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STRICTIFICATION TENSOR PRODUCT OF 2-CATEGORIES

BRANKO NIKOLIĆ

Abstract. Given 2-categories $\mathcal{C}$ and $\mathcal{D}$, let $\mathrm{Lax}(\mathcal{C},\mathcal{D})$ denote the 2-category of lax functors, lax natural transformations and modifications, and $[\mathcal{C},\mathcal{D}]_{\mathrm{lnt}}$ its full sub-2-category of (strict) 2-functors. We give two isomorphic constructions of a 2-category $\mathcal{C}\boxtimes\mathcal{D}$ satisfying $\mathrm{Lax}(\mathcal{C},\mathrm{Lax}(\mathcal{D},\mathcal{E}))\cong[\mathcal{C}\boxtimes\mathcal{D},\mathcal{E}]_{\mathrm{lnt}}$, hence generalising the case of the free distributive law $1\boxtimes 1$. We also discuss dual constructions.

1. Introduction

Monads (aka triples, standard constructions) are given by a category C, an endofunctor $F : C \to C$ and two natural transformations $\eta : 1_C \Rightarrow F$ and $\mu : F^2 \Rightarrow F$, satisfying unit and associativity axioms [8]. Their use is ubiquitous and the most common one is describing a (possibly complicated) algebraic structure as Eilenberg-Moore (EM) algebras [8] on a category of simpler ones. An EM algebra is given by a map $TX \to X$ compatible with μ and η. With algebra morphisms, they form a category $\mathrm{EM}(T)$. The full subcategory of $\mathrm{EM}(T)$ consisting of free algebras is (up to equivalence) usually denoted $\mathrm{KL}(T)$. A typical example is the Abelian group monad on the category of sets taking a set S to the set of elements of the free Abelian group on S.

A distributive law [1] consists of two different monads on the same category satisfying a compatibility condition. Then their composite is a new monad. A typical example is the Abelian group monad together with the monoid monad producing the ring monad, hence the name.

Monads are in fact definable in an arbitrary bicategory $\mathcal{E}$ [9], just by replacing words "functor" with arrow and natural transformation by 2-cell. For example, in the bicategory of spans, monads are precisely (small) categories [2]. A morphism between a monad T on X and S on Y, consists of an arrow $X \xrightarrow{F} Y$ and a "crossing" 2-cell $S\circ F \overset{\sigma}{\Rightarrow} F\circ T$ which is compatible with unit and multiplication for both monads. A morphism between monad morphisms F and G, consists of a 2-cell $F \overset{\alpha}{\Rightarrow} G$ compatible with crossing 2-cells. These form the 2-category of monads in $\mathcal{E}$, called $\mathrm{Mnd}(\mathcal{E})$. Now, a distributive law in $\mathcal{E}$ has a short description as a monad in $\mathrm{Mnd}(\mathcal{E})$. Various duals are expressible using dualities of

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2-categories, for instance, the 2-category of comonads is defined as $\mathrm{Cmd}(\mathcal{E}) = \mathrm{Mnd}(\mathcal{E}^{\mathrm{co}})^{\mathrm{co}}$, mixed distributive laws as $\mathrm{Cmd}(\mathrm{Mnd}(\mathcal{E}))$. Since objects of $\mathcal{E}$ are no longer categories, we have no access to their elements, and cannot form an EM-category; but we can use the 2-dimensional universal property of lax limit to obtain, if exists, an EM-object $\mathrm{EM}(T)$, also denoted C^T. The main topic of [6] is completion of $\mathcal{E}$ under these limits. Dually, lax colimits give KL(T), also denoted C_T.

The free monad [7] is a 2-category FM which classifies monads; that is, the 2-category of strict functors, lax natural transformations and modifications $[\mathrm{FM}, \mathcal{E}]_{\mathrm{lnt}}$ is isomorphic to $\mathrm{Mnd}(\mathcal{E})$. It is given by the suspension of the opposite of the algebraist's category of simplices, $\Delta_{\mathrm{a}}^{\mathrm{op}}$ with ordinal sum as the monoidal structure. We will use it a lot, so we review its definition and some properties in Appendix A. The free mixed distributive law (FMDL) was constructed by Street [12], and is a special case of the construction presented here.

A lax functor [2] (aka morphism) between bicategories generalises the notion of a (strict) 2-functor, by relaxing the conditions of preservation of the unit and composition of arrows. Instead, a lax functor $F : \mathcal{D} \to \mathcal{E}$ is equipped with comparison maps

$$\eta_D : 1_{FD} \Rightarrow F(1_D) \text{ and } \mu_{dd'} : F(d') \circ F(d) \Rightarrow F(d' \circ d)$$

for each object D of $\mathcal{D}$, and composable pair (d, d') of arrows in $\mathcal{D}$. These are required to satisfy unit and associativity laws, and μ is required to be natural in d and d'. The special case of $\mathcal{D} = 1$, that is, if $\mathcal{D}$ has only one 0/1/2-cell, then giving a lax functor exactly corresponds to giving a monad in $\mathcal{E}$. A lax functor from the chaotic category[1] on a set X corresponds to a category enriched in $\mathcal{E}$. Another example, lax functors from $\mathbb{I}(:= 0 \to 1)$ into Span correspond to choosing two categories and a module (aka profunctor, distributor) between them. Lax natural transformations $F \overset{\sigma}{\Rightarrow} G$ between two such functors consist of arrows $FD \xrightarrow{\sigma_D} GD$, for each $D \in \mathcal{D}$, and $Gd \circ \sigma_D \xrightarrow{\sigma_d} \sigma'_D \circ Fd$, for each $D \xrightarrow{d} D'$ in $\mathcal{D}$, natural in d and compatible with η and μ. Finally a modification $\sigma \xrightarrow{m} \tau$ consists of 2-cells $\sigma_D \overset{m_D}{\Longrightarrow} \tau_D$, for each D, compatible with σ. These form a 2-category $\mathrm{Lax}(\mathcal{D}, \mathcal{E})$. The choice of directions gives an isomorphism of 2-categories $\mathrm{Lax}(1, \mathcal{E}) \cong \mathrm{Mnd}(\mathcal{E})$, and by the definition of (free) distributive law (FDL) we have $\mathrm{Lax}(1, \mathrm{Lax}(1, \mathcal{E})) \cong [\mathrm{FDL}, \mathcal{E}]_{\mathrm{lnt}}$.

Our goal is, given 2-categories $\mathcal{C}$ and $\mathcal{D}$, to construct a 2-category $\mathcal{C} \boxtimes \mathcal{D}$ that is "free", in the sense that it strictifies the lax functors, so that

$$\mathrm{Lax}(\mathcal{C}, \mathrm{Lax}(\mathcal{D}, \mathcal{E})) \cong [\mathcal{C} \boxtimes \mathcal{D}, \mathcal{E}]_{\mathrm{lnt}}. \tag{1}$$

The variables C, c, γ used to describe cells in $\mathcal{C}$ (similarly for D, d and δ in $\mathcal{D}$), have sources and targets according to the diagram 2.

$$\begin{tikzcd} C \arrow[r, bend left=50, "c"] \arrow[r, "\bar{c}"] \arrow[r, bend right=50, "\bar{\bar{c}}"'] \arrow[r, phantom, shift left=3, "\gamma \Downarrow"] \arrow[r, phantom, shift right=3, "\bar{\gamma} \Downarrow"] & C' \arrow[r, bend left=50, "c'"] \arrow[r, "\bar{c}'"] \arrow[r, bend right=50, "\bar{\bar{c}}'"'] \arrow[r, phantom, shift left=3, "\gamma' \Downarrow"] \arrow[r, phantom, shift right=3, "\bar{\gamma}' \Downarrow"] & C'' \end{tikzcd} \tag{2}$$

[1] That is, the category having exactly one arrow in each hom.

Horizontal composition is denoted by $\circ$ and vertical by $\bullet$.

2. Tensor product via computads

We begin by fully unpacking the LHS of (1), which involves familiar, but numerous axioms - there are eighteen axioms for an object (lax functor) B, five axioms for an arrow (lax natural transformation) $b : B \to B'$, and two axioms for a 2-cell (modification) $\beta : b \Rightarrow \bar{b}$. Then we review the definition of computads [10] which play the same role for 2-categories as graphs do for usual categories - they are part of a monadic adjunction. We then proceed to construct a computad $\mathcal{G}$ to give a convenient generator-relation description of the tensor product.

2.1. Unpacking. An object B of $\mathrm{Lax}(\mathcal{C}, \mathrm{Lax}(\mathcal{D}, \mathcal{E}))$ assigns to each $C \in \mathcal{C}$ a lax functor $BC : \mathcal{D} \to \mathcal{E}$, which amounts to giving the following data[2] in $\mathcal{E}$:

- for each D an object $BCD \in \mathcal{E}$
- for each d an arrow $BCd : BCD \to BCD'$
- for each δ a 2-cell $BC\delta : BCd \Rightarrow BC\bar{d}$, functorially

$$BC1_d = 1_{BCd} \tag{3}$$
$$BC(\bar{\delta} \bullet \delta) = BC\bar{\delta} \bullet BC\delta \tag{4}$$

- (f1) for each D a unit comparison 2-cell $\eta_{BC1_D} : 1_{BCD} \Rightarrow BC1_D$
- (f1) for each composable pair (d, d') a composition comparison 2-cell $\mu_{BCdd'} : (BCd') \circ (BCd) \Rightarrow (BCd' \circ d)$,
  satisfying unit and associativity axioms,

$$\mu \bullet (1 \circ \eta) = 1 = \mu \bullet (\eta \circ 1) \tag{5}$$
$$\mu \bullet (1 \circ \mu) = \mu \bullet (\mu \circ 1) \tag{6}$$

  together with a naturality condition,

$$\mu_{BC\bar{d}\bar{d}'} \bullet (BC\delta' \circ BC\delta) = BC(\delta' \circ \delta) \bullet \mu_{Cdd'}. \tag{7}$$

Also, B assigns to each $c : C \to C'$ a lax natural transformation $Bc : BC \to BC'$ consisting of:

- arrows $BcD : BCD \to BC'D$

[2] The bullet points marked with (**f1**), (**f2**), (**t1**) and (**t2**) contain 2-cells and axioms that need to be reversed when considering dual constructions in Proposition 2.3.

- $\bullet_{(\mathbf{t1})}$ 2-cells $\sigma_{Bcd} : BC'd \circ BcD \Rightarrow BcD' \circ BCd$,
  with the two axioms expressing compatibility with unit and composition,

$$\sigma \bullet (\eta \circ 1) = 1 \circ \eta \tag{8}$$
$$\sigma \bullet (\mu \circ 1) = (1 \circ \mu) \bullet (\sigma \circ 1) \bullet (1 \circ \sigma) \tag{9}$$

  and one expressing naturality,

$$\sigma_{Bc\bar{d}} \bullet (BC'\delta \circ 1_{BcD}) = (1_{BcD'} \circ BC\delta) \bullet \sigma_{Bcd}\,. \tag{10}$$

Finally, B assigns (functorially) to each 2-cell $\gamma : c \to \bar{c}$ a modification $B\gamma : Bc \Rightarrow B\bar{c}$, which in $\mathcal{E}$ means:

- 2-cells $B\gamma D : BcD \Rightarrow B\bar{c}D$,
  satisfying the modification axiom,

$$\sigma_{B\bar{c}d} \bullet (1_{BC'd} \circ B\gamma D) = (B\gamma D' \circ 1_{BCd}) \bullet \sigma_{Bcd} \tag{11}$$

  and the functoriality condition

$$B1_c D = 1_{BcD} \tag{12}$$
$$B(\bar{\gamma} \bullet \gamma)D = B\bar{\gamma}D \bullet B\gamma D\,. \tag{13}$$

Being a lax functor, B has to provide the unit and composition comparison modifications given by data:

- $\bullet_{(\mathbf{f2})}$ unit 2-cells $\eta_{B1_C D} : 1_{BCD} \Rightarrow B1_C D$
- $\bullet_{(\mathbf{f2})}$ composition 2-cells $\mu_{Bcc'D} : (Bc'D) \circ (BcD) \Rightarrow (Bc' \circ cD)$
  which, in addition to the naturality condition

$$\mu_{B\bar{c}\bar{c}'D} \bullet (B\gamma' D \circ B\gamma D) = B(\gamma' \circ \gamma)D \bullet \mu_{Bcc'D} \tag{14}$$

  and modification axiom,

$$\sigma \bullet (1 \circ \eta) = \eta \circ 1 \tag{15}$$
$$\sigma \bullet (1 \circ \mu) = (\mu \circ 1) \bullet (1 \circ \sigma) \bullet (\sigma \circ 1) \tag{16}$$

  satisfy the unit and associativity axioms (5)-(6).

An arrow $b : B \to B'$, being a lax transformation between lax functors B and B', assigns to each $C \in \mathcal{C}$ a lax transformation $bC : BC \to B'C$ and to each $c : C \to C'$ a modification $\sigma_{bc} : B'c \circ bC \Rightarrow bC' \circ Bc$, which means the following data in $\mathcal{E}$:

- 1-cells $bCD : BCD \to B'CD$

- $\bullet_{(\mathbf{t1})}$ 2-cells $\sigma_{bCd} : B'Cd \circ bCD \Rightarrow bCD' \circ BCd$
- $\bullet_{(\mathbf{t2})}$ 2-cells $\sigma_{bcD} : B'cD \circ bCD \Rightarrow bC'D \circ BcD$,
  subject to naturality

$$\sigma_{b\bar{c}D} \bullet (B'\gamma D \circ 1_{bCD}) = (1_{bC'D} \circ B\gamma D) \bullet \sigma_{bcD} \tag{17}$$
$$\sigma_{bC\bar{d}} \bullet (B'C\delta \circ 1_{bCD}) = (1_{bCD'} \circ BC\delta) \bullet \sigma_{bCd} \tag{18}$$

lax transformation

$$\sigma \bullet (\eta \circ 1) = 1 \circ \eta \tag{19}$$
$$\sigma \bullet (\mu \circ 1) = (1 \circ \mu) \bullet (\sigma \circ 1) \bullet (1 \circ \sigma) \tag{20}$$

and a modification

$$(1 \circ \sigma) \bullet (\sigma \circ 1) \bullet (1 \circ \sigma) = (\sigma \circ 1) \bullet (1 \circ \sigma) \bullet (\sigma \circ 1) \tag{21}$$

axioms.

A 2-cell $\beta : b \to \bar{b}$ in $\mathrm{Lax}(\mathcal{C}, \mathrm{Lax}(\mathcal{D}, \mathcal{E}))$, being a modification, assigns to each $C \in \mathcal{C}$ a modification $\beta C : bC \Rightarrow \bar{b}C$, which in $\mathcal{E}$ means

- 2-cells $\beta CD : bCD \Rightarrow \bar{b}CD$, with modification axioms,

$$\sigma_{\bar{b}cD} \bullet (1_{B'cD} \circ \beta CD) = (\beta C'D \circ 1_{BcD}) \bullet \sigma_{bcD} \tag{22}$$
$$\sigma_{\bar{b}Cd} \bullet (1_{B'Cd} \circ \beta CD) = (\beta CD' \circ 1_{BCd}) \bullet \sigma_{bCd}\,. \tag{23}$$

2.2. Dual cases. Denote by $(\mathrm{Op})\mathrm{Lax}_{(\mathrm{op})}(\mathcal{D}, \mathcal{E})$ the 2-category of (op)lax functors (first op), (op)lax natural transformations (subscript op) and modifications.

2.3. Proposition. *There are isomorphisms:*

$$\mathrm{Lax}_{\mathrm{op}}(\mathcal{D}, \mathcal{E}) \cong \mathrm{Lax}(\mathcal{D}^{\mathrm{op}}, \mathcal{E}^{\mathrm{op}})^{\mathrm{op}} \tag{24}$$
$$\mathrm{OpLax}_{\mathrm{op}}(\mathcal{D}, \mathcal{E}) \cong \mathrm{Lax}(\mathcal{D}^{\mathrm{co}}, \mathcal{E}^{\mathrm{co}})^{\mathrm{co}} \tag{25}$$
$$\mathrm{Lax}(\mathcal{C}, \mathrm{Lax}_{\mathrm{op}}(\mathcal{D}, \mathcal{E})) \cong \mathrm{Lax}_{\mathrm{op}}(\mathcal{D}, \mathrm{Lax}(\mathcal{C}, \mathcal{E})) \tag{26}$$
$$\mathrm{Lax}(\mathcal{C}, \mathrm{OpLax}_{\mathrm{op}}(\mathcal{D}, \mathcal{E})) \cong \mathrm{OpLax}_{\mathrm{op}}(\mathcal{D}, \mathrm{Lax}(\mathcal{C}, \mathcal{E}))\,. \tag{27}$$

Proof. Data and axioms for the LHS of (24) (resp. (25)) are obtained from the beginning of Section 2.1 until the equation (13), by ignoring the letter B in all the names, and reversing the direction of 2-cells for data marked by **(t1)** (resp. **(f1)** or **(t1)**). On the other hand, the data and axioms for the RHS of (24) (resp. (25)) have reversed sources and targets of arrows (resp. 2-cells), compared to the diagram (2), but they also live in $\mathcal{E}^{\mathrm{op}}$ (resp. $\mathcal{E}^{\mathrm{co}}$), rather than $\mathcal{E}$; interpreted in $\mathcal{E}$, they have reversed 2-cells marked by (**t1**) (resp. (**f1**) or (**t1**)). A possibly easier way to see this is to draw string diagrams in $\mathcal{E}^{\mathrm{op}}$ (resp. $\mathcal{E}^{\mathrm{co}}$), and then flip them horizontally (resp. vertically).

To prove (26), observe that the data and axioms in Section 2.1, with **(t1)** 2-cells reversed (LHS), and second and third letter in all labels formally swapped, corresponds to the same data and axioms when C (resp. c, γ) is substituted for D (resp. d, δ), and vice versa, and then **(t2)** 2-cells are reversed (RHS).

Similarly, in (27) reversing **(f1)** and **(t1)** 2-cells, followed by swapping positions in labels, leads the same result as swapping variables and then reversing 2-cells marked by **(f2)** and **(t2)**.

Once the directions for data are fixed, all axioms are determined in a unique way, and there is no need to analyse them separately. ■

2.4. Corollary. *There are isomorphism:*

$$\mathrm{OpLax}(\mathcal{D},\mathcal{E}) \cong \mathrm{Lax}(\mathcal{D}^{\mathrm{co\,op}},\mathcal{E}^{\mathrm{co\,op}})^{\mathrm{co\,op}} \tag{28}$$

$$\mathrm{OpLax}(\mathcal{C},\mathrm{Lax}_{\mathrm{op}}(\mathcal{D},\mathcal{E})) \cong \mathrm{Lax}_{\mathrm{op}}(\mathcal{D},\mathrm{OpLax}(\mathcal{C},\mathcal{E}))\,. \tag{29}$$

2.5. Corollary. *There are isomorphism:*

$$[\mathcal{D},\mathcal{E}]_{\mathrm{ont}} \cong [\mathcal{D}^{\mathrm{op}},\mathcal{E}^{\mathrm{op}}]^{\mathrm{op}}_{\mathrm{lnt}} \tag{30}$$

$$[\mathcal{D},\mathcal{E}]_{\mathrm{ont}} \cong [\mathcal{D}^{\mathrm{co}},\mathcal{E}^{\mathrm{co}}]^{\mathrm{co}}_{\mathrm{lnt}} \tag{31}$$

$$[\mathcal{C},[\mathcal{D},\mathcal{E}]_{\mathrm{ont}}]_{\mathrm{lnt}} \cong [\mathcal{D},[\mathcal{C},\mathcal{E}]_{\mathrm{lnt}}]_{\mathrm{ont}}\,. \tag{32}$$

2.6. Reviewing computads. The content of this part is taken from [10]. We describe the major ideas and leave out the details.

2.7. Definition. *([10], with a technical modification[3]) A computad $\mathcal{G}$ consists of a graph $|\mathcal{G}|$ (providing a set of objects $|\mathcal{G}|_0$ and a set of generating arrows $|\mathcal{G}|_1$), and for each pair of objects $G, G' \in |\mathcal{G}|_0$ a graph $\mathcal{G}(G,G')$ with a set nodes[4] $\mathcal{G}(G,G')_0 = (\mathcal{F}|\mathcal{G}|)(G,G')$ and a set of edges denoted $\mathcal{G}(G,G')_1$ (providing generating 2-cells).*

A computad morphism assigns all the data, respecting sources and targets, forming a category Cmp.

There is a free 2-category $\mathcal{FG}$ on the computad $\mathcal{G}$ that has the same objects as $\mathcal{G}$. Arrows between G and G' are "paths" between G and G'; that is, elements of $\mathcal{G}(G,G')_0$. To define 2-cells, it is not enough to take the free category on $\mathcal{G}(G,G')$ since it does not take whiskering into account. Instead, consider the set of whiskered generating 2-cells

$$\begin{aligned}\mathcal{G}^1(G,G') = \{(p,\alpha,p')|p &\in \mathcal{G}(G,X)_0,\\ \alpha &\in \mathcal{G}(X,X')_1,\\ p' &\in \mathcal{G}(X',G')_0\}\,.\end{aligned}$$

[3]We take all paths between two objects to be the nodes of $\mathcal{G}(G,G')$; that is, $\mathcal{G}(G,G')_0 = (\mathcal{F}|\mathcal{G}|)(G,G')$.

[4]$\mathcal{F}|\mathcal{G}|$ is the free category on a graph $|\mathcal{G}|$

Finally, to impose the middle of four interchange, take the set of whiskered pairs

$$\begin{aligned}\mathcal{G}^2(G,G') = \{(p,\alpha,p',\alpha',p'')|p \in \mathcal{G}(G,X)_0,\\ \alpha \in \mathcal{G}(X,X')_1,\\ p' \in \mathcal{G}(X',X'')_0\\ \alpha' \in \mathcal{G}(X'',X''')_1,\\ p'' \in \mathcal{G}(X''',G')_0\}\end{aligned}$$

and form a coequalizer in Cat to obtain the hom $(\mathcal{FG})(G,G')$

$$\mathcal{FG}^2(G,G') \rightrightarrows \mathcal{FG}^1(G,G') \to (\mathcal{FG})(G,G') \tag{33}$$

where the two parallel arrows are the two obvious ways to compose whiskered α with whiskered α'; see [10] for details and the rest of the construction.

Given a 2-category $\mathcal{E}$, the underlying computad $\mathcal{UE}$ has the underlying graph obtained from the underlying category of $\mathcal{E}$; that is, $|\mathcal{UE}| = \mathcal{U}|\mathcal{E}|$, and the hom graphs have edges $(\mathcal{UE})(E,E')(p,p') = \mathcal{E}(E,E')(\circ p, \circ \bar{p})$, where $\circ p$ denotes the arrow in $\mathcal{E}$ obtained by composing the path p in $\mathcal{E}$. Assignments $\mathcal{F}$ and $\mathcal{U}$ extend to morphisms and form an adjunction, giving a bijection between arrows in Cmp and 2-Cat

$$T : \mathcal{G} \to \mathcal{UE} \quad \leftrightarrow \quad \hat{T} : \mathcal{FG} \to \mathcal{E}\,. \tag{34}$$

Intuitively, the 2-category $\mathcal{FUE}$ is the 2-category of pasting diagrams in $\mathcal{E}$, and the counit of the adjunction is the operation of actual pasting to obtain a (2-)cell in $\mathcal{E}$.

2.8. The tensor product computad. The goal is to construct a computad $\mathcal{G}$ which has data analogous to the one in Section 2.1, and then to impose further identification of 2-cells in $\mathcal{FG}$, analogous to the axioms (3)-(16). Consider the computad $\mathcal{G}$ defined by the following data:

- a set $|\mathcal{G}|_0 = \mathrm{Ob}\mathcal{C} \times \mathrm{Ob}\mathcal{D}$ of nodes, whose elements are denoted $C \boxtimes D$
- the set $|\mathcal{G}|_1((C,D),(C',D'))$ of edges consists of arrows in $\mathcal{C}(C,C')$ if $D = D'$, denoted $c \boxtimes D$, and arrows in $\mathcal{D}(D,D')$ if $C = C'$, denoted $C \boxtimes d$, otherwise it is empty. The concatenation of $c \boxtimes D$ and $C' \boxtimes d$, as an arrow in the free category on $|\mathcal{G}|$, will be denoted by $\{C \boxtimes D \xrightarrow{c\boxtimes D} C' \boxtimes D \xrightarrow{C'\boxtimes d} C' \boxtimes D'\}$, and the empty path on $C \boxtimes D$ by $\{C \boxtimes D\}$. When the meaning is clear from the context we omit the tensor product character. A concise way of expressing the collection of edges is as a disjoint union

$$|\mathcal{G}|_1((C,D),(C',D')) = \mathcal{C}(C,C') \times \delta_{DD'} + \delta_{CC'} \times \mathcal{D}(D,D'), \tag{35}$$

  with δ_{XY} being an empty set when $X \neq Y$ and singleton $\{X\}$ when $X = Y$.
- 2-cells

– for each object C of $\mathcal{C}$ and 2-cell $\delta : d \Rightarrow \bar{d}$ in $\mathcal{D}$,

$$C \boxtimes \delta : \{CD \xrightarrow{Cd} CD'\} \Rightarrow \{CD \xrightarrow{C\bar{d}} CD'\} \tag{36}$$

– for each object D of $\mathcal{D}$ and 2-cell $\gamma : c \Rightarrow \bar{c}$ in $\mathcal{C}$,

$$\gamma \boxtimes D : \{CD \xrightarrow{cD} C'D\} \Rightarrow \{CD \xrightarrow{\bar{c}D} C'D\} \tag{37}$$

–$_{(\mathbf{f1})}$ for each $(C, D) \in |\mathcal{G}|_0$, the unit comparisons

$$\mathbf{id}_{C1_D} : \{CD\} \Rightarrow \{CD \xrightarrow{C1_D} CD\} \tag{38}$$

–$_{(\mathbf{f2})}$ for each $(C, D) \in |\mathcal{G}|_0$, the unit comparisons

$$\mathbf{id}_{1_C D} : \{CD\} \Rightarrow \{CD \xrightarrow{1_C D} CD\} \tag{39}$$

–$_{(\mathbf{f1})}$ for each $C \in \mathcal{C}$ and composable pair (d, d') in $\mathcal{D}$, a composition comparison

$$\mathbf{comp}_{Cdd'} : \{CD \xrightarrow{Cd} CD' \xrightarrow{Cd'} CD''\} \Rightarrow \{CD \xrightarrow{C\boxtimes(d'\circ d)} CD''\} \tag{40}$$

–$_{(\mathbf{f2})}$ for each $D \in \mathcal{D}$ and composable pair (c, c') in $\mathcal{C}$, a composition comparison

$$\mathbf{comp}_{cc'D} : \{CD \xrightarrow{cD} C'D \xrightarrow{c'D} C''D\} \Rightarrow \{CD \xrightarrow{(c'\circ c)\boxtimes D} C''D\} \tag{41}$$

–$_{(\mathbf{t1})}$ for each pair of 1-cells (c, d),

$$\mathbf{swap}_{cd} : \{CD \xrightarrow{cD} C'D \xrightarrow{C'd} C'D'\} \Rightarrow \{CD \xrightarrow{Cd} CD' \xrightarrow{cD'} C'D'\}. \tag{42}$$

The 2-category $\mathcal{C} \boxtimes_{cmp} \mathcal{D}$ is obtained from $\mathcal{FG}$, the free 2-category on the computad $\mathcal{G}$, by imposing identifications:

- preservation of identity 2-cells

$$C \boxtimes 1_d = 1_{C\boxtimes d} \tag{43}$$
$$1_c \boxtimes D = 1_{c\boxtimes D} \tag{44}$$

- distributivity of $\boxtimes$ over vertical composition

$$(C \boxtimes \delta') \bullet (C \boxtimes \delta) = C \boxtimes (\delta' \bullet \delta) \tag{45}$$
$$(\gamma' \boxtimes D) \bullet (\gamma \boxtimes D) = (\gamma' \bullet \gamma) \boxtimes D \tag{46}$$

- compatibility with the composition comparison 2-cells

$$\mathbf{comp}_{C\bar{d}\bar{d}'} \bullet (C \boxtimes \delta' \circ C \boxtimes \delta) = C \boxtimes (\delta' \circ \delta) \bullet \mathbf{comp}_{Cdd'} \tag{47}$$
$$\mathbf{comp}_{\bar{c}\bar{c}'D} \bullet (\gamma' \boxtimes D \circ \gamma \boxtimes D) = (\gamma' \circ \gamma) \boxtimes D \bullet \mathbf{comp}_{cc'D} \tag{48}$$

- compatibility with the swapping 2-cells

$$\mathbf{swap}_{\bar{c}\bar{d}} \bullet (C' \boxtimes \delta \circ \gamma \boxtimes D) = (\gamma \boxtimes D' \circ C \boxtimes \delta) \bullet \mathbf{swap}_{cd} \tag{49}$$

- unit and associativity laws

$$\mathbf{comp} \bullet (1 \circ \mathbf{id}) = 1 \ \& \ \mathbf{comp} \bullet (\mathbf{id} \circ 1) = 1 \tag{50}$$
$$\mathbf{comp} \bullet (\mathbf{comp} \circ 1) = \mathbf{comp} \bullet (1 \circ \mathbf{comp}) \tag{51}$$

- compatibility of swapping with unit and composition

$$\mathbf{swap} \bullet (1 \circ \mathbf{id}) = \mathbf{id} \circ 1 \tag{52}$$
$$\mathbf{swap} \bullet (\mathbf{id} \circ 1) = 1 \circ \mathbf{id} \tag{53}$$
$$\mathbf{swap} \bullet (1 \circ \mathbf{comp}) = (\mathbf{comp} \circ 1) \bullet (1 \circ \mathbf{swap}) \bullet (\mathbf{swap} \circ 1) \tag{54}$$
$$\mathbf{swap} \bullet (\mathbf{comp} \circ 1) = (1 \circ \mathbf{comp}) \bullet (\mathbf{swap} \circ 1) \bullet (1 \circ \mathbf{swap})\,. \tag{55}$$

2.9. Proposition. *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be 2-categories, $\mathcal{C} \boxtimes_{cmp} \mathcal{D}$ the 2-category defined above, then there is an isomorphism*

$$\mathrm{Lax}(\mathcal{C}, \mathrm{Lax}(\mathcal{D}, \mathcal{E})) \cong [\mathcal{C} \boxtimes_{\mathrm{cmp}} \mathcal{D}, \mathcal{E}]_{\mathrm{lnt}}\,. \tag{56}$$

Proof. The data for $\mathcal{G}$ and identifications when forming $\mathcal{C} \boxtimes_{cmp} \mathcal{D}$ correspond exactly to data and laws (3)-(16) for $B \in \mathrm{Lax}(\mathcal{C}, \mathrm{Lax}(\mathcal{D}, \mathcal{E}))$ in the Section 2.1. So, giving B corresponds to giving a computad map $B_{\mathrm{cmp}} : \mathcal{G} \to \mathcal{U}\mathcal{E}$ such that the strict 2-functor $\hat{B}_{\mathrm{cmp}} : \mathcal{F}\mathcal{G} \to \mathcal{E}$ respects the identifications (43)-(55), which corresponds to giving a strict 2-functor $\hat{B} : \mathcal{C} \boxtimes_{\mathrm{cmp}} \mathcal{D} \to \mathcal{E}$.

Define $\mathcal{E}^{\mathcal{D}} := [\mathcal{D}, \mathcal{E}]_{\mathrm{ont}}$. From (32) we get the following isomorphism

$$[\mathcal{F}\mathcal{G}, \mathcal{E}]^{\mathcal{J}}_{\mathrm{lnt}} \cong [\mathcal{F}\mathcal{G}, \mathcal{E}^{\mathcal{J}}]_{\mathrm{lnt}}\,. \tag{57}$$

In particular, we have a bijection on objects, so for a free arrow $\mathcal{J} = \mathbb{I}(:= 0 \to 1)$, (resp. free 2-cell $\mathcal{J} = \mathbb{D}(:= 0 \overset{\to}{\underset{\to}{\Downarrow}} 1)$), we get a bijection between arrows (resp. 2-cells) of $[\mathcal{F}\mathcal{G}, \mathcal{E}]_{\mathrm{lnt}}$ and 2-functors $\mathcal{F}\mathcal{G} \to \mathcal{E}^{\mathcal{I}}$ (resp. $\mathcal{F}\mathcal{G} \to \mathcal{E}^{\mathcal{D}}$).

Consider a lax natural transformation between 2-functors respecting identifications (43)-(55) (as above)

$$\hat{b}_{\mathrm{cmp}} : \hat{B}_{\mathrm{cmp}} \Rightarrow \hat{B}'_{\mathrm{cmp}} : \mathcal{F}\mathcal{G} \to \mathcal{E}\,. \tag{58}$$

It corresponds to a 2-functor

$$\hat{b}^{\mathrm{curry}}_{\mathrm{cmp}} : \mathcal{F}\mathcal{G} \to \mathcal{E}^{\mathbb{I}} \tag{59}$$

which corresponds to a lax natural transformation $b : B \Rightarrow B'$ - the correspondence goes as follows

$$\mathcal{G} \xrightarrow{b^{\text{curry}}_{\text{cmp}}} \mathcal{U}\mathcal{E}^{\mathbb{I}} \tag{60}$$

$$C \boxtimes D \mapsto bCD \tag{61}$$

$$c \boxtimes D,\, C \boxtimes d \mapsto \sigma_{bcD},\, \sigma_{bCd} \tag{62}$$

$$\gamma \boxtimes D,\, C \boxtimes \delta \mapsto (B\gamma D, B'\gamma D),\, (BC\delta, B'C\delta) \tag{63}$$

$$\mathbf{id}_{-},\, \mathbf{comp}_{-},\, \mathbf{swap}_{-} \mapsto (\eta_{B-}, \eta_{B'-}),\, (\mu_{B-}, \mu_{B'-}),\, (\sigma_{B-}, \sigma_{B'-})\,. \tag{64}$$

The RHS of (63) (resp. (64)) being 2-cells of $\mathcal{E}^{\mathbb{I}}$ is equivalent to (17) and (18) (resp. (19), (20) and (21)). The 2-functor $\hat{b}^{\text{curry}}_{\text{cmp}}$ respects identifications (43)-(55) because its source and target do, and so it also corresponds to a 2-functor

$$\hat{b}^{\text{curry}} : \mathcal{C} \boxtimes_{\text{cmp}} \mathcal{D} \to \mathcal{E}^{\mathbb{I}} \tag{65}$$

which is equivalently a lax natural transformation

$$\hat{b} : \hat{B} \Rightarrow \hat{B}' : \mathcal{C} \boxtimes_{\text{cmp}} \mathcal{D} \to \mathcal{E}\,. \tag{66}$$

Similarly, a modification

$$\hat{\beta}_{\text{cmp}} : \hat{b}_{\text{cmp}} \to \hat{\bar{b}}_{\text{cmp}} : \hat{B}_{\text{cmp}} \Rightarrow \hat{B}'_{\text{cmp}} : \mathcal{F}\mathcal{G} \to \mathcal{E} \tag{67}$$

corresponds to a 2-functor

$$\hat{\beta}^{\text{curry}}_{\text{cmp}} : \mathcal{F}\mathcal{G} \to \mathcal{E}^{\mathbb{D}} \tag{68}$$

which corresponds to a modification $\beta : b \to \bar{b}$ via

$$\mathcal{G} \xrightarrow{\beta^{\text{curry}}_{\text{cmp}}} \mathcal{U}\mathcal{E}^{\mathbb{D}} \tag{69}$$

$$C \boxtimes D \mapsto \beta CD \tag{70}$$

$$c \boxtimes D,\, C \boxtimes d \mapsto (\sigma_{bcD}, \sigma_{\bar{b}cD}),\, (\sigma_{bCd}, \sigma_{\bar{b}Cd}) \tag{71}$$

$$\gamma \boxtimes D,\, C \boxtimes \delta \mapsto (B\gamma D, B'\gamma D),\, (BC\delta, B'C\delta) \tag{72}$$

$$\mathbf{id}_{-},\, \mathbf{comp}_{-},\, \mathbf{swap}_{-} \mapsto (\eta_{B-}, \eta_{B'-}),\, (\mu_{B-}, \mu_{B'-}),\, (\sigma_{B-}, \sigma_{B'-})\,. \tag{73}$$

The RHS of (71) being 1-cells of $\mathcal{E}^{\mathbb{D}}$ is equivalent to modification axioms (22) and (23). The RHS of (72) and (73) being 2-cells of $\mathcal{E}^{\mathbb{D}}$, and $\hat{\beta}^{\text{curry}}_{\text{cmp}}$ respecting identifications (43)-(55), are just componentwise properties of $\hat{b}^{\text{curry}}_{\text{cmp}}$ and $\hat{\bar{b}}^{\text{curry}}_{\text{cmp}}$. ∎

2.10. Dual strictifications. Notice that all the data and identifications for $\mathcal{G}(=: \mathcal{G}^{\mathcal{CD}}_{\text{lax}})$, apart from those involving **swap**, are invariant (up to relabelling) with respect to exchanging $\mathcal{C}$ and $\mathcal{D}$. However, if we exchange $\mathcal{C}$ and $\mathcal{D}$ and consider oplax natural transformations at the same time, we arrive at an isomorphic computad $\mathcal{G}^{\mathcal{DC}}_{oplax} \cong \mathcal{G}^{\mathcal{CD}}_{\text{lax}}$, the isomorphism consisting of exchanging the two positions in all the labels. All the identifications are isomorphic as well. This directly leads us to observe

2.11. Corollary. *There is an isomorphism*

$$\mathcal{C} \boxtimes \mathcal{D} \cong (\mathcal{D}^{\mathrm{op}} \boxtimes \mathcal{C}^{\mathrm{op}})^{\mathrm{op}} . \tag{74}$$

Proof. The computad $\mathcal{G}^{\mathcal{DC}}_{oplax}$, with its identifications, generates a 2-category strictifying $\mathrm{Lax}_{\mathrm{op}}(\mathcal{D}, \mathrm{Lax}_{\mathrm{op}}(\mathcal{C}, \mathcal{E}))$. On the other hand,

$$\begin{aligned} \mathrm{Lax}_{\mathrm{op}}(\mathcal{D}, \mathrm{Lax}_{\mathrm{op}}(\mathcal{C}, \mathcal{E})) &\overset{(24)}{\cong} \mathrm{Lax}(\mathcal{D}^{\mathrm{op}}, \mathrm{Lax}(\mathcal{C}^{\mathrm{op}}, \mathcal{E}^{\mathrm{op}}))^{\mathrm{op}} && (75)\\ &\overset{(56)}{\cong} [\mathcal{D}^{\mathrm{op}} \boxtimes \mathcal{C}^{\mathrm{op}}, \mathcal{E}^{\mathrm{op}}]^{\mathrm{op}}_{\mathrm{lnt}} && (76)\\ &\overset{(30)}{\cong} [(\mathcal{D}^{\mathrm{op}} \boxtimes \mathcal{C}^{\mathrm{op}})^{\mathrm{op}}, \mathcal{E}]_{\mathrm{ont}} . && (77) \end{aligned}$$

■

2.12. Corollary. *Given 2-categories $\mathcal{C}$ and $\mathcal{D}$ there are isomorphisms*

$$\begin{aligned} \mathrm{Lax}_{\mathrm{op}}(\mathcal{C}, \mathrm{Lax}_{\mathrm{op}}(\mathcal{D}, \mathcal{E})) &\cong [\mathcal{D} \boxtimes \mathcal{C}, \mathcal{E}]_{\mathrm{ont}} && (78)\\ \mathrm{OpLax}_{\mathrm{op}}(\mathcal{C}, \mathrm{OpLax}_{\mathrm{op}}(\mathcal{D}, \mathcal{E})) &\cong [(\mathcal{C}^{\mathrm{co}} \boxtimes \mathcal{D}^{\mathrm{co}})^{\mathrm{co}}, \mathcal{E}]_{\mathrm{ont}} && (79)\\ \mathrm{OpLax}(\mathcal{C}, \mathrm{OpLax}(\mathcal{D}, \mathcal{E})) &\cong [(\mathcal{D}^{\mathrm{co}} \boxtimes \mathcal{C}^{\mathrm{co}})^{\mathrm{co}}, \mathcal{E}]_{\mathrm{lnt}} . && (80) \end{aligned}$$

When $\mathcal{C} = \mathcal{D} = 1$, we get free distributive laws between monads with opmorphisms (opfunctors in [9]), between comonads with opmorphisms and between comonads with morphisms, respectively.

Now we consider strictification for the case when one of the homs has oplax functors - $\mathrm{Lax}(\mathcal{C}, \mathrm{OpLax}(\mathcal{D}, \mathcal{E}))$. Consider a computad $\mathcal{G}_m$, obtained from $\mathcal{G}$ by reversing 2-cells marked by (**f1**) and changing identifications accordingly. It generates a mixed tensor product $\mathcal{C} \boxtimes^{\mathrm{m}}_{\mathrm{cmp}} \mathcal{D}$, which analogously to Proposition 2.9 and Corollary 2.11 satisfies Corollary 2.13.

2.13. Corollary. *There are isomorphisms:*

$$\begin{aligned} \mathrm{Lax}(\mathcal{C}, \mathrm{OpLax}(\mathcal{D}, \mathcal{E})) &\cong [\mathcal{C} \boxtimes^{\mathrm{m}}_{\mathrm{cmp}} \mathcal{D}, \mathcal{E}]_{\mathrm{lnt}} && (81)\\ \mathcal{C} \boxtimes^{\mathrm{m}}_{\mathrm{cmp}} \mathcal{D} &\cong (\mathcal{D}^{\mathrm{co}} \boxtimes^{\mathrm{m}}_{\mathrm{cmp}} \mathcal{C}^{\mathrm{co}})^{\mathrm{co}} . && (82) \end{aligned}$$

The cases based on this one are:

$$\begin{aligned} \mathrm{OpLax}_{\mathrm{op}}(\mathcal{C}, \mathrm{Lax}_{\mathrm{op}}(\mathcal{D}, \mathcal{E})) &\cong [\mathcal{D} \boxtimes^{\mathrm{m}}_{\mathrm{cmp}} \mathcal{C}, \mathcal{E}]_{\mathrm{ont}} && (83)\\ \mathrm{Lax}_{\mathrm{op}}(\mathcal{C}, \mathrm{OpLax}_{\mathrm{op}}(\mathcal{D}, \mathcal{E})) &\cong [(\mathcal{C}^{\mathrm{op}} \boxtimes^{\mathrm{m}}_{cmp} \mathcal{D}^{\mathrm{op}})^{\mathrm{op}}, \mathcal{E}]_{\mathrm{ont}} && (84)\\ \mathrm{OpLax}(\mathcal{C}, \mathrm{Lax}(\mathcal{D}, \mathcal{E})) &\cong [(\mathcal{D}^{\mathrm{op}} \boxtimes^{\mathrm{m}}_{cmp} \mathcal{C}^{\mathrm{op}})^{\mathrm{op}}, \mathcal{E}]_{\mathrm{lnt}} . && (85) \end{aligned}$$

Finally, when the two homs have different choice for the direction of natural transformations, there is no strictification tensor product, mainly because we have to choose a type of natural transformation for the strict hom. For example, note that the objects $B \in \mathrm{Lax}(\mathcal{C}, \mathrm{Lax}_{\mathrm{op}}(\mathcal{D}, \mathcal{E}))$ correspond to the objects $B \in [\mathcal{D} \boxtimes \mathcal{C}, \mathcal{E}]_{\mathrm{(l)(o)nt}}$ but crossings in the former allow[5] $c \circ b \circ d \Rightarrow d \circ b \circ c$ while crossings of the latter allow $c \circ d \circ b \Rightarrow b \circ d \circ c$ for lax and $b \circ c \circ d \Rightarrow d \circ c \circ b$ for oplax natural transformations, suggesting that this case cannot be strictified. In a similar way, $\mathrm{Lax}(\mathcal{C}, \mathrm{OpLax}_{\mathrm{op}}(\mathcal{D}, \mathcal{E}))$ does not permit strictifications.

3. Simplicial approach

3.1. Bénabou construction of the 2-category of paths. Let $\mathcal{C}$ be a 2-category and $\mathcal{C}^\dagger$ the 2-category of "paths" in $\mathcal{C}$, consisting of the same objects as $\mathcal{C}$, and arrows between C and C' are strict 2-functors p representing paths in $\mathcal{C}$ between C and C'; that is,

$$[n] \xrightarrow{p} \mathcal{C}, \quad p(0) = C, \quad p(n) = C', \tag{86}$$

where $[n]$ is an object of $\Delta_{\perp\top}$, for details see Appendix A. Denote by[6] $(p)_i$ the i^{th} component in the path

$$(p)_i = p\left((i-1) \to i\right) . \tag{87}$$

The identity is a path of zero length on C:

$$[0] \xrightarrow{0_C} \mathcal{C} \tag{88}$$

$$0 \mapsto C \tag{89}$$

and composition is given by "concatenation",

$$(n', p') \circ (n, p) = (n + n', p + p') \tag{90}$$

where $(p + p')_i = (p)_i$ if $i \leqslant n$ and $(p + p')_i = (p')_{i-n}$ otherwise. This composition is strictly associative and unital.

Finally, 2-cells between (n, p) and $(\bar{n}, \bar{p})$, are pairs (ξ, α) where $\xi : [\bar{n}] \to [n]$ is a morphism in $\Delta_{\perp\top}$ and α is an identity on components, oplax-natural transformation, shortly icon, introduced in [5]:

$$\alpha : p \circ \xi \Rightarrow \bar{p}, . \tag{91}$$

with 2-cell components on identity arrows restricted to $\alpha_{1_i} = 1_{1_{\bar{p}(i)}}$, which is true for general (op)lax transformations between normal lax functors. So, α is determined by $\bar{n}$ components on non-identity arrows:

$$\alpha_i := \alpha_{(i-1)\to i} : (p \circ \xi)\left((i-1) \to i\right) \Rightarrow (\bar{p})_i \, . \tag{92}$$

Note that if $\xi(i) = \xi(i-1)$ then the source of the corresponding component of α is the identity; that is, $\alpha_i : 1_{p\xi(i)} \Rightarrow (\bar{p})_i$. The identity 2-cells is given by $1_{(n,p)} = (1_{[n]}, 1_p)$. The

[5] which is a shorter notation for $B'cD' \circ bCD' \circ BCd \Rightarrow B'C'd \circ bC'D \circ BcD$

[6] We reserve p_i, without brackets, to mean the length of the image as in (177).

vertical composite of (ξ, α) and $(\bar{\xi}, \bar{\alpha})$ is obtained by pasting, as in the diagram (93).

$$\begin{tikzcd}
{[n]} \arrow[drr, "p"] \arrow[dr, phantom, "\alpha\Downarrow"] & & \\
{[\bar{n}]} \arrow[u, "\xi"] \arrow[rr, "\bar{p}"] \arrow[dr, phantom, "\bar{\alpha}\Downarrow"] & & \mathcal{C} \\
{[\bar{\bar{n}}]} \arrow[u, "\bar{\xi}"] \arrow[urr, "\bar{\bar{p}}"'] & &
\end{tikzcd} \tag{93}$$

The horizontal composition is concatenation, analogous to the one for path (1-cells), $(\xi', \alpha') \circ (\xi, \alpha) = (\xi + \xi', \alpha + \alpha')$, where $(\alpha + \alpha')_i = \alpha_i$ if $i \leqslant n$, and $(\alpha + \alpha')_i = \alpha'_{i-n}$ otherwise.

3.2. Tensor product simplicially. We proceed to describe our main result: a model $\mathcal{C} \boxtimes_{sim} \mathcal{D}$ for the strictification tensor product and then show that it is isomorphic to $\mathcal{C} \boxtimes_{\mathrm{cmp}} \mathcal{D}$.

Objects of $\mathcal{C} \boxtimes_{sim} \mathcal{D}$ are pairs $(C; D)$ with $C \in \mathcal{C}$ and $D \in \mathcal{D}$.

An arrow in $\mathcal{C} \boxtimes_{sim} \mathcal{D}$ is a sextuple $(n, p, r; m, q, s)$. It consists of a path in $\mathcal{C}$ of length n, a path in $\mathcal{D}$ of length m

$$p : [n] \to \mathcal{C}, \quad q : [m] \to \mathcal{D} \tag{94}$$

and a way to combine them into a string of length $n + m$; that is, a shuffle

$$[n] \xleftarrow{r} [n + m] \xrightarrow{s} [m] \tag{95}$$

where r and s satisfy a compatibility condition (191) saying that one increases if and only if the other one does not.

The identity (empty path) on $(C; D)$ is defined by taking $m = n = 0$, $r = s = 1_{[0]}$, and p and q pick the objects C and D. Composition is defined by path concatenation, formally expressed as tensor product of shuffles.

Below is an example of a 1-cell $\{c_1, d_1, c_2, c_3, d_2\} : (C_1, D_1) \to (C_4, D_3)$ in $\mathcal{C} \boxtimes_{\mathrm{sim}} \mathcal{D}$. Here, $n = 3$, $m = 2$, $r : [5] \to [3]$ and $s : [5] \to [2]$ give the coordinates of the corresponding node in the path, and $p : [3] \to \mathcal{C}$ and $q : [2] \to \mathcal{D}$ are the obvious functors producing the paths $\{c_i\}_{i=1}^{3}$ and $\{d_i\}_{i=1}^{2}$ in $\mathcal{C}$ and $\mathcal{D}$.

$$\begin{array}{cccccc}
 & C_1 & \xrightarrow{c_1} C_2 & \xrightarrow{c_2} & C_3 & \xrightarrow{c_3} C_4 \\
D_1 & C_1D_1 & \xrightarrow{c_1} C_2D_1 & & & \\
{\scriptstyle d_1}\downarrow & & {\scriptstyle d_1}\downarrow & & & \\
D_2 & & C_2D_2 & \xrightarrow{c_2} & C_3D_2 & \xrightarrow{c_3} C_4D_2 \\
{\scriptstyle d_2}\downarrow & & & & & {\scriptstyle d_2}\downarrow \\
D_3 & & & & & C_4D_3
\end{array} \tag{96}$$

A 2-cell

$$(\xi, \alpha; \rho, \beta) : (n, p, r; m, q, s) \to (\bar{n}, \bar{p}, \bar{r}; \bar{m}, \bar{q}, \bar{s}) \tag{97}$$

consists of:

- a shuffle morphism, that is functors $\xi : [\bar{n}] \to [n]$, $\rho : [\bar{m}] \to [m]$ preserving the first and the last element and satisfying, for all $\bar{i} \leqslant \bar{n} + \bar{m}$,

$$\min r^{-1}(\xi \bar{r} \bar{i}) \leqslant \max s^{-1}(\rho \bar{s} \bar{i}) \tag{98}$$

  a condition ensuring that there are no swaps of arrows from $\mathcal{C}$ and $\mathcal{D}$ in the wrong direction. The condition (98) is an explicitly written condition for the existence of the natural transformation (194)

- path 2-cells, that is, icons $\alpha : p \circ \xi \Rightarrow \bar{p}$ and $\beta : q \circ \rho \Rightarrow \bar{q}$, as defined in section 3.1.

Below is an example of a 2-cell.

$$\begin{array}{c} C_1D_1 \xrightarrow[c_1]{} C_2D_1 \xrightarrow[d_1]{} C_2D_2 \xrightarrow[c_2]{} C_3D_2 \xrightarrow[c_3]{} C_4D_2 \xrightarrow[d_2]{} C_4D_3 \\ \\ C_1D_1 \xrightarrow{\bar{c}_1} C_2D_1 \xrightarrow{\bar{d}_1} C_2D_3 \xrightarrow{\bar{c}_2} C_2D_3 \xrightarrow{\bar{c}_3} C_4D_3 \end{array} \tag{99}$$

The above diagram represents two 1-cells and data of ξ and ρ, and what remains is to specify icon components $\alpha_1 : c_1 \Rightarrow \bar{c}_1$, $\alpha_2 : 1_{C_2} \Rightarrow \bar{c}_2$, $\alpha_3 : c_3 \circ c_2 \Rightarrow \bar{c}_3$ in $\mathcal{C}$ and $\beta_1 : d_2 \circ d_1 \Rightarrow \bar{d}_1$ in $\mathcal{D}$.

Vertical composition and whiskerings are defined componentwise as in **Shuff**, $\mathcal{C}^\dagger$ and $\mathcal{D}^\dagger$.

3.3. As a limit. The category $\mathcal{C} \boxtimes_{sim} \mathcal{D}$ is a limit of the following diagram in 2-Cat.

$$\begin{array}{ccccccccc} \mathcal{C}^\dagger & \to & \Sigma\Delta_{\perp\top} & \leftarrow & \text{FDL} & \to & \Sigma\Delta_{\perp\top} & \leftarrow & \mathcal{D}^\dagger \\ C & \mapsto & * & \leftarrow\!\shortmid & * & \mapsto & * & \leftarrow\!\shortmid & D \\ (n,p) & \mapsto & [n] & \leftarrow\!\shortmid & (n,m,s,r) & \mapsto & [m] & \leftarrow\!\shortmid & (m,q) \\ (\xi,\alpha) & \mapsto & \xi & \leftarrow\!\shortmid & (\xi,\rho,\gamma) & \mapsto & \rho & \leftarrow\!\shortmid & (\rho,\beta) \end{array}$$

3.4. Isomorphism between two constructions. This part is about proving the following proposition.

3.5. Theorem. *There is an isomorphism*

$$\mathcal{C} \boxtimes_{sim} \mathcal{D} \cong \mathcal{C} \boxtimes_{cmp} \mathcal{D}\,. \tag{100}$$

We shall define a computad morphism $T : \mathcal{G} \to \mathcal{U}(\mathcal{C} \boxtimes_{sim} \mathcal{D})$, show that the induced strict 2-functor $\hat{T} : \mathcal{F}\mathcal{G} \to \mathcal{C} \boxtimes_{sim} \mathcal{D}$ respects the identifications (43)-(55), and that any other 2-functor $\hat{V} : \mathcal{FG} \to \mathcal{E}$ respecting them factors uniquely through $\hat{T}$. Then, from the universal property of $\mathcal{C} \boxtimes_{cmp} \mathcal{D}$ it will follow that $\mathcal{C} \boxtimes_{cmp} \mathcal{D} \cong \mathcal{C} \boxtimes_{sim} \mathcal{D}$.

The computad morphism $T : \mathcal{G} \to \mathcal{U}(\mathcal{C} \boxtimes_{sim} \mathcal{D})$ is defined on nodes by

$$T(C \boxtimes D) = (C; D)\,, \tag{101}$$

on edges by

$$T(C \boxtimes d) = (0, \{C\}, \sigma_0^1; 1, \{D \xrightarrow{d} D'\}, 1_{[1]}) \tag{102}$$
$$T(c \boxtimes D) = (1, \{C \xrightarrow{c} C'\}, 1_{[1]}; 0, \{D\}, \sigma_0^1)\,, \tag{103}$$

on 2-cells inherited from $\mathcal{C}$ and $\mathcal{D}$ by

$$\begin{aligned} T(C \boxtimes \delta) = (1_{[0]}, \{\}; 1_{[1]}, \{\delta\}) : (0, \{C\}, \sigma_0^1; 1, \{D \xrightarrow{d} D'\}, 1_{[1]}) \\ \Rightarrow (0, \{C\}, \sigma_0^1; 1, \{D \xrightarrow{\bar{d}} D'\}, 1_{[1]}) \end{aligned} \tag{104}$$
$$\begin{aligned} T(\gamma \boxtimes D) = (1_{[1]}, \{\gamma\}; 1_{[0]}, \{\}) : (1, \{C \xrightarrow{c} C'\}, 1_{[1]}; 0, \{D\}, \sigma_0^1) \\ \Rightarrow (1, \{C \xrightarrow{\bar{c}} C'\}, 1_{[1]}; 0, \{D\}, \sigma_0^1) \end{aligned} \tag{105}$$

and on the comparison and swapping 2-cells by

$$\begin{aligned} T(\mathbf{id}_{1_C D}) = (\sigma_0^1, \{1_{1_C}\}; 1_{[0]}, \{\}) : (0, \{C\}, 1_{[0]}; 0, \{D\}, 1_{[0]}) \\ \Rightarrow (1, \{C \xrightarrow{1_C} C\}, 1_{[1]}; 0, \{D\}, \sigma_0^1) \end{aligned} \tag{106}$$
$$\begin{aligned} T(\mathbf{id}_{C 1_D}) = (1_{[0]}, \{\}; \sigma_0^1, \{1_{1_D}\}) : (0, \{C\}, 1_{[0]}; 0, \{D\}, 1_{[0]}) \\ \Rightarrow (0, \{C\}, \sigma_0^1; 1, \{D \xrightarrow{1_D} D\}, 1_{[1]}) \end{aligned} \tag{107}$$

$$\begin{aligned} T(\mathbf{comp}_{c,c',D}) = &(\partial_1^2, \{1_{c' \circ c}\}; 1_{[0]}, \{\}) : \\ &(2, \{C \xrightarrow{c} C' \xrightarrow{c'} C''\}, 1_{[2]}; 0, \{D\}, !_{[2]\to[0]}) \\ &\Rightarrow (1, \{C \xrightarrow{c' \circ c} C''\}, 1_{[2]}; 0, \{D\}, \sigma_0^1) \end{aligned} \tag{108}$$
$$\begin{aligned} T(\mathbf{comp}_{C,d,d'}) = &(1_{[0]}, \{\}; \partial_1^2, \{1_{d' \circ d}\}) : \\ &(0, \{C\}, !_{[2]\to[0]}; 2, \{D \xrightarrow{d} D' \xrightarrow{d'} D''\}, 1_{[2]}) \\ &\Rightarrow (0, \{C\}, \sigma_0^1; 1, \{D \xrightarrow{d' \circ d} D''\}, 1_{[1]}) \end{aligned} \tag{109}$$

$$\begin{aligned} T(\mathbf{swap}_{c,d}) = &(1_{[1]}, \alpha = \{1_c\}; 1_{[1]}, \beta = \{1_d\}) : \\ &(1, \{C \xrightarrow{c} C'\}, \sigma_1^2; 1, \{D \xrightarrow{d} D'\}, \sigma_0^2) \\ &\Rightarrow (1, \{C \xrightarrow{c} C'\}, \sigma_0^2; 1, \{D \xrightarrow{d} D'\}, \sigma_1^2)\,. \end{aligned} \tag{110}$$

To check that the last 2-cell is the valid one, write equation (194) as

$$L\sigma_1^2 \circ 1 \circ \sigma_0^2 = \partial_2^2 \circ \sigma_0^2 \Rightarrow \partial_0^2 \circ \sigma_1^2 = R\sigma_0^2 \circ 1 \circ \sigma_1^2\,. \tag{111}$$

The cells on the RHS of (104)-(110) will be called *elementary 2-cells.*

To see that the induced strict 2-functor respects identifications (43)-(55), note that $T(\mathbf{id})$, $T(\mathbf{comp})$, and $T(\mathbf{swap})$ have trivial icon components, while the definition of T on other parts of the computad have trivial components in **Shuff**, and that the composition of 2-cells in $\mathcal{C} \boxtimes_{sim} \mathcal{D}$ is done independently in each of the components.

Given a computad map $V : \mathcal{G} \to \mathcal{UE}$, such that $\hat{V} : \mathcal{FG} \to \mathcal{E}$ respects the identifications (43)-(55), form the following assignments $W : \mathcal{C} \boxtimes_{sim} \mathcal{D} \to \mathcal{E}$ on objects

$$W(C;D) = V(C \boxtimes D) \tag{112}$$

and on elementary arrows

$$W(0, \{C\}, \sigma_0^1; 1, \{D \xrightarrow{d} D'\}, 1_{[1]}) = W(T(C \boxtimes d)) = V(C \boxtimes d) \tag{113}$$

$$W(1, \{C \xrightarrow{c} C'\}, 1_{[1]}; 0, \{D\}, \sigma_0^1) = W(T(c \boxtimes D)) = V(c \boxtimes D)\,. \tag{114}$$

Since every shuffle can be written uniquely as a sum of shuffles of unit length, the above assignment determines assignment on all 1-cells; given $(n, p, r; m, q, s)$, assign to it the composite given by (115).

$$W(n,p,r;m,q,s) = \circ^1_{i=n+m} \begin{cases} V((p)_i \boxtimes qsi), \text{ if } s_i = 0 \\ V(pri \boxtimes (q)_i), \text{ if } r_i = 0 \end{cases} \tag{115}$$

When $n = m = 0$ we get that W preserves identities; that is,

$$W(1_{(C;D)}) = 1_{W(C;D)}\,. \tag{116}$$

Also, W preserves composition

$$\begin{aligned}
&W(n',p',r';m',q',s') \circ W(n,p,r;m,q,s) = \\
&\circ^1_{i'=n'+m'} \begin{cases} V((p')_{i'} \boxtimes q's'i'), \text{ if } s'_{i'} = 0 \\ V(p'r'i' \boxtimes (q')_{i'}), \text{ if } r'_{i'} = 0 \end{cases} \circ^1_{i=n+m} \begin{cases} V((p)_i \boxtimes qsi), \text{ if } s_i = 0 \\ V(pri \boxtimes (q)_i), \text{ if } r_i = 0 \end{cases} &(117)\\
&= \circ^1_{i=n'+m'+n+m} \begin{cases} V((p')_i \boxtimes q's'i), \text{ if } s'_i = 0, \text{ and } i > n+m \\ V(p'r'i \boxtimes (q')_i), \text{ if } r'_i = 0, \text{ and } i > n+m \\ V((p)_i \boxtimes qsi), \text{ if } s_i = 0, \text{ and } i \leqslant n+m \\ V(pri \boxtimes (q)_i), \text{ if } r_i = 0, \text{ and } i \leqslant n+m \end{cases} &(118)\\
&= W(n'+n, p'+p, r'+r; m'+m, q'+q, s'+s) &(119)\\
&= W((n',p',r';m',q',s') \circ (n,p,r;m,q,s))\,. &(120)
\end{aligned}$$

Hence, it is a functor on the underlying categories.

The requirement that $WT = V$ determines the assignment on identities

$$T(1_g) = 1_{Tg} \tag{121}$$

on elementary 2-cells $T\pi$

$$W(T\pi) = V(\pi) \tag{122}$$

and similarly on whiskered elementary 2-cells

$$W(Tg'' \circ T\pi \circ Tg) := Vg'' \circ V\pi \circ Vg = V(g'' \circ \pi \circ g) \tag{123}$$

where $T\pi$ is an elementary 2-cell and Tg and Tg' are 1-cells.

Given any 2-cell $(\xi, \alpha; \rho, \beta)$, as in (97), choose a decomposition into whiskered elementary 2-cells in the following order, starting from the target 1-cell,

- elementary β, $j = \bar{m}, .., 1$

$$J_j = 1 \circ T(\bar{p}\bar{r}j \boxtimes \beta_j) \circ 1 \tag{124}$$
$$= (1_{[\bar{n}]}, \{1_{p_1}, .., 1_{p_{\bar{n}}}\}; 1_{[\bar{m}]}, \{1_{q_1}, .., \beta_j, .., 1_{q_{\bar{n}}}\}) \tag{125}$$

- elementary α, $i = \bar{n}, .., 1$

$$I_i = 1 \circ T(\alpha_i \boxtimes \bar{q}\bar{s}i) \circ 1 \tag{126}$$
$$= (1_{[\bar{n}]}, \{1_{p_1}, .., \alpha_i, .., 1_{p_{\bar{n}}}\}; 1_{[\bar{m}]}, \{1_{q_1}, .., 1_{q_{\bar{n}}}\}) \tag{127}$$

- comparisons in $\mathcal{D}$, $j = \bar{m}, .., 1$
  - if $\bar{\rho}_j = 0$ then
  $$L_{j,1} = 1 \circ T(\mathbf{id}) \circ 1 =: L_j^{(\mathbf{id})} \tag{128}$$
  - if $\bar{\rho}_j \geqslant 2$, $k = \bar{\rho}_j - 1, ..., 1$
  $$L_{j,k} = 1 \circ T(\mathbf{comp}) \circ 1 =: L_{j,k}^{(\mathbf{comp})} \tag{129}$$

    This order corresponds to left bracketing.
  - if $\bar{\rho}_j = 1$ then $L_{j,1} = 1$, and can be ignored.

- comparisons in $\mathcal{C}$, $i = \bar{n}, .., 1$
  - if $\bar{\xi}_i = 0$ then
  $$K_{i,1} = 1 \circ T(\mathbf{id}) \circ 1 =: K_j^{(\mathbf{id})} \tag{130}$$
  - if $\bar{\xi}_j \geqslant 2$, $k = \bar{\xi}_j - 1, ..., 1$
  $$K_{i,k} = 1 \circ T(\mathbf{comp}) \circ 1 =: K_{j,k}^{(\mathbf{comp})} \tag{131}$$

    This order corresponds to left bracketing.

  - if $\bar{\xi}_j = 1$ then $K_{j,1} = 1$, and can be ignored.

- crossings - the remaining 2-cell to decompose has trivial icon components as well as trivial ξ and ρ. In the relation tables - which define the two shuffles - elementary crossings correspond to switching ones to zeros, or, going backwards, switching zeros to ones. Let (x, y) be the coordinates of the corresponding crossings, order them by $x-y$ and then (if the $x-y$ value is the same) by $x+y$. Our backward decomposition starts with the last crossing in the table. Denote them by S_i.

Now, define

$$W(\xi, \alpha; \rho, \beta) = \circ_i W(J_i) \circ_i W(I_i) \circ_{i,j} W(L_{i,j}) \circ_{i,j} W(K_{i,j}) \circ_i W(S_i) \tag{132}$$

Given a composable pair of 2-cells, the composite of their images under W, $W(\bar{\xi}, \bar{\alpha}; \bar{\rho}, \bar{\beta}) \circ W(\xi, \alpha; \rho, \beta)$, is equal to

$$\begin{aligned} &\circ_i W(\bar{J}_i) \circ_i W(\bar{I}_i) \circ_{i,j} W(\bar{L}_{i,j}) \circ_{i,j} W(\bar{K}_{i,j}) \circ_i W(\bar{S}_i) \\ &\circ_i W(J_i) \circ_i W(I_i) \circ_{i,j} W(L_{i,j}) \circ_{i,j} W(K_{i,j}) \circ_i W(S_i) \end{aligned} \tag{133}$$

which need not be in the canonical form. The assignment on the composite 2-cell

$$(\xi \circ \bar{\xi}, \bar{\alpha} \bullet (\alpha \circ \bar{\xi}); \rho \circ \bar{\rho}, \bar{\beta} \bullet (\beta \circ \bar{\rho})) \tag{134}$$

is in the canonical form, and the two are equal which we show by "bubble-sorting" the decomposition (133). In each step one of two cases can happen:

- the output (target of the elementary part) of the first 2-cell does not overlap with the input (source of the elementary part) of the second 2-cell. Then we can write the vertical composite of their images as

$$\begin{aligned} &W(Tg_5 \circ T\bar{g}_4 \circ Tg_3 \circ T\pi_2 \circ Tg_1) \\ &\bullet W(Tg_5 \circ T\pi_1 \circ Tg_3 \circ Tg_2 \circ Tg_1) \\ &= V(g_5 \circ \pi_1 \circ g_3 \circ \pi_2 \circ g_1) = \\ &W(Tg_5 \circ T\pi_1 \circ Tg_3 \circ T\bar{g}_2 \circ Tg_1) \\ &\bullet W(Tg_5 \circ Tg_4 \circ Tg_3 \circ T\pi_2 \circ Tg_1) \end{aligned} \tag{135}$$

  meaning that we can change the order of their composition after suitably changing the whiskering 1-cells.

- the output of the first 2-cell overlaps with the input of the second 2-cell. Depending on which elementary 2-cells meet, do an operation according to the following table.

$1^{st}\backslash 2^{nd}$	$\bar{J}$	$\bar{I}$	$\bar{L}^{(\mathbf{id})}$	$\bar{L}^{(\mathbf{comp})}$	$\bar{K}^{(\mathbf{id})}$	$\bar{K}^{(\mathbf{comp})}$	$\bar{S}$
J	(45)	$\perp$	$\perp$	(47)	$\perp$	$\perp$	$\perp$/(49)
I	$\perp$	(46)	$\perp$	$\perp$	$\perp$	(48)	(49)/$\perp$
$L^{(\mathbf{id})}$	R	$\perp$	$\perp$	(50)	$\perp$	$\perp$	$\perp$/(53)
$L^{(\mathbf{comp})}$	R	$\perp$	$\perp$	R/(51)	$\perp$	$\perp$	$\perp$/(55)
$K^{(\mathbf{id})}$	$\perp$	R	$\perp$	$\perp$	$\perp$	(50)	(52)/$\perp$
$K^{(\mathbf{comp})}$	$\perp$	R	$\perp$	$\perp$	$\perp$	R/(51)	(54)/$\perp$
S	$\perp/R$	$R/\perp$	$\perp$	$\perp/\perp/R$	$\perp$	$R/\perp/\perp$	$R/\perp/\perp$

(136)

If the first 2-cell has n outputs and the second 2-cell has m inputs, there are $n+m-1$ ways to match them. When different, these cases are separated by "/". The symbol $\perp$ denotes that matching is not possible for that case, and R denotes that the matching is possible, but the order is already correct (lower triangle). Finally, an equation number tells us to apply $\hat{T}$ to both sides, and substitute the LHS, which appears in the composition, with the RHS. Each step changes the decomposition of the 2-cell, and the fact that $\hat{V}$ preserves relations ensures that the composite in $\mathcal{E}$ does not change.

This proves that W is functorial on homs.

A 2-cell in $\mathcal{C} \boxtimes \mathcal{D}$, obtained by whiskering, has the same elementary 2-cells in its decomposition as the original 2-cell. Hence, the two different composites

$$(WT\bar{g}' \circ W(\xi, \alpha; \rho, \beta)) \bullet (W(\xi', \alpha'; \rho', \beta') \circ WTg) \tag{137}$$

and

$$(W(\xi', \alpha'; \rho', \beta') \circ WT\bar{g}) \bullet (WTg' \circ W(\xi, \alpha; \rho, \beta)) \tag{138}$$

necessarily bubble-sort to $W((\xi', \alpha'; \rho', \beta') \circ (\xi, \alpha; \rho, \beta))$. This completes the proof that W is a 2-functor.

The functor $\hat{T}$ is bijective on objects and arrows, and surjective on 2-cells, so W is the unique 2-functor satisfying $\hat{V} = W\hat{T}$.

3.6. Mixed tensor product. The case covering the free mixed distributive law, strictifying $\mathrm{Lax}(\mathcal{C}, \mathrm{OpLax}(\mathcal{D}, \mathcal{E}))$, produces $\mathcal{C} \boxtimes^{\mathrm{m}}_{\mathrm{sim}} \mathcal{D}$ that has the same objects and arrows as $\mathcal{C} \boxtimes \mathcal{D}$, and 2-cells differ by changing the direction of $\rho : [m] \to [\bar{m}]$ to accommodate comultiplication and counit, change in icon $\beta : q \Rightarrow \bar{q}\rho : [m] \to \mathcal{D}$, with the restriction for crossings taking a slightly different form

$$Lr \circ \xi \circ \bar{r} \Rightarrow Rs \circ R\rho \circ \bar{s}\,. \tag{139}$$

With a proof following the same steps as the non-mixed case, we state the following proposition.

3.7. PROPOSITION. *There is an isomorphism*

$$\mathcal{C} \boxtimes^{\mathrm{m}}_{\mathrm{sim}} \mathcal{D} \cong \mathcal{C} \boxtimes^{\mathrm{m}}_{\mathrm{cmp}} \mathcal{D} \,. \tag{140}$$

4. Some properties and an example

4.1. LAX MONOIDAL STRUCTURE. In this section we will recall the universal property of the lax Gray tensor product [4], and use it together with the Bénabou construction of paths from Section 3.1 to describe a lax monoidal structure on the category of 2-categories and lax functors.

Let L2-Cat denote the category of (small) 2-categories and lax functors, while 2-Cat denotes denote the subcategory of L2-Cat consisting of strict 2-functors. The inclusion $i_0 : \text{2-Cat} \hookrightarrow \text{L2-Cat}$ has a left adjoint:

- There is an assignment on objects $(-)^\dagger : \text{L2-Cat} \hookrightarrow \text{2-Cat}$ (the Bénabou strictification construction, Section 3.1)
- For each $\mathcal{C}$ there is an universal L2-Cat arrow (lax functor) $\eta_\mathcal{C} : \mathcal{C} \to \mathcal{C}^\dagger$, meaning, each lax functor $F : \mathcal{C} \to \mathcal{D}$ gives rise to a unique strict functor
$$s_0 F : \mathcal{C}^\dagger \to \mathcal{D} \tag{141}$$
satisfying $s_0 F \circ \eta_\mathcal{C} = F$.

In fact, one could define a computad presentation of $(-)^\dagger$ and analogously to proofs of Proposition 2.9 and Theorem 3.5 show that

$$\mathrm{Lax}(\mathcal{C}, \mathcal{E}) \cong [\mathcal{C}^\dagger, \mathcal{E}]_{\mathrm{lnt}} \,. \tag{142}$$

The lax Gray tensor product, $\otimes_l : \text{2-Cat} \times \text{2-Cat} \to \text{2-Cat}$, is a tensor product for the internal hom $[-,-]_{\mathrm{lnt}}$, that is

$$[\mathcal{C}, [\mathcal{D}, \mathcal{E}]_{\mathrm{lnt}}]_{\mathrm{lnt}} \cong [\mathcal{C} \otimes_l \mathcal{D}, \mathcal{E}]_{\mathrm{lnt}} \,. \tag{143}$$

The left hand side of Eq. (1) can be transformed

$$\mathrm{Lax}(\mathcal{C}, \mathrm{Lax}(\mathcal{D}, \mathcal{E})) \overset{(142)}{\cong} [\mathcal{C}^\dagger, [\mathcal{D}^\dagger, \mathcal{E}]_{\mathrm{lnt}}]_{\mathrm{lnt}} \tag{144}$$
$$\overset{(143)}{\cong} [\mathcal{C}^\dagger \otimes_l \mathcal{D}^\dagger, \mathcal{E}]_{\mathrm{lnt}} \tag{145}$$

leading to the third description of the tensor product[7]

$$\mathcal{C} \boxtimes \mathcal{D} \cong \mathcal{C}^\dagger \otimes_l \mathcal{D}^\dagger \,. \tag{146}$$

From 2-monadic point of view, the $\otimes_l$ is a pseudo algebra on 2-Cat for the monoidal category 2-monad on CAT. The adjunction $(-)^\dagger \dashv i_0$ induces a lax algebra structure on L2-Cat given by

$$\boxtimes_n(\mathcal{C}_1, \dots, \mathcal{C}_n) := \mathcal{C}_1^\dagger \otimes_{\mathrm{l}} \dots \otimes_{\mathrm{l}} \mathcal{C}_n^\dagger \,. \tag{147}$$

[7]The lax Gray product $\otimes_l$ is defined via its universal property, and the explicit description involves relations and quotienting. Our direct description, explained in Section 3, involves no quotienting.

4.2. Generalization of the composite monad. There is an obvious 2-functor $L : \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{C} \times \mathcal{D}$ that forgets shuffles and composes paths. It has a right adjoint R in the 2-category of 2-categories, lax functors and icons:

$$\mathcal{C} \times \mathcal{D} \xrightarrow{R} \mathcal{C} \boxtimes \mathcal{D} \tag{148}$$
$$(C, D) \mapsto C \boxtimes D \tag{149}$$
$$(c, d) \mapsto CD \xrightarrow{Cd} CD' \xrightarrow{cD'} C'D' \tag{150}$$
$$(\gamma, \delta) \mapsto (\gamma \boxtimes D') \circ (C \boxtimes \delta) \tag{151}$$

with identity and composition comparison maps

$$! : 1_{C\boxtimes D} \Rightarrow CD \xrightarrow{C1_D} CD \xrightarrow{1_C D} CD \tag{152}$$
$$(\partial_1^2, 1; \partial_1^2, 1) : CD \xrightarrow{Cd} CD' \xrightarrow{cD'} C'D' \xrightarrow{C'd'} C'D'' \xrightarrow{c'D''} C''D'' \tag{153}$$
$$\Rightarrow CD \xrightarrow{C(d' \circ d)} CD'' \xrightarrow{(c' \circ c)D''} C''D''\,. \tag{154}$$

The composite $L \circ R$ is just the identity functor $1_{\mathcal{C}\times\mathcal{D}}$, while the unit of the adjunction is an icon

$$\eta : 1_{\mathcal{C}\boxtimes\mathcal{D}} \Rightarrow R \circ L \tag{155}$$

assigning to each arrow $(n, p, r; m, q, s)$ in $\mathcal{C} \boxtimes \mathcal{D}$ a 2-cell

$$(!_{[1]\to[n]}, 1_{\circ p}; !_{[1]\to[m]}, 1_{\circ q}) : (n, p, r; m, q, s) \Rightarrow (1, \circ p, \sigma_0^2; 1, \circ q, \sigma_1^2)\,. \tag{156}$$

Whiskering η on the left (resp. right) by L (resp. R) gives the identity on L (resp. R), proving the adjunction axioms.

Any strict functor $\hat{B} : \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{E}$ can be precomposed with R to give a lax functor

$$\hat{B} \circ R : \mathcal{C} \times \mathcal{D} \to \mathcal{E}\,. \tag{157}$$

This generalizes the notion of a composite monad induced by a distributive law.

4.3. Parametrizing parametrization of categories. Take $\mathcal{C}$ and $\mathcal{D}$ to be just categories (seen as locally discrete 2-categories), and[8] $\mathcal{E}$ = Span.

The bicategory of spans is equivalent to the bicategory of matrices, which is in turn a full subcategory of[9] Mod. Each strict functor $\hat{B} : \mathcal{C} \boxtimes \mathcal{D} \to$ Span is, in particular, a normal lax functor, so we can use the Bénabou construction [11] (after forgetting 2-cells) to obtain a category $\tilde{B}_{\text{nerve}}$ parametrised over $\mathcal{C} \boxtimes \mathcal{D}$. Explicitly, $\tilde{B}_{\text{nerve}}$ has objects over $C \boxtimes D$ given by the set BCD. Arrows over $C \boxtimes d$ and $c \boxtimes D$ are elements of spans BCd and BcD respectively, and they generate arrows over arbitrary paths, which are, due to composition in Span, composable tuples.

[8] Instead of Span one can take a strict version with objects sets X, Y... and arrows cocontinuous functors Set$/X \to$ Set$/Y$ which are determined by the assignment of singletons.

[9] Consisting of categories and modules (aka profunctors or distributors)

The 2-cells that we have temporarily forgotten are mapped to span morphisms. In particular, the images $\hat{B}\eta_p$ of the unit of the adjunction (155) give a unique way of "composing" arbitrary arrows in $\tilde{B}_{\text{nerve}}$, resulting in an arrow over a path in $\mathcal{C} \boxtimes \mathcal{D}$ of the form $CD \xrightarrow{Cd} CD' \xrightarrow{cD} C'D'$. The image of this assignment forms a category $\tilde{B}$ whose composition is concatenation in $\tilde{B}_{\text{nerve}}$ followed by applying (the unique) appropriate $B\eta$. Uniqueness guarantees the identity and associativity laws.

Explicitly, $\tilde{B}$ with the same objects as $\tilde{B}_{\text{nerve}}$, and arrows between $X \in B(C \boxtimes D)$ and $X' \in B(C' \boxtimes D')$ are elements of $B(CD \xrightarrow{Cd} CD' \xrightarrow{cD} C'D')$, denoted by pairs (g, f),. The identity is

$$1_X = (1^D_X, 1^C_X), \text{ with} \tag{158}$$
$$1^D_X := (B\mathbf{id}_{C1_D})(X) \tag{159}$$
$$1^C_X := (B\mathbf{id}_{1_C D})(X) \tag{160}$$

and composition is given by

$$(g', f') \circ (g, f) = B((\mathbf{comp} \circ \mathbf{comp}) \bullet (1 \circ \mathbf{swap} \circ 1))(g', f', g, f)\,. \tag{161}$$

For each object $D \in \mathcal{D}$ we get a subcategory $\pi_D\tilde{B}$ parametrized by $\mathcal{C}$ - an object X over C is an element of BCD, and arrow $f : X \to X'$ over c is an element of BcD, which can be identified with an arrow $(1^D_X, f)$ of $\tilde{B}$. Similarly, each object $C \in \mathcal{C}$ gives a subcategory $\pi_C\tilde{B}$, parametrized by $\mathcal{D}$. Furthermore, each arrow (g, f) in $\tilde{B}$ can be decomposed as

$$(1_{D'}, f) \circ (g, 1_C) \tag{162}$$

or as

$$(g, 1_{C'}) \circ (1_D, f)\,. \tag{163}$$

A. Simplices, intervals and shuffles

The *algebraist's delta*, denoted by Δ_a, is the full subcategory of **Cat** consisting of categories $\langle n\rangle$ whose objects are numbers $0, ..., n-1$ and 1-cells are unique $i \to j$ when $i \leqslant j$. The empty category is denoted $\langle 0\rangle$. Arrows between $\langle n\rangle$ and $\langle n'\rangle$ are functors; that is, order preserving functions, generated by face and degeneracy maps

$$\sigma^n_i : \langle n+1\rangle \to \langle n\rangle,\ i = 0, \ldots, n-1 \tag{164}$$
$$\partial^n_i : \langle n\rangle \to \langle n+1\rangle,\ i = 0, \ldots, n \tag{165}$$

which can be presented in a diagram

$$\begin{tikzcd}
\langle 0\rangle \arrow[r, "\partial^0_0"] & \langle 1\rangle \arrow[r, shift left=4, "\partial^1_1"] \arrow[r, shift right=4, "\partial^1_0"'] & \langle 2\rangle \arrow[l, "\sigma^1_0"'] \arrow[r, shift left=8, "\partial^2_2"] \arrow[r, "\partial^2_1"] \arrow[r, shift right=8, "\partial^2_0"'] & \langle 3\rangle \arrow[l, shift right=4, "\sigma^2_1"'] \arrow[l, shift left=4, "\sigma^2_0"] & \cdots
\end{tikzcd} \tag{166}$$

A natural transformation between f and $\bar{f}$, if one exists, is unique and witnesses that $fi \leqslant \bar{f}i$ for all i, turning $\Delta_a[\langle n\rangle, \langle n'\rangle]$ into a poset. The 2-category Δ_a is equipped with a strict monoidal structure, the ordinal sum $\oplus$.

A.1. Intervals - free monoid. Denote by $\Delta_{\perp\top}$ the subcategory of Δ_a, called the category of intervals, consisting of relabelled objects

$$[n] := \langle n+1\rangle,\ n = 0, 1, ... \tag{167}$$

and 1-cells that preserve the first and the last element; it is generated by the arrows from the inside of the diagram (166), represented by the bold part of

$$\begin{tikzcd} \cdot \arrow[r] & \mathbf{[0]} & \mathbf{[1]} \arrow[l, "\sigma_0^1"'] & \mathbf{[2]} \arrow[l, shift right=2, "\sigma_1^2"'] \arrow[l, shift left=2, "\sigma_0^2"] & \cdots \\ & \mathbf{[0]} \arrow[r, shift left=3] \arrow[r, shift right=3] & \mathbf{[1]} \arrow[r, "\partial_1^2"] \arrow[r, shift left=6] \arrow[r, shift right=6] & \mathbf{[2]} & \end{tikzcd} \tag{168}$$

It is clear that suspension (moving nodes to the left) gives an isomorphism

$$\Delta_{\perp\top}^{op} \cong \Delta_a \tag{169}$$
$$[n] = \langle n+1\rangle \mapsto \langle n\rangle \tag{170}$$
$$\sigma_i^n \mapsto \partial_i^{n-1},\ i = 0, \dots, n-1 \tag{171}$$
$$\partial_i^n \mapsto \sigma_{i-1}^{n-1},\ i = 1, \dots, n-1 \tag{172}$$

The tensor product on $\Delta_{\perp\top}$ is inherited from the ordinal sum under the isomorphism (169), and has the interpretation of path concatenation;

$$\xi : [n] \to [m] \tag{173}$$
$$\xi' : [n'] \to [m'] \tag{174}$$

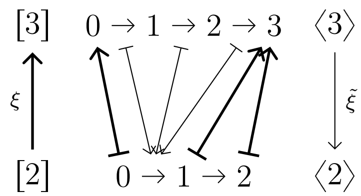
concatenate to

$$\xi + \xi' : [n+n'] \to [m+m'] \tag{175}$$
$$i \mapsto \begin{cases} \xi(i), & \text{if } i \leqslant n \\ \xi'(i-n), & \text{otherwise.} \end{cases} \tag{176}$$

In particular, every such 1-cell ξ can be decomposed

$$\xi = \sum_{i=1}^{n} ! : [1] \to [\xi_i], \text{ with } \sum_{i=1}^{n} \xi_i = m. \tag{177}$$

The image of ξ under the isomorphism is an order preserving function that takes ξ_i points in $\langle m\rangle$ to $i \in \langle n\rangle$. An example of the isomorphism, for $n = 2$ and $m = 3$ can be visualized as



(178)

The embedding $\Delta_{\perp\top} \hookrightarrow \Delta_a$ is a monoidal functor with comparison maps representing

$$\langle 0\rangle \xrightarrow{\partial_0^0} \langle 1\rangle = [0] \tag{179}$$

$$[n] \oplus [n'] = \langle n+n'+2\rangle \xrightarrow{z_{n,n'} := \sigma_n^{n+n'+1}} \langle n+n'+1\rangle = [n] + [n'] \tag{180}$$

There is a functor

$$\Delta_{\perp\top}^{op} \xrightarrow{L} \Delta_a \tag{181}$$

$$[n] = \langle n+1\rangle \mapsto \langle n+1\rangle \tag{182}$$

$$\sigma_i^n \mapsto \partial_{i+1}^n,\ i = 0, \ldots, n-1 \tag{183}$$

$$\partial_i^n \mapsto \sigma_i^n,\ i = 1, \ldots, n-1 \tag{184}$$

assigning to each 1-cell in $\Delta_{\perp\top}$ its left adjoint (Galois connection) in Δ_a. Explicitly, for $\xi : [n] \to [m]$,

$$L(\xi) : \langle m+1\rangle \to \langle n+1\rangle \tag{185}$$

$$i \mapsto \min\{j | i \leqslant \xi(j)\}. \tag{186}$$

The functor L is oplax monoidal, with the same comparison maps (179)-(180), but the naturality holds up to a 2-cell

$$L(\xi + \xi') \circ z_{m,m'} \Rightarrow z_{n,n'} \circ (L\xi \oplus L\xi'). \tag{187}$$

Dually, there is a lax monoidal functor $\Delta_{\perp\top}^{op} \xrightarrow{R} \Delta_a$ assigning right adjoints, with a 2-cell

$$R(\xi + \xi') \circ z_{m,m'} \Leftarrow z_{n,n'} \circ (R\xi \oplus R\xi'). \tag{188}$$

The free 2-category containing a monad [7] is obtained as the suspension of the monoidal category of intervals,

$$\mathrm{FM} := \Sigma\Delta_{\perp\top}. \tag{189}$$

A.2. Shuffles - free distributive law. A shuffle of $\langle n\rangle$ and $\langle m\rangle$ in Δ_a is defined to be a pair of complement embeddings $\langle n\rangle \to \langle n+m\rangle \leftarrow \langle m\rangle$. Shuffles in $\Delta_{\perp\top}$ are inherited via the isomorphism (169) and have the following explicit description:

$$[n] \xleftarrow{r} [n+m] \xrightarrow{s} [m] \tag{190}$$

with the constraint

$$r_i + s_i = 1\,. \tag{191}$$

The numbers r_i and s_i are lengths (either 0 or 1 in this case) of the image of the i^{th} subinterval of $[n+m]$, as in (177). The condition (191) states that each subinterval maps to an interval of length 1 either in $[n]$ or in $[m]$.

An equivalent description of a shuffle is given by a relation of "appearing before in the shuffle"

$$\langle m\rangle^{\text{op}} \times \langle n\rangle \xrightarrow{l} \langle 2\rangle\,. \tag{192}$$

The same relation can be interpreted as a shuffle of segments $[n]$ and $[m]$, for example

$$\begin{array}{c|ccccccc}
[2]\backslash[3] & 0 & \longrightarrow & 1 & \longrightarrow & 2 & \longrightarrow & 3 \\
\hline
0 & & \longrightarrow & & & & & \\
\downarrow & & 0 & \Big\downarrow & 1 & & 1 & \\
1 & & & & \longrightarrow & & \longrightarrow & \\
\downarrow & & 0 & & 0 & & 0 & \Big\downarrow \\
2 & & & & & & &
\end{array} \tag{193}$$

A shuffle morphism $(\xi, \rho) : (n, m, s, r) \to (\bar{n}, \bar{m}, \bar{s}, \bar{r})$ consists of 1-cells $\xi : [\bar{n}] \to [n]$ and $\rho : [\bar{m}] \to [m]$ in $\Delta_{\perp\top}$, such that the following 2-cell in Δ_a exists

$$Lr \circ \xi \circ \bar{r} \Rightarrow Rs \circ \rho \circ \bar{s}\ . \tag{194}$$

When $\xi = 1_{[n]}$ and $\rho = 1_{[m]}$, the condition (194) is equivalent to the fact that the induced relations $l, \bar{l} : \langle m\rangle^{\text{op}} \times \langle n\rangle \to \langle 2\rangle$ satisfy $l \leqslant \bar{l}$, or that the $\bar{l}$ path in the table (193) appears to the down-left of the l path.

Shuffles and their morphisms form a category **Shuff** with the identity morphism $(1_{[n]}, 1_{[m]})$ and composition $(\xi \circ \bar{\xi}, \rho \circ \bar{\rho})$ for which the condition (194) is obtained by pasting

$$\begin{tikzcd}
 & {[\bar{\bar{n}}]} \arrow[r, "\bar{\xi}"] & {[\bar{n}]} \arrow[r, "1"] \arrow[dr, "L\bar{r}"'] \arrow[r, phantom, shift right=4, "\eta\Downarrow"] & {[\bar{n}]} \arrow[r, "\xi"] & {[n]} \arrow[dr, bend left, "Lr"] & \\
{[\bar{\bar{n}} + \bar{\bar{m}}]} \arrow[ur, bend left, "\bar{\bar{r}}"] \arrow[dr, bend right, "\bar{\bar{s}}"'] & \Downarrow & & {[\bar{n} + \bar{m}]} \arrow[u, "\bar{r}"'] \arrow[dr, "\bar{s}"] & \Downarrow & {[n + m]} \\
 & {[\bar{\bar{m}}]} \arrow[r, "\bar{\rho}"'] & {[\bar{m}]} \arrow[ur, "R\bar{s}"] \arrow[r, "1"'] \arrow[r, phantom, shift left=4, "\epsilon\Downarrow"] & {[\bar{m}]} \arrow[r, "\rho"'] & {[m]} \arrow[ur, bend right, "Rs"'] &
\end{tikzcd} \tag{195}$$

Shuff inherits a tensor product from $\Delta_{\perp\top}$ which (algebraically) follows from

$$L(r+r')\circ(\xi+\xi')\circ(\bar r+\bar r')\circ z \overset{(180)}{=} L(r+r')\circ z\circ(\xi\oplus\xi')\circ(\bar r\oplus\bar r') \tag{196}$$

$$\overset{(187)}{\Rightarrow} z\circ(Lr\oplus Lr')\circ(\xi\oplus\xi')\circ(\bar r\oplus\bar r') \tag{197}$$

$$\overset{(194)}{\Rightarrow} z\circ(Rs\oplus Rs')\circ(\rho\oplus\rho')\circ(\bar s\oplus\bar s') \tag{198}$$

$$\overset{(188)}{\Rightarrow} R(s+s')\circ z\circ(\rho\oplus\rho')\circ(\bar s\oplus\bar s') \tag{199}$$

$$\overset{(180)}{=} R(s+s')\circ(\rho+\rho')\circ(\bar s+\bar s')\circ z \tag{200}$$

but can also be seen as "direct summing"[10] the relation tables, for example the shuffle (193) can be interpreted as $([2]\xleftarrow{\sigma_1^3}[3]\xrightarrow{\sigma_0^2\circ\sigma_2^3}[1])+([1]\xleftarrow{\sigma_1^2}[2]\xrightarrow{\sigma_0^2}[1])$.

The free 2-category containing a distributive law is obtained as the suspension of the monoidal category of shuffles,

$$\mathrm{FDL} := \Sigma\mathbf{Shuff}. \tag{201}$$

A.3. Mixed shuffle morphisms - free mixed distributive law. The category of mixed shuffles **MShuff** can be obtained by slightly modifying the construction of **Shuff**; the ρ component of the mixed shuffle morphism has the opposite direction $\rho:[m]\to[\bar m]$, and the existence condition (194) becomes

$$Lr\circ\xi\circ\bar r\Rightarrow Rs\circ R\rho\circ\bar s. \tag{202}$$

The 2-category containing a free mixed distributive law (FMDL) is obtained as the suspension of the monoidal category of mixed shuffles,

$$\mathrm{FMDL} := \Sigma\mathbf{MShuff}. \tag{203}$$

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Mathematics department, Macquarie University,
NSW 2109, Australia
Email: branko.nikolic@ipb.ac.rs

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