Theory and Applications of Categories, Vol. 34, No. 36, 2019, pp. 1165–1178.

A NOTE ON INTERNAL OBJECT ACTION REPRESENTABILITY OF 1-CAT GROUPS AND CROSSED MODULES

PAKO RAMASU

Abstract. The category of 1-cat groups, which is equivalent to the category of crossed modules, has internal object actions which are representable (by internal automorphism groups). Moreover, it is known that the crossed module, corresponding to the representing object $[X] = \underline{Aut}(X)$ associated with a 1-cat group X, must be isomorphic to the Norrie actor of the crossed module corresponding to X. We recall the description of $\underline{Aut}(X)$ from the author's PhD thesis, and construct that isomorphism explicitly.

1. Introduction

There are various contexts, unified in [BJK2005a] (see also [BJK2005b]) based on old ideas of G. M. Kelly, where one can consider the set $Act(B, X)$ of actions of B on X, and, moreover, make $Act(-, X)$ a functor $\mathbf{C}^{op} \to \mathbf{Sets}$, where $\mathbf{C}$ is the suitable category of *acting objects*. We then might have

$$Act(-, X) \approx hom(-, [X]), \tag{1}$$

for some object $[X]$ in $\mathbf{C}$, and if it is the case for each object X from the category of objects on which the objects of $\mathbf{C}$ act, we say that the actions are representable. This notion of *representable actions* was also introduced in [BJK2005a], where the main example of *internal object actions* came from the general theory of semidirect products [BJ1998]. The context we need here is what is described in Subsection 4.4 of [BJK2005a], which is:

•

- Our ground category $\mathbf{C}$ is the category of internal groups in a cartesian closed category $\mathbf{E}$ with finite limits, and we assume $\mathbf{C}$ to be semi-abelian in the sense of [JMT2002].

- An action of an object B of $\mathbf{C}$ on an object X of $\mathbf{C}$ can be identified with a morphism $h : B \times X \to X$ satisfying the usual conditions, which, expressed in terms of generalized elements with $h(b, x)$ written as bx, are:

$$0x = 0, \quad b_1(b_2x) = (b_1 + b_2)x, \quad b(x_1 + x_2) = bx_1 + bx_2; \tag{2}$$

Received by the editors 2019-03-20 and, in final form, 2019-10-16.
Transmitted by Tim Van der Linden. Published on 2019-10-30.
2010 Mathematics Subject Classification: 18G55, 18D15, 18D05, 20L05.
Key words and phrases: crossed module, action of an object, 1-cat group, internal automorphism, actor.

note that we are using additive notation, even though our groups are not necessarily abelian.

- In this case $[X] = \underline{Aut}(X)$, the internal automorphism group of X. As mentioned in [BJK2005a], it is constructed in a straightforward way; however, it involves long calculations, whose details can be found in [BCM2014].

More specifically, we are going to consider the case of **E** being the category of all (small) categories, which allows us to describe **C** either as: •

- the category of crossed modules (see e.g. [BS1976]), or as:
- the category of 1-cat groups [L1982].

In my PhD Thesis [R2015], I was using, with help of George Janelidze, who was my PhD supervisor then, the second description to calculate $[X] = \underline{Aut}(X)$ fully (and even to extend it to n-cat groups, for an arbitrary natural n). The result, recalled in Section 2 below, looks very different from what K. Norrie calls the *actor of a crossed module* in [N1990], and what is recalled in our Sections 3. Nevertheless, as follows from an observation in [BJK2005a], and in fact also from the results of [N1990], there should be no difference. More precisely, the crossed module corresponding to our calculated $[X] = \underline{Aut}(X)$ should be isomorphic to the Norrie actor of crossed module corresponding to X. Constructing this isomorphism explicitly (see Section 4) is the purpose of the present paper. What seems to make this construction interesting is that the two descriptions, of the internal automorphism group and of the Norrie actor, come from very different sources, namely from the theory of cartesian closed categories and from homotopical algebra (for the latter see references in [N1990], especially [L1979] and [W1948]).

2. The 1-cat presentation

Let $M = \{1, s, t\}$ be the monoid in which

$$st = t \quad \text{and} \quad ts = s, \tag{3}$$

and which also implies that s and t are idempotents. A 1-cat group X can be described as an M-group X with

$$(sx_1 = 0 = tx_2) \Rightarrow x_1 + x_2 = x_2 + x_1, \tag{4}$$

for all $x_1, x_2 \in X$, in the additive notation. Since the category of 1-cat groups can be identified with the category of internal groups in a cartesian closed category (of all small categories), each 1-cat group X has its internal automorphism group $\underline{Aut}(X)$, which is also a 1-cat group. Long but routine calculations made in [R2015] show that $\underline{Aut}(X)$ can

be presented as the 1-cat group of maps
$\alpha : M \times X \to X$ with

$$m\alpha(m', x) = \alpha(mm', mx), \tag{5}$$

$$\alpha(m, x_1 + x_2) = \alpha(m, x_1) + \alpha(m, x_2), \tag{6}$$

$$\alpha(m, -) : X \to X \quad \text{is a bijection,} \tag{7}$$

$$\alpha(1, x) = \alpha(1, tx) - \alpha(s, tx) + \alpha(s, x), \tag{8}$$

$$\alpha(1, x) = \alpha(t, x) - \alpha(t, sx) + \alpha(1, sx), \tag{9}$$

for all $m, m' \in M$ and $x_1, x_2 \in X$; the M-action on $\underline{Aut}(X)$ is defined by

$$(m'\alpha)(m, x) = \alpha(mm', x), \tag{10}$$

while the addition on $\underline{Aut}(X)$ is defined by

$$(\alpha + \beta)(m, x) = \alpha(m, \beta(m, x)), \tag{11}$$

making the second projection $M \times X \to X$ the zero element of $\underline{Aut}(X)$.

3. The Norrie presentation

Recall that a crossed module is triple (K, B, ∂), in which B and K are groups with B acting on K, and $\partial : K \to B$ is a group homomorphism with

$$\partial(bk) = b + \partial(k) - b, \tag{12}$$

$$\partial(k_1)k_2 = k_1 + k_2 - k_1, \tag{13}$$

for all $b \in B$ and $k_1, k_2 \in K$.
For example, a 1-cat group X presented as in Section 2 determines a crossed module (K_X, B_X, ∂_X) in which:

$$K_X = \{x \in X | sx = 0\}, \tag{14}$$

$$B_X = \{x \in X | sx = x\} = \{x \in X | tx = x\}, \tag{15}$$

$$B_X \text{ acts on } K_X \text{ via } bk = b + k - b, \tag{16}$$

$$\partial_X : K_X \to B_X \text{ is defined by } \partial_X(k) = tk. \tag{17}$$

The Norrie actor $A(K, B, \partial) = (D(B, K), Aut(K, B, \partial), \Delta)$ [N1990] of a crossed module (K, B, ∂) is described as follows: •

- $Aut(K,B,\partial)$ is the ordinary automorphism group of (K,B,∂), that is,

$$Aut(K,B,\partial)=\{(\kappa,\beta)\in Aut(K)\times Aut(B)|\beta\partial=\partial\kappa \text{ and } \forall_{k\in K}\ \forall_{b\in B}\ \kappa(bk)=\beta(b)\kappa(k)\}; \tag{18}$$

- a map $d:B\to K$ is called a derivation (= a crossed homomorphism) if

$$d(b_1+b_2)=d(b_1)+b_1d(b_2), \tag{19}$$

for all $b_1,b_2\in B$, and such derivations form an additive (not necessarily commutative) monoid $Der(B,K)$, whose addition is defined by

$$(d_1+d_2)(b)=d_1(\partial d_2(b)+b)+d_2(b)=d_1\partial d_2(b)+d_2(b)+d_1(b) \tag{20}$$

(where the second equality follows from
$d_1(\partial d_2(b)+b)+d_2(b)=d_1\partial d_2(b)+d_2(b)+d_1(b)-d_2(b)+d_2(b))$
which makes the zero map $B\to K$ the zero element of $Der(B,K)$;

- $D(B,K)$ is defined as the group of invertible elements of the monoid $Der(B,K)$;

- the action of $Aut(K,B,\partial)$ on $D(B,K)$ is defined by

$$(\kappa,\beta)d=\kappa d\beta^{-1}; \tag{21}$$

- the homomorphism $\Delta:D(B,K)\to Aut(K,B,\partial)$ is defined by
$\Delta(d)=(\Delta_1(d),\Delta_2(d))$, where

$$\Delta_1(d)(k)=d\partial(k)+k\ and\quad \Delta_2(d)(b)=\partial d(b)+b. \tag{22}$$

Note: using the maps Δ_1 and Δ_2 one obtains nicer forms of (19), namely

$$(d_1+d_2)(b)=d_1\Delta_2(d_2)(b)+d_2(b)=\Delta_1(d_1)d_2(b)+d_1(b). \tag{23}$$

4. The isomorphism

In this section we fix: •

- a 1-cat group X presented as an M-group, as in Section 2;
- the corresponding crossed module $(K_X,B_X,\partial_X)=(K,B,\partial)$;
- the Norrie actor $A(K,B,\partial)=(D(B,K),Aut(K,B,\partial),\Delta)$ of (K,B,∂);
- the crossed module corresponding to the 1-cat group $\underline{Aut}(X)$, which will be denoted by $(\underline{K},\underline{B},\underline{\partial})$.

And our aim is to construct an isomorphism

$$(D(B,K),Aut(K,B,\partial),\Delta)\approx(\underline{K},\underline{B},\underline{\partial}). \tag{24}$$

4.1. LEMMA.

(a) $\underline{K} = \{\alpha \in \underline{Aut}(X) | \forall_{x\in X}\ \alpha(s,x) = x\}$;

(b) $\underline{B} = \{\alpha \in \underline{Aut}(X) | \forall_{x\in X}\ \alpha(1,x) = \alpha(s,x) = \alpha(t,x)\}$;

(c) $\underline{B}$ *acts on* $\underline{K}$ *via* $\underline{bk}(m,x) = \underline{b}(m, \underline{k}(m, \underline{b}(m,-)^{-1}(x)))$;

(d) $\underline{\partial} : \underline{K} \to \underline{B}$ *is defined by* $\underline{\partial}(\underline{k})(m,x) = \underline{k}(t,x)$.

PROOF.

(a) According to (14), $\underline{K} = \{\alpha \in \underline{Aut}(X) | s\alpha = 0\}$, and we calculate, for any $m \in M$:
$(s\alpha)(m,x) = \alpha(ms,x)$ (by (10))
$= \alpha(s,x)$ (since $ms = s$ for every $m \in M$).
Since 0 of $\underline{Aut}(X)$ is the second projection $M \times X \to X$, as mentioned at the end of Section 2, this implies that $s\alpha = 0$ is equivalent to $\forall_{x\in X}\ \alpha(s,x) = x$.

(b) For $\alpha \in \underline{Aut}(X)$, we have:
$\alpha \in \underline{B} \iff s\alpha = \alpha$ (by (15))
$\iff \forall_{m\in M}\ \forall_{x\in X}\ \alpha(ms,x) = \alpha(m,x)$ (by (10))
$\iff \forall_{m\in M}\ \forall_{x\in X}\ \alpha(s,x) = \alpha(m,x)$ (again since $ms = s$ for every $m \in M$)
$\iff \forall_{x\in X}\ \alpha(1,x) = \alpha(s,x) = \alpha(t,x)$.

(c) follows from (11) and (16).

(d) follows from (17), (10), and the fact that $mt = t$ for every $m \in M$. ■

4.2. LEMMA. *The formula*

$$f(\alpha)(x) = \alpha(1,x) - x \tag{25}$$

defines a group homomorphism $f : \underline{K} \to D(B,K)$.

PROOF. We need to prove the following:

(a) for every $\alpha \in \underline{K}$ and $x \in B$, $\alpha(1,x) - x$ belongs to K.

(b) for every $\alpha \in \underline{K}$, the map $f(\alpha) : B \to K$ defined by (25) is a derivation, that is, it belongs to $Der(B,K)$;

(c) $f(\alpha_1 + \alpha_2) = f(\alpha_1) + f(\alpha_2)$ for all $\alpha_1, \alpha_2 \in \underline{K}$;

(d) the derivations of the form $f(\alpha)$ above are invertible, that is, they belong to $D(B,K)$.

Proof of (a): We have
$s(\alpha(1,x) - x) = s\alpha(1,x) - sx$ (since X is an M-group)
$= \alpha(s,sx) - sx$ (by (5))
$= sx - sx$ (by Lemma 4.1(a))
$= 0$.

Proof of (b): According to (19), we need to prove that

$$\alpha(1, x_1 + x_2) - (x_1 + x_2) = \alpha(1, x_1) - x_1 + x_1(\alpha(1, x_2) - x_2), \tag{26}$$

for all $x_1, x_2 \in B$. We have:
$\alpha(1, x_1 + x_2) - (x_1 + x_2) = \alpha(1, x_1) + \alpha(1, x_2) - (x_1 + x_2)$ (by (6))
$= \alpha(1, x_1) + \alpha(1, x_2) - x_2 - x_1$,
and so to prove (26) is to show that

$$x_1 + \alpha(1, x_2) - x_2 - x_1 = x_1(\alpha(1, x_2) - x_2),$$

which follows from (16).

Proof of (c): For $\alpha_1, \alpha_2 \in \underline{K}$ and $x \in B$ we have:
$f(\alpha_1 + \alpha_2)(x) = (\alpha_1 + \alpha_2)(1, x) - x$ (by definition of f)
$= \alpha_1(1, \alpha_2(1, x)) - x$ (by (11))
$= \alpha_1(1, t\alpha_2(1, x)) - \alpha_1(s, t\alpha_2(1, x)) + \alpha_1(s, \alpha_2(1, x)) - x$ (by (8) used for $\alpha_2(1, x)$ instead of x)
$= \alpha_1(1, t\alpha_2(1, x)) - t\alpha_2(1, x) + \alpha_2(1, x) - x$ (by Lemma 4.1(a))

and

$(f(\alpha_1) + f(\alpha_2))(x) = f(\alpha_1)(tf(\alpha_2)(x) + x) + f(\alpha_2)(x)$ (by the first equality of (20) and (17))
$= f(\alpha_1)(t(\alpha_2(1, x) - x) + x) + \alpha_2(1, x) - x$ (by the definition of f)
$= f(\alpha_1)(t\alpha_2(1, x)) + \alpha_2(1, x) - x$ (since $tx = x$, by (15))
$= \alpha_1(1, t\alpha_2(1, x)) - t\alpha_2(1, x) + \alpha_2(1, x) - x$ (by the definition of f).

That is, $f(\alpha_1 + \alpha_2) = f(\alpha_1) + f(\alpha_2)$ for all $\alpha_1, \alpha_2 \in \underline{K}$.

Proof of (d): Since $\underline{K}$ is a group, having proved (c) we only need to prove $f(0) = 0$. For, just note: •

- $\underline{K}$ is a subgroup of $\underline{Aut}(X)$;
- the 0 of $\underline{Aut}(X)$ the second projection $M \times X \to X$, as mentioned at the end of Section 2;
- therefore $f(0)(x) = x - x = 0$;
- the 0 of $D(B, K)$is the zero map $B \to K$, as mentioned below (20). ∎

Let us put $f(\alpha) = d$ and try to recover $\alpha \in \underline{K}$ from d. For, take an arbitrary each $x \in X$, and observe:

•

(a) since α is in $\underline{K}$, $\alpha(s, x) = x$ (by Lemma 4.1(a));

(b) $\alpha(1, sx) = d(sx) + sx$ and $\alpha(1, tx) = d(tx) + tx$ (by (25));

(c) $\alpha(t, sx) = \alpha(t, tsx) = t\alpha(1, sx)$ (by (5));

(d) $\alpha(t, sx) = td(sx) + tsx = td(sx) + sx$ (by (c) and (b));

(e) $\alpha(1, x) = \alpha(1, tx) - \alpha(s, tx) + \alpha(s, x) = d(tx) + tx - tx + x = d(tx) + x$ (where the first equality is (8), while the second one follows from (b) and (a));

(f) $\alpha(t, x) = \alpha(1, x) - \alpha(1, sx) + \alpha(t, sx) = d(tx) + x - sx - d(sx) + td(sx) + sx$ $= d(tx) - d(sx) + td(sx) + x - sx + sx = d(tx) - d(sx) + td(sx) + x$ (where: the first equality follows from (9); the second one follows from (e), (b), and (d); and the third one from (4), since $s(x - sx) = 0 = t(-d(sx) + td(sx))$).

Denoting α by $g(d)$, this gives:

$$g(d)(m, x) = \begin{cases} d(tx) + x, & \text{if} \quad m = 1; \\ x & \text{if} \quad m = s; \\ d(tx) - d(sx) + td(sx) + x, & \text{if} \quad m = t. \end{cases} \tag{27}$$

We can also rewrite this as

$$\begin{cases} g(d)(s, x) = x, \\ g(d)(m, x) = d(tx) - d(sx) + md(sx) + x, \quad \text{if} \quad m \neq s. \end{cases} \tag{28}$$

4.3. Lemma. *The pair* (28) *of formulas defines a map* $g : D(B, K) \to \underline{K}$.

Proof. We need to prove that the map $\alpha : M \times X \to X$, defined by

$$\alpha(m, x) = \begin{cases} d(tx) + x, & \text{if} \quad m = 1; \\ x & \text{if} \quad m = s; \\ d(tx) - d(sx) + td(sx) + x, & \text{if} \quad m = t; \end{cases} \tag{29}$$

belongs to $\underline{Aut}(X)$, that is, it satisfies (5)-(9); if so, then it will belong to $\underline{K}$ by Lemma 4.1(a). Note also, that verifying (5)-(7) we exclude the trivial case $m = s$.

Verification of (5). Here the case $m = 1$ is also trivial, and so we only need to consider the case $m = t$. We have:

$t\alpha(1, x) = t(d(tx) + x) = td(tx) + tx = d(tx) - d(tx) + td(tx) + tx$
$= d(ttx) - d(stx) + td(stx) + tx = \alpha(t, tx) = \alpha(t1, tx)$;

$t\alpha(s,x) = ts = \alpha(s,tx) = \alpha(ts,tx)$;

$t\alpha(t,x) = td(tx) - td(sx) + ttd(sx) + tx = td(tx) + tx = \alpha(t,tx) = \alpha(tt,tx)$,
(where the equality $td(stx) + tx = \alpha(t,tx)$ already has appeared in the first calculation of this proof).

Verification of (6). First note that, for all b_1 and b_2 in
$B = B_X = \{x \in X | sx = x\} = \{x \in X | tx = x\}$, we have

$$d(b_1 + b_2) = d(b_1) + b_1 + d(b_2) - b_1 \tag{30}$$

in X, as follows from (19) and (16). Then, suppose $m = 1$; we have:
$\alpha(1, x_1 + x_2) = d(tx_1 + tx_2) + x_1 + x_2 = d(tx_1) + tx_1 + d(tx_2) - tx_1 + x_1 + x_2$ (by (30))
$= d(tx_1) + tx_1 - tx_1 + x_1 + d(tx_2) + x_2$ (since $sd(tx_2) = 0 = t(-tx_1 + x_1)$)
$= d(tx_1) + x_1 + d(tx_2) + x_2 = \alpha(1, x_1) + \alpha(1, x_2)$.

Now suppose $m = t$; we have:
$\alpha(t, x_1 + x_2) = d(tx_1 + tx_2) - d(sx_1 + sx_2) + td(sx_1 + sx_2) + x_1 + x_2$
$= (d(tx_1) + tx_1 + d(tx_2) - tx_1) - (d(sx_1) + sx_1 + d(sx_2) - sx_1) + t(d(sx_1) + sx_1 + d(sx_2) - sx_1) + x_1 + x_2$ (by (30))
$= d(tx_1) + tx_1 + d(tx_2) - tx_1 + sx_1 - d(sx_2) - sx_1 - d(sx_1) + td(sx_1) + sx_1 + td(sx_2) - sx_1 + x_1 + x_2$
(using $ts = s$ twice)
$= d(tx_1) - d(sx_1) + td(sx_1) + tx_1 + d(tx_2) - tx_1 + sx_1 - d(sx_2) - sx_1 + sx_1 + td(sx_2) - sx_1 + x_1 + x_2$
(since $s(tx_1 + d(tx_2) - tx_1 + sx_1 - d(sx_2) - sx_1) = 0 = t(-d(sx_1) + td(sx_1)))$
$= d(tx_1) - d(sx_1) + td(sx_1) + tx_1 + d(tx_2) - tx_1 + sx_1 - d(sx_2) + td(sx_2) - sx_1 + x_1 + x_2$
$= d(tx_1) - d(sx_1) + td(sx_1) + tx_1 + d(tx_2) - tx_1 + sx_1 - sx_1 + x_1 - d(sx_2) + td(sx_2) + x_2$
(since $s(-sx_1 + x_1) = 0 = t(-d(sx_2) + td(sx_2)))$
$= d(tx_1) - d(sx_1) + td(sx_1) + tx_1 + d(tx_2) - tx_1 + x_1 - d(sx_2) + td(sx_2) + x_2$
$= d(tx_1) - d(sx_1) + td(sx_1) + tx_1 - tx_1 + x_1 + d(tx_2) - d(sx_2) + td(sx_2) + x_2$
(since $s(d(tx_2)) = 0 = t(-tx_1 + x_1))$
$= d(tx_1) - d(sx_1) + td(sx_1) + x_1 + d(tx_2) - d(sx_2) + td(sx_2) + x_2 = \alpha(t, x_1) + \alpha(t, x_2)$.

Verification of (7). Let $\beta : M \times X \to X$ be the map defined in the same way as α but with d replaced with its inverse e in the group $D(B, K)$. We have:
$\alpha(1, \beta(1, x)) = \alpha(1, e(tx) + x) = d(t(e(tx) + x)) + e(tx) + x$
$= d(te(tx) + tx) + e(tx) + x = (d + e)(tx) + x$ (by the first equality of (20) and (17))
$= g(d + e)(1, x)$ (by (27))
$= x$ (which easily follows from $d + e = 0$),
and similarly $\beta(1, \alpha(1, x)) = x$, which proves that $\alpha(1, -) : X \to X$ is a bijection.

Next, we have:
$\alpha(t, \beta(t, x)) = \alpha(t, e(tx) - e(sx) + te(sx) + x)$
$= d(t(e(tx) - e(sx) + te(sx) + x)) - d(s(e(tx) - e(sx) + te(sx) + x)) + td(s(e(tx) - e(sx) +$

$te(sx)+x))+e(tx)-e(sx)+te(sx)+x$
$=d(te(tx)-te(sx)+te(sx)+tx)-d(se(tx)-se(sx)+te(sx)+sx)+td(se(tx)-se(sx)+te(sx)+sx)+e(tx)-e(sx)+te(sx)+x$
$=d(te(tx)+tx)-d(te(sx)+sx)+td(te(sx)+sx)+e(tx)-e(sx)+te(sx)+x$ (using the fact that $se(tx)=0=se(sx)$ twice)
$=d(te(tx)+tx)+e(tx)-e(sx)-d(te(sx)+sx)+t(d(te(sx)+sx))+te(sx)+x$ (since $s(+e(tx)-e(sx))=0=t(-d(te(sx)+sx)+t(d(te(sx)+sx))))$
$=d(te(tx)+tx)+e(tx)-(d(te(sx)+sx)+e(sx))+t(d(te(sx)+sx)+e(sx))+x$
$=(d+e)(tx)-(d+e)(sx)+t(d+e)(sx)+x$ (by the first equality of (20) and (17))
$=g(d+e)(t,x)$ (by (27))
$=x$ (since $d+e=0$ again).
After that we conclude that $\alpha(m,-):X\to X$ is a bijection in the same way as we did for $\alpha(1,-)$.

Verification of (8) and (9) . We have:
$\alpha(1,tx)-\alpha(s,tx)+\alpha(s,x)=d(tx)+tx-tx+x=d(tx)+x=\alpha(1,x),$

and

$\alpha(t,x)-\alpha(t,sx)+\alpha(1,sx)=d(tx)-d(sx)+td(sx)+x-(d(tsx)-d(ssx)+td(ssx)+sx)+d(tsx)+sx$
$=d(tx)-d(sx)+td(sx)+x-sx-td(sx)+d(sx)+sx$
$=d(tx)-d(sx)+td(sx)-td(sx)+d(sx)+x-sx+sx$ (since $s(x-sx)=0=t(-td(sx)+d(sx)))$
$=d(tx)+x=\alpha(1,x).$ ■

4.4. Lemma. *The maps $f:\underline{K}\to D(B,K)$ and $g:D(B,K)\to\underline{K}$, of Lemmas 4.2 and 4.3, are inverse of each other.*

Proof. $gf=1_{\underline{K}}$ by the definition of g. For fg, any $d\in D(B,K)$, and any $x\in B$, we have:

$((fg))(d)(x)=(f(g(d))(x)=g(d)(1,x)-x=d(tx)+x-x$ (by (27))
$=d(tx)=d(x)$ (since x is in B).

That is, $fg(d)=d$ (for any $d\in D(B,K)$), and so $fg=1_{D(B,K)}$. ■

4.5. Lemma. *The assignment*

$$p(\alpha)=(p_1(\alpha),p_2(\alpha)),\ \textit{where}\ p_1(\alpha)(k)=\alpha(1,k)\ \textit{and}\quad p_2(\alpha)(b)=\alpha(1,b), \tag{31}$$

defines a map $p:\underline{B}\to Aut(K,B,\partial)$.

Proof. Let us write $p(\alpha)=(\kappa,\beta)$. First of all, for $k\in K$, we have

$s(\kappa(k)) = s\alpha(1,k) = \alpha(s,sk)$ (by (5))
$= \alpha(s,0)$ (by (14))
$= 0$ (the fact that $\alpha(s,-)$ preserves zero follows from (6)),

and, for $b \in B$, we have

$s\beta(b) = s\alpha(1,b) = \alpha(s,sb)$ (as above)
$= \alpha(s,b)$ (by (15))
$= \alpha(1,b)$ (by Lemma 4.1(b))
$= \beta(b)$;
that is, $\kappa(k)$ and $\beta(b)$ belong to K and B, respectively. After that (6) (together with (11)) and (7) tell us that (κ,β) belongs to $Aut(K) \times Aut(B)$, and we only need to prove that $\beta\partial = \partial\kappa$ and $\kappa(bk) = \beta(b)\kappa(k)$ for all $k \in K$ and $b \in B$. We have:

$\beta\partial(k) = \beta(tk)$ (by (17))
$= \alpha(1,tk)$ (by (31))
$= \alpha(t,tk)$ (by Lemma 4.1(b))
$= t\alpha(1,k)$ (by (5))
$= \partial\kappa(k)$,

$\kappa(bk) = \alpha(1,bk) = \alpha(1,b+k-b)$ (by (16))
$= \alpha(1,b) + \alpha(1,k) - \alpha(1,b)$ (by (6))
$= \beta(b) + \kappa(k) - \beta(b) = \beta(b)\kappa(k)$ (by (16)),

and so (κ,β) belongs to $Aut(K,B,\partial)$ as desired. ∎

4.6. Lemma. *The map $p : \underline{B} \to Aut(K,B,\partial)$ is a group homomorphism.*

Proof. For α_1 and α_2 in $\underline{B}$, k in K, and b in B, we have to prove that
$p_1(\alpha_1+\alpha_2)(k) = p_1(\alpha_1)(p_1(\alpha_2)(k))$ and $p_2(\alpha_1+\alpha_2)(b) = p_2(\alpha_1)(p_2(\alpha_2)(b))$,
that is, to prove that
$(\alpha_1+\alpha_2)(1,k) = \alpha_1(1,\alpha_2(1,k))$ and $(\alpha_1+\alpha_2)(1,b) = \alpha_1(1,\alpha_2(1,b))$,
but both of these equalities are special cases of (11). ∎

4.7. Lemma. *The formula*

$$q(\kappa,\beta)(m,x) = \kappa(x-sx) + \beta(sx) \tag{32}$$

defines a map $q : Aut(K,B,\partial) \to \underline{B}$.

Proof. First of all, the expression $\kappa(x-sx)+\beta(sx)$ is well-defined, which follows from (14) and (15), since $s(x-sx) = 0$ and $s(sx) = sx$. Next, it is independent of m, as needed according to Lemma 4.1(b). After that we write $\alpha(m,x) = \kappa(x-sx)+\beta(sx)$, and we have to prove that conditions (5)-(9) are satisfied.

Verification of (5). The case $m = 1$ is trivial. We have
$s\alpha(m', x) = s\kappa(x - sx) + s\beta(sx)$
$= \beta(sx)$ (by (14) and (15), since $\kappa(x - sx) \in K$ and $\beta(sx) \in B$)
$= \kappa(sx - ssx) + \beta(ssx) = \alpha(sm', sx)$;

$t\alpha(m', x) = t\kappa(x - sx) + t\beta(sx)$
$= \partial\kappa(x - sx) + \beta(sx)$ (using (17) and the fact that $\beta(sx)$ is in B)
$= \beta\partial(x - sx) + \beta(sx)$ (since (κ, β) is in $Aut(K, B, \partial)$)
$= \beta(tx - sx) + \beta(sx)$ (by (17))
$= \beta(tx) - \beta(sx) + \beta(sx)$ (since β is in $Aut(B)$, and both tx and sx are in B)
$= \beta(tx) = \kappa(tx - stx) + \beta(stx) = \alpha(tm', tx)$.

Verification of (6). We have
$\alpha(m, x_1+x_2) = \kappa(x_1+x_2-s(x_1+x_2))+\beta(s(x_1+x_2)) = \kappa(x_1+x_2-sx_2-sx_1)+\beta(sx_1+sx_2)$
$= \kappa(x_1 - sx_1 + sx_1 + x_2 - sx_2 - sx_1) + \beta(sx_1) + \beta(sx_2)$ (since both sx_1 and sx_2 are in B, we can indeed write $\beta(sx_1 + sx_2) = \beta(sx_1) + \beta(sx_2)$)
$= \kappa(x_1 - sx_1 + (sx_1)(x_2 - sx_2)) + \beta(sx_1) + \beta(sx_2)$ (by (16))
$= \kappa(x_1-sx_1)+\kappa((sx_1)(x_2-sx_2))+\beta(sx_1)+\beta(sx_2)$ (since both x_1-sx_1 and $(sx_1)(x_2-sx_2)$) are in K, we can indeed write $\kappa(x_1 - sx_1 + (sx_1)(x_2 - sx_2)) = \kappa(x_1 - sx_1) + \kappa((sx_1)(x_2 - sx_2)))$
$= \kappa(x_1 - sx_1) + \beta(sx_1)\kappa(x_2 - sx_2) + \beta(sx_1) + \beta(sx_2)$ (since (κ, β) is in $Aut(K, B, \partial)$)
$= \kappa(x_1 - sx_1) + \beta(sx_1) + \kappa(x_2 - sx_2) - \beta(sx_1) + \beta(sx_1) + \beta(sx_2)$ (by (16))
$= \kappa(x_1 - sx_1) + \beta(sx_1) + \kappa(x_2 - sx_2) + \beta(sx_2)$
$= \alpha(m, x_1) + \alpha(m, x_2)$.

Verification of (7). We define a map $\gamma : X \to X$ by $\gamma(x) = \kappa^{-1}(x - sx) + \beta^{-1}(sx)$, and calculate
$\gamma(\alpha(m, x)) = \kappa^{-1}(\alpha(m, x) - s\alpha(m, x)) + \beta^{-1}(s\alpha(m, x))$
$= \kappa^{-1}(k(x - sx) + \beta(sx) - s(\kappa(x - sx) + \beta(sx))) + \beta^{-1}(s(\kappa(x - sx) + \beta(sx)))$
$= \kappa^{-1}(k(x - sx)) + \beta^{-1}(\beta(sx))$ (since $s\kappa(x - sx) = 0$ and $s\beta(sx) = \beta(sx)$, which follows from (14) and 15), respectively)
$= x - sx + sx$
$= x$,
which tells us that the composite $\gamma\alpha(m, -)$ is the identity map of X; similarly, so is $\alpha(m, -)\gamma$. This proves (7).

Verification of (8) and (9): just use the fact that $\alpha(m, x)$ is independent of m. ∎

4.8. Lemma. *The maps* $p : \underline{B} \to Aut(K, B, \partial)$ *and* $q : Aut(K, B, \partial) \to \underline{B}$*, of Lemmas 4.5 and 4.7, are inverse of each other.*

Proof. To prove that pq is the identity map of $Aut(K, B, \partial)$ is to prove that, for each $(\kappa, \beta) \in Aut(K, B, \partial)$, we have

$$p_1 q(\kappa, \beta) = \kappa \text{ and } p_2 q(\kappa, \beta) = \beta,$$

which means that, for each $k \in K$ and each $b \in B$, we have

$$\kappa(k - sk) + \beta(sk) = \kappa(k) \text{ and } \kappa(b - sb) + \beta(sb) = \beta(b).$$

The first of these two formulas follows from (14), while the second one follows from (15).

To prove that qp is the identity map of $\underline{B}$ is to prove that, for each $\alpha \in \underline{B}$, we have

$$qp(\alpha) = \alpha,$$

which means that, for each $(m, x) \in M \times X$, we have

$$\alpha(1, x - sx) + \alpha(1, sx) = \alpha(m, x).$$

Indeed,
$\alpha(1, x - sx) + \alpha(1, sx) = \alpha(1, x)$ (by (6)), and $\alpha(1, x) = \alpha(m, x)$ (by Lemma 4.1(b)). ■

4.9. Lemma. *The diagram*

$$\begin{array}{ccc} \underline{K} & \xrightarrow{\partial} & \underline{B} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ D(B, K) & \xrightarrow{\Delta} & Aut(K, B, \partial) \end{array} \tag{33}$$

commutes.

Proof. To prove that diagram (33) commutes, is to prove that, for each $\alpha \in \underline{K}$, we have

$$p_1(\underline{\partial}(\alpha)) = \Delta_1(f(\alpha)) \text{ and } p_2(\underline{\partial}(\alpha)) = \Delta_2(f(\alpha)),$$

which means that, for each $k \in K$ and each $b \in B$, we have

$$\alpha(t, k) = f(\alpha)\partial(k) + k \text{ and } \alpha(t, b) = \partial f(\alpha)(b) + b,$$

or, equivalently (by (17) and our definition of f),

$$\alpha(t, k) = \alpha(1, tk) - tk + k \text{ and } \alpha(t, b) = t\alpha(1, b) - tb + b.$$

The first of these two formulas follows from (8) and Lemma 4.1(a), while the second one follows from (5) and $tb = b$ (by (15)). ■

4.10. Lemma. *For every* $\underline{k} \in \underline{K}$ *and every* $\underline{b} \in \underline{B}$*, we have* $f(\underline{bk}) = p(\underline{b})f(\underline{k})$*.*

Proof. For any $x \in B$, we have
$(p(\underline{b})f(\underline{k}))(x) = (p_1(\underline{b})f(\underline{k})p_2(\underline{b})^{-1})(x)$ (by (21))
$= (p_1(\underline{b})f(\underline{k}))(\underline{b}(1,-)^{-1}(x))$ (by (31))
$= p_1(\underline{b})(\underline{k}(1,\underline{b}(1,-)^{-1}(x)) - \underline{b}(1,-)^{-1}(x))$ (by (25))
$= \underline{b}(1,\underline{k}(1,\underline{b}(1,-)^{-1}(x))) - \underline{b}(1,-)^{-1}(x)$ (by (31))
$= \underline{b}(1,\underline{k}(1,\underline{b}(1,-)^{-1}(x))) - \underline{b}(1,\underline{b}(1,-)^{-1}(x))$ (by (6))
$= \underline{b}(1,\underline{k}(1,\underline{b}(1,-)^{-1}(x))) - x = \underline{bk}(1,x) - x$ (by Lemma 4.1(c))
$= f(\underline{bk})(x)$ (by (25)). ∎

Putting these lemmas together, we obtain the desired isomorphism (24). More precisely, we obtain

4.11. Theorem. *Let* f, g, p*, and* q *be defined as in Lemmas* 4.2*,* 4.3*,* 4.5*, and* 4.7*, respectively. Then*

$$(f,p) : (\underline{K}, \underline{B}, \underline{\partial}) \longrightarrow (D(B,K), Aut(K,B,\partial), \Delta)$$

is an isomorphism of crossed modules, whose inverse is (g,q)*.* ∎

Acknowledgements

The results in this paper formed part of my PhD thesis [R2015] written at University of Cape Town. I would like to thank my former supervisor Prof. G. Janelidze for his support and guidance during my period of study.

References

[BCM2014] F. Borceux, M. M. Clementino, and A. Montoli, On the representability of actions for topological algebras, in Categorical methods in algebra and topology, Special Volume in Honour of Manuela Sobral, Textos de Matemática 46, Departamento de Matemática, Faculdade de Ciências e Tecnologia da Universidade de Coimbra, 2014, 41-66

[BJ1998] D. Bourn and G. Janelidze, Protomodularity, descent, and semidirect products, Theory and Applications of Categories 4, 1998, 37-46

[BJK2005a] F. Borceux, G. Janelidze, and G. M. Kelly, Internal object actions, Commentationes Mathematicae Universitatis Carolinae, 46, 2, 2005, 235-255

[BJK2005b] F. Borceux, G. Janelidze, and G. M. Kelly, On the representability of actions in a semi-abelian category, Theory and Applications of Categories, 14, 11, 2005, 244-286

[BS1976] R. Brown and C. B. Spencer, G-groupoids, crossed modules and the fundamental groupoid of a topological group, Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen, Series A 79 = Indagationes Mathematicae 38, 1976, 296-302

[JMT2002] G. Janelidze, L. Márki, and W. Tholen, Semi-abelian categories, Journal of Pure and Applied Algebra 168, 2002, 367-386

[L1982] J.-L. Loday, Spaces with finitely many non-trivial homotopy groups, Journal of Pure and Applied Algebra 24, 1982, 179-202

[L1979] A. S.-T. Lue, Semi-complete crossed modules and holomorphs of groups, Bulletin of the London Mathematical Society 11, 1979, 8-16

[N1990] K. Norrie, Actions and automorphisms of crossed modules, Bulletin de la Société Mathématique de France 118, 1990, 129-146

[R2015] P. Ramasu, Internal monoid actions in a cartesian closed category and higher-dimensional group automorphisms, PhD Thesis, University of Cape Town 2015

[W1948] J. H. C. Whitehead, On operators in relative homotopy groups, Annals of Mathematics, Second series 49, 3, 1948, 610-640

Department of Mathematics and Statistical Sciences, Botswana International University of Science and Technology
Private Bag 16, Palapye, Botswana
Email: ramasup@biust.ac.bw

This article may be accessed at http://www.tac.mta.ca/tac/