Theory and Applications of Categories, Vol. 34, No. 25, 2019, pp. 714–735.

MONIC SKELETA, BOUNDARIES, AUFHEBUNG, AND THE MEANING OF 'ONE-DIMENSIONALITY'

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ABSTRACT. Let $\mathcal{E}$ be a topos. If l is a level of $\mathcal{E}$ with monic skeleta then it makes sense to consider the objects in $\mathcal{E}$ that have *l-skeletal boundaries*. In particular, if $p : \mathcal{E} \to \mathcal{S}$ is a pre-cohesive geometric morphism then its centre (that may be called *level* 0) has monic skeleta. Let *level* 1 be the Aufhebung of level 0. We show that if level 1 has monic skeleta then the quotients of 0-separated objects with 0-skeletal boundaries are 1-skeletal. We also prove that in several examples (such as the classifier of non-trivial Boolean algebras, simplicial sets and the classifier of strictly bipointed objects) every 1-skeletal object is of that form.

1. The meaning of 'one-dimensionality'

The purpose of this paper is to explore the possibility of an elementary characterization of '1-dimensional spaces' in certain 'gros' toposes. The specific form of the characterization is motivated by [2] where, among other things, the extension of intuitionistic logic determined by the set of polyhedra of dimension $\leq d$ is identified for each $d \in \mathbb{N}$. In fact, readers will easily see that our definition of a subobject having l-skeletal boundary is closely related with the formula BD_1 op. cit.

To outline the contents of the paper in more detail we first need to say what we understand by 'dimension 1'. In brief, we will use the dimension theory for objects in toposes of spaces outlined in Section II of [7]. In the second paragraph there Lawvere says that:

> 'One-dimensional', like 'connected', is actually a philosophical concept, related to the minimal Hegelian level of figures which must be considered within an arbitrary space in order to determine that space's connectedness.

and in page 9, after defining the partial order of levels or 'dimensions' he states a theorem making the above remark rigorous. We will need that result so we recall the necessary material to state it.

Received by the editors 2019-06-18 and, in final form, 2019-09-03.
Transmitted by RJ Wood. Published on 2019-09-05.
2010 Mathematics Subject Classification: 18B25, 18F20.
Key words and phrases: Topos theory, Axiomatic Cohesion.

Let $\mathcal{E}$ be a topos. A *level* of $\mathcal{E}$ is a string of adjoints

$$\begin{tikzcd} \mathcal{E} \arrow[d, "l^*" description] \\ \mathcal{L} \arrow[u, bend left=60, "l_!"] \arrow[u, bend right=60, "l_*"'] \end{tikzcd} \qquad l_! \dashv l^* \dashv l_*$$

with fully faithful $l_!, l_* : \mathcal{L} \to \mathcal{E}$. Equivalently, a level of $\mathcal{E}$ is an essential subtopos of $\mathcal{E}$. We sometimes use the subtopos notation by writing, for example, that a subtopos $j : \mathcal{E}_j \to \mathcal{E}$ is a level.

If l is a level of $\mathcal{E}$ then, for each X in $\mathcal{E}$, the counit $l_!(l^*X) \to X$ is called the *l-skeleton* of X. The object X is said to be *l-skeletal* if its skeleton is an isomorphism. Readers are invited to think of $l_! : \mathcal{E}_l \to \mathcal{E}$ as the subcategory of objects X such that $\dim X \leq l$. Of course, the objects X such that the unit $X \to l_*(l^*X)$ is an isomorphism (monomorphism) are the l-sheaves (l-separated objects).

The levels of $\mathcal{E}$ may be partially ordered as subtoposes. That is, m is *above* l if and only if l_* factors through m_*; which is equivalent to l^* factoring through m^*, or to $l_!$ factoring through $m_!$.

A subtopos $m : \mathcal{E}_m \to \mathcal{E}$ is *way above* level l if both subcategories $l_!, l_* : \mathcal{E}_l \to \mathcal{E}$ factor through m_*. In other words, the subtopos m is way above level l if and only if, m is above l (as subtoposes) and also $l_! : \mathcal{E}_l \to \mathcal{E}$ factors through $m_* : \mathcal{E}_m \to \mathcal{E}$.

1.1. DEFINITION. The *Aufhebung* of level l is the least level of $\mathcal{E}$ that is way above l.

For further details see, for example, [7], [8] and references therein. See also [5] where the poset of levels of any Grothendieck topos is shown to be complete. Page 9 of [7] states that "The basic idea is simply to identify dimensions with levels and then try to determine what the general dimensions are in particular examples." This is not an easy task in general. Lawvere's work on graphic monoids develops a testing ground where some of the complications disappear. The unpublished [18] presents important early calculations involving the topos of simplicial sets. (It is stated there that the motivation stems from 1985 seminars by Lawvere at SUNY Buffalo.) See also [6] where the work in [18] is summarized and extended. Problem 4 in [10] proposes to express, in combinatorial or number theoretic terms, the way below relation in the poset of levels of the classifier of non-trivial Boolean algebras.

Let us go back to the meaning of 'dimension 1'. Since we are mainly interested in toposes 'of spaces' we fix a pre-cohesive geometric morphism $p : \mathcal{E} \to \mathcal{S}$. Recall [11] that this means that the adjunction $p^* \dashv p_*$ extends to a string $p_! \dashv p^* \dashv p_* \dashv p^!$ such that $p^* : \mathcal{S} \to \mathcal{E}$ is fully faithful, $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite products and (Nullstellensatz) the canonical $p_* \to p_!$ is epic. Equivalently, p is local, hyperconnected, essential and the leftmost adjoint $p_!$ preserves finite products. The intuition, supported by experience, is that $\mathcal{E}$ is a category of spaces and that $\mathcal{S}$ is a category of sets; that the functor $p_* : \mathcal{E} \to \mathcal{S}$ sends a space to the associated set of points and that $p^* : \mathcal{S} \to \mathcal{E}$ sends a set to the associated discrete space; the functor $p_!$ sends a space to its associated set of connected components. See [7, 9] and also [11].

Of course, the centre $p_* \dashv p^! : \mathcal{S} \to \mathcal{E}$ of the local p is a level; let us call it *level 0* so that the full subcategory of 0-skeletal objects is $p^* : \mathcal{S} \to \mathcal{E}$. To aid the intuitive discussion, we will use *discrete* as a synonym of 0-skeletal. Level 1 is defined as the Aufhebung of level 0. We may now return to the initial quotation and continue with one from page 9 in the same paper:

> Because of the special feature of dimension zero of having a components functor to it (usually there is no analogue of that functor in higher dimensions), the definition of dimension one is equivalent to the quite plausible condition: the smallest dimension such that the set of components of an arbitrary space is the same as the set of components of the skeleton at that dimension of the space

In other words:

1.2. Theorem. [Lawvere] *For any level l of $\mathcal{E}$ above level 0, l is way above 0 if and only if, for every X in $\mathcal{E}$, $p_!(l_!(l^*X)) \to p_!X$ is an isomorphism (where $l_!(l^*X) \to X$ is the l-skeleton of X).*

Proof. Once the framework is in place, the proof is not too difficult. See p. 19 of [15]. ■

So, if level 1 exists then a level l is above 1 if and only if $p_!$ inverts the l-skeleton of every object. Alternatively, as Lawvere says "more pictorially: if two points of any space can be connected by anything, then they can be connected by a curve. Here of course by 'curve' we mean any figure in (i.e. map to) the given space whose domain is one-dimensional."

The main contribution of the present paper is an elementary characterization of the 1-skeletal objects in several examples. This is a natural continuation of the characterization of 0-skeletal (i.e. discrete) objects in locally connected pre-cohesive geometric morphisms given in [15]; namely, an object is 0-skeletal if and only if it is decidable.

1.3. Definition. A subobject $u : U \to X$ in $\mathcal{E}$ is said to have *discrete boundary* if the subobject $u \vee (u \Rightarrow \beta_X)$ is the whole of X, where β is the (monic) counit of $p^* \dashv p_*$.

Notice that if the posets of subobjects in $\mathcal{E}$ were coHeyting (as in a presheaf topos, for example) then $\top_X \leq u \vee (u \Rightarrow \beta_X)$ would be equivalent to $\partial u \leq \beta_X$, where $\partial u = (\top_X / u) \wedge u$, $\top_X$ is the top subobject of X and $(-)/u \dashv u \vee (-)$.

1.4. Definition. An object of $\mathcal{E}$ has *discrete boundaries* if all its subobjects do.

For instance, every object in the pre-cohesive topos $\widehat{\Delta_1}$ of reflexive graphs has discrete boundaries. So, in this case, an object is 1-skeletal if and only if it has discrete boundaries; but this is not true in general. For instance, in the pre-cohesive topos $\widehat{\Delta}$ of simplicial sets, every 1-skeletal object has discrete boundaries but the converse is not true.

1.5. Example. [A simplicial set with discrete boundaries that is not 1-skeletal] The three monomorphisms $[1] \to [2]$ in Δ induce subobjects $\Delta(-,[1]) \to \Delta(-,[2])$ in $\widehat{\Delta}$ that we can picture as the edges of a solid triangle. Denote their join by $B \to \Delta(-,[2])$. If we take the pushout

$$\begin{tikzcd} B \arrow[r, "!"] \arrow[d, "b"'] & 1 \arrow[d] \\ \Delta(-,[2]) \arrow[r] & X \end{tikzcd}$$

then X has discrete boundaries but it is not 1-skeletal because it has a non-degenerate figure $\Delta(-,[2]) \to X$. This figure is so 'singular' that it cannot be distinguished from the point $1 \to X$ if tested from lower dimensional figures.

So, as already mentioned, objects with discrete boundaries need not be 1-skeletal but, in a sense, this is a phenomenon due to very singular figures. In fact, if a simplicial set is 0-separated ($\neg\neg$-separated in this case) and has discrete boundaries then it is 1-skeletal. Actually, it follows from our results that a simplicial set X is 1-skeletal if and only if there is a 0-separated object S with discrete boundaries and an epimorphism $S \to X$. We stress that, as it is a quotient of an object with discrete boundaries, X has discrete boundaries too.

Part of the relation between 1-skeletal objects and those with discrete boundaries may be proved in elementary terms. A level l is said to have *monic skeleta* if the counit $l_!(l^*X) \to X$ is monic for every X. We will show (Corollary 4.5) that if level 1 exists and has monic skeleta then every quotient of a 0-separated object with discrete boundaries is 1-skeletal.

Although it is certainly a restriction, it is not completely unnatural to pay special attention to levels with monic skeleta. In some of the more important examples (including simplicial sets) used in algebraic topology, all levels have monic skeleta. It is also intuitive to picture the 'n-dimensional part' of an object as a subobject. Moreover, notice that Definition 1.3 rests on the fact that β is monic so it is natural to extend the idea to levels with monic skeleta.

In Section 2 we recall some of our main examples and show that their levels have monic skeleta. In Section 3 we introduce objects with l-skeletal boundaries for an arbitrary level l with monic skeleta and prove some of their basic properties. In Section 4 we concentrate on objects with 0-skeletal boundaries (i.e. with discrete boundaries) and relate them with level 1. The rest of the sections restrict to pre-cohesive presheaf toposes. In Section 5 we show that the full subcategory of objects with discrete boundaries in such a topos is coreflective and we identify the representable objects there. The corresponding objects in the site are called *edge types* and the level that they induce is studied in Section 6. Under certain conditions we show, in Section 7, that that level coincides with level 1, and that 1-skeletal objects coincide with the quotients of 0-separated objects with discrete boundaries. The last two sections deal with an extra example: a pre-cohesive topos embedding affine spaces. The main issue about it is to prove that level 1 has monic skeleta. In Section 8 we prove some simple auxiliary facts about absolute pushouts and in

Section 9 we prove that the 1-skeletal objects in the extra example are again the quotients of 0-separated objects with discrete boundaries.

2. Levels with monic skeleta in presheaf toposes

In this section we illustrate levels with monic skeleta using some familiar presheaf toposes. Absolute pushouts will play a relevant role and we assume that the reader is familiar with the notion of *absolute colimit* as introduced in [16].

Let $\psi : \mathcal{C} \to \mathcal{D}$ be a full and faithful functor between small categories so that the induced geometric morphism $\psi : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ is a level. It is well-known that $\psi_!$ is the left Kan extension along f so, for every P in $\widehat{\mathcal{C}}$, $\psi_! P = \int^C PC \times \mathcal{D}(-, \psi C)$ and hence, for D in $\mathcal{D}$, $(\psi_! P)D$ may be presented using the following coequalizer diagram

$$\coprod_{C,C'} PC \times \mathcal{C}(C', C) \times \mathcal{D}(D, \psi C') \rightrightarrows \coprod_C PC \times \mathcal{D}(D, \psi C) \longrightarrow (\psi_! P)D$$

where the parallel maps are the evident natural ones. See, for example, [13]. If $f : D \to \psi C$ in $\mathcal{D}$ and $x \in PC$ then the resulting element in $(\psi_! P)D$ is denoted by $x \otimes f$.

The counit $\beta = \beta_X : \psi_!(\psi^* X) \to X$ is determined as suggested by the following diagram

$$\begin{tikzcd} \coprod_{C,C'} X(\psi C) \times \mathcal{C}(C', C) \times \mathcal{D}(D, \psi C') \arrow[r, shift left] \arrow[r, shift right] & \coprod_C X(\psi C) \times \mathcal{D}(D, \psi C) \arrow[r] \arrow[dr] & (\psi_!(\psi^* X))D \arrow[d, "\beta"] \\ & & XD \end{tikzcd}$$

for each X in $\widehat{\mathcal{D}}$ and D in $\mathcal{D}$, where the diagonal function is induced by the action of X. So, for $x \otimes f \in (\psi_!(\psi^* X))D$ as above, $\beta(x \otimes f) = x \cdot f \in XD$.

2.1. Proposition. *Let $\mathcal{D}$ be equipped with a factorization system* (E, M) *such that*

1. *Every $D \to \psi C$ in* M *is in the image of $\psi : \mathcal{C} \to \mathcal{D}$.*
2. *Every $\psi C \to D$ in* E *is in the image of $\psi : \mathcal{C} \to \mathcal{D}$.*
3. *Pushouts of spans in* E *exist and are absolute.*

Then level ψ has monic skeleta.

Proof. First fix an object P in $\widehat{\mathcal{C}}$ and D in $\mathcal{D}$. Let $x \otimes f \in (\psi_! P)D$ with $f : D \to \psi C$. By hypothesis, f factors as $f = me$ with e in E and m in M. Also by hypothesis, $m = \psi n$ for some $n : C' \to C$ in $\mathcal{C}$. Then $x \otimes f = x \otimes ((\psi n)e) = (x \cdot n) \otimes e$ so every element of $(\psi_! P)D$ is of the form $x \otimes e$ with $e : D \to \psi C$ in E.

We need to show that the counit $\beta : \psi_!(\psi^* X) \to X$ is monic for every object X in $\widehat{\mathcal{C}}$. To prove that $\beta : (\psi_!(\psi^* X))D \to XD$ is injective let $x \otimes e, x' \otimes e' \in (\psi_!(\psi^* X))D$ be such that $\beta(x \otimes e) = \beta(x' \otimes e')$. That is, $x \cdot e = x' \cdot e'$. Without loss of generality we may assume that the maps $e : D \to \psi C$ and $e' : D \to \psi C'$ are in E. The pushout of e and e' exists by hypothesis and the maps in the colimiting cone are also in E by the general theory of factorization systems. Moreover, also by hypothesis, the maps in the colimiting cone are in the image of ψ. So we have an absolute pushout as on the left below

$$\begin{tikzcd} D \arrow[r, "e'"] \arrow[d, "e"'] & \psi C' \arrow[d, "\psi d'"] \\ \psi C \arrow[r, "\psi d"'] & \psi B \end{tikzcd} \qquad \begin{tikzcd} D \arrow[r, "e'"] \arrow[d, "e"'] & \psi C' \arrow[d, "\psi d'"] \arrow[ddr, bend left, "x'"] & \\ \psi C \arrow[r, "\psi d"'] \arrow[drr, bend right, "x"'] & \psi B \arrow[dr, dotted, "y"] & \\ & & X \end{tikzcd}$$

for some $d : C \to B$ and $d' : C' \to B$ in $\mathcal{C}$. The Yoneda embedding $\mathcal{D} \to \widehat{\mathcal{D}}$ must preserve the pushout so we have a diagram in $\widehat{\mathcal{D}}$ as on the right above, inducing a $y \in X(\psi B) = (\psi^* X)B$ satisfying the evident equations. We can then calculate:

$$x \otimes e = (y \cdot d) \otimes e = y \otimes ((\psi d)e) = y \otimes ((\psi d')e') = (y \cdot d') \otimes e' = x' \otimes e'$$

completing the proof. ∎

2.2. Example. [Levels in the classifier of non-trivial Boolean algebras] Let $\mathbb{F}$ be the category of non-empty finite sets. It has epic/monic factorizations and, by Proposition 2.3 in [17], pushouts of epimorphisms are absolute. If we let $\mathbb{F}_n \to \mathbb{F}$ be the subcategory of finite sets of cardinality at most $n \in \mathbb{N}$ then it is clear that Proposition 2.1 applies so the induced level $\widehat{\mathbb{F}_n} \to \widehat{\mathbb{F}}$ has monic skeleta.

2.3. Example. [Levels in the topos of simplicial sets] Let Δ be the usual site for the topos of simplicial sets. Section II.3.2 in [3] states that pushouts of epimorphisms in Δ exist and shows that the Yoneda embedding $\Delta \to \widehat{\Delta}$ preserves them. By Theorem 2.1 in [16], pushouts of epimorphisms in Δ are absolute. If we let $\Delta_n \to \Delta$ be the usual truncation then, clearly, Proposition 2.1 applies so the induced level $\widehat{\Delta_n} \to \widehat{\Delta}$ has monic skeleta.

Gabriel and Zisman say that the fact that $\Delta \to \widehat{\Delta}$ preserves pushouts of epimorphisms is “another interpretation of Eilenberg-Zilber lemma”. Similarly, Proposition 3.1 in [17] looks like a variant of the EZ-Lemma for $\widehat{\mathbb{F}}$.

2.4. Example. [The classifier of strictly bipointed objects] Consider the geometric theory presented by two constants $0, 1$ and the sequent $0 = 1 \vdash \perp$. Its classifying topos may be described as $\mathbf{Set}^{\mathcal{A}} = \widehat{\mathcal{A}^{\mathrm{op}}}$ where $\mathcal{A}$ is the category of free ‘bipointed sets’ generated by a finite set. In other words, $\mathcal{A}$ is the category of strictly bipointed finite sets and functions between them that preserve the distinguished points. It is easy to check that the forgetful functor $U : \mathcal{A} \to \mathbb{F}$ creates pullbacks. A classical result says that non-empty intersections are absolute. (See p. 86 in [16] and Remark 2.5 in [17].) So intersections are absolute in $\mathcal{A}$. Therefore, pushouts of epimorphisms are absolute in $\mathcal{A}^{\mathrm{op}}$. Again, Proposition 2.1 applies to the obvious truncations.

It is tempting to attempt to characterize the small categories $\mathcal{C}$ such that every level of $\widehat{\mathcal{C}}$ has monic skeleta. It might also be interesting to compare these with the *elegant Reedy categories* defined in [1]. Anyway, I believe it is folklore that in the above examples all subtoposes are induced by full subcategories of the respective sites. We include a proof below.

2.5. Proposition. *Let $\mathcal{D}$ be a small category. If every map in $\mathcal{D}$ factors as a split epic followed by a split monic then every level of $\widehat{\mathcal{D}}$ is induced by a full subcategory of $\mathcal{D}$. If, moreover, every object of $\mathcal{D}$ has a finite set of subobjects then every subtopos is a level of that form.*

Proof. Let $\mathcal{E} \to \widehat{\mathcal{D}}$ be a level of $\widehat{\mathcal{D}}$. By Theorem 4.4 in [5], it is determined by an idempotent ideal $\mathcal{I}$ of $\mathcal{D}$. Let $\mathcal{C} \to \mathcal{D}$ be the full subcategory of those objects C such that the identity on C is in $\mathcal{I}$. We want to show that the subtopos $\mathcal{E} \to \widehat{\mathcal{D}}$ coincides with the subtopos $\widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ induced by the inclusion of $\mathcal{C}$ in $\mathcal{D}$. It is enough to show that $\mathcal{I} = \mathcal{J}$ where $\mathcal{J}$ is the idempotent ideal of $\mathcal{D}$ consisting of the maps that factor through some object in the subcategory $\mathcal{C}$. Certainly, $\mathcal{J} \subseteq \mathcal{I}$ because if $f = gh$ with $g : C \to D$, $h : D' \to C$ and C in $\mathcal{C}$ then, as $id_C \in \mathcal{I}$, $f = g(id_C)h$ must also be in the two-sided ideal $\mathcal{I}$.

To prove $\mathcal{I} \subseteq \mathcal{J}$, let $f : D \to F$ be in $\mathcal{I}$. By hypothesis we can factor f as $f = me$ with $m : E \to F$ split monic and $e : D \to E$ split-epic. As $\mathcal{I}$ is a two-sided ideal, the result of pre-composing f with a section of e and post-composing with a retraction for m is also in $\mathcal{I}$. That is, the identity on E is in $\mathcal{I}$. So f is in $\mathcal{J}$. This completes the proof that every level is induced by a full subcategory of the site.

Assume now that every object of $\mathcal{D}$ has a finite set of subobjects. Since every map factors as split-epi followed by a mono, every sieve on an object is generated by a family of monomorphisms. Our present finiteness assumption implies that the ordered set of sieves on any object of $\mathcal{D}$ satisfies the descending chain condition. In this case, it follows from Remark 4.9 in [5] that every subtopos of $\widehat{\mathcal{D}}$ is essential. ∎

It follows that all the subtoposes of $\widehat{\mathbb{F}}$, $\widehat{\Delta}$ and $\widehat{\mathcal{A}}$ are levels with monic skeleta. In fact, they are all of the form discussed in Examples 2.2, 2.3 and 2.4 respectively.

There seems to be room for improving Proposition 2.5. Notice that idempotence of the relevant ideals is not used in the proof. In fact, it would seem that one could consider more general factorization systems satisfying some ‘well-foundedness’ condition. Compare with the proof of the EZ-Lemma in Section II.3.1 of [3], with Proposition 3.1 in [17], and also with Proposition 4.10 in [5] and Lemma C2.2.21 in [4]. Compare, moreover, with Proposition 4 in [8] which proves that, in a graphic category, any two split epimorphisms with common domain have an absolute pushout.

3. Skeletal boundaries

Let $\mathcal{E}$ be a topos and let $l : \mathcal{E}_l \to \mathcal{E}$ be a level with monic skeleta $\ell_X : l_!(l^*X) \to X$. So, for any subobject $u : U \to X$ we may build the implication $u \Rightarrow \ell_X : (U \Rightarrow \ell_X) \to X$.

3.1. Definition. A subobject $u : U \to X$ has *l-skeletal boundary* if $\top_X \leq u \vee (u \Rightarrow \ell_X)$ as subobjects of X.

As already suggested after Definition 1.3, notice that if the Heyting algebra of subobjects of X were coHeyting then $\top \leq u \vee (u \Rightarrow \ell)$ would be equivalent to $\partial u = (\top/u) \wedge u \leq \ell$. (If there is no risk of confusion we sometimes drop sub-indexes.)

3.2. Definition. An object X in $\mathcal{E}$ has *l-skeletal boundaries* if every subobject of X has l-skeletal boundary.

If we let $(l+1)$ be the Aufhebung of level l then, roughly, and with caution, we may sometimes think of $(l+1)$-skeletal objects as having l-skeletal boundaries.

3.3. Example. [The objects with $-\infty$-skeletal boundaries] There is a smallest level of $\mathcal{E}$ that we denote by $-\infty : \mathcal{E}_{-\infty} \to \mathcal{E}$. As a category, $\mathcal{E}_{-\infty}$ is terminal and the leftmost adjoint $\mathcal{E}_{-\infty} \to \mathcal{E}$ is the full subcategory determined by the initial object of $\mathcal{E}$ so level $-\infty$ has monic skeleta. Indeed, the $-\infty$-skeleton of X is the unique $0 \to X$. Then, a subobject $u : U \to X$ has $(-\infty)$-skeletal boundary if and only if u is complemented. So an object X has $(-\infty)$-boundaries if and only if the Heyting algebra of subobjects of X is Boolean.

Geometric intuition suggests that monos and epis do not raise dimension.

3.4. Lemma. *For any $f : X \to Y$ in $\mathcal{E}$ the following hold:*

1. *If f is epic and X has l-skeletal boundaries then so does Y.*
2. *Assuming that ℓ is mono-cartesian, if f is monic and Y has l-skeletal boundaries then so does X.*

Proof. Assume that f is epic and let $v : V \to Y$ be monic. As $\ell_X \leq f^*\ell_Y$, we also have $(f^*v \Rightarrow \ell_X) \leq (f^*v \Rightarrow f^*\ell_Y)$. So we can calculate:

$$\top \leq f^*v \vee (f^*v \Rightarrow \ell_X) \leq f^*v \vee (f^*v \Rightarrow f^*\ell_Y) = f^*(v \vee (v \Rightarrow \ell_Y))$$

and, as f is epic, $\top \leq v \vee (v \Rightarrow \ell_Y)$ as subobjects of Y.

Assume now that f and $u : U \to X$ are monic. Then so is the composite $fu : U \to Y$ and, by hypothesis, $\top_Y \leq (fu) \vee ((fu) \Rightarrow \ell_Y)$. Also, $f^*(fu) = u$ and, as ℓ is mono-cartesian, $f^*\ell_Y = \ell_X$ as subobjects of X. Then

$$\top_X = f^*\top_Y \leq f^*((fu) \vee ((fu) \Rightarrow \ell_Y)) = f^*(fu) \vee (f^*(fu) \Rightarrow f^*\ell_Y) = u \vee (u \Rightarrow \ell_X)$$

so X has ℓ-skeletal boundaries. ∎

It follows from [5], but it is also easy to see that ℓ_X is the least l-dense subobject of X. The following related remark is very simple but we need it explicitly.

3.5. Lemma. *For any monic $u : U \to X$ in $\mathcal{E}$, u is l-dense if and only if $\ell_X = u \wedge (u \Rightarrow \ell_X)$. In other words, there exists a (necessarily unique map) $l_!(l^*X) \to U$ making the triangle on the left below commute*

$$\begin{tikzcd} l_!(l^*X) \arrow[r] \arrow[dr, "\ell"'] & U \arrow[d, "u"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} l_!(l^*X) \arrow[r] \arrow[d] \arrow[dr, "\ell"] & U \arrow[d, "u"] \\ (U \Rightarrow \ell) \arrow[r, "u\Rightarrow\ell"'] & X \end{tikzcd}$$

if and only if there is a pullback square as on the right above. In this case, the subobjects $l_!(l^*X) \to (U \Rightarrow \ell_X)$ and $l_!(l^*X) \to U$ coincide with the l-skeleta of $(U \Rightarrow \ell)$ and U respectively.

Proof. We have that $\ell_X \le u \Rightarrow \ell_X$ and $u \wedge (u \Rightarrow \ell_X) \le \ell_X$ from the basic theory of Heyting algebras and, from these, the equivalence in the first part of the statement follows. Moreover, if we apply l^* to the square in the statement then we get the pullback on the left below

$$\begin{tikzcd} l^*(l_!(l^*X)) \arrow[r] \arrow[d] \arrow[dr, "l^*\ell"] & l^*U \arrow[d, "l^*u"] \\ l^*(U \Rightarrow \ell) \arrow[r, "l^*(u\Rightarrow\ell)"'] & l^*X \end{tikzcd} \qquad \begin{tikzcd} (l_!l^*l_!l^*)X \arrow[r, "\cong"] \arrow[d, "\ell"', "\cong"] & l_!(l^*U) \arrow[d, "\ell"] \\ l_!(l^*X) \arrow[r] & U \end{tikzcd}$$

and, as $l^*\ell$ is an iso, it follows that l^*u is an iso. Then the naturality square on the right above shows that $l_!(l^*X) \to U$ coincides with the l-skeleton of U as subobjects of U. Similarly for $(U \Rightarrow \beta)$. ■

The sum of spaces of a given dimension should not raise dimension.

3.6. Lemma. *Coproducts of objects with l-skeletal boundaries have l-skeletal boundaries.*

Proof. Using extensivity it is not difficult to prove that for all subobjects u, u' of X and v, v' of Y,

$$(u + v) \Rightarrow (u' + v') \quad = \quad (u \Rightarrow u') + (v \Rightarrow v')$$

as subobjects of $X + Y$. Similarly, for two families of subobjects $(u_i : U_i \to X_i \mid i \in I)$ and $(v_i : V_i \to X_i \mid i \in I)$ we have

$$\left(\sum_{i\in I} u_i\right) \Rightarrow \left(\sum_{i\in I} v_i\right) \quad = \quad \sum_{i\in I}(u_i \Rightarrow v_i)$$

as subobjects of $\sum_{i\in I} X_i$. In particular, as $\ell_{\sum_i X_i} = \sum_i \ell_{X_i}$, we have that

$$\left(\sum_{i\in I} u_i\right) \Rightarrow \ell_{\sum_i X_i} = \sum_{i\in I}(u_i \Rightarrow \ell_{X_i})$$

as subobjects of $\sum_{i\in I} X_i$. By (infinitary) extensivity, any subobject of $\sum_{i\in I} X_i$ is of the form $\sum_{i\in I} u_i$ for some family $(u_i : U_i \to X_i \mid i \in I)$ of subobjects. So, if each X_i has l-skeletal boundaries then so does $\sum_{i\in I} X_i$. ■

4. Discrete boundaries and level 1

Fix a pre-cohesive $p : \mathcal{E} \to \mathcal{S}$ with (monic) counit β. The centre $p_* \dashv p^! : \mathcal{S} \to \mathcal{E}$ of p will be called *level 0* so that $\beta_X : p^*(p_*X) \to X$ is the 0-skeleton of X. Then an object has 0-skeletal boundaries if and only if it has discrete boundaries in the sense of Definition 1.4. That is, if and only if $\top_X \leq u \vee (u \Rightarrow \beta_X)$ for every subobject u of X.

Let $m : \mathcal{E}_m \to \mathcal{E}$ be a level of $\mathcal{E}$ way above 0.

4.1. Lemma. *For every m-dense subobject $u : U \to X$ in $\mathcal{E}$, $p_!u : p_!U \to p_!X$ is an iso.*

Proof. Consider the naturality square determined by the counit of $m_! \dashv m^*$ and the subobject u, and then apply $p_!$ to obtain the commutative square

$$\begin{tikzcd}
p_!(m_!(m^*U)) \arrow[r] \arrow[d, "p_!(m_!(m^*u))"'] & p_!U \arrow[d, "p_!u"] \\
p_!(m_!(m^*X)) \arrow[r] & p_!X
\end{tikzcd}$$

in $\mathcal{S}$. The left vertical map is an iso because u is m-dense by hypothesis and the horizontal maps are isos by Theorem 1.2. ■

Geometric intuition suggests that m-dense subobjects with discrete domain are special.

4.2. Lemma. *If Y is 0-separated and $v : p^*A \to Y$ is an m-dense subobject then v is an isomorphism.*

Proof. As m is way above 0, $p_!v : p_!(p^*A) \to p_!Y$ is an iso by Lemma 4.1 so we can take the composite

$$Y \xrightarrow{\sigma} p^*(p_!Y) \xrightarrow{p^*(p_!v)^{-1}} p^*(p_!(p^*A)) \xrightarrow{p^*\tau} p^*A$$

where σ and τ are the unit and counit of $p_! \dashv p^*$. Let us denote it by $r : Y \to p^*A$. It is easy to check that v is a section of r so $vrv = v : p^*A \to Y$. As m is above 0 and v is m-dense, v is also 0-dense. Since Y is 0-separated, the equality $vrv = v$ implies that $vr = id : Y \to Y$. That is, r is the inverse of v. ■

The next result is probably the key technical result since it is where objects with discrete boundaries interact with Theorem 1.2 via Lemma 4.2.

4.3. Lemma. *Let $u : U \to X$ be m-dense. If X is 0-separated and has discrete boundaries then u is an isomorphism.*

Proof. As u is m-dense by hypothesis and m is above 0, u is also 0-dense. In other words, $\beta_X \leq u$ as subobjects of X. Then the following square

$$\begin{tikzcd}
p^*(p_*X) \arrow[r] \arrow[d] & U \arrow[d, "u"] \\
U \Rightarrow \beta \arrow[r, "u\Rightarrow\beta"'] & X
\end{tikzcd}$$

is a pullback by Lemma 3.5 so the left vertical map $p^*(p_*X) \to (U \Rightarrow \ell)$ is an m-dense subobject of $(U \Rightarrow \beta)$, which is 0-separated because it is a subobject of X. Therefore, the left vertical map is an iso by Lemma 4.2 and, as X has l-skeletal boundaries, the square is a pushout, so u is an iso. ∎

We may rephrase Lemma 4.3 as follows: If X is 0-separated and has discrete boundaries then the m-skeleton $m_!(m^*X) \to X$ is epic. So, if we further assume that m has monic skeleta then 0-separated objects with discrete boundaries are m-skeletal.

4.4. Proposition. *If level m is way above 0 then every quotient of a 0-separated object with discrete boundaries has epic m-skeleton.*

Proof. Let X be 0-separated and have discrete boundaries and let $f : X \to Y$ be epic. If $v : V \to Y$ is an m-dense subobject then the pullback f^*v is an m-dense subobject of X, so it is an iso by Lemma 4.3. That is, f factors through v and so the monic v must be epic. In other words, the only m-dense subobject of Y is the whole of Y so the m-skeleton of Y must be epic. ∎

So, if m has monic skeleta then every quotient of a 0-separated object with discrete boundaries is m-skeletal.

4.5. Corollary. *If level 1 exists and has monic skeleta then every quotient of a 0-separated object with discrete boundaries is 1-skeletal.*

5. Presheaves with discrete boundaries

Let $\mathcal{C}$ be a small category such that every object has a point so that $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is pre-cohesive. In this case, level 0 coincides with the subtopos of sheaves for the $\neg\neg$ topology [11]. For any X in $\widehat{\mathcal{C}}$, the figures of X in the subobject $\beta : p^*(p_*X) \to X$ will be called *points* for brevity. For any subobject $u : U \to X$ in $\widehat{\mathcal{C}}$, the subobject $u \Rightarrow \beta : (U \Rightarrow \beta) \to X$ may be described as follows

$$(U \Rightarrow \beta)C = \{x \in XC \mid \text{for all } f : B \to C, x \cdot f \in UB \text{ implies } x \cdot f \text{ is a point}\} \subseteq XC$$

for each C in $\mathcal{C}$, using the standard description of implication in presheaf toposes; see for example [13].

5.1. Lemma. *For any X in $\widehat{\mathcal{C}}$, C in $\mathcal{C}$ and $x \in XC$ the following are equivalent:*

1. *For every $f : B \to C$ in $\mathcal{C}$, either $x \cdot f$ is a point or there is a $g : C \to B$ such that $x \cdot f \cdot g = x$.*
2. *For every subobject $u : U \to X$, the figure $x : \mathcal{C}(-, C) \to X$ factors through the join $u \vee (u \Rightarrow \beta) : U \vee (U \Rightarrow \beta) \to X$.*

Proof. To prove that the first item implies the second let $u : U \to X$ be a subobject. We show that if $x \notin UC \subseteq XC$ then $x \in (U \Rightarrow \beta)C \subseteq XC$. Indeed, let $f : B \to C$ be such that $x \cdot f \in UB$. By hypothesis, $x \cdot f$ is a point or there is a $g : C \to B$ such that $x \cdot f \cdot g = x$. Assume that there is such a g. Then, as u is a subpresheaf and $x \cdot f \in UB$, $x = x \cdot f \cdot g \in UC$ which contradicts the current hypothesis. That is, if $x \notin UC \subseteq XC$ then, for all $f : B \to C$, $x \cdot f \in UB$ implies $x \cdot f$ is a point. Therefore, $x \in UC$ or $x \in (U \Rightarrow \beta)C$.

To prove that the second item implies the first we introduce some notation. For an arbitrary B in $\mathcal{C}$ and $y \in XB$, the image of the corresponding $\mathcal{C}(-, B) \to X$ will be denoted by $u_y : U_y \to X$ so that $z \in XA$ is in $U_y A \subseteq XA$ if and only if there is a $g : A \to B$ such that $y \cdot g = z$.

Assume now that the second item holds and let $f : B \to C$ in $\mathcal{C}$. Then it must be the case that $x \in U_{x \cdot f} C$ or $x \in (U_{x \cdot f} \Rightarrow \beta)C$. The first disjunct holds if and only if there is a $g : C \to B$ such that $x \cdot f \cdot g = x$. So it remains to prove that the second disjunct implies that $x \cdot f$ is a point. Now $x \in (U_{x \cdot f} \Rightarrow \beta)C$ if and only if for every $h : A \to C$, $x \cdot h \in U_{x \cdot f} A$ implies that $x \cdot h$ is a point. That is, the existence of a $g : A \to B$ such that $x \cdot h = x \cdot f \cdot g$ implies that $x \cdot h$ is a point. Taking $h = f$ and $g = id$ we obtain that $x \cdot f$ is a point. ■

If the equivalent conditions of Lemma 5.1 hold then x will be called an *edge* of X. For each C in $\mathcal{C}$ let $\mathrm{e}_{X,C} : (\mathrm{E}X)C \to XC$ be the subset of edges in X.

5.2. Lemma. *For any $x \in (\mathrm{E}X)C \subseteq XC$ and $f : B \to C$, $x \cdot f \in (\mathrm{E}X)B \subseteq XB$.*

Proof. Let $g : A \to B$ be a map in $\mathcal{C}$. We need to prove that $(x \cdot f) \cdot g$ is a point or there is an $h : B \to A$ such that $(x \cdot f) \cdot g \cdot h = x \cdot f$. As x is an edge by hypothesis, $x \cdot (f \cdot g)$ is a point or there is an $k : C \to A$ such that $x \cdot (f \cdot g) \cdot k = x$. In the second case, taking $h = kf$, we get that $(x \cdot f) \cdot g \cdot h = x \cdot f$. ■

In other words, we have a subpresheaf $\mathrm{e}_X : \mathrm{E}X \to X$ of edges and it is such that $\beta_X \leq \mathrm{e}_X$.

5.3. Proposition. *For every X in $\widehat{\mathcal{C}}$, X has discrete boundaries if and only if $\mathrm{e}_X : \mathrm{E}X \to X$ is an iso. Moreover, $\mathrm{e}_X : \mathrm{E}X \to X$ is universal from the subcategory of objects with discrete boundaries to X.*

Proof. It follows from the second item of Lemma 5.1 that, for every subobject $u : U \to X$, $\mathrm{e}_X \leq u \vee (u \Rightarrow \beta_X)$. So, if e_X is the top subobject then X has discrete boundaries. On the other hand, if X has discrete boundaries then every $x \in XC$ is in the subset $(\mathrm{E}X)C \subseteq XC$. So, indeed, X has discrete boundaries if and only if e_X is an iso.

As the action of X restricts to $\mathrm{E}X$, it follows from the first item of Lemma 5.1 that every figure of $\mathrm{E}X$ is an edge so $e_{\mathrm{E}} : \mathrm{E}(\mathrm{E}X) \to \mathrm{E}X$ is an iso and hence, $\mathrm{E}X$ has discrete boundaries. It remains to show that $\mathrm{e}_X : \mathrm{E}X \to X$ is universal. For this, notice that any map $\psi : W \to X$ in $\widehat{\mathcal{C}}$ sends edges of W to edges of X so, if W has discrete boundaries then ψ must factor through $\mathrm{e}_X : \mathrm{E}X \to X$. ∎

In other words, the full subcategory of objects with discrete boundaries is coreflective. We next study representable presheaves with discrete boundaries. For brevity, we say that a map $f : B \to C$ in $\mathcal{C}$ is *constant* if it factors through a map $1 \to C$.

5.4. Lemma. *A map $e : C \to D$ in $\mathcal{C}$ is an edge of $\mathcal{C}(-, D)$ if and only if for every $f : B \to C$, ef is constant or there is a $g : C \to B$ such that $efg = e$. So, if e is monic then, e is an edge of $\mathcal{C}(-, D)$ if and only if its domain C is such that every map with codomain C is constant or has a section.*

Proof. The first part of the statement is a direct rephrasing of the definition of edge in the present particular case. If e is monic then, $efg = e$ is equivalent to $fg = id$. So it remains to show that ef constant is equivalent to f constant. One direction is trivial, so assume that ef is constant. That is, there exists a commutative diagram as below

$$\begin{tikzcd} B \arrow[r] \arrow[d, "f"'] & 1 \arrow[d] \arrow[dl, dotted] \\ C \arrow[r, "e"'] & D \end{tikzcd}$$

for some point $1 \to D$. As every object of $\mathcal{C}$ has a point, the top map is split epic. Orthogonality of split epimorphisms against monomorphisms implies that f is constant. ∎

We can now derive a characterization of representables with discrete boundaries.

5.5. Lemma. *For any object C in $\mathcal{C}$, the following are equivalent:*

1. *The representable $\mathcal{C}(-, C)$ in $\widehat{\mathcal{C}}$ has discrete boundaries.*
2. *For every $f : B \to C$ in $\mathcal{C}$, f is constant or f has a section.*

In this case, a sieve on C contains the identity or consists of points.

Proof. By Lemma 5.4, the second item is equivalent to $id_C : C \to C$ being a edge as a figure of $\mathcal{C}(-, C)$. So, if $\mathcal{C}(-, C)$ has discrete boundaries then id_C must be a edge by Proposition 5.3 and the second item holds. Conversely, if the second item holds, then id_C is in the subobject of $\mathcal{C}(-, C)$ determined by the edges. So the subobject must be the whole of $\mathcal{C}(-, C)$. ∎

An object C in $\mathcal{C}$ satisfying the equivalent conditions of will be called an *edge type*.

6. The level induced by edge types

Let $\mathcal{C}$ be a small category with terminal object and such that every object has a point so that $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is pre-cohesive. Let $\mathbf{e} : \mathcal{C}_{\mathrm{e}} \to \mathcal{C}$ be the full subcategory of edge types and $\mathbf{e} : \widehat{\mathcal{C}_{\mathrm{e}}} \to \widehat{\mathcal{C}}$ be the induced level.

6.1. Proposition. *Level* $\mathbf{e}$ *is above level* 0 *and every* $\mathbf{e}$*-skeletal object has discrete boundaries.*

Proof. Level 0 is induced by the full subcategory $\mathcal{C}_0 \to \mathcal{C}$ consisting of the terminal object. This subcategory factors through $\mathcal{C}_{\mathrm{e}} \to \mathcal{C}$ so level $\mathbf{e}$ is above level 0.

For any G in $\widehat{\mathcal{C}_{\mathrm{e}}}$ we show that every figure of $\mathbf{e}_! G$ is an edge. So let C in $\mathcal{C}$ and $x \otimes k \in (\mathbf{e}_! G)C$ where $k : C \to D$ with D an edge type and $x \in GD$. To prove that $x \otimes k$ is an edge of $\mathbf{e}_! G$ let $f : B \to C$. By Lemma 5.5, $kf : B \to D$ is constant or it has a section.

If kf is constant then $kf = d!$ for a point $d : 1 \to D$ which, of course, is a map in $\mathcal{C}_{\mathrm{e}}$ so

$$(x \otimes k) \cdot f = x \otimes (d!) = (x \cdot d) \otimes ! = ((x \cdot d) \otimes id_1) \cdot ! \in p^*(p_*(\mathbf{e}_! G)) = p^*((\mathbf{e}_! G)1)$$

showing that, in this case, $(x \otimes k) \cdot f$ is a point. On the other hand, if kf has a section $s : D \to B$ then, if we let $g = sk : C \to B$ then $(x \otimes k) \cdot f \cdot g = (x \otimes k)$. Therefore, $x \otimes k$ is an edge by the first item of Lemma(/definition) 5.1. That is, every figure of $\mathbf{e}_! G$ is an edge so it has discrete boundaries by Proposition 5.3. ∎

The canonical transformation $\theta : p_* \to p_!$ is epic (Nullstellensatz). So, for any X in $\widehat{\mathcal{C}}$, the set $p_! X$ is a quotient of $p_* X = X1$ and we write the elements of $p_! X$ as $x \otimes 1$ with $x \in X1$. The associated equivalence relation on $p_* X$ may be described as follows.

6.2. Lemma. *For every* X *in* $\widehat{\mathcal{C}}$*, and* $x, x' \in X1$*,* $x \otimes 1 = x' \otimes 1$ *if and only if there is a zig-zag in* $\mathcal{C}$ *as below*

$$\begin{tikzcd}
& C_1 & & C_2 & & & C_n & \\
1 \arrow[ur, "f_{1,l}"] & & 1 \arrow[ul, "f_{1,r}"'] \arrow[ur, "f_{2,l}"] & & 1 \arrow[ul, "f_{2,r}"'] \arrow[r, dotted, no head] & 1 \arrow[ur, "f_{n,l}"] & & 1 \arrow[ul, "f_{n,r}"']
\end{tikzcd}$$

and an indexed family $(x_i \in XC_i \mid 1 \leq i \leq n)$ *such that* $x = x_1 \cdot f_{1,l}$*,* $x' = x_n \cdot f_{n,r}$ *and, for every* $1 \leq i \leq n-1$*,* $x_i \cdot f_{i,r} = x_{i+1} \cdot f_{i+1,l}$*.*

Proof. This is well-known. See, for example, Lemma 5.7 in [14]. ∎

6.3. LEMMA. *If the terminal object is the only edge type with a unique point then* $\mathbf{e} : \widehat{\mathcal{C}_{\mathrm{e}}} \to \widehat{\mathcal{C}}$ *is below any level that is way above* 0.

PROOF. Let $j : \mathbf{Sh}(\mathcal{C}, J) \to \widehat{\mathcal{C}}$ be a level. Then, for any X in $\widehat{\mathcal{C}}$, the image of the counit $j_!(j^*X) \to X$ is the least j-dense subobject of X. If j is way above 0 then $p_!$ inverts every j-dense subobject by Lemma 4.1. In particular, it must invert the J-covering sieves when considered as subobjects of representables. So consider a J-covering sieve S on an edge type E. As j is above 0, S contains all the points of E. As $p_!(\mathcal{C}(-, E)) = 1$, $p_!S = 1$ so, by Lemma 6.2, either E has a unique point or there is a commutative diagram

$$\begin{tikzcd} & E & \\ 1 \arrow[ur, "a"] \arrow[r, "f_l"'] & C \arrow[u, "f"] & 1 \arrow[ul, "b"'] \arrow[l, "f_r"] \end{tikzcd}$$

with $a, b : 1 \to E$ distinct points and f (in S) connecting them. In this case, f is not constant and, as E is an edge type, f has a section and so S is the maximal sieve. On the other hand, if E has a unique point then it must be terminal by hypothesis.

Altogether, if level j is way above 0 then edge types are J-irreducible and so $\mathbf{e} : \widehat{\mathcal{C}_{\mathrm{e}}} \to \widehat{\mathcal{C}}$ is below $j : \mathbf{Sh}(\mathcal{C}, J) \to \widehat{\mathcal{C}}$. ∎

We say that *points (in* $\mathcal{C}$*) separate maps with edge type codomain* if for every edge type E and maps $f, g : C \to E$ such that $fc = gc$ for every $c : 1 \to C$, $f = g$. For instance, if 1 is a separator in $\mathcal{C}$ then points separate maps so, in particular, maps with edge type codomain.

6.4. LEMMA. *If points in* $\mathcal{C}$ *separate maps with edge type codomain then* 1 *is the only edge type with a unique point and* $\mathbf{e}$*-skeletal objects are quotients of* 0*-separated objects with discrete boundaries.*

PROOF. It is easy to check that if points in $\mathcal{C}$ separate maps with codomain C then $\mathcal{C}(-, C)$ is $\neg\neg$-separated (i.e. 0-separated). If, moreover, C has a unique point then C must be terminal.

The standard construction of $\mathbf{e}_!$ provides an epimorphism

$$\textstyle\sum_{E \in \mathcal{C}_{\mathrm{e}}} GE \times \mathcal{C}(-, E) \longrightarrow \mathbf{e}_! G$$

in $\widehat{\mathcal{C}}$ so $\mathbf{e}_! G$ is a quotient of a sum of objects that are 0-separated and have discrete boundaries. It is well-known that separated objects are closed under coproduct and, by Lemma 3.6, so are objects with discrete boundaries. Therefore, $\mathbf{e}_! G$ is a quotient of a 0-separated object with discrete boundaries. ∎

7. Level 1 in presheaf toposes with enough edge types

Let $\mathcal{C}$ be a small category with terminal object and such that every object has a point so that $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is pre-cohesive. In this section we give a sufficient condition for $\mathbf{e} : \widehat{\mathcal{C}_{\mathrm{e}}} \to \widehat{\mathcal{C}}$ to be level 1 of p. We have already seen sufficient conditions that guarantee that level $\mathbf{e}$ is 'low dimensional'. We now need conditions that ensure that it is way above 0. In other words, it remains to guarantee that there are enough figures of edge type "in order to determine that space's connectedness."

We say that an object C in $\mathcal{C}$ is *edge-wise connected* if for every $u, v : 1 \to C$ there is an edge type B and a map $f : B \to C$ such that both u and v factor through f.

7.1. Lemma. *If every object of $\mathcal{C}$ is edge-wise connected then $\widehat{\mathcal{C}_{\mathrm{e}}} \to \widehat{\mathcal{C}}$ is way above* 0.

Proof. We already know that $\mathbf{e}$ is above 0 (Proposition 6.1). The topology J on $\mathcal{C}$ corresponding to level $\widehat{\mathcal{C}_{\mathrm{e}}} \to \widehat{\mathcal{C}}$ sends C in $\mathcal{C}$ to the set of sieves on C that contain all the maps whose domain is an edge type. The fact that every object is edge-wise connected implies that every J-covering sieve is connected, so $p^* : \mathbf{Set} \to \widehat{\mathcal{C}}$ factors through the inclusion $\mathbf{Sh}(\mathcal{C}, J) = \widehat{\mathcal{C}_{\mathrm{e}}} \to \widehat{\mathcal{C}}$. ■

We summarize part of the above as follows.

7.2. Proposition. *If every object of $\mathcal{C}$ is edge-wise connected and* 1 *is the only edge type with exactly one point then $\widehat{\mathcal{C}_{\mathrm{e}}} \to \widehat{\mathcal{C}}$ is level* 1. *Moreover,* 1*-skeletal objects have discrete boundaries.*

Proof. By Lemma 7.1 level $\widehat{\mathcal{C}_{\mathrm{e}}} \to \widehat{\mathcal{C}}$ is way above 0 and by Lemma 6.3 it is the least one such. Proposition 6.1 implies that 1-skeletal objects have discrete boundaries. ■

The following variant of Proposition 7.2 gives, under stronger hypotheses, a characterization of 1-skeletal objects.

7.3. Proposition. *Let $\mathcal{C}$ be such that its points separate maps with edge type codomain and every object is edge-wise connected then $\widehat{\mathcal{C}_{\mathrm{e}}} \to \widehat{\mathcal{C}}$ is level* 1. *If, moreover, this level has monic skeleta then* 1*-skeletal objects coincide with quotients of* 0*-separated objects with discrete boundaries.*

Proof. Lemma 6.4 and Proposition 7.2 imply that $\widehat{\mathcal{C}_{\mathrm{e}}} \to \widehat{\mathcal{C}}$ is level 1 and that 1-skeletal objects must be quotients of 0-separated objects with discrete boundaries. Corollary 4.5 completes the proof. ■

By the results in Section 2, Proposition 7.3 applies to the toposes discussed there. The issue of monic skeleta deserves further analysis.

7.4. Lemma. *If for every pair of split epimorphisms $C \to E$ and $C \to E'$ with edge types E and E' there exists an absolute pushout*

$$\begin{tikzcd} C \arrow[r] \arrow[d] & E' \arrow[d] \\ E \arrow[r] & D \end{tikzcd}$$

with D an edge type, then $\widehat{\mathcal{C}}_{\mathrm{e}} \to \widehat{\mathcal{C}}$ has monic skeleta.

PROOF. Using a variation of Proposition 2.1. ■

The fact that E and E' are edge types suggests the possibility of simpler sufficient conditions. We propose one in the next section.

8. Absolute pushouts of edge types

The following is surely folklore or well-known.

8.1. LEMMA. *If the following two squares*

$$\begin{tikzcd} E' \arrow[r, "s"] \arrow[d, "d'"'] & C \arrow[r, "e'"] \arrow[d, "e"'] & E' \arrow[d, "d'"] \\ D \arrow[r, "t"'] & E \arrow[r, "d"'] & D \end{tikzcd}$$

commute, the horizontal composites are identities and e is epic then the right square is a pushout. So, if e is split epic then the pushout is absolute.

PROOF. It should follow as an application of Proposition 5.5 in [16] but we give a direct proof. Let the following diagram

$$\begin{tikzcd} C \arrow[r, "e'"] \arrow[d, "e"'] & E' \arrow[d, "g"] \\ E \arrow[r, "f"'] & B \end{tikzcd}$$

commute. Let $h = ft : D \to B$ and, using that the two squares commute, calculate

$$hde = ftd'e' = fese' = ge'se' = ge' = fe$$

to conclude that $hd = f$ because e is epic. Similarly, using that e' is epic it is easy to show that $hd' = g$. So the right square in the statement is indeed a pushout. If e is split epic then all the hypotheses are preserved by any functor, so the result follows. ■

We are interested in the following particular cases.

8.2. LEMMA. *Let $e : C \to E$ be split epic and let $e' : C \to E'$ be a map with a section $s : E' \to C$. Then the following hold:*

1. *If $es : E' \to E$ is constant then the following square*

$$\begin{tikzcd} C \arrow[r, "e'"] \arrow[d, "e"'] & E' \arrow[d, "!"] \\ E \arrow[r, "!"'] & 1 \end{tikzcd}$$

is an absolute pushout.

2. If $e = ese' : E' \to E$ then the following square

$$\begin{tikzcd} C \arrow[r, "e'"] \arrow[d, "e"'] & E' \arrow[d, "es"] \\ E \arrow[r, "id"'] & E \end{tikzcd}$$

is an absolute pushout.

Proof. To prove the first item observe that the two squares on the left below commute

$$\begin{tikzcd} E' \arrow[r, "s"] \arrow[d, "!"'] & C \arrow[r, "e'"] \arrow[d, "e"] & E' \arrow[d, "!"] \\ 1 \arrow[r] & E \arrow[r, "!"'] & 1 \end{tikzcd} \qquad \begin{tikzcd} E' \arrow[r, "s"] \arrow[d, "es"'] & C \arrow[r, "e'"] \arrow[d, "e"] & E' \arrow[d, "es"] \\ E \arrow[r, "id"'] & E \arrow[r, "id"'] & E \end{tikzcd}$$

and apply Lemma 8.1. To prove the second consider the diagram on the right above and apply Lemma 8.1 again. ■

Let $\mathcal{C}$ be a category with terminal object and such that every object has a point.

8.3. Lemma. *If E and E' are edge types then, for every $f : E' \to E$, f is constant or it is an isomorphism.*

Proof. As E is an edge type, f is constant or it has a section; so we concentrate on the second case. Let $s : E \to E'$ be a section of f. Since E' is an edge type, s is constant or it has a section. If the monic s is constant then E is subterminal, but E is also the codomain of the constant fs so E must be terminal. Hence, in this case, f is constant. On the other hand, if s has a section, then s must be an isomorphism and then f is an iso too. ■

In particular, every endomorphism of edge types is constant or is an automorphism. This is, of course, what happens in the combinatorial examples of Section 2, but it is also reminiscent of the affine line.

8.4. Proposition. *If every reflexive pair $f, g : C \to E$ with edge type E extends to a split coequalizer then $\widehat{\mathcal{C}}_{\mathrm{e}} \to \widehat{\mathcal{C}}$ has monic skeleta.*

Proof. We use Lemma 7.4 so consider a pair of split epis $e : C \to E$ and $e' : C \to E'$ with edge types E and E'. Let $s : E' \to C$ be a section of e'. By Lemma 8.3 the composite $es : E' \to E$ is constant or it is an isomorphism. If it is constant then the pushout of e and e' exists and is absolute by Lemma 8.2. On the other hand, if es is an iso then $e, ese' : C \to E$ have $s(es)^{-1} : E \to C$ as a common section. So, by hypothesis, we have a split coequalizer

$$\begin{tikzcd} C \arrow[r, shift left=2, "ese'"] \arrow[r, shift right=2, "e"'] & E \arrow[l, "t" description] \arrow[r, "q"] & Q \arrow[l, bend left, "u"] \end{tikzcd}$$

with $qu = id_Q$, $et = id_E$ and $ese't = uq$. As the parallel pair has a common section, the long rectangle below

$$\begin{tikzcd} C \arrow[r, "e'"] \arrow[d, "e"'] & E' \arrow[r, "s"] \arrow[d, "qes"] & C \arrow[r, "e"] & E \arrow[d, "q"] \\ E \arrow[r, "q"'] & Q \arrow[rr, "id"'] & & Q \end{tikzcd}$$

is a pushout. Since the horizontal maps of the inner short rectangle are isos, the inner left square is a pushout. Moreover, as split coequalizers are absolute, this pushout is also absolute. Finally, by Lemma 8.3, the composite $uq : E \to E$ is an isomorphism or it is constant. So Q is iso to E or it is terminal. In any case, it is an edge type. Altogether, Lemma 7.4 applies. ∎

9. Vector spaces and Affine spaces

The purpose of this section is to prove that the category of affine spaces satisfies the hypothesis of Proposition 8.4. We freely use the notation in Chapter XII of [12]. Fix a field F and let $\mathcal{V}$ be the category of finite dimensional vector spaces over F. It has a terminal object and every object has a (unique) point. As an object of $\mathcal{V}$, F is the only edge type other than the terminal object.

9.1. Lemma. *Every reflexive pair $f, g : V \to E$ in $\mathcal{V}$ with edge type E extends to a split coequalizer.*

Proof. As we have already mentioned, $E = 1$ or $E = F$. We need only concentrate on the second case. So consider a reflexive pair $f, g : V \to F$. Reflexivity implies that $\dim V \geq 1$. If $\dim V = 1$ then the common section must be an iso so $f = g$ and the pair f, g trivially extends to a split coequalizer. So assume that $\dim V \geq 2$. As f has a section by hypothesis, $\dim(\ker f) = \dim V - 1 \geq 1$. Similarly for g. We claim that $f = g$ or there is a map $t : F \to \ker f$ such that the composite

$$F \xrightarrow{t} \ker f \longrightarrow V \xrightarrow{g} F$$

is an isomorphism. Indeed, we have that $\ker f \subseteq \ker g$ or there is a $v \in \ker f$ such that $v \notin \ker g$. If $\ker f \subseteq \ker g$ then, as domain and codomain have the same dimension, the inclusion is an isomorphism, and $f = g$. On the other hand, v determines a map $F \to \ker f$ such that the composite $F \to F$ above is not constant. Pre-composing with its inverse we obtain an $h : F \to V$ such that $gh = id$ and fh is constant (i.e. it is the zero map). Therefore, h witnesses that the pair f, g is contractible and so, its coequalizer is split. ∎

Assume now that the field F is of characteristic not 2 and let $\mathcal{A}$ be the category of affine spaces (over F) as defined in Chapter XII of [12]. Again, $\mathcal{A}$ has a terminal object and every object has a point. In contrast with $\mathcal{V}$, 1 is the only edge type in $\mathcal{A}$ with only one point. The only other edge type in $\mathcal{A}$ is the affine line $F^\flat$.

9.2. PROPOSITION. *The essential subtopos induced by the inclusion $\mathcal{A}_e \to \mathcal{A}$ has monic skeleta and is level* 1 *of the pre-cohesive $\widehat{\mathcal{A}} \to$ **Set**. Moreover,* 1*-skeletal objects coincide with quotients of* 0*-separated objects with discrete boundaries.*

PROOF. Let $(-)^\# : \mathcal{A} \to \mathcal{V}$ be the ‘trace’ functor that sends an affine space P to the associated vector space $P^\#$ of translations. Consider a reflexive pair $f, g : P \to E$ in $\mathcal{A}$ with edge type E and common section $s : E \to P$. It is clear from its definition that the trace functor preserves edge types so, by Lemma 9.1, the reflexive pair $f^\#, g^\# : P^\# \to E^\#$ extends to a split coequalizer

$$\begin{tikzcd} P^\# \arrow[r, shift left=4, "f^\#"] \arrow[r, shift right=4, "g^\#"'] & E^\# \arrow[l, "t" description] \arrow[r, "q"] & Q \arrow[l, bend left, "u"] \end{tikzcd}$$

in $\mathcal{V}$ with such that $qu = id_Q$, $f^\# t = id_{E^\#}$ and $g^\# t = uq$. Consider now the affine space $Q^\flat$ determined by Q and choose a point $x_0 \in E$. By Theorem XII.4 in [12] there exists a unique $\overline{q} : E \to Q^\flat$ in $\mathcal{A}$ such that $\overline{q}x_0 = 0 \in Q^\flat$. Similarly, there is a unique $\overline{u} : Q^\flat \to E$ with $\overline{u}0 = x_0$. Moreover, there is a unique $\overline{t} : E \to P$ such that $\overline{t}x_0 = sx_0$. In other words, we have a diagram

$$\begin{tikzcd} P \arrow[r, shift left=3, "f"] \arrow[r, shift right=3, "g"'] & E \arrow[l, "\overline{t}" description] \arrow[r, "\overline{q}"] & Q^\flat \arrow[l, bend left, "\overline{u}"] \end{tikzcd}$$

in $\mathcal{A}$. We prove below that the diagram is a split coequalizer.

1. $\overline{q}f = \overline{q}g$:

$$\overline{q}(f(w + sx_0)) = \overline{q}(f^\# w + f(sx_0)) = \overline{q}(f^\# w + x_0) = q(f^\# w) + 0 =$$
$$= q(g^\# w) + 0 = \overline{q}((g^\# w) + x_0) = \overline{q}(g^\# w + g(sx_0)) = \overline{q}(g(w + sx_0))$$

for every translation w of P.

2. $\overline{q}\ \overline{u} = id_{Q^\flat}$:

$$\overline{q}(\overline{u}(w + 0)) = \overline{q}(u(w) + x_0) = q(u(w)) + \overline{q}x_0 = w + 0$$

for every translation w of $Q^\flat$.

3. $f\ \overline{t} = id_E$ and $g\ \overline{t} = \overline{u}\ \overline{q}$:

$$f(\overline{t}(w + x_0)) = f(tw + sx_0) = f^\#(tw) + f(sx_0) = w + x_0$$

$$g(\overline{t}(w+x_0)) = g(tw+sx_0) = g^\#(tw)+g(sx_0) = u(qw)+x_0 = \overline{u}(qw+0) = \overline{u}(\overline{q}(w+x_0))$$

for every translation w of E.

Proposition 8.4 implies that $\widehat{\mathcal{A}_e} \to \widehat{\mathcal{A}}$ has monic skeleta and Proposition 7.3 ends the proof. ∎

The pre-cohesive $\widehat{\mathcal{A}} \to \mathbf{Set}$ seems an interesting example deserving further study.

Acknowledgements

I would like to thank F. W. Lawvere for explaining several of his ideas about the Aufhebung. I also want to thank V. Marra for his suggestion to consider bounded depth formulas [2]. Much of the work was done during a visit to the Università di Bologna in 2018, with the support of C. Smith and funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 690974.

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