, which is clearly the case. ∎

4.11. ∗-OBJECTS. We conclude this section by describing rather explicitly the monoidal category

$$\ast\text{-}\mathbf{Obj}\big(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_\ast, j_\ast\big) \in \mathbf{MCat} \tag{30}$$

of ∗-objects in the involutive monoidal category of symmetric sequences. Given any ∗-object $(\ast : X \to J_\ast X) \in \ast\text{-}\mathbf{Obj}\big(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_\ast, j_\ast\big)$, we consider its components at $(\underline{c}, t) \in \Sigma_{\mathfrak{C}} \times \mathfrak{C}$ and observe that this is precisely the same data as a symmetric sequence with values in $\ast\text{-}\mathbf{Obj}(\mathbf{M}, J, j)$, which is a cocomplete closed symmetric monoidal category, cf. Proposition 3.17 and Remark 2.22. Similarly, one observes that a morphism in (30) is the same data as a morphism in $\mathbf{SymSeq}_{\mathfrak{C}}(\ast\text{-}\mathbf{Obj}(\mathbf{M}, J, j))$, which means that these two categories are canonically isomorphic. We now show that this isomorphism is compatible with the monoidal structures.

4.12. PROPOSITION. *The canonical identification above defines an isomorphism*

$$\ast\text{-}\mathbf{Obj}\big(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_\ast, j_\ast\big) \;\cong\; \mathbf{SymSeq}_{\mathfrak{C}}\big(\ast\text{-}\mathbf{Obj}(\mathbf{M}, J, j)\big) \tag{31}$$

of monoidal categories.

PROOF. It remains to prove that our canonical isomorphism of categories is monoidal, i.e. that tensor products and units are preserved up to coherent isomorphisms. Given two objects $\ast : X \to J_\ast X$ and $\ast' : Y \to J_\ast Y$ in $\ast\text{-}\mathbf{Obj}\big(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_\ast, j_\ast\big)$, their tensor product reads as

$$\big(\ast : X \to J_\ast X\big) \circ \big(\ast' : Y \to J_\ast Y\big) \;=\; \Big(X \circ Y \xrightarrow{\;\ast \circ \ast'\;} J_\ast X \circ J_\ast Y \xrightarrow{\;(J_{\ast 2})_{X,Y}\;} J_\ast(X \circ Y) \Big) \quad .$$

By a brief calculation one shows that the composed morphism on the right-hand side of this equation is induced by functoriality of coends and **Set**-tensoring via the family of maps

$$X\left(\begin{smallmatrix}t\\\underline{a}\end{smallmatrix}\right) \otimes \bigotimes_{i=1}^m Y\left(\begin{smallmatrix}a_i\\\underline{b}_i\end{smallmatrix}\right) \xrightarrow{\;\ast \otimes \otimes_{i=1}^m \ast'\;} JX\left(\begin{smallmatrix}t\\\underline{a}\end{smallmatrix}\right) \otimes \bigotimes_{i=1}^m JY\left(\begin{smallmatrix}a_i\\\underline{b}_i\end{smallmatrix}\right) \xrightarrow{\;J_2^m\;} J\Big(X\left(\begin{smallmatrix}t\\\underline{a}\end{smallmatrix}\right) \otimes \bigotimes_{i=1}^m Y\left(\begin{smallmatrix}a_i\\\underline{b}_i\end{smallmatrix}\right)\Big) \tag{32}$$

Notice that (32) is the tensor product $\big(\ast : X\left(\begin{smallmatrix}t\\\underline{a}\end{smallmatrix}\right) \to JX\left(\begin{smallmatrix}t\\\underline{a}\end{smallmatrix}\right)\big) \otimes \bigotimes_{i=1}^m \big(\ast' : Y\left(\begin{smallmatrix}a_i\\\underline{b}_i\end{smallmatrix}\right) \to JY\left(\begin{smallmatrix}a_i\\\underline{b}_i\end{smallmatrix}\right)\big)$ in $\ast\text{-}\mathbf{Obj}(\mathbf{M}, J, j)$. Because J preserves coends and the **Set**-tensoring, we obtain the natural isomorphism relating the tensor products on both sides of (31). A similar construction provides the isomorphism relating the units. ∎

5. Colored ∗-operads

Let $\mathfrak{C} \in \mathbf{Set}$ be any non-empty set. We briefly recall the concept of $\mathfrak{C}$-colored operads.

5.1. DEFINITION. The category of $\mathfrak{C}$-*colored operads* with values in a cocomplete closed symmetric monoidal category $\mathbf{M}$ is the category of monoids (cf. Definition 3.4) in the monoidal category $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ (cf. (18), (19) and (20)), i.e.

$$\mathbf{Op}_{\mathfrak{C}}(\mathbf{M}) \; := \; \mathbf{Mon}\big(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})\big) \quad .$$

Using the concepts and techniques that we have developed so far in this paper, the above definition admits the following natural generalization to involutive category theory.

5.2. DEFINITION. The category of $\mathfrak{C}$-*colored* ∗-*operads* with values in a cocomplete involutive closed symmetric monoidal category $(\mathbf{M}, J, j)$ is the category of ∗-monoids (cf. Definition 3.23) in the involutive monoidal category $(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*)$ (cf. Theorem 4.7), i.e.

$$\text{∗-}\mathbf{Op}_{\mathfrak{C}}(\mathbf{M}, J, j) \; := \; \text{∗-}\mathbf{Mon}\big(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*\big) \quad .$$

5.3. REMARK. It is worth to specialize Remark 3.24 to the present case. We observe that a $\mathfrak{C}$-colored ∗-operad is a quadruple $(\mathcal{O}, \gamma, \mathbb{1}, *)$ consisting of a $\mathfrak{C}$-colored symmetric sequence $\mathcal{O} \in \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ and three $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$-morphisms $\gamma : \mathcal{O} \circ \mathcal{O} \to \mathcal{O}$ (called *operadic composition*), $\mathbb{1} : I_{\circ} \to \mathcal{O}$ (called *operadic unit*) and $* : \mathcal{O} \to J_*\mathcal{O}$ (called ∗-*involution*), which satisfy the following conditions:

(1) $(\mathcal{O}, \gamma, \mathbb{1})$ is a monoid in $(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), \circ, I_{\circ})$, i.e. the diagrams

$$
\begin{array}{ccc}
(\mathcal{O} \circ \mathcal{O}) \circ \mathcal{O} \xrightarrow{\;\cong\;} \mathcal{O} \circ (\mathcal{O} \circ \mathcal{O}) \xrightarrow{\;\mathrm{id}\circ\gamma\;} \mathcal{O} \circ \mathcal{O} \\
{\scriptstyle \gamma\circ\mathrm{id}} \downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow {\scriptstyle \gamma} \\
\mathcal{O} \circ \mathcal{O} \xrightarrow{\qquad\qquad \gamma \qquad\qquad} \mathcal{O}
\end{array}
\qquad
\begin{array}{ccc}
I_{\circ} \circ \mathcal{O} \xrightarrow{\;\mathbb{1}\circ\mathrm{id}\;} \mathcal{O} \circ \mathcal{O} \xleftarrow{\;\mathrm{id}\circ\mathbb{1}\;} \mathcal{O} \circ I_{\circ} \\
{\scriptstyle \cong} \searrow \quad \downarrow {\scriptstyle \gamma} \quad \swarrow {\scriptstyle \cong} \\
\mathcal{O}
\end{array}
$$

in $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ commute;

(2) $* : \mathcal{O} \to J_*\mathcal{O}$ is a ∗-object in $(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*)$, i.e. the diagram

$$
\begin{array}{ccc}
\mathcal{O} & \xrightarrow{\;*\;} & J_*\mathcal{O} \\
& {\scriptstyle (j_*)_{\mathcal{O}}} \searrow & \downarrow {\scriptstyle J_**} \\
& & J_*^2\mathcal{O}
\end{array}
$$

in $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ commutes;

(3) these two structures are compatible, i.e. the diagrams

$$
\begin{array}{ccc}
I_{\circ} & \xrightarrow{\;\mathbb{1}\;} & \mathcal{O} \\
{\scriptstyle J_*0} \downarrow & & \downarrow {\scriptstyle *} \\
J_*I_{\circ} & \xrightarrow{\;J_*\mathbb{1}\;} & J_*\mathcal{O}
\end{array}
\qquad
\begin{array}{ccccc}
\mathcal{O} \circ \mathcal{O} & \xrightarrow{\;*\circ*\;} & J_*\mathcal{O} \circ J_*\mathcal{O} & \xrightarrow{\;(J_{*2})_{\mathcal{O},\mathcal{O}}\;} & J_*(\mathcal{O} \circ \mathcal{O}) \\
{\scriptstyle \gamma} \downarrow & & & & \downarrow {\scriptstyle J_*\gamma} \\
\mathcal{O} & & \xrightarrow{\qquad\qquad * \qquad\qquad} & & J_*\mathcal{O}
\end{array}
$$

in $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ commute.

In particular, there exist two equivalent interpretations of a colored $*$-operad: The first option is to regard $(\mathcal{O}, \gamma, \mathbb{1})$ as an ordinary $\mathfrak{C}$-colored operad valued in $(\mathbf{M}, J, j)$, equipped with an operad morphism $* : (\mathcal{O}, \gamma, \mathbb{1}) \to \mathbf{Mon}(J_*)(\mathcal{O}, \gamma, \mathbb{1})$. The second option is to regard $* : \mathcal{O} \to J_*\mathcal{O}$ as a $*$-object in $(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*)$, equipped with the structure of a monoid consisting of the $*$-morphisms $\gamma : (* : \mathcal{O} \to J_*\mathcal{O}) \circ (* : \mathcal{O} \to J_*\mathcal{O}) \to (* : \mathcal{O} \to J_*\mathcal{O})$ and $\mathbb{1} : (J_{*0} : I_\circ \to J_*I_\circ) \to (* : \mathcal{O} \to J_*\mathcal{O})$.

5.4. PROPOSITION. *The category of $\mathfrak{C}$-colored $*$-operads with values in a cocomplete involutive closed symmetric monoidal category $(\mathbf{M}, J, j)$ is isomorphic to the category of $\mathfrak{C}$-colored operads with values in the cocomplete closed symmetric monoidal category $*$-$\mathbf{Obj}(\mathbf{M}, J, j)$, i.e. there exists an isomorphism*

$$*\text{-}\mathbf{Op}_{\mathfrak{C}}(\mathbf{M}, J, j) \;\cong\; \mathbf{Op}_{\mathfrak{C}}\big(*\text{-}\mathbf{Obj}(\mathbf{M}, J, j)\big)$$

of categories.

PROOF. This is proven by the following chain of $\mathbf{Cat}$-isomorphisms

$$\begin{aligned}
\text{-}\mathbf{Op}_{\mathfrak{C}}(\mathbf{M}, J, j) \;&=\; \mathbf{Mon}\big(\text{-}\mathbf{Obj}\big(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*\big)\big) \\
&\cong\; \mathbf{Mon}\big(\mathbf{SymSeq}_{\mathfrak{C}}\big(*\text{-}\mathbf{Obj}(\mathbf{M}, J, j)\big)\big) \;=\; \mathbf{Op}_{\mathfrak{C}}\big(*\text{-}\mathbf{Obj}(\mathbf{M}, J, j)\big) \quad,
\end{aligned}$$

where in the first step we used Definitions 5.2 and 3.23, in the second step Proposition 4.12 and in the last step Definition 5.1. ∎

5.5. REMARK. Proposition 5.4 may be summarized by the following slogan: Colored $*$-operads are the same things as colored operads in $*$-objects. We would like to stress that this result, whose proof relies on the whole spectrum of techniques for involutive category theory developed in [Jac12] and in the previous sections of the present paper, does not make the definition of operads as $*$-monoids unnecessary. Being able to switch between these two equivalent perspectives on colored $*$-operads is valuable for various reasons. On the one hand, when interpreted as ordinary colored operads in $*$-$\mathbf{Obj}(\mathbf{M}, J, j)$, it is straightforward to transfer structural results and techniques from ordinary operad theory to involutive operad theory. On the other hand, when interpreted according to Definition 5.2 as $*$-monoids, it is relatively easy to equip known examples of ordinary colored operads with a suitable $*$-involution, see Section 7 for a specific class of examples. Moreover, this perspective relates to the involutive monoid and monad theory initiated in [Jac12], see also Section 6 below.

We shall now study the behavior of colored $*$-operads under changing the underlying set of colors. Let $f : \mathfrak{C} \to \mathfrak{D}$ be a map between non-empty sets. By Theorem 4.10, we obtain an involutive monoidal functor $(f^*, \mathrm{id}_{f^*J_*}) : (\mathbf{SymSeq}_{\mathfrak{D}}(\mathbf{M}), J_*, j_*) \to (\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*)$. As a consequence of 2-functoriality of $*$-$\mathbf{Mon} : \mathbf{IMCat} \to \mathbf{Cat}$ (cf. Definition 3.23) and the definition of colored $*$-operads (cf. Definition 5.2), we obtain

5.6. PROPOSITION. *For every map $f : \mathfrak{C} \to \mathfrak{D}$ between non-empty sets, there exists a functor*

$$f^* := *\text{-}\mathbf{Mon}(f^*, \mathrm{id}_{f^*J_*}) \ : \ *\text{-}\mathbf{Op}_{\mathfrak{D}}(\mathbf{M}, J, j) \ \longrightarrow \ *\text{-}\mathbf{Op}_{\mathfrak{C}}(\mathbf{M}, J, j) \quad ,$$

which we call the pullback functor.

Using the pullback functor, we may define the category of ∗-operads with varying colors.

5.7. DEFINITION. We denote by $*\text{-}\mathbf{Op}(\mathbf{M}, J, j)$ the category of *colored ∗-operads* with values in $(\mathbf{M}, J, j)$. The objects are pairs $(\mathfrak{C}, \mathcal{O})$ consisting of a non-empty set $\mathfrak{C} \in \mathbf{Set}$ and a $\mathfrak{C}$-colored ∗-operad $\mathcal{O} \in *\text{-}\mathbf{Op}_{\mathfrak{C}}(\mathbf{M}, J, j)$. The morphisms are pairs $(f, \phi) : (\mathfrak{C}, \mathcal{O}) \to (\mathfrak{D}, \mathcal{P})$ consisting of a map $f : \mathfrak{C} \to \mathfrak{D}$ between non-empty sets and a $*\text{-}\mathbf{Op}_{\mathfrak{C}}(\mathbf{M}, J, j)$-morphism $\phi : \mathcal{O} \to f^*\mathcal{P}$.

5.8. REMARK. There exists a projection functor $\pi : *\text{-}\mathbf{Op}(\mathbf{M}, J, j) \to \mathbf{Set}$, given explicitly by $(\mathfrak{C}, \mathcal{O}) \mapsto \mathfrak{C}$, whose fiber $\pi^{-1}(\mathfrak{C})$ over $\emptyset \neq \mathfrak{C} \in \mathbf{Set}$ is isomorphic to the category $*\text{-}\mathbf{Op}_{\mathfrak{C}}(\mathbf{M}, J, j)$ of $\mathfrak{C}$-colored ∗-operads.

6. ∗-algebras over colored ∗-operads

A convenient description of algebras over colored operads is in terms of algebras over their associated monads. Let us briefly review the relevant constructions before generalizing them to the setting of involutive categories.

Let $\mathfrak{C} \in \mathbf{Set}$ be a non-empty set of colors. Recall that the category of $\mathfrak{C}$-*colored objects* with values in $\mathbf{M}$ is the functor category $\mathbf{M}^{\mathfrak{C}}$. We may equivalently regard $\mathbf{M}^{\mathfrak{C}}$ as the full subcategory of $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ consisting of all functors $X : \Sigma_{\mathfrak{C}} \times \mathfrak{C} \to \mathbf{M}$ such that $X\binom{t}{\underline{c}} = \emptyset$, for all $(\underline{c}, t) \in \Sigma_{\mathfrak{C}} \times \mathfrak{C}$ with length $|\underline{c}| \geq 1$. We introduce the notation $X_t := X\binom{t}{\emptyset}$, for all $t \in \mathfrak{C}$.

Given any $\mathfrak{C}$-colored operad $\mathcal{O} \in \mathbf{Op}_{\mathfrak{C}}(\mathbf{M})$, the endofunctor $\mathcal{O} \circ (-) : \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) \to \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ restricts to an endofunctor

$$\mathcal{O} \circ (-) \ : \ \mathbf{M}^{\mathfrak{C}} \ \longrightarrow \ \mathbf{M}^{\mathfrak{C}} \tag{33}$$

on the category of colored objects. Because $\mathcal{O}$ is by definition a monoid in $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$, with multiplication γ and unit $\mathbb{1}$, it follows that (33) canonically carries the structure of a *monad* in the category $\mathbf{M}^{\mathfrak{C}}$. We refer to [MacL98, Chapter VI] for details on monad theory. Concretely, the structure natural transformations $\gamma : \mathcal{O} \circ (\mathcal{O} \circ (-)) \to \mathcal{O} \circ (-)$ and $\mathbb{1} : \mathrm{Id}_{\mathbf{M}^{\mathfrak{C}}} \to \mathcal{O} \circ (-)$, which we denote with abuse of notation by the same symbols as the operadic composition and unit, are given by the components

$$\begin{array}{ccc}
\mathcal{O} \circ (\mathcal{O} \circ X) \xrightarrow{\gamma_X} \mathcal{O} \circ X & X \xrightarrow{\mathbb{1}_X} \mathcal{O} \circ X & \\
{\scriptstyle\cong}\uparrow \quad\quad {\scriptstyle\gamma\circ\mathrm{id}} \nearrow & {\scriptstyle\cong}\uparrow \quad\quad {\scriptstyle\mathbb{1}\circ\mathrm{id}} \nearrow & \\
(\mathcal{O} \circ \mathcal{O}) \circ X & I_\circ \circ X &
\end{array} \tag{34}$$

for all $X \in \mathbf{M}^{\mathfrak{C}}$.

6.1. DEFINITION. The category $\mathbf{Alg}(\mathcal{O})$ of *algebras over a $\mathfrak{C}$-colored operad* $\mathcal{O} \in \mathbf{Op}_{\mathfrak{C}}(\mathbf{M})$ is the category of algebras over the monad $\mathcal{O} \circ (-) : \mathbf{M}^{\mathfrak{C}} \to \mathbf{M}^{\mathfrak{C}}$. Concretely, an object of $\mathbf{Alg}(\mathcal{O})$ is a pair (A, α) consisting of an object $A \in \mathbf{M}^{\mathfrak{C}}$ and an $\mathbf{M}^{\mathfrak{C}}$-morphism $\alpha : \mathcal{O} \circ A \to A$ such that $\alpha (\mathcal{O} \circ \alpha) = \alpha \, \gamma_A$ and $\alpha \, \mathbb{1}_A = \mathrm{id}_A$. An $\mathbf{Alg}(\mathcal{O})$-morphism $\varphi : (A, \alpha) \to (B, \beta)$ is an $\mathbf{M}^{\mathfrak{C}}$-morphism $\varphi : A \to B$ that preserves the structure maps, i.e. $\beta (\mathcal{O} \circ \varphi) = \varphi \, \alpha$.

The assignment of the categories of algebras to colored operads is functorial

$$\mathbf{Alg} : \mathbf{Op}(\mathbf{M})^{\mathrm{op}} \longrightarrow \mathbf{Cat} \qquad (35)$$

with respect to the category $\mathbf{Op}(\mathbf{M})$ of colored operads with varying colors. Concretely, given any $\mathbf{Op}(\mathbf{M})$-morphism $(f, \phi) : (\mathfrak{C}, \mathcal{O}) \to (\mathfrak{D}, \mathcal{P})$, i.e. a map of non-empty sets $f : \mathfrak{C} \to \mathfrak{D}$ together with an $\mathbf{Op}_{\mathfrak{C}}(\mathbf{M})$-morphism $\phi : \mathcal{O} \to f^*\mathcal{P}$, we define a functor

$$(f, \phi)^* := \mathbf{Alg}(f, \phi) : \mathbf{Alg}(\mathcal{P}) \longrightarrow \mathbf{Alg}(\mathcal{O}) \qquad (36)$$

by setting

$$(f, \phi)^*(A, \alpha) := \left(f^*A, \ \mathcal{O} \circ f^*A \xrightarrow{\phi \circ \mathrm{id}} f^*\mathcal{P} \circ f^*A \xrightarrow{(f_2^*)_{\mathcal{P},A}} f^*(\mathcal{P} \circ A) \xrightarrow{f^*\alpha} f^*A \right) \ ,$$

for all $\mathcal{P}$-algebras $(A, \alpha : \mathcal{P} \circ A \to A) \in \mathbf{Alg}(\mathcal{P})$. (The natural transformation f_2^* was defined in (28).) Furthermore, as a consequence of the adjoint lifting theorem [Bor94, Chapter 4.5], it follows that the functor $(f, \phi)^*$ admits a left adjoint (called *operadic left Kan extension*), i.e. we obtain an adjunction

$$(f, \phi)_! : \mathbf{Alg}(\mathcal{O}) \rightleftarrows \mathbf{Alg}(\mathcal{P}) : (f, \phi)^* \quad , \qquad (37)$$

for every $\mathbf{Op}(\mathbf{M})$-morphism $(f, \phi) : (\mathfrak{C}, \mathcal{O}) \to (\mathfrak{D}, \mathcal{P})$. See for example [BM07, BSW17] for further details and also [BSW17] for applications of these adjunctions to quantum field theory.

We develop now a generalization of these definitions and constructions to the setting of involutive categories. Let $(\mathbf{M}, J, j)$ be a cocomplete involutive closed symmetric monoidal category. The involutive analog of the category of $\mathfrak{C}$-colored objects is obtained by using the exponential 2-functor (24) to form $(\mathbf{M}, J, j)^{\mathrm{triv}(\mathfrak{C})} \in \mathbf{ICat}$. Notice that the full subcategory embedding $\mathbf{M}^{\mathfrak{C}} \hookrightarrow \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ can be equipped with an obvious involutive structure, thus providing an $\mathbf{ICat}$-isomorphism between $(\mathbf{M}, J, j)^{\mathrm{triv}(\mathfrak{C})}$ and the involutive category obtained by restricting the involutive structure on $(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), J_*, j_*)$ to the full subcategory $\mathbf{M}^{\mathfrak{C}} \subseteq \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$. In the following we shall always suppress this isomorphism and identify the involutive categories

$$(\mathbf{M}^{\mathfrak{C}}, J_*, j_*) \cong (\mathbf{M}, J, j)^{\mathrm{triv}(\mathfrak{C})} \quad .$$

Given a $\mathfrak{C}$-colored ∗-operad $\mathcal{O} \in *\text{-}\mathbf{Op}_{\mathfrak{C}}(\mathbf{M}, J, j)$ in the sense of Definition 5.2 (see also Remark 5.3 for a more explicit description), we obtain an involutive endofunctor

$$\big(\mathcal{O} \circ (-), \nu\big) : (\mathbf{M}^{\mathfrak{C}}, J_*, j_*) \longrightarrow (\mathbf{M}^{\mathfrak{C}}, J_*, j_*) \tag{38}$$

with the natural transformation $\nu : \mathcal{O} \circ J_*(-) \to J_*(\mathcal{O} \circ (-))$ defined by the components

$$\mathcal{O} \circ J_* X \xrightarrow{\quad \nu_X \quad} J_*\big(\mathcal{O} \circ X\big)$$

with the lower path $\mathcal{O} \circ J_* X \xrightarrow{*\circ \mathrm{id}} J_*\mathcal{O} \circ J_* X \xrightarrow{(J_{*2})_{\mathcal{O},X}} J_*(\mathcal{O} \circ X)$

for all $X \in \mathbf{M}^{\mathfrak{C}}$, where $* : \mathcal{O} \to J_*\mathcal{O}$ denotes the ∗-involution on $\mathcal{O}$.

6.2. PROPOSITION. *Given any $\mathfrak{C}$-colored ∗-operad $(\mathcal{O}, \gamma, \mathbb{1}, *) \in *\text{-}\mathbf{Op}_{\mathfrak{C}}(\mathbf{M}, J, j)$, the components given in (34) define involutive natural transformations $\gamma : (\mathcal{O} \circ (-), \nu) (\mathcal{O} \circ (-), \nu) \to (\mathcal{O} \circ (-), \nu)$ and $\mathbb{1} : (\mathrm{Id}_{\mathbf{M}^{\mathfrak{C}}}, \mathrm{id}_{J_*}) \to (\mathcal{O} \circ (-), \nu)$ for the involutive endofunctor (38). In the terminology of [Jac12, Definition 7], the triple $\big((\mathcal{O} \circ (-), \nu), \gamma, \mathbb{1}\big)$ is an involutive monad in $(\mathbf{M}^{\mathfrak{C}}, J_*, j_*)$.*

PROOF. This statement is analogous [Jac12, Example 3 (i)] and may be proven by a slightly lengthy diagram chase argument. ∎

The category of algebras $\mathbf{Alg}(\mathcal{O})$ (cf. Definition 6.1) over (the underlying colored operad of) a $\mathfrak{C}$-colored ∗-operad $\mathcal{O} \in *\text{-}\mathbf{Op}_{\mathfrak{C}}(\mathbf{M}, J, j)$ can be equipped with a canonical involutive structure

$$\big(\mathbf{Alg}(\mathcal{O}), J_{\mathcal{O}}, j_{\mathcal{O}}\big) \in \mathbf{ICat} \quad , \tag{39}$$

see also [Jac12, Proposition 3] for a similar construction. Concretely, the endofunctor $J_{\mathcal{O}} : \mathbf{Alg}(\mathcal{O}) \to \mathbf{Alg}(\mathcal{O})$ acts on objects $(A, \alpha) \in \mathbf{Alg}(\mathcal{O})$ as

$$J_{\mathcal{O}}\big(A, \alpha\big) := \Big(J_* A, \; \mathcal{O} \circ J_* A \xrightarrow{*\circ \mathrm{id}} J_*\mathcal{O} \circ J_* A \xrightarrow{(J_{*2})_{\mathcal{O},A}} J_*\big(\mathcal{O} \circ A\big) \xrightarrow{J_*\alpha} J_* A \Big)$$

and on morphisms as J_*. The natural transformation $j_{\mathcal{O}} : \mathrm{Id}_{\mathbf{Alg}(\mathcal{O})} \to J_{\mathcal{O}}^2$ is defined by the components $j_{\mathcal{O}(A,\alpha)} := j_{*A}$, for all $(A, \alpha) \in \mathbf{Alg}(\mathcal{O})$. This allows us to introduce the concept of ∗-algebras over colored ∗-operads.

6.3. DEFINITION. The category of *∗-algebras* over $\mathcal{O} \in *\text{-}\mathbf{Op}_{\mathfrak{C}}(\mathbf{M}, J, j)$ is defined by evaluating the 2-functor $*\text{-}\mathbf{Obj} : \mathbf{ICat} \to \mathbf{Cat}$ (cf. (3)) on the involutive category of $\mathcal{O}$-algebras (39), i.e.

$$*\text{-}\mathbf{Alg}(\mathcal{O}) := *\text{-}\mathbf{Obj}\big(\mathbf{Alg}(\mathcal{O}), J_{\mathcal{O}}, j_{\mathcal{O}}\big) \quad .$$

6.4. REMARK. Unpacking this definition, we obtain that a $*$-algebra over $\mathcal{O}$ is a triple $(A, \alpha, *_A) \in *\text{-}\mathbf{Alg}(\mathcal{O})$ consisting of a $\mathfrak{C}$-colored object $A \in \mathbf{M}^{\mathfrak{C}}$ and two $\mathbf{M}^{\mathfrak{C}}$-morphisms $\alpha : \mathcal{O} \circ A \to A$ and $*_A : A \to J_*A$, which satisfy the following conditions:

(1) $(A, \alpha) \in \mathbf{Alg}(\mathcal{O})$ is an algebra over the $\mathfrak{C}$-colored operad $\mathcal{O}$;

(2) $(*_A : A \to J_*A) \in *\text{-}\mathbf{Obj}(\mathbf{M}^{\mathfrak{C}}, J_*, j_*)$ is a $*$-object in the involutive category $(\mathbf{M}^{\mathfrak{C}}, J_*, j_*)$;

(3) these two structures are compatible, i.e. the diagram

$$
\begin{array}{ccc}
\mathcal{O} \circ A & \xrightarrow{\quad\quad\quad\quad\quad \alpha \quad\quad\quad\quad\quad} & A \\
{\scriptstyle \mathcal{O} \circ *_A}\downarrow & & \downarrow{\scriptstyle *_A} \\
\mathcal{O} \circ J_*A \xrightarrow[*\circ\mathrm{id}]{} J_*\mathcal{O} \circ J_*A \xrightarrow[(J_{*2})_{\mathcal{O},A}]{} J_*(\mathcal{O} \circ A) \xrightarrow[J_*\alpha]{} & & J_*A
\end{array}
$$

in $\mathbf{M}^{\mathfrak{C}}$ commutes.

A $*$-algebra morphism $\varphi : (A, \alpha, *_A) \to (B, \beta, *_B)$ is an $\mathbf{M}^{\mathfrak{C}}$-morphism $\varphi : A \to B$ preserving the structure maps and $*$-involutions, i.e. $\beta\,(\mathcal{O} \circ \varphi) = \varphi\,\alpha$ and $*_B\,\varphi = (J_*\varphi)\,*_A$.

Similarly to (35), we observe that the assignment of the involutive categories of algebras to colored $*$-operads is functorial

$$\mathbf{Alg} : *\text{-}\mathbf{Op}(\mathbf{M}, J, j)^{\mathrm{op}} \longrightarrow \mathbf{ICat} \tag{40}$$

with respect to the category $*\text{-}\mathbf{Op}(\mathbf{M}, J, j)$ of colored $*$-operads with varying colors (cf. Definition 5.7). Concretely, this functor assigns to a $*\text{-}\mathbf{Op}(\mathbf{M}, J, j)$-morphism $(f, \phi) : (\mathbf{C}, \mathcal{O}) \to (\mathbf{D}, \mathcal{P})$ the involutive functor

$$\left((f, \phi)^*, \mathrm{id}_{(f,\phi)^* J_\mathcal{P}}\right) : \left(\mathbf{Alg}(\mathcal{P}), J_\mathcal{P}, j_\mathcal{P}\right) \longrightarrow \left(\mathbf{Alg}(\mathcal{O}), J_\mathcal{O}, j_\mathcal{O}\right) \quad, \tag{41}$$

which is given by equipping the pullback functor (36) with the trivial involutive structure $\mathrm{id}_{(f,\phi)^* J_\mathcal{P}} : (f, \phi)^* J_\mathcal{P} \to J_\mathcal{O}\,(f, \phi)^* = (f, \phi)^* J_\mathcal{P}$. (Showing that $J_\mathcal{O}\,(f, \phi)^* = (f, \phi)^* J_\mathcal{P}$ requires a brief check.) As a consequence of (40) and (2-)functoriality of $*\text{-}\mathbf{Obj} : \mathbf{ICat} \to \mathbf{Cat}$ (cf. (3)), we obtain that also the assignment of the categories of $*$-algebras (cf. Definition 6.3) to colored $*$-operad is functorial

$$*\text{-}\mathbf{Alg} : *\text{-}\mathbf{Op}(\mathbf{M}, J, j)^{\mathrm{op}} \longrightarrow \mathbf{Cat} \quad. \tag{42}$$

Given any $*\text{-}\mathbf{Op}(\mathbf{M}, J, j)$-morphism $(f, \phi) : (\mathfrak{C}, \mathcal{O}) \to (\mathfrak{D}, \mathcal{P})$, we denote the corresponding functor simply by

$$(f, \phi)^* := *\text{-}\mathbf{Alg}(f, \phi) : *\text{-}\mathbf{Alg}(\mathcal{P}) \longrightarrow *\text{-}\mathbf{Alg}(\mathcal{O}) \quad.$$

Concretely, it is given by evaluating the 2-functor $*\text{-}\mathbf{Obj} : \mathbf{ICat} \to \mathbf{Cat}$ given in (3) on the involutive functor (41).

6.5. REMARK. Recalling Proposition 5.4, there exists an isomorphism

$$\ast\text{-}\mathbf{Op}_{\mathfrak{C}}(\mathbf{M}, J, j) \;\cong\; \mathbf{Op}_{\mathfrak{C}}\big(\ast\text{-}\mathbf{Obj}(\mathbf{M}, J, j)\big)$$

between the category of colored ∗-operads with values in $(\mathbf{M}, J, j)$ and the category of ordinary colored operads with values in $\ast\text{-}\mathbf{Obj}(\mathbf{M}, J, j)$. This isomorphism clearly extends to the categories of colored (∗-)operads with varying colors. As a consequence, there exists a second option for assigning categories of ∗-algebras to colored ∗-operads, which is given by the lower path in the diagram

$$\ast\text{-}\mathbf{Op}(\mathbf{M}, J, j)^{\mathrm{op}} \xrightarrow{\quad \ast\text{-}\mathbf{Alg} \quad} \mathbf{Cat} \tag{43}$$

(with diagonal arrow $\cong$ to $\mathbf{Op}\big(\ast\text{-}\mathbf{Obj}(\mathbf{M}, J, j)\big)^{\mathrm{op}}$, vertical arrow $\cong$, and $\mathbf{Alg}$)

where $\mathbf{Alg}$ denotes the functor given in (35). Similarly to [Jac12, Proposition 3], one can prove that the diagram (43) commutes up to a natural isomorphism, hence the second option for assigning the categories of ∗-algebras is equivalent to our original definition in (42).

We would like to emphasize that the main reason why the diagram in (43) commutes is that the conditions (1-3) in Remark 6.4 admit two equivalent interpretations: The first option is to regard $(A, \alpha) \in \mathbf{Alg}(\mathcal{O})$ as an algebra over the $\mathfrak{C}$-colored operad $\mathcal{O}$ and $\ast_A : (A, \alpha) \to J_{\mathcal{O}}(A, \alpha)$ as an $\mathbf{Alg}(\mathcal{O})$-morphism. One observes that $\big(\ast_A : (A, \alpha) \to J_{\mathcal{O}}(A, \alpha)\big) \in \ast\text{-}\mathbf{Obj}\big(\mathbf{Alg}(\mathcal{O}), J_{\mathcal{O}}, j_{\mathcal{O}}\big)$ is a ∗-object in the involutive category $\big(\mathbf{Alg}(\mathcal{O}), J_{\mathcal{O}}, j_{\mathcal{O}}\big)$, which recovers our original Definition 6.3 and hence the upper path in the diagram (43). The second option is to regard $(\ast_A : A \to J_{\ast}A) \in \ast\text{-}\mathbf{Obj}(\mathbf{M}, J, j)^{\mathfrak{C}}$ as a $\mathfrak{C}$-colored object in $\ast\text{-}\mathbf{Obj}(\mathbf{M}, J, j)$ and $\alpha : (\ast : \mathcal{O} \to J_{\ast}\mathcal{O}) \circ (\ast_A : A \to J_{\ast}A) \to (\ast_A : A \to J_{\ast}A)$ as a $\ast\text{-}\mathbf{Obj}(\mathbf{M}, J, j)^{\mathfrak{C}}$-morphism. One observes that this defines an algebra over $\mathcal{O}$, regarded as an object in $\mathbf{Op}_{\mathfrak{C}}(\ast\text{-}\mathbf{Obj}(\mathbf{M}, J, j))$, which recovers the lower path in the diagram (43).

We conclude this section by noticing that (41) equips the right adjoint functor $(f, \phi)^{\ast} : \mathbf{Alg}(\mathcal{P}) \to \mathbf{Alg}(\mathcal{O})$ of the adjunction (37) with an involutive structure. Hence, applying Proposition 2.13, we obtain a canonical involutive structure $\lambda_{(f, \phi)} : (f, \phi)_! \, J_{\mathcal{O}} \to J_{\mathcal{P}} \, (f, \phi)_!$ on the left adjoint functor $(f, \phi)_! : \mathbf{Alg}(\mathcal{O}) \to \mathbf{Alg}(\mathcal{P})$ together with an involutive adjunction

$$\big((f, \phi)_!, \lambda_{(f, \phi)}\big) : \big(\mathbf{Alg}(\mathcal{O}), J_{\mathcal{O}}, j_{\mathcal{O}}\big) \rightleftarrows \big(\mathbf{Alg}(\mathcal{P}), J_{\mathcal{P}}, j_{\mathcal{P}}\big) : \big((f, \phi)^{\ast}, \mathrm{id}_{(f, \phi)^{\ast} \, J_{\mathcal{P}}}\big) \tag{44}$$

Because 2-functors preserve adjunctions, we may apply the 2-functor $\ast\text{-}\mathbf{Obj} : \mathbf{ICat} \to \mathbf{Cat}$ to the involutive adjunction (44) in order to obtain an adjunction

$$(f, \phi)_! : \ast\text{-}\mathbf{Alg}(\mathcal{O}) \rightleftarrows \ast\text{-}\mathbf{Alg}(\mathcal{P}) : (f, \phi)^{\ast} \tag{45}$$

between the categories of ∗-algebras. Summing up, we have proven

6.6. THEOREM. *Associated to every $*$-$\mathbf{Op}(\mathbf{M}, J, j)$-morphism $(f, \phi) : (\mathfrak{C}, \mathcal{O}) \to (\mathfrak{D}, \mathcal{P})$, there is an involutive adjunction (44) between the involutive categories of algebras and an adjunction (45) between the categories of $*$-algebras.*

7. Algebraic quantum field theory $*$-operads

As an application of the concepts and techniques developed in this paper, we study the family of colored operads arising in algebraic quantum field theory [BSW17] within the setting of involutive category theory. The main motivation for promoting these colored operads to colored $*$-operads is due to quantum physics: A quantum mechanical system is described not only by an associative and unital algebra over $\mathbb{C}$, but rather by an associative and unital $*$-algebra A over $\mathbb{C}$. Here the relevant type of $*$-algebras is the reversing one, i.e. $(a\,b)^* = b^*\,a^*$. The additional structure given by the complex anti-linear $*$-involution is essential for quantum physics: It enters the GNS construction that is crucial to recover the usual probabilistic interpretation of quantum theory in terms of Hilbert spaces.

Throughout this section we let $(\mathbf{M}, J, j)$ be any cocomplete involutive closed symmetric monoidal category. In traditional quantum field theory, one would choose the example given by complex vector spaces $(\mathbf{Vec}_{\mathbb{C}}, \overline{(-)}, \mathrm{id}_{\mathrm{Id}_{\mathbf{Vec}_{\mathbb{C}}}})$, see Examples 2.4, 2.19 and 3.11 for details. More modern approaches to quantum *gauge* theories, however, have lead to the concept of *homotopical quantum field theory* and crucially rely on using different and richer target categories, such as chain complexes and other monoidal model categories, see e.g. [BSS15, BS17, BSW17, BSW18, Yau18] for algebraic quantum field theory and also [CG17] for similar developments in factorization algebras. Hence, it is justified to present our constructions with this high level of generality.

Let us provide a very brief review of the algebraic quantum field theory operads constructed in [BSW17]. We refer to this paper for more details and the physical motivations.

7.1. DEFINITION. An *orthogonality relation* on a small category $\mathbf{C}$ is a subset $\perp \subseteq \mathrm{Mor}\,\mathbf{C}_{\mathrm{t}} \times_{\mathrm{t}} \mathrm{Mor}\,\mathbf{C}$ of the set of pairs of $\mathbf{C}$-morphisms with coinciding target that is symmetric, i.e. $(f_1, f_2) \in \perp$ implies $(f_2, f_1) \in \perp$, and stable under post- and pre-composition, i.e. $(f_1, f_2) \in \perp$ implies $(g f_1, g f_2) \in \perp$ and $(f_1 h_1, f_2 h_2) \in \perp$ for all composable $\mathbf{C}$-morphisms g, h_1 and h_2. We call elements $(f_1, f_2) \in \perp$ *orthogonal pairs* and also write $f_1 \perp f_2$. A pair $(\mathbf{C}, \perp)$ consisting of a small category $\mathbf{C}$ and an orthogonality relation $\perp$ on $\mathbf{C}$ is called an *orthogonal category.*

7.2. EXAMPLE. On the terminal category $\mathbf{C} = \{\bullet\}$ there exist precisely two different orthogonality relations, namely $\perp = \emptyset$ and $\perp = \{(\mathrm{id}_\bullet, \mathrm{id}_\bullet)\}$. The corresponding orthogonal categories $(\{\bullet\}, \emptyset)$ and $(\{*\}, \{(\mathrm{id}_\bullet, \mathrm{id}_\bullet)\})$ will be used below to illustrate our constructions for the simplest possible examples.

7.3. EXAMPLE. The following is the prime example of an orthogonal category, see e.g. [BFV03, BSW17] for the details. Let $\mathbf{Loc}$ be the category of globally hyperbolic Lorentzian manifolds (of a fixed dimension ≥ 2) with morphisms given by causally convex and open

isometric embeddings. Two morphisms $f_1 : M_1 \to M$ and $f_2 : M_2 \to M$ to a common Lorentzian manifold M are declared to be orthogonal, $f_1 \perp f_2$, if and only if their images are causally disjoint subsets of M, i.e. there exists no causal curve connecting $f_1(M_1)$ and $f_2(M_2)$. The resulting orthogonal category $(\mathbf{Loc}, \perp)$ describes the physical concept of *spacetimes* (in the sense of Einstein's general relativity) and their causal relations. It provides the foundation for formulating locally covariant algebraic quantum field theory [BFV03].

Another related example is obtained by the following construction: Choosing any globally hyperbolic Lorentzian manifold $M \in \mathbf{Loc}$, consider the over category $\mathbf{Loc}/M$ together with the forgetful functor $U : \mathbf{Loc}/M \to \mathbf{Loc}$. The orthogonality relation $\perp$ on $\mathbf{Loc}$ pulls back under U to an orthogonality relation $\perp_M$ on $\mathbf{Loc}/M$. Explicitly, two morphisms g_1 and g_2 to a common target in $\mathbf{Loc}/M$ are orthogonal with respect to $\perp_M$ if and only if $U(g_1) \perp U(g_2)$ in $(\mathbf{Loc}, \perp)$. The resulting orthogonal category $(\mathbf{Loc}/M, \perp_M)$ describes causally convex open subsets of the fixed globally hyperbolic Lorentzian manifold M (interpreted physically as the universe) and their causal relations. It provides the foundation for formulating Haag-Kastler type algebraic quantum field theories [HK64].

Let $(\mathbf{C}, \perp)$ be an orthogonal category and denote by $\mathbf{C}_0$ the set of objects of $\mathbf{C}$. To define the algebraic quantum field theory operad associated to $(\mathbf{C}, \perp)$ it is convenient to introduce the following notations: Given $\underline{c} = (c_1, \ldots, c_n) \in \Sigma_{\mathbf{C}_0}$ and $t \in \mathbf{C}$, we denote by $\mathbf{C}(\underline{c}, t) := \prod_{i=1}^{n} \mathbf{C}(c_i, t)$ the product of Hom-sets. Its elements will be denoted by symbols like $\underline{f} = (f_1, \ldots, f_n) \in \mathbf{C}(\underline{c}, t)$. The following definition is due to [BSW17].

7.4. DEFINITION. Let $(\mathbf{C}, \perp)$ be an orthogonal category. The *algebraic quantum field theory operad* of type $(\mathbf{C}, \perp)$ with values in $\mathbf{M}$ is the $\mathbf{C}_0$-colored operad $\mathcal{O}_{(\mathbf{C}, \perp)} \in \mathbf{Op}_{\mathbf{C}_0}(\mathbf{M})$ defined as follows:

(a) For any $(\underline{c}, t) \in \Sigma_{\mathbf{C}_0} \times \mathbf{C}_0$, we set

$$\mathcal{O}_{(\mathbf{C}, \perp)} \binom{t}{\underline{c}} := \left(\Sigma_{|\underline{c}|} \times \mathbf{C}(\underline{c}, t) \right) / \sim_{\perp} \otimes I \in \mathbf{M} \quad,$$

where the equivalence relation is as follows: $(\sigma, \underline{f}) \sim_{\perp} (\sigma', \underline{f}')$ if and only if (1) $\underline{f} = \underline{f}'$ and (2) the right permutation $\sigma \sigma'^{-1} : \underline{f} \sigma^{-1} \to \underline{f} \sigma'^{-1}$ is generated by transpositions of adjacent orthogonal pairs.

(b) For any $\Sigma_{\mathbf{C}_0} \times \mathbf{C}_0$-morphism $\sigma' : (\underline{c}, t) \to (\underline{c} \sigma', t)$, we set

$$\mathcal{O}_{(\mathbf{C}, \perp)}(\sigma') : \mathcal{O}_{(\mathbf{C}, \perp)} \binom{t}{\underline{c}} \longrightarrow \mathcal{O}_{(\mathbf{C}, \perp)} \binom{t}{\underline{c} \sigma'}$$

to be the $\mathbf{M}$-morphism induced by the map of sets $[\sigma, \underline{f}] \mapsto [\sigma \sigma', \underline{f} \sigma']$ via functoriality of the $\mathbf{Set}$-tensoring.

(c) The operadic composition is determined by the $\mathbf{M}$-morphisms

$$\gamma : \mathcal{O}_{(\mathbf{C}, \perp)} \binom{t}{\underline{a}} \otimes \bigotimes_{i=1}^{m} \mathcal{O}_{(\mathbf{C}, \perp)} \binom{a_i}{\underline{b}_i} \longrightarrow \mathcal{O}_{(\mathbf{C}, \perp)} \binom{t}{\underline{b}_1 \otimes \cdots \otimes \underline{b}_m}$$

induced by the maps of sets

$$[\sigma, \underline{f}] \otimes \bigotimes_{i=1}^{m} [\sigma_i, \underline{g}_i] \longmapsto \left[\sigma(\sigma_1, \ldots, \sigma_m), \underline{f}(\underline{g}_1, \ldots, \underline{g}_m)\right] \quad .$$

Here $\sigma(\sigma_1, \ldots, \sigma_m) = \sigma\langle|\underline{b}_{\sigma^{-1}(1)}|, \ldots, |\underline{b}_{\sigma^{-1}(m)}|\rangle \, (\sigma_1 \oplus \cdots \oplus \sigma_m)$ denotes the group multiplication in $\Sigma_{|\underline{b}_1|+\cdots+|\underline{b}_m|}$ of the corresponding block permutation and block sum permutation, and $\underline{f}(\underline{g}_1, \ldots, \underline{g}_m) = (f_1 \, g_{11}, \ldots, f_m \, g_{m|\underline{b}_m|})$ is given by composition of $\mathbf{C}$-morphisms.

(d) The operadic unit is determined by the $\mathbf{M}$-morphisms

$$\mathbb{1} : I \longrightarrow \mathcal{O}_{(\mathbf{C}, \perp)} \binom{t}{t}$$

induced by the maps of sets $\bullet \mapsto (e, \mathrm{id}_t)$, where $e \in \Sigma_1$ is the group unit, via functoriality of the $\mathbf{Set}$-tensoring.

The following results are proven in [BSW17].

7.5. THEOREM. *For any orthogonal category* $(\mathbf{C}, \perp)$, *Definition 7.4 defines a* $\mathbf{C}_0$-*colored operad* $\mathcal{O}_{(\mathbf{C}, \perp)} \in \mathbf{Op}_{\mathbf{C}_0}(\mathbf{M})$. *Furthermore, there exists an isomorphism*

$$\mathbf{Alg}(\mathcal{O}_{(\mathbf{C}, \perp)}) \cong \mathbf{Mon}(\mathbf{M})^{(\mathbf{C}, \perp)}$$

between the category of $\mathcal{O}_{(\mathbf{C}, \perp)}$-*algebras and the category of* $\perp$-*commutative functors from* $\mathbf{C}$ *to* $\mathbf{Mon}(\mathbf{M})$. *Concretely, the latter is the full subcategory of the functor category* $\mathbf{Mon}(\mathbf{M})^{\mathbf{C}}$ *consisting of all functors* $\mathfrak{A} : \mathbf{C} \to \mathbf{Mon}(\mathbf{M})$ *for which the diagrams*

$$\begin{array}{ccc}
\mathfrak{A}(c_1) \otimes \mathfrak{A}(c_2) & \xrightarrow{\mathfrak{A}(f_1) \otimes \mathfrak{A}(f_2)} & \mathfrak{A}(c) \otimes \mathfrak{A}(c) \\
{\scriptstyle \mathfrak{A}(f_1) \otimes \mathfrak{A}(f_2)} \downarrow & & \downarrow {\scriptstyle \mu_c^{\mathrm{op}}} \\
\mathfrak{A}(c) \otimes \mathfrak{A}(c) & \xrightarrow[\mu_c]{} & \mathfrak{A}(c)
\end{array} \tag{46}$$

in $\mathbf{M}$ *commute, for all orthogonal pairs* $f_1 \perp f_2$. *Here* μ_c *(respectively* μ_c^{op}*) denotes the (opposite) multiplication in the monoid* $\mathfrak{A}(c)$.

7.6. EXAMPLE. Recalling the orthogonal categories from Example 7.2, one easily observes that the family of colored operads in Definition 7.4 includes the associative operad and the commutative operad as very special cases. Concretely, $\mathcal{O}_{(\{\bullet\}, \emptyset)} = \mathsf{As}$ is the associative operad and hence $\mathbf{Alg}(\mathcal{O}_{(\{\bullet\}, \emptyset)}) \cong \mathbf{Mon}(\mathbf{M})$ is the category of monoids in $\mathbf{M}$. Similarly, $\mathcal{O}_{(\{\bullet\}, \{(\mathrm{id}_\bullet, \mathrm{id}_\bullet)\})} = \mathsf{Com}$ is the commutative operad and hence $\mathbf{Alg}(\mathcal{O}_{(\{\bullet\}, \{(\mathrm{id}_\bullet, \mathrm{id}_\bullet)\})}) \cong \mathbf{CMon}(\mathbf{M})$ is the category of commutative monoids in $\mathbf{M}$.

7.7. EXAMPLE. More interestingly, the colored operad $\mathcal{O}_{(\mathbf{Loc},\perp)}$ associated to the orthogonal category $(\mathbf{Loc}, \perp)$ from Example 7.3 describes locally covariant algebraic quantum field theories in the sense of [BFV03], i.e. $\mathbf{Alg}(\mathcal{O}_{(\mathbf{Loc},\perp)}) \cong \mathbf{QFT}(\mathbf{Loc})$ is the category of such theories. The $\perp$-commutativity property in (46) formalizes the Einstein causality axiom, which states that observables localized in causally disjoint subsets commute with each other. The colored operad $\mathcal{O}_{(\mathbf{Loc}/M,\perp_M)}$ associated to the over category describes Haag-Kastler type algebraic quantum field theories [HK64] on the fixed Lorentzian manifold $M \in \mathbf{Loc}$, i.e. $\mathbf{Alg}(\mathcal{O}_{(\mathbf{Loc}/M,\perp_M)}) \cong \mathbf{QFT}(M)$ is the category of such theories. By Theorem 7.5, these are characterized as pre-cosheaves of monoids on M satisfying Einstein causality.

We will now endow $\mathcal{O}_{(\mathbf{C},\perp)}$ with the structure of a colored ∗-operad. According to Remark 5.3, this amounts to equipping the symmetric sequence underlying $\mathcal{O}_{(\mathbf{C},\perp)}$ with the structure of a ∗-object in the involutive monoidal category $(\mathbf{SymSeq}_{\mathbf{C}_0}(\mathbf{M}), J_*, j_*)$ that is compatible with the operadic compositions and units. Let us define a $\mathbf{SymSeq}_{\mathbf{C}_0}(\mathbf{M})$-morphism $* : \mathcal{O}_{(\mathbf{C},\perp)} \to J_*\mathcal{O}_{(\mathbf{C},\perp)}$ by setting, for all $(\underline{c}, t) \in \Sigma_{\mathbf{C}_0} \times \mathbf{C}_0$,

$$
\begin{array}{ccc}
\mathcal{O}_{(\mathbf{C},\perp)}\binom{t}{\underline{c}} & \xrightarrow{\quad * \quad} & J\mathcal{O}_{(\mathbf{C},\perp)}\binom{t}{\underline{c}} \\
\Big\| & & \Big\downarrow{\scriptstyle \cong} \\
\big(\Sigma_{|\underline{c}|} \times \mathbf{C}(\underline{c}, t)\big)/\!\!\sim_\perp \otimes\, I & \xrightarrow[\rho_{|\underline{c}|} \otimes J_0]{} & \big(\Sigma_{|\underline{c}|} \times \mathbf{C}(\underline{c}, t)\big)/\!\!\sim_\perp \otimes\, JI
\end{array}
\tag{47}
$$

to be the $\mathbf{M}$-morphism induced by the map of sets $\rho_{|\underline{c}|} : [\sigma, \underline{f}] \mapsto [\rho_{|\underline{c}|}\sigma, \underline{f}]$, where $\rho_{|\underline{c}|} \in \Sigma_{|\underline{c}|}$ is the order-reversal permutation from Example 2.5, and the $\mathbf{M}$-morphism $J_0 : I \to JI$. (For the right vertical arrow recall that J is self-adjoint, hence it preserves the $\mathbf{Set}$-tensoring.) Evidently, (47) is equivariant with respect to the action of permutations given in Definition 7.4 (b), hence it defines a $\mathbf{SymSeq}_{\mathbf{C}_0}(\mathbf{M})$-morphism. It is, moreover, straightforward to verify that $(* : \mathcal{O}_{(\mathbf{C},\perp)} \to J_*\mathcal{O}_{(\mathbf{C},\perp)}) \in *\text{-}\mathbf{Obj}(\mathbf{SymSeq}_{\mathbf{C}_0}(\mathbf{M}), J_*, j_*)$ is a ∗-object by using that $\rho_{|\underline{c}|}^2 = e$ is the identity permutation and that $j : \mathrm{Id}_\mathbf{M} \to J^2$ is by hypothesis a monoidal natural transformation.

7.8. PROPOSITION. *Endowing the colored operad $(\mathcal{O}_{(\mathbf{C},\perp)}, \gamma, \mathbb{1}) \in \mathbf{Op}_{\mathbf{C}_0}(\mathbf{M})$ from Definition 7.4 with the $*$-involution $* : \mathcal{O}_{(\mathbf{C},\perp)} \to J_*\mathcal{O}_{(\mathbf{C},\perp)}$ defined in (47) yields a colored $*$-operad*

$$
\big(\mathcal{O}_{(\mathbf{C},\perp)}, \gamma, \mathbb{1}, *\big) \in *\text{-}\mathbf{Op}_{\mathbf{C}_0}(\mathbf{M}, J, j) \quad .
$$

PROOF. It remains to check the compatibility conditions in Remark 5.3 (3). This is a straightforward calculation using standard permutation group properties. ■

Let us now study the ∗-algebras over the colored ∗-operad $\mathcal{O}_{(\mathbf{C},\perp)} \in *\text{-}\mathbf{Op}_{\mathbf{C}_0}(\mathbf{M}, J, j)$ defined in Proposition 7.8. Using the explicit description explained in Remark 6.4, these are triples $(A, \alpha, *_A)$ consisting of an algebra (A, α) over $\mathcal{O}_{(\mathbf{C},\perp)}$ together with a compatible ∗-involution $*_A : A \to J_*A$. Using Theorem 7.5 to identify (A, α) with a $\perp$-commutative

functor $\mathfrak{A} : \mathbf{C} \to \mathbf{Mon(M)}$, the $*$-involution $*_A : A \to J_*A$ is identified with a family of $\mathbf{M}$-morphisms

$$*_c : \mathfrak{A}(c) \longrightarrow J\mathfrak{A}(c) \quad ,$$

for all $c \in \mathbf{C}$. As a consequence of Remark 6.4 (3), such family has to satisfy the following basic conditions:

(1) *Compatibility with monoid structure:* For all $c \in \mathbf{C}$,

$$\mathfrak{A}(c) \otimes \mathfrak{A}(c) \xrightarrow{*_c \otimes *_c} J\mathfrak{A}(c) \otimes J\mathfrak{A}(c) \xrightarrow{J_{2\mathfrak{A}(c),\mathfrak{A}(c)}} J\big(\mathfrak{A}(c) \otimes \mathfrak{A}(c)\big) \qquad (48)$$

$$\downarrow{\mu_c} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{J\mu_c^{\mathrm{op}}}$$

$$\mathfrak{A}(c) \xrightarrow{\qquad\qquad\qquad *_c \qquad\qquad\qquad} J\mathfrak{A}(c)$$

where μ_c (respectively μ_c^{op}) is the (opposite) multiplication on $\mathfrak{A}(c) \in \mathbf{Mon(M)}$, and

$$I \xrightarrow{\quad J_0 \quad} JI \qquad\qquad (49)$$

$$\downarrow{\eta_c} \qquad\qquad \downarrow{J\eta_c}$$

$$\mathfrak{A}(c) \xrightarrow{\quad *_c \quad} J\mathfrak{A}(c)$$

where η_c is the unit on $\mathfrak{A}(c) \in \mathbf{Mon(M)}$.

(2) *Compatibility with functor structure:* For all $\mathbf{C}$-morphisms $f : c \to c'$,

$$\mathfrak{A}(c) \xrightarrow{\quad *_c \quad} J\mathfrak{A}(c) \qquad\qquad (50)$$

$$\downarrow{\mathfrak{A}(f)} \qquad\qquad \downarrow{J\mathfrak{A}(f)}$$

$$\mathfrak{A}(c') \xrightarrow{\quad *_{c'} \quad} J\mathfrak{A}(c')$$

7.9. EXAMPLE. To illustrate the behavior of these $*$-involutions, consider the orthogonal category $(\{\bullet\}, \emptyset)$ from Examples 7.2 and 7.6. Then $\mathcal{O}_{(\{\bullet\},\emptyset)} = \mathsf{As}$ is the associative operad and Proposition 7.8 defines a $*$-operad structure on it. For later convenience, let us denote the corresponding category of $*$-algebras by

$$*\text{-}\mathbf{Mon}_{\mathrm{rev}}(\mathbf{M}, J, j) := *\text{-}\mathbf{Alg}(\mathcal{O}_{(\{\bullet\},\emptyset)}) \quad . \qquad\qquad (51)$$

Using our concrete description from above, an object in this category is a quadruple $(A, \mu, \eta, *)$ consisting of a monoid $(A, \mu, \eta) \in \mathbf{Mon(M)}$ together with a $*$-involution $* : A \to JA$ satisfying the compatibility conditions (48) and (49). (The conditions in (50) are vacuous because we consider the discrete category $\{\bullet\}$ in this example.) Comparing these structures to $*$-monoids, cf. Remark 3.24, we observe that they are very similar, up to the

appearance of the opposite multiplication in (48). This order-reversal of the multiplication under ∗-involution, which results from our ∗-operad structure (47), motivates our notation ∗-$\mathbf{Mon}_{\mathrm{rev}}(\mathbf{M}, J, j)$.

As a very concrete example, and referring back to Example 3.25, let us consider the involutive symmetric monoidal category $(\mathbf{Vec}_{\mathbb{C}}, \overline{(-)}, \mathrm{id}_{\mathrm{Id}_{\mathbf{Vec}_{\mathbb{C}}}})$ from Example 3.11. In this case (51) describes the category of order-reversing associative and unital ∗-algebras over $\mathbb{C}$, i.e. $(a\,b)^* = b^*\,a^*$, which is of major relevance for (traditional) quantum physics.

7.10. REMARK. We would like to mention that (47) is not the only possible ∗-involution on the colored operad $\mathcal{O}_{(\mathbf{C}, \perp)}$. For example, we could replace the order-reversal permutations $\rho_{|\underline{c}|}$ in (47) by the identity permutations e. This would define another colored ∗-operad structure on $\mathcal{O}_{(\mathbf{C}, \perp)}$ that differs from our choice above. The ∗-algebras for this alternative choice do not describe order-reversing ∗-involutions. In particular, ∗-algebras over $\mathcal{O}_{(\{\bullet\}, \emptyset)} = \mathsf{As}$ for this choice of ∗-involution are non-reversing ∗-monoids as in Remark 3.24. Hence, our general framework for (colored) ∗-operads is sufficiently flexible to capture both reversing and non-reversing ∗-involutions on monoids, which correspond to different choices of ∗-operad structures on the same underlying operad $\mathcal{O}_{(\{\bullet\}, \emptyset)} = \mathsf{As}$.

In general, we have the following explicit characterization of ∗-algebras over the colored ∗-operad $\mathcal{O}_{(\mathbf{C}, \perp)} \in$ ∗-$\mathbf{Op}_{\mathbf{C}_0}(\mathbf{M}, J, j)$ defined in Proposition 7.8.

7.11. PROPOSITION. *For any orthogonal category* $(\mathbf{C}, \perp)$, *there exists an isomorphism*

$$∗\text{-}\mathbf{Alg}(\mathcal{O}_{(\mathbf{C}, \perp)}) \;\cong\; ∗\text{-}\mathbf{Mon}_{\mathrm{rev}}(\mathbf{M}, J, j)^{(\mathbf{C}, \perp)}$$

between the category of ∗-algebras over $\mathcal{O}_{(\mathbf{C}, \perp)}$ *and the category of* $\perp$-*commutative functors from* $\mathbf{C}$ *to the category of order-reversing ∗-monoids in* $(\mathbf{M}, J, j)$, *cf. Example 7.9.*

PROOF. This is an immediate consequence of Theorem 7.5 together with (48), (49) and (50). ∎

7.12. EXAMPLE. Applying this result to Example 7.7, we observe that the category of ∗-algebras over the colored ∗-operad $\mathcal{O}_{(\mathbf{Loc}, \perp)}$ is the category of locally covariant algebraic quantum field theories endowed with ∗-involutions, ∗-$\mathbf{Alg}(\mathcal{O}_{(\mathbf{Loc}, \perp)}) \cong$ ∗-$\mathbf{QFT}(\mathbf{Loc})$. The order-reversing nature of the ∗-involutions is precisely what is needed in quantum physics [HK64, BFV03]. In complete analogy, the category of ∗-algebras over the colored ∗-operad $\mathcal{O}_{(\mathbf{Loc}/M, \perp_M)}$ is the category of Haag-Kastler type algebraic quantum field theories on the Lorentzian manifold M endowed with ∗-involutions, ∗-$\mathbf{Alg}(\mathcal{O}_{(\mathbf{Loc}/M, \perp_M)}) \cong$ ∗-$\mathbf{QFT}(M)$.

We conclude this section with some further remarks on constructions and results that are of interest in quantum field theory.

Change of orthogonal category adjunctions: The assignment $(\mathbf{C}, \perp) \mapsto \mathcal{O}_{(\mathbf{C}, \perp)}$ of our colored ∗-operads is functorial

$$\mathcal{O} : \mathbf{OrthCat} \longrightarrow ∗\text{-}\mathbf{Op}(\mathbf{M}, J, j)$$

on the category of orthogonal categories, where a morphism $F : (\mathbf{C}, \perp) \to (\mathbf{C}', \perp')$ is a functor preserving the orthogonality relations in the sense of $F(\perp) \subseteq \perp'$. Together with Theorem 6.6, this implies

7.13. COROLLARY. *Associated to every* $\mathbf{OrthCat}$-*morphism* $F : (\mathbf{C}, \perp) \to (\mathbf{C}', \perp')$ *there is an adjunction*

$$\mathcal{O}_{F!} : *\text{-}\mathbf{Alg}(\mathcal{O}_{(\mathbf{C}, \perp)}) \overset{\longrightarrow}{\longleftarrow} *\text{-}\mathbf{Alg}(\mathcal{O}_{(\mathbf{C}', \perp')}) : \mathcal{O}_{F^*} \quad .$$

7.14. REMARK. Such adjunctions have plenty of quantum field theoretic applications, see e.g. [BSW17] and also [BDS17] for concrete examples. The results of this section show that these adjunctions are also available in the involutive setting, which is crucial to describe the order-reversing associative and unital $*$-algebras appearing in quantum field theory.

States and the GNS construction: Building on the results in [Jac12], we shall briefly explain the GNS construction for order-reversing $*$-monoids and $*$-algebraic quantum field theories with values in an arbitrary cocomplete involutive closed symmetric monoidal category $(\mathbf{M}, J, j)$. This requires some preparatory definitions and terminology.

7.15. DEFINITION.

(a) A *state* (also called *algebraic state*) on an order-reversing $*$-monoid $(A, \mu, \eta, *) \in *\text{-}\mathbf{Mon}_{\mathrm{rev}}(\mathbf{M}, J, j)$ is a $*\text{-}\mathbf{Obj}(\mathbf{M}, J, j)$-morphism

$$\omega : \big(* : A \to JA\big) \longrightarrow \big(J_0 : I \to JI\big) \quad .$$

To simplify notation, we shall write $\omega : A \to I$ for a state on A.

(b) Given any object $V \in \mathbf{M}$, define the following $*$-object structure

$$
\begin{array}{ccc}
JV \otimes V & \xrightarrow{\;*_{JV \otimes V}\;} & J(JV \otimes V) \\
{\scriptstyle \tau_{JV,V}} \downarrow & & \uparrow {\scriptstyle J_{2\,JV,V}} \\
V \otimes JV & \xrightarrow[\;j_V \otimes \mathrm{id}\;]{} & J^2 V \otimes JV
\end{array}
$$

on $JV \otimes V$. An *inner product space* in $(\mathbf{M}, J, j)$ is a pair $(V, \langle \cdot, \cdot \rangle)$ consisting of an object $V \in \mathbf{M}$ and a $*\text{-}\mathbf{Obj}(\mathbf{M}, J, j)$-morphism

$$\langle \cdot, \cdot \rangle : \big(*_{JV \otimes V} : JV \otimes V \to J(JV \otimes V)\big) \longrightarrow \big(J_0 : I \to JI\big) \quad .$$

To simplify notation, we shall write $\langle \cdot, \cdot \rangle : JV \otimes V \to I$ for an inner product.

(c) A $*$-*representation* of an order-reversing $*$-monoid $(A, \mu, \eta, *) \in *\text{-}\mathbf{Mon}_{\mathrm{rev}}(\mathbf{M}, J, j)$ on an inner product space $(V, \langle \cdot, \cdot \rangle)$ in $(\mathbf{M}, J, j)$ is a left (A, μ, η)-module structure

$\ell : A \otimes V \to V$ on V that is compatible with the inner product, i.e. the diagram

$$
\begin{array}{ccc}
JV \otimes A \otimes V & \xrightarrow{\mathrm{id} \otimes \ell} & JV \otimes V \\
{\scriptstyle \tau_{JV,A} \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \langle \cdot, \cdot \rangle} \\
A \otimes JV \otimes V & & \\
{\scriptstyle * \otimes \mathrm{id} \otimes \mathrm{id}} \downarrow & & \\
JA \otimes JV \otimes V \xrightarrow{J_{2\,A,V} \otimes \mathrm{id}} J(A \otimes V) \otimes V \xrightarrow{J\ell \otimes \mathrm{id}} JV \otimes V \xrightarrow{\langle \cdot, \cdot \rangle} I
\end{array}
$$

in $\mathbf{M}$ commutes.

7.16. REMARK. Notice that there is no concept of *positivity* for a state $\omega : A \to I$ or an inner product $\langle \cdot, \cdot \rangle : JV \otimes V \to I$ in an arbitrary involutive symmetric monoidal category $(\mathbf{M}, J, j)$. That is why Definition 7.15 does not take this property into account. For certain examples, e.g. the involutive category $(\mathbf{Vec}_{\mathbb{C}}, \overline{(-)}, \mathrm{id}_{\mathrm{Id}_{\mathbf{Vec}_{\mathbb{C}}}})$ of complex vector spaces, one may select positive states and positive inner products by imposing additional conditions on the states and inner product spaces in the sense of Definition 7.15. Concretely, a state $\omega : A \to \mathbb{C}$ is positive if $\omega(a^* a) \geq 0$, for all $a \in A$, and an inner product $\langle \cdot, \cdot \rangle : \overline{V} \otimes V \to \mathbb{C}$ is positive if $\langle v, v \rangle \geq 0$, for all $v \in V$.

The GNS construction for order-reversing $*$-monoids in $(\mathbf{M}, J, j)$ is as follows.

7.17. PROPOSITION. *Let $\omega : A \to I$ be a state on $(A, \mu, \eta, *) \in *\text{-}\mathbf{Mon}_{\mathrm{rev}}(\mathbf{M}, J, j)$. Then*

$$
\begin{array}{ccc}
JA \otimes A & \xrightarrow{\langle \cdot, \cdot \rangle} & I \\
{\scriptstyle * \otimes \mathrm{id}} \uparrow {\scriptstyle \cong} & & \uparrow {\scriptstyle \omega} \\
A \otimes A & \xrightarrow{\mu} & A
\end{array}
$$

defines an inner product space structure on the underlying object $A \in \mathbf{M}$. Moreover, $\ell = \mu : A \otimes A \to A$ defines a $*$-representation of $(A, \mu, \eta, *)$ on $(A, \langle \cdot, \cdot \rangle)$.

PROOF. This is an elementary diagram chase using in particular the property (48) for order-reversing $*$-monoids. ∎

7.18. EXAMPLE. This concept of states and $*$-representations generalizes immediately to $*$-algebraic quantum field theory. Let $(\mathbf{C}, \perp)$ be any orthogonal category and $\mathfrak{A} \in *\text{-}\mathbf{Alg}(\mathcal{O}_{(\mathbf{C}, \perp)})$ a $*$-algebra over the corresponding algebraic quantum field theory $*$-operad. By Proposition 7.11, $\mathfrak{A}$ is a $\perp$-commutative functor $\mathfrak{A} : \mathbf{C} \to *\text{-}\mathbf{Mon}_{\mathrm{rev}}(\mathbf{M}, J, j)$ with values in the category of order-reversing $*$-monoids. The usual concept of states considered in algebraic quantum field theory is point-wise, see e.g. [HK64, BFV03]. Concretely, we define a state on $\mathfrak{A}$ to be a family $\omega_c : \mathfrak{A}(c) \to I$ of states in the sense of Definition 7.15, for all objects $c \in \mathbf{C}$, such that $\omega_{c'} \mathfrak{A}(f) = \omega_c$, for every $\mathbf{C}$-morphism $f : c \to c'$. Applying the GNS construction from Proposition 7.17, we obtain a family of inner product

spaces $(\mathfrak{A}(c), \langle \cdot, \cdot \rangle_c)$ and a family of $*$-representations that are functorial in c. In case $\mathbf{C}$ has a terminal object $t \in \mathbf{C}$, e.g. $\mathbf{C} = \mathbf{Loc}/M$ from Example 7.3, then every choice of state $\omega_t : \mathfrak{A}(t) \to I$ on the corresponding order-reversing $*$-monoid defines a state on $\mathfrak{A}$ via pullback $\omega_c := \omega_t \mathfrak{A}(\exists! : c \to t)$ along the unique $\mathbf{C}$-morphism to t. The GNS representation for $\omega_t : \mathfrak{A}(t) \to I$ then defines a $*$-representation of $\mathfrak{A}$ on a common inner product space $(\mathfrak{A}(t), \langle \cdot, \cdot \rangle_t)$. Such $*$-representations are typically used for Haag-Kastler type algebraic quantum field theories on $\mathbf{Loc}/M$, cf. [HK64].

$\mathcal{E}_\infty$-**resolution and homotopy algebraic quantum field theories:** The results of this section generalize to *homotopy algebraic quantum field theories* [BSW18]. These are homotopy algebras over the colored operad $\mathcal{O}_{(\mathbf{C}, \perp)}$ in the symmetric monoidal *model* category $\mathbf{Ch}_\mathbb{C}$ of chain complexes of complex vector spaces. Concretely, we shall discuss the Σ-cofibrant resolution $w : \mathcal{O}_{(\mathbf{C}, \perp)} \otimes \mathcal{E}_\infty \to \mathcal{O}_{(\mathbf{C}, \perp)}$ obtained by the component-wise tensoring of the colored operad $\mathcal{O}_{(\mathbf{C}, \perp)}$ and the Barratt-Eccles operad $\mathcal{E}_\infty$ from [BF04]. Algebras over the colored operad $\mathcal{O}_{(\mathbf{C}, \perp)} \otimes \mathcal{E}_\infty$ play a prominent role in formalizing quantum gauge theories, see [BSW18] for details.

As a first step, we shall equip the *simplicial* Barratt-Eccles operad $\mathcal{E}_\infty^{\mathbf{sSet}}$ with a $*$-structure. Transfer along the normalized chains functor $N_* : \mathbf{sSet} \to \mathbf{Ch}_\mathbb{C}$ then will define a $*$-structure on the operad $\mathcal{E}_\infty = N_*(\mathcal{E}_\infty^{\mathbf{sSet}})$ in $\mathbf{Ch}_\mathbb{C}$. Recall from e.g. [BF04] that the simplicial set of n-ary operations in $\mathcal{E}_\infty^{\mathbf{sSet}}$ is the nerve of the action groupoid $\Sigma_n//\Sigma_n$. Explicitly, $\mathcal{E}_\infty^{\mathbf{sSet}}(n)_k := \Sigma_n^{\times k+1}$ is the set of n-ary operations of degree k. Consider now the trivial involutive symmetric monoidal category $(\mathbf{sSet}, \mathrm{Id}_{\mathbf{sSet}}, \mathrm{id}_{\mathrm{Id}_{\mathbf{sSet}}})$ of simplicial sets. We endow $\mathcal{E}_\infty^{\mathbf{sSet}}$ with a $*$-involution similar to that on the associative operad As in Example 7.9, see also (47). Explicitly, we define $*_\mathcal{E} : \mathcal{E}_\infty^{\mathbf{sSet}} \to \mathcal{E}_\infty^{\mathbf{sSet}}$ as the map that sends a tuple $(\sigma_0, \ldots, \sigma_k) \in \Sigma_n^{\times n+1}$ to $(\rho_n \sigma_0, \ldots, \rho_n \sigma_k) \in \Sigma_n^{\times n+1}$, where $\rho_n \in \Sigma_n$ is the order-reversal permutation from Example 2.5. Clearly, this provides a $*$-object structure on the underlying symmetric sequence, whose compatibility with the operadic composition and unit follows from elementary properties of the permutation group.

Consider now the involutive symmetric monoidal category $(\mathbf{Ch}_\mathbb{C}, \overline{(-)}, \mathrm{id}_{\mathrm{Id}_{\mathbf{Ch}_\mathbb{C}}})$ of chain complexes of complex vector spaces, obtained similarly to Examples 2.4 and 3.11. We equip the symmetric monoidal normalized chains functor $N_* : \mathbf{sSet} \to \mathbf{Ch}_\mathbb{C}$ with the structure of an involutive symmetric monoidal functor $(N_*, \nu) : (\mathbf{sSet}, \mathrm{Id}_{\mathbf{sSet}}, \mathrm{id}_{\mathrm{Id}_{\mathbf{sSet}}}) \to (\mathbf{Ch}_\mathbb{C}, \overline{(-)}, \mathrm{id}_{\mathrm{Id}_{\mathbf{Ch}_\mathbb{C}}})$ by declaring $\nu_X : N_*(X) \to \overline{N_*(X)}$ to act by complex conjugation on $\mathbb{C}$-valued chains in a simplicial set X. We define the Barratt-Eccles $*$-operad $\mathcal{E}_\infty$ in $\mathbf{Ch}_\mathbb{C}$ by applying the involutive symmetric monoidal functor (N_*, ν) to the $*$-operad $\mathcal{E}_\infty^{\mathbf{sSet}}$ in simplicial sets. Combining this with the colored $*$-operad structure from Proposition 7.8, one immediately obtains the following result.

7.19. PROPOSITION. *The component-wise tensor product of the $*$-involutions on $\mathcal{O}_{(\mathbf{C}, \perp)}$ and $\mathcal{E}_\infty$ defines a colored $*$-operad structure on $\mathcal{O}_{(\mathbf{C}, \perp)} \otimes \mathcal{E}_\infty$.*

7.20. REMARK. Similarly to Remark 7.10, the $*$-involution on the Barratt-Eccles operad $\mathcal{E}_\infty$ considered above is not the only one. For example, one could replace order-reversal permutations by identity permutations. Our choice is motivated by the fact that every

∗-algebra over $\mathcal{E}_\infty$ (in our sense) has an underlying order-reversing differential graded ∗-algebra. This is a consequence of the evident ∗-operad inclusion $\mathsf{As} \to \mathcal{E}_\infty$, where As carries the order-reversing ∗-structure from Example 7.9.

Acknowledgments

References

[BD98] J. C. Baez and J. Dolan, "Higher-dimensional algebra III: n-categories and the algebra of opetopes," Adv. Math. **135**, no. 2, 145–206 (1998).

[BM09] E. J. Beggs and S. Majid, "Bar categories and star operations," Algebr. Represent. Theory **12**, 103 (2009) [arXiv:math/0701008 [math.QA]].

[BDS17] M. Benini, C. Dappiaggi and A. Schenkel, "Algebraic quantum field theory on spacetimes with timelike boundary," Annales Henri Poincaré **19**, no. 8, 2401 (2018) [arXiv:1712.06686 [math-ph]].

[BS17] M. Benini and A. Schenkel, "Quantum field theories on categories fibered in groupoids," Commun. Math. Phys. **356**, no. 1, 19 (2017) [arXiv:1610.06071 [math-ph]].

[BSS15] M. Benini, A. Schenkel and R. J. Szabo, "Homotopy colimits and global observables in Abelian gauge theory," Lett. Math. Phys. **105**, no. 9, 1193 (2015) [arXiv:1503.08839 [math-ph]].

[BSW17] M. Benini, A. Schenkel and L. Woike, "Operads for algebraic quantum field theory," arXiv:1709.08657 [math-ph].

[BSW18] M. Benini, A. Schenkel and L. Woike, "Homotopy theory of algebraic quantum field theories," *to appear in Letter in Mathematical Physics* [arXiv:1805.08795 [math-ph]].

[BF04] C. Berger and B. Fresse, "Combinatorial operad actions on cochains," Math. Proc. Cambridge Philos. Soc. **137**, no. 1, 135–174 (2004) [arXiv:math/0109158 [math.AT]].

[BM07] C. Berger and I. Moerdijk, "Resolution of coloured operads and rectification of homotopy algebras," in: A. Davydov, M. Batanin, M. Johnson, S. Lack and A. Neeman (eds.), *Categories in algebra, geometry and mathematical physics*, Contemp. Math. **431**, 31–58, American Mathematical Society, Providence, RI (2007).

[Bor94] F. Borceux, *Handbook of categorical algebra 2: Categories and structures*, Encyclopedia of Mathematics and its Applications **51**, Cambridge University Press, Cambridge (1994).

[BFV03] R. Brunetti, K. Fredenhagen and R. Verch, "The generally covariant locality principle: A new paradigm for local quantum field theory," Commun. Math. Phys. **237**, 31 (2003) [math-ph/0112041].

[CG17] K. Costello and O. Gwilliam, *Factorization algebras in quantum field theory. Vol. 1*, New Mathematical Monographs **31**, Cambridge University Press, Cambridge (2017).

[Egg11] J. M. Egger, "On involutive monoidal categories," Theory and Applications of Categories **25**, no. 14, 368 (2011).

[GJ17] N. Gambino and A. Joyal, "On operads, bimodules and analytic functors," Mem. Amer. Math. Soc. **249**, no. 1184 (2017) [arXiv:1405.7270 [math.CT]].

[HK64] R. Haag and D. Kastler, "An algebraic approach to quantum field theory," J. Math. Phys. **5**, 848 (1964).

[Jac12] B. Jacobs, "Involutive Categories and Monoids, with a GNS-Correspondence," Found. Phys. **42**, no. 7, 874 (2012) [arXiv:1003.4552 [cs.LO]].

[KS74] G. M. Kelly and R. Street, "Review of the elements of 2-categories," in: G. M. Kelly (ed.), *Category Seminar*, Proceedings of the Sydney Category Theory Seminar 1972/1973, Lecture Notes in Mathematics **420**, Springer Verlag, Berlin (1974).

[MacL98] S. Mac Lane, *Categories for the working mathematician*, Graduate Texts in Mathematics, Springer Verlag, New York (1998).

[WY18] D. White and D. Yau, "Bousfield localization and algebras over colored operads," Appl. Categor. Struct, **26**, 153 (2018) [arXiv:1503.06720 [math.AT]].

[Yau16] D. Yau, *Colored operads*, Graduate Studies in Mathematics **170**, American Mathematical Society, Providence, RI (2016).

[Yau18] D. Yau, "Homotopical Quantum Field Theory," arXiv:1802.08101 [math-ph].

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