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DISTRIBUTIVE LAWS BETWEEN THE THREE GRACES

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ABSTRACT. By the Three Graces we refer, following J.-L. Loday, to the algebraic operads $\mathcal{A}ss$, $\mathcal{C}om$, and $\mathcal{L}ie$, each generated by a single binary operation; algebras over these operads are respectively associative, commutative associative, and Lie. We classify all distributive laws (in the categorical sense of Beck) between these three operads. Some of our results depend on the computer algebra system Maple, especially its packages `LinearAlgebra` and `Groebner`.

All algebras are equal, but some algebras are more equal than others.

1. Introduction

As the epigraph indicates[1], some algebras are more important than others. Experience teaches us that the most common classes of algebras are the Three Graces[2] — associative, commutative associative, and Lie — together with other classes of algebras that combine these in a specific way. The algebras in these three classes are representations of the quadratic Koszul operads denoted $\mathcal{A}ss$, $\mathcal{C}om$, and $\mathcal{L}ie$, or created from these operads using quadratic homogeneous distributive laws (the precise meaning of this phrase will be explained in §2). Examples of structures combining two of these operads or their operadic suspensions are the following:

- Poisson algebras, omnipresent in classical mechanics [4, 5, 26, 27, 28, 33, 37]

- Gerstenhaber algebras [3, 13, 21, 25, 32]

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[1] The allusion is to a famous quotation from George Orwell's satire *Animal Farm.*

[2] This terminology originated with J.-L. Loday, referring in particular to the famous painting *Les Trois Grâces*, a Renaissance masterpiece by Lucas Cranach the Elder. Since 2011 it has been in the collection of the Musée du Louvre in Paris. It depicts the *charites* or daughters of Zeus from classical Greek mythology: Aglaea (meaning elegance or splendor), Euphrosyne (mirth or happiness), and Thalia (youth or beauty).

- Batalin-Vilkovisky algebras [2, 18, 22, 24, 43, 53]
- e_n-algebras and the little cubes operad from homotopy theory [7, 12, 17, 40, 46, 47]

Notice that the Lie bracket of the last three structures is shifted.

The motivation for the present article is to investigate whether there are other combinations of the Three Graces via such a distributive law, beyond the well-known examples. It turned out that there are, up to isomorphism, only the classical, well-known distributive laws, plus the trivial and truncated ones. Since classifying distributive laws amounts to solving hundreds of quadratic equations, we found it fascinating that for the Three Graces this huge system has only a small finite number of solutions. This kind of rigidity which the Three Graces possess might be another reason why they are more equal than others. Although the results of this article might not surprise everyone, we thought that at some point of the history of mankind this analysis had to be made[3].

The existence of this paper was greatly facilitated by advances in computer-assisted mathematics, and in particular the computer algebra system Maple; worksheets written by the first author expressly for this project were used to extend hand calculations of the second author dating from some 20 years ago.

In Section 2 we recall Jon Beck's definition of distributive laws [6] along with its operadic translation [16, 36]. In the subsequent sections we classify all homogeneous operadic distributive laws between the Three Graces. The last section classifies distributive laws between associative and magmatic multiplications. It points to the fact that, while outside the realm of the Three Graces various bizarre-looking distributive laws exist, they may turn out to be isomorphic to the expected ones. Classifying all possible distributive laws is difficult, but to verify whether a given formula induces a distributive law is relatively simple. We did so by hand in Sections 3 and 7, believing it might elucidate the meaning of coherence of distributive laws.

Let us close this introduction by formulating the following

Problem. Characterize pairs of operads for which there exists only a finite number of non-isomorphic distributive laws between them.

Any two of the Three Graces form such a pair as does, according to Section 7, also the pair of operads for associative and magmatic multiplications. In a sequel to this paper we intend to perform a similar analysis for bialgebras.

Acknowledgment. We are indebted to Vladimir Dotsenko for explaining to us that the Eulerian substitution (13) brings one of our bizarre distributive laws to the standard truncated one. Also the suggestions of an anonymous referee were very helpful.

[3]In the context of the present paper we found it interesting that, according to [19, 20], one of the Three Graces — the operad $\mathcal{L}ie$ — has the property that the variety of its algebras is the only variety of non-associative algebras which is locally algebraically cartesian closed.

2. Distributive laws

2.1. Background. In this section we recall basic facts about distributive laws, closely following the work of Fox and the second author [16]; see also the original paper by Beck [6] and the works of Street [48] and Lack [31]. We will assume working knowledge of operads and their various versions. Suitable references are the monographs [8, 39, 34] complemented with [35] and the original source [23]. All algebraic objects will be defined over a ground field $\Bbbk$ of characteristic 0, and the basic category will be the monoidal category of $\mathbb{Z}$-graded vector spaces with the Koszul sign rule. Loosely speaking, a distributive law relates operations of two types, in the sense that it rearranges multiple applications of these operations in such a way that operations of the first type are applied first, followed by those of the second type. Moreover, this rearrangement must be done in a way that is coherent in the categorical sense.

2.2. Example. Poisson algebras have two operations: the Lie bracket $[a, b]$, and the commutative associative multiplication $a \cdot b$, which are related by the derivation law:

$$[a \cdot b, c] = a \cdot [b, c] + [a, c] \cdot b. \tag{1}$$

On the left side we see the operation of the second type, namely $a \cdot b$, multiplied by c using the operation of the first type, while in each term on the right side we first apply the Lie bracket and then the operation of the second type. By repeated application of equation (1) regarded as a directed (left to right) rewriting rule, we may convert any monomial, involving some number of occurrences of the first and second operations, into a sum of terms where all of the Lie brackets have been applied first. Coherence means that equation (1) does not introduce any 'unexpected relations'; in other words, the free Poisson algebra generated by a vector space X is naturally isomorphic [45, Lemma 1] to the free commutative associative algebra on the free Lie algebra generated by X:

$$\mathbf{Pois}(X) \cong \mathbf{Com}(\mathbf{Lie}(X)).$$

Distributive laws are ordered: equation (1) is a distributive law of a Lie multiplication over a commutative associative multiplication; we denote this by

$$\mathcal{D} : \mathcal{L}ie(\mathcal{C}om) \rightsquigarrow \mathcal{C}om(\mathcal{L}ie).$$

2.3. Definition. *Let us recall the precise definition introduced by Beck [6]. Assume that $T_1 = (T_1, \mu_1, \eta_1)$ and $T_2 = (T_2, \mu_2, \eta_2)$ are monads (formerly called triples) on a category $\mathtt{C}$. A distributive law guarantees that for every T_2-algebra A in $\mathtt{C}$, the object $T_2(A) \in \mathtt{C}$ has the structure of a T_1-algebra in a very explicit way. More precisely, a distributive law is a natural transformation*

$$\lambda : T_1T_2 \to T_2T_1, \tag{2}$$

such that, for every T_2-algebra $A = (A, \alpha \colon T_2(A) \to A)$, the object $T_2(A) \in \mathtt{C}$ is a T_1-algebra with structure morphism

$$T_1T_2A \xrightarrow{\lambda} T_2T_1A \xrightarrow{T_2\alpha} T_2A.$$

This imposes certain conditions on λ whose explicit form can be found in [6]; see also [16, §3]. In this situation, the endofunctor $T = T_2T_1$ is again a monad, with structure transformations

$$\mu = T_2\mu_1 \circ \mu_2T_1^2 \circ T_2\lambda T_1, \qquad \eta = \eta_1 \circ \eta_2 \circ T_1.$$

The equality

$$T(X) = T_2\big(T_1(X)\big), \qquad X \in \mathtt{C},$$

may be interpreted as saying that the free T-algebra on X is (as an object of $\mathtt{C}$) naturally isomorphic to the free T_2-algebra generated by the free T_1-algebra on X.

2.4. Example. We know one example of a distributive law from elementary school. If $\mathtt{C}$ is the category of sets, T_1 the commutative monoid monad, and T_2 the abelian group monad, then the equation $x(a + b) = xa + xb$ generates a natural transformation $T_1T_2 \to T_2T_1$ taking a product of sums to a sum of products. The algebras for the combined monad $T = T_2T_1$ are commutative rings.

2.5. Setting of this article. We restrict ourselves to monads given by the free $\mathcal{P}$-algebra functor for a binary quadratic finitely generated operad $\mathcal{P}$. Moreover, the distributive laws we consider will be given by very specific data. Before we give a precise definition, we need to establish some notational conventions; we write Σ_n for the symmetric group on n letters.

2.6. Notation. If E is a vector space which is also a Σ_2-module, then $\mathcal{F}(E)$ denotes the free operad generated by E placed in arity 2. For a Σ_3-submodule $R \subseteq \mathcal{F}(E)(3)$, we write $\langle\, E; R\,\rangle$ for the quotient $\mathcal{F}(E)/(R)$ of the free operad $\mathcal{F}(E)$ modulo the operad ideal (R) generated by R.

Suppose that the Σ_2-module E has an invariant decomposition $E = E_1 \oplus E_2$. This induces the decomposition

$$\mathcal{F}(E)(3) = \mathcal{F}(E)(3)_{11} \oplus \mathcal{F}(E)(3)_{12} \oplus \mathcal{F}(E)(3)_{21} \oplus \mathcal{F}(E)(3)_{22},$$

where $\mathcal{F}(E)(3)_{ij}$ is the Σ_3-invariant subspace of $\mathcal{F}(E)(3)$ generated by the compositions of the form $\mu(1, \nu)$ and $\mu(\nu, 1)$ with $\mu \in E_i$ and $\nu \in E_j$ for $i, j = 1, 2$. Notice that $\mathcal{F}(E)(3)_{ii}$ can be identified with the image of the map $F(E_i)(3) \to \mathcal{F}(E)(3)$ induced by the inclusion $E_i \subseteq E$. Let us consider a Σ_3-invariant map

$$\mathcal{D}\colon \mathcal{F}(E)(3)_{12} \longrightarrow \mathcal{F}(E)(3)_{21}. \tag{3}$$

Every such map defines a Σ_3-submodule $R_{\mathcal{D}} \subseteq \mathcal{F}(E)(3)$ generated by elements of the form $x - \mathcal{D}(x)$ for $x \in \mathcal{F}(E)(3)_{12}$.

Let $\mathcal{P} = \langle\, E; R\,\rangle$ be a binary quadratic operad for which there exists a Σ_2-module decomposition $E = E_1 \oplus E_2$, a Σ_3-equivariant linear map $\mathcal{D}\colon \mathcal{F}(E)(3)_{12} \to \mathcal{F}(E)(3)_{21}$, and Σ_3-invariant subsets $R_i \subseteq \mathcal{F}(E)(3)_{ii}$, $i = 1, 2$, such that $R = R_1 \oplus R_{\mathcal{D}} \oplus R_2$. In other words, the operad $\mathcal{P}$ has the presentation

$$\mathcal{P} = \langle\, E_1 \oplus E_2; R_1 \oplus R_{\mathcal{D}} \oplus R_2\,\rangle. \tag{4}$$

We consider the suboperads $\mathcal{P}_i = \langle E_i; R_i \rangle \subseteq \mathcal{P}$ for $i = 1, 2$. For $1 \le s \le l \le n$, and a sequence $m_1, \dots, m_l \ge 1$ with $m_1 + \cdots + m_l = n$, we write $\mathcal{P}(n)_l$ for the Σ_n-submodule of the free product[4] $\mathcal{P}_2 * \mathcal{P}_1$ generated by the elements of the form $\mu(\nu_1, \dots, \nu_l)$ for $\mu \in \mathcal{P}_2(l)$ and $\nu_s \in \mathcal{P}_1(m_s)$. The inclusions $\mathcal{P}_i \subseteq \mathcal{P}$ $(i = 1, 2)$ induce, for any $n \ge 2$, an equivariant linear map

$$\xi(n) \colon \bigoplus_{1 \le l \le n} \mathcal{P}(n)_l \longrightarrow \mathcal{P}(n).$$

2.7. Definition. *We say that the map $\mathcal{D}$ of equation (3) is an (operadic homogeneous binary quadratic) distributive law of $\mathcal{P}_1$ over $\mathcal{P}_2$ if the map $\xi(n)$ is an isomorphism for every $n \ge 2$. We express this fact by writing $\mathcal{D} \colon \mathcal{P}_1(\mathcal{P}_2) \rightsquigarrow \mathcal{P}_2(\mathcal{P}_1)$.*

We denote by T_i $(i = 1, 2)$ the free $\mathcal{P}_i$-operad monad acting on the category of Σ-modules. From [36, Proposition 2.6] we know that a distributive law in the sense of Definition 2.7 determines, in a very explicit way, a distributive law (2) in the sense of Beck, namely $\lambda \colon T_1T_2 \to T_2T_1$, for which the combined monad $T = T_2T_1$ is the monad for $\mathcal{P}$-algebras. Of course, not all distributive laws in the sense of Beck are distributive laws in the sense of Definition 2.7: see Example 2.4, which is not even 'operadic' since x appears twice in the right hand side.

2.8. Remark. One sometimes says more precisely that the map in (3) satisfying the condition of Definition 2.7 is a *rewriting rule* defining a distributive law between the associated monads. Rewriting rules are often conveniently expressed in the form of an equation such as (1) whose left hand side belongs to $\mathcal{F}(E)(3)_{12}$ and right hand side to $\mathcal{F}(E)(3)_{21}$.

The adjectives *binary quadratic* in Definition 2.7 mean that the distributive law involves binary quadratic operads and is therefore determined by its behavior inside $\mathcal{F}(E)(3)$; from this it follows that the resulting operad (4) is again binary quadratic. Quadratic operads have their Koszul duals, and therefore we have the following result.

2.9. Lemma. [16, Lemma 9.3] *In the situation of Definition 2.7 one has the following canonical dual binary quadratic homogeneous distributive law of $\mathcal{P}_2^!$ over $\mathcal{P}_1^!$,*

$$\mathcal{D}^! \colon \mathcal{P}_2^!(\mathcal{P}_1^!) \rightsquigarrow \mathcal{P}_1^!(\mathcal{P}_2^!),$$

such that the resulting combined operad is the Koszul dual of the operad (4).

The adjective *homogeneous* in Definition 2.7 means that the distributive law preserves the bigrading of the free operad $\mathcal{F}(E_1 \oplus E_2)$ given by the number of operations first from E_1 and then from E_2. Therefore the resulting combined binary quadratic operad (4) is also bigraded, and hence free $\mathcal{P}$-algebras are also bigraded. As a consequence, the operadic cohomology of $\mathcal{P}$-algebras can be calculated as the cohomology of a bicomplex combining $\mathcal{P}_1$- and $\mathcal{P}_2$-cochains; see [16, Theorem 10.2].

[4] I.e. of the coproduct in the category of unital operads.

2.10. Example. An 'archetypal' distributive law in the sense of Definition 2.7 is equation (1) which combines Lie and commutative associative algebras into Poisson algebras. A particular *inhomogeneous* binary quadratic operadic distributive law is that which describes associative algebras as algebras with two operations, a commutative nonassociative multiplication $- \cdot -$ and a Lie bracket $[-,-]$, with the relations

$$[x, y \cdot z] = [x, y] \cdot z + y \cdot [x, z], \qquad [y, [x, z]] = (x \cdot y) \cdot z - x \cdot (y \cdot z).$$

This law is indeed not homogeneous, since on the left side of the second equation we see a term of bidegree $(0, 2)$, i.e., with no instance of the multiplication $- \cdot -$ but two instances of $[-,-]$, while the terms on the right hand side are of bidegree $(2, 0)$.

In general, defining a transformation λ as in equation (2), and verifying that it is indeed a distributive law, is a difficult problem; however, *operadic* distributive laws are determined by a very small set of data of essentially finitary nature. Moreover, verifying the required property (Definition 2.7) boils down to a finite calculation.

2.11. Theorem. [36, Theorem 2.3] *The map $\xi(n)$ is an isomorphism for all $n \geq 2$ if and only if it is an isomorphism for $n = 4$.*

The image of the map $\xi(4)$ is spanned by elements of $\mathcal{P}(4)$ that can be written as $\mu(\nu_1, \dots, \nu_l)$ for some $\mu \in \mathcal{P}_2(l)$ and $\nu_s \in \mathcal{P}_1(m_s)$, $m_1, \dots, m_l \geq 1$, $m_1 + \cdots + m_l = 4$, where the composition now very crucially takes place inside $\mathcal{P}$. Since, by iterated applications of the distributive law, each element of $\mathcal{P}(4)$ can be brought to that form, $\xi(4)$ is always an epimorphism of finite-dimensional spaces. The condition of Theorem 2.11 is therefore equivalent to the equality

$$\dim \bigoplus_{1 \leq l \leq 4} \mathcal{P}(4)_l = \dim \mathcal{P}(4).$$

The map $\xi(4)$ need not be a monomorphism, since the distributive law can be applied in several different ways, leading to potentially different preimages of a given element of $\mathcal{P}(4)$. Let us illustrate it on an explicit example.

We assume for simplicity that our operads are non-Σ and concentrated in degree zero; the general case has the same essential features. Let $\{a_s\}$ resp. $\{b_t\}$ be a basis of the vector space E_1 resp. E_2, where s and t run over some finite sets. Then the set $\{a_s(b_t, 1), a_s(1, b_t)\}$ forms a basis of $\mathcal{F}(E)(3)_{12}$, and $\{b_t(a_s, 1), b_t(1, a_s)\}$ a basis of $\mathcal{F}(E)(3)_{21}$. The map in (3) is thus determined by a finite set of parameters $\{A^{uv}_{st}, B^{uv}_{st}, C^{uv}_{st}, D^{uv}_{st}\}$ in the ground field $\Bbbk$, via the equations

$$\mathcal{D}(a_s(b_t, 1)) = A^{uv}_{st} b_u(a_v, 1) + B^{uv}_{st} b_u(1, a_v),$$
$$\mathcal{D}(a_s(1, b_t)) = C^{uv}_{st} b_u(a_v, 1) + D^{uv}_{st} b_u(1, a_v),$$

where the summation over repeated indexes is assumed. We are going to use the distributive law to bring the element $a_s(b_i, b_j) \in \mathcal{P}(4)$ into a form manifestly in the image of $\xi(4)$. We calculate

$$a_s(b_i, b_j) = a_s(b_i, 1)(1, 1, b_j) = A^{uv}_{si} b_u(a_v, 1)(1, 1, b_j) + B^{uv}_{si} b_u(1, a_v)(1, 1, b_j)$$

$$
\begin{aligned}
&= A^{uv}_{si} b_u(1, b_j)(a_v, 1, 1) + B^{uv}_{si} b_u(1, a_v(1, b_j))\\
&= A^{uv}_{si} b_u(1, b_j)(a_v, 1, 1) + B^{uv}_{si} C^{xy}_{vj} b_u(1, b_x(a_y, 1)) + B^{uv}_{si} D^{xy}_{vj} b_u(1, b_x(1, a_y))\\
&= A^{uv}_{si} b_u(1, b_j)(a_v, 1, 1) + B^{uv}_{si} C^{xy}_{vj} b_u(1, b_x)(1, a_y, 1) + B^{uv}_{si} D^{xy}_{vj} b_u(1, b_x)(1, 1, a_y).
\end{aligned}
$$

We may however do the calculation differently, namely

$$
\begin{aligned}
a_s(b_i, b_j) &= a_s(1, b_j)(b_i, 1, 1) = C^{uv}_{sj} b_u(a_v, 1)(b_i, 1, 1) + D^{uv}_{sj} b_u(1, a_v)(b_i, 1, 1)\\
&= C^{uv}_{sj} b_u(a_v(b_i, 1), 1) + D^{uv}_{sj} b_u(b_i, 1)(1, 1, a_v)\\
&= C^{uv}_{sj} A^{xy}_{vi} b_u(b_x(a_y, 1), 1) + C^{uv}_{sj} B^{xy}_{vi} b_u(b_x(1, a_y), 1) + D^{uv}_{sj} b_u(b_i, 1)(1, 1, a_v)\\
&= C^{uv}_{sj} A^{xy}_{vi} b_u(b_x, 1)(a_y, 1, 1) + C^{uv}_{sj} B^{xy}_{vi} b_u(b_x, 1)(1, a_y, 1) + D^{uv}_{sj} b_u(b_i, 1)(1, 1, a_v).
\end{aligned}
$$

Notice that all sums above are finite. Both expressions in the last lines of the displays represent the same element of $\mathcal{P}(4)$ in the image of $\xi(4)$. But they may also be interpreted inside $\mathcal{P}(4)_3$ as expressions defining elements L_{sij} resp. R_{sij} such that

$$\xi(L_{sij}) = \xi(R_{sij}).$$

Requiring that $\xi(4)$ is a monomorphism therefore needs that

$$L_{sij} = R_{sij}$$

in $\mathcal{P}(4)_3$. The last equation is obviously equivalent to a finite set of quadratic equations without constant terms indexed by a basis of $\mathcal{P}(4)_3$, for the structure parameters $\{A^{uv}_{st}, B^{uv}_{st}, C^{uv}_{st}, D^{uv}_{st}\}$.

The analysis for other types of elements of $\mathcal{P}(4)$ leads to the same type of conclusion, i.e. to a finite set of quadratic equations without constant terms in the structure parameters of the distributive law, as we will also see in the second part of the article. In particular, taking $\mathcal{D}$ to be identically zero always gives a distributive law, the *trivial* one.

The discussion in this section makes clear the prominent role played by operadic homogeneous binary quadratic distributive laws. In the rest of this article we will deal exclusively with such distributive laws, and will therefore omit the adjectives *operadic homogeneous binary quadratic* and speak simply about *distributive laws*.

2.12. Case studies. In the following sections we describe all distributive laws between the Three Graces. Since the correspondence $\mathcal{D} \longmapsto \mathcal{D}^!$ of Lemma 2.9 is clearly one-to-one, it translates the classification problem for distributive laws of the type $\mathcal{P}_1(\mathcal{P}_2) \rightsquigarrow \mathcal{P}_2(\mathcal{P}_1)$ into an equivalent problem for distributive laws of the type $\mathcal{P}^!_2(\mathcal{P}^!_1) \rightsquigarrow \mathcal{P}^!_1(\mathcal{P}^!_2)$. It therefore suffices to consider the seven cases in the first column of the following table:

type of distributive laws	the dual type
$\underline{\mathcal{A}ss}(\underline{\mathcal{A}ss}) \rightsquigarrow \underline{\mathcal{A}ss}(\underline{\mathcal{A}ss})$	the same
$\mathcal{A}ss(\mathcal{A}ss) \rightsquigarrow \mathcal{A}ss(\mathcal{A}ss)$	the same
$\mathcal{L}ie(\mathcal{A}ss) \rightsquigarrow \mathcal{A}ss(\mathcal{L}ie)$	$\mathcal{A}ss(\mathcal{C}om) \rightsquigarrow \mathcal{C}om(\mathcal{A}ss)$
$\mathcal{C}om(\mathcal{A}ss) \rightsquigarrow \mathcal{A}ss(\mathcal{C}om)$	$\mathcal{A}ss(\mathcal{L}ie) \rightsquigarrow \mathcal{L}ie(\mathcal{A}ss)$
$\mathcal{C}om(\mathcal{C}om) \rightsquigarrow \mathcal{C}om(\mathcal{C}om)$	$\mathcal{L}ie(\mathcal{L}ie) \rightsquigarrow \mathcal{L}ie(\mathcal{L}ie)$
$\mathcal{C}om(\mathcal{L}ie) \rightsquigarrow \mathcal{L}ie(\mathcal{C}om)$	the same
$\mathcal{L}ie(\mathcal{C}om) \rightsquigarrow \mathcal{C}om(\mathcal{L}ie)$	the same

3. Distributive laws $\mathcal{A}ss(\mathcal{A}ss) \rightsquigarrow \mathcal{A}ss(\mathcal{A}ss)$

In this section we describe all distributive laws of the associative operad over itself. We will analyze first the versions living in the world of nonsymmetric operads[5] where distributive laws are given by formulas without permutation of variables, and then we move to the general case. The main result, Theorem 3.6, states that there are only three non-isomorphic distributive laws — the trivial one, the truncated one, and the one for nonsymmetric Poisson algebras (see Remark 3.3 below).

3.1. Non-Σ version. In this subsection we prove:

3.2. Theorem. *The only distributive laws between two associative multiplications that do not involve permutations of variables are given by*

$$\begin{array}{llcllcl}
(a) & (x \circ y) \bullet z & = & 0, & x \bullet (y \circ z) & = & 0\\
(b) & (x \circ y) \bullet z & = & 0, & x \bullet (y \circ z) & = & (x \bullet y) \circ z\\
(c) & (x \circ y) \bullet z & = & x \circ (y \bullet z), & x \bullet (y \circ z) & = & 0\\
(d) & (x \circ y) \bullet z & = & x \circ (y \bullet z), & x \bullet (y \circ z) & = & (x \bullet y) \circ z
\end{array}$$

3.3. Remark. Distributive law (a) is the trivial one. Distributive law (d) describes structures studied by the second author in [36], where they were called 'nonsymmetric Poisson algebras'. The corresponding distributive law was written as

$$\langle x \cdot y, z\rangle = x \cdot \langle y, z\rangle, \ \langle x, y \cdot z\rangle = \langle x, y\rangle \cdot z,$$

which is indeed a nonsymmetric form of equation (1). The same structures were later called $As^{(2)}$-algebras in [54].

Proof of Theorem 3.2. To save space, we will omit in this proof the symbol $\circ$ and write $\cdot$ instead of $\bullet$. We will also omit parentheses whenever the meaning is clear. We therefore write for example $xy \cdot z$ instead of $(x \circ y) \bullet z$.

Let $\underline{\mathcal{BB}}$ be the free nonsymmetric operad generated by two binary operations denoted xy and $x \cdot y$. We use the following ordered basis for $\underline{\mathcal{BB}}(3)$ consisting of eight monomials:

$$(xy)z, \quad x(yz), \quad (x \cdot y) \cdot z, \quad x \cdot (y \cdot z), \quad xy \cdot z, \quad x \cdot yz, \quad (x \cdot y)z, \quad x(y \cdot z).$$

[5] Sometimes also called non-Σ operads.

We identify quadratic relations with row vectors of coefficients with respect to this basis. Consider the ideal $\mathbf{I} \subset \underline{\mathcal{BB}}$ generated by the subspace $R = \mathbf{I}(3) \subset \underline{\mathcal{BB}}(3)$ which is the row space of the following matrix:

$$[R] = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & 1 & c & d \end{bmatrix}$$

Row 1 expresses the associativity of xy. Row 2 expresses the associativity of $x \cdot y$. Rows 3 and 4 express two relations which may also be written as rewriting rules:

$$\begin{array}{lcl} xy \cdot z + a\,(x \cdot y)z + b\,x(y \cdot z) \equiv 0 & \text{or} & xy \cdot z \longrightarrow -a\,(x \cdot y)z - b\,x(y \cdot z), \\ x \cdot yz + c\,(x \cdot y)z + d\,x(y \cdot z) \equiv 0 & \text{or} & x \cdot yz \longrightarrow -c\,(x \cdot y)z - d\,x(y \cdot z). \end{array}$$

These rules allow us to eliminate binary trees[6] with root operation $x \cdot y$ by replacing them by linear combinations of binary trees with root operation xy. Let us denote the four relations corresponding to the four rows of $[R]$ as follows:

$$\begin{aligned} \alpha_1(x,y,z) &= (xy)z - x(yz), \\ \alpha_2(x,y,z) &= (x \cdot y) \cdot z - x \cdot (y \cdot z), \\ \beta_1(x,y,z) &= xy \cdot z + a\,(x \cdot y)z + b\,x(y \cdot z), \\ \beta_2(x,y,z) &= x \cdot yz + c\,(x \cdot y)z + d\,x(y \cdot z). \end{aligned}$$

Let $\rho(x,y,z)$ represent any of these four relations. Then $\rho(x,y,z)$ has ten cubic (arity 4) consequences, namely

$$\begin{array}{lllll} \rho(wx,y,z), & \rho(w \cdot x,y,z), & \rho(w,xy,z), & \rho(w,x \cdot y,z), & \rho(w,x,yz), \\ \rho(w,x,y \cdot z), & \rho(w,x,y)z, & \rho(w,x,y) \cdot z, & w\rho(x,y,z), & w \cdot \rho(x,y,z). \end{array} \tag{5}$$

Altogether the four relations α_1, α_2, β_1, β_2 have 40 cubic consequences which span the subspace $RR = \mathbf{I}(4) \subset \underline{\mathcal{BB}}(4)$. The subspace RR may be identified with the row space of the 40×40 matrix $[RR]$: the rows correspond to the consequences of the four quadratic relations (ordered in some convenient way), and the columns correspond to the monomial basis of $\underline{\mathcal{BB}}(4)$ ordered first by association type as follows:

$$\begin{array}{lll} ((w \star_1 x) \star_2 y) \star_3 z, & (w \star_1 (x \star_2 y)) \star_3 z, & (w \star_1 x) \star_2 (y \star_3 z), \\ w \star_1 ((x \star_2 y) \star_3 z), & w \star_1 (x \star_2 (y \star_3 z)). & \end{array} \tag{6}$$

Within each association type, the sequence $\star_1\star_2\star_3$ represents one of the eight sequences of operation symbols; we order these as follows, where the vertical line $|$ represents the operation symbol for xy:

$$\star_1\star_2\star_3 \quad = \quad |||, \quad ||\cdot, \quad |\cdot|, \quad |\cdot\cdot, \quad \cdot||, \quad \cdot|\cdot, \quad \cdot\cdot|, \quad \cdot\cdot\cdot, \tag{7}$$

$$
\begin{bmatrix}
+ & . & . & . & . & . & . & . & - & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & + & . & . & . & - & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & + & . & . & . & . & . & . & . & . & . & . & - & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & + & . & . & . & . & . & . & . & . & . & . & - & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & + & . & . & . & . & - & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & + & . & . & . & . & . & . & . & . & . & - & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
+ & - & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & + & - & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & + & - & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & + & - & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & + & . & . & . & . & - & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & + & . & . & . & . & . & . & . & . & - & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & + & . & . & . & . & . & . & . & . & . & . & - & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & + & . & . & . & . & . & . & . & . & . & . & - & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & + & . & . & . & . & . & . & - & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & + & . & . & . & - & . & . & . & . \\
. & . & + & - & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & + & - & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & + & - & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & + & - & . & . & . & . \\
. & . & . & . & a & . & . & . & . & b & . & . & . & . & . & . & . & . & . & . & + & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & a & . & . & . & . & . & . & . & . & b & . & . & . & . & . & . & . & . & . & . & . & . & . & . & + & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & a & . & . & . & . & . & . & . & . & . & . & b & . & . & . & . & + & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & a & . & . & . & . & . & . & . & . & . & . & b & . & . & . & . & . & . & . & . & . & . & . & . & + & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & a & . & . & . & . & . & . & b & . & . & . & . & . & . & . & . & . & . & + & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & a & . & . & . & b & . & . & . & . & . & . & . & . & . & . & . & . & . & + & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & + & . & a & b & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & + & . & a & b & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & + & . & a & b & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & + & . & a & b \\
. & . & . & . & c & . & . & . & . & d & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & + & . & . & . & . & . & . & . & . & . & . & . \\
. & . & c & . & . & . & . & . & . & . & . & d & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & + & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & c & . & . & . & . & . & . & . & . & . & . & d & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & + & . & . & . & . & . & . & . \\
. & . & . & c & . & . & . & . & . & . & . & . & . & . & d & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & + & . \\
. & . & . & . & . & . & . & . & . & . & c & . & . & . & . & . & . & d & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & + & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & c & . & . & . & d & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & + \\
. & . & . & . & . & + & c & d & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & + & c & d & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & + & c & d & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & + & c & d
\end{bmatrix}
$$

Figure 1: Matrix $[RR]$: cubic consequences of quadratic relations

The matrix $[RR]$ has entries in the set $\{0, 1, -1, a, b, c, d\}$ and hence may be regarded as a matrix over the polynomial ring $\Bbbk[a, b, c, d]$. This matrix is displayed in Figure 1 with dot, $+, -$ for $0, 1, -1$ respectively.

To understand how the rank of $[RR]$ depends on the parameters a, b, c, d we first use elementary row and column operations to compute a partial Smith form as described in [8, Chapter 8]. Roughly speaking, we repeatedly move entries equal to ± 1 to the upper left diagonal of the matrix, change their signs if necessary, and then use each resulting diagonal 1 to eliminate the entries below and to the right, continuing until the lower right block no longer contains a nonzero scalar. When this computation terminates, we have reduced $[RR]$ to the block-diagonal matrix $\mathrm{diag}(I_{32}, L)$, which is row-column equivalent to $[RR]$ and hence has the same rank as $[RR]$, where L is an 8×8 matrix over $\Bbbk[a, b, c, d]$ which has two zero rows and two zero columns. After deleting these superfluous rows and

---

[6] We use the standard bijection between monomials and rooted trees, cf. Remark 3.10.

columns, we obtain this 6×6 matrix:

$$L' = \begin{bmatrix} -ad & -b^2-b & a^2-ac & 0 & 0 & 0 \\ 0 & bd+d & -ac-a & 0 & 0 & 0 \\ ad & bd-d^2 & c^2+c & 0 & 0 & 0 \\ 0 & 0 & 0 & -b^2-b & -ab-a & -a^2-ab \\ 0 & 0 & 0 & -ad & 0 & ad \\ 0 & 0 & 0 & -cd-d^2 & -cd-d & -c^2-c \end{bmatrix}$$

In order for the map representing the distributive law to be an isomorphism, it is necessary and sufficient that $[RR]$ have rank 32, or equivalently that L' be the zero matrix. Consider the set G consisting of the nonzero entries of L'. We compute a Gröbner basis for the ideal $J \subset \Bbbk[a,b,c,d]$ generated by G with respect to the deglex monomial order determined by $a \prec b \prec c \prec d$. This Gröbner basis for J consists of the polynomials a, d, $b(b{+}1)$, $c(c{+}1)$. Hence J is a zero-dimensional ideal whose zero set consists of exactly four points:

$$(a,b,c,d) \;=\; (0,0,0,0), \quad (0,0,-1,0), \quad (0,-1,0,0), \quad (0,-1,-1,0).$$

These solutions correspond to the following pairs of rewriting rules

$$\begin{array}{llll} (a) & xy\cdot z \longrightarrow 0, & x\cdot yz \longrightarrow 0 \\ (b) & xy\cdot z \longrightarrow 0, & x\cdot yz \longrightarrow (x\cdot y)z \\ (c) & xy\cdot z \longrightarrow x(y\cdot z), & x\cdot yz \longrightarrow 0 \\ (d) & xy\cdot z \longrightarrow x(y\cdot z), & x\cdot yz \longrightarrow (x\cdot y)z \end{array}$$

which give the four nonsymmetric laws $\underline{\mathcal{A}ss}(\underline{\mathcal{A}ss}) \rightsquigarrow \underline{\mathcal{A}ss}(\underline{\mathcal{A}ss})$ of Theorem 3.2. ■

3.4. General version. In this subsection we generalize Theorem 3.2 by allowing permutations of variables:

3.5. Theorem. *The only distributive laws between two associative multiplications are the four laws of Theorem 3.2 together with the following three:*

$$\begin{array}{lll} (e) & (x\circ y)\bullet z \;=\; 0, & x\bullet(y\circ z) \;=\; y\circ(x\bullet z) \\ (f) & (x\circ y)\bullet z \;=\; (x\bullet z)\circ y, & x\bullet(y\circ z) \;=\; 0 \\ (g) & (x\circ y)\bullet z \;=\; (x\bullet z)\circ y, & x\bullet(y\circ z) \;=\; y\circ(x\bullet z). \end{array}$$

The proof is postponed to the end of this subsection. We note that the rewriting rule $(x\circ y)\bullet z \;=\; (x\bullet z)\circ y$ states that the right multiplications $-\circ y$ and $-\bullet z$ commute; similarly, $x\bullet(y\circ z) \;=\; y\circ(x\bullet z)$ states that the left multiplications $y\circ -$ and $x\bullet -$ commute.

Let us denote by $\mathcal{A}_a, \ldots \mathcal{A}_g$ the operads defined by distributive laws (a)–(g) of Theorems 3.2 and 3.5 (in the given order). It turns out that these operads fall into three isomorphism classes: $\{\mathcal{A}_a\}$, $\{\mathcal{A}_b, \mathcal{A}_c, \mathcal{A}_e, \mathcal{A}_f\}$, and $\{\mathcal{A}_d, \mathcal{A}_g\}$. The corresponding isomorphisms are given by changing one or both multiplications into the opposite, that is $\circ \mapsto \circ^{\mathrm{op}}$

and/or $\bullet \mapsto \bullet^{\rm op}$. It is easy to verify that one gets the following isomorphism diagrams:

$$
\begin{tikzcd}
\mathcal{A}_b \arrow[r, "\circ\mapsto\circ^{\rm op}", "\cong"'] \arrow[d, "\bullet\mapsto\bullet^{\rm op}"', "\cong"] & \mathcal{A}_e \arrow[d, "\cong"', "\bullet\mapsto\bullet^{\rm op}"] \\
\mathcal{A}_f \arrow[r, "\circ\mapsto\circ^{\rm op}", "\cong"'] & \mathcal{A}_c
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{A}_d \arrow[r, "\circ\mapsto\circ^{\rm op}", "\cong"'] \arrow[d, "\bullet\mapsto\bullet^{\rm op}"', "\cong"] & \mathcal{A}_g \arrow[d, "\cong"', "\bullet\mapsto\bullet^{\rm op}"] \\
\mathcal{A}_g \arrow[r, "\circ\mapsto\circ^{\rm op}", "\cong"'] & \mathcal{A}_d.
\end{tikzcd}
$$

One therefore has:

3.6. Theorem. *There are precisely three isomorphism classes of distributive laws between two associative multiplications, namely*

- *the trivial law (a),*
- *the truncated law represented by rewriting rules (b), (c), (e) or (f), and*
- *the law for nonsymmetric Poisson algebras represented by (d) or (g).*

3.7. Remark. Note that the operads $\mathcal{A}_a$, $\mathcal{A}_b$, $\mathcal{A}_c$, and $\mathcal{A}_d$ defined by distributive laws (a)–(d) of Theorem 3.2 are mutually nonisomorphic in the category of non-Σ operads. Therefore in the category of algebras over nonsymmetric operads there are four different distributive laws between two associative multiplications.

Theorem 3.6 has the following simple but very interesting consequence:

3.8. Corollary. *Up to isomorphism, the only distributive law between two associative multiplications in the monoidal category of sets is that of nonsymmetric Poisson algebras.*

3.9. Example. Let us verify 'by hand' that (e) indeed determines a distributive law. We must check that it is compatible with the associativity of $\bullet$ and $\circ$. We also need to check that the result of repeated applications of (e) does not depend on their order. Theorem 2.11 tells us that it suffices to consider only expressions involving four variables.

Compatibility with the associativity of $\circ$. The associativity of $\circ$ means that

$$\big((y \circ z) \circ w\big) = \big(y \circ (z \circ w)\big),$$

for arbitrary symbols y, z, w. Thus, for a symbol x, one has

$$x \bullet \big((y \circ z) \circ w\big) = x \bullet \big(y \circ (z \circ w)\big). \tag{8}$$

The compatibility with associativity means that both sides of this equation remain equal after we apply, possibly repeatedly, rule (e) to them. For the left side of (8) we get

$$x \bullet \big((y \circ z) \circ w\big) = (y \circ z) \circ (x \bullet w),$$

while the right hand side becomes

$$x \bullet \big(y \circ (z \circ w)\big) = y \circ \big(x \bullet (z \circ w)\big) = y \circ \big(z \circ (x \bullet w)\big).$$

So we need to check whether

$$(y \circ z) \circ (x \bullet w) = y \circ \big(z \circ (x \bullet w)\big).$$

This equality follows from the associativity of $\circ$. We need to do the same analysis for

$$\big((x \circ y) \circ z\big) \bullet w = \big(x \circ (y \circ z)\big) \bullet w.$$

In this case (e) turns both sides into 0.

Compatibility with the associativity of $\bullet$. We need to consider three equations implied by the associativity of $\bullet$. The first one is

$$(x \bullet y) \bullet (z \circ w) = x \bullet \big(y \bullet (z \circ w)\big).$$

Modifying the left hand side using (e) gives

$$(x \bullet y) \bullet (z \circ w) = z \circ \big((x \bullet y) \bullet w\big),$$

while for the right hand side we obtain

$$x \bullet \big(y \bullet (z \circ w)\big) = x \bullet \big(z \circ (y \bullet w)\big) = z \circ \big(x \bullet (y \bullet w)\big).$$

However thanks to the associativity of $\bullet$ we have

$$z \circ \big((x \bullet y) \bullet w\big) = z \circ \big(x \bullet (y \bullet w)\big).$$

The next equation to analyze is

$$\big(x \bullet (y \circ z)\big) \bullet w = x \bullet \big((y \circ z) \bullet w\big).$$

The left side expands as

$$\big(x \bullet (y \circ z)\big) \bullet w = \big(y \circ (x \bullet z)\big) \bullet w = 0,$$

while the right side is seen to be zero immediately. The last equation to be considered is

$$\big((x \circ y) \bullet z\big) \bullet w = (x \circ y) \bullet (z \bullet w).$$

But applying (e) turns both sides immediately to zero.

Independence of order. All expressions featured above offered at most one way to apply (e). This is not true for

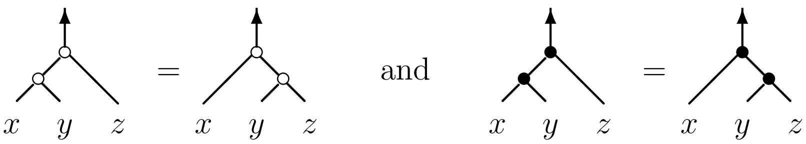
$$(x \circ y) \bullet (z \circ w).$$

Applying the first rule of (e) first, with $(z \circ w)$ instead of z, turns it into zero, while applying the second rule of (e) first we get

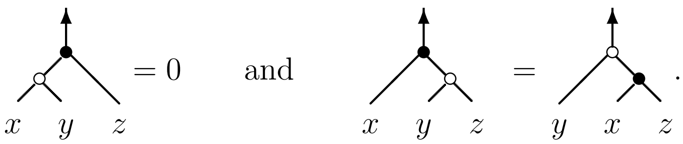
$$(x \circ y) \bullet (z \circ w) = z \circ \big((x \circ y) \bullet w\big),$$

which is zero again, by the first rule of (e). It is not difficult to see that the above finite number of cases was all we needed to check, and thus the verification that (e) defines a distributive law is finished.

3.10. Remark. The above calculations can be visualized by labelled planar rooted trees. Representing the $\circ$-multiplication by a white vertex with two inputs and one output, and the $\bullet$-multiplication by a similar black vertex, the associativity of $\circ$ and $\bullet$ can be depicted as



while rule (e) reads



A pictorial verification of the compatibility of rule (e) with equation (8) is shown in Figure 2; the remaining (and in fact easier) cases can be verified similarly.

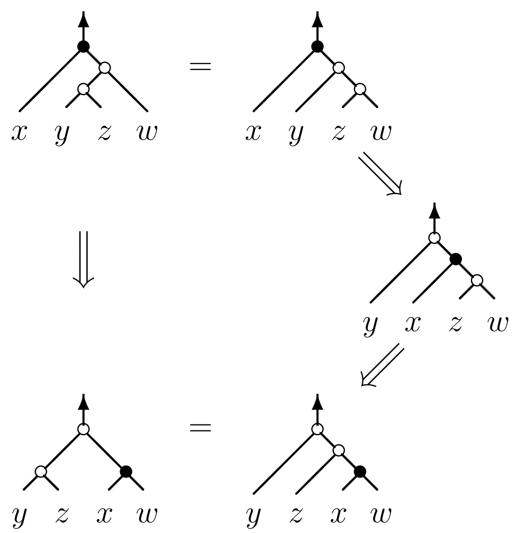


Figure 2: Tree diagrams for compatibility proof

Proof of Theorem 3.5. We use the same conventions regarding the notation for the $\circ$ and $\bullet$ products as in the proof of Theorem 3.2. The method for the symmetric case is essentially the same as for the nonsymmetric case, although the matrices and the number of parameters are six times larger. Let $\mathcal{BB}$ be the free symmetric operad generated by two binary operations denoted xy and $x \cdot y$. We use the following ordered basis for $\mathcal{BB}(3)$ consisting of 48 monomials:

$$
\begin{array}{llll}
(x^\sigma y^\sigma)z^\sigma, & x^\sigma(y^\sigma z^\sigma), & (x^\sigma \cdot y^\sigma) \cdot z^\sigma, & x^\sigma \cdot (y^\sigma \cdot z^\sigma), \\
x^\sigma y^\sigma \cdot z^\sigma, & x^\sigma \cdot y^\sigma z^\sigma, & (x^\sigma \cdot y^\sigma) z^\sigma, & x^\sigma (y^\sigma \cdot z^\sigma).
\end{array}
$$

The permutations $\sigma \in S_3$ permuting the arguments x, y, z (not the positions) are in lexicographical order. We identify quadratic relations with row vectors of coefficients with respect to this basis. Consider the ideal $\mathbf{I} \subset \mathcal{BB}$ generated by the subspace $R = \mathbf{I}(3)$ which is the row space of the following block matrix:

$$[R] = \begin{bmatrix} I_6 & -I_6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & I_6 & -I_6 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & I_6 & \cdot & A & B \\ \cdot & \cdot & \cdot & \cdot & \cdot & I_6 & C & D \end{bmatrix} \tag{9}$$

We write I_6 and dot for the 6×6 identity and zero matrices, together with

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_2 & a_1 & a_5 & a_6 & a_3 & a_4 \\ a_3 & a_4 & a_1 & a_2 & a_6 & a_5 \\ a_5 & a_6 & a_2 & a_1 & a_4 & a_3 \\ a_4 & a_3 & a_6 & a_5 & a_1 & a_2 \\ a_6 & a_5 & a_4 & a_3 & a_2 & a_1 \end{bmatrix}, \tag{10}$$

and similarly for B, C and D. Thus $[R]$ contains 24 parameters. We point out that rows 1, 7, 13, 19 generate the row space of $[R]$ as an S_3-module: rows 1 and 7 represent associativity for operations xy and $x \cdot y$; rows 13 and 19 represent the rewriting rules which show how to express a binary tree with operation $x \cdot y$ at the root as a linear combination of binary trees with operation xy at the root:

$$\begin{aligned} & xy \cdot z + a_1(x \cdot y)z + a_2(x \cdot z)y + a_3(y \cdot x)z + a_4(y \cdot z)x + a_5(z \cdot x)y + a_6(z \cdot y)x \\ & \quad + b_1 x(y \cdot z) + b_2 x(z \cdot y) + b_3 y(x \cdot z) + b_4 y(z \cdot x) + b_5 z(x \cdot y) + b_6 z(y \cdot x) \equiv 0, \\ & x \cdot yz + c_1(x \cdot y)z + c_2(x \cdot z)y + c_3(y \cdot x)z + c_4(y \cdot z)x + c_5(z \cdot x)y + c_6(z \cdot y)x \\ & \quad + d_1 x(y \cdot z) + d_2 x(z \cdot y) + d_3 y(x \cdot z) + d_4 y(z \cdot x) + d_5 z(x \cdot y) + d_6 z(y \cdot x) \equiv 0. \end{aligned}$$

Let $\rho(x, y, z)$ be the relation represented by one of the rows 1, 7, 13, 19. Each of these four relations has ten cubic consequences as in equation (5), for a total of 40 relations which generate the S_4-module $RR = \mathbf{I}(4) \subset \mathcal{BB}(4)$. Each of these 40 relations has 24 permutations, for a total of 960 relations which span RR as a subspace of $\mathcal{BB}(4)$. If we apply the 24 permutations of w, x, y, z to the 40 nonsymmetric monomials in equations (6)-(7) then we obtain 960 monomials which form an ordered basis of $\mathcal{BB}(4)$. Thus we can represent RR as the row space of a 960×960 matrix $[RR]$ whose entries belong to

$$\{0, \pm 1\} \cup X, \qquad \text{where} \qquad X = \{a_k, b_k, c_k, d_k \mid 1 \leq k \leq 6\}.$$

Thus $[RR]$ may be regarded as a matrix over the polynomial ring $\Bbbk[X]$ with 24 variables. As in the nonsymmetric case, we compute a partial Smith form for $[RR]$ and obtain a block diagonal matrix $\operatorname{diag}(I_{768}, L)$ where L has size 192×192 and contains no nonzero scalar entries. The set of nonzero entries of L contains 575 polynomials, all of which have total degree 1 or 2 in the variables X. From this large set of ideal generators we obtain a deglex Gröbner basis of only 28 polynomials:

$$
\begin{gathered}
a_1,\ \ a_3,\ \ a_4,\ \ a_5,\ \ a_6,\ \ b_2,\ \ b_3,\ \ b_4,\ \ b_5,\ \ b_6,\ \ c_2,\ \ c_3,\ \ c_4,\ \ c_5,\ \ c_6,\ \ d_1,\ \ d_2,\ \ d_4,\ \ d_5,\ \ d_6,\\
a_2^2+a_2,\ \ a_2b_1,\ \ a_2c_1,\ \ b_1^2+b_1,\ \ b_1d_3,\ \ c_1^2+c_1,\ \ c_1d_3,\ \ d_3^2+d_3.
\end{gathered}
$$

From this we easily determine that the ideal is zero-dimensional and that its zero set consists of the following seven points:

	a_1	a_2	a_3	a_4	a_5	a_6	b_1	b_2	b_3	b_4	b_5	b_6	c_1	c_2	c_3	c_4	c_5	c_6	d_1	d_2	d_3	d_4	d_5	d_6
1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
2	·	·	·	·	·	·	·	·	·	·	·	·	-1	·	·	·	·	·	·	·	·	·	·	·
3	·	·	·	·	·	·	-1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
4	·	·	·	·	·	·	-1	·	·	·	·	·	-1	·	·	·	·	·	·	·	·	·	·	·
5	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	-1	·	·	·
6	·	-1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
7	·	-1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	-1	·	·	·

For points 1–4, the matrices A, B, C, D from equations (9)-(10) are as follows:

$$
\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -I_6 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -I_6 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -I_6 \\ -I_6 & 0 \end{bmatrix}.
$$

The corresponding distributive laws are simply the symmetrizations of the four laws from the nonsymmetric case. Points (5)-(7) give new symmetric distributive laws which have no analogue in the nonsymmetric case. Consider these (negative) permutation matrices:

$$
P = \begin{bmatrix} \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \end{bmatrix}, \qquad Q = \begin{bmatrix} \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \end{bmatrix}.
$$

Then points 5–7 correspond to

$$
\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix}, \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}.
$$

These solutions correspond respectively to (all permutations of) these rewriting rules:

$$
\begin{array}{lll}
5: & xy \cdot z \longrightarrow 0, & x \cdot yz \longrightarrow y(x \cdot z) \\
6: & xy \cdot z \longrightarrow (x \cdot z)y, & x \cdot yz \longrightarrow 0 \\
7: & xy \cdot z \longrightarrow (x \cdot z)y, & x \cdot yz \longrightarrow y(x \cdot z).
\end{array}
$$

These are the three remaining distributive laws of Theorem 3.5. ∎

4. Distributive laws $\mathcal{C}om(\mathcal{A}ss) \rightsquigarrow \mathcal{A}ss(\mathcal{C}om)$

4.1. Theorem. *The only distributive law $\mathcal{C}om(\mathcal{A}ss) \rightsquigarrow \mathcal{A}ss(\mathcal{C}om)$ is the trivial one.*

Proof. We write ab for the associative operation, and $a \cdot b$ for the commutative associative operation. Commutativity implies that we need to consider only six association types in arity 3, which we order as follows:

$$* \cdot * \cdot * = (* \cdot *) \cdot *, \qquad (**) \cdot *, \qquad (* \cdot *)*, \qquad *** = (**)*, \qquad *(* \cdot *), \qquad *(**).$$

Similarly, we need consider only 25 association types in arity 4; in the following ordered list we include all the parentheses:

$$\begin{array}{lllll}
((* \cdot *) \cdot *) \cdot *, & ((**) \cdot *) \cdot *, & ((* \cdot *)*) \cdot *, & ((**)*) \cdot *, & (*(* \cdot *)) \cdot *, \\
(*(**)) \cdot *, & (* \cdot *) \cdot (* \cdot *), & (* \cdot *) \cdot (**), & (**) \cdot (**), & ((* \cdot *) \cdot *)*, \\
((**) \cdot *)*, & ((* \cdot *)*)*, & ((**)*)*, & (*(* \cdot *))*, & (*(**))*, \\
(* \cdot *)(* \cdot *), & (* \cdot *)(**), & (**)(* \cdot *), & (**)(**), & *((* \cdot *) \cdot *), \\
*((**) \cdot *), & *((* \cdot *)*), & *((**)*), & *(*(* \cdot *)), & *(*(**)).
\end{array}$$

The number of distinct association types for a sequence of n arguments with two associative binary operations, one commutative and one noncommutative, is sequence A276277 in the Online Encyclopedia of Integer Sequences (oeis.org):

$$1,\ 2,\ 6,\ 25,\ 111,\ 540,\ 2736,\ 14396,\ 77649,\ 427608,\ 2392866,\ 13570386,\ 77815161,\ \dots$$

Applying all permutations to the arguments, and ignoring duplications which follow from commutativity, we obtain 27 distinct multilinear monomials in arity 3, ordered as follows:

$$\begin{array}{lllllllll}
(a \cdot b) \cdot c, & (a \cdot c) \cdot b, & (b \cdot c) \cdot a, & (ab) \cdot c, & (ac) \cdot b, & (ba) \cdot c, & (bc) \cdot a, & (ca) \cdot b, & (cb) \cdot a, \\
(a \cdot b)c, & (a \cdot c)b, & (b \cdot c)a, & (ab)c, & (ac)b, & (ba)c, & (bc)a, & (ca)b, & (cb)a, \\
a(b \cdot c), & b(a \cdot c), & c(a \cdot b), & a(bc), & a(cb), & b(ac), & b(ca), & c(ab), & c(ba).
\end{array}$$

Similarly, we obtain 405 distinct multilinear monomials of arity 4. The number of distinct multilinear monomials with two associative binary operations, one commutative and one noncommutative, is the sextuple factorial, sequence A011781 in the OEIS:

$$\prod_{k=0}^{n-1} (6k+3) = 1,\ 3,\ 27,\ 405,\ 8505,\ 229635,\ 7577955,\ 295540245,\ 13299311025,\ \dots$$

Figure 3 displays the matrix whose row space is the S_3-submodule generated by three quadratic relations: associativity for ab, associativity for $a \cdot b$, and the relation expressing the reduction of a monomial of the form $(ab) \cdot c$ to a linear combination of permutations of the monomial $(a \cdot b)c$.

The S_4-module generated by the consequences of the three quadratic relations has size 540×405. Its partial Smith form consists of an identity matrix of size 330 and a lower

$$
\left[\begin{array}{ccccccccccccccccccccccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & - & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & - & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & - & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & - & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & - & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & - \\
\hline
+ & \cdot & - & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & - & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
- & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & a_1 & a_2 & a_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_1 & b_2 & b_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & a_2 & a_1 & a_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_1 & b_3 & b_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & a_1 & a_3 & a_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_2 & b_1 & b_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & a_2 & a_3 & a_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_3 & b_1 & b_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & a_3 & a_1 & a_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_2 & b_3 & b_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & a_3 & a_2 & a_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_3 & b_2 & b_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right]
$$

Figure 3: Associative-commutative quadratic relation matrix

right block L of size 210×75. The matrix L contains 56 distinct nonzero polynomials of degrees 1 and 2; replacing each by its monic form gives the following 43 polynomials:

$$
\begin{gathered}
a_3,\ b_2,\ a_2^2,\ a_3^2,\ b_1^2,\ b_2^2,\ a_1b_3,\ a_1(a_2{+}1),\ a_1(a_1{+}a_2{+}a_3{+}b_1{+}b_2{+}b_3),\ a_2a_1,\ a_2a_3,\ a_2b_2, \\
a_2(a_2{+}1),\ a_2(a_3{+}b_2),\ a_3a_1,\ a_3b_1,\ a_3(a_2{+}b_1{+}1),\ b_1b_2,\ b_1b_3,\ b_1(a_3{+}b_2),\ b_1(b_1{+}1), \\
b_2b_3,\ b_2(a_2{+}b_1{+}1),\ b_3(b_1{+}1),\ b_3(a_1{+}a_2{+}a_3{+}b_1{+}b_2{+}b_3),\ a_1(a_2{-}a_1),\ a_1(a_3{-}a_1), \\
a_1(a_3{-}a_2),\ a_1(b_2{-}b_1),\ b_3(a_3{-}a_2),\ b_3(b_2{-}b_1),\ b_3(b_3{-}b_1),\ b_3(b_3{-}b_2),\ a_2b_1{+}a_3^2, \\
a_2b_1{+}b_2^2,\ a_1a_2{+}a_3b_3,\ a_1b_1{+}a_2b_3,\ a_1b_2{+}a_3b_3,\ a_1b_2{+}b_1b_3,\ a_2^2{+}a_3b_2{+}a_2,\ a_3b_2{+}b_1^2{+}b_1, \\
a_1a_3{+}a_2b_3{+}b_3,\ a_1b_1{+}b_2b_3{+}a_1.
\end{gathered}
$$

One easily verifies that the deglex Gröbner basis for the ideal generated by these polynomials consists of the six variables $a_1, a_2, a_3, b_1, b_2, b_3$ and this completes the proof. ∎

5. Distributive laws $\mathcal{L}ie(\mathcal{A}ss) \rightsquigarrow \mathcal{A}ss(\mathcal{L}ie)$

The methods in this case are very similar to the case $\mathcal{C}om(\mathcal{A}ss) \rightsquigarrow \mathcal{A}ss(\mathcal{C}om)$ except that instead of a commutative associative operation we have a Lie bracket: an anticommutative operation satisfying the Jacobi identity. This requires keeping track of sign changes that occur as a result of anticommutativity when calculating normal forms of the monomials in consequences and permutations of various quadratic and cubic relations.

5.1. THEOREM. *The only distributive law $\mathcal{L}ie(\mathcal{A}ss) \rightsquigarrow \mathcal{A}ss(\mathcal{L}ie)$ is the trivial one. By Koszul duality, the same conclusion holds for $\mathcal{A}ss(\mathcal{C}om) \rightsquigarrow \mathcal{C}om(\mathcal{A}ss)$.*

5.2. NON-EXAMPLE. One is tempted to relax the commutativity of the associative multiplication of Poisson algebras, keeping other axioms unchanged, as done e.g. in [1].

We show that in this case the derivation rule (1) does not define a distributive law $\mathcal{L}ie(\mathcal{A}ss) \rightsquigarrow \mathcal{A}ss(\mathcal{L}ie)$, so we suspect that these naïve noncommutative Poisson algebras are ill-behaved. More specifically, we show that the rule (1) is not compatible with the anticommutativity of $[-,-]$. Let us consider the equation

$$[ab, cd] = -[cd, ab]. \tag{11}$$

Expanding its left side using (1) twice gives

$$[ab, cd] = a[b, cd] + [a, cd]b = ac[b, d] + a[b, c]d + c[a, d]b + [a, c]db,$$

while the right side results in

$$\begin{aligned}-[cd, ab] = -c[d, ab] - [c, ab]d &= -ca[d, b] - c[d, a]b - a[c, b]d - [c, a]bd \\ &= ca[b, d] + c[a, d]b + a[b, c]d + [a, c]bd.\end{aligned}$$

The compatibility of (1) with (11) would require the equality

$$ac[b, d] + c[a, d]b + a[b, c]d + [a, c]db = ca[b, d] + c[a, d]b + a[b, c]d + [a, c]bd,$$

which is the same as

$$(ac - ca)[b, d] + [a, c](db - bd) = 0.$$

One however cannot expect this to be true in general unless $ac = ca$ and $db = bd$. If we denote the commutator of the associative multiplication by $\{-,-\}$ then we obtain

$$\{a, c\}[b, d] = [a, c]\{b, d\}, \tag{12}$$

which can be found e.g. in [14, Lemma 1.1], in [51, Lemma 1.1] or in [52, Theorem 1]. Theodore Voronov informed us, referring to a rare 1932 book[7] by Fok, that (12) was first obtained by Dirac, who used it to motivate his argument that in quantum mechanics, the 'quantum Poisson bracket' has to be proportional to the commutator of the operators. As shown in [14, Theorem 1.2], a similar statement holds for prime noncommutative Poisson algebras – their bracket is always proportional to the commutator of the associative multiplication.

5.3. Remark. We advise the reader that there are other structures called 'noncommutative Poisson algebras' in the literature. The structure in [29, 30] combines Leibniz and associative algebras via the derivation rule (1); it is therefore of type $\mathcal{L}ei(\mathcal{A}ss) \rightsquigarrow \mathcal{A}ss(\mathcal{L}ei)$. The structure in [11] is defined as a Poisson algebra on the abelization $A/[A, A]$ of an associative algebra A. Other generalizations include double Poisson algebras [49, 50] equipped with a 'double bracket' $A \otimes A \to A \otimes A$, or a twisted version in the physics paper [44].

Close in nature to Poisson algebras are *Gerstenhaber algebras* whose Lie bracket has an odd degree, usually $+1$ or -1, depending on the conventions. They naturally appear

[7]Cf. formula (14), page 41, of the second edition [15] of that book.

as the structure of the Hochschild cohomology of associative algebras. Noncommutative versions of Gerstenhaber algebras, their role in deformation theory and their relation to the commutative ones, were considered in [41, 42]. The Gerstenhaber analog of (12) can be found in [41, Remark 1.9]. Sergei Merkulov in the same paper conjectured that the operad $\mathcal{G}erst$ for noncommutative Gerstenhaber algebras is Koszul. As Vladimir Dotsenko informed us, the conjecture is still open, but we expect the contrary, believing that the Gerstenhaber analog of the 'unexpected' relation (12) may play the same role as the equation implied by the 'fake pentagon' [38, page 380] does for the non-Koszulity of the operad for anti-associative algebras.

6. The remaining cases

In this section we analyze the remaining three types of distributive laws between the Three Graces.

6.1. Theorem. *For $\mathcal{C}om(\mathcal{C}om) \rightsquigarrow \mathcal{C}om(\mathcal{C}om)$ we obtain only the trivial distributive law.*

Proof. The calculations are similar to those discussed in detail in previous sections, so we provide only a brief outline. The number of distinct association types in arity n for two commutative operations is sequence OEIS A226909; see also [10]:

$$1,\ 2,\ 4,\ 14,\ 44,\ 164,\ 616,\ 2450,\ 9908,\ 41116,\ 173144,\ 739884,\ 3196344,\ 13944200,\ \dots.$$

For arities 3 and 4, these types are as follows:

$$\begin{array}{l}
(**)*,\quad (*\cdot *)*,\quad (**)\cdot *,\quad (*\cdot *)\cdot *;\\
((**)*)*,\quad ((*\cdot *)*)*,\quad ((**)\cdot *)*,\quad ((*\cdot *)\cdot *)*,\quad (**)(**),\\
(**)(*\cdot *),\quad (*\cdot *)(*\cdot *),\quad ((**)*)\cdot *,\quad ((*\cdot *)*)\cdot *,\quad ((**)\cdot *)\cdot *,\\
((*\cdot *)\cdot *)\cdot *,\quad (**)\cdot (**),\quad (**)\cdot (*\cdot *),\quad (*\cdot *)\cdot (*\cdot *).
\end{array}$$

The number of distinct multilinear monomials is the quadruple factorial (OEIS A001813):

$$\frac{(2n)!}{n!} = 1,\ 2,\ 12,\ 120,\ 1680,\ 30240,\ 665280,\ 17297280,\ 518918400,\ 17643225600,\ \dots.$$

For arity 3, these monomials are as follows (in lex order):

$(ab)c$, $(ac)b$, $(bc)a$, $(a{\cdot}b)c$, $(a{\cdot}c)b$, $(b{\cdot}c)a$, $(ab){\cdot}c$, $(ac){\cdot}b$, $(bc){\cdot}a$, $(a{\cdot}b){\cdot}c$, $(a{\cdot}c){\cdot}b$, $(b{\cdot}c){\cdot}a$.

Using these monomials, associativity has the form

$$(ab)c-(bc)a, \qquad\qquad (a\cdot b)\cdot c-(b\cdot c)\cdot a.$$

The most general distributive law relating the operations is as follows, where x_1, x_2, x_3 are free parameters:

$$x_1(ab)\cdot c + x_2(ac)\cdot b + x_3(bc)\cdot a - (a\cdot b)c.$$

Applying all permutations of the variables a, b, c to these three relations, and expressing the relations as row vectors of coefficients, we obtain this matrix:

$$\begin{bmatrix} 1 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot \\ \cdot & \cdot & \cdot & x_1 & x_2 & x_3 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & x_2 & x_3 & x_1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & x_3 & x_1 & x_2 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

We compute the consequences in arity 4 of these nine relations I in arity 3. If we write ω_1, ω_2 for the two operations then for each I we obtain $I \circ_k \omega_j$ $(k = 1, 2, 3;\ j = 1, 2)$ and $\omega_j \circ_k I$ $(j, k = 1, 2)$ where $\circ_k$ denotes operadic partial composition. Each term of each consequence must be straightened using commutativity to convert the underlying monomial to one of the 120 normal forms in arity 4. Each quadratic relation I produces 10 cubic consequences for a total of 30; applying all permutations of the four variables a, b, c, d we obtain altogether 360 cubic relations, which we store in a 360×120 matrix R with entries 0, 1, -1, x_1, x_2, x_3. Following [8], we compute a partial Smith form

$$\begin{bmatrix} I_{105} & 0 \\ 0 & B \end{bmatrix},$$

where the lower right block B contains the following nonzero entries:

$$\begin{aligned} &x_2^2,\ x_2x_3,\ x_3x_1,\ -x_1^2,\ -x_2^2,\ -x_2x_3,\ -x_3x_1,\ x_2 - x_3,\ x_3 - x_2,\ -x_3^2 - x_3,\ x_3^2 + x_3, \\ &-x_1x_2 - x_1,\ x_1x_2 + x_1,\ -x_3^2 - x_2,\ x_3^2 + x_2,\ -x_2x_3 - x_3,\ x_2x_3 + x_3,\ -x_2x_3 - x_2, \\ &x_2x_3 + x_2,\ -x_2x_3 + x_3^2,\ x_2x_3 - x_3^2,\ -x_2^2 + x_3^2,\ -x_2^2 + x_2x_3,\ x_2^2 - x_2x_3,\ -x_1x_2 + x_1x_3, \\ &x_1x_2 - x_1x_3,\ -x_1x_2 - x_1x_3 - x_1,\ x_1x_2 + x_1x_3 + x_1,\ -x_1^2 - x_1x_2 - x_1x_3, \\ &-x_1^2 - x_1x_2 + x_1x_3,\ -x_1^2 + x_1x_2 - x_1x_3,\ x_1^2 - x_1x_2 + x_1x_3,\ x_1^2 + x_1x_2 + x_1x_3. \end{aligned}$$

The ideal in $\Bbbk[x_1, x_2, x_3]$ generated by these polynomials has Gröbner basis x_1, x_2, x_3. ∎

6.2. THEOREM. *For $\mathcal{C}om(\mathcal{L}ie) \rightsquigarrow \mathcal{L}ie(\mathcal{C}om)$ we obtain only the trivial distributive law.*

PROOF. Very similar to the proof of Theorem 6.1. ∎

6.3. THEOREM. *The only nontrivial distributive law $\mathcal{L}ie(\mathcal{C}om) \rightsquigarrow \mathcal{C}om(\mathcal{L}ie)$ is that for Poisson algebras.*

The theorem is a particular case of the classification of generalized distributive laws between $\mathcal{L}ie$ and $\mathcal{C}om$ given in [9].

7. Associative-magmatic laws

In this final section we analyze distributive laws $\mathcal{A}ss(\mathcal{M}ag) \rightsquigarrow \mathcal{M}ag(\mathcal{A}ss)$ between associative and magmatic (no axioms) multiplications denoted $\bullet$ resp. $\circ$.

7.1. THEOREM. *There are only two non-isomorphic distributive laws between the associative and magmatic multiplication, the trivial one and the truncated one represented by the rewriting rules (b), (c), (e) or (f) of Theorems 3.2 and 3.5.*

PROOF. Maple found the following rewriting rules, with $\alpha, \gamma \in \Bbbk$ arbitrary parameters for which the square roots in the formulas exist, and $\iota = \sqrt{-1}$:

1) $(x\circ y)\bullet z = 0, \quad x\bullet(y\circ z) = 0,$
2) $(x\circ y)\bullet z = 0,$
$x\bullet(y\circ z) = \frac{1}{2}(x\bullet y)\circ z + \frac{1}{2}\iota\,(x\bullet z)\circ y + \frac{1}{2}\,y\circ(x\bullet z) - \frac{1}{2}\iota\,z\circ(x\bullet y),$
3) $(x\circ y)\bullet z = \frac{1}{2}(x\bullet z)\circ y + \frac{1}{2}\iota\,(y\bullet z)\circ x + \frac{1}{2}\,x\circ(y\bullet z) - \frac{1}{2}\iota\,y\circ(x\bullet z),$
$x\bullet(y\circ z) = 0,$
4) $(x\circ y)\bullet z = 0,$
$x\bullet(y\circ z) = -\gamma\,(x\bullet y)\circ z + \sqrt{\gamma^2+\gamma}\,(x\bullet z)\circ y + (\gamma+1)\,y\circ(x\bullet z) - \sqrt{\gamma^2+\gamma}\,z\circ(x\bullet y),$
5) $(x\circ y)\bullet z = -\alpha\,(x\bullet z)\circ y + \sqrt{\alpha^2+\alpha}\,(y\bullet z)\circ x + (\alpha+1)\,x\circ(y\bullet z) - \sqrt{\alpha^2+\alpha}\,y\circ(x\bullet z),$
$x\bullet(y\circ z) = 0.$

Law 1) is the trivial one. Laws 4) and 5) are isomorphic, via the replacement $\bullet \mapsto \bullet^{\mathrm{op}}$ of the $\bullet$-product by the opposite one. Law 2) is obtained from 4) by substituting $\gamma = -\frac{1}{2}$ and, likewise, the substitution $\alpha = -\frac{1}{2}$ brings 5) into 3).

The proof will therefore be finished if we show that 4) is isomorphic to the truncated distributive law. The following method, suggested by Vladimir Dotsenko, is based on the substitution

$$\gamma = \frac{1}{t^2-1},\ t \neq \pm 1. \tag{13}$$

Notice that its inverse can be written as

$$t = \frac{\sqrt{\gamma^2+\gamma}}{\gamma},$$

thus for any $\gamma \neq 0$ for which $\sqrt{\gamma^2+\gamma}$ exists one has t fulfilling (13). It is straightforward to verify that the replacement

$$x \circ y \mapsto x \circ y + t(y \circ x)$$

brings 4) into the truncated rule

$$(x \circ y) \bullet z = 0,\ x \bullet (y \circ z) = (x \circ y) \bullet z.$$

If $\gamma = 0$, in which case the substitution (13) cannot be used, then 4) becomes another (but isomorphic) truncated rule

$$(x \circ y) \bullet z = 0,\ \ x \bullet (y \circ z) = y \circ (x \bullet z).$$

This finishes the proof. ∎

7.2. Remark. The reason for including the above proof instead of just referring to the result of Maple calculation was to show that, outside the realm of Three Graces, various bizarre-looking distributive laws, such as 4) or 5), may exist. Since one of the operads — in this case $\mathcal{M}ag$ — may have a huge group of automorphisms, these weird laws may however turn to be isomorphic to mild and expected ones.

Theorem 7.1 has the following obvious but surprising

7.3. Corollary. *In the cartesian monoidal category of sets, there are no distributive laws of type* $\mathcal{A}ss(\mathcal{M}ag) \rightsquigarrow \mathcal{M}ag(\mathcal{A}ss)$.

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