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FREE GLOBULARLY GENERATED DOUBLE CATEGORIES I

JUAN ORENDAIN

ABSTRACT. This is the first part of a two paper series studying free globularly generated double categories. In this first installment we introduce the free globularly generated double category construction. The free globularly generated double category construction canonically associates to every bicategory together with a possible category of vertical morphisms, a double category fixing this set of initial data in a free and minimal way. We use the free globularly generated double category to study length, free products, and problems of internalization. We use the free globularly generated double category construction to provide formal functorial extensions of the Haagerup standard form construction and the Connes fusion operation to inclusions of factors of not-necessarily finite Jones index.

1. Introduction

Double categories were introduced by Ehresmann in [11]. Bicategories were later introduced by Bénabou in [3]. Both double categories and bicategories express the notion of a higher categorical structure of second order, each with its advantages and disadvantages. Double categories and bicategories relate in different ways.

Every double category admits an underlying bicategory, its horizontal bicategory. The horizontal bicategory HC of a double category C 'flattens' C by discarding vertical morphisms and only considering globular squares. There are several structures transferring vertical information on a double category to its horizontal bicategory, e.g. connection pairs [5], thin structures [6], and foldings and cofoldings [7] among others. A great deal of information about a double category can be reduced to information about its horizontal bicategory under the assumption of the existence of such structures, see [13] for example.

Bicategories on the other hand 'lift' to double categories through several different constructions, examples of which are the Ehresmann double category of quintets construction [12] for 2-categories, the double category of adjoints construction [17], the double category of spans construction [9], the construction of framed bicategories through monoidal fibrations of [18] and the construction of the double category of semisimple von Neumann

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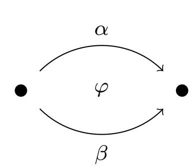
algebras and finite morphisms of [1,2]. These construction follows different methods and the resulting double categories express different aspects of the bicategories they lift. In all cases one starts with a bicategory as initial set of data together with a choice of vertical morphisms, which serve as a 'direction' towards which one lifts. In all cases one ends up with a double category having relevant information about the initial bicategory and the collection of vertical morphisms, and relating to this initial set of data through horizontalization.

We are interested general constructions of the type described above. This is the first part of a two paper series studying the free globularly generated double category construction. The free globularly generated double category construction canonically associates to every bicategory, together with a direction towards which to lift, i.e. together with a category of vertical morphisms, a double category. This double category fixes the initial set of data and is minimal with respect to this property. In this paper we provide a detailed construction of the free globularly generated double category associated to a decorated bicategory and we apply this construction to problems of existence of internalizations and to the concept of length of a double category. We now present a more detailed account of the contents of this paper.

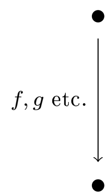
1.1. The problem of existence of internalizations. Given a bicategory $\mathcal{B}$ we will say that a category $\mathcal{B}^*$ is a decoration of $\mathcal{B}$ if the collection of 0-cells of $\mathcal{B}$ is equal to the collection of objects of $\mathcal{B}^*$. In this case we say that the pair $(\mathcal{B}^*, \mathcal{B})$ is a decorated bicategory. We think of decorated bicategories as bicategories together with an orthogonal direction, provided by the corresponding decoration, towards which to lift $\mathcal{B}$ to a double category. Given a double category C the pair (C_0, HC) where C_0 is the category of objects of C, is a decorated bicategory. We write H^*C for this decorated bicategory. We call H^*C the decorated horizontalization of C. We consider the following problem.

1.2. Problem. *Let $(\mathcal{B}^*, \mathcal{B})$ be a decorated bicategory. Find double categories C such that $H^*C = \mathcal{B}$.*

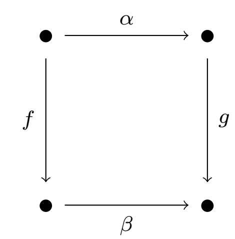
Given a decorated bicategory $(\mathcal{B}^*, \mathcal{B})$ we say that a solution C to Problem 1.2 for $(\mathcal{B}^*, \mathcal{B})$ is an internalization of $(\mathcal{B}^*, \mathcal{B})$. We are thus interested in finding internalizations to decorated bicategories. Pictorially, we are interested in the following situation: Given a set of 2-dimensional cells of the form



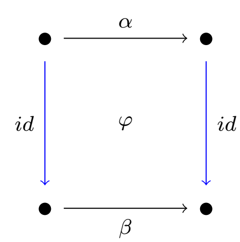
forming a bicategory, and given a collection of vertical arrows of the form:



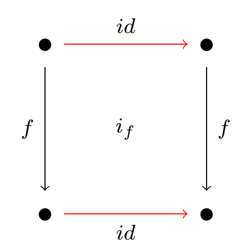
forming a category, such that the endpoints of these arrows are the same as the vertices of the above globular diagrams, we consider boundaries of hollow squares of the form:



formed by horizontal edges of globular diagrams and decoration arrows. Identifying globular diagrams as above with squares of the form:



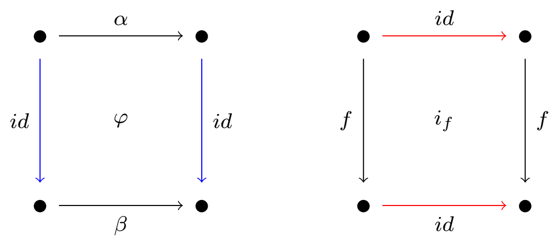
and formally associating to every vertical arrow as above a unique identity square as:



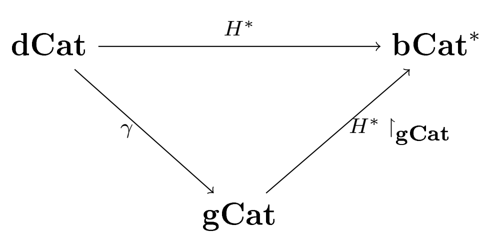
Problem 1.2 asks about coherent ways to fill boundaries of hollow squares as above in such a way that globular and identity squares defined by the set of initial data above are fixed and such that the resulting structure forms a double category. We regard such problems as formal versions of arguments of 'filling squares' classically considered in nonabelian algebraic topology, see [4]. One of the author's motivations for studying such problems is the problem of existence of a compatible pair of tensor functors L^2 and $\boxtimes$, associating

to every von Neumann algebra A its Haagerup standard form $L^2(A)$, and associating to every horizontally compatible pair of Hilbert bimodules ${}_AH_B, {}_BK_C$ the corresponding fusion Hilbert bimodule ${}_AH \boxtimes_B K_C$ respectively. Such compatible pair of functors should provide the pair formed by the category of von Neumann algebras and their morphisms and the category of Hilbert bimodules and equivariant intertwining operators with the structure of a category internal to tensor categories. The methods developed in the present series of papers provide partial solutions to this problem.

1.3. A case for globularly generated double categories. The situation described above motivated the author to introduce the concept of globularly generated double category in [16]. We say that a double category is globularly generated if it is generated by its collection of globular squares. Pictorially, a double category C is globularly generated if every square in C can be written as horizontal and vertical compositions of squares of the form:



Given a double category C we write γC for the sub-double category of C generated by squares as above. γC is globularly generated, it satisfies the equation $H^*C = H^*\gamma C$ and it is the minimal sub-double category of C satisfying this equation. We call γC the globularly generated piece of C. A double category C is globularly generated if and only if there are no proper sub-double categories D of C such that $H^*C = H^*D$. Globularly generated double categories are thus precisely the minimal solutions to Problem 1.2. This can be expressed categorically as follows: Write **dCat**, **gCat** and **bCat*** for the category of double categories and double functors, for the subcategory of **dCat** generated by globularly generated double categories, and for the category of decorated bicategories and decorated pseudofunctors respectively. The globularly generated piece construction extends to a reflector (2-reflector in fact) γ of **dCat** on **gCat**. It is not difficult to see that this implies that γ is in fact a Grothendieck fibration. Moreover, the decorated horizontal bicategory construction extends to a functor H^* from **dCat** to **bCat***, which by the comments above is easily seen to be constant on the fibers of γ. We obtain a commutative triangle:

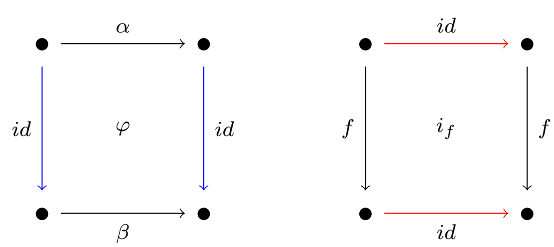


We thus think of double categories as being parametrized, or bundled, by globularly generated double categories. The relevant information about Problem 1.2 is contained in the bases of this fibration. We summarize this by saying that finding solutions to Problem 1.2 is equivalent to finding globularly generated solutions. We believe this justifies the study of globularly generated double categories.

Globularly generated double categories admit intrinsic structure that makes them, to some extent, easy to describe. The category of squares C_1 of a globularly generated double category C admits an expression as a limit $\lim V_C^k$ of a chain of categories $V_C^1 \subseteq V_C^2 \subseteq \dots$ defined inductively by setting V_C^1 as the subcategory of C_1 generated by squares as above, and by setting V_C^k as the subcategory of C generated by horizontal compositions of morphisms in V_C^{k-1} for every $k > 1$. We call this chain of categories the vertical filtration of C. The vertical filtration allows us to define numerical invariants for double categories. We say that a square φ in a globularly generated double category C is of length k, $\ell\varphi$ in symbols, if φ is a morphism in V_C^k but not a morphism in V_C^{k-1}. We define the length of a double category C, which we write ℓC, as the supremum of lengths of squares in γC. The only examples of globularly generated double categories studied so far, i.e. trivial double categories and globularly generated pieces of double categories of bordisms, algebras, and von Neumann algebras, are all of length 1.

1.4. Free globularly generated double categories. We will, from now on, denote a decorated bicategory $(\mathcal{B}^*, \mathcal{B})$ simply by $\mathcal{B}$. The free globularly generated double category construction associates to every decorated bicategory $\mathcal{B}$ a globularly generated double category $Q_{\mathcal{B}}$ in such a way that $Q_{\mathcal{B}}$ fixes the data of $\mathcal{B}$ and such that the only relations satisfied by the squares of $Q_{\mathcal{B}}$ are those relations coming from relations satisfied by the 2-cells of $\mathcal{B}$ and the morphisms of $\mathcal{B}^*$.

The intuitive idea behind the free globularly generated double category construction is as follows: Suppose we are provided with a decorated bicategory $\mathcal{B}$. We wish to construct, from the data of $\mathcal{B}$ alone, a double category C satisfying the equation $H^*C = \mathcal{B}$, and we wish to do this in a minimal way. As outlined above we thus wish to construct a globularly generated double category C satisfying the equation $H^*C = \mathcal{B}$. Such double category has $\mathcal{B}^*$ as category of objects, has the collection of 1-cells $\mathcal{B}_1$ of $\mathcal{B}$ as collection of horizontal morphisms, and all its squares can be expressed as a finite sequence of vertical and horizontal compositions of squares of the form:



with φ being a 2-cell in $\mathcal{B}$ and f being a morphism in $\mathcal{B}^*$. Moreover, the vertical filtration of C provides a way to organize these expressions into strata indicating some measure of complexity.

The free globularly generated double category construction formally reproduces the above situation. We begin by formally associating to every vertical morphism $f : a \longrightarrow b$ in the decoration $\mathcal{B}^*$ of the decorated bicategory we are provided with, a square of the form:

$$\begin{array}{ccc} a & \xrightarrow{id} & a \\ {\scriptstyle f}\downarrow & i_f & \downarrow{\scriptstyle f} \\ b & \xrightarrow[id]{} & b \end{array}$$

and we formally associate to every 2-cell in $\mathcal{B}$ of the form:

$$a \;\begin{array}{c} \xrightarrow{\alpha} \\ \varphi \\ \xrightarrow[\beta]{} \end{array}\; b$$

a square of the form:

$$\begin{array}{ccc} a & \xrightarrow{\alpha} & b \\ {\scriptstyle id}\downarrow & \varphi & \downarrow{\scriptstyle id} \\ a & \xrightarrow[\beta]{} & b \end{array}$$

Having done this, we consider the path category generated by these squares. These are squares of length 1. We will write F_1 for this category. Inductively we define a category F_k as the path category of the collection of formal horizontal compositions of squares in F_{k-1}, assuming F_{k-1} has been defined. This provides a collection of squares with free horizontal and vertical composition rules. Dividing by a suitable relation we would obtain a free double category in the sense of [8]. Since we wish to fix the data provided by $\mathcal{B}$ we divide the structure we obtain by a finer equivalence relation R_∞ and thus obtain a double category $Q_{\mathcal{B}}$ such that $Q_{\mathcal{B}}$ is globularly generated. If we choose R_∞ carefully enough, the category of objects of $Q_{\mathcal{B}}$ will be $\mathcal{B}^*$ and the collection of horizontal morphisms of $Q_{\mathcal{B}}$ will be the collection of 1-cells of $\mathcal{B}$.

We prove that thus defined the globularly generated double category $Q_{\mathcal{B}}$ associated to a decorated bicategory $\mathcal{B}$ does not necessarily provide a solution to Problem 1.2 for $\mathcal{B}$. The only obstruction for this is that through composition operations in $Q_{\mathcal{B}}$ we may inadvertently construct new globular squares not already in $\mathcal{B}$. We provide conditions on decorated bicategories $\mathcal{B}$ that guarantee that $Q_{\mathcal{B}}$ provides solutions to problem 1.2 for $\mathcal{B}$. Moreover, in the case in which $Q_{\mathcal{B}}$ does not provide solutions to Problem 1.2 for $\mathcal{B}$ we prove that if we modify $\mathcal{B}$ enough, we can construct a decorated bicategory for which the free globularly generated double category does provide solutions to Problem 1.2.

The free globularly generated double category construction provides a method for explicitly constructing double categories satisfying certain conditions. We use the free globularly generated double category construction to provide examples of double categories with non-trivial length and examples of double categories with infinite length. We relate the free globularly generated double category construction with the free product of groups and monoids, and with the free double category construction of [8]. Further, we apply free globularly generated double categories to provide formal solutions to the problem of existence of functorial extensions of both the Haagerup standard form and the Connes fusion operation.

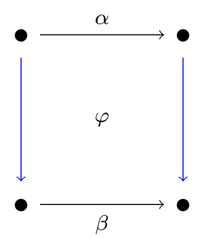
Finally, in the second part of the present series of papers we provide an interpretation of the globularly generated double category construction as the object function of a functor Q from $\mathbf{bCat}^*$ to $\mathbf{gCat}$ satisfying the equation:

$$Q \dashv H^* \restriction_{\mathbf{gCat}}$$

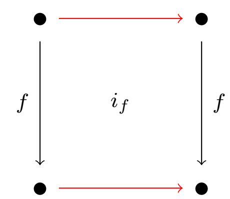
thus completing the diagram expressing the fact that H^* factors through fibers of $\mathbf{dCat}$ modulo the globularly generated piece fibration γ presented above. Now, the restriction $H^* \restriction_{\mathbf{gCat}}$ is faithful and thus the above result provides in particular the globularly generated double category construction with the structure of a free object in $\mathbf{gCat}$. We will thus interpret free globularly generated double categories as sets of generators for bases mod γ of solutions to Problem 1.2.

1.5. Conventions. We follow the usual conventions for the theory of bicategories and double categories, with a few exceptions. The word double category will always mean pseudo double category. We will write $\mathcal{B}_0, \mathcal{B}_1$, and $\mathcal{B}_2$ for the collections of 0-, 1-, and 2-cells of a bicategory $\mathcal{B}$ and we will write C_0, C_1 for the category of objects and vertical arrows and the category of horizontal arrows and squares of a double category C respectively. We will write horizontal identities and compositions as i and $*$. We will write vertical compositions as word concatenation. We will write λ, ρ and A for left and right identity transformations and associators of both bicategories and double categories. As above, we will denote a decorated bicategory $(\mathcal{B}^*, \mathcal{B})$ simply by $\mathcal{B}$. Thus when we say that $\mathcal{B}$ is a decorated bicategory the letter $\mathcal{B}$ will denote both a decorated bicategory and its underlying bicategory. We will write $\mathcal{B}^*$ for the decoration of a decorated bicategory $\mathcal{B}$. For most of the paper we will interpret decorated bicategories as decorated horizontalizations of double categories, we will thus sometimes call the 0-. 1-, and 2-cells of the underlying bicategory of a decorated bicategory $\mathcal{B}$ the objects, the horizontal morphisms and the globular squares of $\mathcal{B}$ and we will call the morphisms of the decoration $\mathcal{B}^*$ of $\mathcal{B}$ the vertical morphisms of $\mathcal{B}$. Pictorially we will represent vertical identity endomorphisms by blue arrows and horizontal identity endomorphisms by red arrows as was done above.

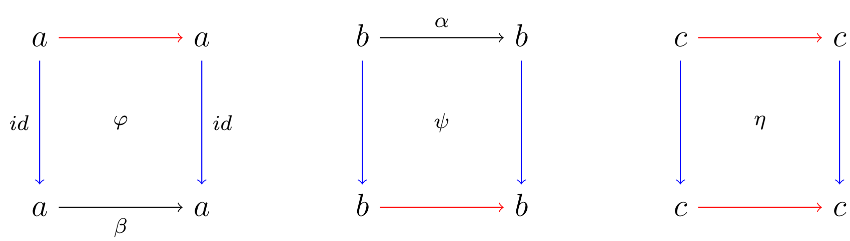
Squares of the form:



will thus represent globular squares, squares of the form:



will represent horizontal identities and squares of the form:



will represent a globular square from the horizontal identity i_a of the object a to the horizontal morphism β, a globular square from the horizontal morphism α to i_b and a globular endomorphism of i_c respectively.

1.6. Contents of the paper. We now sketch the contents of the paper. In section 2 we present a detailed construction of the free globularly generated double category associated to a decorated bicategory. We do this in several steps and our construction not only yields a free globularly generated double category but a free vertical filtration which will allow us to associate numerical invariants to decorated bicategories. In section 3 we study relations between the free globularly generated double category construction and Problem 1.2. We provide conditions on decorated bicategories that ensure that the corresponding free globularly generated double category is an internalization and in situations in which this is not the case we introduce a method under which one can always extend a decorated bicategory to a decorated bicategory for which the free globularly generated double category is an internalizations. In section 4 we apply the free globularly generated double category construction to provide examples of double categories with non-trivial length. In section 5 we study the free globularly generated double category in the case of deloopings of monoidal categories decorated by deloopings of groups. We prove that the free globularly generated double category associated to decorated bicategories of this type are always of length 1. Finally, in section 6 we apply a modification of the free globularly generated double category to provide compatible formal functorial extensions of the Haagerup standard form construction and the Connes fusion operation to certain linear categories of Hilbert spaces.

2. The free globularly generated double category

In this section we introduce the free globularly generated double category construction. The free globularly generated double category construction canonically associates a globularly generated double category to every decorated bicategory. The strategy behind the construction is to emulate the internal structure defined by the vertical and horizontal filtrations in abstract globularly generated double categories in order to obtain, from the data of a decorated bicategory alone, a globularly generated double category. The construction of the free globularly generated double category is rather involved and we divide it into several steps. We begin with a few preliminary definitions and results.

2.1. Preliminaries: Evaluations. Let X and Y be sets. Let $s, t : X \longrightarrow Y$ be functions. Let $x_1, ..., x_n$ be a sequence in X. We will say that $x_1, ..., x_k$ is compatible with respect to s and t if the equation $tx_{i+1} = sx_i$ holds for every $1 \leq i \leq k-1$. Equivalently, $x_1, ..., x_k$ is compatible with respect to s and t if $x_1, ..., x_n$ is a composable sequence of morphisms in the free category generated by X with s and t as domain and codomain functions respectively. Given a compatible sequence $x_1, ..., x_k$ in X, we call any way of writing the word $x_k...x_1$ following an admissible parenthesis pattern, an evaluation of $x_1, ..., x_k$. Equivalently, the evaluations of a compatible sequence $x_1, ..., x_k$ are different ways of writing the word $x_k...x_1$ composing elements of $x_1, ..., x_k$ two by two in the free category generated by X, with s, t as domain and codomain functions. For example, (yx) is the only evaluation of the two term compatible sequence x, y and $(x(yz)), ((xy)z)$ are the two evaluations of the compatible three term sequence x, y, z. We will write $X_{s,t}$ for the set of evaluations of finite sequences of elements of X, compatible with respect to s and t.

Given functions $s, t : X \longrightarrow Y$, we write $\tilde{s}$ and $\tilde{t}$ for the functions $\tilde{s}, \tilde{t} : X_{s,t} \longrightarrow Y$ defined as follows: Given an evaluation Φ of a compatible sequence $x_1, ..., x_k$ in X we make $\tilde{s}\Phi$ and $\tilde{t}\Phi$ to be equal to sx_1 and tx_k respectively. Observe that the values $\tilde{s}\Phi$ and $\tilde{t}\Phi$ do not depend on the particular evaluation Φ of $x_1, ..., x_k$. Given a pair of compatible sequences $x_1, ..., x_k$ and $x_{k+1}, ..., x_n$ in X, such that the 2 term sequence x_k, x_{k+1} is compatible, and given evaluations Φ and Ψ of $x_1, ..., x_k$ and $x_{k+1}, ..., x_n$, the equation $\tilde{t}\Phi = \tilde{s}\Psi$ is satisfied and the concatenation of Ψ and Φ defines an evaluation of the sequence $x_1, ..., x_k, x_{k+1}, ..., x_n$. We denote the concatenation of Φ and Ψ satisfying the conditions above by $\Psi *_{s,t} \Phi$. This operation defines a function from $X_{s,t} \times_Y X_{s,t}$ to $X_{s,t}$

where the fibration in $X_{s,t} \times_Y X_{s,t}$ is taken with respect to the pair $\tilde{s}, \tilde{t}$. We write $*_{s,t}$ for this function.

Now, given sets X, X', Y, and Y', and functions $s, t : X \longrightarrow Y$ and $s', t' : X' \longrightarrow Y'$, we say that a pair of functions $\varphi : X \longrightarrow X'$ and $\phi : Y \longrightarrow Y'$ is compatible if the following two squares commute

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X' \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ Y & \xrightarrow{\phi} & Y' \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\varphi} & X' \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ Y & \xrightarrow{\phi} & Y' \end{array}$$

Given sets X, X', Y, and Y', functions $s, t : X \longrightarrow Y$ and $s', t' : X' \longrightarrow Y'$, and a compatible pair of functions $\varphi : X \longrightarrow X'$ and $\phi : Y \longrightarrow Y'$, if $x_1, ..., x_k$ is a sequence in X, compatible with respect to s, t then the sequence $\varphi x_1, ..., \varphi x_k$ is compatible with respect to s', t'. Moreover, given an evaluation Φ of a compatible sequence $x_1, ..., x_k$, the same parenthesis pattern defining the evaluation Φ defines an evaluation of the compatible sequence $\varphi x_1, ..., \varphi x_k$. We write $\mu_{\varphi,\phi}\Phi$ for this evaluation. We write $\mu_{\varphi,\phi}$ for the function from $X_{s,t}$ to $X'_{s',t'}$ associating the evaluation $\mu_{\varphi,\phi}\Phi$ to every evaluation Φ in $X_{s,t}$. The proof of the following lemma is straightforward.

2.2. LEMMA. *Let X, X', Y, and Y' be sets. Let $s, t : X \longrightarrow Y$ and $s', t' : X' \longrightarrow Y'$ be functions. Let $\varphi : X \longrightarrow X'$ and $\phi : Y \longrightarrow Y'$ be functions such that the pair φ, ϕ is compatible. In that case the function $\mu_{\varphi,\phi}$ associated to the pair φ, ϕ satisfies the following conditions*

1. *The following two squares commute*

$$\begin{array}{ccc} X_{s,t} & \xrightarrow{\mu_{\varphi,\phi}} & X'_{s',t'} \\ {\scriptstyle \tilde{s}}\downarrow & & \downarrow{\scriptstyle \tilde{t}} \\ Y & \xrightarrow{\phi} & Y' \end{array} \qquad \begin{array}{ccc} X_{s,t} & \xrightarrow{\mu_{\varphi,\phi}} & X'_{s',t'} \\ {\scriptstyle \tilde{s}}\downarrow & & \downarrow{\scriptstyle \tilde{t}} \\ Y & \xrightarrow{\phi} & Y' \end{array}$$

2. *The following square commutes*

$$\begin{array}{ccc} X_{s,t} \times_Y X_{s,t} & \xrightarrow{\mu_{\varphi,\phi} \times_\phi \mu_{\varphi,\phi}} & X'_{s',t'} \times_Y X'_{s',t'} \\ {\scriptstyle *_{s,t}}\downarrow & & \downarrow{\scriptstyle *_{s',t'}} \\ X_{s,t} & \xrightarrow{\mu_{\varphi,\phi}} & X'_{s',t'} \end{array}$$

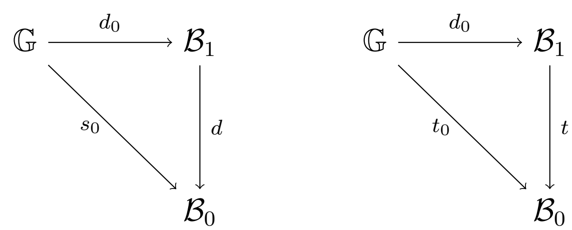
2.3. Preliminaries: Notational conventions. Let $\mathcal{B}$ be a decorated bicategory. Let α be a vertical morphism in $\mathcal{B}$. We will write i_α for the singleton $\{\alpha\}$. We call i_α the formal horizontal identity of α. We write $\mathbb{G}$ for the union of the collection of globular squares in $\mathcal{B}$ and the collection of formal horizontal identities of vertical morphisms of $\mathcal{B}$. We will adopt the following notational conventions for the elements of $\mathbb{G}$.

1. Let Φ be a globular square in $\mathcal{B}$. We write $d_0\Phi, c_0\Phi$ for the domain and codomain of Φ in $\mathcal{B}$. Let α be a vertical morphism in $\mathcal{B}$. Let a and b be the domain and the codomain, in $\mathcal{B}^*$ of α. We write $d_0 i_\alpha$ and $c_0 i_\alpha$ for the horizontal identities id_a and id_b of a and b in $\mathcal{B}$. We write d_0 and c_0 for the functions from $\mathbb{G}$ to $\mathcal{B}_1$ associating $d_0\Phi$ and $c_0\Phi$ to every element Φ of $\mathbb{G}$.

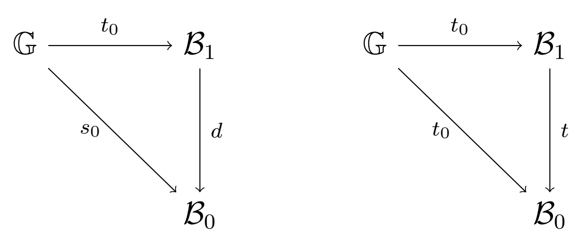
2. Let Φ be a globular square in $\mathcal{B}$. We write $s_0\Phi$ and $t_0\Phi$ for the source and target of Φ in $\mathcal{B}$. Let α be a vertical morphism in $\mathcal{B}$. In that case we write $s_0 i_\alpha$ and $t_0 i_\alpha$ for the morphism α. We write s_0 and t_0 for the functions from $\mathbb{G}$ to the collection of vertical morphisms of $\mathcal{B}$ associating $s_0\Phi$ and $t_0\Phi$ to every element Φ of $\mathbb{G}$.

The functions d_0, c_0, s_0, and t_0 defined above are easily seen to be related by the following conditions

1. The following two triangles commute



2. The following two triangles commute



Given a decorated bicategory $\mathcal{B}$, we denote by p the function from the collection of evaluations $\mathcal{B}_{1_{d,c}}$ of $\mathcal{B}_1$, with respect to the pair formed by the domain and the codomain functions d, c in the bicategory underlying $\mathcal{B}$, to the set of horizontal morphisms $\mathcal{B}_1$ of $\mathcal{B}$ defined as follows: for every composable sequence $f_1, ..., f_k$ of horizontal morphisms in $\mathcal{B}$ and for every evaluation Φ of $f_1, ..., f_k$, the image $p\Phi$ of Φ under p is equal to the horizontal composition of the sequence $f_1, ..., f_k$ in $\mathcal{B}$ following the parenthesis pattern defining Φ. We call p the projection associated to $\mathcal{B}_1$.

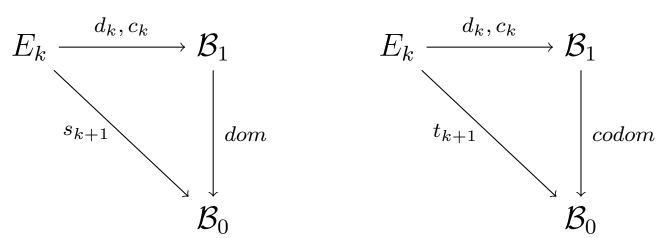
2.4. The main construction: Inductive step. Given a decorated bicategory $\mathcal{B}$ we will write E_1 for the collection of evaluations $\mathbb{G}_{s_0,t_0}$ of $\mathbb{G}$ with respect to the pair of functions s_0, t_0. We denote by s_1 and t_1 the functions $\tilde{s}_0$ and $\tilde{t}_0$. Thus defined s_1 and t_1 are functions from E_1 to $\mathrm{Hom}_{\mathcal{B}^*}$. We write $*_1$ for the operation $*_{s_1,t_1}$ on E_1. Finally, we write d_1 for the composition $p\mu_{d_0,id}$, and we write c_1 for the composition $p\mu_{c_0,id}$. Thus defined d_1 and c_1 are functions from E_1 to the set of horizontal morphisms $\mathcal{B}_1$ of $\mathcal{B}$. The following theorem is the first step towards the free globularly generated double category construction.

2.5. Theorem. *Let $\mathcal{B}$ be a decorated bicategory. There exists a pair of sequences of triples (E_k, d_k, c_k) and (F_k, s_{k+1}, t_{k+1}), such that for each k, E_k is a set containing E_1, d_k, c_k are functions from E_k to $\mathcal{B}_1$ extending the functions d_1 and c_1 defined above, F_k is a category having $\mathcal{B}_1$ as collection of objects, and s_{k+1}, t_{k+1} are functors from F_k to $\mathcal{B}^*$. The pair of sequences (E_k, d_k, c_k) and (F_k, s_{k+1}, t_{k+1}) satisfies the following conditions:*

1. *For every k, Hom_{F_k} is contained in E_{k+1}. Moreover, E_{k+1} is equal to the set of evaluations $Hom_{F_k s_{k+1},t_{k+1}}$ of Hom_{F_k} with respect to the pair formed by the morphism functions of s_{k+1} and t_{k+1}.*

2. *For every k, E_k is contained in Hom_{F_k}. Moreover, F_k is equal to the free category generated by E_k with functions d_k and c_k as domain and codomain functions respectively. The restriction of the morphism functions of s_{k+1} and t_{k+1} to E_1 are equal to the functions s_1 and t_1 defined above.*

3. *For every positive integer k the following triangles commute:*



The conditions 1-3 above determine the pair of sequences of triples (E_k, d_k, c_k) and (F_k, s_{k+1}, t_{k+1})

Proof. Let $\mathcal{B}$ be a decorated bicategory. We wish to construct a pair of sequences of triples (E_k, d_k, c_k) and (F_k, s_{k+1}, t_{k+1}) with k running through the collection of all positive integers, such that for each positive integer k, E_k is a set extending to the set E_1 associated to $\mathcal{B}$, such that d_k and c_k are functions from E_k to the collection $\mathcal{B}_1$ of horizontal morphisms of $\mathcal{B}$ extending the functions d_1 and c_1, such that F_k is a category having $\mathcal{B}_1$ as set of objects and s_{k+1}, t_{k+1} are functors from F_k to $\mathcal{B}^*$. Moreover, we wish to define the pair of sequences (E_k, d_k, c_k) and (F_k, s_{k+1}, t_{k+1}) in such a way that conditions 1-3 above are satisfied.

We proceed by induction on k. We begin by defining the triple (F_1, s_2, t_2). We make F_1 to be the free category generated by E_1 with respect to d_1, c_1. The functions s_1 and t_1 are compatible with d_1 and c_1 and thus admit a unique extension to functors from F_1 to the decoration $\mathcal{B}^*$ of $\mathcal{B}$. We make s_2 and t_2 be the corresponding functorial extensions of s_1 and t_1. We write $\bullet_1$ for the composition operation in F_1.

Let now k be a positive integer strictly greater than 1. Assume that we have extended the definition of the triple (E_1, d_1, c_1) and the definition of the triple (F_1, s_2, t_2) of the previous paragraph, to a sequence of pairs of triples (E_m, d_m, c_m) and (F_m, s_{m+1}, t_{m+1}) for every $m \leq k$, where E_m is assumed to be a set containing E_1, d_m and c_m are assumed to be functions from E_m to $\mathcal{B}_1$ extending d_1 and c_1, F_m is assumed to be a category containing F_1 as subcategory, and s_{m+1} and t_{m+1} are assumed to be functors from F_m to $\mathcal{B}^*$ extending s_2 and t_2 respectively. Moreover, we assume that the pair of sequences of triples (E_m, d_m, c_m) and (F_m, s_{m+1}, t_{m+1}) satisfies conditions 1-3 above.

We now wish to extend the definition of the pair of sequences of triples (E_m, d_m, c_m) and (F_m, s_{m+1}, t_{m+1}) to the definition of a pair (E_k, d_k, c_k) and (F_k, s_{k+1}, t_{k+1}) satisfying the conditions of the theorem. We begin with the definition of the triple (E_k, d_k, c_k). We make E_k to be the collection of evaluations $\mathrm{Hom}_{F_{k-1} s_k, t_k}$ of $\mathrm{Hom}_{F_{k-1}}$ with respect to the pair formed by the morphism functions of s_k and t_k. We write s_{k+1} and t_{k+1} for the extensions $\tilde{s}_k$ and $\tilde{t}_k$, to E_k, of the morphism functions of s_k and t_k. We denote by $*_k$ the concatenation operation $*_{s_{k+1}, t_{k+1}}$ in E_k, with respect to s_{k+1} and t_{k+1}. We now make the function d_k to be the composition $p\mu_{d_{k-1}, id}$ of the function associated to the pair formed by the domain function d_{k-1} in F_{k-1} and the identity function in the collection of horizontal morphisms $\mathcal{B}_1$ of $\mathcal{B}$ and the projection p associated to the collection of horizontal morphisms $\mathcal{B}_1$ of $\mathcal{B}$. We make c_k to be the composition $p\mu_{c_{k-1}, id}$ of the function associated to the pair formed by the codomain function in F_{k-1} and the identity function in the collection of horizontal morphisms $\mathcal{B}_1$ of $\mathcal{B}$, and the projection p associated to the collection of horizontal morphisms $\mathcal{B}_1$ of $\mathcal{B}$. The functions d_k and c_k are well defined. Thus defined d_k and c_k are functions from E_k to $\mathcal{B}_1$ satisfying condition 3 of the theorem by lemma 3.1. We now define the triple (F_k, s_{k+1}, t_{k+1}). We make the category F_k to be the free category generated by E_k, with d_k and c_k as domain and codomain functions. The collection of objects of F_k is thus $\mathcal{B}_1$. We write $\bullet_k$ for the composition operation in F_k. By the fact that the functions d_k, c_k, s_k and t_k satisfy the condition 3 of the theorem it follows that the pairs formed by s_k and t_k together with the domain and codomain functions defined on $\mathcal{B}_1$ admit unique extensions to functors from F_k to $\mathcal{B}^*$. We make s_{k+1} and t_{k+1} to be these functors. Thus defined the triple (F_k, s_{k+1}, t_{k+1}) satisfies the conditions of the theorem.

It is obvious that conditions 1-3 of the theorem determine the pair of sequences (E_k, d_k, c_k) and (F_k, s_{k+1}, t_{k+1}). This concludes the proof. ■

2.6. Observation. *Let $\mathcal{B}$ be a decorated bicategory. Let m, k be positive integers such that m is less than or equal to k. In that case, as defined above, E_m is contained in E_k, d_m and c_m are equal to the restrictions to E_k of d_k and c_k respectively, and the concatenation operation $*_m$ is equal to the restriction to E_m of $*_k$. Moreover, F_m is a subcategory of F_k*

and the functors s_{m+1}, t_{m+1} are equal to the restrictions to F_m of s_{k+1} and t_{k+1} respectively.

2.7. Observation. *Let $\mathcal{B}$ be a decorated bicategory. Let k be a positive integer. The following two squares commute where q_1^k and q_2^k denote the left and the right projections of $E_k \times_{Hom_{\mathcal{B}^*}} E_k$ onto E_k respectively.*

$$\begin{array}{ccc} E_k \times_{Hom_{\mathcal{B}^*}} E_k & \xrightarrow{*_k} & E_k \\ \downarrow{\scriptstyle q_1^k} & & \downarrow{\scriptstyle s_k} \\ E_k & \xrightarrow[s_k]{} & Hom_{\mathcal{B}^*} \end{array} \qquad \begin{array}{ccc} E_k \times_{Hom_{\mathcal{B}^*}} E_k & \xrightarrow{*_k} & E_k \\ \downarrow{\scriptstyle q_2^k} & & \downarrow{\scriptstyle t_k} \\ E_k & \xrightarrow[s_k]{} & Hom_{\mathcal{B}^*} \end{array}$$

2.8. The main construction: Taking limits. As the next step in the free globularily generated double category construction we apply a limiting procedure to the pieces of structure obtained in Theorem 2.5. We define an equivalence relation R_∞ such that after taking quotients modulo R_∞ we will obtain the necessary relations defining a double category. We keep the notation from Theorem 2.5.

2.9. Notation. Let $\mathcal{B}$ be a decorated bicategory. We write E_∞ for $\bigcup_{k=1}^{\infty} E_k$. We write d_∞ and c_∞ for $\lim d_k$ and $\lim c_k$ respectively. Thus defined d_∞ and c_∞ are functions from E_∞ to $\mathcal{B}_1$. We write $*_\infty$ for $\lim *_k$. Thus defined $*_\infty$ is a function from $E_\infty \times_{\mathcal{B}^*} E_\infty$ to E_∞. We write F_∞ for the category $\lim F_k$. The collection of objects of F_∞ is the collection of horizontal morphisms $\mathcal{B}_1$ of $\mathcal{B}$, and the collection of morphisms of F_∞ is E_∞. The domain and codomain functions of F_∞ are d_∞ and c_∞. We write s_∞, t_∞ for $\lim s_k$ and $\lim_k$ respectively. Thus defined s_∞ and t_∞ are functors from F_∞ to $\mathcal{B}^*$. Finally, we write $\bullet_\infty$ for the composition operation of F_∞. Thus defined $\bullet_\infty$ is equal to $\lim \bullet_k$ where for every k $\bullet_k$ is the composition operation of F_k.

2.10. Definition. *Let $\mathcal{B}$ be a decorated bicategory. We write R_∞ for the equivalence relation generated by the following relations defined on E_∞:*

1. *Let Φ_i, Ψ_i, $i = 1, 2$ be morphisms in F_∞ such that the pairs Φ_1, Φ_2 and Ψ_1, Ψ_2 are compatible with respect to the pair s_∞, t_∞ and such that the pairs Φ_i, Ψ_i, $i = 1, 2$ are both compatible with respect to the pair d_∞, c_∞. We identify the compositions:*

$$(\Psi_2 *_\infty \Psi_1) \bullet_\infty (\Phi_2 *_\infty \Phi_1) \text{ and } (\Psi_2 \bullet_\infty \Phi_2) *_\infty (\Psi_1 \bullet_\infty \Phi_1)$$

2. *Let Φ and Ψ be globular squares of $\mathcal{B}$ such that the pair Φ, Ψ is compatible with respect to s_∞, t_∞. We identify $\Psi \bullet_\infty \Phi$ with the vertical composition $\Psi\Phi$ of Φ and Ψ in $\mathcal{B}$. Moreover, if α and β are vertical morphisms in $\mathcal{B}$ such that the pair α, β is composable in $\mathcal{B}^*$, then we identify $i_\beta \bullet_\infty i_\alpha$ with $i_{\beta\alpha}$.*

3. *Let Φ and Ψ be globular squares of $\mathcal{B}$ such that the pair Φ, Ψ is compatible with respect to s_∞, t_∞. We identify $\Psi *_\infty \Phi$ with the horizontal composition $\Psi * \Phi$ in $\mathcal{B}$.*

4. *Let Φ be a morphism in F_∞. We identify Φ with the compositions*

$$\lambda_{c_\infty \Phi} \bullet_\infty (\Phi *_\infty i_{t_\infty \Phi}) \bullet_\infty \lambda^{-1}_{d_\infty \Phi} \text{ and } \rho_{c_\infty \Phi} \bullet_\infty (i_{s_\infty \Phi} *_\infty \Phi) \bullet_\infty \rho^{-1}_{d_\infty \Phi}$$

where λ and ρ denote the left and right identity transformations of the bicategory underlying $\mathcal{B}$.

5. *Let Φ, Ψ, Θ be elements of E_∞ such that the triple Φ, Ψ, Θ is compatible with respect to the pair s_∞, t_∞. In that case we identify the compositions:*

$$A_{c_\infty \Phi, c_\infty \Psi, c_\infty \Theta} \bullet_\infty [\Theta *_\infty (\Psi *_\infty \Phi)] \text{ and } [(\Theta *_\infty \Psi) *_\infty \Phi] \bullet_\infty A_{d_\infty \Phi, d_\infty \Psi, d_\infty \Theta}$$

where A denotes the associator of the bicategory underlying $\mathcal{B}$.

2.11. LEMMA. *Let $\mathcal{B}$ be a decorated bicategory. R_∞ is compatible with the domain, codomain, and composition operation functions d_∞, c_∞, and $\bullet_\infty$ of F_∞.*

PROOF. Let $\mathcal{B}$ be a decorated bicategory. We wish to prove that the equivalence relation R_∞ is compatible with domain, codomain, and composition operation functions d_∞, c_∞, and $\bullet_\infty$ defining the category structure on F_∞.

Let Φ_i, Ψ_i, $i = 1, 2$ be morphisms in F_∞ such that the pairs Φ_1, Φ_2 and Ψ_1, Ψ_2 are compatible with respect to source and target functors s_∞, t_∞ of F_∞ and such that the pairs Φ_i, Ψ_i, $i = 1, 2$ are compatible with respect to domain and codomain functions d_∞, c_∞. In that case the domain

$$d_\infty(\Psi_1 *_\infty \Psi_1) \bullet_\infty (\Phi_2 *_\infty \Phi_1)$$

of $(\Psi_1 *_\infty \Psi_1) \bullet_\infty (\Phi_2 *_\infty \Phi_1)$ is equal to the domain $d_\infty(\Phi_2 *_\infty \Phi_1)$ of $(\Phi_2 *_\infty \Phi_1)$ which in turn is equal to the composition $d_\infty \Phi_2 d_\infty \Phi_1$ of the domains $d_\infty \Phi_1$ and $d_\infty \Phi_2$ of Φ_1 and Ψ_1. Now, the domain

$$d_\infty(\Psi_2 \bullet_\infty \Phi_2) *_\infty (\Psi_1 \bullet_\infty \Phi_1)$$

of the composition $(\Psi_1 \bullet_\infty \Phi_1) *_\infty (\Psi_2 \bullet_\infty \Phi_2)$ is equal to the composition

$$d_\infty(\Psi_2 \bullet_\infty \Phi_2) d_\infty(\Psi_1 \bullet_\infty \Phi_1)$$

of the domain $d_\infty(\Psi_1 \bullet_\infty \Phi_1)$ of the composition $\Psi_1 \bullet_\infty \Phi_1$ and the domain $d_\infty(\Psi_2 \bullet_\infty \Phi_2)$ of the composition $\Psi_2 \bullet_\infty \Phi_2$. The domain $d_\infty(\Psi_1 \bullet_\infty \Phi_1)$ of the composition $\Psi_1 \bullet_\infty \Phi_1$ is equal to the domain $d_\infty \Phi_1$ of Φ_1 and the domain $d_\infty(\Psi_2 \bullet_\infty \Phi_2)$ of the composition $\Psi_2 \bullet_\infty \Phi_2$ is equal to the domain $d_\infty \Phi_2$ of Φ_2. Thus the domain

$$d_\infty(\Psi_2 \bullet_\infty \Phi_2) *_\infty (\Psi_1 \bullet_\infty \Phi_1)$$

of the composition $(\Psi_2 \bullet_\infty \Phi_2) *_\infty (\Psi_1 \bullet_\infty \Phi_1)$ is equal to the composition $d_\infty d_\infty \Phi_2 d_\infty \Phi_1$ of the domain $d_\infty \Phi_1$ of Φ_1 and the domain $d_\infty \Phi_2$ of Φ_2. We conclude that equivalence relation 1 in the definition of R_∞ is compatible with respect to the domain function d_∞ of F_∞. A similar computation proves that relation 1 in the definition of R_∞ is compatible with respect to the codomain function c_∞ of F_∞.

Let now Φ, Ψ be elements of $\mathbb{G}$. suppose pair Φ, Ψ is compatible with respect to the domain and codomain functions d_∞ and c_∞ of F_∞. In that case the domain and codomain $d_\infty \Psi \bullet_\infty \Phi$ and $\Psi \bullet_\infty \Phi$ of the composition $\Psi \bullet_\infty \Phi$ are equal to the domain $d_\infty \Phi$ of Φ and the codomain $c_\infty \Psi$ of Ψ respectively. The domain and codomain $d_\infty \Psi\Phi$ and $c_\infty \Psi\Phi$ of the vertical composition $\Psi\Phi$ of Φ and Ψ is equal to the domain $d_\infty \Phi$ of Φ and the codomain $c_\infty \Psi$ of Ψ. We conclude that relation 2 in the definition of R_∞ is compatible with the domain and codomain functions d_∞ and c_∞ of F_∞.

Let Φ and Ψ be globular squares in $\mathcal{B}$. Suppose that the pair Φ, Ψ is compatible with respect to the morphism functions of functors s_∞ and t_∞. In that case the domain $d_\infty \Psi *_\infty \Phi$ and the codomain $c_\infty \Psi *_\infty \Phi$ of the horizontal composition $\Psi *_\infty \Phi$ of Φ and Ψ are equal to the compositions, in $\mathcal{B}$, $d_\infty \Psi d_\infty \Phi$ and $c_\infty \Psi d_\infty \Phi$ respectively. Now, the domain $d_\infty \Psi * \Phi$ and the codomain $c_\infty \Psi * \Phi$ of the horizontal composition, in $\mathcal{B}$, of Φ and Ψ is equal to the compositions $d_\infty \Phi d_\infty \Psi$ and $c_\infty \Phi c_\infty \Psi$ respectively. This proves that relation 3 in the definition of R_∞ is compatible with the domain and codomain functions d_∞ and c_∞ of F_∞.

Let now Φ be a general morphism in F_∞. In that case the domain

$$d_\infty \lambda_{c_\infty \Phi} \bullet_\infty (\Phi *_\infty i_{t_\infty \Phi}) \bullet_\infty \lambda^{-1}_{d_\infty \Phi}$$

of the composition $\lambda_{c_\infty \Phi} \bullet_\infty (\Phi *_\infty i_{t_\infty \Phi}) \bullet_\infty \lambda^{-1}_{d_\infty \Phi}$ is equal to the domain $d_\infty \lambda^{-1}_{d_\infty \Phi}$ of $\lambda^{-1}_{d_\infty \Phi}$, which is equal to the domain $d_\infty \Phi$ of Φ. Similarly the domain

$$d_\infty \rho_{c_\infty \Phi} \bullet_\infty (i_{s_\infty \Phi} *_\infty \Phi) \bullet_\infty \rho^{-1}_{d_\infty \Phi}$$

of the composition $\rho_{c_\infty \Phi} \bullet_\infty (i_{s_\infty \Phi} *_\infty \Phi) \bullet_\infty \rho^{-1}_{d_\infty \Phi}$ is equal to the domain $d_\infty \rho^{-1}_{d_\infty \Phi}$ of $\rho^{-1}_{d_\infty \Phi}$, which is equal to the domain $d_\infty \Phi$ of Φ. We conclude, from this that relation 4 in the definition of R_∞ is compatible with the domain function d_∞ of F_∞. An analogous computation proves that relation 4 in the definition of R_∞ is compatible with the codomain function c_∞ of F_∞.

Let Φ, Ψ, Θ be general morphisms in F_∞. Suppose that the triple Φ, Ψ, Θ is compatible with respect to the morphism functions of s_∞ and t_∞. In that case the domain

$$d_\infty A_{c_\infty \Phi, c_\infty \Psi, c_\infty \Theta} \bullet_\infty [\Theta *_\infty (\Psi *_\infty \Phi)]$$

is equal to the composition $A_{c_\infty \Phi, c_\infty \Psi, c_\infty \Theta} \bullet_\infty [\Theta *_\infty (\Psi *_\infty \Phi)]$ is equal to domain $d_\infty \Theta *_\infty (\Psi *_\infty \Phi)$ of the composition $\Theta *_\infty (\Psi *_\infty \Phi)$ which in turn is equal to the composition $d_\infty \Theta d_\infty \Psi d_\infty \Phi$ in $\mathcal{B}$. Now, the domain

$$d_\infty [(\Theta *_\infty \Psi) *_\infty \Phi] \bullet_\infty A_{d_\infty \Phi, d_\infty \Psi, d_\infty \Theta}$$

is equal to the domain $d_\infty A_{d_\infty\Phi,d_\infty\Psi,d_\infty\Theta}$ of the associator $A_{d_\infty\Phi,d_\infty\Psi,d_\infty\Theta}$ associated to the triple Φ,Ψ,Θ, which is, by definition, equal to the composition $d_\infty\Theta d_\infty\Psi d_\infty\Phi$ in $\mathcal{B}$. We conclude that relation 5 in the definition of R_∞ is compatible with the domain function d_∞ of F_∞. An analogous computation proves that relation 5 in the definition of R_∞ is compatible with the codomain function c_∞ of F_∞.

Finally, the fact that the equivalence relation R_∞ is compatible with respect to the composition function $\bullet_\infty$ on F_∞ follows from the fact that F_∞ is the limit of a sequence of free categories. ■

2.12. The main construction: Dividing by R_∞. As the next step of the free globularly generated double category construction we divide the category F_∞ defined in 2.9 by the equivalence relation R_∞ and we prove that the structure thus obtained is compatible with the rest of the pieces of structure in Theorem 2.5.

2.13. Definition. *Let $\mathcal{B}$ be a decorated bicategory. We write V_∞ for the quotient category F_∞/R_∞. We keep writing d_∞, c_∞, and $\bullet_\infty$ for the domain, codomain, and composition operation functions in V_∞. We write H_∞ for the collection of morphisms of V_∞. Thus defined H_∞ is equal to the quotient E_∞/R_∞ of the collection of morphisms E_∞ of F_∞ modulo R_∞.*

2.14. Lemma. *Let $\mathcal{B}$ be a decorated bicategory. In that case the source and target functors s_∞ and t_∞, and the horizontal composition functor $*_\infty$ are all compatible with R_∞.*

Proof. Let $\mathcal{B}$ be a decorated bicategory. We wish to prove that in that the source and target functors s_∞ and t_∞, and the horizontal composition functor $*_\infty$ defined on F_∞ associated to $\mathcal{B}$ are compatible with R_∞.

Let first Φ_i,Ψ_i, $i=1,2$ be morphisms in F_∞ such that the pairs Φ_i,Ψ_i, $i=1,2$ are compatible with respect to the domain and codomain functions d_∞ and c_∞ in F_∞ and such that the pairs Φ_1,Φ_2 and Ψ_1,Ψ_1 are compatible with respect to the source and target functors s_∞,t_∞ in F_∞. In that case the source

$$s_\infty(\Psi_2 *_\infty \Psi_1)\bullet_\infty(\Phi_2 *_\infty \Phi_1)$$

of composition $(\Psi_2 *_\infty \Psi_1)\bullet_\infty(\Phi_2 *_\infty \Phi_1)$ is equal to the composition

$$s_\infty(\Psi_2 *_\infty \Psi_1)s_\infty(\Phi_2 *_\infty \Phi_1)$$

in the decoration $\mathcal{B}^*$ of $\mathcal{B}$ of the vertical morphisms $s_\infty(\Phi_2 *_\infty \Phi_1)$ and $s_\infty(\Psi_2 *_\infty \Psi_1)$. The source $s_\infty(\Phi_2 *_\infty \Phi_1)$ of the concatenation $\Phi_2 *_\infty \Phi_1$ is equal to the source $s_\infty\Phi_1$ of Φ_1 and the source $s_\infty(\Psi_2 *_\infty \Psi_1)$ of the concatenation $\Psi_2 *_\infty \Psi_1$ is equal to the source $s_\infty\Psi_2$. The source

$$s_\infty(\Psi_2 *_\infty \Psi_1)\bullet_\infty(\Phi_2 *_\infty \Phi_1)$$

of composition $(\Psi_2 *_\infty \Psi_1) \bullet_\infty (\Phi_2 *_\infty \Phi_1)$ is thus equal to the composition $s_\infty\Psi_1 s_\infty\Phi_1$ in $\mathcal{B}^*$. Now the source

$$s_\infty(\Psi_2 \bullet_\infty \Phi_2) *_\infty (\Psi_1 \bullet_\infty \Phi_1)$$

of the concatenation $(\Psi_2 \bullet_\infty \Phi_2) *_\infty (\Psi_1 \bullet_\infty \Phi_1)$ is equal to the source $s_\infty(\Psi_1 \bullet_\infty \Phi_1)$ of composition $\Psi_1 \bullet_\infty \Phi_1$. Now, the source $s_\infty(\Psi_1 \bullet_\infty \Phi_1)$ is equal to the composition $s_\infty\Psi_1 s_\infty\Phi_1$ of the source $s_\infty\Phi_1$ and $s_\infty\Psi_1$ in $\mathcal{B}^*$. This proves that the source functor s_∞ is compatible with respect to relation 1 in the definition of R_∞. An analogous argument proves that the target functor t_∞ in F_∞ is compatible with respect to relation 1 in the definition of R_∞.

Let now Φ and Ψ be morphisms in $\mathbb{G}$ such that the pair Φ, Ψ is compatible with respect to the domain and codomain functions d_∞ and c_∞ in F_∞. Suppose first that Φ and Ψ are globular morphisms in $\mathcal{B}$ such that the domain in $\mathcal{B}$ of the domain and codomain of Φ and Ψ in $\mathcal{B}$ respectively are equal to the object a in $\mathcal{B}$. In that case the source $s_\infty\Psi \bullet_\infty \Phi$ of the composition $\Psi \bullet_\infty \Phi$ is equal to the composition $s_\infty\Psi s_\infty\Phi$ of $s_\infty\Phi$ and $s_\infty\Phi$ in $\mathcal{B}^*$. Now the source $s_\infty\Psi$ of Φ and the source $s_\infty\Psi$ of Ψ are both equal to the identity endomorphism id_a of the object a in the decoration $\mathcal{B}^*$ of $\mathcal{B}$. Now the domain in $\mathcal{B}$ of the vertical composition $\Psi \bullet \Phi$ in $\mathcal{B}$ is equal to the domain of Φ in $\mathcal{B}$ and thus the domain of the domain in $\mathcal{B}$ of the vertical composition $\Psi \bullet \Phi$ of Φ and Ψ is equal to the object a of $\mathcal{B}$. It follows, from this, that the source $s_\infty\Psi \bullet \Phi$ of the globular morphism in $\mathcal{B}$ formed as the vertical composition $\Psi \bullet \Phi$ of Φ and Ψ in $\mathcal{B}$ is equal to the identity endomorphism id_a of the object a in $\mathcal{B}$. The source functor s_∞ in F_∞ is thus compatible with the restriction to the collection of globular morphisms of $\mathcal{B}$ of relation 2 in the definition of R_∞. Suppose now that the morphisms Φ and Ψ are formal horizontal identities i_α and i_β respectively, of a composable pair of vertical morphisms α, β in $\mathcal{B}$. In this case the source $s_\infty i_\beta \bullet_\infty i_\alpha$ of the vertical composition $i_\beta \bullet_\infty i_\alpha$ of i_α and i_β is equal to the composition $s_\infty i_\beta s_\infty i_\alpha$ of $s_\infty i_\alpha$ and i_β in $\mathcal{B}^*$. The source $s_\infty i_\alpha$ is equal to the morphism α and the source $s_\infty i_\beta$ of i_β is equal to the morphism β. We conclude that the source $s_\infty i_\beta \bullet_\infty i_\alpha$ of the composition $i_\beta \bullet_\infty i_\alpha$ is equal to the composition $\beta\alpha$ of α and β in $\mathcal{B}^*$. Finally, the source $s_\infty i_{\beta\alpha}$ of the formal horizontal identity $i_{\beta\alpha}$ of the composition $\beta\alpha$ is equal to the composition $\beta\alpha$. This proves that the source functor s_∞ is compatible with the restriction to the collection of formal horizontal identities of relation 2 in the definition of R_∞. We conclude that the source functor s_∞ is compatible with respect to relation 2 in the definition of R_∞. An analogous argument proves that the target functor t_∞ in F_∞ is compatible with respect to relation 2 in the definition of R_∞.

Let now Φ and Ψ be globular morphisms in $\mathcal{B}$ such that the pair Φ, Ψ is compatible with respect to the source and the target functors s_∞, t_∞ in F_∞. The source $s_\infty\Psi *_\infty \Phi$ of the concatenation $\Psi *_\infty \Phi$ of Φ and Ψ is equal to the source $s_\infty\Phi$ which is equal to the identity endomorphism in $\mathcal{B}^*$ of the domain of the domain in $\mathcal{B}$ of Φ. Now, the source $s_\infty\Psi * \Phi$ of the globular morphism in $\mathcal{B}$ formed as the horizontal composition $\Psi * \Phi$ in $\mathcal{B}$ of Φ and Ψ is equal to the identity endomorphism in $\mathcal{B}^*$ of the domain of the domain in $\mathcal{B}$ of the composition $\Psi * \Phi$. The domain of the domain of the horizontal composition $\Phi * \Phi$

is equal to the domain of the domain of Φ. We conclude that the source $s_\infty\Psi * \Phi$ of the horizontal composition $\Psi * \Phi$ is equal to the identity endomorphism in $\mathcal{B}^*$ of the domain of the domain in $\mathcal{B}$ of Φ. The source functor s_∞ in $\mathcal{B}$ is thus compatible with relation 3 in the definition of R_∞. An identical argument proves that the target functor t_∞ in F_∞ is compatible with relation 3 in the definition of R_∞.

Let now Φ be a general morphism in F_∞. In that case the source

$$s_\infty\lambda_{c_\infty\Phi} \bullet_\infty (\Phi *_\infty i_{t_\infty\Phi}) \bullet_\infty \lambda^{-1}_{d_\infty\Phi}$$

of the composition $\lambda_{c_\infty\Phi} \bullet_\infty (\Phi *_\infty i_{t_\infty\Phi}) \bullet_\infty \lambda^{-1}_{d_\infty\Phi}$ is equal to the composition

$$s_\infty\lambda_{c_\infty\Phi}s_\infty(\Phi *_\infty i_{t_\infty\Phi})s_\infty\lambda^{-1}_{d_\infty\Phi}$$

in $\mathcal{B}^*$ of $s_\infty\lambda_{c_\infty\Phi}$, $s_\infty(\Phi *_\infty i_{t_\infty\Phi})$, and $s_\infty\lambda^{-1}_{d_\infty\Phi}$. Now, since $\lambda_{c_\infty\Phi}$ and $\lambda_{d_\infty\Phi}$ are globular, the composition

$$s_\infty\lambda_{c_\infty\Phi}s_\infty(\Phi *_\infty i_{t_\infty\Phi})s_\infty\lambda^{-1}_{d_\infty\Phi}$$

is equal to the source $s_\infty\Phi *_\infty i_{t_\infty\Phi}$ of $\Phi *_\infty i_{t_\infty\Phi}$, which is equal to the source $s_\infty\Phi$ of Φ. We conclude that the source functor s_∞ on F_∞ is compatible with relation 3 in the definition of R_∞. An analogous argument proves that the target functor t_∞ is compatible with relation 4 in the definition of R_∞. Further, an analogous argument proves that the source and target functors s_∞ and t_∞ in F_∞ are compatible with relation 5 in the definition of R_∞. ∎

2.15. The main construction: Observations on lemma 2.11.
Before presenting the definition of the free globularly generated double category we present a few preliminary observations on lemma 2.11.

2.16. Observation. *Let $\mathcal{B}$ be a decorated bicategory. By lemma 3.9 the functors s_∞ and t_∞ descend to functors from V_∞ to $\mathcal{B}^*$. We keep denoting these functors by s_∞ and t_∞. Moreover, the composition operation function $*_\infty$ descends to a function from $H_\infty \times_{Hom_{\mathcal{B}^*}} H_\infty$ to H_∞ such that, by relation 1 in the definition of R_∞, together with the composition operation function for horizontal morphisms in $\mathcal{B}$ forms a functor from $V_\infty \times_{\mathcal{B}^*} V_\infty$ to V_∞. We denote this functor by $*_\infty$.*

2.17. Observation. *Let $\mathcal{B}$ be a decorated bicategory. Let k be a positive integer. In that case the relation R_∞ restricts to an equivalence relation in E_k. We denote by H_k the quotient E_k/R_∞ of E_k modulo R_∞. Moreover, R_∞ restricts to an equivalence relation on the collection of morphisms Hom_{F_k} of F_k. The relation R_∞ is compatible with the domain and codomain functions d_k and c_k of F_k and is thus compatible with the category structure of F_k. We denote by V_k the quotient F_k/R_∞ of F_k modulo R_∞ and keep denoting by d_k, c_k, and $\bullet_k$ the domain, the codomain, and the composition operation functions in V_k. The functors s_{k+1} and t_{k+1} are compatible with R_∞ and thus induce functors from V_k to the decoration $\mathcal{B}^*$ of $\mathcal{B}$. We keep denoting these functors by s_{k+1} and t_{k+1} respectively.*

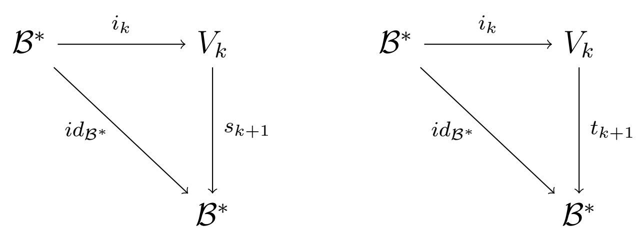
Finally, observe that the function $*_k$ *is compatible with* R_∞ *and thus defines a function from the set of all morphisms of* $V_k \times_{\mathcal{B}^*} V_k$ *to the set of morphisms of* V_k. *This function, together with the composition operation function for horizontal morphisms in* $\mathcal{B}$ *forms a functor from* $V_k \times_{\mathcal{B}^*} V_k$ *to* V_k. *We keep denoting this functor by* $*_k$.

2.18. Observation. *Let* $\mathcal{B}$ *be a decorated bicategory. Let* m, k *be positive integers such that* m *is less than or equal to* k. *In that case* H_m *is contained in* H_k, *the category* V_m *is a subcategory of the category* V_k, *the functors* s_{m+1} *and* t_{m+1} *are restrictions to* V_m *of functors* s_{k+1} *and* t_{k+1}, *and the functor* $*_m$ *is the restriction to* $V_m \times_{\mathcal{B}^*} V_m$ *of the functor* $*_k$. *Moreover, the category* V_∞ *is equal to the limit* $\lim V_k$ *of the sequence* V_k, *the collection of morphisms* H_∞ *of* V_∞ *is equal to the union* $\bigcup_{k=1}^\infty H_k$ *of the sequence* H_k, *and the functors* s_∞, t_∞ *and* $*_\infty$ *are the limits of the sequences of functors* s_k, t_k *and* $*_k$ *respectively.*

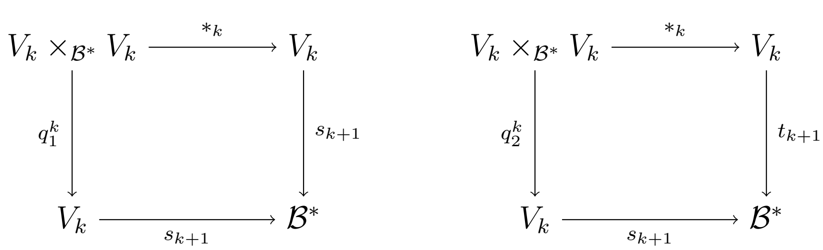
2.19. Notation. Let $\mathcal{B}$ be a decorated bicategory. The pair formed by the function associating the horizontal identity i_a to every object a of $\mathcal{B}$ and the function associating the formal horizontal identity i_α to every vertical morphism α in $\mathcal{B}$ defines a functor from the decoration $\mathcal{B}^*$ of $\mathcal{B}$ to the category V_∞ associated to $\mathcal{B}$. We denote this functor by i_∞. For every positive integer k we denote the codomain restriction to the category V_k of the functor i_∞ by i_k. Thus defined, i_k is a functor from the decoration $\mathcal{B}^*$ of $\mathcal{B}$ to the category V_k associated to $\mathcal{B}$ for every positive integer k.

2.20. Lemma. *Let* $\mathcal{B}$ *be a decorated bicategory. Let* k *be a positive integer. The functors* s_{k+1}, t_{k+1}, i_k, *and* $*_k$ *satisfy the following two conditions:*

1. *The following two triangles commute*



2. *The following two squares commute, where* q_1^k, q_2^k *denote the left and right projection functors from* $V_k \times_{\mathcal{B}^*} V_k$ *to* V_k *respectively.*



PROOF. Let $\mathcal{B}$ be a decorated bicategory. Let k be a positive integer. We wish to prove that conditions 1 and 2 above are satisfied.

We begin by proving that the functors s_{k+1}, t_{k+1}, and i_k satisfy condition 1 above. Let α be a vertical morphism in $\mathcal{B}$. The formal horizontal identity i_α associated to α, that is, the image $i_k\alpha$ of α under the functor i_k, is a morphism in V_1. From this and from the fact that the sequences s_{k+1} and t_{k+1} satisfy the conditions of proposition 3.2 it follows that the commutativity of triangles in 1 is equivalent to the commutativity of the following triangles:

$$\begin{array}{ccc} \mathcal{B}^* & \xrightarrow{i_1} & V_1 \\ & \searrow^{id_{V_1}} & \downarrow s_2 \\ & & \mathcal{B}^* \end{array} \qquad \begin{array}{ccc} \mathcal{B}^* & \xrightarrow{i_1} & V_1 \\ & \searrow^{id_{V_1}} & \downarrow t_2 \\ & & \mathcal{B}^* \end{array}$$

which follows directly from the definition of the functions d_1 and c_1. We now prove that the functors s_{k+1}, t_{k+1}, and $*_k$ satisfy condition 2 above. The commutativity of squares in 2 when evaluated on morphisms of $H_k \times_{\mathrm{Hom}_{\mathcal{B}^*}} H_k$ follows from observation 3.4. The general commutativity of the squares in condition 2 follows from this and from the fact that all edges involved are functors. This concludes the proof of the lemma. ■

The following corollary follows directly from the previous lemma by taking limits.

2.21. COROLLARY. *Let $\mathcal{B}$ be a decorated bicategory. The functors s_∞, t_∞, i, and $*_\infty$ satisfy the following two conditions:*

1. *The following two triangles commute:*

$$\begin{array}{ccc} \mathcal{B}^* & \xrightarrow{i} & V_\infty \\ & \searrow^{id_{\mathcal{B}^*}} & \downarrow s_\infty \\ & & \mathcal{B}^* \end{array} \qquad \begin{array}{ccc} \mathcal{B}^* & \xrightarrow{i} & V_\infty \\ & \searrow^{id_{\mathcal{B}^*}} & \downarrow t_\infty \\ & & \mathcal{B}^* \end{array}$$

2. *The following two squares commute, where q_1^∞, q_2^∞ denote the left and right projection functors from $V_\infty \times_{\mathcal{B}^*} V_\infty$ to V_∞ respectively.*

$$\begin{array}{ccc} V_\infty \times_{\mathcal{B}^*} V_\infty & \xrightarrow{*_\infty} & V_\infty \\ {\scriptstyle q_1^\infty}\downarrow & & \downarrow{\scriptstyle s_\infty} \\ V_\infty & \xrightarrow[s_\infty]{} & \mathcal{B}^* \end{array} \qquad \begin{array}{ccc} V_\infty \times_{\mathcal{B}^*} V_\infty & \xrightarrow{*_\infty} & V_\infty \\ {\scriptstyle q_2^\infty}\downarrow & & \downarrow{\scriptstyle t_\infty} \\ V_\infty & \xrightarrow[s_{k+1}]{} & \mathcal{B}^* \end{array}$$

2.22. The main construction: The definition. Let $\mathcal{B}$ be a decorated bicategory. We will denote by $Q_{\mathcal{B}}$ the pair formed by the decoration $\mathcal{B}^*$ of $\mathcal{B}$ and the category V_∞ associated to $\mathcal{B}$. The following theorem says that we can endow the pair $Q_{\mathcal{B}}$ with the structure of a globularly generated double category.

2.23. Theorem. *Let $\mathcal{B}$ be a decorated bicategory. The pair $Q_{\mathcal{B}}$ together with functors s_∞, t_∞, i, the functor $*_\infty$, and the collection of left and right identity transformations, and associator of $\mathcal{B}$, is a double category. Moreover, with this structure, the double category $Q_{\mathcal{B}}$, is globularly generated.*

Proof. Let $\mathcal{B}$ be a decorated bicategory. We wish to prove in this case that the pair $Q_{\mathcal{B}}$, together with the functors s_∞, t_∞, i, the functor $*_\infty$, and the collection of left and right identity transformations and associator of $\mathcal{B}$ is a globularly generated double category.

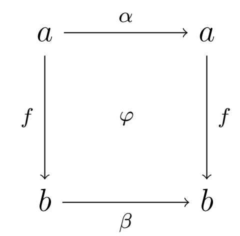
The formal horizontal identity functor i and the horizontal composition functor $*_\infty$ are compatible with the functors s_∞ and t_∞ by corollary 3.15. The collections of left identity transformations and right identity transformations of $\mathcal{B}$ form a natural transformation from $*_\infty(is_\infty \times id_{V_\infty})$ to the identity endofunctor id_{V_∞} of V_∞ and a natural transformation from $*_\infty(id_{V_\infty} \times it_\infty)$ to the identity endofunctor id_{V_∞} of V_∞ respectively by the fact that morphisms in V_∞ satisfy relation 4 in the definition of R_∞. The collection of associators of $\mathcal{B}$ forms a natural transformation from the composition $*_\infty(*_\infty \times id_{V_\infty})$ to the composition $*_\infty(id_{V_\infty} \times *_\infty)$ by the fact that morphisms in V_∞ satisfy relation 5 in the definition of R_∞. The left and right identity and the associator relations for $Q_{\mathcal{B}}$ again follow from the fact that morphisms in V_∞ satisfy relations 4 and 5 in the definition of R_∞. The fact that the pair formed by the functor i and functor $*$ satisfies Mc Lane's triangle and pentagon relations with respect to the left and right identity transformations and associator follows from the fact that the components of the left and right identity transformations and associator satisfy Mac Lane's axioms for the bicategory $\mathcal{B}$. This proves that $Q_{\mathcal{B}}$ with the structure described is a double category. A straightforward induction argument proves that for every positive integer k every morphism of V_k is globularly generated in $Q_{\mathcal{B}}$, from which it follows that the double category $Q_{\mathcal{B}}$ is globularly generated. This concludes the proof of the theorem. ∎

2.24. Definition. *Let $\mathcal{B}$ be a decorated bicategory. We call the globularly generated double category $Q_{\mathcal{B}}$ the free globularly generated double category associated to $\mathcal{B}$.*

Lemma 2.20 provides the free globularly generated double category $Q_{\mathcal{B}}$ associated to a decorated bicategory $\mathcal{B}$ with a filtration $\{V_k\}$ of its category of morphisms $Q_{\mathcal{B}_1}$. We call this filtration the **free vertical filtration** associated to $Q_{\mathcal{B}}$. We use this filtration to define numerical invariants for $Q_{\mathcal{B}}$. Given a square φ in $Q_{\mathcal{B}}$ we say that φ is of **free length** k, $\ell_{free}\varphi = k$ in symbols, if φ is a morphism in V_k and φ is not a morphism in V_{k-1}. Further, we say that $Q_{\mathcal{B}}$ has free vertical lenght $k \in \mathbb{N} \cup \{\infty\}$, $\ell_{free}Q_{\mathcal{B}} = k$ in symbols, if k is the supremum of all free vertical lengths of squares in $Q_{\mathcal{B}}$. The free vertical filtration $\{V_k\}$ of $Q_{\mathcal{B}}$ might differ from the vertical filtration $\left\{V_k^{Q_{\mathcal{B}}}\right\}$ associated to $Q_{\mathcal{B}}$ as a globularly generated double category in [16]. The free length $\ell_{free}\varphi$ of a square φ in

$Q_{\mathcal{B}}$thus might differ from the length $\ell\varphi$ of φ and correspondingly the free length $\ell_{free}Q_{\mathcal{B}}$ of $Q_{\mathcal{B}}$ might differ from the length $\ell Q_{\mathcal{B}}$ of $Q_{\mathcal{B}}$ as a globularly generated double category. In section 3 and section 4 we study situations in which the free vertical filtration and the usual filtration of a free globularly generated double category coincide.

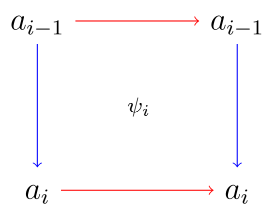
Using arguments analogous as those employed in the proof of [16, Lemma 4.2] it is easily proven that every free length 1 square φ in the free globularily generated double category associated to a decorated bicategory $\mathcal{B}$ admitting a pictorial representation as:



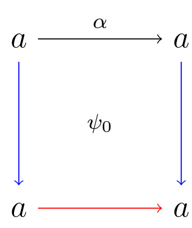
admits a factorization as a vertical composition of the form:

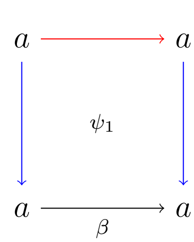
$$\psi_k \bullet_\infty i_{f_k} \bullet_\infty \psi_{k-1} \dots \psi_1 \bullet_\infty i_{f_1} \bullet_\infty \psi_0$$

where $f_i : a_{i-1} \longrightarrow a_i$ is a morphisms in $\mathcal{B}^*$ for every $1 \leq i \leq k$, ψ_i is a globular square, in $\mathcal{B}$ of the form:



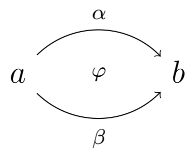
for every $1 \leq i \leq k-1$, and where ψ_0, ψ_k are globular squares of the form:





We will make strong use of this fact in the rest of the paper.

Let k be a field. We will understand for a k-linear decorated bicategory a decorated bicategory $\mathcal{B}$ such that both the underlying bicategory and the decoration of $\mathcal{B}$ are endowed with k-linear structures, where we understand for a k-linear structure on a bicategory $\mathcal{B}$ a structure of k-vector space for the set of 2-cells of the form:



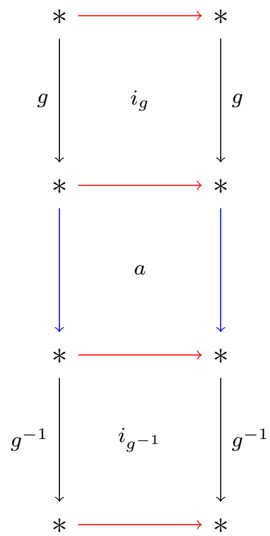
for any pair of 1-cells α, β in $\mathcal{B}$ fitting in a diagram as above, in such a way that all the corresponding structure and coherence data is k-linear. When the decorated bicategory $\mathcal{B}$ is endowed with a linear structure, the free globularly double category construction can be modified, in the obvious way, such that the resulting double category, which we denote $Q^k_{\mathcal{B}}$, is endowed with the structure of a category internal to k-linear categories. We study this modification of the globularly generated double category construction in the context of categorical aspects of the representation theory of von Neumann algebras in section 6.

Finally, it is natural to expect relations between the free globularly generated double category construction and the free double category construction of Dawson and Paré [8]. Let G be a reflexive double graph. Write $\mathcal{B}_G$ for the decorated horizontalization $H^*F(D)$ of the free double category generated by D. From the way the globularily generated double category was constructed it is easily seen that the free double category $F(D)$ generated by D and $Q_{\mathcal{B}_D}$ are related through the equation $Q_{\mathcal{B}_D} = \gamma F(D)$.

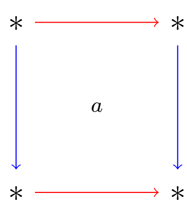
3. Free globularly generated internalizations

In this section we study situations in which the free globularly generated double category construction provides solutions to Problem 1.2. The following example shows that the free globularly generated double category construction does not always provide solutions to this problem. Given a monoid M we write ΩM for the delooping category of M, i.e. ΩM is the category with a single object $*$ whose monoid of endomorphisms $End_{\Omega M}(*)$ is M. Given a monoidal category D we write $2D$ for the delooping bicategory of D, i.e. ΩD is the single object bicategory whose monoidal category of endomorphisms is D. Observe that given a monoid M the delooping category ΩM of M admits the structure of strict monoidal category if and only if M is commutative by the Eckman-Hilton argument [10].

3.1. Example. Let G be a group such that $G \neq \{1\}$. Let A be an abelian group such that $A \neq \{0\}$. Let $\mathcal{B}$ be the decorated bicategory whose underlying bicategory is the single 0-cell and single 1-cell 2-category $2\Omega A$ and whose decoration $\mathcal{B}^*$ is the delooping category ΩG. The free globularly generated double category $Q_{\mathcal{B}}$ associated to $\mathcal{B}$ does not provide solutions to Problem 1.2 for $\mathcal{B}$. To see this consider the square:



in $Q_{\mathcal{B}}$, where g is any element of G such that $g \neq 1$ and where a is an element of A such that $a \neq 0$. We denote this square by φ. Thus defined φ satisfies the equations $t_\infty \varphi = s_\infty \varphi = g^{-1}g = 1$ and is thus globular in $Q_{\mathcal{B}}$. The only globular squares of $\mathcal{B}$ are the squares of the form:



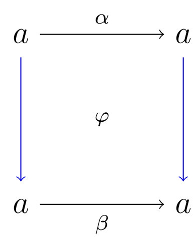
for $a \in A$. The square φ represents the word $g^{-1}ag$ in the free product $G * A$, which is not an element of A. We conclude that φ is a globular square in $H^*Q_{\mathcal{B}}$ not contained in $\mathcal{B}$ and thus that $Q_{\mathcal{B}}$ does not provide a solution to Problem 1.2 for $\mathcal{B}$.

We now provide conditions under which the free globularly generated double category $Q_{\mathcal{B}}$ associated to a decorated bicategory $\mathcal{B}$ does provide a solution to Problem 1.2. We say that a category $\mathcal{B}^*$ is reduced when the only left or right invertible morphisms of $\mathcal{B}$ are identities. Examples of reduced categories are delooping categories ΩM where M is a monoid without non-trivial left or right invertible (in particular M is a reduced monoid) categories associated to partially ordered sets, e.g. Open(X) for a topological space X, and path categories associated to graphs. The following proposition says that the free globularly generated double category associated to a decorated bicategory with reduced decoration provides solutions to Problem 1.2.

3.2. Proposition. Let $\mathcal{B}$ be a decorated bicategory. If $\mathcal{B}^*$ is reduced then the equation $H^*Q_{\mathcal{B}} = \mathcal{B}$ holds.

Proof. Let $\mathcal{B}$ be a decorated bicategory. Assume that $\mathcal{B}^*$ is reduced. We wish to prove that in this case the equation $H^*Q_{\mathcal{B}} = \mathcal{B}$ holds.

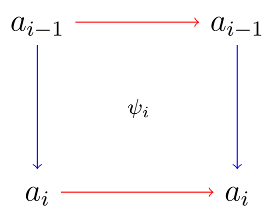
We proceed by induction on k to prove that every globular square in V_k is a globular square in $\mathcal{B}$. We begin by proving the statement for $k = 1$. Let φ be a globular square in V_1. By [16, Lemma 4.4] if φ is not a horizontal endomorphism of $Q_{\mathcal{B}}$ then φ is a globular square in $\mathcal{B}$. We thus assume that φ is a horizontal endomorphism in $Q_{\mathcal{B}}$. Represent φ pictorially as:



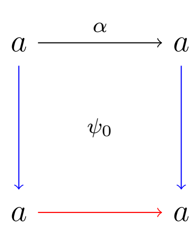
In that case φ can be written as a vertical composition, in $Q_{\mathcal{B}}$ of the form:

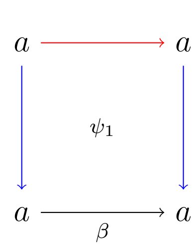
$$\psi_k \bullet_\infty i_{f_k} \bullet_\infty \psi_{k-1} \dots \psi_1 \bullet_\infty i_{f_1} \bullet_\infty \psi_0$$

where $f_i : a_{i-1} \longrightarrow a_i$ is a morphisms in $\mathcal{B}^*$ for every $1 \leq i \leq k$, ψ_i is a globular square, in $\mathcal{B}$ of the form:



for every $1 \leq i \leq k-1$, and where ψ_0, ψ_k are globular squares of the form:





From the fact that φ is globular it follows that the composition $f_k \dots f_1$ is equal to id_a. By the fact that $\mathcal{B}^*$ is reduced it follows that $a_i = a$ and $f_i = id_a$ for every $1 \leq i \leq k$. We conclude that φ is a vertical composition, in $Q_{\mathcal{B}}$ of globular squares of $\mathcal{B}$ and thus is a globular square in $\mathcal{B}$.

Let $k > 1$. Suppose that the result is true for all positive integers m such that $m < k$, i.e. suppose that every globular square in V_m is a globular square in $\mathcal{B}$ for every $m < k$. Let φ be a globular square in V_k. We prove that φ is a globular square in $\mathcal{B}$. Assume first that $\varphi \in H_k$. In that case φ admits a decomposition as $\psi_n *_\infty \cdots *_\infty \psi_1$ where ψ_i is a square in V_{k-1} for every i. The horizontal composition of non-globular squares is never globular, thus in the above case ψ_i is globular for every i. By the induction hypothesis φ is in this case horizontal composition of globular squares in $\mathcal{B}$ and is thus a globular square in $\mathcal{B}$. Now suppose that φ is a general square of V_k. In that case φ admits a decomposition as vertical composition $\psi_n \bullet_\infty \cdots \bullet_\infty \psi_1$ where φ_i is a globular square in H_k for every i. By the above argument every ψ_i is a globular square in $\mathcal{B}$ and thus φ is a globular square in $\mathcal{B}$. This concludes the proof of the proposition.

■

In the cases in which the free globularly generated double category $Q_\mathcal{B}$ associated to a decorated bicategory $\mathcal{B}$ is not an internalization of $\mathcal{B}$ we can always associate to $\mathcal{B}$ a larger decorated bicategory for which the free globularly generated double category construction does provide solutions to Problem 1.2. To see we first prove the following proposition.

3.3. Proposition. Let $\mathcal{B}$ be a decorated bicategory. In that case the equation $Q_{H^*Q_\mathcal{B}} = Q_\mathcal{B}$ holds.

Proof. Let $\mathcal{B}$ be a decorated bicategory. We wish to prove that the equation $Q_{H^*Q_\mathcal{B}} = Q_\mathcal{B}$ holds.

The categories of objects of $Q_\mathcal{B}$ and $Q_{H^*Q_\mathcal{B}}$ are both equal to $\mathcal{B}^*$. The collections of horizontal morphisms of $Q_\mathcal{B}$ and $Q_{H^*Q_\mathcal{B}}$ are both equal to $\mathcal{B}_1$. The collection of squares of $Q_\mathcal{B}$ is clearly contained in $Q_{H^*Q_\mathcal{B}}$. To prove the proposition we thus need to prove that every square of $Q_{H^*Q_\mathcal{B}}$ is a square in $Q_\mathcal{B}$. For every positive integer k we will write $\tilde{V}_k$ and $\tilde{H}_k$ for the category V_k associated to $H^*Q_\mathcal{B}$ and for the set H_k associated to $H^*Q_\mathcal{B}$ in lemma 2.20. We prove, by induction on k, that every square in $\tilde{V}_k$ is a square in $Q_\mathcal{B}$.

Let φ be a square in $\tilde{V}_1$. In that case φ admits a decomposition as:

$$\varphi = \psi_n \bullet_\infty i_{f_n} \bullet_\infty \psi_{n-1} \ldots \psi_1 \bullet_\infty i_{f_1} \bullet_\infty \psi_0$$

where $\psi_0, ..., \psi_n$ and $f_1, ..., f_n$ are as in the proof of proposition 3.2. Observe that each i_{f_j} is a square in V_1 and each ψ_j is a square in some V_{k_j} and thus is a square of $Q_\mathcal{B}$ for every i. φ is thus a square in $Q_\mathcal{B}$.

Let k be a positive integer such that $k > 1$. Suppose that the result is true for every $m \leq k$. We prove that every square in $\tilde{V}_k$ is a square in $Q_\mathcal{B}$. Let φ be a square in $\tilde{V}_k$. Suppose first that $\varphi \in \tilde{H}_k$. In that case φ admits a decomposition as $\varphi = \psi_n *_\infty \cdots *_\infty \psi_1$ where $\psi_1, ..., \psi_n$ are squares in $\tilde{V}_{k-1}$ and thus are squares in $Q_\mathcal{B}$. The square φ is thus a square in $Q_\mathcal{B}$. Suppose now that φ is a general square in $\tilde{V}_k$. In that case φ admits a decomposition as $\varphi = \psi_n \bullet_\infty \cdots \bullet_\infty \psi_1$ where ψ_i is a square in $\tilde{H}_k$ and is thus a square in $Q_\mathcal{B}$. The square φ is thus a square in $Q_\mathcal{B}$. This concludes the proof of the proposition. ■

Proposition 3.3 says that the operation of taking the free globularly generated double category is idempotent, i.e. stops at order 2. We have the following immediate corollary.

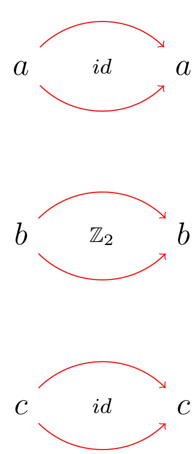
3.4. COROLLARY. *Let $\mathcal{B}$ be a decorated bicategory. In that case $Q_{\mathcal{B}}$ is an internalization of $H^*Q_{\mathcal{B}}$.*

Given a decorated bicategory $\mathcal{B}$ we call the decorated bicategory $H^*Q_{\mathcal{B}}$ the **saturation** of $\mathcal{B}$. We say that a decorated bicategory $\mathcal{B}$ is **saturated** whenever $\mathcal{B}$ is equal to its saturation $H^*Q_{\mathcal{B}}$. While the free globularly generated double category $Q_{\mathcal{B}}$ might not always provide a solution to Problem 1.2 for the decorated bicategory $\mathcal{B}$ provided as set of initial conditions, the free globularly generated double category $Q_{H^*Q_{\mathcal{B}}}$ always provides a solution to Problem 1.2 for the saturation $H^*Q_{\mathcal{B}}$ of $\mathcal{B}$. We compute saturations of certain decorated bicategories in the following sections. Observe that if a decorated bicategory $\mathcal{B}$ is saturated then the vertical filtration and the free vertical filtration of $Q_{\mathcal{B}}$ coincide and thus the free vertical lenght and the usual vertical length of squares in $Q_{\mathcal{B}}$ and of $Q_{\mathcal{B}}$ itself coincide. Decorated bicategories with reduced decorations are saturated by proposition 3.2.

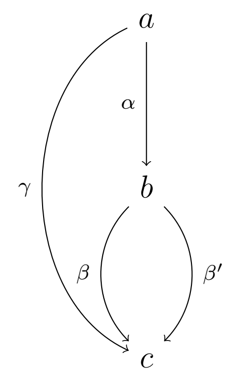
4. Length

In this section we apply the free globularly generated double category construction to provide examples of double categories of non-trivial length. All the examples of double categories considered in [16], i.e. trivial double categories, and double categories of bordisms, algebras and von Neumann algebras are proven to be of length 1. The following example proves that the concept of length of a double category is non-trivial by explicitly constructing a double category of length equal to 2.

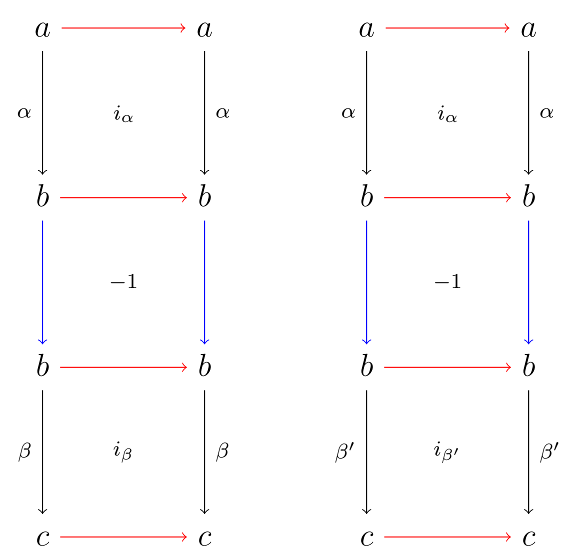
4.1. EXAMPLE. Let $\mathcal{B}$ denote the following 2-category: $\mathcal{B}$ has three objects a, b, c and only identity horizontal 1-cells. All 2-cells in $\mathcal{B}$ will be identities except for one vertical endomorphism cell of i_b. This cell, together with the identity 2-cell of i_b will form the group $\mathbb{Z}_2$ under both horizontal and vertical composition. Pictorially $\mathcal{B}$ is represented by the diagram:



Now decorate $\mathcal{B}$ with the following category $\mathcal{B}^*$: $\mathcal{B}^*$ has non-identity morphism $\alpha : a \longrightarrow b, \beta, \beta' : b \longrightarrow c$ and $\gamma : a \longrightarrow c$ satisfying the relation $\beta\alpha = \gamma = \beta'\alpha$. We represent $\mathcal{B}^*$ pictorially as:



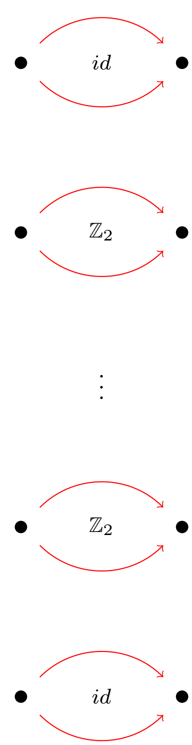
We claim that $\ell Q_{\mathcal{B}} = 2$. Observe first that since $\mathcal{B}^*$ is reduced, it is enough to prove that $\ell_{free} Q_{\mathcal{B}} = 2$. We exhibit a pair of horizontally composable squares φ, ψ in $Q_{\mathcal{B}}$ of vertical length 1 such that $\varphi *_\infty \psi$ is not a morphism in V_1. Write φ and ψ for the squares pictorially represented as:



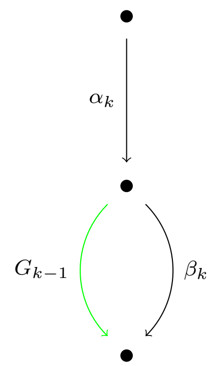
Thus defined φ, ψ satisfy the equation $t_\infty \varphi = s_\infty \varphi = \gamma$ and thus are horizontally composable in $Q_{\mathcal{B}}$. We prove that $\varphi *_\infty \psi$ is not a morphism in V_1. To do this we first observe that $s_\infty \varphi *_\infty \psi = t_\infty \varphi *_\infty \psi = \gamma$. The only squares in V_1 with source and target equal to γ are i_γ and the squares φ and ψ. To see that $\varphi *_\infty \psi$ is not equal to any of these three squares in $Q_{\mathcal{B}}$ observe that while $\varphi *_\infty \varphi = i_\gamma$ and $\psi *_\infty \psi = i_\gamma$, $\varphi *_\infty \psi$ satisfies the relations $\varphi *_\infty (\varphi *_\infty \psi) = \psi$ and $(\varphi *_\infty \psi) *_\infty \psi = \varphi$. From this and from the obvious fact that $\varphi *_\infty \psi$ is not equal to i_γ it follows that $\varphi *_\infty \psi$ does not have vertical length equal to 1. $\ell Q_{\mathcal{B}}$ thus satisfies the inequality $\ell Q_{\mathcal{B}} \geq 2$, but it is obvious from the definition of $\mathcal{B}$ and $\mathcal{B}^*$ that $\ell Q_{\mathcal{B}} \leq 2$. We conclude that $\ell Q_{\mathcal{B}} = 2$.

The above example shows that the concept of vertical length of a double category is not trivial. We explain how to extend the construction presented in example 4.1 to a sequence of saturated decorated bicategories $\mathcal{B}_k$ such that $\ell Q_{\mathcal{B}} = k$ for every k.

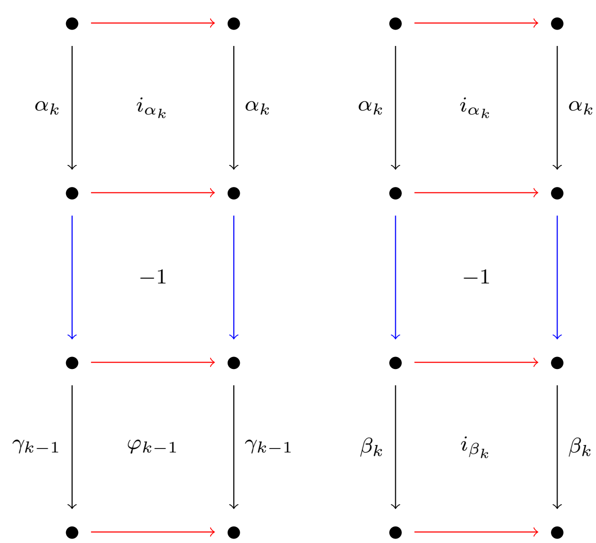
Let k be a positive integer. We make the underlying bicategory of $\mathcal{B}_k$ to be the $k+2$ vertex/2-cell version of the 2-category employed in the construction of example 4.1. The underlying 2-category of $\mathcal{B}_k$ is represented by a vertical sequence of $k+2$ diagrams of the form:



We define the decoration $\mathcal{B}_k^*$ of $\mathcal{B}_k$. We make $\mathcal{B}_k^*$ to be generated by the graph G_k, which we define inductively as follows: We make G_1 be the graph generated by the arrows α, β, β' defining the category $\mathcal{B}^*$ in example 4.1. Let $k > 1$. Assuming the graph G_{k-1} has been defined, we make the graph G_k to be the graph pictorially represented by the diagram:



where the green arrow represents the graph G_{k-1}. It easily proven that thus defined the graph G_k has $k+2$ vertices and exactly $k+1$ paths of maximal length $k+1$. Let $\mathcal{B}_k^*$ be the category generated by G_k by identifying the maximal paths in each of the G_m for $m \leq k$. Thus defined G_k has a unique maximal path, which we denote by γ_k. Observe that $\mathcal{B}_1^*$ is the category $\mathcal{B}^*$ of example 4.1. Now, assume the existence of a square φ_{k-1} in $\mathcal{B}_{k-1}$ of length $k-1$ having γ_{k-1} as source and target. Write ψ_k, ψ_k' to be the following two squares of $Q_{\mathcal{B}_k}$:



Thus defined ψ_k, ψ_k' are of length $k-1$ and by arguments similar to those presented in example 4.1 the horizontal composition $\psi_k *_\infty \psi_k'$ is of length k. We write φ_k for this square. The free globularly generated double category $Q_{\mathcal{B}_k}$ is thus of length $\geq k$ for every k. It is easily seen that $\ell Q_{\mathcal{B}_k}$ is in fact equal to k for every k.

Finally, observe that if $\mathcal{B}_\infty$ is the limit $\lim \mathcal{B}_k$, i.e. $\mathcal{B}_\infty$ is equal to the limit of diagram of 2-categories $\mathcal{B}_k$, decorated by the limit of the diagram of categories $\mathcal{B}_k^*$, then $\ell Q_{\mathcal{B}_\infty} = \infty$.

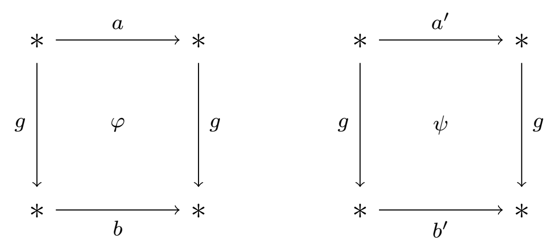
5. Group decorations

In this section we study free globularly generated double categories associated to monoidal categories decorated by groups. We prove that the free globularly generated double category associated to any such decorated bicategory has free lenght equal to 1. Moreover, we prove that in this case the free globularly generated double category construction specializes to the free product operation of groups. We use this to provide explicit descriptions for saturations of such decorated bicategories. We begin by proving the following proposition.

5.1. PROPOSITION. Let G be a group. Let D be a monoidal category. If we write $\mathcal{B}$ for the decorated bicategory $(\Omega G, 2D)$ then $\ell_{free}Q_{\mathcal{B}} = 1$.

PROOF. Let G be a group. Let D be a monoidal category. We wish to prove that the free globularly generated double category $Q_{\mathcal{B}}$ associated to $\mathcal{B} = (\Omega G, 2D)$ is such that $\ell_{free}Q_{\mathcal{B}} = 1$.

We prove that V_1 is closed under $*_\infty$. Let φ, ψ be squares in V_1 such that $t\varphi = s\psi$. If φ, ψ are globular squares in $\mathcal{B}$ then $\varphi *_\infty \psi$ is a globular square in $\mathcal{B}$ and thus is a square in V_1. We thus assume that ϕ, ψ are not globular squares in $\mathcal{B}$. By results of [16] φ, ψ are horizontal endomorphisms. Represent φ and ψ pictorially as:



where a, a', b and b' are objects in D and $g \in G$. Write φ and ψ as vertical compositions of the form

$$\varphi = \varphi_{k+1} \bullet_\infty i_{g_k} \bullet_\infty \cdots \bullet_\infty i_{g_1} \bullet_\infty \varphi_0$$

and

$$\psi = \psi_{s+1} \bullet_\infty i_{g'_s} \bullet_\infty \cdots \bullet_\infty i_{g'_1} \bullet_\infty \psi_0$$

where $g_1, ..., g_k, g'_1, ..., g'_s$ are elements of G such that $g_1 \dots g_k = g = g'_1 \dots g'_s$, where $\varphi_1, ..., \varphi_k, \psi_1, ..., \psi_k \in End_D(1)$, where φ_0, ψ_0 are morphisms, in D, from a to 1 and from a' to 1 respectively, and where $\varphi_{k+1}, \psi_{k+1}$ are morphisms, in D, from 1 to b and b' respectively. We refer to these decompositions as equations 1 and 2. We make the above decompositions of φ and ψ horizontally compatible. Write g_1 as $g_1 g^{-1} g = g_1 g^{-1}(g'_1 \dots g'_s)$. Using this write i_{g_1} as $i_{g_1 g^{-1}}(i_{g'_s} \bullet_\infty \cdots \bullet_\infty i_{g'_1})$. Inserting an identity endomorphism in between each $i_{g'_i}$ and $i_{g'_{i+1}}$ in the above decomposition we obtain a decomposition of $i_{g_1} \bullet_\infty \varphi_0$ as:

$$i_{g_1} \bullet_\infty \varphi_0 = i_{g_1 g^{-1}} \bullet_\infty (i_{g'_s} \bullet_\infty id_{i_{g'_s}} \bullet_\infty \cdots \bullet_\infty id_{i_{g'_1}} \bullet_\infty i_{g'_1} \bullet id_{i_{g'_s}}) \bullet_\infty \varphi_0$$

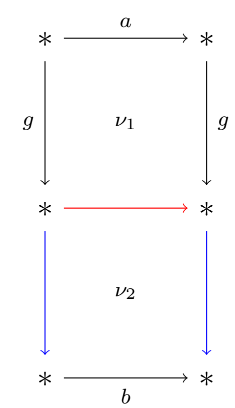
Write η for the vertical composition

$$\varphi_{k+1} \bullet_\infty i_{g_k} \cdots \bullet_\infty i_{g_2} \bullet_\infty \varphi_1$$

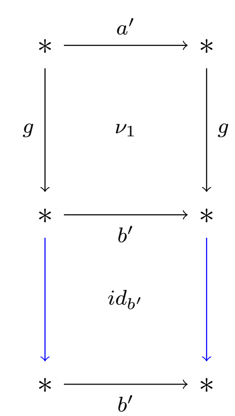
obtained from decomposition 1 by removing the first two terms from right to left. Substituting in decomposition 1 we obtain a decomposition of φ as a vertical composition of the form:

$$(\eta \bullet_\infty i_{g_1 g^{-1}}) \bullet_\infty (i_{g'_s} \bullet_\infty id_{i_{g'_s}} \bullet_\infty \cdots \bullet_\infty id_{i_{g'_1}} \bullet_\infty i_{g'_1} \bullet_\infty \varphi_0)$$

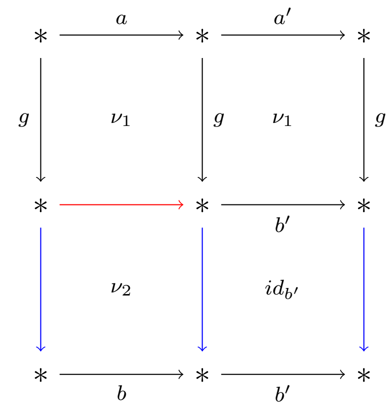
If we write ν_1, ν_2 for the expression on the first and second parenthesis above respectively we obtain a pictorial representation of φ as:



Writing ψ as $i_{id_{b'}} \bullet_\infty \psi$ we obtain a pictorial representation of ψ as:



The horizontal composition $\varphi *_\infty \psi$ thus admits a pictorial representation as:



By the way ν_1 was defined the horizontal composition of the two upper squares in the above diagram is equal to

$$\psi = \psi_{s+1} \bullet_\infty i_{g'_s} \bullet_\infty \cdots \bullet_\infty i_{g'_1} \bullet_\infty (\varphi_0 *_\infty \psi_0)$$

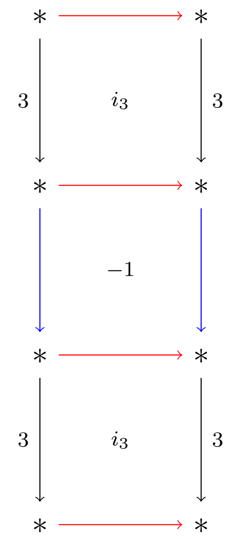
which is clear of free vertical length 1. The horizontal composition of the two bottom squares of the above diagram is clearly of free vertical length 1 and thus $\varphi *_\infty \psi$ is of vertical length 1. We conclude that $\ell_{free} Q_{\mathcal{B}} = 1$ as desired. ∎

We use proposition 5.1 to relate the free globularly generated double category construction to the free product operation between groups. Moreover, we provide an explicit description of saturations of single object 2-categories decorated by deloopings of groups. This is the content of the following corollary.

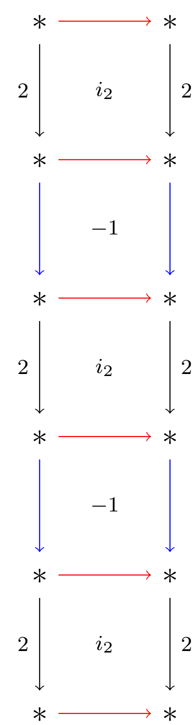
5.2. Corollary. *Let G, A be groups. Suppose A is abelian. Let $\mathcal{B}$ denote the decorated category $(\Omega G, 2\Omega A)$. In that case $Q_{\mathcal{B}}$ has ΩG as category of objects and $\Omega(G * A)$ as category of squares. Moreover, the saturation $H^* Q_{\mathcal{B}}$ of $\mathcal{B}$ has the subgroup of $G * A$ of words $a_k g_k \ldots a_i g_1$ such that $g_k \ldots g_1 = 1$ as groupoid of globular squares.*

The following example shows that the assumption of G being a group is essential for proposition 5.1. We show the existence of a single object bicategory $\mathcal{B}$ decorated by a reduced monoid such that $Q_{\mathcal{B}}$ has squares of length equal to 2.

5.3. Example. Let $\mathcal{B}$ be the decorated bicategory $(\Omega(\mathbb{N} \setminus \{1\}), 2\Omega\mathbb{Z}_2)$. From proposition 3.2 and from the fact that $\mathbb{N} \setminus \{1\}$ is a reduced monoid it follows that $\mathcal{B}$ is saturated. We claim that $Q_{\mathcal{B}}$ admits squares of length equal to 2. To see this let φ be the following quare:



and let ψ be the square:



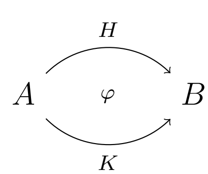
Thus defined both φ and ψ are horizontal endomorphisms in V_1 such that $t_\infty \varphi = 6 = s_\infty \psi$. By the fact that both 2 and 3 are irreducible in $\mathbb{N} \setminus \{1\}$ it easily follows that $\varphi *_\infty \psi$ and any horizontal composition of $\varphi *_\infty \psi$ with itself are not morphisms in V_1 and are thus of free length ≥ 2. Clearly both $\varphi *_\infty \psi$ and horizontal composition of $\varphi *_\infty \psi$ with itself are of free length ≤ 2 and thus are of free length exactly 2.

Observe that the arguments of subdividing squares of free length 1 employed in the proof of proposition 5.1 can easily be modified to prove that the free globularly generated double category associated to any monoidal category decorated by $\Omega\mathbb{N}$ has vertical length 1. Moreover, observe that in the case in which a decorated bicategory $\mathcal{B}$ is of the form $(\Omega M, A)$ for monoids/algebras M, A where A is commutative, then the first term of the free vertical filtration V_1 of $Q_\mathcal{B}$ is equal to the delooping $\Omega(M * A)$ of $M * A$.

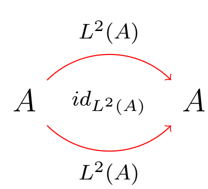
6. von Neumann algebras

In this section we study applications of the free globularly generated double category construction to the problem of existence of functorial extensions of the Haagerup standard form construction and the Connes fusion operation, see [1]. We prove that the bicategory of factors, Hilbert bimodules, and intertwining operators, decorated by not-necessarily finite index inclusions is saturated. This provides extensions of the Haagerup standard form construction and the Connes fusion operation, on the category of factors and not-necessarily finite index inclusions and a certain linear category properly containing the category of Hilbert spaces and bounded operators. These functors are compatible in the sense that they form the structure data of a category internal to linear categories internalizing the decorated bicategory of factors. We apply the saturation process introduced in section 3 to the problem of extending the Haagerup standard form construction and the Connes fusion operation, to functors on a category of general (not-necessarily factors) von Neumann algebras and general (not-necessarily finite index) von Neumann algebra morphisms.

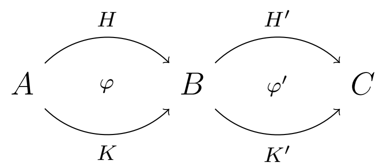
Our construction is as follows: We write $\mathbf{Mod}^{fact}$ for the bicategory whose 2-cells are of the form:



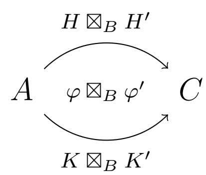
where A, B are factors, H, K are A-B left-right Hilbert bimodules over A, B and where φ is a bounded intertwiner from H to K. The horizontal identity cells in $\mathbf{Mod}^{fact}$ are of the form:



where A is a factor and where $L^2(A)$ denotes the Haagerup standard form of A, see [14,1]. Given two horizontally compatible 2-cells in $\mathbf{Mod}^{fact}$ of the form:



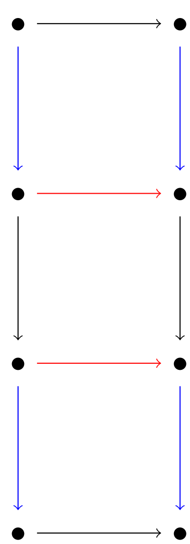
the horizontal composition of φ and φ' in $\mathbf{Mod}^{fact}$ is the 2-cell:



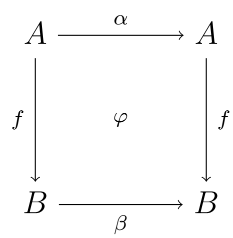
where $H \boxtimes_B H', K \boxtimes_B K'$ and $\varphi \boxtimes_B \varphi'$ denote the Connes fusion of H and H', of K and K' and of φ and φ' respectively. Thus defined $\mathbf{Mod}^{fact}$ is linear (C^* tensor in fact). We write $\mathbf{vN}^{fact}$ for the category whose objects are factors and whose morphisms are (possibly infinite index) von Neumann algebra morphisms. Thus defined $\mathbf{vN}^{fact}$ is linear. The pair $(\mathbf{vN}^{fact}, \mathbf{Mod}^{fact})$ is thus a linear decorated bicategory. We write W^*_{fact} for this decorated bicategory. We prove the following proposition.

6.1. Proposition. The linear decorated bicategory W^*_{fact} is saturated and moreover, the equation $\ell Q^{\mathbb{C}}_{W^*_{fact}} = 1$ holds.

Proof. We wish to prove that W^*_{fact} is saturated and that it satisfies the equation $\ell Q^{\mathbb{C}}_{W^*_{fact}} = 1$. We prove that every square in V_1 is a multiple of a square admitting a pictorial representation as:



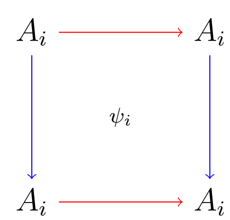
Let φ be a square in $Q^{\mathbb{C}}_{W^*_{fact}}$ of free length 1. Represent φ pictorially as:



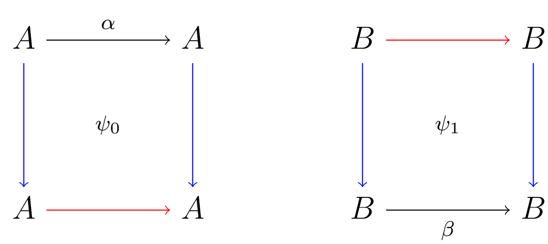
Write φ as a vertical composition of the form

$$\psi_{k+1} \bullet_\infty i_{f_k} \bullet_\infty \psi_{k-1} \bullet_\infty \cdots \bullet_\infty \psi_1 \bullet_\infty i_{f_1} \bullet_\infty \psi_0$$

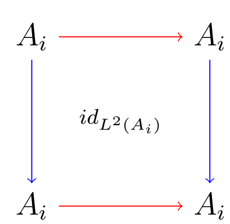
where f_i is morphism from a factor A_{i-1} to a factor A_i, where f admits a decomposition as $f = f_k \dots f_1$, and where ψ_i is a square of the form:



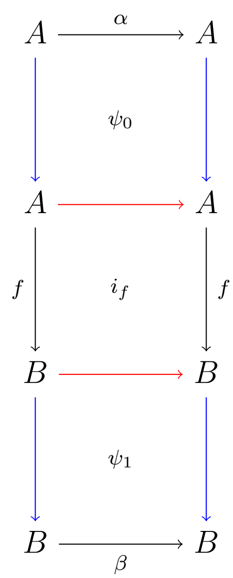
for every $1 \leq i \leq k$ and where ψ_0, ψ_{k+1} are squares of the form:



Let $1 \leq i \leq k$. From the fact that A_i is a factor it follows that the algebra of endomorphisms $End_{W^*_{fact}}(L^2(A_i))$ of A_i is 1-dimensional and thus is equal to $\mathbb{C} id_{L^2(A_i)}$. From this it follows there exists a $\lambda_i \in \mathbb{C}$ such that the square ψ_i is equal to λ_i times the square:



From this we conclude that φ is equal to $\Pi_{i=1}^{k}\lambda_i$ times the square:



This proves our claim. Observe that a square of the form above is globular if and only if it is a square in W^*_{free}. This proves that $H^*Q^{\mathbb{C}}_{W^*_{free}} = W^*_{free}$. The equation $\ell Q^{\mathbb{C}}_{W^*_{free}} = 1$ follows from the fact that the horizontal composition of two squares admitting pictorial representations as above admits a pictorial representation as above. This concludes the proof of the proposition. ∎

The category of squares $Q^{\mathbb{C}}_{W^*_{free_1}}$ of $Q^{\mathbb{C}}_{W^*_{free}}$ is thus a linear category whose objects are Hilbert bimodules between factors, whose morphisms are either usual intertwining operators between Hilbert bimodules or formal compositions as described in the proof of proposition 6.1. The function associating to every von Neumann algebra A its Haagerup standard form $L^2(A)$ admits an extension, as the horizontal identity functor of $Q_{W^*_{free}}$, to a linear functor

$$L^2 : \mathbf{vN}^{fact} \longrightarrow Q^{\mathbb{C}}_{W^*_{free_1}}$$

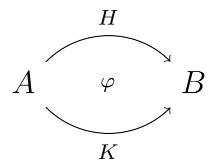
The functor (on the left and right entries) associating to every compatible pair of Hilbert bimodules H, K or intertwining operators φ, ψ their Connes fusion $H \boxtimes K$ or $\varphi \boxtimes \psi$ respectively, admits an extension (to the bottom variable), as the horizontal composition functor of $Q_{W^*_{fact}}$, to a linear functor

$$\boxtimes_{\bullet} : Q^{\mathbb{C}}_{W^*_{free_1}} \times_{\mathbf{vN}^{fact}} Q^{\mathbb{C}}_{W^*_{free_1}} \longrightarrow Q^{\mathbb{C}}_{W^*_{free_1}}$$

Moreover, these two linear functors are compatible in the sense that they provide $Q^{\mathbb{C}}_{W^*_{free}}$ with the structure of a category internal to linear categories.

The techniques employed in the proof of proposition 6.1 do not apply to the bicategory of general, i.e. non-necessarily factor, von Neumann algebras nor even to semisimple von

Neumann algebras. Through corollary 3.4 we obtain weaker versions of proposition 6.1 for the case of von Neumann algebras with not-necessarily trivial center. Write **Mod** for the linear bicategory whose 2-cells are of the form:



where A, B are now general von Neumann algebras, H, K are left-right Hilbert bimodules over A, B, and φ is an intertwining operator from H to K. The horizontal identity and horizontal composition on **Mod** are defined in analogy to those defining $\mathbf{Mod}^{fact}$. Write **vN** for the linear category of von Neumann algebras and general (not-necessarily finite) von Neumann algebra morphisms. The pair $(\mathbf{vN}, \mathbf{Mod})$ is a decorated linear category. We write W^* for this decorated bicategory. Write $\tilde{W}^*$ for the saturation of W^*. In that case the category of morphisms $Q^{\mathbb{C}}_{\tilde{W}^*_1}$ of $Q^{\mathbb{C}}_{\tilde{W}^*}$ is a linear category, whose objects are Hilbert bimodules over general von Neumann algebras, and whose morphisms contain the usual intertwining operators in **Mod**. The function associating to every von Neumann algebra A its Haagerup standard form $L^2(A)$ extends, as the horizontal identity functor of $Q^{\mathbb{C}}_{\tilde{W}^*}$, to a linear functor

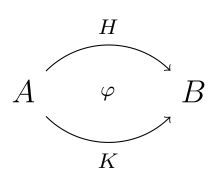
$$L^2 : \mathbf{vN} \longrightarrow Q^{\mathbb{C}}_{\tilde{W}^*_1}$$

and the Connes fusion bifunctor $\boxtimes$ extends, as the horizontal composition functor of $Q^{\mathbb{C}}_{\tilde{W}^*}$, to a linear functor

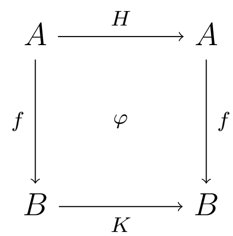
$$\boxtimes_\bullet : Q^{\mathbb{C}}_{\tilde{W}^*_1} \times_{\mathbf{vN}} Q^{\mathbb{C}}_{\tilde{W}^*_1} \longrightarrow Q^{\mathbb{C}}_{\tilde{W}^*_1}$$

Moreover, these functors are compatible in the sense that they provide $Q^{\mathbb{C}}_{\tilde{W}^*}$ with the structure of a linear double category.

In [1,2] a solution to Problem 1.2 is presented for the decorated bicategory whose 2-cells are of the form:



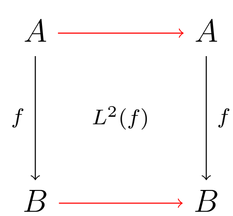
where A, B are factors (more generally A, B are semisimple) H, K are left-right Hilbert bimodules over A, B and where φ is an intertwiner operator from H to K, and whose decoration is the category of factors and finite index inclusions. The horizontal identity and the horizontal composition functors, i.e. the corresponding functorial extensions of the Haagerup standard form construction and the Connes fusion operation, are defined making strong use of the Kosaki theory of minimal conditional expectations of finite index subfactors [15]. We write BDH for this double category. We ask how the double category $Q^{\mathbb{C}}_{W^*_{fact}}$ described in the proof of proposition 6.1 and BDH are related. We consider the sub-double category of $Q^{\mathbb{C}}_{W^*_{fact}}$ generated by globular squares and the squares of the form:



where f is an inclusion of finite Jones index. We write $Q^{\mathbb{C}}_{W^*_{fin}}$ for this double category and we write W^*_{fin} for $H^*Q^{\mathbb{C}}_{W^*_{fin}}$. We have the following equation:

$$H^*Q_{W^*_{fin}} = W^*_{fin} = H^*BDH$$

and thus $Q^{\mathbb{C}}_{W^*_{fin}}$ and BDH have the same category of objects, the same collection of horizontal morphisms, and the same collection of horizontal and globular squares. It is natural to expect some higher relation between the squares of $Q^{\mathbb{C}}_{W^*_{fin}}$ and the squares of γBDH to hold. It is easily seen that certain relations that hold in γBDH do not hold on $Q^{\mathbb{C}}_{W^*_{fin}}$, e.g. change of base algebra. This makes it obvious that the double categories γBDH and $Q^{\mathbb{C}}_{W^*_{fin}}$ are non-equivalent. There is an obvious strict tensor double functor π from $Q^{\mathbb{C}}_{W^*_{fin}}$ to γBDH such that π restricts to the identity on $H^*Q^{\mathbb{C}}_{W^*_{fin}}$. This double functor preserves squares of the form:



Since both $Q^{\mathbb{C}}_{W^*_{fin}}$ to γBDH are generated by both $H^*Q^{\mathbb{C}}_{W^*_{fin}}$ and the set of squares as above, the double functor π is unique with respect to its value on $H^*Q^{\mathbb{C}}_{W^*_{fin}}$ and is surjective on squares. We study double functors of this form and the way they relate free globularly generated double categories to globularly generated internalizations in the second installment of the present series of papers.

The constructions presented above have an obvious drawback. All the categories of von Neumann algebras and all the bicategories of Hilbert bimodules we have considered are symmetric monoidal. We wish for the corresponding free globularly generated double categories and thus for the corresponding functorial extensions of the Haagerup standard form construction and the Connes fusion operation to be symmetric monoidal. It is not obvious how to extend the combined coherence data of both the decoration and the underlying bicategory of a symmetric monoidal bicategory into coherence data for the obvious choice of monoidal structure on the free globularly generated double category construction. These questions will be explored elsewhere.

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Centro de Ciencias Matemáticas - UNAM Campus Morelia
Antigua Carretera a Pátzcuaro 8701, Morelia, Michoacán, México C.P. 58089

Email: `jorendain@matmor.unam.mx`

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