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LEFT-INVARIANT VECTOR FIELDS ON A LIE 2-GROUP

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ABSTRACT. A Lie 2-group G is a category internal to the category of Lie groups. Consequently it is a monoidal category and a Lie groupoid. The Lie groupoid structure on G gives rise to the Lie 2-algebra $\mathbb{X}(G)$ of multiplicative vector fields, see [2]. The monoidal structure on G gives rise to a left action of the 2-group G on the Lie groupoid G, hence to an action of G on the Lie 2-algebra $\mathbb{X}(G)$. As a result we get the Lie 2-algebra $\mathbb{X}(G)^G$ of left-invariant multiplicative vector fields.

On the other hand there is a well-known construction that associates a Lie 2-algebra $\mathfrak{g}$ to a Lie 2-group G: apply the functor $\mathsf{Lie} : \mathsf{LieGp} \to \mathsf{LieAlg}$ to the structure maps of the category G. We show that the Lie 2-algebra $\mathfrak{g}$ is isomorphic to the Lie 2-algebra $\mathbb{X}(G)^G$ of left invariant multiplicative vector fields.

1. Introduction

Recall that a strict Lie 2-group G is a category internal to the category LieGp of Lie groups (the notions of internal categories, functors and natural transformations are recalled in Definition 2.3). Thus G is a category whose collection of objects is a Lie group G_0, the collection of morphisms is a Lie group G_1 and all the structure maps: source s, target t, unit $1 : G_0 \to G_1$ and composition $* : G_1 \times_{s,G_0,t} G_1 \to G_1$ are maps of Lie groups.

There is a well-known functor $\mathsf{Lie} : \mathsf{LieGp} \to \mathsf{LieAlg}$ from the category of Lie groups to the category of Lie algebras. The functor Lie assigns to a Lie group H its tangent space at the identity $\mathfrak{h} = T_eH$. The Lie bracket on $\mathfrak{h}$ is defined by the identification of T_eH with the Lie algebra of left-invariant vector fields on the Lie group H. To a map $f : H \to L$ of Lie groups the functor Lie assigns the differential $T_ef : T_eH \to T_eL$, which happens to be a Lie algebra map. Consequently given a Lie 2-group $G = \{G_1 \rightrightarrows G_0\}$ we can apply the functor Lie to all the structure maps of G and obtain a (strict) Lie 2-algebra $\mathfrak{g} = \{\mathfrak{g}_1 \rightrightarrows \mathfrak{g}_0\}$.

On the other hand, any Lie 2-group happens to be a Lie groupoid. In fact, it is an action groupoid [1, Proposition 32] (see also Corollary 2.7 below). Hepworth in [6] pointed out that any Lie groupoid K possesses a *category* $\mathbb{X}(K)$ of vector fields (and not just a vector space). The objects of this category are well-known multiplicative vector fields of Mackenzie and Xu [9]. Multiplicative vector fields on a Lie groupoid naturally form a Lie algebra. It was shown in [2] that the space of morphisms of $\mathbb{X}(K)$ is a Lie algebra as well, and moreover $\mathbb{X}(K)$ is a strict Lie 2-algebra (that is, a category internal to Lie

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algebras). One may expect that for a Lie 2-group G one can define the Lie 2-algebra $\mathbb{X}(G)^G$ of left-invariant vector fields on G and that this Lie 2-algebra is isomorphic to the Lie 2-algebra $\mathfrak{g}$. But what does it mean for a Lie 2-group to act on its Lie 2-algebra? And what does it mean to be left-invariant for such an action? We proceed by analogy with ordinary Lie groups.

A Lie group H acts on itself by left multiplication: for any $x \in H$ we have a diffeomorphism $L_x : H \to H$, $L_x(a) := xa$. These diffeomorphisms, in turn, give rise to a representation

$$\lambda : H \to \mathrm{GL}(\mathcal{X}(H))$$

of the group H on the vector space $\mathcal{X}(H)$ of vector fields on the Lie group H. Namely, for each $x \in H$, the linear map $\lambda(x) : \mathcal{X}(H) \to \mathcal{X}(H)$ is defined by

$$\lambda(x)v := TL_x \circ v \circ L_{x^{-1}}$$

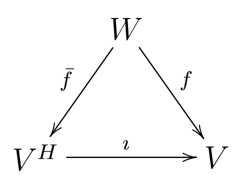
for all vector fields $v \in \mathcal{X}(H)$. Next recall that given a representation $\rho : H \to \mathrm{GL}(V)$ of a Lie group H on a vector space V the space V^H of H-fixed vectors is usually defined by

$$V^H = \{v \in V \mid \rho(x)v = v \text{ for all } x \in H\}.$$

The space V^H has the following universal property: for any linear map $f : W \to V$ (where W is some vector space) so that

$$\rho(x) \circ f = f$$

for all $x \in H$, there is a unique linear map $\bar{f} : W \to V^H$ so that the diagram



commutes. Here $\imath : V^H \hookrightarrow V$ is the inclusion map. If we view the group H as a category BH with one object $*$ and $\mathsf{Hom}_{BH}(*, *) = H$, then the representation $\rho : H \to \mathrm{GL}(V)$ can be viewed as the functor $\rho : BH \to \mathsf{Vect}$ (where Vect is the category of vector spaces and linear maps) with $\rho(*) = V$. From this point of view the vector space V^H of H-fixed vectors "is" the limit of the functor ρ:

$$V^H = \lim(\rho : BH \to \mathsf{Vect}).$$

Consequently the vector space $\mathcal{X}(H)^H$ of left-invariant vector fields on a Lie group H is the limit of the functor $\lambda : BH \to \mathsf{Vect}$ with $\lambda(*) = \mathcal{X}(H)$ and $\lambda(x)v = TL_x \circ v \circ L_{x^{-1}}$ for all $x \in H, v \in \mathcal{X}(H)$:

$$\mathcal{X}(H)^H = \lim(\lambda : BH \to \mathsf{Vect}).$$

Now consider a Lie 2-group G. Each object x of G gives rise to a functor $L_x : G \to G$ which is given on an arrow $b \xleftarrow{\sigma} a$ of G by

$$L_x(b \xleftarrow{\sigma} a) = x \cdot b \xleftarrow{1_x \cdot \sigma} x \cdot a.$$

Here $\cdot$ denotes both multiplications: in the group G_0 and in the group G_1. The symbol 1_x stands for the identity arrow at the object x. For any arrow $y \xleftarrow{\gamma} x$ of G there is a natural transformation

$$L_\gamma : L_x \Rightarrow L_y.$$

The component of L_γ at an object a of G is defined by

$$L_\gamma(a) = y \cdot a \xleftarrow{\gamma \cdot 1_a} x \cdot a.$$

The proof that L_γ is in fact a natural transformation is not completely trivial; see Lemma 3.8.

Next recall that there is a tangent (2-)functor $T : \mathsf{LieGpd} \to \mathsf{LieGpd}$ from the category of Lie groupoids to itself. This functor is an extension of the tangent functor $T : \mathsf{Man} \to \mathsf{Man}$ on the category of manifolds. On objects T assigns to a Lie groupoid K its tangent groupoid TK. On morphisms T assigns to a functor $f : K \to K'$ the derivative $Tf : TK \to TK'$. To a natural transformation $\alpha : f \Rightarrow f'$ between two functors $f, f' : K \to K'$ the functor T assigns the derivative $T\alpha$ (note that a natural transformation α is, in particular, a smooth map $\alpha : K_0 \to K_1'$, so $T\alpha : TK_0 \to TK_1'$ makes sense). Note also that the projection functors $\pi_K : TK \to K$ assemble into a (2-)natural transformation $\pi : T \Rightarrow \mathrm{id}_{\mathsf{LieGpd}}$.

Given an object x of a Lie 2-group G there is a functor $\lambda(x) : \mathbb{X}(G) \to \mathbb{X}(G)$ from the category of vector fields on the Lie groupoid G to itself (see Lemma 3.12 and the discussion right after it). It is defined as follows: given a multiplicative vector field $v : G \to TG$, the value of $\lambda(x)$ on v is given by

$$\lambda(x)\,(v) := TL_x \circ v \circ L_{x^{-1}}.$$

The value of $\lambda(x)$ on a morphism $\alpha : v \Rightarrow w$ (i.e., on a natural transformation between the two functors) of $\mathbb{X}(G)$ is the composite

$$TG \xleftarrow{TL_x} TG \;\underset{w}{\overset{v}{\Downarrow \alpha}}\; G \xleftarrow{L_{x^{-1}}} G\,.$$

That is,

$$\lambda(x)\,(\alpha) := TL_x\, \alpha\, L_{x^{-1}},$$

the whiskering of the natural transformation α by the functors TL_x and $L_{x^{-1}}$. Note that $\lambda(x) \circ \lambda(x^{-1}) = \mathrm{id}_{\mathbb{X}(G)} = \lambda(x^{-1}) \circ \lambda(x)$. And, more generally, $\lambda(x) \circ \lambda(y) = \lambda(x \cdot y)$ for all objects x, y of the Lie 2-group G. For any arrow $y \xleftarrow{\gamma} x$ in the category G we have a natural transformation $\lambda(\gamma) : \lambda(x) \Rightarrow \lambda(y)$: its component

$$\lambda(\gamma)v : \lambda(x)v \Rightarrow \lambda(y)v$$

at a multiplicative vector field v is given by the composite

$$TG \underset{TL_x}{\overset{TL_y}{\leftleftarrows}} \!\!{\scriptstyle \Downarrow TL_\gamma}\; TG \xleftarrow{v} G \underset{L_{x^{-1}}}{\overset{L_{y^{-1}}}{\leftleftarrows}} \!\!{\scriptstyle \Downarrow L_{\gamma^{-1}}}\; G\,.$$

We can always think of a Lie 2-algebra $\mathbb{X}(G)$ as a 2-vector space (i.e., a category internal to the category of vector spaces) by forgetting the Lie brackets. A 2-vector space has a strict 2-group of automorphisms. By definition the objects of this 2-group are strictly invertible functors internal to the category of vector spaces and the morphisms are natural isomorphisms (also internal to the category of vector spaces). We denote the 2-group of automorphisms of $\mathbb{X}(G)$ by $\mathrm{GL}(\mathbb{X}(G))$. The functors $\lambda(x)$ and the natural transformations $\lambda(\gamma)$ described above assemble into a single homomorphism of 2-groups $\lambda : G \to \mathrm{GL}(\mathbb{X}(G))$ (i.e., a functor internal to the category of groups), which we can think of as the "left regular representation" of the Lie 2-group G on its category of vector fields $\mathbb{X}(G)$. The main result of the paper may now be stated as follows.

1.1. Theorem. *Let G be a (strict) Lie 2-group, $\mathfrak{g}$ its Lie 2-algebra obtained by applying the* Lie *functor to its structure maps, $\mathbb{X}(G)$ the Lie 2-algebra of multiplicative vector fields, and $\lambda : G \to \mathrm{GL}(\mathbb{X}(G))$ the representation of G on the 2-vector space $\mathbb{X}(G)$ of multiplicative vector fields which arises from the left multiplication as described above. There is a natural 1-morphism $p : \mathfrak{g} \to \mathbb{X}(G)$ of Lie 2-algebras which is fully faithful and injective on objects. Hence the image $p(\mathfrak{g})$ of the functor p is a full Lie 2-subalgebra of $\mathbb{X}(G)$.*

Moreover the inclusion $p(\mathfrak{g}) \overset{i}{\hookrightarrow} \mathbb{X}(G)$ is the strict conical 2-limit of the functor $\lambda : G \to \mathrm{GL}(\mathbb{X}(G))$. Hence the Lie 2-algebra $\mathfrak{g}$ is isomorphic to the Lie 2-algebra $\mathbb{X}(G)^G := \lim(\lambda : G \to \mathrm{GL}(\mathbb{X}(G)))$ of left-invariant vector fields on the Lie 2-group G.

Related work. Higher Lie theory is a well-developed subject. The ideas go back to the work of Quillen [11] and Sullivan [15] on rational homotopy theory. The problem of associating a Lie 2-algebra to a strict Lie 2-group is, of course, solved by applying a Lie functor to the Lie 2-group. In fact a much harder problem has been solved by Ševera who introduced a Lie-like functors that go from Lie n-groups to L_∞-algebras and from Lie n-groupoids to dg-manifolds [12, 13]. In particular one can use Ševera's method to differentiate weak Lie 2-groups [7].

An even harder problem is that of integration. We note the work of Crainic and Fernandes [3], Getzler [4], Henriques [5] and Ševera and Širaň [14].

Outline of the paper.
In Section 2 we fix our notation, which unfortunately is considerable. We recall the definitions internal categories, of 2-groups, Lie 2-groups, Lie 2-algebras and 2-vector spaces. We then recall the interaction of composition and multiplications in a Lie 2-group and the fact that any Lie 2-group is a Lie groupoid. We discuss the category of vector fields $\mathbb{X}(K)$ on a Lie groupoid K and the fact that this category is naturally a Lie 2-algebra. In particular we discuss the origin of the Lie bracket on the space of morphisms of $\mathbb{X}(K)$.

In Section 3 we discuss the 2-group of automorphisms of a category. We define an action of a 2-group on a category and express the action in terms of a 1-morphism of 2-groups. We show that the multiplication of a Lie 2-group G leads to an action $L : G \to \mathrm{Aut}(G)$ of the group on itself by smooth (internal) functors and natural isomorphisms. We show that an action of a Lie 2-group G on a Lie groupoid K by smooth (internal) functors and natural isomorphisms leads to a representation of G on the on the 2-vector space $\mathbb{X}(K)$ of vector fields on K. In particular left multiplication $L : G \to \mathrm{Aut}(G)$ leads to a representation $\lambda : G \to \mathrm{GL}(\mathbb{X}(G))$ of a Lie 2-group G on its 2-vector space of vector fields. Various results of this section may well be known to experts. I don't know of suitable references.

In Section 4 for a Lie 2-group G we construct a 1-morphism of Lie 2-algebras $p : \mathfrak{g} \to \mathbb{X}(G)$ which is fully faithful and injective on objects. Consequently the image $p(\mathfrak{g})$ is a full Lie 2-subalgebra of the Lie 2-algebra of vector fields $\mathbb{X}(G)$.

Finally in Section 5 we show that the inclusion $i : p(\mathfrak{g}) \hookrightarrow \mathbb{X}(G)$ is a strict conical 2-limit of the left regular representation $\lambda : G \to \mathrm{GL}(\mathbb{X}(G))$.

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2. Background and notation

2.1. Notation. Given a natural transformation $\alpha : f \Rightarrow g$ between a pair of functors $f, g : \mathsf{A} \to \mathsf{B}$ we denote the component of α at an object a of A either as α_a or as $\alpha(a)$, depending on readability.

2.2. Notation. Given a category C we denote its collection of objects by C_0 and its collection of morphisms by C_1. The source and target maps of the category C are denoted by $s, t : \mathsf{C}_1 \to \mathsf{C}_0$, respectively. The unit map from objects to morphisms is denoted by $1 : \mathsf{C}_0 \to \mathsf{C}_1$. We write

$$* : \mathsf{C}_1 \times_{s,\mathsf{C}_0,t} \mathsf{C}_1 \to \mathsf{C}_1, \qquad (\sigma, \gamma) \mapsto \sigma * \gamma$$

to denote composition in the category C. Here and elsewhere

$$\mathsf{C}_2 := \mathsf{C}_1 \times_{s,\mathsf{C}_0,t} \mathsf{C}_1 = \{(\gamma_2, \gamma_1) \in G_1 \times G_1 \mid s(\gamma_2) = t(\gamma_1)\}$$

denotes the fiber product of the maps $s : \mathsf{C}_1 \to \mathsf{C}_0$ and $t : \mathsf{C}_1 \to \mathsf{C}_0$.

In this paper there are many Lie 2-algebras, compositions and multiplications. For the reader's convenience we summarize our notation below. Some of the notation has already been introduced above. The explanation of the rest follows the summary.

Summary of notation.

$s, t : \mathsf{C}_1 \to \mathsf{C}_0$	The source and target maps of a category C.
$* : \mathsf{C}_1 \times_{s,\mathsf{C}_0,t} \mathsf{C}_1 \to \mathsf{C}_1$	The composition map of a category C.
$1 : \mathsf{C}_0 \to \mathsf{C}_1$	The unit map of a category C.
$1_x \in \mathsf{C}_1$	the value of the unit map $1 : \mathsf{C}_0 \to \mathsf{C}_1$ on an object x of C.
$g\alpha f$	the whiskering of a natural transformation $\alpha : k \Rightarrow h$ by functors g and f: $$g\alpha f = \mathsf{D} \xleftarrow{g} \mathsf{C} \overset{k}{\underset{h}{\Leftarrow\!\!\!=}}\!{\scriptstyle \alpha}\; \mathsf{B} \xleftarrow{f} \mathsf{A}\,.$$
$G = \{G_1 \rightrightarrows G_0\}$	a Lie 2-group with the Lie group G_0 of objects and G_1 of morphisms.
$e_0 \in G_0$, $e_1 \in G_1$	the multiplicative identities in the Lie groups G_0 and G_1 respectively.
$\mathfrak{g} = \{\mathfrak{g}_1 \rightrightarrows \mathfrak{g}_0\}$	the Lie 2-algebra of a Lie 2-group G obtained by applying the Lie functor to the objects, morphisms and the structure maps of G: $\mathfrak{g}_0 = T_{e_0}G_0$, $\mathfrak{g}_1 = T_{e_1}G_1$.
$\mathscr{L}(G)$	the Lie 2-algebra of a Lie 2-group G whose objects are the left-invariant vector fields on the Lie group G_0 and morphisms are the left-invariant vector fields on the Lie group G_1. It is isomorphic to $\mathfrak{g}$.
$\mathcal{X}(M)$	the Lie algebra of vector fields on a manifold M.
$\cdot$ or m	the multiplication of the Lie 2-group G. We may view m as a functor. It has components $m_1 : G_1 \times G_1 \to G_1$ and $m_0 : G_0 \times G_0 \to G_0$. We may abbreviate m_1 and m_0 as m.
$\bullet$ or $Tm : TG \times TG \to TG$	The derivative of the multiplication functor $m : G \times G \to G$.

$\star : TK_1 \times_{TK_0} TK_1 \to TK$	the composition in the tangent groupoid TK of a Lie groupoid K; $\star$ is the derivative of the composition $$* : K_1 \times_{K_0} K_1 \to K_1.$$
$\mathbb{X}(K)$	the Lie 2-algebra of vector fields on a Lie groupoid K or the 2-vector space underlying the Lie 2-algebra.
$\mathcal{L}_z : Z \to Z$	left multiplication diffeomorphism of a Lie group Z defined by an element $z \in Z$: $\mathcal{L}_z(z') = zz'$ for all $z' \in Z$.
$L_x : G \to G$	the *functor* from a Lie 2-group G to itself defined by the left multiplication by an object x of G.
$L_\gamma : L_x \Rightarrow L_y$	the natural transformation between two left multiplication functors defined by an arrow $x \xrightarrow{\gamma} y$ in a Lie 2-group G.
$\lambda(x) : \mathbb{X}(G) \to \mathbb{X}(G)$	the 1-morphism of the 2-vector space $\mathbb{X}(G)$ induced by an object x of G. It is induced by the left-multiplications functors $TL_x : TG \to TG$ and $L_{x^{-1}} : G \to G$.
$\lambda(\gamma) : \lambda(x) \Rightarrow \lambda(y)$	the 2-morphism of the Lie 2-algebra $\mathbb{X}(G)$ induced by an arrow $x \xrightarrow{\gamma} y$ in the Lie 2-group G.

2.3. Definition. *Recall that given a category* C *with finite limits one can talk about* categories internal to C *[10]. Namely a category C internal to the category* C *consists of two objects C_1, C_0 of* C *together with a five morphisms of* C*) $s, t : C_1 \to C_0$ (source, target), $1 : C_0 \to C_1$ (unit) and composition/multiplication $* : C_1 \times_{C_0} C_1 \to C_1$ satisfying the usual equations. Similarly, given two categories internal to* C *there exist* internal functors *between them. Internal functors consists of pairs of morphisms of* C *satisfying the appropriate equations. And given two internal functors one can talk about* internal natural transformations *between the functors. The categories* C *of interest to us include groups, vector spaces, Lie groups and Lie algebras. The resulting internal categories are called* 2-groups *(also known as cat-groups, categorical groups, gr-categories and categories with a group structure), Baez-Crans* 2-vector spaces*,* Lie 2-groups *and* Lie 2-algebras*, respectively.*

We note that in particular a Lie 2-group G has a Lie group G_0 of objects, a Lie group G_1 of morphisms and all the structure maps: source $s : G_1 \to G_0$, target $t : G_1 \to G_0$, unit $1 : G_0 \to G_1$ and composition $* : G_1 \times_{s,G_0,t} G_1 \to G_1$ are maps of Lie groups (the Lie group structure on $G_2 := G_1 \times_{s,G_0,t} G_1 \to G_1$ is discussed below). We denote the multiplicative identity of the group G_0 by e_0. Since $1 : G_0 \to G_1$ is a map of Lie groups, the multiplicative identity e_1 of G_1 satisfies

$$e_1 = 1_{e_0}.$$

We denote the Lie group multiplications on G_1 and G_0 by m_1 and m_0 respectively. Since the category of Lie groups has transverse fiber products, the fiber product $G_2 = G_1 \times_{s,G_0,t} G_1$ is a Lie group. We denote the multiplication on this group by m_2. If we identify G_2 with the Lie subgroup of $G_1 \times G_1$:

$$G_2 = \{(\sigma, \gamma) \in G_1 \times G_1 \mid s(\sigma) = t(\gamma)\},$$

then the multiplication m_2 is given by the formula

$$m_2((\sigma_2, \gamma_2), (\sigma_1, \gamma_1)) = (m_1(\sigma_2, \sigma_1), m_1(\gamma_2, \gamma_1)).$$

Alternatively, using the infix notation $\cdot$ for the multiplications the formula above amounts to

$$(\sigma_2, \gamma_2) \cdot (\sigma_1, \gamma_1) = (\sigma_2 \cdot \sigma_1, \gamma_2 \cdot \gamma_1).$$

The following lemma is well-known to experts and is easy to prove. None the less it is crucial for many computations in the paper.

2.4. Lemma. *Let $G = \{G_1 \rightrightarrows G_0\}$ be a Lie 2-group with the composition $* : G_2 = G_1 \times_{G_0} G_1 \to G_1$ and multiplication $m_1 : G_1 \times G_1 \to G_1$, $(\gamma, \sigma) \mapsto \gamma \cdot \sigma$. Then*

$$(\sigma_2 * \sigma_1) \cdot (\gamma_2 * \gamma_1) = (\sigma_2 \cdot \gamma_2) * (\sigma_1 \cdot \gamma_1), \tag{2.1}$$

for all $(\sigma_2, \sigma_1), (\gamma_2, \gamma_1) \in G_2 = G_1 \times_{s,G_0,t} G_1$.

Proof. Since the composition $* : G_2 \to G_1$ is a Lie group homomorphism,

$$*((\sigma_2, \sigma_1) \cdot (\gamma_2, \gamma_1)) = (*(\sigma_2, \sigma_1)) \cdot (*(\gamma_2, \gamma_1)). \tag{2.2}$$

On the other hand

$$(\sigma_2, \sigma_1) \cdot (\gamma_2, \gamma_1) = (\sigma_2 \cdot \gamma_2, \sigma_1 \cdot \gamma_1) \tag{2.3}$$

while

$$(*(\sigma_2, \sigma_1)) \cdot (*(\gamma_2, \gamma_1)) \equiv (\sigma_2 * \sigma_1) \cdot (\gamma_2 * \gamma_1) \tag{2.4}$$

when we switch from the prefix to infix notation. Similarly,

$$*(\sigma_2 \cdot \gamma_2, \sigma_1 \cdot \gamma_1) \equiv (\sigma_2 \cdot \gamma_2) * (\sigma_1 \cdot \gamma_1). \tag{2.5}$$

Therefore

$$(\sigma_2 \cdot \gamma_2) * (\sigma_1 \cdot \gamma_1) = (\sigma_2 * \sigma_1) \cdot (\gamma_2 * \gamma_1).$$

∎

2.5. Corollary. *The multiplications $m_i : G_i \times G_i \to G_i$, $i = 0, 1$ on a Lie 2-group G assemble into a functor $m : G \times G \to G$.*

Proof. Omitted. ∎

Equation (2.1) also implies that the multiplication functor $m : G \times G \to G$ and the composition homomorphism $* : G_1 \times_{G_0} G_1 \to G_1$ in a Lie 2-group G are closely related. In fact they determine each other [10]. For the convenience of the reader we recall a proof that the multiplication functor m determines the composition homomorphism $*$:

2.6. Lemma. *For any two composable arrows σ, γ of a Lie 2-group G with $s(\sigma) = b = t(\gamma)$*

$$\sigma * \gamma = \gamma \cdot 1_{b^{-1}} \cdot \sigma.$$

Here as before $s, t : G_1 \to G_0$ are the source and target maps, $1_{b^{-1}}$ denotes the unit arrow at the object b^{-1} of G, $\cdot$ stands for the multiplication m_1 on the space of arrows G_1 of the Lie 2-group G and $* : G_1 \times_{G_0} G_1 \to G_1$ is the composition homomorphism.

Proof. We follow the proof in [10, p. 186]. Note that since $1 : G_0 \to G_1$ is a homomorphism, the inverse 1_b^{-1} of 1_b with respect to the multiplication m_1 is $1_{b^{-1}}$. We compute

$$\begin{aligned}
\sigma * \gamma &= ((1_b \cdot (1_b^{-1} \cdot \sigma)) * (\gamma \cdot (1_b^{-1} \cdot 1_b)) \\
&= (1_b * \gamma) \cdot ((1_b^{-1} \cdot \sigma) * (1_b^{-1} \cdot 1_b)) \quad \text{by (2.1)} \\
&= \gamma \cdot (1_b^{-1} * 1_b^{-1}) \cdot (\sigma * 1_b) \qquad \text{by (2.1) again} \\
&= \gamma \cdot 1_{b^{-1}} \cdot \sigma \quad \text{since } 1_x * 1_x = 1_x \text{ for all } x \in G_0 \text{ and } (1_b)^{-1} = 1_{b^{-1}}.
\end{aligned}$$

∎

Lemma 2.6 has a well-known corollary: any Lie 2-group is a Lie groupoid. In fact we can be more precise:

2.7. Corollary. *A Lie 2-group G is isomorphic, as a category internal to the category of manifolds, to the action groupoid $\{K \times G_0 \rightrightarrows G_0\}$ where K is the kernel of the source map $s : G_1 \to G_0$ and the action of K on G_0 is given by*

$$k \diamond x := t(k) \cdot x$$

for all $(k, x) \in K \times G_0$. As before $t : G_1 \to G_0$ is the target map.

Proof Sketch of proof. The isomorphism of categories $\varphi : G \to \{K \times G_0 \to G_0\}$ is defined to be identity on objects. On arrows φ is given by

$$\varphi_1(y \xleftarrow{\gamma} x) = (\gamma \cdot 1_{x^{-1}}, x).$$

∎

2.8. Remark. The same argument shows that any 2-group (i.e., a category internal to the category of groups) is an action groupoid.

We next recall the definitions of the 2-categories of Lie 2-algebras and of 2-vector spaces.

2.9. Definition. *Lie 2-algebras naturally form a strict 2-category* Lie2Alg*. The objects of this 2-category are Lie 2-algebras, the 1-morphisms are functors internal to the category* LieAlg *of Lie algebras and 2-morphisms are internal natural transformations.*

2.10. Definition. *2-vector spaces naturally form a strict 2-category* 2Vect*. The objects of this 2-category are 2-vector spaces. The 1-morphisms of* 2Vect *are internal functors and 2-morphisms are internal natural transformations.*

2.11. Remark. There is an evident forgetful functor U : Lie2Alg $\to$ 2Vect. We will suppress this functor in our notation and will use the same symbol for a Lie 2-algebra and its image under the functor U, that is, its underlying 2-vector space.

2.12. The Lie 2-algebra $\mathbb{X}(K)$ of multiplicative vector fields on a Lie groupoid K.

In this subsection we recall some of the results of [2]. We start by recalling the definition of the *category* of multiplicative vector fields $\mathbb{X}(K)$ on a Lie groupoid K, which is due to Hepworth [6].

2.13. Definition. *A* multiplicative vector field *on a Lie groupoid $K = \{K_1 \rightrightarrows K_0\}$ is a functor $v : K \to TK$ so that $\pi_K \circ v = \mathrm{id}_K$. A* morphism *(or an* arrow*) from a multiplicative vector field v to a multiplicative vector field w is a natural transformation $\alpha : v \Rightarrow w$ so that $\pi_K(\alpha(x)) = 1_x$ for any object x of the groupoid K.*

Multiplicative vector fields and morphisms between them are easily seen to form a category: the composite of two morphisms $\alpha : v \Rightarrow w$ and $\beta : w \Rightarrow u$ is the natural transformation $\beta \circ_v \alpha$, where $\circ_v$ denotes the vertical composition of natural transformations. That is, for any object $x \in K_0$

$$(\beta \circ_v \alpha)(x) = \beta(x) \star \alpha(x)$$

where as before $\star : TK_1 \times_{TK_0} TK_1 \to TK_1$ is the derivative of the composition $* : K_1 \times_{K_0} K_1 \to K_1$. Since $\pi_K : TK \to K$ is functor,

$$\pi_K(\beta(x) \star \alpha(x)) = \pi_K(\beta(x)) * \pi_K(\alpha(x)) = 1_x * 1_x = 1_x$$

for all $x \in K_0$. Hence $\beta \circ_v \alpha$ is a morphism from v to u.

It is not hard to see that the collection $\mathbb{X}(K)_0$ multiplicative vector fields form a vector space [9]. It is a little harder to see that $\mathbb{X}(K)_0$ is a Lie algebra (*op. cit.*). However, the

Lie bracket on $\mathbb{X}(K)_0$ is easy to describe. A multiplicative vector field $u : K \to TK$ is, in particular, a pair of ordinary vector fields:

$$u = (u_0 : K_0 \to TK_0, u_1 : K_1 \to TK_1).$$

The bracket on $\mathbb{X}(K)_0$ is defined by

$$[(u_0, u_1), (v_0, v_1)] := ([u_0, v_0], [u_1, v_1]).$$

To see that the definition makes sense one checks that $([u_0, v_0], [u_1, v_1])$ is a functor from K to TK; see [9].

The space of arrows $\mathbb{X}(K)_1$ is a Lie algebra as well and the structure maps of the category $\mathbb{X}(K)$ are Lie algebra maps [2]. In other words the category $\mathbb{X}(K)$ underlies a Lie 2-algebra.

The bracket on the elements of $\mathbb{X}(K)_1$ ultimately comes from the Lie bracket on the vector fields on the manifold K_1 [2]. But the relationship is not direct since the elements of $\mathbb{X}(K)_1$ are not vector fields. In more detail, write an arrow $\alpha \in \mathbb{X}(K)_1$ as

$$\alpha = (\alpha - \mathbb{1}_{\mathbb{s}(\alpha)}) + \mathbb{1}_{\mathbb{s}(\alpha)},$$

where $\mathbb{1} : \mathbb{X}(K)_0 \to \mathbb{X}(K)_1$ is the unit map and $\mathbb{s} : \mathbb{X}(K)_1 \to \mathbb{X}(K)_0$ is the source map of the category $\mathbb{X}(K)$. Recall that for a multiplicative vector field X, the morphism $\mathbb{1}_X : X \Rightarrow X$ is defined by

$$\mathbb{1}_X(x) = T1(X_0(x))$$

for all $x \in K_0$. The multiplicative vector field $\mathbb{s}(\alpha)$ satisfies

$$(\mathbb{s}(\alpha))_0(x) = Ts(\alpha(x))$$

for all $x \in K_0$, where on the right hand side $s : K_1 \to K_0$ is, as before, the source map for the Lie groupoid K. Then

$$Ts(\alpha - \mathbb{1}_{\mathbb{s}(\alpha)}) = 0,$$

hence $\alpha - \mathbb{1}_{\mathbb{s}(\alpha)}$ is a section of the Lie algebroid $A_K \to K_0$ of the Lie groupoid K.

Recall that the Lie bracket on the space of sections $\Gamma(A_K)$ of the Lie algebroid A_K is constructed by embedding $\Gamma(A_K)$ into the space of vector fields on K_1 as right-invariant vector fields. That is, one constructs a map

$$j : \Gamma(A_K) \to \mathcal{X}(K_1)$$

by setting

$$j(\sigma)\,(\gamma) := TR_\gamma\,(\sigma(t(\gamma))) \tag{2.6}$$

for all $\gamma \in K_1$. The map $R_\gamma : s^{-1}(t(\gamma)) \to K_1$ is defined by composition with γ on the right:

$$R_\gamma(\mu) := \mu * \gamma$$

for all $\mu \in K_1$ with $s(\mu) = t(\gamma)$.

We now recall the construction of a Lie algebra structure on the space $\mathbb{X}(K)_1$ (see [2] where the details of the construction are phrased somewhat differently). Define

$$J : \mathbb{X}(K)_1 \to \mathcal{X}(K_1)$$

by setting

$$J(\alpha) := j(\alpha - \mathbb{1}_{\mathbb{s}(\alpha)}) + \mathbb{s}(\alpha)_1. \tag{2.7}$$

The map J is injective and its image happens to be closed under the Lie bracket. So for $\alpha, \beta \in \mathbb{X}(K)_1$ we can (and do) define the Lie bracket $[\alpha, \beta]$ to be the unique element of $\mathbb{X}(K)_1$ with

$$J([\alpha, \beta]) = [J(\alpha), J(\beta)].$$

One checks that the category $\mathbb{X}(K)$ of multiplicative vector fields with the Lie algebra structures on the spaces of objects and morphisms does form a Lie 2-algebra; see [2].

3. Actions and representations of Lie 2-groups

The goal of this section is to construct a representation $\lambda : G \to \mathrm{GL}(\mathbb{X}(G))$ of a Lie group G on its 2-vector space $\mathbb{X}(G)$ of vector fields induced by the action of G on itself by left multiplication. This is the representation briefly described in the introduction. We start by recalling some well-known material about actions of Lie 2-groups. Recall that a 2-group is a category internal to the category of groups and a homomorphism of 2-groups is a functor internal to the category of groups (cf. Definition 2.3).

3.1. Definition. [the 2-group $\mathrm{Aut}(K)$] *Let K be a Lie groupoid. The 2-group* $\mathrm{Aut}(K)$ *of automorphisms of K is defined as follows.*

The group of objects $\mathrm{Aut}(K)_0$ *consists of strictly invertible smooth (i.e., internal) functors $f : K \to K$. The group operation on* $\mathrm{Aut}(K)_0$ *is the composition of functors. The group of morphisms* $\mathrm{Aut}(K)_1$ *is the group of (smooth) natural isomorphisms under* vertical *composition. The composition homomorphism* $* : \mathrm{Aut}(K)_1 \times_{\mathrm{Aut}(K)_0} \mathrm{Aut}(K)_1 \to \mathrm{Aut}(K)_1$ *is the* horizontal *composition of natural isomorphisms. There are also evident source, target and unit maps:*

$$s(f \overset{\alpha}{\Rightarrow} g) = f, \quad t(f \overset{\alpha}{\Rightarrow} g) = g, \quad 1(f) = (f \overset{\mathrm{id}_f}{\Rightarrow} f).$$

Note that the component of id_f *at an object $x \in K_0$ is*

$$\mathrm{id}_f(x) = 1_{f(x)},$$

the unit arrow on the object $f(x)$ of K.

3.2. Definition. *A (strict left)* action *of a Lie 2-group G on a Lie groupoid K is a functor $\mathbf{a}: G \times K \to K$ so that the two diagrams*

$$\begin{array}{ccc} G\times G\times K & \xrightarrow{\mathrm{id}_G\times \mathbf{a}} & G\times K \\ \Big\downarrow{\scriptstyle m\times \mathrm{id}_K} & & \Big\downarrow{\scriptstyle \mathbf{a}} \\ G\times K & \xrightarrow[\mathbf{a}]{} & K \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} G\times K & \xrightarrow{\mathbf{a}} & K \\ \Big\uparrow{\scriptstyle e\times \mathrm{id}_K} & \diagup\!\!\diagup & \\ K & & \end{array} \tag{3.1}$$

commute. Here as before $m: G\times G \to G$ is the multiplication functor. The functor $e\times \mathrm{id}_K$ is defined by $(e\times \mathrm{id}_K)(\sigma) = (e_1,\sigma)$ for all arrows σ of K, where as before $e_1 \in G_1$ is the multiplicative identity.

3.3. Notation. Given an action $\mathbf{a}: G\times K \to K$ a Lie 2-group G on a Lie groupoid K it will be convenient at times to abbreviate $\mathbf{a}(x,b)$ as $x\cdot b$ for any two objects x of G and b of K. Similarly we abbreviate $\mathbf{a}(\gamma,\sigma)$ as $\gamma\cdot\sigma$ for arrows γ of G and σ of K.

3.4. Remark. In the notation above the fact that $\mathbf{a}: G\times K\to K$ preserves the composition of arrows translates into

$$(\gamma_2 * \gamma_1)\cdot(\sigma_2 * \sigma_1) = (\gamma_2\cdot\sigma_2) * (\gamma_1\cdot\sigma_1) \tag{3.2}$$

for any two pairs of composable arrows $(\gamma_2,\gamma_1) \in G_1 \times_{G_0} G_1$ and $(\sigma_2,\sigma_1)\in K_1\times_{K_0} K_1$.

3.5. Lemma. *An action $\mathbf{a}: G\times K\to K$ of a Lie 2-group G on a Lie groupoid K gives rise to a homomorphism of 2-groups*

$$\hat{\mathbf{a}}: G \to \mathrm{Aut}(K). \tag{3.3}$$

In particular for each object $x\in G_0$ there is a functor $\hat{\mathbf{a}}(x): K\to K$ satisfying

$$\hat{\mathbf{a}}(x)\,(b \xleftarrow{\sigma} a) := x\cdot b \xleftarrow{1_x\cdot\sigma} x\cdot a$$

for all arrows $b\xleftarrow{\sigma} a$ of the groupoid K. And for each arrow $y \xleftarrow{\gamma} y$ of G there is a natural transformation $\hat{\mathbf{a}}(\gamma): \hat{\mathbf{a}}(x) \Rightarrow \hat{\mathbf{a}}(y)$ satisfying

$$\hat{\mathbf{a}}(\gamma)(b) = \gamma\cdot 1_b$$

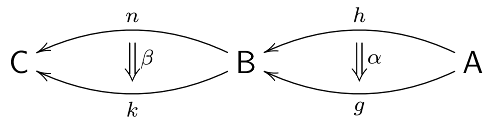
for all objects b of K.

3.6. Remark. The functor (3.3) is a homomorphism of 2-groups if and only if

1. $\hat{\mathbf{a}}(e_0 \xleftarrow{e_1} e_0) = \;(\mathrm{id}_K \overset{1_{\mathrm{id}_K}}{\Leftarrow} \mathrm{id}_K)$ and

2. $\hat{\mathbf{a}}(\gamma_2\cdot\gamma_1) = \hat{\mathbf{a}}(\gamma_2)\circ_{hor}\hat{\mathbf{a}}(\gamma_1)$ for any pairs of arrows γ_2,γ_1 of G. (Here as before $\cdot$ denotes the multiplication in the Lie group G_1.)

3.7. Remark. Recall that given four functors and two natural transformations as below



the component $(\beta \circ_{hor} \alpha)(a)$ of the horizontal composition of β and α at an object $a \in \mathsf{A}_0$ is given by

$$(\beta \circ_{hor} \alpha)(a) = \beta_{g(a)} * n(\alpha_a)$$

where $* : \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \to \mathsf{C}_1$ is the composition in the category C.

Proof of Lemma 3.5. Since $\mathbf{a} : G \times K \to K$ is a functor, for any two composable arrows σ_2, σ_1 in K and for any object x of G

$$\mathbf{a}(1_x, \sigma_2 * \sigma_1) = \mathbf{a}((1_x, \sigma_2) * (1_x, \sigma_1)) = \mathbf{a}(1_x, \sigma_2) * \mathbf{a}(1_x, \sigma_1).$$

We also have $\mathbf{a}(1_x, \sigma_2 * \sigma_1) = \mathbf{a}((1_x, \sigma_2) * (1_x, \sigma_1))$ and $\mathbf{a}(1_x, \sigma_2) * \mathbf{a}(1_x, \sigma_1) = \hat{\mathbf{a}}(x)(\sigma_2) * \hat{\mathbf{a}}(x)(\sigma_1)$. Hence

$$\hat{\mathbf{a}}(x)(\sigma_2 * \sigma_1) = \hat{\mathbf{a}}(x)(\sigma_2) * \hat{\mathbf{a}}(x)(\sigma_1).$$

We conclude that $\hat{\mathbf{a}}(x)$ is a functor for all objects x of the 2-group G.

To check that for an arrow $x \xrightarrow{\gamma} y$ in G, $\hat{\mathbf{a}}(\gamma)$ is a natural transformation from the functor $\hat{\mathbf{a}}(x)$ to the functor $\hat{\mathbf{a}}(y)$ we need to check that for any arrow $b \xleftarrow{\sigma} a$ in K

$$\hat{\mathbf{a}}(\gamma)(b) * \hat{\mathbf{a}}(x)(\sigma) = \hat{\mathbf{a}}(y)(\sigma) * \hat{\mathbf{a}}(\gamma)(b). \tag{3.4}$$

Now

$$\begin{aligned} \hat{\mathbf{a}}(\gamma)(b) * \hat{\mathbf{a}}(x)(\sigma) &= (\gamma \cdot 1_b) * (1_x \cdot \sigma) \\ &= (\gamma * 1_x) \cdot (1_b * \sigma) \qquad \text{(since } a \text{ is a functor)} \\ &= \gamma \cdot \sigma. \end{aligned}$$

Similarly

$$\hat{\mathbf{a}}(y)(\sigma) * \hat{\mathbf{a}}(\gamma)(b) = \gamma \cdot \sigma$$

as well. Hence (3.4) holds and $\hat{\mathbf{a}}(\gamma)$ *is* a natural transformation. Since K is a groupoid $\hat{\mathbf{a}}(\gamma)$ is a natural isomorphism.

It is easy to see that $\hat{\mathbf{a}}(e_0)$ is the identity functor id_K and that $\hat{\mathbf{a}}(e_1)$ is the identity natural isomorphism 1_{id_K}.

To prove that $\hat{\mathbf{a}}$ is a homomorphism of 2-groups it remains to check that

$$\hat{\mathbf{a}}(\gamma_2 \cdot \gamma_1) = \hat{\mathbf{a}}(\gamma_2) \circ_{hor} \hat{\mathbf{a}}(\gamma_1) \tag{3.5}$$

for all arrows γ_2, γ_1 of G. This is a computation. Fix an object a of K. Then

$$\begin{aligned} \hat{\mathbf{a}}(\gamma_2) \circ_{hor} \hat{\mathbf{a}}(\gamma_1) &= \hat{\mathbf{a}}(\gamma_2)(\hat{\mathbf{a}}(\gamma_1)a) * (\hat{\mathbf{a}}(x_2)\,(\hat{\mathbf{a}}(\gamma_1)a) \qquad \text{(by Remark 3.7)} \\ &= (\gamma_2 \cdot 1_{y_2 \cdot a}) * (1_{x_2} \cdot (\gamma_1 \cdot 1_a)) \qquad \text{(by definition of } \hat{\mathbf{a}}) \\ &= \gamma_2 \cdot (\gamma_1 \cdot 1_a) = (\gamma_2 \cdot]\gamma_1) \cdot 1_a \quad \text{(since the left diagram in (3.1) commutes)} \\ &= \hat{\mathbf{a}}(\gamma_2 \cdot \gamma_1)(a). \end{aligned}$$

■

3.8. Corollary. *For any Lie 2-group G there is a homomorphism of 2-groups*

$$L : G \to \mathrm{Aut}(G), \qquad (x \xrightarrow{\gamma} y) \mapsto (L_x \overset{L_\gamma}{\Rightarrow} L_y) \tag{3.6}$$

where the smooth functors $L_x : G \to G$ are defined by

$$L_x(\sigma) = 1_x \cdot \sigma$$

and the natural isomorphisms $L_\gamma : L_x \Rightarrow L_y$ are defined by

$$L_\gamma(a) = \gamma \cdot 1_a$$

for all objects a of G. Here $\cdot$ denotes the multiplication in the group G_0 and in the group G_1.

Proof. The multiplication functor $m : G \times G \to G$ is an action of the Lie 2-group G on the Lie groupoid G. Now apply Lemma 3.5. ∎

3.9. Lemma. *Let G be a 2-group and K a Lie groupoid. A homomorphism $\rho : G \to \mathrm{Aut}(K)$ induces a homomorphism*

$$T\rho : G \to \mathrm{Aut}(TK), \qquad T\rho(x \xrightarrow{\gamma} y) = T\rho(x) \overset{T\rho(\gamma)}{\Rightarrow} T\rho(y).$$

Proof. The homomorphism $T\rho$ is obtained by composing the functor ρ with the tangent 2-functor T : LieGpd $\to$ LieGpd. ∎

3.10. Notation. We denote the 2-vector space underlying the Lie 2-algebra of vector fields on a Lie groupoid K by the same symbol $\mathbb{X}(K)$.

3.11. Definition. [the 2-group $\mathrm{GL}(V)$] *Let V be a 2-vector space. We define the 2-group $\mathrm{GL}(V)$ of automorphisms of a 2-vector space V as follows. The group of objects $\mathrm{GL}(V)_0$ consists of strictly invertible linear (i.e., internal) functors $f : V \to V$. The group operation on $\mathrm{GL}(V)_0$ is the composition of functors. The group of morphisms $\mathrm{GL}(V)_1$ is the group of internal natural isomorphisms under vertical composition. The composition homomorphism $* : \mathrm{GL}_1 \times_{\mathrm{GL}_0} \mathrm{GL}_1 \to \mathrm{GL}_1$ is the horizontal composition of natural isomorphisms. There are also evident source, target and unit maps:*

$$s(f \overset{\alpha}{\Rightarrow} g) = f, \quad t(f \overset{\alpha}{\Rightarrow} g) = g, \quad 1(f) = (f \overset{\mathrm{id}_f}{\Rightarrow} f).$$

3.12. Lemma. *Let G be a Lie 2-group and K a Lie groupoid. A homomorphism $\varphi : G \to \mathrm{Aut}(K)$ of 2-groups (i.e., a functor internal to the category of groups) gives rise to a homomorphism of 2-groups*

$$\Phi : G \to \mathrm{GL}(\mathbb{X}(K)),$$

a representation of the 2-group G on the 2-vector space of vector fields on the Lie groupoid K.

PROOF. As a first step given an object x of G we would like to define a functor $\Phi(x) : \mathbb{X}(K) \to \mathbb{X}(K)$ by setting

$$\Phi(x)(v \overset{\alpha}{\Rightarrow} w) = TK \xleftarrow{T\varphi(x)} TK \underset{w}{\overset{v}{\Leftarrow\!\Downarrow\alpha}} K \xleftarrow{\varphi(x^{-1})} G\,.$$

for all arrows $v \overset{\alpha}{\Rightarrow} w$ in the 2-vector space $\mathbb{X}(K)$. An object of $\mathbb{X}(K)$ is a functor $v : K \to TK$ with $\pi \circ v = \mathrm{id}_K$. Since $\varphi(x^{-1})$ and $T\varphi(x)$ are functors,

$$\Phi(x)v := T\varphi(x) \circ v \circ \varphi(x^{-1})$$

is a functor. Moreover

$$\begin{aligned} \pi \circ (\Phi(x)v) &= \pi \circ T\varphi(x) \circ v \circ \varphi(x^{-1}) \\ &= \varphi(x) \circ \pi \circ v \circ \varphi(x^{-1}) \qquad \text{since } \pi \circ T\varphi = \varphi \circ \pi \\ &= \varphi(x) \circ \mathrm{id}_K \circ \varphi(x^{-1}) = \mathrm{id}_K. \end{aligned}$$

Hence $\Phi(x)v$ is an object of $\mathbb{X}(K)$ for all $x \in G_0$ and all $v \in \mathbb{X}(K)_0$.

An arrow in $\mathbb{X}(K)$ from an object v to an object w is a natural transformation $\alpha : v \Rightarrow w$ with $\pi\alpha = 1_{\mathrm{id}_K}$. Now since $\Phi(x)\alpha$ is obtained from a natural transformation α by whiskering with functors (namely $\Phi(x)\alpha = T\varphi(x)\alpha\varphi(x^{-1})$), $\Phi(x)\alpha$ is a natural transformation from $\Phi(x)v$ to $\Phi(x)w$. Additionally

$$\begin{aligned} \pi(\Phi(x)\alpha) &= \pi T\varphi(x)\alpha\varphi(x^{-1}) \\ &= \varphi(x)\pi\alpha\varphi(x^{-1}) \\ &= \varphi(x)1_{\mathrm{id}_K}\varphi(x^{-1}) = 1_{\mathrm{id}_K}. \end{aligned}$$

Hence $\Phi(x)\alpha$ is an arrow in the 2-vector space $\mathbb{X}(K)$. Finally the purported functor $\Phi(x)$ preserves composition of arrows because whiskering by functors commutes with the vertical composition of natural transformations. We conclude that $\Phi(x) : \mathbb{X}(K) \to \mathbb{X}(K)$ is a well-defined functor.

Since the components $T\varphi(x)_0 : TK_0 \to TK_0$ and $T\varphi_1 : TK_1 \to TK_1$ are fiberwise linear, for any scalars $c, d \in \mathbb{R}$ and any two multiplicative vector fields $v, w : K \to TK$

$$\Phi(x)(cv + dw) = c\Phi(x)v + d\Phi(x)w.$$

Similarly for any two arrows $\alpha_1 : v_1 \Rightarrow w_1$, $\alpha_2 : v_2 \Rightarrow w_2$, any two scalars $c_1, c_2 \in \mathbb{R}$ and any object a of K

$$\begin{aligned} \Phi(x)(c_1\alpha_1 + c_2\alpha_2)\,(a) &= T\varphi(x)(c_1\alpha_1(\varphi(x^{-1})(a) + c_2\alpha_2(\varphi(x^{-1})(a)) \\ &= c_1(T\varphi(x)\alpha_1\varphi(x^{-1}))(a) + c_2(T\varphi(x)\alpha_2\varphi(x^{-1}))(a) \\ &= (c_1\Phi(x)\alpha_1 + c_2\Phi(x)\alpha_2)(a). \end{aligned}$$

We conclude that $\Phi(x)$ is a 1-morphism of 2-vector spaces.

Given an arrow $x \overset{\gamma}{\leftarrow} y$ in G we would like to define a natural transformation $\Phi(\gamma) : \Phi(x) \Rightarrow \Phi(y)$ by setting

$$\Phi(\gamma)v := \; TK \underset{T\varphi(x)}{\overset{T\varphi(y)}{\underset{\Downarrow T\varphi(\gamma)}{\longleftarrow\!\!\!\longleftarrow}}} TK \xleftarrow{v} K \underset{\varphi(x^{-1})}{\overset{\varphi(y^{-1})}{\underset{\Downarrow \varphi(\gamma^{-1})}{\longleftarrow\!\!\!\longleftarrow}}} K$$

for all multiplicative vector fields $v : K \to TK$. By construction $\Phi(\gamma)v$ is a natural transformation from $T\varphi(x) \circ v \circ \varphi(x^{-1}) = \Phi(x)v$ to $T\varphi(y) \circ v \circ \varphi(y^{-1}) = \Phi(y)v$. Moreover

$$\begin{aligned} \pi(\Phi(\gamma)v) &= \pi T\varphi(\gamma)v\varphi(\gamma^{-1}) \\ &= \varphi(\gamma)(\pi \circ v)\varphi(\gamma^{-1}) \\ &= \varphi(\gamma) \circ_{vert} 1_{\mathrm{id}_K} \circ_{vert} \varphi(\gamma^{-1}) \\ &= 1_{\mathrm{id}_K} \qquad (\text{ since } \varphi \text{ is a homomorphism}). \end{aligned}$$

We conclude that for any multiplicative vector field v the natural transformation $\Phi(\gamma)v$ is an arrow in the 2-vector space $\mathbb{X}(K)$.

It is easy to check that $\Phi(\gamma) : \mathbb{X}(K)_0 \to \mathbb{X}(K)_1$ is linear. We now check that $\Phi(\gamma)$ is an actual natural transformation from $\Phi(x)$ to $\Phi(y)$. That is, we check that for any arrow $v \overset{\alpha}{\Rightarrow} w$ in $\mathbb{X}(K)$ the diagram

$$\begin{array}{ccc} \Phi(x)w & \overset{\Phi(x)\alpha}{\Longleftarrow} & \Phi(x)v \\ {\scriptstyle \Phi(\gamma)w}\Downarrow & & \Downarrow{\scriptstyle \Phi(\gamma)v} \\ \Phi(y)w & \overset{\Phi(y)\alpha}{\Longleftarrow} & \Phi(y)v \end{array}$$

commutes in the category $\mathbb{X}(K)$. By definition the composition of the arrows $\Phi(\gamma)w$ and $\Phi(x)\alpha$ is the vertical composition of the diagrams

$$TK \xleftarrow{T\varphi(x)} TK \underset{w}{\overset{v}{\underset{\Downarrow \alpha}{\longleftarrow\!\!\!\longleftarrow}}} K \xleftarrow{\varphi(x^{-1})} G$$

and

$$TK \underset{T\varphi(x)}{\overset{T\varphi(y)}{\underset{\Downarrow T\varphi(\gamma)}{\longleftarrow\!\!\!\longleftarrow}}} TK \xleftarrow{w} K \underset{\varphi(x^{-1})}{\overset{\varphi(y^{-1})}{\underset{\Downarrow \varphi(\gamma^{-1})}{\longleftarrow\!\!\!\longleftarrow}}} K$$

which is

$$T\varphi(\gamma) \circ_{hor} \alpha \circ_{hor} \varphi(\gamma^{-1}).$$

Similarly

$$\Phi(y)\alpha \circ_{vert} \Phi(\gamma)\alpha = T\varphi(\gamma) \circ_{hor} \alpha \circ_{hor} \varphi(\gamma^{-1})$$

as well. Therefore $\Phi(\gamma) : \Phi(x) \Rightarrow \Phi(y)$ is a 2-morphism of 2-vector spaces.

We finish the proof by checking that Φ is a homomorphism of 2-groups. Clearly $\Phi(e_0) = \mathrm{id}_{\mathbb{X}(K)}$ and $\Phi(1_{e_0}) = 1_{\mathrm{id}_{\mathbb{X}(K)}}$. For any two objects x_2, x_1 of G $\varphi(x_2 \cdot x_1) = \varphi(x_2) \circ \varphi(x_1)$ and $T\varphi(x_2 \cdot x_1) = T\varphi(x_2) \circ T\varphi(x_1)$ Consequently for any multiplicative vector field v

$$\begin{aligned} \Phi(x_2 \cdot x_1)v &= T(\varphi(x_2 \cdot x_1) \circ v \circ \varphi((x_2 \cdot x_1)^{-1}) \\ &= T\varphi(x_2) \circ T\varphi(x_1) \circ v \circ \varphi(x_2^{-1}) \circ \varphi(x_1^{-1}) \\ &= \Phi(x_2)(\Phi(x_1)v). \end{aligned}$$

Checking that $\Phi(\gamma_2 \cdot \gamma_1) = \Phi(\gamma_2) \circ_{hor} \Phi(\gamma_1)$ is a bit more involved. Note first that

$$\varphi(\gamma_2 \cdot \gamma_1) = \varphi(\gamma_2) \circ_{hor} \varphi(\gamma_1)$$

since φ is a homomorphism (1-morphism) of 2-groups. Similarly

$$T\varphi(\gamma_2 \cdot \gamma_1) = T\varphi(\gamma_2) \circ_{hor} T\varphi(\gamma_1).$$

Recall that the arrows in the category $\mathrm{GL}(\mathbb{X}(K))$ are natural isomorphisms, and that the composition of arrows in $\mathrm{GL}(\mathbb{X}(K))$ is the vertical composition. Hence by Remark 3.7 for any object u of $\mathbb{X}(K)$,

$$(\Phi(\gamma_2) \circ_{hor} \Phi(\gamma_1))(u) = (\Phi(\gamma_2)(\Phi(y_1)u)) \circ_{vert} (\Phi(x_2)(\Phi(\gamma_1)u)).$$

Since

$$(\Phi(x_2)(\Phi(\gamma_1)u) \quad = \quad TK \xleftarrow{T\varphi(x_2)} TK \underset{T\varphi(x_1)}{\overset{T\varphi(y_1)}{\Longleftarrow}}_{\Downarrow T\varphi(\gamma_1)} TK \xleftarrow{u} K \underset{\varphi(x_1^{-1})}{\overset{\varphi(y_1^{-1})}{\Longleftarrow}}_{\Downarrow \varphi(\gamma_1^{-1})} K \xleftarrow{\varphi(x_2^{-1})} K$$

and

$$\Phi(\gamma_2)(\Phi(y_1)u) \quad = \quad TK \underset{T\varphi(y_2)}{\overset{T\varphi(x_2)}{\Longleftarrow}}_{\Downarrow T\varphi(\gamma_2)} TK \xleftarrow{T\varphi(y_1)} TK \xleftarrow{u} K \xleftarrow{\varphi(y_1^{-1})} K \underset{\varphi(x_2^{-1})}{\overset{\varphi(y_2^{-1})}{\Longleftarrow}}_{\Downarrow \varphi(\gamma_2^{-1})} K\,,$$

$$\begin{aligned} (\Phi(\gamma_2)(\Phi(y_1)u)) \circ_{vert} (\Phi(x_2)(\Phi(\gamma_1)u)) &= (T\varphi(\gamma_2) \circ_{hor} T\varphi(\gamma_1))\, u \left(\varphi(\gamma_1^{-1}) \circ_{hor} \varphi(\gamma_2^{-1})\right) \\ &= (T\varphi)(\gamma_2 \cdot \gamma_1)\, u\, \varphi((\gamma_2 \cdot \gamma_1)^{-1}) \\ &= \Phi(\gamma_2 \cdot \gamma_1)u. \end{aligned}$$

We conclude that $\Phi(\gamma_2 \cdot \gamma_1) = \Phi(\gamma_2) \circ_{hor} \Phi(\gamma_1)$ for all arrows γ_2, γ_1 of the Lie 2-group G. It now follows that $\Phi : G \to \mathrm{GL}(\mathbb{X}(K))$ is a homomorphism of 2-groups. ∎

We are now in position to construct the representation $\lambda : G \to \mathrm{GL}(\mathbb{X}(G))$ of a Lie 2-group G on its 2-vector space $\mathbb{X}(G)$ of vector fields coming from the multiplication on the left.

3.13. LEMMA. *Left multiplication on a Lie 2-group G induces a homomorphism of 2-groups*

$$\lambda : G \to \mathrm{GL}(\mathbb{X}(G))$$

from G to the 2-group $\mathrm{GL}(\mathbb{X}(G))$ of automorphisms of the 2-vector space of vector fields on the Lie groupoid G. For each object x of G, $\lambda(x) : \mathbb{X}(G) \to \mathbb{X}(G)$ is a linear functor with

$$\lambda(x)\,(v \overset{\alpha}{\Rightarrow} w) = TG \xleftarrow{TL_x} TG \;\substack{\xleftarrow{\;v\;} \\ \Downarrow \alpha \\ \xleftarrow[\;w\;]{}}\; G \xleftarrow{L_{x^{-1}}} G$$

for each arrow $v \overset{\alpha}{\Rightarrow} w$ of G. Here as before $L : G \to \mathrm{Aut}(G)$ is the homomorphism of 2-groups induced by multiplication on the left. For each arrow $x \xrightarrow{\gamma} y$ of G, $\lambda(\gamma) : \lambda(x) \to \lambda(y)$ is a natural isomorphism with

$$\lambda(\gamma)v = TG \;\substack{\xleftarrow{\;TL_y\;} \\ \Downarrow TL_\gamma \\ \xleftarrow[\;TL_x\;]{}}\; TG \xleftarrow{v} G \;\substack{\xleftarrow{\;L_{y^{-1}}\;} \\ \Downarrow L_{\gamma^{-1}} \\ \xleftarrow[\;L_{x^{-1}}\;]{}}\; G$$

for all objects v of the 2-vector space $\mathbb{X}(G)$.

PROOF. by Corollary 3.8 multiplication on G gives rise to a homomorphism of 2-groups $L : G \to \mathrm{Aut}(G)$. By Lemma 3.12 the homomorphism L gives rise to the homomorphism $\lambda : G \to \mathrm{GL}(\mathbb{X}(G))$. ■

4. A map of Lie 2-algebras $p : \mathfrak{g} \to \mathbb{X}(G)$

Recall that to a Lie 2-group $G = \{G_1 \rightrightarrows G_0\}$ one can associate a Lie 2-algebra $\mathfrak{g} = \{\mathfrak{g}_1 \rightrightarrows \mathfrak{g}_0\}$ by applying the Lie functor to the Lie group G_0 of objects, the Lie group G_1 of morphisms and to the structure maps of G. That is, $\mathfrak{g}_0 = T_{e_0}G_0$, $\mathfrak{g}_1 = T_{e_1}G_1$ and so on. In this section we prove:

4.1. THEOREM. *Let G be a Lie 2-group and $\mathfrak{g}$ the associated Lie 2-algebra There is a morphism of Lie 2-algebras $p : \mathfrak{g} \to \mathbb{X}(G)$ from the Lie 2-algebra $\mathfrak{g}$ to the Lie 2-algebra of multiplicative vector fields $\mathbb{X}(G)$. Moreover the functor p is injective on objects and fully faithful.*

The theorem has an immediate corollary:

4.2. COROLLARY. *The image $p(\mathfrak{g})$ of the functor $p : \mathfrak{g} \to \mathbb{X}(G)$ is a full Lie 2-subalgebra of the Lie 2-algebra of vector fields $\mathbb{X}(G)$. This 2-subalgebra $p(\mathfrak{g})$ is isomorphic to $\mathfrak{g}$.*

We construct p as a composite of two of functors.

4.3. A Lie 2-algebra $\mathscr{L}(G)$ associated to a Lie 2-group G.
Recall that to define a Lie bracket on the tangent space at the identity T_eH of a Lie group H one identifies T_eH with the space of left-invariant vector fields $\mathcal{X}(H)^H$ on H.

Similarly given a Lie 2-group G we define the Lie 2-algebra $\mathscr{L}(G)$ as follows. We define the Lie algebra of objects $\mathscr{L}(G)_0$ of $\mathscr{L}(G)$ to be the Lie algebra of left-invariant vector fields $\mathcal{X}(G_0)^{G_0}$ on the Lie group G_0. We define the Lie algebra of morphisms $\mathscr{L}(G)_1$ to be the Lie algebra $\mathcal{X}(G_1)^{G_1}$ of left-invariant vector fields on the Lie group G_1. The source map $\mathbf{s} : \mathscr{L}(G)_1 \to \mathscr{L}(G)_0$ is defined by setting the source of a vector field $\alpha \in \mathscr{L}(G)_1$ to be the unique left-invariant vector field $v \in \mathscr{L}(G)_0$ which is $s : G_1 \to G_0$ related to α. The target map $\mathbf{t} : \mathscr{L}(G)_1 \to \mathscr{L}(G)_0$ is defined similarly. The unit map $\mathbf{1} : \mathscr{L}(G)_0 \to \mathscr{L}(G)_1$ is defined by setting $\mathbf{1}_u$ to be the unique left-invariant vector field on G_1 which is $1 : G_0 \to G_1$ related to $u \in \mathscr{L}(G)_0$. The composition

$$\circledast : \mathscr{L}(G)_1 \times_{\mathscr{L}(G)_0} \mathscr{L}(G)_1 \to \mathscr{L}(G)_1$$

is defined pointwise; it is induced by the composition

$$\star : TG_1 \times_{TG_0} TG_1 \to TG_1$$

in the tangent groupoid TG (recall that $\star = T*$, where $* : G_1 \times_{G_0} G_1 \to G_1$ is the composition in G). Thus $\circledast$ is defined by

$$(\alpha \circledast \beta)(\gamma) := \alpha(\gamma) \star \beta(\gamma)$$

for all arrows $\gamma \in G_1$. Routine computations establish that $\mathscr{L}(G)$ is indeed a Lie 2-algebra.

There is an evident functor

$$\ell : \mathfrak{g} = \{T_{e_1}G_1 \rightrightarrows T_{e_0}G_0\} \to \mathscr{L}(G).$$

which sends a vector $v \in \mathfrak{g}_0 = T_{e_0}G_0$ to the corresponding left-invariant vector field $\ell(v)$ on the Lie group G_0 and an arrow $\alpha : v \to w \in \mathfrak{g}_1 = T_{e_1}G_1$ to the corresponding left-invariant vector field $\ell(\alpha)$ on the Lie group G_1. By definition of the Lie brackets on $\mathfrak{g}_0$ and on $\mathfrak{g}_1$ the maps $\ell : \mathfrak{g}_0 \to \mathscr{L}(G)_0$, $\ell : \mathfrak{g}_1 \to \mathscr{L}(G)_1$ are Lie algebra maps. On the other hand ℓ is also an isomorphism of categories — its inverse is given by evaluation at the identities:

$$\ell^{-1}(\beta : u \to u') = \beta(e_1) : u(e_0) \to u'(e_0).$$

We next construct a functor $q : \mathscr{L}(G) \to \mathbb{X}(G)$. Given a left-invariant vector field u on the Lie group G_0 we define a multiplicative vector field $q(u)$ as follows. We take the object part $q(u)_0 : G_0 \to TG_0$ to be u:

$$q(u)_0 := u. \tag{4.1}$$

We define $q(u)_1 : G_1 \to TG_1$ by setting

$$q(u)_1(\gamma) = (T\mathcal{L}_\gamma \circ T1)\,(u(e_0)) \tag{4.2}$$

for all $\gamma \in G_1$. Here and elsewhere in the paper $\mathcal{L}_\gamma : G_1 \to G_1$ is the left multiplication by γ and $T\mathcal{L}_\gamma : TG_1 \to TG_1$ is its derivative. It is clear that both $q(u)_0$ and $q(u)_1$ are vector fields. It is less clear that $q(u) : G \to TG$ is a functor.

4.4. Lemma. *For any vector $u \in T_{e_0}G_0$ the vector field $q(u)_1 : G_1 \to TG_1$ defined by (4.4) preserves composition of arrows:*

$$q(u)_1(\gamma_2 * \gamma_1) = (q(u)_1(\gamma_2)) \star (q(u)_1(\gamma_1))\,. \tag{4.3}$$

for all composable arrows $(\gamma_2, \gamma_1) \in G_1 \times_{G_0} G_1$. Here as before $*$ is the composition in the Lie groupoid G and $\star$ is the composition in the tangent groupoid TG.

4.5. Remark. For a manifold M and a point $q \in M$ we write (q, X) for the tangent vector $X \in T_qM$. With this notation it is easy to see that

$$T\mathcal{L}_\gamma(\sigma, X) = Tm((\gamma, 0), (\sigma, X))$$

for all $\gamma, \sigma \in G_1$, $X \in T_\sigma G_1$. Note also that since the composition $\star = T* : T(G_1 \times_{G_0} G_1) \to TG_1$ is fiberwise linear,

$$(\gamma_2, 0) \star (\gamma_1, 0) = (\gamma_2 * \gamma_1, 0).$$

Proof of Lemma 4.4. Recall that the tangent functor $T : \mathsf{Man} \to \mathsf{Man}$ extends to a 2-functor $T : \mathsf{LieGpd} \to \mathsf{LieGpd}$ on the 2-category of Lie groupoids. As a special case (any Lie group is a Lie groupoid with one object) the functor T induces a functor on the category LieGp of Lie groups. Consequently for a Lie 2-group G its tangent groupoid TG is a Lie 2-group as well. The unit map of the groupoid TG is the derivative $T1$ of the unit map $1 : G_0 \to G_1$. The interchange law (see Lemma 2.4) in the case of TG reads:

$$Tm((\mu_2 \star \mu_1), (\nu_2 \star \nu_1)) = Tm(\mu_2, \nu_2) \star Tm(\mu_1, \nu_1)) \tag{4.4}$$

for all composable pairs $(\mu_2, \mu_1), (\nu_2, \nu_1) \in TG_1 \times_{TG_0} TG_1$. Now take $\nu_2 = \nu_1 = T1(u(e_0))$ which we abbreviate as $\mathbf{1}$. Then (4.4) reads:

$$Tm((\mu_2 \star \mu_1), (\mathbf{1} \star \mathbf{1})) = (Tm(\mu_2, \mathbf{1})) \star (Tm(\mu_1, \mathbf{1})). \tag{4.5}$$

Then by definition of $q(u)_1$, for any $\gamma \in G_1$

$$q(u)_1(\gamma) = T\mathcal{L}_\gamma\,\mathbf{1} = Tm((\gamma, 0), \mathbf{1}).$$

Therefore

$$\begin{aligned} q(u)_1\,(\gamma_2 * \gamma_1) &= T\mathcal{L}_{\gamma_2*\gamma_1}(\mathbf{1}) \\ &= Tm((\gamma_2 * \gamma_1, 0), \mathbf{1}) \\ &= Tm((\gamma_2, 0) \star (\gamma_1, 0), \mathbf{1} \star \mathbf{1}) \\ &= Tm((\gamma_2, 0), \mathbf{1}) \star Tm((\gamma_1, 0), \mathbf{1}) \qquad \text{by (4.5)} \\ &= T\mathcal{L}_{\gamma_2}(\mathbf{1}) \star T\mathcal{L}_{\gamma_1}(\mathbf{1}) \\ &= (q(u)_1(\gamma_2)) \star (q(u)_1(\gamma_1))\,. \end{aligned}$$

■

It is easy to see that $Ts \circ q(u)_1 = q(u)_0 \circ s$, $Tt \circ q(u)_1 = q(u)_0 \circ t$ and $q(u)_1 \circ 1 = T1 \circ q(u)_0$. We conclude that $q(u) = (q(u)_0, q(u)_1) : G \to TG$ is a multiplicative vector field for any left-invariant vector field u on the Lie group G_0. We thus have constructed the functor q on objects.

An arrow $v \xrightarrow{\alpha} u$ in $\mathscr{L}(G)$ is a vector field $\alpha : G_1 \to TG_1$ which is source map s related to v and target map t related to u. Define $q(\alpha) : G_0 \to TG_1$ by

$$q(\alpha) = \alpha \circ 1, \tag{4.6}$$

where as before $1 : G_0 \to G_1$ is the unit map. We need to check that $q(\alpha)$ is an arrow in the category $\mathbb{X}(G)$ from $q(v)$ to $q(u)$. That is, we need to check that $q(\alpha)$ is a natural transformation from $q(v)$ to $q(u)$ with

$$\pi_G(q(\alpha)\,(x)) = 1_x \tag{4.7}$$

for all $x \in G_0$.

Checking that (4.7) holds is easy. Since α is a vector field on G_1,

$$\pi_{G_1}(\alpha(\gamma)) = \gamma$$

for all $\gamma \in G_1$. In particular $\pi_{G_1}(\alpha(1_x)) = 1_x$ for all $x \in G_0$, which implies (4.7).

We now check that $q(\alpha)$ is in fact a natural transformation from the functor $q(v)$ to the functor $q(u)$. Since α is s-related to v

$$Ts(q(\alpha)\,(x)) = Ts(\alpha(1_x)) = v_0(s(1_x)) = v_0(x)$$

for all $x \in G_0$. Since $q(v)_0 = v_0$ it follows that the source of the putative natural transformation $q(\alpha) : G_0 \to TG_1$ is $q(v)$. Similarly the target of $q(\alpha)$ is $q(u)$. It remains to check that $q(\alpha)$ is actually a natural transformation: that is, for any arrow $x \xrightarrow{\gamma} y$ in G, the diagram

$$\begin{array}{ccc} q(v)(x) & \xrightarrow{q(v)(\gamma)} & q(v)(y) \\ {\scriptstyle q(\alpha)(x)}\downarrow & & \downarrow{\scriptstyle q(\alpha)(y)} \\ q(u)(x) & \xrightarrow[q(u)(\gamma)]{} & q(u)(y) \end{array}$$

commutes in the category TG, i.e.,

$$q(\alpha)(y) \star q(v)(\gamma) = q(y)(\gamma) \star q(\alpha)(x). \tag{4.8}$$

By definition of $q(\alpha)$,

$$q(\alpha)(y) = \alpha(1_y).$$

Since α is left-invariant

$$\alpha(1_y) = T\mathcal{L}_{1_y}(\alpha(e_1)) = Tm((1_y, 0), (1_{e_0}, \alpha(1_{e_0}))).$$

Similarly

$$q(\alpha)(x) = Tm((1_x, 0), (1_{e_0}, \alpha(1_{e_0}))).$$

On the other hand,

$$q(v)(\gamma) = T\mathcal{L}_\gamma(\mathbf{1}_{v_0(e_0)}) = Tm((\gamma, 0), (1_{e_0}, \mathbf{1}_{v_0(e_0)})).$$

Similarly,

$$q(u)(\gamma) = Tm((\gamma, 0), (1_{e_0}, \mathbf{1}_{u_0(e_0)})).$$

Now

$$\begin{aligned} q(\alpha)(y) \star p(v)(\gamma) &= Tm((1_y, 0), (1_{e_0}, \alpha(1_{e_0}))) \star Tm((\gamma, 0), ((1_{e_0}, v_1(1_{e_0}))) \\ &= Tm((1_y, 0) \star (\gamma, 0), (1_{e_0}, \alpha(1_{e_0})) \star (1_{e_0}, \mathbf{1}_{v_0(e_0)})) \\ &= Tm((1_y * \gamma, 0), (1_{e_0}, \alpha(1_{e_0}))) \\ &= Tm((\gamma, 0), (1_{e_0}, \alpha(1_{e_0}))). \end{aligned}$$

Similarly,

$$\begin{aligned} q(u)(\gamma) \star q(\alpha)(x) &= Tm((\gamma, 0), (1_{e_0}, \mathbf{1}_{u(1_{e_0})})) \bullet Tm((1_x, 0), (1_{e_0}, \alpha(1_{e_0}))) \\ &= Tm((\gamma, 0) \star (1_x, 0), (1_{e_0}, \mathbf{1}_{u(1_{e_0})})) \star (1_{e_0}, \alpha(1_{e_0})) \\ &= Tm((\gamma, 0), (1_{e_0}, \alpha(1_{e_0}))). \end{aligned}$$

It follows that (4.8) holds. Hence $q(\alpha)$ is an arrow in $\mathbb{X}(G)$ from $q(v)$ to $q(u)$.

It is not hard to check that the map $q : \mathscr{L}(G) \to \mathbb{X}(G)$ constructed above is in fact a functor. We need to check that q is a map of Lie 2-algebras. For this it suffices to check that $q_0 : \mathscr{L}(G)_0 \to \mathbb{X}(G)_0$ and $q_1 : \mathscr{L}(G)_1 \to \mathbb{X}(G)_1$ are Lie algebra maps.

Recall that the Lie bracket of two multiplicative vector fields $X = (X_0, X_1)$ and $Y = (Y_0, Y_1)$ is given by

$$[X, Y] := ([X_0, Y_0], [X_1, Y_1]).$$

It follow from the definition of the functor q on objects that for any two vector fields $u, v \in \mathscr{L}(G)_0$

(i) $[q(u)_0, q(v)_0] = q([u, v])_0$ and

(ii) $q([u,v])_1$ is the unique left-invariant vector field on G_1 which is 1-related to $[u,v]$. Hence
$$q([u,v])_1 = [q(u)_1, q(v)_1].$$

We conclude that
$$q : \mathscr{L}(G)_0 \to \mathbb{X}(G)_0$$
is a map of Lie algebras.

We next check that $q : \mathscr{L}(G)_1 \to \mathbb{X}(G)_1$ is also a map of Lie algebras. Recall the construction of a Lie algebra structure on the space $\mathbb{X}(G)_1$ starts with the injective linear map $j : \Gamma(A_G) \to \mathcal{X}(K_1)$ that maps the section of the Lie algebroid $A_G \to G_0$ to the corresponding right-invariant vector field (see (2.6)). We then embed $\mathbb{X}(G)_1$ into the space of vector fields $\mathcal{X}(G_1)$ by the map J (see (2.7)) and give $\mathbb{X}(G)_1$ the induced Lie algebra structure: for $\alpha, \beta \in \mathbb{X}(G)_1$ their bracket $[\alpha, \beta]$ is the unique element of the vector space $\mathbb{X}(G)_1$ with
$$J([\alpha,\beta]) = [J(\alpha), J(\beta)].$$

4.6. Lemma. *(We use the notation developed above.) For any left-invariant vector field $\alpha \in \mathscr{L}(G) \equiv \mathcal{X}(G_1)^{G_1}$*
$$J(q(\alpha)) = \alpha.$$
Hence $q : \mathscr{L}(G)_1 \to \mathbb{X}(G)_1$ is a Lie algebra map.

Proof. Since G is a Lie 2-group, for any $(\sigma, \gamma) \in G_1 \times_{G_0} G_1$
$$R_\gamma(\sigma) = \sigma * \gamma = \gamma \cdot (1_{t(\gamma)})^{-1} \cdot \sigma$$
by Lemma 2.6. Therefore for any curve $\sigma(\tau)$ in G_1 lying entirely in a fiber of the source map $s : G_1 \to G_0$ with $\sigma(0) = 1_y$ for some $y \in G_0$
$$TR\gamma(\dot\sigma(0)) = \left.\frac{d}{d\tau}\right|_0 \sigma(\tau) * \gamma = \left.\frac{d}{dt}\right|_0 \gamma \cdot (1_y)^{-1} \cdot \sigma(\tau) = T\mathcal{L}_{\gamma\cdot(1_y)^{-1}}(\dot\sigma(0)).$$

It follows that for any section $\zeta \in \Gamma(A_G)$ of the algebroid, $j(\zeta) \in \mathcal{X}(G_1)$ is given by
$$j(\zeta)\,(\gamma) = T\mathcal{L}_{\gamma\cdot(1_{t(\gamma)})^{-1}}(\zeta(t(\gamma))). \tag{4.9}$$

Now, for any arrow $\alpha : u \to v$ in $\mathscr{L}(G)_1$,
$$q(\alpha) = \alpha \circ 1 : q(u) \Rightarrow q(v).$$

Hence for any arrow $y \xrightarrow{\gamma} x$ in the Lie groupoid G
$$\begin{aligned} J(q(\alpha))\,(\gamma) &= j(q(\alpha) - \mathbb{1}_{q(u)}) + q(u)_1(\gamma) \\ &= T\mathcal{L}_{\gamma\cdot(1_y)^{-1}}(\alpha(1_y) - T1(u(y))) + T\mathcal{L}_\gamma(q(u)_1(e_1)) \\ &\qquad \text{(by (4.9) and left-invariance of } q(u)_1) \\ &= \alpha(\gamma\cdot(1_y)^{-1}\cdot 1_y) - (T\mathcal{L}_{\gamma\cdot(1_y)^{-1}} \circ T1 \circ T\mathcal{L}_y)\, u(e_0) + T\mathcal{L}_\gamma(T1(u(e_0))). \end{aligned}$$

Now for any $z \in G_0$

$$\left(\mathcal{L}_{\gamma\cdot(1_y)^{-1}} \circ 1 \circ \mathcal{L}_y\right)(z) = \gamma \cdot (1_y)^{-1} \cdot 1_{yz} = \gamma \cdot (1_y)^{-1} \cdot 1_y \cdot 1_z,$$

where the last equality holds since $1 : G_0 \to G_1$ is a homomorphism. Hence

$$\mathcal{L}_{\gamma\cdot(1_y)^{-1}} \circ 1 \circ \mathcal{L}_y = \mathcal{L}_\gamma \circ 1.$$

and consequently

$$T\mathcal{L}_{\gamma\cdot(1_y)^{-1}} \circ T1 \circ T\mathcal{L}_y = T\mathcal{L}_\gamma \circ T1.$$

It follows that

$$J(q(\alpha))\,(\gamma) = \alpha(\gamma)$$

for all $\gamma \in G_1$ and all $\alpha \in \mathscr{L}(G)_1$.

Now by definition of the bracket on the vector space $\mathbb{X}(G)_1$, for any $\alpha, \beta \in \mathscr{L}(G)_1$ the bracket $[q(\alpha), q(\beta)]$ is the unique element of $\mathbb{X}(G)_1$ such that

$$J([q(\alpha), q(\beta)]) = [J(q(\alpha)), J(q(\beta))].$$

On the other hand

$$J(q[\alpha, \beta])) = [\alpha, \beta] = [J(q(\alpha)), J(q(\beta))]$$

as well. Hence,

$$q([\alpha, \beta]) = [q(\alpha), q(\beta)]$$

for all $\alpha, \beta \in \mathcal{L}(G)_1$. ∎

We conclude that the functor

$$q : \mathscr{L}(G) \to \mathbb{X}(G)$$

from the category $\mathscr{L}(G)$ of left-invariant vector fields on the Lie 2-group G to the category $\mathbb{X}(G)$ of multiplicative vector fields on the Lie groupoid G is a 1-morphism of Lie 2-algebras. By construction q is fully faithful and is injective on objects. We now define $p : \mathfrak{g} \to \mathbb{X}(G)$ to be the composite

$$p := q \circ \ell.$$

By construction p is fully faithful and injective on objects.

5. Universal properties of the inclusion $i : p(\mathfrak{g}) \hookrightarrow \mathbb{X}(G)$

As before G denotes a Lie 2-group and $\mathbb{X}(G)$ the Lie 2-algebra of multiplicative vector fields on the Lie groupoid G.

5.1. Lemma. *A multiplicative vector field $u = (u_0, u_1) : G \to TG$ on a Lie 2-group G satisfies*

$$\lambda(\gamma)(u) = 1_u \tag{5.1}$$

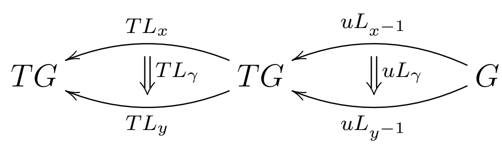
for an arrow γ of G if and only if for all $z \in G_0$

$$u_1(1_z) = T\mathcal{L}_\gamma(u_1(\mathcal{L}_{\gamma^{-1}}(1_z))). \tag{5.2}$$

As before $\mathcal{L}_\sigma : G_1 \to G_1$ denotes the multiplication on the left by $\sigma \in G_1$ and $\lambda : G \to \mathrm{GL}(\mathbb{X}(G))$ is the action of G on its vector fields induced by left multiplication (see Lemma 3.13).

Proof. The proof is a computation.

Recall that for an arrow $x \xrightarrow{\gamma} y \in G_1$ the u-component of the natural transformation $\lambda(\gamma) : \lambda(x) \Rightarrow \lambda(y)$ is defined to be the composite



Hence for any object $z \in G_0$

$$\begin{aligned} (\lambda(\gamma)(u))(z) &= ((TL_\gamma) \circ_{hor} (uL_{\gamma^{-1}}))(z) \\ &= (TL_\gamma)((u_0 \circ L_{y^{-1}})(z)]) \star (TL_x)(u_1(L_{\gamma^{-1}}(z)) \qquad \text{(by Remark 3.7)} \end{aligned}$$

where $\star$ is the composition in the Lie groupoid TG. For any tangent vector $\dot{a} \in T_aG_0$

$$TL_\gamma(a, \dot{a}) = Tm((\gamma, 0), (1_a, T1(\dot{a})))$$

Hence

$$(TL_\gamma)((u_0 \circ L_{y^{-1}})(z)) = Tm((\gamma, 0), (1_{y^{-1}z}, T1\, u_1(1_{y^{-1}z}))).$$

For any tangent vector $\dot{\sigma} \in T_\sigma G_1$

$$TL_x(\sigma, \dot{\sigma}) = Tm((1_x, 0), (\sigma, \dot{\sigma})).$$

Hence

$$(TL_x)(u_1(L_{\gamma^{-1}}(z)) = Tm((1_x, 0), (\gamma^{-1}1_z, u_1\gamma^{-1}1_z)).$$

Recall that since $Tm : TG \times TG \to TG$ is a functor, for any two pairs of composable arrows $((\sigma_2, \dot{\sigma}_2), (\sigma_1, \dot{\sigma}_1)), ((\sigma_4, \dot{\sigma}_4), (\sigma_3, \dot{\sigma}_3)) \in TG_1 \times_{TG_0} TG_1$

$$Tm((\sigma_2, \dot{\sigma}_2) \star (\sigma_1, \dot{\sigma}_1)), ((\sigma_4, \dot{\sigma}_4) \star (\sigma_3, \dot{\sigma}_3)) = Tm((\sigma_2, \dot{\sigma}_2), (\sigma_4, \dot{\sigma}_4)) \star Tm((\sigma_1, \dot{\sigma}_1)) \star (\sigma_3, \dot{\sigma}_3))$$

Hence

$$\begin{aligned}
(\lambda(\gamma)(u))\,(z) &= (TL_\gamma)\,((u_0 \circ L_{y^{-1}})(z)]) \star (TL_x)(u_1(L_{\gamma^{-1}}(z)) \\
&= Tm((\gamma, 0), (1_{y^{-1}z}, T1\, u_1(1_{y^{-1}z}))) \star Tm((1_x, 0), (\sigma, \dot\sigma)) \\
&= Tm((\gamma, 0) \star (1_x, 0), (1_{y^{-1}z}, u_1(1_{y^{-1}z})) \star (\gamma^{-1}1_x, u_1(\gamma^{-1}1_x))) \\
&= Tm((\gamma * 1_x, 0), (1_{y^{-1}z} * (\gamma^{-1}1_x), T1(u_0(y^{-1}z)) \star u_1(\gamma^{-1}1 - x))) \\
&= Tm((\gamma, 0), (\gamma^{-1}1_z, u_1(\gamma^{-1}1_z)).
\end{aligned}$$

Now, for any $\dot\sigma \in T_\sigma G_1$,

$$T\mathcal{L}_\gamma(\sigma, \dot\sigma) = Tm((\gamma, 0), (\sigma, \dot\sigma)),$$

where, as before $\mathcal{L}_\gamma : G_1 \to G_1$ is the left multiplication by γ and $T\mathcal{L}_\gamma : TG_1 \to TG_1$ is its derivative. Therefore

$$(\lambda(\gamma)(u))\,(z) = T\mathcal{L}_\gamma(u_1(\mathcal{L}_{\gamma^{-1}}1_z)).$$

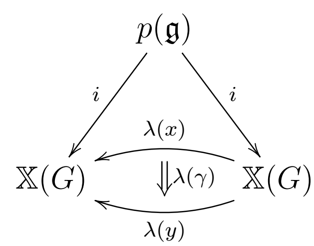
Since the z component of $1_u : u \Rightarrow u$ is $T1(u_0(z))$ and since $T1(u_0(z)) = u_1(1_z)$ (because $u : G \to TG$ is a functor), the result now follows:

$$T\mathcal{L}_\gamma(u_1(\mathcal{L}_{\gamma^{-1}}1_z)) = u_1(1_z).$$

∎

5.2. Theorem. *Let G be a Lie 2-group, $\mathfrak{g}$ the associated tangent Lie 2-algebra, $p : \mathfrak{g} \to \mathbb{X}(G)$ is the map of 2-vector spaces constructed in Theorem 4.1, $i : p(\mathfrak{g}) \hookrightarrow \mathbb{X}(G)$ the inclusion of 2-vector spaces and $\lambda : G \to \mathrm{Aut}(\mathbb{X}(G))$ is the action of the Lie 2-group G on its Lie 2-algebra of multiplicative vector fields by left multiplication (see Lemma 3.13).*

1. *The diagram*



(5.3)

commutes for any choice of arrow $x \xrightarrow{\gamma} y \in G_1$. That is,

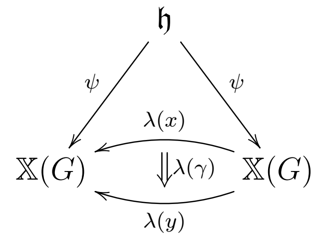
$$\lambda(x) \circ i = i$$

for all $x \in G_0$ and

$$\lambda(\gamma)i = 1_i$$

for all $\gamma \in G_1$ (here $\lambda(\gamma)i$ is the whiskering of the natural transformation $\lambda(\gamma)$ by the functor i).

2. For any map $\psi : \mathfrak{h} \to \mathbb{X}(G)$ of 2-vector spaces such that the diagram



(5.4)

commutes for all choices of arrows $x \xrightarrow{\gamma} y \in G_1$ there exists a unique map of 2-vector spaces $\bar{\psi} : \mathfrak{h} \to p(\mathfrak{g})$ so that

$$\psi = i \circ \bar{\psi}.$$

In other words $i : p(\mathfrak{g}) \to \mathbb{X}(G)$ is a (strict conical) limit of the functor $\lambda : G \to \mathrm{GL}(\mathbb{X}(G))$.

5.3. Remark. In the course of the proof we realize the limit of the functor $\lambda : G \to \mathrm{GL}(\mathbb{X}(G))$ explicitly as a sub 2-vector space of the 2-vector space $\mathbb{X}(G)$ cut out by equations.

Proof. We argue first

(i) For any multiplicative vector field $u = (u_0, u_1) : G \to TG$

$$\lambda(x)u = u \quad \text{for all } x \in G_0 \qquad \text{and} \qquad \lambda(\gamma)u = 1_u \quad \text{for all } \gamma \in G_1 \tag{5.5}$$

if and only if

$$u = p(u(e_0)). \tag{5.6}$$

(ii) For any morphism $\alpha : u \Rightarrow v$ in the category $\mathbb{X}(G)$

$$\lambda(x)(\alpha) = \alpha \text{ for all } x \in G_0 \tag{5.7}$$

if and only if

$$\alpha = p(\alpha(e_0)). \tag{5.8}$$

Proof Proof of (i). By Lemma 5.1 $\lambda(\gamma))(u) = 1_u$ if and only if (5.2) holds for all $z \in G_0$. It is easy to see that (5.2) is equivalent to the vector field u_1 on the Lie group G_1 being left-invariant.

On the other hand $\lambda(x)u = u$ for all $x \in G_0$ translates into

$$T\mathcal{L}_x \circ u_0 \circ \mathcal{L}_{x^{-1}} = u_0$$
$$\text{and}$$
$$T\mathcal{L}_{1_x} \circ u_1 \circ \mathcal{L}_{1_x^{-1}} = u_1.$$

Thus (5.5) implies that u_0 and u_1 are both left-invariant. Moreover, since u is multiplicative and $e_1 = 1(e_0)$, $u_1(e_1) = T1u_0(e_0)$. Hence (5.5) implies (5.6).

Conversely, suppose $(u_0, u_1) = p(a)$ for some $a \in \mathfrak{g}_0$. By construction of the functor p, $a = u_0(e_0)$. Moreover u_1 is a left-invariant vector field on G_1 with $u_1(e_1) = T1(u_0(e_0))$. Hence equation (5.2) hold for all $z \in G_0$ and all $\gamma \in G_1$, which implies that $\lambda(\gamma)u = 1_u$ for all $\gamma \in G_1$. It also implies that

$$T\mathcal{L}_{1_x} \circ u_1 \circ \mathcal{L}_{1_{x^{-1}}} = u_1 \tag{5.9}$$

for all $x \in G_1$. On the other hand, by construction of the functor p the vector field u_0 on G_0 is left-invariant. Hence

$$T\mathcal{L}_x \circ u_0 \circ \mathcal{L}_{x^{-1}} = u_0 \tag{5.10}$$

for all $x \in G_0$. Therefore $\lambda(x)u = u$ for all $x \in G_0$. We conclude that if $u = p(a)$ then (5.5) holds. This finishes our proof of (i). ■

Proof Proof of (ii). By definition of the functor $\lambda(x)$,

$$(\lambda(x)\alpha)(z) = TL_x(\alpha(x^{-1}z))$$

for all $z \in G_0$. By definition of the functor L_x on arrows,

$$TL_x(\alpha(x^{-1}z)) = T\mathcal{L}_{1_z}(\alpha(x^{-1}z)$$

where as before $\mathcal{L}_{1_z}$ is left multiplication by $1_z \in G_1$. Thus if $\lambda(x)\alpha = \alpha$ then

$$\alpha(x) = T\mathcal{L}_{1_x}(\alpha(e_0))$$

Hence (5.7) implies (5.8).

Conversely, if $\alpha = p(b)$ for some $b \in \mathfrak{g}_1$ then $\alpha(x) = T\mathcal{L}_{1_x}b$ for all $x \in G_0$ and $\alpha(e_0) = T\mathcal{L}_{1_{e_0}}b = b$. This finishes our proof of (ii). ■

The proof of part (1) of the theorem is now easy. By (i), for any object $a \in \mathfrak{g}_0$, and any object $x \in G_0$,

$$\lambda(x)(p(a)) = p(a).$$

By (ii), for any object $b \in \mathfrak{g}_1$ and any arrow $\gamma \in G_1$

$$\lambda(\gamma)(p(b)) = p(b).$$

Hence (5.3) commutes.

Now suppose $\psi : \mathfrak{h} \to \mathbb{X}(G)$ is a map of 2-vector spaces making the diagram (5.4) commute. Then for any object X of $\mathfrak{h}$

$$\lambda(x)\psi(X) = \psi(X) \quad \text{for all } x \in G_0 \qquad \text{and} \qquad \lambda(\gamma)\psi(X) = 1_{\psi(X)} \quad \text{for all } \gamma \in G_1$$

Consequently by (i)

$$\psi(X) = p(\psi(X)(e_0)).$$

The commutativity of (5.4) also implies that

$$\lambda(x)(\psi(Y)) = \psi(Y) \text{ for all } x \in G_0$$

for any arrow Y in $\mathfrak{h}$. Then by (ii)

$$\psi(Y) = p(\psi(Y)\,(e_0)).$$

We conclude that the image of $\psi : \mathfrak{h} \to \mathbb{X}(G)$ is contained in $p(\mathfrak{g})$ and the result follows. ■

We are now in position to prove our main result by putting together all the work we have already done.

PROOF OF THEOREM 1.1. By Lemma 3.13 the action of the Lie 2-group G on itself by multiplication on the left gives rise to a homomorphism of 2-groups $\lambda : G \to \mathrm{GL}(\mathbb{X}(G))$. By Theorem 4.1 we have a 1-morphism of Lie 2-algebras $p : \mathfrak{g} \to \mathbb{X}(G)$ which is fully faithful and injective on objects. In particular $p : \mathfrak{g} \to p(\mathfrak{g})$ is an isomorphism of Lie 2-algebras.

On the other hand by Theorem 5.2, the 2-vector space $p(\mathfrak{g})$ underlying the Lie 2-algebra $p(\mathfrak{g})$ is a limit of the functor $\lambda : G \to \mathrm{GL}(\mathbb{X}(G))$. Hence it makes sense to say that $p(\mathfrak{g})$ is the 2-vector space $\mathbb{X}(G)^G$ of left-invariant vector fields. As we remarked previously $p(\mathfrak{g})$ is also a Lie 2-subalgebra of $\mathbb{X}(G)$ which is isomorphic to the Lie 2-algebra $\mathfrak{g}$. ∎

References

[1] J. Baez and A. Lauda, Higher-dimensional algebra. V. 2-groups, *Theory Appl. Categ.* **12** (2004), 423–491.

[2] D. Berwick-Evans and E. Lerman, Lie 2-algebras of vector fields arXiv.org/abs/1609.03944.

[3] M. Crainic, and R. Loja Fernandes, Integrability of Lie brackets, *Ann. of Math.* (2) **157** (2003), no. 2, 575–620.

[4] E. Getzler, Lie theory for nilpotent L_∞-algebras, *Ann. of Math.* (2) **170** (2009), 271–301.

[5] A. Henriques, Integrating L_∞-algebras, *Comp. Math.* **144** (2008), 1017–1045.

[6] R. Hepworth, Vector fields and flows on differentiable stacks, *Theory Appl. Categ.* **22** (2009), 542–587.

[7] B. Jurčo, C. Sämann and M. Wolf Semistrict higher gauge theory, *J. High Energy Phys.* (2015), no. 4, 087, front matter+66 pp.

[8] K.C.H. Mackenzie, *General theory of Lie groupoids and Lie algebroids*, London Mathematical Society Lecture Note Series **213**, Cambridge University Press, Cambridge, 2005. xxxviii+501 pp.

[9] K.C.H. Mackenzie and P. Xu, Classical lifting properties and multiplicative vector fields, *Quart. J. Math. Oxford(2),* **49** (1998), 59–85.

[10] S. Mac Lane, *Categories for the Working Mathematician*, second edition, Springer-Verlag, New York, 1998.

[11] D. Quillen, Rational homotopy theory, *Ann. of Math.* (2) **90** (1969), 205–295.

[12] P. Ševera, L_∞-algebras as 1-jets of simplicial manifolds, `arxiv:math/0612349`.

[13] P. Ševera L_∞-algebras as first approximations, XXVI Workshop on Geometrical Methods in Physics, 199–204, AIP Conf. Proc., **956**, Amer. Inst. Phys., Melville, NY, 2007.

[14] P. Ševera and M. Širaň, Integration of differential graded manifolds, *IMRN*, rnz004 (2019), https://doi.org/10.1093/imrn/rnz004, 1–46.

[15] D. Sullivan, Infinitesimal computations in topology, *Publ. math. de l'I.H.E.S.* **47** (1977), 269–331.

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