THE DUAL OF COMPACT ORDERED SPACES IS A VARIETY

MARCO ABBADINI

Abstract. In a recent paper (2018), D. Hofmann, R. Neves and P. Nora proved that the dual of the category of compact ordered spaces and monotone continuous maps is a quasi-variety—not finitary, but bounded by $\aleph_1$. An open question was: is it also a variety? We show that the answer is affirmative. We describe the variety by means of a set of finitary operations, together with an operation of countably infinite arity, and equational axioms. The dual equivalence is induced by the dualizing object $[0, 1]$.

1. Introduction

Compact ordered spaces were introduced by L. Nachbin, and they are to topology and partial order what compact Hausdorff spaces are to topology. A *compact ordered space* $(X, \leq, \tau)$ consists of a compact space (X, τ) equipped with a partial order $\leq$ so that the set

$$\{(x, y) \in X \times X \mid x \leq y\}$$

is closed in $X \times X$ with respect to the product topology (see [Nachbin, 1965] for a standard reference); we are interested in the category **PosComp** of compact ordered spaces with monotone continuous maps.

The goal of this paper is to establish for **PosComp** a result which is known to hold for the category **CompHaus** of compact Hausdorff spaces with continuous maps—namely, that the dual category is a variety (not finitary, but bounded by $\aleph_1$). As far as compact Hausdorff spaces are concerned, we recall some historical details: in 1969, Duskin proved that the functor $\hom(-, [0, 1]) \colon \mathbf{CompHaus}^{\mathrm{op}} \to \mathbf{Set}$ is monadic [Duskin, 1969]; Isbell presented a set of primitive operations of **CompHaus**$^{\mathrm{op}}$, using finitely many finitary operations, along with an operation of countably infinite arity [Isbell, 1982]; finally, Marra and Reggio provided finitely many axioms to axiomatize the variety **CompHaus**$^{\mathrm{op}}$ [Marra and Reggio, 2017].

These results were a source of motivation for the algebraic study of the dual of **PosComp** in [Hofmann et al., 2018]: the authors proved that **PosComp**$^{\mathrm{op}}$ is a quasi-variety—not finitary, but bounded by $\aleph_1$—leaving as open the following question.

Is **PosComp**$^{\mathrm{op}}$ also a variety?

Our main result is that the answer is affirmative, as stated in the following theorem.

1.1. THEOREM. [Main result] *The dual of* **PosComp** *is equivalent to a variety of algebras.*

The proof of our main result is at times inspired by [Hofmann et al., 2018], but does not depend on their results. In this paper, under the term *variety of algebras*, we admit the so called varieties of *infinitary* algebras, whose operations may have infinite arity (see [Słomiński, 1959] for varieties of infinitary algebras). The variety in Theorem 1.1 will be denoted by $\mathbf{MC}_\infty$, where $\mathbf{M}$ stands for "monotone" and $\mathbf{C}$ stands for "continuous". We will give a set of primitive operations (of countable arity) and a set of equational axioms for $\mathbf{MC}_\infty$.

The set $[0, 1]$ is both a compact ordered space, with the canonical order and the euclidean topology, and an MC_∞-algebra, in a natural way. In fact, we present the duality between **PosComp** and $\mathbf{MC}_\infty$ as induced by the dualizing object $[0, 1]$. This dual equivalence coincides, essentially, with the duality available in [Hofmann et al., 2018]: the main difference is that, on the algebraic side, we consider a slightly different set of primitive operations, that facilitates us to state the axioms in an equational form.

We will show $\mathbf{MC}_\infty = \mathrm{ISP}([0, 1])$, where P denotes the closure under products, S denotes the closure under subalgebras, and I denotes the closure under isomorphisms. Moreover, we will see that the operations of $\mathbf{MC}_\infty$, interpreted in $[0, 1]$, are precisely the monotone continuous maps from a power of $[0, 1]$ to $[0, 1]$. In other words, $\mathbf{MC}_\infty$ is the category of algebras of the varietal theory (in the sense of [Linton, 1966]) whose objects are powers of $[0, 1]$ and whose morphisms are the monotone continuous maps.

1.2. THE STRATEGY. Our strategy to prove the duality follows the lines of [Marra and Reggio, 2017], which, to the best of our knowledge, used this strategy for the first time in a similar context—namely, to obtain a finite equational axiomatization of the dual of **CompHaus**. We prove that the dual of **PosComp** is a variety via the following steps.

1. We obtain a dual adjunction between the category of preordered topological spaces and a finitary variety **MC**, to be defined. This dual adjunction is induced by the dualizing object $[0, 1]$.

2. We characterize the objects which are fixed by the adjunction. On the topological side, the fixed objects are precisely the compact ordered spaces. On the algebraic side, the fixed objects are the archimedean Cauchy complete MC-algebras. Hence, a duality is established between the full subcategories of compact ordered spaces and archimedean Cauchy complete MC-algebras.

3. We show that the full subcategory of archimedean Cauchy complete MC-algebras is isomorphic to an infinitary variety $\mathbf{MC}_\infty$, obtained by adding a term δ of countably infinite arity to the language of **MC**, together with some new appropriate equational axioms. The term δ is intended to map *enough* Cauchy sequences to their limit. The forgetful functor $\mathbf{MC}_\infty \to \mathbf{MC}$, restricted at codomain, gives the desired isomorphism.

To show that every compact ordered space is fixed by the adjunction, we use an analogue of Urysohn's Lemma. To show that every archimedean Cauchy complete MC-algebra is fixed, we use the Subdirect Representation Theorem, which applies since **MC** is finitary, and an analogue of Stone-Weierstrass Theorem. The Subdirect Representation Theorem is used to show that every archimedean MC-algebra is mapped injectively by the unit of the adjunction. The analogue of Stone-Weierstrass Theorem is used to show that if the algebra is Cauchy complete, then it is mapped surjectively by the unit of the adjunction.

2. The category **PreTop** of preordered topological spaces

2.1. DEFINITION. *A preordered topological space* $(X, \leq, \tau)$ *consists of a set* X*, a preorder* $\leq$ *on* X *and a topology* τ *on* X*.*

When no confusion arises, we write X instead of $(X, \leq, \tau)$. We denote with **PreTop** the category whose objects are preordered topological spaces and whose morphisms from A to B are the monotone (i.e. $x \leq y \Rightarrow f(x) \leq f(y)$) continuous functions $f \colon A \to B$.

It is well known that the forgetful functors from the category of topological spaces and from the category of preordered sets to the category of sets are both topological, in the sense of [Adámek et al., 2006, Definition 21.1]. The forgetful functor $U_{\mathbf{PreTop}} \colon \mathbf{PreTop} \to \mathbf{Set}$ is topological, too; indeed, every $U_{\mathbf{PreTop}}$-structured source

$$\left(X \xrightarrow{f_i} U_{\mathbf{PreTop}}(X_i, \leq_i, \tau_i) \right)_{i \in I}$$

admits a unique $U_{\mathbf{PreTop}}$-initial lift

$$\left((X, \leq, \tau) \xrightarrow{f_i} (X_i, \leq_i, \tau_i) \right)_{i \in I},$$

where $\leq$ is defined by

$$x \leq y \Leftrightarrow \forall i \in I \ f_i(x) \leq f_i(y),$$

and τ is the topology generated by

$$\{ f_i^{-1}(O_i) \mid i \in I, O_i \in \tau_i \}.$$

It is well known that topological functors lift limits (in particular, products) uniquely [Adámek et al., 2006, Proposition 21.15]. For a family $(X_i, \leq_i)$ of preordered spaces, the *product preorder* on $\prod_{i \in I} X_i$ is the preorder $\leq$ defined by

$$x \leq y \iff \forall i \in I \ \pi_i(x) \leq_i \pi_i(y),$$

where π_i is the projection onto the i-th coordinate. For a family $(X_i, \tau_i)_{i \in I}$ of topological spaces, the *product topology* on $\prod_{i \in I} X_i$ is the topology generated by

$$\{\pi_i^{-1}(O_i) \mid i \in I, O_i \in \tau_i\}.$$

Finally, for a family $(X_i, \tau_i, \leq_i)_{i \in I}$ of preordered topological spaces, the unique $U_{\mathbf{PreTop}}$-initial lift of the $U_{\mathbf{PreTop}}$-structured source $\left(\prod_{i \in I} X_i \xrightarrow{\pi_i} U_{\mathbf{PreTop}}(X_i, \leq_i, \tau_i)\right)_{i \in I}$ is

$$\left(\left(\prod_{i \in I} X, \leq, \tau\right) \xrightarrow{\pi_i} (X_i, \leq_i, \tau_i)\right)_{i \in I},$$

where $\leq$ is the product preorder, and τ is the product topology. Moreover, this is a categorical product in **PreTop**. Unless otherwise stated, when referring to the set-theoretic product of preordered topological spaces as a preordered topological space, we implicitly assume that the preorder is the product preorder and the topology is the product topology. Then, if I is a set, and $f : [0,1]^I \to [0,1]$ is a monotone and continuous function, we have the following: for every preordered topological space X, f is an internal operation on $\hom_{\mathbf{PreTop}}(X, [0,1])$, meaning that, for every I-indexed family $(g_i)_{i \in I}$ of monotone continuous functions from X to $[0,1]$, the function $X \to [0,1]$, $x \mapsto f((g_i(x))_{i \in I})$ is monotone and continuous, as well.

3. The variety **MC**

We define some operations on $[0,1]$. For $a, b \in [0,1]$, $a \vee b$ and $a \wedge b$ denote, respectively, the supremum and the infimum of $\{a, b\}$, $a \oplus b := \min\{a+b, 1\}$, and $a \odot b := \max\{a+b-1, 0\}$. Moreover, for each $\lambda \in [0,1]$, the constant symbol λ denotes λ itself.

3.1. REMARK. Each of these operations ($\vee$, $\wedge$, $\oplus$, $\odot$, and, for every $\lambda \in [0,1]$, the constant function λ) is monotone and continuous with respect to the product order and product topology.

Note that we do not consider the function $\neg : [0,1] \to [0,1]$ $a \mapsto 1 - a$, since it is not monotone.

We define a finitary variety **MC** of algebras of type $\mathcal{L} = \{\oplus, \odot, \vee, \wedge, 0, 1\} \cup \{\lambda \mid \lambda \in [0,1]\}$. Specifically, an algebra A belongs to **MC** (and we say that A is an *MC-algebra*) if it satisfies the following identities, which, as one may verify, are all satisfied by $[0,1]$.

1. $\langle A, \vee, \wedge, 0, 1 \rangle$ is a distributive bounded lattice.

   (a) $a \vee b = b \vee a$.

   (b) $a \wedge b = b \wedge a$.

   (c) $a \vee (b \vee c) = (a \vee b) \vee c$.

   (d) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$.

(e) $a \vee (a \wedge b) = a$.

(f) $a \wedge (a \vee b) = a$.

(g) $a \vee 0 = a$.

(h) $a \wedge 1 = a$.

(i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

(j) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

2. $\langle A, \oplus, 0 \rangle$ is a commutative monoid, with absorbing element 1.

(a) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

(b) $(a \oplus b) = (b \oplus a)$.

(c) $a \oplus 0 = a$.

(d) $a \oplus 1 = 1$.

3. $\langle A, \odot, 1 \rangle$ is a commutative monoid, with absorbing element 0.

(a) $(a \odot b) \odot c = a \odot (b \odot c)$.

(b) $(a \odot b) = (b \odot a)$.

(c) $a \odot 1 = a$.

(d) $a \odot 0 = 0$.

4. $\oplus$ and $\odot$ distribute over $\vee$ and $\wedge$.

(a) $(a \vee b) \oplus c = (a \oplus c) \vee (b \oplus c)$.

(b) $(a \wedge b) \oplus c = (a \oplus c) \wedge (b \oplus c)$.

(c) $(a \vee b) \odot c = (a \odot c) \vee (b \odot c)$.

(d) $(a \wedge b) \odot c = (a \odot c) \wedge (b \odot c)$.

5. $(a \oplus b) \odot c \leq a \oplus (b \odot c)$.

6. For each $\lambda \in [0, 1]$, we have the axiom $a \leq (a \odot (1 - \lambda)) \oplus \lambda$.

7. For each $\lambda \in [0, 1]$, we have the axiom $a \geq (a \oplus \lambda) \odot (1 - \lambda)$.

8. For every $n, m \in \{0, 1, 2, \dots\}$, we have the axiom

$$a \wedge (b \oplus \underbrace{(c \odot \lambda) \oplus \cdots \oplus (c \odot \lambda)}_{n \text{ times}}) \leq (a \odot \underbrace{(c \oplus \lambda) \odot \cdots \odot (c \oplus \lambda)}_{m \text{ times}}) \vee b.$$

9. For $\alpha, \beta, \gamma \in [0, 1]$ such that $\alpha \vee \beta = \gamma$ in $[0, 1]$, we have the axiom $\alpha \vee \beta = \gamma$.

10. For $\alpha, \beta, \gamma \in [0, 1]$ such that $\alpha \wedge \beta = \gamma$ in $[0, 1]$, we have the axiom $\alpha \wedge \beta = \gamma$.

11. For $\alpha, \beta, \gamma \in [0,1]$ such that $\alpha \oplus \beta = \gamma$ in $[0,1]$, we have the axiom $\alpha \oplus \beta = \gamma$.

12. For $\alpha, \beta, \gamma \in [0,1]$ such that $\alpha \odot \beta = \gamma$ in $[0,1]$, we have the axiom $\alpha \odot \beta = \gamma$.

For $\lambda \in [0,1]$, we write $x \ominus \lambda$ for $x \odot (1-\lambda)$. In $[0,1]$, $x \ominus \lambda = \max\{x - \lambda, 0\}$. We remark that we allow the notation $x \ominus \lambda$ only when λ is a constant symbol in $[0,1]$.

4. The dual adjunction between **PreTop** and **MC**

Let X be a preordered topological space. We set

$$\mathrm{C}(X) := \hom_{\mathbf{PreTop}}(X, [0,1]) =$$

$$= \{f \colon X \to [0,1] \mid f \text{ is monotone and continuous}\}.$$

Since, by Remark 3.1, the interpretation in $[0,1]$ of every MC-operation is monotone and continuous, $\mathrm{C}(X)$ is an MC-algebra with pointwise defined operations. For each $x \in X$, we set

$$\mathrm{ev}_x \colon \mathrm{C}(X) \longrightarrow [0,1]$$
$$a \longmapsto a(x).$$

Let $A \in \mathbf{MC}$. Set $\mathrm{Max}(A) := \hom_{\mathbf{MC}}(A, [0,1])$. The motivation for this name stems from the fact that the set of morphisms from an MC-algebra A to $[0,1]$ is in bijection with the set of maximal congruences on A; this follows from the fact that $[0,1]$ is the only simple algebra, as it will be proved in Corollary 7.18. For each $a \in A$, we set

$$\mathrm{ev}_a \colon \mathrm{Max}(A) \longrightarrow [0,1]$$
$$x \longmapsto x(a).$$

For all $x, y \in \mathrm{Max}(A)$, set $x \leq y$ if, and only if, for all $a \in A$, $\mathrm{ev}_a(x) \leq \mathrm{ev}_a(y)$, i.e., $x(a) \leq y(a)$. Let τ be the smallest topology on $\mathrm{Max}(A)$ that contains $\mathrm{ev}_a^{-1}(O)$ (i.e., $\{x \in \mathrm{Max}(A) \mid x(a) \in O\}$) for every $a \in A$ and O open subset of $[0,1]$.

In [Porst and Tholen, 1991, Section 1-C], some properties are discussed that are sufficient to establish a dual adjunction induced by a dualizing object. These properties are expressed in terms of the existence of certain initial lifts, and in our case these properties hold. Indeed, let $\mathrm{U}_{\mathbf{PreTop}} \colon \mathbf{PreTop} \to \mathbf{Set}$ and $\mathrm{U}_{\mathbf{MC}} \colon \mathbf{MC} \to \mathbf{Set}$ denote the forgetful functors; by the results discussed in section 2, for every $A \in \mathbf{MC}$, $\left((\mathrm{Max}(A), \leq, \tau) \xrightarrow{\mathrm{ev}_a} [0,1]\right)_{a \in A}$ is the unique $\mathrm{U}_{\mathbf{PreTop}}$-initial lift of its image under $\mathrm{U}_{\mathbf{PreTop}}$. Moreover, since for every preordered topological space X the operations in $\mathrm{C}(X)$ are pointwise defined, we have that

$$\left(\mathrm{C}(X) \xrightarrow{\mathrm{ev}_x} [0,1]\right)_{x \in X}$$

(where $C(X)$ denotes the MC-algebra whose underlying set is $\hom_{\textbf{PreTop}}(X, [0, 1])$ and with pointwise defined operations) is the unique $U_{\textbf{MC}}$-initial lift of its image under $U_{\textbf{MC}}$.

Therefore, we have a dual adjunction between **PreTop** and **MC** induced by the dualizing object $[0, 1]$, that we now make explicit. In accordance with [Porst and Tholen, 1991], we find a more natural choice to use *contravariant* functors between **PosComp** and **MC** rather than *covariant* ones between **PosComp**$^{\text{op}}$ and **MC**, or between **PosComp** and **MC**$^{\text{op}}$. This choice seems to us more natural in the context of dual adjunctions induced by a dualizing object, because it respects the symmetry between the two involved categories (**PosComp** and **MC**, in our case). Since we are considering contravariant functors, we end up using *two units*, rather than a unit and a counit.

The assignment C on the objects may be extended on arrows so that C becomes a contravariant functor: for a morphism $g \colon X \to Y$ in **PreTop**, we set

$$C(g) \colon C(Y) \longrightarrow C(X)$$
$$a \longmapsto a \circ g.$$

Analogously, the assignment Max on the objects may be extended on arrows so that Max becomes a contravariant functor: for a morphism $f \colon A \to B$ in **MC**, we set

$$\text{Max}(f) \colon \text{Max}(B) \longrightarrow \text{Max}(A)$$
$$x \longmapsto x \circ f.$$

The adjunction is given as follows. Let $X \in \textbf{PreTop}$ and $A \in \textbf{MC}$. To each morphism $g \colon X \to \text{Max}(A)$ in **PreTop** we associate the following morphism in **MC**:

$$\hat{g} \colon A \longrightarrow C(X)$$
$$a \longmapsto \text{ev}_a \circ g; \ x \mapsto (g(x))(a).$$

To each morphism $f \colon A \to C(X)$ in **MC** we associate the following morphism in **PreTop**:

$$\check{f} \colon X \longrightarrow \text{Max}(A)$$
$$x \longmapsto \text{ev}_x \circ f; \ a \mapsto (f(a))(x).$$

For $X \in \textbf{PreTop}$, the unit at X is

$$\eta_X \colon X \longrightarrow \text{MaxC}(X)$$
$$x \longmapsto (\text{ev}_x \colon C(X) \to [0, 1]; a \mapsto a(x)).$$

For $A \in \textbf{MC}$, the unit at A is

$$\varepsilon_A \colon A \longrightarrow \text{CMax}(A)$$
$$a \longmapsto (\text{ev}_a \colon \text{Max}(A) \to [0, 1]; x \mapsto x(a)).$$

5. Fixed objects on the geometrical side

Let us recall the definition of compact ordered space.

5.1. DEFINITION. *A compact ordered space* $(X, \leq, \tau)$ *consists of a compact space* (X, τ) *equipped with a partial order* $\leq$ *so that the set*

$$\{(x, y) \in X \times X \mid x \leq y\}$$

is closed in $X \times X$ *with respect to the product topology.*

A standard reference is [Nachbin, 1965]. We recall that every compact ordered space is Hausdorff [Nachbin, 1965, Proposition 2, Chapter 1, p. 27]. We denote with **PosComp** the category of compact ordered spaces with monotone continuous maps.

The goal of this section is to prove the following.

5.2. THEOREM. *Let* X *be a preordered topological space. The following conditions are equivalent.*

1. *The unit* $\eta_X \colon X \to \mathrm{MaxC}(X)$ *is an isomorphism.*

2. *There exists an MC-algebra* A *such that* X *and* $\mathrm{Max}(A)$ *are isomorphic preordered topological spaces.*

3. X *is a compact ordered space.*

5.3. REMARK. The implication [(1)$\Rightarrow$(2)] in Theorem 5.2 is immediate: take $A = \mathrm{C}(X)$.

5.4. $\mathrm{Max}(A)$ IS A COMPACT ORDERED SPACE. In this subsection, we prove the implication [(2)$\Rightarrow$(3)] of Theorem 5.2, i.e., for every $A \in \mathbf{MC}$, $\mathrm{Max}(A)$ is a compact ordered space. We need the following lemmas and remarks.

5.5. LEMMA. *Let* X *be a compact ordered space, and let* Y *be a closed subset of* X. *Then* Y, *equipped with the topology and the order induced by* X, *is a compact ordered space.*

PROOF. Since Y is a closed subspace of a compact space, Y is compact. Clearly, the partial order on X induces a partial order on Y. The product topology on $Y \times Y$ coincides with the subspace topology on $Y \times Y$ as subspace of $X \times X$. Since $\{(x, y) \in X \times X \mid x \leq y\}$ is a closed subset of $X \times X$, $\{(x, y) \in Y \times Y \mid x \leq y\} = \{(x, y) \in X \times X \mid x \leq y\} \cap Y \times Y$ is closed in $Y \times Y$. ∎

5.6. LEMMA. *Let* $(X_i)_{i \in I}$ *be a family of compact ordered spaces. Then,* $\prod_{i \in I} X_i$, *equipped with the product topology and product order, is a compact ordered space.*

PROOF. By Tychonoff's theorem, $\prod_{i \in I} X_i$ is compact. Let us consider the bijection $\phi \colon (\prod_{i \in I} X_i) \times (\prod_{i \in I} X_i) \to \prod_{i \in I}(X_i \times X_i); ((a_i)_{i \in I}, (b_i)_{i \in I}) \mapsto (a_i, b_i)_{i \in I}$. The function ϕ is a homeomorphism, and the image under ϕ of the set

$$\left\{ \left((x_i)_{i \in I}, (y_i)_{i \in I} \right) \in \left(\prod_{i \in I} X_i \right) \times \left(\prod_{i \in I} X_i \right) \mid (x_i)_{i \in I} \leq (y_i)_{i \in I} \right\}$$

is $\prod_{i \in I} \{(x_i, y_i) \in X_i \times X_i \mid x_i \leq_i y_i\}$, which is closed. ∎

5.7. REMARK. For every set A, $[0,1]^A$ (with the product order and product topology) is a compact ordered space.

If X a Hausdorff space, then the diagonal of $X \times X$ is closed; as a consequence, we have the following.

5.8. REMARK. Let Y be a Hausdorff space, let X be a topological space, and let $f, g \colon X \to Y$ be continuous functions. Then, $\{x \in X \mid f(x) = g(x)\}$ is closed.

We can now prove the implication $[(2) \Rightarrow (3)]$ of Theorem 5.2.

5.9. THEOREM. *For $A \in \mathbf{MC}$, $\mathrm{Max}(A)$ is a compact ordered space.*

PROOF. $\mathrm{Max}(A) = \hom_{\mathbf{MC}}(A, [0,1])$ is a subset of $[0,1]^A$. By Remark 5.7, $[0,1]^A$ (with the product order and product topology) is a compact ordered space. The topology on $\mathrm{Max}(A)$ coincides with the induced topology on $\mathrm{Max}(A)$ as a subspace of $[0,1]^A$; moreover, the order on $\mathrm{Max}(A)$ coincides with the order induced by $[0,1]^A$. By Lemma 5.5, it is enough to show that $\mathrm{Max}(A)$ is closed. The idea is that $\mathrm{Max}(A)$ is closed because it is defined by equations, which express the preservations of primitive operation symbols of $\mathbf{MC}$. To make this precise, let $\mathcal{L}$ denote the set of primitive operation symbols of $\mathbf{MC}$. For each $h \in \mathcal{L}$, we denote with $\mathrm{ar}(h)$ the arity of h; moreover, we denote with h_A the interpretation of h in A, and by $h_{[0,1]}$ the interpretation of h in $[0,1]$. For $a \in A$, we denote with $\pi_a \colon [0,1]^A \to [0,1]$ the projection onto the a-th coordinate (which is continuous). We have

$$\mathrm{Max}(A) = \hom_{\mathbf{MC}}(A, [0,1]) =$$

$$= \{x \colon A \to [0,1] \mid \forall h \in \mathcal{L} \ \forall a_1, \ldots, a_{\mathrm{ar}(h)} \in A$$

$$x(h_A(a_1, \ldots, a_{\mathrm{ar}(h)})) = h_{[0,1]}(x(a_1), \ldots, x(a_{\mathrm{ar}(h)}))\} =$$

$$= \bigcap_{h \in \mathcal{L}} \bigcap_{a_1, \ldots, a_{\mathrm{ar}(h)} \in A} \{x \colon A \to [0,1] \mid$$

$$x(h_A(a_1, \ldots, a_{\mathrm{ar}(h)})) = h_{[0,1]}(x(a_1), \ldots, x(a_{\mathrm{ar}(h)}))\} =$$

$$= \bigcap_{h \in \mathcal{L}} \bigcap_{a_1, \ldots, a_{\mathrm{ar}(h)} \in A} \{x \in [0,1]^A \mid \pi_{h_A(a_1, \ldots, a_{\mathrm{ar}(h)})}(x) = h_{[0,1]}(\pi_{a_1}(x), \ldots, \pi_{a_{\mathrm{ar}(h)}}(x))\}.$$

By Remark 3.1, $h_{[0,1]}$ is continuous; therefore, the function from $[0,1]^A$ to $[0,1]$ which maps x to $h_{[0,1]}(\pi_{a_1}(x), \ldots, \pi_{a_{\mathrm{ar}(h)}}(x))$ is continuous. Since $[0,1]$ is Hausdorff, by Remark 5.8, $\{x \in [0,1]^A \mid \pi_{h_A(a_1, \ldots, a_{\mathrm{ar}(h)})}(x) = h_{[0,1]}(\pi_{a_1}(x), \ldots, \pi_{a_{\mathrm{ar}(h)}}(x))\}$ is closed. ∎

5.10. THE UNIT $\eta_X \colon X \to \mathrm{MaxC}(X)$ IS INJECTIVE. We now turn to the proof of the implication $[(3) \Rightarrow (1)]$ in Theorem 5.2, which states that, if X is a compact ordered space, $\eta_X \colon X \to \mathrm{MaxC}(X)$ is an isomorphism. Our source of inspiration is [Hofmann and Nora, 2018]. The results we will obtain in the present section may be seen, essentially, as specific cases of the results available in [Hofmann and Nora, 2018]; nevertheless, for reasons of

presentation, we provide independent proofs here. In this subsection, we prove that, if X is a compact ordered space, then $\eta_X \colon X \to \mathrm{MaxC}(X)$ is injective; this result is essentially due to L. Nachbin.

5.11. DEFINITION. *For X a partially ordered set, we call* upper *an upward closed subset of X, and* lower *a downward closed one.*

For X a partial ordered set, and $x \in X$, we set $\downarrow x := \{z \in X \mid z \leq x\}$ and $\uparrow x := \{z \in X \mid x \leq z\}$. The following is well known.

5.12. LEMMA. *Let X be a compact ordered space, and let $x \in X$. Then $\downarrow x$ is the smallest closed lower subset of X that contains x and $\uparrow x$ is the smallest closed upper subset of X that contains x.*

PROOF. Let us prove that $\downarrow x$ is closed. Set $D := \{(u, v) \in X \times X \mid u \leq v\}$. D is closed by definition of compact ordered space. Moreover, since any compact ordered space is Hausdorff, every point of X is closed. Hence $D \cap (X \times \{x\}) = \{(z, x) \mid z \in X : z \leq x\}$ is closed. Since the projection $\pi_1 \colon X \times X \to X$ onto the first coordinate is closed, $\pi_1(\{(z, x) \mid z \in X : z \leq x\}) = \{z \in X \mid z \leq x\} = \downarrow x$ is closed. Analogously for $\uparrow x$. The rest of the statement is straightforward to prove. ∎

5.13. PROPOSITION. [Ordered version of Urysohn's Lemma] *Let X be a compact ordered space, let A be a closed lower subset, and let B be a closed upper subset, with $A \cap B = \emptyset$. Then there exists a monotone and continuous function $\psi \colon X \to [0, 1]$ such that, for every $x \in A$, $\psi(x) = 0$, and, for every $x \in B$, $\psi(x) = 1$.*

PROOF. See [Nachbin, 1965, Chapter I, Theorem 1, p. 30]. ∎

5.14. COROLLARY. *Let X be a compact ordered space, and let $x, y \in X$ such that $x \not\geq y$. Then there exists a monotone and continuous function $\psi \colon X \to [0, 1]$ such that $\psi(x) = 0$ and $\psi(y) = 1$.*

PROOF. Set $\downarrow x \cap \uparrow y = \emptyset$. By Lemma 5.12, $\downarrow x$ is a closed lower subset and $\uparrow y$ is a closed upper subset. Therefore we may apply Proposition 5.13 with $A = \downarrow x$ and $B = \uparrow y$. ∎

5.15. COROLLARY. *Let X be a compact ordered space, and let $x, y \in X$. Suppose that, for every $\psi \colon X \to [0, 1]$ monotone and continuous, $\psi(x) \leq \psi(y)$. Then $x \leq y$.*

A consequence of Corollary 5.15 is the fact that every compact ordered space embeds into a power of $[0, 1]$.

5.16. PROPOSITION. *For every X compact ordered space, η_X is injective.*

PROOF. Let $x, y \in X$. Suppose $x \neq y$. Then, either $x \not\geq y$ or $y \not\geq x$. Suppose, without loss of generality, $x \not\geq y$. Then, by Corollary 5.14, there exists $\psi \in \mathrm{C}(X)$ such that $\psi(x) = 0$ and $\psi(y) = 1$. Therefore, $(\eta_X(x))(\psi) = \mathrm{ev}_x(\psi) = \psi(x) = 0 \neq 1 = \psi(y) = \mathrm{ev}_y(\psi) = (\eta_X(y))(\psi)$. Thus, $\eta_X(x) \neq \eta_X(y)$. ∎

5.17. THE UNIT $\eta_X \colon X \to \mathrm{MaxC}(X)$ IS SURJECTIVE. We continue the path that allows us to prove the implication [(3)⇒(1)] in Theorem 5.2: if X is a compact ordered space, $\eta_X \colon X \to \mathrm{MaxC}(X)$ is an isomorphism. In this subsection, we prove that, if X is a compact ordered space, then $\eta_X \colon X \to \mathrm{MaxC}(X)$ is surjective.

Let X be a compact ordered space, and let $\Phi \colon \mathrm{C}(X) \to [0,1]$ be an MC-morphism, i.e. $\Phi \in \mathrm{MaxC}(X)$. The goal is to find $x \in X$ such that $\Phi = \mathrm{ev}_x$. For every $\psi \in \mathrm{C}(X)$, set $\mathcal{Z}(\psi) := \{x \in X \mid \psi(x) = 0\}$. Moreover, set $\widetilde{\mathcal{Z}}(\Phi) := \bigcap_{\psi \in \mathrm{C}(X) : \Phi(\psi)=0} \mathcal{Z}(\psi)$. We shall prove that $\widetilde{\mathcal{Z}}(\Phi)$ has a maximum element x, and that $\Phi = \mathrm{ev}_x$. Set $\mathcal{A}(\Phi) := \bigcap_{\psi \in \mathrm{C}(X)} \{y \in X \mid \psi(y) \le \Phi(\psi)\}$.

5.18. LEMMA. $\mathcal{A}(\Phi) = \widetilde{\mathcal{Z}}(\Phi)$.

PROOF. Let us prove ($\subseteq$). Let $y \in \mathcal{A}(\Phi)$. Then for every $\psi \in \mathrm{C}(X)$, we have $\psi(y) \le \Phi(\psi)$. Therefore, if $\Phi(\psi) = 0$, then $\psi(y) = 0$, i.e., $\psi \in \mathcal{Z}(\Phi)$. Hence, $y \in \widetilde{\mathcal{Z}}(\Phi)$.

Let us prove ($\supseteq$). Let $x \in \widetilde{\mathcal{Z}}(\Phi)$. Let $\psi \in \mathrm{C}(X)$. We shall prove $\psi(x) \le \Phi(\psi)$. Set $\psi' := \psi \ominus \Phi(\psi)$. Then $\Phi(\psi') = \Phi(\psi) \ominus \Phi(\psi) = 0$. Since $x \in \widetilde{\mathcal{Z}}(\Phi)$, we have $\psi'(x) = 0$, i.e. $\psi(x) \ominus \Phi(\psi)$., i.e. $\psi(x) \le \Phi(\psi)$. ∎

5.19. LEMMA.

1. $\widetilde{\mathcal{Z}}(\Phi)$ is a closed lower subset of X.

2. For every $\psi \in \mathrm{C}(X)$, $\sup_{y \in \mathcal{A}(\Phi)} \psi(y) \le \Phi(\psi)$.

PROOF.

1. For every $\psi \in \mathrm{C}(X)$, $\mathcal{Z}(\psi)$ is closed, hence $\widetilde{\mathcal{Z}}(\Phi)$ is closed.

   Suppose $y \in \widetilde{\mathcal{Z}}(\Phi)$ and $x \le y$. Then, for every $\psi \in \mathrm{C}(X)$ such that $\Phi(\psi) = 0$, we have $y \in \mathcal{Z}(\psi)$, i.e. $\psi(y) = 0$. Since ψ is monotone, we have $\psi(x) = 0$, i.e. $x \in \mathcal{Z}(\psi)$. Therefore, $x \in \widetilde{\mathcal{Z}}(\Phi)$.

2. It follows from the fact that, for every $\psi \in \mathrm{C}(X)$ and $y \in \mathcal{A}(\Phi)$, $\psi(y) \le \Phi(\psi)$.

∎

The following is inspired by [Hofmann and Nora, 2018, Proposition 6.12].

5.20. PROPOSITION. *Let X be a compact ordered space and let $\Phi \colon \mathrm{C}(X) \to [0,1]$ be an MC-morphism. Then, for all $\psi \in \mathrm{C}(X)$, we have*

$$\Phi(\psi) = \sup_{y \in \widetilde{\mathcal{Z}}(\Phi)} \psi(y).$$

PROOF. Let $\psi \in C(X)$. We already know

$$\sup_{y \in \widetilde{\mathcal{Z}}(\Phi)} \psi(y) \overset{\text{Lem. 5.18}}{=} \sup_{y \in \mathcal{A}(\Phi)} \psi(y) \overset{\text{Lem. 5.19}}{\leq} \Phi(\psi).$$

Let us set $\lambda := \sup_{y \in \widetilde{\mathcal{Z}}(\Phi)} \psi(y)$. We shall prove $\Phi(\psi) \leq \lambda$. Let $\varepsilon > 0$. We shall prove $\Phi(\psi) \leq \lambda + \varepsilon$. Set

$$U := \{y \in X \mid \psi(y) < \lambda + \varepsilon\}.$$

Clearly, U is open and, by definition of λ, $\widetilde{\mathcal{Z}}(\Phi) \subseteq U$. Let $x \in X \setminus \widetilde{\mathcal{Z}}(\Phi)$. There is some $\widetilde{\psi} \in C(X)$ with $\Phi(\widetilde{\psi}) = 0$ and $\widetilde{\psi}(x) \neq 0$. Let $n \in \mathbb{N}$ be such that $n(\widetilde{\psi}(x)) \geq 1$. Set $\psi' := \underbrace{\widetilde{\psi} \oplus \cdots \oplus \widetilde{\psi}}_{n \text{ times}}$. Then $\Phi(\psi') = 0$ and $\psi'(x) = 1$. For every $\psi' \in C(X)$ we set

$$s(\psi') := \{x \in X \mid \psi'(x) > 1 - \varepsilon\}.$$

By the considerations above,

$$X = U \cup \bigcup_{\psi' \in C(X) : \Phi(\psi') = 0} s(\psi');$$

since X is compact, we find $\psi_1, \ldots, \psi_n \in C(X)$ with $\Phi(\psi_i) = 0$ and

$$X = U \cup s(\psi_1) \cup \cdots \cup s(\psi_n).$$

Therefore, for all $x \in X$, either $x \in U$, i.e., $\psi(x) < \lambda + \varepsilon$, or there exists $j \in \{1, \ldots, n\}$ such that $x \in s(\psi_j)$, i.e., $\psi_j(x) > 1 - \varepsilon$. Hence,

$$\psi \odot (1 - \varepsilon) \leq ((\lambda \oplus \varepsilon) \odot (1 - \varepsilon)) \vee \psi_1 \vee \cdots \vee \psi_n \leq \lambda \vee \psi_1 \vee \cdots \vee \psi_n.$$

Therefore,

$$\Phi(\psi) \odot (1 - \varepsilon) \leq \lambda \vee \Phi(\psi_1) \vee \cdots \vee \Phi(\psi_n) = \lambda \vee 0 \vee \cdots \vee 0 = \lambda.$$

Hence $\Phi(\psi) \leq (\Phi(\psi) \odot (1 - \varepsilon)) \oplus \varepsilon \leq \lambda \oplus \varepsilon$.   ∎

5.21. DEFINITION. *A nonempty subset A of a topological space X is irreducible if $A \subseteq B \cup C$ for closed subsets B and C implies $A \subseteq B$ or $A \subseteq C$.*

5.22. NOTATION. Let X be a compact ordered space, and let τ be the topology on X. We denote with $\tau^{\sharp}$ the topology of all upper open subsets of X.

A subset $B \subseteq X$ is upper if, and only if, its complement $X \setminus B$ is lower; hence the closed subsets of $(X, \tau^{\sharp})$ are precisely the closed lower subsets of (X, τ).

5.23. DEFINITION. *Let X be a compact ordered space and let τ be the topology on X. We say that $A \subseteq X$ is $\sharp$-irreducible if it is irreducible in $(X, \tau^{\sharp})$, i.e., for all closed lower subsets A_1, A_2 of (X, τ) with $A \subseteq A_1 \cup A_2$, one has $A \subseteq A_1$ or $A \subseteq A_2$.*

5.24. PROPOSITION. *Let X be a compact ordered space and $\Phi\colon C(X) \to [0,1]$ an MC-morphism. Then*

1. $\widetilde{\mathcal{Z}}(\Phi) \neq \emptyset$.

2. $\widetilde{\mathcal{Z}}(\Phi)$ *is $\sharp$-irreducible.*

PROOF.

1. $1 = \Phi(1) = \sup_{x \in \widetilde{\mathcal{Z}}(\Phi)} 1$.

2. Let $A_1, A_2 \subseteq X$ be closed upper subsets with $\widetilde{\mathcal{Z}}(\Phi) \subseteq A_1 \cup A_2$. We shall prove that either $\widetilde{\mathcal{Z}}(\Phi) \subseteq A_1$ or $\widetilde{\mathcal{Z}}(\Phi) \subseteq A_2$. Suppose, by way of contradiction, $x \in \widetilde{\mathcal{Z}}(\Phi) \setminus A_1$ and $y \in \widetilde{\mathcal{Z}}(\Phi) \setminus A_2$. Then $({\uparrow} x) \cap A_1 = \emptyset$ and $({\uparrow} y) \cap A_2 = \emptyset$. By Proposition 5.13, there exist $\psi_1, \psi_2 \in C(X)$ such that,

   (a) $\psi_1(x) = 1$ and, for all $z \in A_1$, $\psi_1(z) = 0$.

   (b) $\psi_2(y) = 1$ and, for all $z \in A_2$, $\psi_2(z) = 0$.

   Then, for all $z \in \widetilde{\mathcal{Z}}(\Phi)$, $(\psi_1 \odot \psi_2)(z) = \psi_1(z) \odot \psi_2(z) = 0$, since either $\psi_1(z) = 0$ or $\psi_2(z) = 0$.

   Hence, $\Phi(\psi_1 \odot \psi_2) = \sup_{z \in \widetilde{\mathcal{Z}}(\Phi)} \psi_1(z) \odot \psi_2(z) = 0$.

   But, also,
   $$\Phi(\psi_1 \odot \psi_2) = \Phi(\psi_1) \odot \Phi(\psi_2) =$$
   $$= \left(\sup_{z \in \widetilde{\mathcal{Z}}(\Phi)} \psi_1(z) \right) \odot \left(\sup_{z \in \widetilde{\mathcal{Z}}(\Phi)} \psi_2(z) \right) \geq$$
   $$\geq \psi_1(x) \odot \psi_2(y) =$$
   $$= 1 \odot 1 =$$
   $$= 1.$$

   This is a contradiction.

   ∎

5.25. DEFINITION. *We say that a topological space (X, τ) is sober if, for every irreducible closed set C, there exists a unique $x \in X$ such that the closure of $\{x\}$ is C.*

5.26. PROPOSITION. *Let $(X, \leq, \tau)$ be a compact ordered space. Then $(X, \tau^{\sharp})$ is sober.*

PROOF. See Proposition VI.6.11 in [Gierz et al., 2003]. ∎

5.27. REMARK. Let $(X, \leq, \tau)$ be a compact ordered space, and let $x \in X$. The closure of $\{x\}$ in $(X, \tau^\sharp)$ is $\downarrow x$.

5.28. PROPOSITION. *Let X be a compact ordered space, and let $A \subseteq X$ be an irreducible closed lower subset of X. Then there exists a unique $x \in X$ such that $A = \downarrow x$.*

PROOF. By Proposition 5.26 and Remark 5.27.   ∎

5.29. THEOREM. *Let X be a compact ordered space, and let $\Phi \colon \mathrm{C}(X) \to [0,1]$ be a map that preserves the operations $\vee, \wedge, \oplus, \odot$ and every constant $\lambda \in [0,1]$. Then, there exists a unique $x \in X$ such that, for every $\psi \in \mathrm{C}(X)$, $\Phi(\psi) = \psi(x)$, i.e. $\Phi = \mathrm{ev}_x$.*

PROOF. For every $x \in X$, $\eta_X(x) = \mathrm{ev}_x$. Hence, uniqueness follows from injectivity of η_X, which was established in Proposition 5.16 and which—we recall—was a consequence of Corollary 5.14. Concerning existence, by Proposition 5.28, there exists $x \in X$ such that $\widetilde{\mathcal{Z}}(\Phi) = \downarrow x$. Then

$$\Phi(\psi) = \sup_{z \in \widetilde{\mathcal{Z}}(\Phi)} \psi(z) = \psi(x).$$

∎

5.30. COROLLARY. *If X is a compact ordered space, the map $\eta_X \colon X \to \mathrm{MaxC}(X)$ is surjective.*

We may now conclude the proof of Theorem 5.2, which asserted, for X a preordered topological space, the equivalence of the following conditions.

1. The unit $\eta_X \colon X \to \mathrm{MaxC}(A)$ is an isomorphism.

2. There exists an MC-algebra A such that X and $\mathrm{Max}(A)$ are isomorphic preordered topological spaces.

3. X is a compact ordered space.

PROOF OF THEOREM 5.2. $[(1) \Rightarrow (2)]$ By Remark 5.3.
$[(2) \Rightarrow (3)]$ By Theorem 5.9.
$[(3) \Rightarrow (1)]$ By Proposition 5.16, η_X is injective. By Corollary 5.30, η_X is surjective. Every continuous map between compact Hausdorff spaces is closed, and every closed bijective continuous map between topological spaces is a homeomorphism. We are left to show that η_X reflects the order, i.e., for every $x, y \in X$, if $\eta_X(x) \leq \eta_X(y)$, then $x \leq y$. If $\eta_X(x) \leq \eta_X(y)$, then, for every $\psi \in \mathrm{C}(X)$, $(\eta_X(x))(\psi) \leq (\eta_X(y))(\psi)$, i.e., $\psi(x) \leq \psi(y)$. By Corollary 5.15, $x \leq y$.   ∎

6. Fixed objects on the algebraic side: the goal

6.1. DEFINITION. *Let $A \in \mathbf{MC}$ and $x, y \in A$. We set*

$$\uparrow_x^y := \{\lambda \in [0, 1] \mid y \leq x \oplus \lambda\};$$

$$\mathrm{d}^{\uparrow}(x, y) := \inf \uparrow_x^y;$$

and

$$\mathrm{d}(x, y) := \max\{\mathrm{d}^{\uparrow}(x, y), \mathrm{d}^{\uparrow}(y, x)\}.$$

On $[0, 1]$, $\mathrm{d}^{\uparrow}(x, y) = (y - x)^+$ (where $z^+ := \max\{z, 0\}$), and $\mathrm{d}(x, y) = |y - x|$. If X is a set, and L is an MC-subalgebra of $[0, 1]^X$, then, on L, $\mathrm{d}^{\uparrow}(f, g) = \sup_{x \in X}(g(x) - f(x))^+$, and d coincides with the sup metric. We mention that, in $[0, 1]$, $(y - x)^+$ coincides with $y \ominus x$.

6.2. DEFINITION. *Let $A \in \mathbf{MC}$. We say that A is* archimedean *if, for all $x, y \in A$,*

$$\mathrm{d}(x, y) = 0 \Rightarrow x = y.$$

The idea—as we will see—is that $A \in \mathbf{MC}$ is archimedean if, and only if, A is an MC-subalgebra of $[0, 1]^X$, for some set X. For now, we have the following.

6.3. REMARK. If X is a set, and L is an MC-subalgebra of $[0, 1]^X$, then L is archimedean. Indeed, d coincides with the sup metric, that satisfies the implication $\mathrm{d}(x, y) = 0 \Rightarrow x = y$.

In Definition 6.4 below, we define Cauchy sequences, convergence, and Cauchy completeness. These definitions are standard; anyway, one should pay attention to the fact that d is not required to be a metric, because $\mathrm{d}(x, y) = 0 \Rightarrow x = y$ might fail.

6.4. DEFINITION. *Let $A \in \mathbf{MC}$, let $(a_n)_{n \in \mathbb{N}}$ be a sequence in A, and let $a \in A$. We say that $(a_n)_{n \in \mathbb{N}}$ is a* Cauchy sequence *if, for all $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that, for all $n, m \geq k$, $\mathrm{d}(a_n, a_m) < \varepsilon$. We say that $(a_n)_{n \in \mathbb{N}}$ converges to a, or that a is a limit of $(a_n)_{n \in \mathbb{N}}$, if, for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that, for all $m \geq n$, $\mathrm{d}(a_m, a) < \varepsilon$. We say that $(a_n)_{n \in \mathbb{N}}$* converges *if there exists $b \in A$ such that $(a_n)_{n \in \mathbb{N}}$ converges to b. We say that A is* Cauchy complete *if every Cauchy sequence in A converges.*

We remark that a sequence may have more than one limit, since d is not a metric.

6.5. REMARK. On $[0, 1]$ the concepts of Cauchy sequence and convergence to an element in Definition 6.4 coincide with the usual ones with respect to the euclidean distance. In particular, $[0, 1]$ is Cauchy complete. If X is a set, and L is an MC-subalgebra of $[0, 1]^X$, the concepts of Cauchy sequence and convergence to an element in Definition 6.4 coincide with the usual ones with respect to the sup metric.

Our next goal is to prove the following.

6.6. THEOREM. *Let $A \in \mathbf{MC}$. The following conditions are equivalent.*

1. *The unit $\varepsilon_A \colon A \to \mathrm{CMax}(A)$ is an isomorphism.*

2. *There exists a preordered topological space X such that A and $\mathrm{C}(X)$ are isomorphic MC-algebras.*

3. *A is archimedean and Cauchy complete.*

The current section and the following two are intended to prove Theorem 6.6 above.

6.7. REMARK. We give here the proofs of the implications $[(1)\Rightarrow(2)]$ and $[(2)\Rightarrow(3)]$ of Theorem 6.6.
$[(1)\Rightarrow(2)]$ Take $X = \mathrm{Max}(A)$.
$[(2)\Rightarrow(3)]$ A is archimedean because $\mathrm{C}(X)$ is archimedean, by Remark 6.3. Let us prove that $\mathrm{C}(X)$ (and hence A) is Cauchy complete. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{C}(X)$ which is Cauchy with respect to the metric d. Then, there exists a function $f \colon X \to [0,1]$ such that f_n converges to f uniformly. It is well known that the uniform limit of a sequence of continuous functions is continuous. Since, for all $n \in \mathbb{N}$, f_n is continuous, f is continuous. Let us prove that f is monotone. Let $x, y \in X$ with $x \le y$. For all $n \in \mathbb{N}$, f_n is monotone. Therefore, $f(x) = \lim_{n\to\infty} f_n(x) \le \lim_{n\to\infty} f_n(y) = f(y)$.

We are left to prove the implication $[(3)\Rightarrow(2)]$ of Theorem 6.6, i.e., that, for every archimedean Cauchy complete MC-algebra A, the unit $\varepsilon_A \colon A \to \mathrm{CMax}(A)$ is an isomorphism. This implication is a consequence of the following two theorems, whose proofs we conclude, respectively, at the ends of the following two sections.

6.8. THEOREM. *Let $A \in \mathbf{MC}$. The following conditions are equivalent.*

1. *A is archimedean.*

2. *For every $x, y \in A$ with $x \ne y$, there exists an MC-morphism $\varphi \colon A \to [0,1]$ such that $\varphi(x) \ne \varphi(y)$.*

3. *There exists a set X such that A is an MC-subalgebra of $[0,1]^X$.*

4. *The unit $\varepsilon_A \colon A \to \mathrm{CMax}(A)$ is injective.*

6.9. THEOREM. *Let $A \in \mathbf{MC}$. The following conditions are equivalent.*

1. *A is Cauchy complete.*

2. *The unit $\varepsilon_A \colon A \to \mathrm{CMax}(A)$ is surjective.*

7. A is archimedean if, and only if, the unit ε_A is injective

The aim of this section is to prove Theorem 6.8 above. Some of the implications between the four conditions in Theorem 6.8 are relatively easy to prove, and we collect their proofs in the following remark.

7.1. REMARK. [Part of the proof of Theorem 6.8] We prove some of the implications in Theorem 6.8.

[(4)⇒(3)] Immediate.

[(3)⇒(2)] Let $f, g \in A \subseteq [0,1]^X$ be such that $f \neq g$. Then, there exist $z \in X$ such that $f(z) \neq g(z)$. Set

$$\varphi \colon A \longrightarrow [0,1]$$
$$h \longmapsto h(z).$$

The function $\varphi \colon A \to [0,1]$ is clearly an MC-morphism. $\varphi(f) = f(z) \neq g(z) = \varphi(g)$.

[(2)⇒(4)] Let $x, y \in A$ be such that $x \neq y$. We shall prove $\varepsilon_A(x) \neq \varepsilon_A(y)$. By hypothesis, there exists an MC-morphism $\varphi \colon A \to [0,1]$ such that $\varphi(x) \neq \varphi(y)$. Note that $\varphi \in \mathrm{Max}(A)$. We have $(\varepsilon_A(x))(\varphi) = \mathrm{ev}_x(\varphi) = \varphi(x) \neq \varphi(y) = \mathrm{ev}_y(\varphi) = (\varepsilon_A(y))(\varphi)$. This proves $\varepsilon_A(x) \neq \varepsilon_A(y)$.

[(3)⇒(1)] By Remark 6.3.

In Remark 7.1 we have proved that (2), (3) and (4) in Theorem 6.8 are equivalent, and that any of these conditions implies (1). What is missing to prove Theorem 6.8 is the implication [(1)⇒(2)] (for example), i.e., that if A is archimedean, then, for every $x \neq y \in A$, there exists an MC-morphism $\varphi \colon A \to [0,1]$ such that $\varphi(x) \neq \varphi(y)$. The proof of this last implication will take us some effort. The idea is that, since **MC** is a *finitary* variety, we may apply the Subdirect Representation Theorem. The Subdirect Representation Theorem ensures, for every $A \in \mathbf{MC}$, the existence of an injective MC-morphism $\iota \colon A \hookrightarrow \prod_{i \in I} A_i$, where, for all $i \in I$, A_i is subdirectly irreducible. We will show that every subdirectly irreducible algebra B consists essentially of the set $[0,1]$ together with some additional elements, each of which lies "just above" or "just below" one particular $\lambda \in [0,1]$. This allows us to define the morphism $\mathrm{ess} \colon B \to [0,1]$ that "kills the infinitesimals". We will show that the morphism ess preserves d. Composing the map $\iota \colon A \hookrightarrow \prod_{i \in I} A_i$ with the morphisms $\mathrm{ess}_i \colon A_i \to [0,1]$, we obtain a map $h \colon A \to \prod_{i \in I}[0,1]$ that correctly translates the function d on A to the sup metric on $\prod_{i \in I}[0,1]$. Then, if $x \neq y \in A$, and A is archimedean, we have $\mathrm{d}(x,y) \neq 0$, and therefore the sup distance between $h(x)$ and $h(y)$ is not zero. From this we conclude that there is a morphism $\varphi \colon A \to [0,1]$ such that $\varphi(x) \neq \varphi(y)$.

7.2. SUBDIRECTLY IRREDUCIBLE MC-ALGEBRAS. We start the path that will take us to the proof that every archimedean MC-algebra admits enough morphisms towards $[0,1]$, i.e., the implication [(1)⇒(2)] in Theorem 6.8. The main goal of this subsection is to prove the following.

7.3. THEOREM. *Let $A \in \mathbf{MC}$ be subdirectly irreducible. Then, for all $x \in A$, and $\lambda \in [0,1]$, $x \leq \lambda$ or $\lambda \leq x$.*

To a reader that has familiarity with abelian lattice-ordered groups or MV-algebras, Theorem 7.3 might sound analogous to the well-known result that every subdirectly irreducible abelian lattice-ordered group (or MV-algebra) is totally ordered. Note, however,

that Theorem 7.3 does not say "every subdirectly irreducible $A \in \mathbf{MC}$ is totally ordered"; in fact, it says something something weaker. The author does not know whether every subdirectly irreducible $A \in \mathbf{MC}$ is totally ordered; nevertheless, for our purposes, Theorem 7.3 is enough.

The idea for the proof of Theorem 7.3 is the following: if, by contraposition, $x \not\leq \lambda$ and $\lambda \not\leq x$, then we could construct two nonminimal congruences whose intersection is minimal, which shows that A is not subdirectly irreducible. The idea is that these two congruences are the congruences generated, respectively, by $x \sim x \vee \lambda$ and $x \sim x \wedge \lambda$.

Our convention is that we do not consider the trivial algebra (i.e, with exactly an element) as subdirectly irreducible.

7.4. NOTATION. Let $A \in \mathbf{MC}$, and let $\mathcal{L}$ denote the language of $\mathbf{MC}$. For $A \in \mathbf{MC}$, let A^∂ denote the $\mathcal{L}$-algebra that shares the same underlying set with A, and which is such that $\vee_{A^\partial} = \wedge_A$, $\wedge_{A^\partial} = \vee_A$, $\oplus_{A^\partial} = \odot_A$, $\odot_{A^\partial} = \oplus_A$, and, for every $\lambda \in [0, 1]$, $\lambda_{A^\partial} = (1 - \lambda)_A$. We call A^∂ the *order-dual algebra* of A.

Roughly speaking, the dual operation γ^∂ of an operation γ of arity I is given by $\gamma^\partial((x_i)_{i \in I}) := 1 - \gamma((1 - x_i)_{i \in I})$ (this makes sense in $[0, 1]$).

We will use the concept of order-dual algebra only to shorten some proofs. This is made possible by the following lemma.

7.5. LEMMA. *Let $A \in \mathbf{MC}$. Then, $A^\partial \in \mathbf{MC}$.*

PROOF. The only nontrivial part is showing that Axioms (9), (10), (11), (12) "dualize". But this holds because, for every $a, b, c \in [0, 1]$, we have the following properties.

1. $a \wedge b = c \Leftrightarrow (1 - a) \vee (1 - b) = 1 - c$.

2. $a \vee b = c \Leftrightarrow (1 - a) \wedge (1 - b) = 1 - c$.

3. $a \oplus b = c \Leftrightarrow (1 - a) \odot (1 - b) = 1 - c$.

4. $a \odot b = c \Leftrightarrow (1 - a) \oplus (1 - b) = 1 - c$.

∎

7.6. REMARK. $\left(A^\partial\right)^\partial = A$.

7.7. LEMMA. *Let $A \in \mathbf{MC}$. The following properties hold for all $a, b, c, a', b' \in A$.*

1. $a \leq b \Rightarrow a \oplus c \leq b \oplus c$.

2. $a \leq b \Rightarrow a \odot c \leq b \odot c$.

3. $a \leq a \oplus b$.

4. $a \geq a \odot b$.

5. If $a \leq a'$ and $b \leq b'$, then $a \oplus b \leq a' \oplus b'$.

6. If $a \leq a'$ and $b \leq b'$, then $a \odot b \leq a' \odot b'$.

PROOF. (1) Suppose $a \leq b$. Then $a \oplus c = (a \wedge b) \oplus c = (a \oplus c) \wedge (b \oplus c)$.
(3) From $0 \leq b$ we obtain $a \oplus 0 \leq a \oplus b$. Since $a \oplus 0 = a$, $a \leq a \oplus b$.
(5) This follows from (1), taking into account that $\oplus$ is commutative.
Items (2), (4) and (6) in A coincide respectively with items (1), (3) and (5) in the order-dual algebra of A. ∎

7.8. LEMMA. *Let $A \in \mathbf{MC}$ and $x \in A$. For $a, a' \in A$, set $a \sim_{\oplus}^{x} a'$ if, and only if, there exist $n, m \in \mathbb{N}$ such that*

$$a \oplus (\underbrace{x \oplus \cdots \oplus x}_{n \ times}) \geq a'$$

$$a' \oplus (\underbrace{x \oplus \cdots \oplus x}_{m \ times}) \geq a.$$

Then $\sim_{\oplus}^{x}$ is a congruence.

PROOF. We first prove that $\sim_{\oplus}^{x}$ is an equivalence relation. The relation $\sim_{\oplus}^{x}$ is trivially reflexive and symmetric. To prove transitivity, suppose $a \sim_{\oplus}^{x} b \sim_{\oplus}^{x} c$. Then

$$(a \oplus (\underbrace{x \oplus \cdots \oplus x}_{n \ times})) \oplus (\underbrace{x \oplus \cdots \oplus x}_{n' \ times}) \geq b \oplus (\underbrace{x \oplus \cdots \oplus x}_{n' \ times}) \geq c.$$

Analogously for the other inequality.

Suppose $a \sim_{\oplus}^{x} a'$ and $b \sim_{\oplus}^{x} b'$.

1. We shall prove $a \vee b \sim_{\oplus}^{x} a' \vee b'$.

$$(a \vee b) \oplus (\underbrace{x \oplus \cdots \oplus x}_{\max\{n,n'\} \ times}) \geq a' \vee b'.$$

   Analogously for the other inequality.

2. We shall prove $a \wedge b \sim_{\oplus}^{x} a' \wedge b'$.

$$(a \wedge b) \oplus (\underbrace{x \oplus \cdots \oplus x}_{\max\{n,n'\} \ times}) = (a \oplus (\underbrace{x \oplus \cdots \oplus x}_{\max\{n,n'\} \ times})) \wedge (b \oplus (\underbrace{x \oplus \cdots \oplus x}_{\max\{n,n'\} \ times})) \geq a' \wedge b'.$$

   Analogously for the other inequality.

3. We shall prove $a \oplus b \sim_{\oplus}^{x} a' \oplus b'$.

$$(a \oplus b) \oplus (\underbrace{x \oplus \cdots \oplus x}_{(n+n') \ times}) = (a \oplus (\underbrace{x \oplus \cdots \oplus x}_{n \ times})) \oplus (b \oplus (\underbrace{x \oplus \cdots \oplus x}_{n' \ times})) \geq a' \oplus b'.$$

   Analogously for the other inequality.

4. We shall prove $a \odot b \sim_{\oplus}^{x} a' \odot b'$.

$$(a \odot b) \oplus (\underbrace{x \oplus \cdots \oplus x}_{(n+n') \ times}) \geq (a \oplus (\underbrace{x \oplus \cdots \oplus x}_{n \ times})) \odot (b \oplus (\underbrace{x \oplus \cdots \oplus x}_{n' \ times})) \geq a' \odot b'.$$

   Analogously for the other inequality.

∎

7.9. LEMMA. *Let $A \in \mathbf{MC}$, $x \in A$.*
   For $b, b' \in A$, set $b \sim_{\odot}^{x} b'$ if, and only if there exists $n, m \in \mathbb{N}$ such that

$$b \odot (\underbrace{x \odot \cdots \odot x}_{n \; times}) \le b',$$

$$b' \odot (\underbrace{x \odot \cdots \odot x}_{m \; times}) \le b.$$

Then $\sim_{\odot}^{x}$ is a congruence.

PROOF. The relation $\sim_{\odot}^{x}$ is a congruence in A^{∂}, by Lemma 7.8. This implies that $\sim_{\odot}^{x}$ is a congruence in A. ∎

We call *minimal congruence* (or *trivial congruence*) on an algebra A the smallest congruence, which is $\Delta_A := \{(a, a) \mid a \in A\}$.

We are now ready for the proof of Theorem 7.3.

PROOF OF THEOREM 7.3. Suppose, by way of contradiction, $x \not\le \lambda$ and $\lambda \not\le x$. Then, $x \ominus \lambda \ne 0$ and $x \oplus (1 - \lambda) \ne 1$. Indeed, $x \le (x \ominus \lambda) \oplus \lambda$, and $x \ge (x \oplus (1 - \lambda)) \ominus (1 - \lambda)$.

For $a, b \in A$, set $a \sim_{\oplus} b$ if, and only if, there exists $n, m \in \mathbb{N}$ such that

$$a \oplus (\underbrace{(x \ominus \lambda) \oplus \cdots \oplus (x \ominus \lambda)}_{n \text{ times}}) \ge b,$$

$$b \oplus (\underbrace{(x \ominus \lambda) \oplus \cdots \oplus (x \ominus \lambda)}_{m \text{ times}}) \ge a.$$

By Lemma 7.8, $\sim_{\oplus}$ is a congruence.

For $a, b \in A$, set $a \sim_{\odot} b$ if, and only if, there exists $n, m \in \mathbb{N}$ such that

$$a \odot (\underbrace{(x \oplus (1 - \lambda)) \odot \cdots \odot (x \oplus (1 - \lambda))}_{n \text{ times}}) \le b,$$

$$b \odot (\underbrace{(x \oplus (1 - \lambda)) \odot \cdots \odot (x \oplus (1 - \lambda))}_{m \text{ times}}) \le a.$$

By Lemma 7.9, $\sim_{\odot}$ is a congruence.

Since $x \ominus \lambda \ne 0$ and $x \ominus \lambda \sim_{\oplus} 0$, $\sim_{\oplus}$ is not the minimal congruence. Since $x \oplus (1 - \lambda) \ne 1$ and $x \oplus (1 - \lambda) \sim_{\oplus} 1$, $\sim_{\oplus}$ is not the minimal congruence.

We claim that the congruence $\sim_{\oplus} \cap \sim_{\odot}$ is the minimal one. Indeed, let us take $a, b \in A$ such that $a \sim_{\oplus} b$ and $a \sim_{\odot} b$.

Then

$$a \le b \oplus (\underbrace{(x \ominus \lambda) \oplus \cdots \oplus (x \ominus \lambda)}_{m \text{ times}}).$$

and

$$a \odot (\underbrace{(x \oplus (1 - \lambda)) \odot \cdots \odot (x \oplus (1 - \lambda))}_{n \text{ times}}) \le b.$$

Then
$$a = a \wedge (b \oplus (\underbrace{(x \ominus \lambda) \oplus \cdots \oplus (x \ominus \lambda)}_{m \text{ times}})) \leq$$

$$\leq (a \odot (\underbrace{(x \oplus (1 - \lambda)) \odot \cdots \odot (x \oplus (1 - \lambda))}_{n \text{ times}})) \vee b = b.$$

Analogously, $b \leq a$, and therefore $a = b$. Therefore $\sim_\oplus \cap \sim_\odot$ is the minimal congruence. This is a contradiction; indeed, the minimal congruence on a subdirectly irreducible algebra is $\wedge$-irreducible in the lattice of congruences, and therefore the intersection of nonminimal congruence is not minimal in a subdirectly irreducible algebra. ∎

7.10. THE MORPHISM ess FROM A SUBDIRECTLY IRREDUCIBLE MC-ALGEBRA TO $[0, 1]$ THAT KILLS INFINITESIMALS. In Theorem 7.3 we proved that, if $A \in \mathbf{MC}$ is subdirectly irreducible, then, for all $x \in A$, and $\lambda \in [0, 1]$, $x \leq \lambda$ or $\lambda \leq x$. The intuition is that A consists essentially of the set $[0, 1]$, together with some infinitesimals, each of which lies "just above" or "just below" one particular $\lambda \in [0, 1]$. In this subsection we show that one can define the function $\mathrm{ess} \colon A \to [0, 1]$ that "kills the infinitesimals" and that this map is an MC-morphism (see Theorem 7.17).

7.11. LEMMA. *Let $A \in \mathbf{MC}$. Let $\alpha, \beta \in [0, 1]$ be such that $\alpha \neq \beta$. The following conditions are equivalent.*

  1. *A is trivial.*

  2. *$\alpha_A = \beta_A$.*

PROOF. If A is trivial, then, clearly, $\alpha_A = \beta_A$. Suppose $\alpha_A = \beta_A$ and suppose, without loss of generality, $\alpha < \beta$. Then, in A, $\beta \ominus \alpha = \beta \ominus \beta = 0$. Then, for every $n \in \mathbb{N}$, in A, $0 = \underbrace{(\beta \ominus \alpha) \oplus \cdots \oplus (\beta \ominus \alpha)}_{n \text{ times}}$. Let n be big enough so that, in $[0, 1]$, $\underbrace{(\beta \ominus \alpha) \oplus \cdots \oplus (\beta \ominus \alpha)}_{n \text{ times}} = 1$. Then, in A, $0_A = \underbrace{(\beta \ominus \alpha) \oplus \cdots \oplus (\beta \ominus \alpha)}_{n \text{ times}} = 1_A$. Since 0_A is the bottom and 1_A is the top of A, for every $x, y \in A$, $x = y$. ∎

7.12. DEFINITION. *Let $A \in \mathbf{MC}$ and $x \in A$. We set*

$$I_x := \{\lambda \in [0, 1] \mid \lambda \leq x\};$$

$$S_x := \{\lambda \in [0, 1] \mid x \leq \lambda\};$$

$$\mathrm{essinf}\, x := \sup I_x;$$

$$\mathrm{esssup}\, x := \inf S_x.$$

7.13. REMARK. Let $A \in \mathbf{MC}$ and $x \in A$. Then, $\mathrm{essinf}\, x$, calculated in A, is $\mathrm{esssup}\, x$, calculated in A^∂.

7.14. REMARK. I_x is an initial segment of $[0, 1]$, $0 \in I_x$, S_x is a final segment of $[0, 1]$, and $1 \in S_x$. In addition, we note the following facts.

1. If A is trivial, then $I_x = [0, 1] = S_x$, and thus essinf $x = 1$ and esssup $x = 0$.

2. If A is not trivial, then $I_x \cap S_x$ has at most one element, because otherwise there would exists $\alpha \neq \beta$ in $[0, 1]$ such that, in A, $\alpha \leq x \leq \beta \leq x \leq \alpha$, and hence $\alpha_A = \beta_A$; but, since A is non-trivial, this is not possible by Lemma 7.11. Hence, if A is not trivial,
$$\text{essinf } x \leq \text{esssup } x.$$

7.15. LEMMA. *Let $A \in \mathbf{MC}$ be nontrivial. If $x \in A$ is such that, for all $\lambda \in [0, 1]$, $x \leq \lambda$ or $\lambda \leq x$, then*
$$\text{essinf } x = \text{esssup } x.$$
In particular, if A is subdirectly irreducible, then, for all $x \in X$, essinf $x = $ esssup x.

PROOF. By Remark 7.14, essinf $x \leq $ esssup x. Since, by hypothesis, $I_x \cup S_x = [0, 1]$, we conclude essinf $x = $ esssup x. If A is subdirectly irreducible, then, by Theorem 7.3, for all $x \in A$ and $\lambda \in [0, 1]$, $x \leq \lambda$ or $\lambda \leq x$. ∎

7.16. NOTATION. For $A \in \mathbf{MC}$ and $x \in A$, if essinf $x = $ esssup x, we set
$$\text{ess } x := \text{essinf } x = \text{esssup } x.$$

7.17. THEOREM. *Let $A \in \mathbf{MC}$ be such that, for all $x \in A$, essinf $x = $ esssup x (this holds, in particular, if A is subdirectly irreducible). Then, the function*
$$\text{ess} \colon \mathbf{MC} \longrightarrow [0, 1]$$
$$x \longmapsto \text{ess } x$$

is a surjective MC-morphism.

PROOF. For all $x \in A$, recall: $I_x := \{\lambda \in [0, 1] \mid \lambda \leq x\}$, $S_x := \{\lambda \in [0, 1] \mid x \leq \lambda\}$, and ess $x = \sup I_x = \inf S_x$. For every constant symbol $\lambda \in [0, 1]$, ess clearly preserves λ. Let $x, y \in A$ and let $\otimes$ denote any operation amongst $\{\vee, \wedge, \oplus, \odot\}$. We shall show ess$(x \otimes y) = \text{ess } x \otimes \text{ess } y$. For every $U, W \subseteq [0, 1]$, set $U \otimes W := \{\alpha \otimes \beta \mid \alpha \in U, \beta \in W\}$. Since $\otimes \colon [0, 1]^2 \to [0, 1]$ is continuous, for every nonempty $U, V \subseteq [0, 1]$, $\sup(U \otimes W) = (\sup U) \otimes (\sup W)$ and $\sup(U \otimes W) = (\sup U) \otimes (\sup W)$. We shall show ess$(x \otimes y) = \text{ess } x \otimes \text{ess } y$. Now,
$$\text{ess}(x \otimes y) = \sup I_{x \otimes y},$$
$$\text{ess}(x \otimes y) = \inf S_{x \otimes y},$$
$$\text{ess } x \otimes \text{ess } y = (\sup I_x) \otimes (\sup I_y) = \sup(I_x \otimes I_y),$$
$$\text{ess } x \otimes \text{ess } y = (\inf S_x) \otimes (\inf S_y) = \inf(S_x \otimes S_y).$$

Let us take $\lambda \in I_{x \otimes y}$. Then $\lambda \leq x \otimes y$. Therefore, for every $\alpha \in S_x, \beta \in S_y$, $\lambda \leq x \otimes y \leq \alpha \otimes \beta$. This shows $\mathrm{ess}(x \otimes y) = \sup I_{x \otimes y} \leq \inf(S_x \otimes S_y) = \mathrm{ess}\, x \otimes \mathrm{ess}\, y$. Let us now take $\lambda \in S_{x \otimes y}$. Then, $x \otimes y \leq \lambda$. Then, for every $\alpha \in I_x, \beta \in I_y$, we have $\alpha \otimes \beta \leq x \otimes y \leq \lambda$. This shows $\mathrm{ess}\, x \otimes \mathrm{ess}\, y = \sup(I_x \otimes I_y) \leq \inf S_{x \otimes y} = \mathrm{ess}(x \otimes y)$.

The function ess is surjective, because it must preserve every constant symbol $\lambda \in [0, 1]$.

By Lemma 7.15, if A is subdirectly irreducible, then, for all $x \in A$, essinf $x = $ esssup x. ∎

We call an algebra A *simple* if it is not trivial and any proper quotient of A is trivial. From Theorem 7.17, we obtain that $[0, 1]$ is the unique simple MC-algebra, as stated in Corollary 7.18 below. This is similar to Hölder's Theorem for lattice-ordered groups.

7.18. COROLLARY. *Let $A \in \mathbf{MC}$. A is simple if, and only if, it is isomorphic to $[0, 1]$.*

PROOF. $[0, 1]$ is simple. Indeed, if $B \in \mathbf{MC}$, and $\varphi \colon [0, 1] \to B$ is a surjective not-injective MC-morphism, then there exist $\alpha, \beta \in [0, 1]$ such that $\alpha \neq \beta$ and $\varphi(\alpha) = \varphi(\beta)$, i.e., $\alpha_B = \beta_B$. By Lemma 7.11, B is trivial. Hence, $[0, 1]$ is simple, as well as any of its isomorphic copies. Suppose that A is simple. By the subdirect representation theorem, it is isomorphic to a subdirect product $\prod_{i \in I} B_i$ of subdirectly irreducible algebras. Since A is simple, it is not trivial. Hence, $I \neq \emptyset$, and thus there exists a surjective morphism $\varphi \colon A \to B$, with B subdirectly irreducible. By Theorem 7.17, we have a surjective MC-morphism ess$\colon B \to [0, 1]$. Hence, we have a surjective MC-morphism $\psi \colon A \to [0, 1]$, which must be injective since A is simple. Hence ψ is an isomorphism. ∎

Corollary 7.18 implies that the set of morphisms from an MC-algebra A to $[0, 1]$ is in bijection with the set of maximal congruences on A. This explains why we gave the name $\mathrm{Max}(A)$ for the set $\hom_{\mathbf{MC}}(A, [0, 1])$.

7.19. THE MAP ess PRESERVES DISTANCE. The main goal of this subsection is to prove that, for A a nontrivial subdirectly irreducible algebra, the map ess$\colon A \to [0, 1]$ preserves d, i.e., $\mathrm{d}_A(a, b) = \mathrm{d}_{[0,1]}(\mathrm{ess}\, a, \mathrm{ess}\, b)$. We will actually prove, in Lemma 7.26, a slightly stronger statement, i.e., that $\mathrm{d}^\uparrow$ is preserved.

7.20. LEMMA. *Let $A \in \mathbf{MC}$ and $x, y \in A$. Then $y \leq x \oplus \lambda$ if, and only if, $y \ominus \lambda \leq x$.*

PROOF. If $y \leq x \oplus \lambda$, then $y \ominus \lambda \leq (x \oplus \lambda) \ominus \lambda \leq x$. If $y \ominus \lambda \leq x$, then $y \leq (y \ominus \lambda) \oplus \lambda \leq x \oplus \lambda$. ∎

7.21. REMARK. Let $A \in \mathbf{MC}$ and $x, y \in A$. Then, by Remark 7.20, we have

$$\uparrow_x^y := \{\lambda \in [0, 1] \mid y \leq x \oplus \lambda\} = \{\lambda \in [0, 1] \mid y \ominus \lambda \leq x\}.$$

Hence

$$\mathrm{d}^\uparrow(x, y) := \inf \uparrow_x^y = \inf\{\lambda \in [0, 1] \mid y \leq x \oplus \lambda\} = \inf\{\lambda \in [0, 1] \mid y \ominus \lambda \leq x\}.$$

7.22. REMARK. Let $A \in \mathbf{MC}$. Then the set $\uparrow_x^y$ calculated in A equals $\uparrow_y^x$ calculated in A^∂. Thus $\mathrm{d}^\uparrow{}_A(x, y) = \mathrm{d}^\uparrow{}_{A^\partial}(y, x)$.

7.23. REMARK.

1. Let I be a set, and, for each $i \in I$, let $A_i \in \mathbf{MC}$. Let $a, b \in \prod_{i \in I} A_i$. Then, $\mathrm{d}^\uparrow(a, b) = \sup_{i \in I} \mathrm{d}^\uparrow(a_i, b_i)$.

2. Let $A \in \mathbf{MC}$, let B be an MC-subalgebra of A, and let $x, y \in B$. Then $\mathrm{d}^\uparrow(x, y)$ is the same calculated in A and B.

7.24. LEMMA. *Let $A \in \mathbf{MC}$ and $x, y, z \in A$. The following properties hold.*

1. $\mathrm{d}^\uparrow(x, y) \in [0, 1]$.

2. $x \le y \Rightarrow \mathrm{d}^\uparrow(y, x) = 0$.

3. $\mathrm{d}^\uparrow(x, z) \le \mathrm{d}^\uparrow(x, y) \oplus \mathrm{d}^\uparrow(y, z) \le \mathrm{d}^\uparrow(x, y) + \mathrm{d}^\uparrow(y, z)$.

PROOF. (1) and (2) are clear, by definition. To prove (3), let $\lambda_0 \in \uparrow_x^y$ and $\lambda_1 \in \uparrow_y^z$. Then $y \le x \oplus \lambda_0$, $z \le y \oplus \lambda_1$. Then $z \le y \oplus \lambda_1 \le (x \oplus \lambda_0) \oplus \lambda_1 = x \oplus (\lambda_0 \oplus \lambda_1)$. Therefore, $\lambda_0 \oplus \lambda_1 \in \uparrow_x^z$. Therefore, $\mathrm{d}^\uparrow(x, z) = \inf(\uparrow_x^z) \le \inf(\uparrow_x^y \oplus \uparrow_y^z)$. By continuity of $\oplus$, $\inf(\uparrow_x^y \oplus \uparrow_y^z) = \inf(\uparrow_x^y) \oplus \inf(\uparrow_y^z) = \mathrm{d}^\uparrow(x, y) \oplus \mathrm{d}^\uparrow(y, z)$. ∎

7.25. LEMMA. *Let $A \in \mathbf{MC}$. Then*

1. *For every $x \in A$, $\mathrm{d}^\uparrow(x, \mathrm{essinf}\, x) = \mathrm{d}^\uparrow(\mathrm{esssup}\, x, x) = 0$.*

2. *If A is nontrivial, for every $\alpha, \beta \in [0, 1]$, $\mathrm{d}^\uparrow(\alpha_A, \beta_A) = (\beta - \alpha)^+$.*

PROOF.

1. Let $\varepsilon > 0$. Then $x \le \mathrm{esssup}\, x \oplus \varepsilon$. Therefore, $\varepsilon \in \uparrow_{\mathrm{esssup}\, x}^x$. Since it holds for every ε, then $\mathrm{d}^\uparrow(\mathrm{esssup}\, x, x) = \inf \uparrow_{\mathrm{esssup}\, x}^x = 0$. Via the order-dual algebra, $\mathrm{d}^\uparrow(x, \mathrm{essinf}\, x) = 0$ is automatically proven.

2. If A is nontrivial, then $\uparrow_\alpha^\beta = \{\lambda \in [0, 1] \mid \beta_A \le \alpha_A \oplus \lambda\} = \{\lambda \in [0, 1] \mid \lambda \ge (\beta - \alpha)^+\}$.

∎

7.26. LEMMA. *Let $A \in \mathbf{MC}$, and let $x, y \in A$ be such that* $\operatorname{essinf} x = \operatorname{esssup} x$ *and* $\operatorname{essinf} y = \operatorname{esssup} y$. *Then,* $\mathrm{d}^{\uparrow}(x, y) = (\operatorname{ess} y - \operatorname{ess} x)^{+}$.

PROOF. A is nontrivial, because $\operatorname{essinf} x = \operatorname{esssup} x$. Thus, $(\operatorname{ess} y - \operatorname{ess} x)^{+} = \mathrm{d}^{\uparrow}(\operatorname{ess} x, \operatorname{ess} y)$. We are left to prove $\mathrm{d}^{\uparrow}(x, y) = \mathrm{d}^{\uparrow}(\operatorname{ess} x, \operatorname{ess} y)$. We have

$$\mathrm{d}^{\uparrow}(x, y) \leq \mathrm{d}^{\uparrow}(x, \operatorname{ess} x) \oplus \mathrm{d}^{\uparrow}(\operatorname{ess} x, \operatorname{ess} y) \oplus \mathrm{d}^{\uparrow}(\operatorname{ess} y, y) =$$

$$= 0 \oplus \mathrm{d}^{\uparrow}(\operatorname{ess} x, \operatorname{ess} y) \oplus 0 =$$

$$= \mathrm{d}^{\uparrow}(\operatorname{ess} x, \operatorname{ess} y).$$

Moreover,
$$\mathrm{d}^{\uparrow}(\operatorname{ess} x, \operatorname{ess} y) \leq \mathrm{d}^{\uparrow}(\operatorname{ess} x, x) \oplus \mathrm{d}^{\uparrow}(x, y) \oplus \mathrm{d}^{\uparrow}(y, \operatorname{ess} y) =$$

$$= 0 \oplus \mathrm{d}^{\uparrow}(x, y) \oplus 0 =$$

$$= \mathrm{d}^{\uparrow}(x, y).$$

$\blacksquare$

7.27. EVERY ARCHIMEDEAN MC-ALGEBRA IS AN ALGEBRA OF FUNCTIONS. In this subsection, we prove that every archimedean MC-algebra has enough morphisms towards $[0, 1]$ to separate its elements; this completes the proof of Theorem 6.8.

7.28. LEMMA. *Let $A, B \in \mathbf{MC}$ and let $\phi \colon A \to B$ be an MC-morphism. Then, for every* $x, y \in A$,
$$\mathrm{d}^{\uparrow}(\phi(x), \phi(y)) \leq \mathrm{d}^{\uparrow}(x, y).$$

PROOF. Let $\lambda \in \uparrow_{x}^{y}$: $y \leq x \oplus \lambda$. Then $\phi(y) \leq \phi(x) \oplus \lambda$; thus $\lambda \in \uparrow_{\phi(x)}^{\phi(y)}$. Therefore, $\mathrm{d}\uparrow(\phi(x), \phi(y)) = \inf \uparrow_{\phi(x)}^{\phi(y)} \leq \inf \uparrow_{x}^{y} = \mathrm{d}^{\uparrow}(x, y)$. $\blacksquare$

7.29. THEOREM. *Let $A \in \mathbf{MC}$, and let $x, y \in A$. Then,*
$$\mathrm{d}^{\uparrow}(x, y) = \sup_{\varphi \colon A \to [0,1] \; \text{MC-morphism}} (\varphi(y) - \varphi(x))^{+}.$$

PROOF. By the subdirect representation theorem, A is an MC-subalgebra of a product of subdirectly irreducible MC-algebras. Say $\iota \colon A \to \prod_{i \in I} A_i$. For each $i \in I$, consider the projection $\pi_i \colon \prod_{i \in I} A_i \to A_i$ and the morphism $\operatorname{ess}_i \colon A_i \to [0, 1]$ as in Theorem 7.17. Then
$$\mathrm{d}^{\uparrow}(x, y) = \sup_{i \in I} \mathrm{d}^{\uparrow}(\pi_i \iota(x), \pi_i \iota(y)) = \sup_{i \in I}(\operatorname{ess}_i \pi_i \iota(y) - \operatorname{ess}_i \pi_i \iota(x))^{+}.$$

Since $\operatorname{ess}_i \circ \pi_i \circ \iota \colon A \to [0, 1]$ is an MC-morphism, we obtain
$$\mathrm{d}^{\uparrow}(x, y) \leq \sup_{\varphi \colon A \to [0,1] \; \text{MC-morphism}} (\varphi(y) - \varphi(x))^{+}.$$

From Lemma 7.28, we obtain the converse inequality. $\blacksquare$

7.30. LEMMA. *Let $A \in \mathbf{MC}$ and $x, y, z \in A$. The following properties hold.*

  1. $d(x, y) \in [0, 1]$.

  2. $d(x, x) = 0$.

  3. $d(x, z) \leq d(x, y) \oplus d(y, z) \leq d(x, y) + d(y, z)$.

PROOF. (1) and (2) are clear. Let us prove (3).

$$d(x, z) = \max\{d^\uparrow(x, z), d^\uparrow(z, x)\} \leq$$

$$\leq \max\{d^\uparrow(x, y) \oplus d^\uparrow(y, z), d^\uparrow(z, y) \oplus d^\uparrow(y, x)\} \leq$$

$$\leq \max\{d^\uparrow(x, y), d^\uparrow(y, x)\} \oplus \max\{d^\uparrow(y, z), d^\uparrow(z, y)\} =$$

$$= d(x, y) \oplus d(y, z).$$

∎

7.31. REMARK. Let $A \in \mathbf{MC}$. Then A is archimedean if, and only if, (A, d) is a metric space.

7.32. THEOREM. *Let $A \in \mathbf{MC}$, and let $x, y \in A$. Then*

$$d(x, y) = \sup_{\varphi \colon A \to [0,1] \ \text{MC-morphism}} |\varphi(x) - \varphi(y)|.$$

PROOF.

$$d(x, y) = \max\{d^\uparrow(x, y), d^\uparrow(y, x)\} \overset{\text{Thm. } 7.29}{=}$$

$$= \max\left\{ \sup_{\varphi \colon A \to [0,1] \ \text{MC-morph.}} (\varphi(y) - \varphi(x))^+, \sup_{\varphi \colon A \to [0,1] \ \text{MC-morph.}} (\varphi(x) - \varphi(y))^+ \right\} =$$

$$= \sup_{\varphi \colon A \to [0,1] \ \text{MC-morph.}} \max\left\{ (\varphi(y) - \varphi(x))^+, (\varphi(x) - \varphi(y))^+ \right\} =$$

$$= \sup_{\varphi \colon A \to [0,1] \ \text{MC-morph.}} |\varphi(x) - \varphi(y)|.$$

∎

We are ready to prove the implication [(1)⇒(2)] in Theorem 6.8.

7.33. THEOREM. *Let $A \in \mathbf{MC}$ be archimedean, and let $x, y \in A$ with $x \neq y$. Then, there exists an MC-morphism $\varphi \colon A \to [0, 1]$ such that $\varphi(x) \neq \varphi(y)$.*

PROOF. By Theorem 7.32, for every $x, y \in A$, we have

$$d(x, y) = \sup_{\varphi \colon A \to [0,1] \ \text{MC-morphism}} |\varphi(x) - \varphi(y)|.$$

Since A is archimedean, $d(x, y) \neq 0$. Therefore, $\sup_{\varphi \colon A \to [0,1] \ \text{MC-morphism}} |\varphi(x) - \varphi(y)| \neq 0$, and hence there exists $\varphi \colon A \to [0, 1]$ MC-morphism such that $|\varphi(x) - \varphi(y)| \neq 0$, i.e., $\varphi(x) \neq \varphi(y)$.

∎

7.34. COROLLARY. *For every $A \in \mathbf{MC}$, and every $x, y \in A$, we have $\mathrm{d}_A(x, y) = \mathrm{d}_{\mathrm{CMax}(A)}(\mathrm{ev}_x, \mathrm{ev}_y)$.*

The results obtained so far enable us to prove one of the main results of this section—namely, Theorem 6.8.

PROOF OF THEOREM 6.8. By Remark 7.1 and Theorem 7.33.   ∎

We add one additional characterization of archimedean MC-algebras.

7.35. THEOREM. *Let $A \in \mathbf{MC}$. Then the following conditions are equivalent.*

1. *A is archimedean.*

2. *For every $x, y \in A$, $\mathrm{d}^{\uparrow}(x, y) = 0$ implies $y \leq x$.*

PROOF. [(2)⇒(1)] Let $x, y \in A$, and suppose $\mathrm{d}(x, y) = 0$. Then $\mathrm{d}^{\uparrow}(x, y) = 0$ and $\mathrm{d}^{\uparrow}(y, x) = 0$. Hence $y \leq x$ and $x \leq y$. Therefore $x = y$.
[(1)⇒(2)] Let us suppose A is archimedean, and let $x, y \in A$ be such that $\mathrm{d}^{\uparrow}(x, y) = 0$. Then, for every MC-morphism $\varphi \colon A \to [0, 1]$, $\varphi(y) \leq \varphi(x)$; hence $\varphi(y) \vee \varphi(x) = \varphi(x)$, which implies $\varphi(y \vee x) = \varphi(x)$. Hence, for all MC-morphisms $\varphi \colon A \to [0, 1]$, $|\varphi(y \vee x) - \varphi(x)| = 0$. By Theorem 7.32, $\mathrm{d}(y \vee x, x) = 0$. Since A is archimedean, $y \vee x = x$, that is, $y \leq x$.   ∎

8. A is Cauchy complete if, and only if, the unit ε_A is surjective

The aim of this section is to prove Theorem 6.9 above, which states, for any $A \in \mathbf{MC}$, the equivalence of the following conditions.

1. A is Cauchy complete.

2. The unit $\varepsilon_A \colon A \to \mathrm{CMax}(A)$ is surjective.

8.1. REMARK. The implication [(2)⇒(1)] of Theorem 6.9—i.e., if the unit $\varepsilon_A \colon A \to \mathrm{CMax}(A)$ is surjective, then A is Cauchy complete—follows from the fact, observed in Corollary 7.34, that ε_A preserves d. In detail, let $(a_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in A. Then $(\varepsilon_A(a_n))_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathrm{CMax}(A)$. Since $\mathrm{CMax}(A)$ is Cauchy complete (see [(2)⇒(3)] in Theorem 6.6), there exists $f \in \mathrm{CMax}(A)$ such that $(\varepsilon_A(a_n))_{n \in \mathbb{N}}$ converges to f. Since ε_A is surjective, there exists $a \in A$ such that $\varepsilon_A(a) = f$. The sequence $(a_n)_{n \in \mathbb{N}}$ converges to a.

We are left to prove that if A is Cauchy complete, then ε_A is surjective. To do so, we make use of an analogue of Stone-Weierstrass Theorem. Our source of inspiration is [Hofmann and Nora, 2018, Section 7].

8.2. LEMMA. *Let X be a preordered topological space, let $x, y \in X$, let L be an MC-subalgebra of $\mathrm{C}(X)$, and let $\phi \in L$ be such that $\phi(x) < \phi(y)$. Then, there exists $\psi \in L$ and an open neighbourhood U_y of y such that $\psi(x) = 0$ and, for all $z \in U_y$, $\psi(z) = 1$.*

PROOF. There exists $c \in [0,1]$ such that $\varphi(x) < c < \varphi(y)$. Let $n \in \mathbb{N}$ be such that $n(c - \varphi(x)) \geq 1$. Set $\psi := \underbrace{(\varphi \ominus \varphi(x)) \oplus \cdots \oplus (\varphi \ominus \varphi(x))}_{n \text{ times}}$. Set U_y as the pre-image of $(c, 1]$ under φ. Note that $y \in U_y$. We have $\psi(x) = \underbrace{(\varphi(x) \ominus \varphi(x)) \oplus \cdots \oplus (\varphi(x) \ominus \varphi(x))}_{n \text{ times}} = 0$, and, for every $z \in U_y$, $\psi(z) = \underbrace{(\varphi(z) \ominus \varphi(x)) \oplus \cdots \oplus (\varphi(z) \ominus \varphi(x))}_{n \text{ times}} = 1$. ∎

8.3. THEOREM. [Ordered version of Stone-Weierstrass Theorem] *Let X be a preordered topological space, let L be an MC-subalgebra of $\mathrm{C}(X)$, and suppose that, for every $x, y \in X$, if $x \not\geq y$ then there exists $\phi \in L$ such that $\phi(x) < \phi(y)$. If X is compact, then, for every $\psi \in \mathrm{C}(X)$, there exists a sequence $(\phi_n)_{n \in \mathbb{N}}$ in L converging to ψ in the sup metric.*

PROOF. Fix $\varepsilon \in (0, 1]$; we shall find $\phi \in L$ such that $\sup_{x \in X} |\psi(x) - \phi(x)| \leq \varepsilon$. Fix $x \in X$. Set $U := \{z \in X \mid \psi(z) < \psi(x) + \varepsilon\}$. The set U is open. Moreover, for every $y \in X$ such that $y \leq x$, we have $y \in U$ (by monotonicity of ψ); contrapositively, for every $y \in X \setminus U$ we have $x \not\geq y$. Hence, by Lemma 8.2, for every $y \in X \setminus U$ there exists $\alpha_y \in L$ and an open neighbourhood U_y of y such that $\alpha_y(x) = 0$ and, for all $z \in U_y$, $\alpha_y(z) = 1$.

By compactness of X, there exist finitely many elements $y_1, \ldots, y_n \in X \setminus U$ such that $X = U \cup U_{y_1} \cup \cdots \cup U_{y_n}$. Set $\lambda := \psi(x)$, and set $\overline{\lambda} \colon X \to [0,1]$ to be the function constantly equal to λ. Let us define $\phi_x := \alpha_{y_1} \oplus \cdots \oplus \alpha_{y_n} \oplus \overline{\lambda}$. We claim that ϕ_x has the following properties.

a1. $\phi_x(x) = \psi(x)$.

a2. For every $z \in X$, $\phi_x(z) > \psi(z) - \varepsilon$.

Indeed, (a1) holds because, for $1 \leq i \leq n$, we have $\alpha_{y_i}(x) = 0$, and so $\phi_x(x) = \alpha_{y_1}(x) \oplus \cdots \oplus \alpha_{y_n}(x) \oplus \lambda = 0 \oplus \cdots \oplus 0 \oplus \lambda = \lambda = \psi(x)$. We prove (a2) by cases. If $z \in U$, then $\psi(x) = \alpha_{y_1}(z) \oplus \cdots \oplus \alpha_{y_n}(z) \oplus \lambda \geq \lambda = \psi(z) > \psi(z) - \varepsilon$. If $z \in X \setminus U$, there exists $i \in \{1, \ldots, n\}$ such that $z \in U_{y_i}$. Thus, $\phi_x(z) = \alpha_{y_1}(z) \oplus \cdots \oplus \alpha_{y_n}(z) \oplus \lambda = 1 \geq \psi(z) > 1 - \varepsilon$. This settles the claim that (a1) and (a2) hold.

Now x is not fixed anymore. For $x \in X$, set

$$V_x := \{z \in X \mid \phi_x(z) < \psi(z) + \varepsilon\}.$$

The set V_x is open because the functions ϕ_x and ψ are continuous. Moreover $x \in V_x$ because of (a1). Therefore the family $(V_x)_{x \in X}$ is an open cover of X. Again, by compactness of X, there exists a finite subcover $V_{x_1}, \ldots, V_{x_m}$ of X. Define $\phi := \phi_{x_1} \wedge \cdots \wedge \phi_{x_m}$; note that $\phi \in L$. For all $z \in X$ we have the following.

b1. There exists $i \in \{1, \ldots, n\}$ such that $z \in V_{x_i}$. Hence, $\phi(z) = \phi_{x_1}(z) \wedge \cdots \wedge \phi_{x_m}(z) \leq \phi_{x_i}(z) < \psi(z) + \varepsilon$.

b2. By (a2), $\phi(z) = \phi_{x_1}(z) \wedge \cdots \wedge \phi_{x_m}(z) > (\psi(z) - \varepsilon) \wedge \cdots \wedge (\psi(z) - \varepsilon) = \psi(z) - \varepsilon$.

Hence, for all $z \in X$, $\psi(z) - \varepsilon < \phi(z) < \psi(z) + \varepsilon$, which implies $\sup_{x \in A} |\psi(x) - \phi(z)| \leq \varepsilon$. ∎

8.4. THEOREM. *Let $A \in \mathbf{MC}$. If A is Cauchy complete, then*

$$\varepsilon_A \colon A \longrightarrow \mathrm{CMax}(A)$$
$$a \longmapsto \mathrm{ev}_a \colon \mathrm{Max}(A) \to [0,1], \ x \mapsto x(a)$$

is surjective.

PROOF. Set $L \subseteq \mathrm{CMax}(A)$ as the image of A under ε_A. L is an MC-subalgebra of $\mathrm{CMax}(A)$. By the definition of the topology and the order on $\mathrm{Max}(A)$, the hypothesis in Theorem 8.3 are fulfilled, with $X := \mathrm{Max}(A)$. Hence Theorem 8.3 applies: for every $\psi \in \mathrm{CMax}(A)$, there exists a sequence $(\widetilde{a}_n)_{n \in \mathbb{N}}$ in the image of ε_A converging to ψ with respect to the sup metric. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in A such that, for every $n \in \mathbb{N}$, $\mathrm{ev}_{a_n} = \widetilde{a}_n$. Therefore, $(\widetilde{a}_n)$ is a Cauchy sequence with respect to the sup metric. Therefore, for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that, for every $n, m \geq k$, we have $\mathrm{d}(\mathrm{ev}_{a_n}, \mathrm{ev}_{a_m}) < \varepsilon$. Since, by Corollary 7.34, ε_A preserves d, $\mathrm{d}(a_n, a_m) = \mathrm{d}(\mathrm{ev}_{a_n}, \mathrm{ev}_{a_m}) < \varepsilon$, and therefore $(a_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. Since A is Cauchy complete, there exists $a \in A$ such that a_n converges to a. Therefore, for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that, for all $m \geq n$, $\mathrm{d}(a_n, a) < \varepsilon$. Since, by Corollary 7.34, ε_A preserves d, for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that, for all $m \geq n$, $\mathrm{d}(\mathrm{ev}_{a_n}, \mathrm{ev}_a) = \mathrm{d}(a_n, a) < \varepsilon$. Hence, ev_{a_n} converges both to ev_a and ψ. Therefore, $\psi = \mathrm{ev}_a$. ∎

We can now prove Theorem 6.9.

PROOF OF THEOREM 6.9. By Remark 8.1 and Theorem 8.4. ∎

Finally, we can prove Theorem 6.6.

PROOF OF THEOREM 6.6. By Remark 6.7 and Theorems 6.8 and 6.9. ∎

Combining Theorems 5.2 and 6.6, we have that the adjoint contravariant functors C: **PreTop** $\to$ **MC** and Max: **MC** $\to$ **PreTop** restrict to a dual equivalence between **PosComp** and the full subcategory of archimedean Cauchy complete MC-algebras. Hence, we have the following.

8.5. THEOREM. *The dual of* **PosComp** *is equivalent to the full subcategory of* **MC** *given by the archimedean Cauchy complete MC-algebras.*

9. The variety $\mathbf{MC}_\infty$

Up to now, we have proved that the category **PosComp** of compact ordered spaces is dually equivalent to the full subcategory of archimedean Cauchy complete MC-algebras. Our final goal is to show that the full subcategory of archimedean Cauchy complete MC-algebras is isomorphic to a variety. Our strategy to achieve this purpose is analogous to the strategy that, in [Hofmann et al., 2018], Section 3, after Theorem 3.8, was pursued to show that a given category was a quasi-variety. The crucial difference is that we make the axioms equational; in achieving the equational axiomatization, having the operation $\wedge$ amongst the primitive operations has facilitated us.

9.1. ADDING THE INFINITARY "CAUCHY" OPERATION δ. In order to ensure Cauchy completeness, we would like to add an operation δ of countably infinite arity to the class of operations of **MC** that computes the limit of "enough" Cauchy sequences, meaning that convergence of such sequences in an MC-algebra is enough to imply Cauchy completeness (and, at the same time, it is possible to interpret δ in $[0, 1]$ so that it becomes a monotone continuous function from $[0, 1]^{\mathbb{N}}$ to $[0, 1]$ that calculates the limit of such sequences, see [Hofmann et al., 2018, p. 283]).

9.2. DEFINITION. *Let $A \in$ **MC**. A sequence $(a_n)_{n \in \mathbb{N}}$ in A is called HNN-Cauchy if, for every $n \in \mathbb{N}$,*

$$a_n \leq a_{n+1} \leq a_n \oplus \frac{1}{2^n}.$$

This definition is inspired by Lemma 3.9 in [Hofmann et al., 2018]; in fact, "HNN" stands for "Hofmann, Neves, Nora", the authors of the paper.

9.3. LEMMA. *Let $A \in$ **MC**, and let $(a_n)_{n \in \mathbb{N}}$ be an HNN-Cauchy sequence in A. Then, for every $n, m \in \mathbb{N}$, with $n \leq m$, we have*

$$a_n \leq a_m \leq a_n \oplus \frac{1}{2^{n-1}},$$

and therefore $(a_n)_{n \in \mathbb{N}}$ is a Cauchy sequence.

PROOF. The inequality $a_n \leq a_m$ is obtained by induction on m. Moreover,

$$a_m \leq (\ldots (a_n \oplus \frac{1}{2^n}) \oplus \ldots) \oplus \frac{1}{2^{m-1}} \leq a_n \oplus \sum_{i=n}^{\infty} \frac{1}{2^i} = a_n \oplus \frac{1}{2^{n-1}}.$$

Hence,

$$\mathrm{d}(a_n, a_m) = \max\{\mathrm{d}^{\uparrow}(a_n, a_m), \mathrm{d}^{\uparrow}(a_m, a_n)\} = \max\{\mathrm{d}^{\uparrow}(a_n, a_m), 0\} = \mathrm{d}^{\uparrow}(a_n, a_m) \leq \frac{1}{2^{n-1}}.$$

∎

9.4. LEMMA. *For $A \in$ **MC**, the following conditions are equivalent.*

1. *A is Cauchy complete.*

2. *Every HNN-Cauchy sequence in A converges.*

PROOF. [(1)$\Rightarrow$(2)] By Lemma 9.3.
[(2)$\Rightarrow$(1)] Let $(a_n)_{n \in \mathbb{N}}$ be a Cauchy sequence. For every $i \in \mathbb{N}$, let $k_i \in \mathbb{N}$ be such that, for every $n, m \geq k_i$, $\mathrm{d}(a_n, a_m) < \frac{1}{2^{i+1}}$. For each $i \in \mathbb{N}$, set $b_i := a_{k_i}$. Then, for every $i \in \mathbb{N}$, and every $n, m \geq i$, $\mathrm{d}(b_n, b_m) < \frac{1}{2^{i+1}}$. In particular, for each $i \in \mathbb{N}$, $\mathrm{d}(b_i, b_{i+1}) < \frac{1}{2^{i+1}}$. Therefore,

$$b_i \ominus \frac{1}{2^{i+1}} \leq b_{i+1};$$

$$b_{i+1} \ominus \frac{1}{2^{i+1}} \leq b_i.$$

For all $i \in \mathbb{N}$, we set $c_i := b_i \ominus \frac{1}{2^i}$.

Then, for every $i \in \mathbb{N}$,

$$c_i = b_i \ominus \frac{1}{2^i} = \left(b_i \ominus \frac{1}{2^{i+1}} \right) \ominus \frac{1}{2^{i+1}} \leq b_{i+1} \ominus \frac{1}{2^{i+1}} = c_{i+1};$$

$$c_{i+1} = b_{i+1} \ominus \frac{1}{2^{i+1}} \leq b_i \leq \left(b_i \ominus \frac{1}{2^i} \right) \oplus \frac{1}{2^i} = c_i \oplus \frac{1}{2^i}.$$

So, $c_i \leq c_{i+1} \leq c_i \oplus \frac{1}{2^i}$. Hence, $(c_n)_{n \in \mathbb{N}}$ is an HNN-Cauchy sequence, and thus there exists $c \in A$ such that $(c_n)_{n \in \mathbb{N}}$ converges to c.

We have

$$\mathrm{d}(b_i, c) \leq \mathrm{d}(b_i, c_i) + \mathrm{d}(c_i, c) = \mathrm{d}\left(b_i, b_i \ominus \frac{1}{2^i} \right) + \mathrm{d}(c_i, c) \leq \frac{1}{2^i} + \mathrm{d}(c_i, c) \overset{i \to \infty}{\to} 0.$$

Therefore, the sequence $(b_i)_{n \in \mathbb{N}}$ converges to c. The sequence $(a_n)_{n \in \mathbb{N}}$ is a Cauchy sequence that admits a convergent subsequence $(b_i)_{i \in \mathbb{N}}$; by a standard argument, it follows that $(a_n)_{n \in \mathbb{N}}$ converges. ∎

9.5. NOTATION. Inductively on $n \in \mathbb{N}$, we define the term ρ_n of arity $n+1$ in the language of **MC** as follows.

$$\rho_0(x_0) := x_0;$$

$$\text{for } n \in \mathbb{N} \quad \rho_{n+1}(x_0, \ldots, x_{n+1}) := (x_0 \vee \cdots \vee x_{n+1}) \wedge \left(\rho_n(x_0, \ldots, x_n) \oplus \frac{1}{2^n} \right).$$

9.6. LEMMA. *Let $A \in$ **MC**. For every $n \in \mathbb{N}$, and every $x_0, \ldots, x_{n+1} \in A$, the following properties hold.*

$$\rho_n(x_0, \ldots, x_n) \leq \rho_{n+1}(x_0, \ldots, x_{n+1}) \leq \rho_n(x_0, \ldots, x_n) \oplus \frac{1}{2^n}.$$

PROOF. By definition of ρ_n, we have $\rho_n(x_0, \ldots, x_n) \leq x_0 \vee \cdots \vee x_n$ and $\rho_{n+1}(x_0, \ldots, x_{n+1}) \leq \rho_n(x_0, \ldots, x_n) \oplus \frac{1}{2^n}$. As a consequence, $\rho_n(x_0, \ldots, x_n) \leq x_0 \vee \cdots \vee x_n \leq x_0 \vee \cdots \vee x_{n+1}$ and $\rho_n(x_0, \ldots, x_n) \leq \rho_n(x_0, \ldots, x_n) \oplus \frac{1}{2^n}$. Thus, $\rho_n(x_0, \ldots, x_n) \leq (x_0 \vee \cdots \vee x_{n+1}) \wedge \left(\rho_n(x_0, \ldots, x_n) \oplus \frac{1}{2^n} \right) = \rho_{n+1}(x_0, \ldots, x_{n+1})$. ∎

9.7. LEMMA. *Let $A \in$ **MC**, and let $(x_n)_{n \in \mathbb{N}}$ be a sequence in A. The following properties hold.*

1. The sequence $(\rho_n(x_0, \ldots, x_n))_{n \in \mathbb{N}}$ is an HNN-Cauchy sequence.

2. If $(x_n)_{n \in \mathbb{N}}$ is an HNN-Cauchy sequence, then, for all $n \in \mathbb{N}$,

$$\rho_n(x_0, \ldots, x_n) = x_n.$$

PROOF. (1) follows from Lemma 9.6. (2) is proved inductively. The case $n = 0$ is trivial. Inductive step: let $n \in \mathbb{N}$. Then $\rho_{n+1}(x_0, \ldots, x_{n+1}) := (x_0 \vee \cdots \vee x_{n+1}) \wedge \left(\rho_n(x_0, \ldots, x_n) \oplus \frac{1}{2^n} \right) \overset{\text{ind. hyp.}}{=} (x_0 \vee \cdots \vee x_{n+1}) \wedge \left(x_n \oplus \frac{1}{2^n} \right) = x_{n+1} \wedge \left(x_n \oplus \frac{1}{2^n} \right) = x_{n+1}$. ∎

Let $A \in \mathbf{MC}$, and let $(a_n)_{n \in \mathbb{N}}$ be a sequence in A. If A is Cauchy complete, the sequence $(\rho_n(a_0, \ldots, a_n))_{n \in \mathbb{N}}$ admits a limit in A. If, additionally, A is archimedean, (A, d) is a metric space and thus the limit is unique.

9.8. NOTATION. Let $A \in \mathbf{MC}$ be archimedean and Cauchy complete. For every sequence $(a_n)_{n \in \mathbb{N}}$ in A, we set

$$\delta(a_0, a_1, a_2, \ldots) := \lim_{n \to \infty} \rho_n(a_0, \ldots, a_n).$$

The definition of δ takes inspiration from [Hofmann et al., 2018, Section 3]; in fact, the function $\delta \colon [0,1]^{\mathbb{N}} \to [0,1]$ in [Hofmann et al., 2018, Section 3] coincides with the interpretation in $[0,1]$ of what we call δ here.

9.9. REMARK. δ calculates the limit of HNN-Cauchy sequences.

9.10. PROPOSITION. *Let $A \in \mathbf{MC}$ be archimedean and Cauchy complete. The following properties hold.*

1. $\delta(x, x, x, \ldots) = x$.

2. $\delta(x_0, x_1, x_2, \ldots) \leq \delta(x_0 \vee y_0, x_1 \vee y_1, x_2 \vee y_2, \ldots)$.

3. $\delta\left(x \ominus \frac{1}{2^0}, x \ominus \frac{1}{2^1}, x \ominus \frac{1}{2^2}, \ldots\right) = x$.

4. *For all $n \in \mathbb{N}$*

$$\rho_n(x_0, \ldots, x_n) \leq \delta(x_0, x_1, x_2, \ldots) \leq \rho_n(x_0, \ldots, x_n) \oplus \frac{1}{2^{n-1}}.$$

PROOF. Since $A \in \mathbf{MC}$ is archimedean, A is (isomorphic to) a subalgebra of $[0,1]^X$, for some set X. The function $\mathrm{d} \colon A \times A \to [0,1]$ coincides with the sup metric. We recall that, for every sequence $(f_n)_{n \in \mathbb{N}}$ in A, $\delta((f_n)_{n \in \mathbb{N}}) = \lim_{n \to \infty} \rho_n(f_0, \ldots, f_n)$. The convergence is uniform, and therefore pointwise. Hence, it is enough to prove (1), (2), (3) and (4) for $A = [0,1]$.

1. The sequence $(x, x, x, \ldots)$ is HNN-Cauchy; thus $\delta(x, x, x, \ldots) = \lim_{n \to \infty} x = x$.

2. For each $n \in \mathbb{N}$, set $z_n := x_n \vee y_n$. By induction, we show $\rho_n(x_0, \ldots, x_n) \leq \rho_n(z_0, \ldots, z_n)$. Indeed, for $n = 0$, we have $\rho_0(x_0) = x_0 \leq z_0 = \rho_0(z_0)$. Inductive step: let $n \in \mathbb{N}$; then, $\rho_{n+1}(x_0, \ldots, x_{n+1}) = (x_0 \vee \cdots \vee x_{n+1}) \wedge \left(\rho_n(x_0, \ldots, x_n) \oplus \frac{1}{2^n}\right) \overset{\text{ind. hyp.}}{\leq} (z_0 \vee \cdots \vee z_{n+1}) \wedge \left(\rho_n(z_0, \ldots, z_n) \oplus \frac{1}{2^n}\right) = \rho_{n+1}(z_0, \ldots, z_{n+1})$. Hence, we have proved inductively $\rho_n(x_0, \ldots, x_n) \leq \rho_n(z_0, \ldots, z_n)$. Since, in $[0,1]$, $\lim$ is monotone, we have

$$\delta(x_0, x_1, x_2, \ldots) = \lim_{n \to \infty} \rho_n(x_0, \ldots, x_n) \leq$$

$$\leq \lim_{n \to \infty} \rho_n(z_0, \ldots, z_n) =$$

$$= \delta(x_0 \vee y_0, x_1 \vee y_1, x_2 \vee y_2, \ldots).$$

3. Let us prove that $\left(x \ominus \frac{1}{2^0}, x \ominus \frac{1}{2^1}, x \ominus \frac{1}{2^2}, \dots\right)$ is an HNN-Cauchy sequence. Indeed, $x \ominus \frac{1}{2^n} \leq x \ominus \frac{1}{2^{n+1}} \leq x \leq \left(x \ominus \frac{1}{2^n}\right) \oplus \frac{1}{2^n}$.

4. By Lemma 9.7, the sequence $(\rho_n(x_0, \dots, x_n))_{n \in \mathbb{N}}$ is an HNN-Cauchy sequence. By Lemma 9.3, for every $n, m \in \mathbb{N}$, with $n \leq m$, we have

$$\rho_n(x_0, \dots, x_n) \leq \rho_m(x_0, \dots, x_m) \leq \rho_n(x_0, \dots, x_n) \oplus \frac{1}{2^{n-1}}.$$

Fix n and let m tend to ∞.

∎

9.11. THE VARIETY $\mathbf{MC}_\infty$. Recall the inductive definition of the term ρ_n of arity $n+1$ in the language of $\mathbf{MC}$:

$$\rho_0(x_0) := x_0;$$

$$\text{for } n \in \mathbb{N} \quad \rho_{n+1}(x_0, \dots, x_{n+1}) := (x_0 \vee \cdots \vee x_{n+1}) \wedge \left(\rho_n(x_0, \dots, x_n) \oplus \frac{1}{2^n}\right).$$

9.12. DEFINITION. *We define the variety $\mathbf{MC}_\infty$ as the variety obtained from the variety $\mathbf{MC}$ by adding an operation δ of countably infinite arity, together with the following additional axioms.*

1. $\delta(x, x, x, \dots) = x$.

2. $\delta(x_0, x_1, x_2, \dots) \leq \delta(x_0 \vee y_0, x_1 \vee y_1, x_2 \vee y_2, \dots)$.

3. $\delta\left(x \ominus \frac{1}{2^0}, x \ominus \frac{1}{2^1}, x \ominus \frac{1}{2^2}, \dots\right) = x$.

4. For all $n \in \mathbb{N}$

$$\rho_n(x_0, \dots, x_n) \leq \delta(x_0, x_1, x_2, \dots) \leq \rho_n(x_0, \dots, x_n) \oplus \frac{1}{2^{n-1}}.$$

The idea behind this definition is the following. Axioms (1), (2), (3) imply being archimedean (see Proposition 9.14 below); Axiom (4) forces $\delta(x_0, x_1, x_2, \dots)$ to be the limit of $(\rho_n(x_0, \dots, x_n))_{n \in \mathbb{N}}$, and therefore it implies Cauchy completeness (see Proposition 9.16 below).

9.13. GENERAL PROPERTIES OF THE FORGETFUL FUNCTOR $\mathbf{MC}_\infty \to \mathbf{MC}$.

9.14. PROPOSITION. *Let $A \in \mathbf{MC}_\infty$. Then A is archimedean.*

PROOF. Let $x, y \in A$ be such that $\mathrm{d}(x, y) = 0$. We shall show $x = y$. Since $\mathrm{d}(x, y) = \max\{\mathrm{d}^\uparrow(x, y), \mathrm{d}^\uparrow(y, x)\}$, $\mathrm{d}^\uparrow(y, x) = 0$. We recall $\mathrm{d}^\uparrow(y, x) = \inf\{\lambda \in [0, 1] \mid x \ominus \lambda \leq y\} = 0$. Hence, for all $\lambda \in (0, 1]$, we have $x \ominus \lambda \leq y$. Note that (2) in Definition 9.12 implies that δ is monotone. We have

$$x \overset{(3)}{=} \delta\left(x \ominus \frac{1}{2^0}, x \ominus \frac{1}{2^1}, x \ominus \frac{1}{2^2}, \dots\right) \overset{(2)}{\leq} \delta(y, y, y, \dots) \overset{(1)}{=} y.$$

Analogously, one shows that $y \leq x$. Hence, $x = y$.

∎

9.15. THEOREM. *Let $A, B \in \mathbf{MC}_\infty$, and let $\varphi \colon A \to B$ be an MC-morphism. Then, φ preserves δ.*

PROOF. We should prove $\varphi(\delta_A(x_0, x_1, x_2, \dots)) = \delta_B(\varphi(x_0), \varphi(x_1), \varphi(x_2), \dots)$. Since B is archimedean, it is enough to prove

$$\mathrm{d}(\varphi(\delta_A(x_0, x_1, x_2, \dots)), \delta_B(\varphi(x_0), \varphi(x_1), \varphi(x_2), \dots)) = 0.$$

For all $n \in \mathbb{N}$, we have

$$\rho_n(\varphi(x_0), \dots, \varphi(x_n)) \le \varphi(\delta_A(x_0, x_1, x_2, \dots)) \le \rho_n(\varphi(x_0), \dots, \varphi(x_n)) \oplus \frac{1}{2^{n-1}},$$

because φ is an MC-morphism. Moreover, since B is an MC_∞-algebra, we have

$$\rho_n(\varphi(x_0), \dots, \varphi(x_n)) \le \delta_B(\varphi(x_0), \varphi(x_1), \varphi(x_2), \dots) \le \rho_n(\varphi(x_0), \dots, \varphi(x_n)) \oplus \frac{1}{2^{n-1}}.$$

Hence, for all $n \in \mathbb{N}$, we have

$$\varphi(\delta_A(x_0, x_1, x_2, \dots)) \le \delta_B(\varphi(x_0), \varphi(x_1), \varphi(x_2), \dots) \oplus \frac{1}{2^{n-1}}$$

and

$$\delta_B(\varphi(x_0), \varphi(x_1), \varphi(x_2), \dots)) \le \varphi(\delta_A(x_0, x_1, x_2, \dots)) \oplus \frac{1}{2^{n-1}}.$$

Thus,

$$\mathrm{d}(\varphi(\delta_A(x_0, x_1, x_2, \dots)), \delta_B(\varphi(x_0), \varphi(x_1), \varphi(x_2), \dots)) = 0.$$

∎

9.16. PROPOSITION. *Let $A \in \mathbf{MC}_\infty$. Then A is Cauchy complete.*

PROOF. It is enough to prove that every HNN-Cauchy sequence in A converges. Let $(x_n)_{n \in \mathbb{N}}$ be an HNN-Cauchy sequence in A. Then, $\rho_n(x_0, \dots, x_n) = x_n$. Hence, for all $n \in \mathbb{N}$, $x_n \le \delta(x_0, x_1, x_2, \dots) \le x_n \oplus \frac{1}{2^{n-1}}$, which implies $\mathrm{d}(\delta(x_0, x_1, x_2, \dots), x_n) \le \frac{1}{2^{n-1}}$, which implies that $\delta(x_0, x_1, x_2, \dots)$ is a limit for $(x_n)_{n \in \mathbb{N}}$. ∎

We denote with $\mathrm{U}_{\mathbf{MC}_\infty, \mathbf{MC}} \colon \mathbf{MC}_\infty \to \mathbf{MC}$ the forgetful functor.

9.17. THEOREM.

1. $\mathrm{U}_{\mathbf{MC}_\infty, \mathbf{MC}}$ *is full and faithful.*

2. $\mathrm{U}_{\mathbf{MC}_\infty, \mathbf{MC}}$ *is injective on objects: $\mathrm{U}_{\mathbf{MC}_\infty, \mathbf{MC}}(A) = \mathrm{U}_{\mathbf{MC}_\infty, \mathbf{MC}}(B)$ implies $A = B$. This means that every MC-algebra admits at most one MC_∞-structure that extends its MC-structure.*

3. *For $A \in \mathbf{MC}$, there exists $\widetilde{A} \in \mathbf{MC}_\infty$ such that $\mathrm{U}_{\mathbf{MC}_\infty, \mathbf{MC}}(\widetilde{A}) = A$ if, and only if, A is archimedean and Cauchy complete.*

4. *The image of $\mathrm{U}_{\mathbf{MC}_\infty, \mathbf{MC}}$ on objects is closed under isomorphisms.*

5. *The MC-algebra $[0, 1]$ admits a (unique) MC_∞-structure.*

PROOF.

1. The fact that $U_{\mathbf{MC}_\infty,\mathbf{MC}}$ is faithful is trivial. The fact that $U_{\mathbf{MC}_\infty,\mathbf{MC}}$ is full is Theorem 9.15.

2. Suppose $U_{\mathbf{MC}_\infty,\mathbf{MC}}(A) = U_{\mathbf{MC}_\infty,\mathbf{MC}}(B)$. Then, A and B share the same underlying set. Let $\mathrm{Id}\colon U_{\mathbf{MC}_\infty,\mathbf{MC}}(A) \to U_{\mathbf{MC}_\infty,\mathbf{MC}}(B)$ be the identity function (which is an MC-morphism since $U_{\mathbf{MC}_\infty,\mathbf{MC}}(A) = U_{\mathbf{MC}_\infty,\mathbf{MC}}(B)$). Since $U_{\mathbf{MC}_\infty,\mathbf{MC}}$ is full, there exists an MC_∞-morphism $\varphi\colon A \to B$ such that $U_{\mathbf{MC}_\infty,\mathbf{MC}}(\varphi) = \mathrm{Id}$. Then, φ is the identity function, and thus $A = B$.

3. Suppose there exists $\widetilde{A} \in \mathbf{MC}_\infty$ such that $U_{\mathbf{MC}_\infty,\mathbf{MC}}(\widetilde{A}) = A$. By Propositions 9.14 and 9.16, A is archimedean and Cauchy complete. For the converse implication, suppose that A is archimedean and Cauchy complete. Then, by Proposition 9.10, A admits an MC_∞-structure.

4. It is a consequence of (3).

5. $[0, 1]$ is archimedean and Cauchy complete.

∎

9.18. COROLLARY. *The variety $\mathbf{MC}_\infty$ is isomorphic to the full subcategory of $\mathbf{MC}$ given by the archimedean Cauchy complete MC-algebras.*

We can now prove Theorem 1.1, which is our main result: the dual of $\mathbf{PosComp}$ is equivalent to a variety of algebras.

PROOF OF THEOREM 1.1. The dual of $\mathbf{PosComp}$ is equivalent, by Theorem 8.5, to the full subcategory of $\mathbf{MC}$ given by the archimedean Cauchy complete MC-algebras, which is equivalent, by Corollary 9.18, to the variety $\mathbf{MC}_\infty$. Hence, the dual of $\mathbf{PosComp}$ is equivalent to the variety $\mathbf{MC}_\infty$. ∎

10. The variety $\mathbf{MC}_\infty$ and Linton's varietal theories

10.1. LEMMA. *The function $\delta\colon [0,1]^{\mathbb{N}} \to [0,1]$ is monotone and continuous with respect to the product order and product topology.*

PROOF. For every $n \in \mathbb{N}$, we set

$$\widetilde{\rho}_n \colon [0,1]^{\mathbb{N}} \longrightarrow [0,1]$$
$$(x_n)_{n\in\mathbb{N}} \longmapsto \rho_n(x_0, \ldots, x_n).$$

Then, the sequence $(\widetilde{\rho}_n)_{n\in\mathbb{N}}$ tends to δ with respect to the supremum norm, i.e., uniformly. For every $i \in \mathbb{N}$, the projection $\pi_i\colon [0,1]^{\mathbb{N}} \to [0,1]$ onto the i-th coordinate is continuous, and for every $n \in \mathbb{N}$, $\rho_n\colon [0,1]^n \to [0,1]$ is continuous. Therefore, for every $n \in \mathbb{N}$, $\widetilde{\rho}_n\colon [0,1]^{\mathbb{N}} \to [0,1]$ is continuous. Since $(\widetilde{\rho}_n)_{n\in\mathbb{N}}$ converges to δ uniformly, δ is continuous.

For every $n \in \mathbb{N}$, one proves, by induction, that $\rho_n \colon [0,1]^n \to [0,1]$ is monotone, and thus $\widetilde{\rho}_n \colon [0,1]^{\mathbb{N}} \to [0,1]$ is monotone, as well. Since δ is the pointwise limit of $\widetilde{\rho}_n$, δ is monotone, as well.   ■

The primitive operation symbols of $\mathbf{MC}_\infty$ (δ, $\wedge$, $\vee$, $\oplus$, $\odot$, and, for every $\lambda \in [0,1]$, the constant symbol λ) have a natural interpretation in $[0,1]$; indeed, each of them can be viewed as a function from a power of $[0,1]$ to $[0,1]$ itself. As we noticed in Remark 3.1 and in Lemma 10.1, they are monotone and continuous. In fact, the operations of $\mathbf{MC}_\infty$ are *all* monotone continuous functions from some power of $[0,1]$ to $[0,1]$. The following theorems make this statement precise.

10.2. THEOREM. *For each cardinal κ, the set of monotone continuous functions from $[0,1]^\kappa$ to $[0,1]$ coincides with the set of interpretations in $[0,1]$ of MC_∞-terms of arity κ.*

PROOF. Let A be the set of functions $f \colon [0,1]^\kappa \to [0,1]$ for which there exist an MC_∞-term (depending on f) of arity κ whose interpretation in $[0,1]$ is f. Since the interpretation in $[0,1]$ of an MC_∞-term is monotone and continuous by Remark 3.1 and Lemma 10.1, we have $A \subseteq \mathrm{C}([0,1]^\kappa)$. Moreover, A contains, for each $i \in \kappa$, the projection $\pi_i \colon [0,1]^\kappa \to [0,1]$. Then, Theorem 8.3 applies, and we obtain that A is dense in $\mathrm{C}([0,1]^\kappa)$. Furthermore, A is an MC_∞-algebra, and therefore it is Cauchy complete; thus $A = \mathrm{C}([0,1]^\kappa)$.   ■

Roughly speaking, Theorem 10.2 says that the interpretation in $[0,1]$ is a surjective operator from the class of equivalence classes of MC_∞-terms (where the equivalence relation is defined in the standard manner by identifying two terms if their interpretation in each algebra of the variety coincides) to the class of monotone continuous functions from some power of $[0,1]$ to $[0,1]$ itself. One consequence of the Theorem 10.3 below is that this operator is injective, too, and so the equivalence classes of MC_∞-terms are in bijective correspondence with the monotone continuous functions from some power of $[0,1]$ to $[0,1]$ itself. To state the theorem, we recall the standard operators I (closure under isomorphisms), S (closure under subalgebras) and P (closure under products). Moreover, we denote simply with $[0,1]$ the canonical MC_∞-algebra whose underlying set is the unit interval $[0,1]$.

10.3. THEOREM.

$$\mathbf{MC}_\infty = \mathrm{ISP}([0,1]).$$

PROOF. The right-to-left inclusion $\supseteq$ is clear because $\mathbf{MC}_\infty$ is a variety containing $[0,1]$. For the converse inclusion, let $A \in \mathbf{MC}_\infty$. Then A is archimedean, and thus the MC-morphisms towards $[0,1]$ separate the points of A. Since $\mathrm{U}_{\mathbf{MC}_\infty,\mathbf{MC}}$ is full, every MC-morphism from A to $[0,1]$ is also an MC_∞-morphism. Hence, there are enough MC_∞-morphisms from A to $[0,1]$ to separate the points of A, and so A is isomorphic to a subalgebra of a power of $[0,1]$.   ■

10.4. THEOREM. *Let I be a set. The MC_∞-algebra $\mathrm{C}\left([0,1]^I\right)$ is freely generated by the projections $\left(\pi_i\colon \mathrm{C}\left([0,1]^I\right) \to [0,1]\right)_{i\in I}$.*

PROOF. By Theorem 10.3, the free algebra generated by the set I is the set of functions $[0,1]^I \to [0,1]$ that are the evaluation of a term of arity $|I|$. By Theorem 10.2, this set is precisely $\mathrm{C}([0,1]^I)$. ∎

Let us recall, from [Linton, 1966, Section 1], Linton's definition of equational theory, varietal theory, equational category and varietal category. An *equational theory* is a product preserving covariant functor $T\colon \mathbf{Set}^{\mathrm{op}} \to \mathbb{T}$ from the dual of the category of sets to a category $\mathbb{T}$ whose class of objects is put by T in one-one correspondence with the objects of $\mathbf{Set}^{\mathrm{op}}$. One may then identify each object $\widetilde{n}$ of $\mathbb{T}$ with the set $n \in \mathbf{Set}$ such that $T(n) = \widetilde{n}$. The idea behind this definition is that the morphisms in $\mathbb{T}$ from $T(n)$ to $T(m)$ are the $|m|$-tuples of equivalence classes of terms of arity $|n|$ of a certain variety of algebras—where the equivalence relation is defined in the standard manner by identifying two terms if their interpretation in each algebra of the variety coincides—and the composition of morphisms is just the composition of terms (modulo the equivalence relation). From the category $\mathbf{Set}^{\mathbb{T}}$ of set valued functors on $\mathbb{T}$, we single out the full subcategory $\mathbf{Set}^T$ whose objects are the functors $X\colon \mathbb{T} \to \mathbf{Set}$ such that the composite $XT\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ preserves products. One such functor X is called a T-algebra. Any category equivalent to the category $\mathbf{Set}^T$ is called an *equational category*. Evaluation at the object $T(1) \in \mathbb{T}$ provides a faithful functor $U_T\colon \mathbf{Set}^T \to \mathbf{Set}$, the underlying set functor for T-algebras. The equational theory T is called *varietal* if the category $\mathbb{T}$ is locally small, and in this case any category equivalent to $\mathbf{Set}^T$ is said to be a *varietal category*.

Linton's setting generalizes Lawvere's perspective for finitary algebras [Lawvere, 1963] to the infinitary ones (see [Słomiński, 1959]). In fact, every variety of algebras $\mathbf{V}$ is a varietal category: $\mathbf{V}$ is equivalent to the category of T-algebras, where $\mathbb{T}$ is the opposite of the category of free algebras with homomorphisms, and $T\colon \mathbf{Set}^{\mathrm{op}} \to \mathbb{T}$ maps a set I to the free algebra $\mathrm{Free}(I)$ over I. Note that the set of homomorphisms from $\mathrm{Free}(n)$ to $\mathrm{Free}(m)$ is in bijection with the set of $|n|$-tuples of equivalence classes of terms of arity $|m|$, where the equivalence relation is defined in the standard manner by identifying two terms if their interpretation in each algebra of the variety coincides.

10.5. REMARK. The results in this section show that $\mathbf{MC}_\infty$ is the category of algebras of the varietal theory $T\colon \mathbf{Set} \to \mathbb{T}$, where, for each set I, $T(I) = [0,1]^I$, and the morphisms from $T(I)$ to $T(J)$ are the monotone continuous maps from $[0,1]^I$ to $[0,1]^J$. The fact that the concrete varietal category $\mathbf{Set}^T$, i.e. $\mathbf{MC}_\infty$, has a class of primitive operations of countable arity is equivalent to the fact that every continuous map from a power of $[0,1]$ to $[0,1]$ depends on at most countably many coordinates [Mibu, 1944, Theorem 1]. However, in this paper we do not settle the question whether $\mathbf{PosComp}^{\mathrm{op}}$ is equivalent or not to a variety of *finitary* algebras; what we can say is that the functor $\hom(-, [0,1])\colon \mathbf{PosComp} \to \mathbf{Set}$ cannot be naturally isomorphic to the forgetful functor

of a variety of finitary algebras, because the function δ fails to be dependent on at most finitely many coordinates.

11. Conclusions

C and Max establish a dual adjunction between **PreTop** and **MC**, induced by the dualizing object $[0, 1]$. The fixed objects of this adjunction are precisely the objects in the images of the two functors: the fixed objects in **PreTop** are the compact ordered spaces, while the fixed objects in **MC** are the archimedean Cauchy complete algebras. The forgetful functor from the variety $\mathbf{MC}_\infty$ to the full subcategory of **MC** of archimedean Cauchy complete algebras is an isomorphism of categories. Therefore, C and Max restrict to a dual equivalence between **PosComp** and $\mathbf{MC}_\infty$, induced by the dualizing object $[0, 1]$. The main result is Theorem 1.1, i.e., the following.

> The category $\mathbf{PosComp}^{\mathrm{op}}$ is equivalent to a variety of algebras.

The additional results are the description of the variety by means of operations and equational axioms, the description of the dual equivalence via the dualizing object $[0, 1]$, and the extension of the duality to a wider dual adjunction between the category **PreTop** and the finitary variety **MC**.

References

J. Adámek, H. Herrlich, and G. E. Strecker. Abstract and concrete categories: the joy of cats. *Repr. Theory Appl. Categ.*, (17):1–507, 2006. URL http://www.tac.mta.ca/tac/reprints/articles/17/tr17.pdf. Reprint of the 1990 original [Wiley, New York].

J. Duskin. Variations on Beck's tripleability criterion. In *Reports of the Midwest Category Seminar, III*, pages 74–129. Springer, Berlin, 1969.

G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D. S. Scott. *Continuous lattices and domains*, volume 93 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2003. ISBN 0-521-80338-1. doi: 10.1017/CBO9780511542725. URL https://doi.org/10.1017/CBO9780511542725.

D. Hofmann and P. Nora. Enriched Stone-type dualities. *Advances in Mathematics*, 330, 2018. doi: 10.1016/j.aim.2018.03.010.

D. Hofmann, R. Neves, and P. Nora. Generating the algebraic theory of $C(X)$: the case of partially ordered compact spaces. *Theory and Applications of Categories*, 33(12), 2018.

J. Isbell. Generating the algebraic theory of $C(X)$. *Algebra Universalis*, 15(2):153–155, 1982. ISSN 0002-5240. doi: 10.1007/BF02483718. URL `https://doi.org/10.1007/BF02483718`.

F. W. Lawvere. Functorial semantics of algebraic theories. *Proc. Nat. Acad. Sci. U.S.A.*, 50:869–872, 1963. ISSN 0027-8424. doi: 10.1073/pnas.50.5.869. URL `https://doi-org.pros.lib.unimi.it:2050/10.1073/pnas.50.5.869`.

F. E. J. Linton. Some aspects of equational categories. In *Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965)*, pages 84–94. Springer, New York, 1966.

V. Marra and L. Reggio. Stone duality above dimension zero: axiomatising the algebraic theory of C(X). *Adv. Math.*, 307:253–287, 2017. ISSN 0001-8708. doi: 10.1016/j.aim.2016.11.012. URL `https://doi.org/10.1016/j.aim.2016.11.012`.

Y. Mibu. On Baire functions on infinite product spaces. *Proc. Imp. Acad. Tokyo*, 20:661–663, 1944. ISSN 0369-9846. URL `http://projecteuclid.org.pros.lib.unimi.it/euclid.pja/1195572745`.

L. Nachbin. *Topology and order*. Translated from the Portuguese by Lulu Bechtolsheim. Van Nostrand Mathematical Studies, No. 4. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1965.

H.E. Porst and W. Tholen. Concrete dualities. In *Category theory at work (Bremen, 1990)*, volume 18 of *Res. Exp. Math.*, pages 111–136. Heldermann, Berlin, 1991.

J. Słomiński. The theory of abstract algebras with infinitary operations. *Rozprawy Mat.*, 18:67 pp. (1959), 1959. ISSN 0860-2581.

Department of Mathematics Federigo Enriques, Università degli Studi di Milano
via Cesare Saldini 50, 20133 Milano, Italy.
Email: `marco.abbadini@unimi.it`

This article may be accessed at `http://www.tac.mta.ca/tac/`