Theory and Applications of Categories, Vol. 34, No. 8, 2019, pp. 221–242.

CLUSTER-TILTING SUBCATEGORIES IN EXTRIANGULATED CATEGORIES

PANYUE ZHOU AND BIN ZHU

ABSTRACT. Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. We show that certain quotient categories of extriangulated categories are equivalent to module categories by some restriction of functor $\mathbb{E}$, and in some cases, they are abelian. This result can be regarded as a simultaneous generalization of Koenig-Zhu [KZ] and Demonet-Liu [DL]. In addition, we introduce the notion of maximal rigid subcategories in extriangulated categories. Cluster tilting subcategories are obviously strongly functorially finite maximal rigid subcategories, we prove that the converse is true if the 2-Calabi-Yau extriangulated categories admit a cluster tilting subcategories, which generalizes a result of Buan-Iyama-Reiten-Scott [BIRS] and Zhou-Zhu [ZZ].

1. Introduction

Cluster categories associated to finite dimensional hereditary algebras were introduced in [BMRRT] (see [CCS] for type A). These 2-Calabi-Yau triangulated categories arise as orbit categories of derived categories, and provide a categorification of the combinatorics of the cluster algebras introduced in [FZ] by Fomin and Zelevinsky in the acyclic case. They also provide a generalized framework for classical tilting theory, with the cluster tilting objects and their endomorphism rings, the cluster tilted algebras.

Cluster-tilting theory gives a way to construct abelian categories from some triangulated categories and exact categories. Let $\mathcal{T}$ be a cluster-tilting subcategory in a cluster category $\mathscr{C}$. Buan-Marsh-Reiten [BMR] proved that the quotient category $\mathscr{C}/\mathcal{T}[1]$ is equivalent to the category of finitely presented modules $\mathsf{mod}\mathcal{T}$, where $\mathsf{mod}\mathcal{T}$ is abelian. This was proved by Keller-Reiten [KR] in case $\mathscr{C}$ is a 2-Calabi-Yau category, and proved in [KZ] for general case (see also [IY]). Let $\mathcal{B}$ be an exact category with enough projectives and enough injectives, and $\mathcal{T}$ a cluster-tilting subcategory of $\mathcal{B}$. Demonet-Liu [DL] proved that the quotient category $\mathcal{B}/\mathcal{T}$ is equivalent to the category of finitely presented modules $\mathsf{mod}\underline{\mathcal{T}}$, where $\underline{\mathcal{T}}$ is the stable category of $\mathcal{T}$ by projectives and $\mathsf{mod}\underline{\mathcal{T}}$ is abelian.

Cluster-tilting objects (subcategories) are maximal rigid objects (subcategories), the converse is not true in general. The first examples of 2-Calabi-Yau triangulated categories

Bin Zhu:Supported by the NSF of China (Grants No. 11671221) and Panyue Zhou: Supported by the China Postdoctoral Science Foundation (Grant No. 2016M591155) and by the Hunan Provincial Natural Science Foundation of China (Grant No. 2018JJ3205)

Received by the editors 2016-11-21 and, in final form, 2019-03-09.

Transmitted by Steve Lack. Published on 2019-03-14.

2010 Mathematics Subject Classification: 18E30; 18E10.

Key words and phrases: Extriangulated categories; Cluster-tilting subcategories; Rigid subcategories; Maximal rigid subcategories; Quotient categories.

© Panyue Zhou and Bin Zhu, 2019. Permission to copy for private use granted.

in which maximal rigid objects are not cluster-tilting were given in [BIKR, BMV] (see also the example in Section 5 of [KZ]). In any 2-Calabi-Yau triangulated category (or exact stably 2-Calabi-Yau category) with a cluster-tilting subcategory, any functorially finite maximal rigid subcategory is cluster-tilting [ZZ] [BIRS].

Recently, Nakaoka and Palu [NP] introduced the notion of an extriangulated category by extracting those properties of Ext^1 on exact categories and on triangulated categories that seem relevant from the point-of-view of cotorsion pairs. The class of extriangulated categories not only contains exact categories and extension closed subcategories of triangulated categories as examples, but it is also closed under taking some quotients. There are many other examples for extriangulated categories which are neither exact categories nor triangulated categories [NP, ZhZ].

In this paper, we give a common framework for constructing equivalences of categories from the (sub)quotients of extriangulated categories by rigid subcategories to module categories. As an application, this result unifies the work of Koenig-Zhu [KZ] and the work of Demonet-Liu [DL]. We define the notion of maximal rigid subcategories in extriangulated categories. We show that if a 2-Calabi-Yau extriangulated category contains a cluster-tilting subcategory, then any strongly functorially finite maximal rigid subcategory is cluster-tilting. This result can be regarded as a simultaneous generalization of Theorem II.1.8 in [BIRS] and Theorem 2.6 in [ZZ].

The paper is organized as follows. In Section 2, we review some elementary definitions and facts that we need to use later on, including extriangulated category, quotient category and module category. In Section 3, we prove that certain quotient categories of extriangulated categories $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ are equivalent to module categories under some restriction of the functor $\mathbb{E}(-,-)$, see Theorem 3.4 and Theorem 3.13 for more details. In Section 4, we prove that any strongly functorially finite maximal rigid subcategory in a 2-Calabi-Yau extriangulated category is cluster-tilting, see Theorem 4.3.

2. Preliminaries

Throughout the paper, when we say that $\mathscr{C}$ is a category, we always assume that $\mathscr{C}$ is an additive category. All subcategories of a category $\mathscr{C}$ are full subcategories and closed under isomorphisms. For any object $X \in \mathscr{C}$, we denote by $\mathsf{add}\, X$ the subcategory of $\mathscr{C}$ whose objects are direct summands of finite direct sums of finite many copies of X. We denote by $\mathscr{C}(A, B)$ or $\mathrm{Hom}_{\mathscr{C}}(A, B)$ the set of morphisms from A to B in $\mathscr{C}$, and denote by $[\mathscr{X}](A, B)$ the subgroup of $\mathrm{Hom}_{\mathscr{C}}(A, B)$ consisting of morphisms which factor through objects in a subcategory $\mathscr{X}$ of $\mathscr{C}$. The quotient category $\mathscr{C}/[\mathscr{X}]$ of $\mathscr{C}$ by a subcategory $\mathscr{X}$ is the category with the same objects as $\mathscr{C}$ and the space of morphisms from A to B is the quotient of group of morphisms from A to B in $\mathscr{C}$ by the subgroup consisting of morphisms factor through objects in $\mathscr{X}$. We use Ab to denote the category of abelian groups.

Recall that a subcategory $\mathscr{X}$ of an additive category $\mathscr{C}$ is said to be *contravariantly finite* in $\mathscr{C}$ if for every object M of $\mathscr{C}$, there exists some X in $\mathscr{X}$ and a morphism

$f\colon X \to M$ such that for every X' in $\mathscr{X}$ the sequence

$$\mathrm{Hom}_{\mathscr{C}}(X', X) \xrightarrow{f\circ} \mathrm{Hom}_{\mathscr{C}}(X', M) \longrightarrow 0$$

is exact. In this case f is called a right $\mathscr{X}$-approximation. Dually, we define covariantly finite subcategories in $\mathscr{C}$ and left $\mathscr{X}$-approximations. Furthermore, a subcategory of $\mathscr{C}$ is said to be *functorially finite* in $\mathscr{C}$ if it is both contravariantly finite and covariantly finite in $\mathscr{C}$. For more details, we refer to [AR].

Let $\mathscr{C}$ be a category and $g\colon B \to C$ a morphism in $\mathscr{C}$. A *pseudokernel* of g is a morphism $f\colon A \to B$ such that for any $C' \in \mathscr{C}$ the sequence of abelian groups

$$\mathscr{C}(C', A) \xrightarrow{f\circ} \mathscr{C}(C', B) \xrightarrow{g\circ} \mathscr{C}(C', C)$$

is exact. Equivalently, f is a pseudokernel of g if $gf = 0$ and for each morphism $h\colon C' \to B$ such that $gh = 0$ there exists a (not necessarily unique) morphism $p\colon C' \to A$ such that $h = fp$. These properties are subsumed in the following commutative diagram

$$\begin{tikzcd}
 & C' \arrow[dl, dashed, "p"'] \arrow[d, "\forall h"] \arrow[dr, "0"] & \\
A \arrow[r, "f"] & B \arrow[r, "g"] & C
\end{tikzcd}$$

Clearly, a pseudokernel f of g is a kernel of g if and only if f is a monomorphism. The concept of a *pseudocokernel* is defined dually.

A $\mathscr{C}$-module is a contravariant functor $\mathbb{G}\colon \mathscr{C} \to Ab$. Then $\mathscr{C}$-modules form an abelian category $\mathsf{Mod}\mathscr{C}$. By Yoneda's lemma, representable functors are projective objects in $\mathsf{Mod}\mathscr{C}$. We call $M \in \mathsf{Mod}\mathscr{C}$ *coherent* [Au] if there exists an exact sequence

$$\mathrm{Hom}_{\mathscr{C}}(-, C_1) \xrightarrow{\beta\circ} \mathrm{Hom}_{\mathscr{C}}(-, C_0) \longrightarrow M \longrightarrow 0$$

of $\mathscr{C}$-modules with $C_1, C_0 \in \mathscr{C}$. We denote by $\mathsf{mod}\mathscr{C}$ the full subcategory of $\mathsf{Mod}\mathscr{C}$ consisting of coherent $\mathscr{C}$-modules. It is easily checked that $\mathsf{mod}\mathscr{C}$ is closed under cokernels and extensions in $\mathsf{Mod}\mathscr{C}$. Moreover, $\mathsf{mod}\mathscr{C}$ is closed under kernels in $\mathsf{Mod}\mathscr{C}$ if and only if $\mathscr{C}$ has pseudokernels. In this case, $\mathsf{mod}\mathscr{C}$ forms an abelian category (see [Au]).

We recall some basics on extriangulated categories from [NP].

Let $\mathscr{C}$ be an additive category. Suppose that $\mathscr{C}$ is equipped with a biadditive functor $\mathbb{E}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Ab$. For any pair of objects $A, C \in \mathscr{C}$, an element $\delta \in \mathbb{E}(C, A)$ is called an $\mathbb{E}$-*extension*. Thus formally, an $\mathbb{E}$-extension is a triplet (A, δ, C). For any $A, C \in \mathscr{C}$, the zero element $0 \in \mathbb{E}(C, A)$ is called the spilt $\mathbb{E}$-extension.

Let (A, δ, C) be an $\mathbb{E}$-extension. Since $\mathbb{E}$ is a bifunctor, for any $a \in \mathscr{C}(A, A')$ and $c \in \mathscr{C}(C', C)$, we have $\mathbb{E}$-extensions

$$\mathbb{E}(C, a)(\delta) \in \mathbb{E}(C, A') \quad \text{and} \quad \mathbb{E}(c, A)(\delta) \in \mathbb{E}(C', A).$$

We abbreviate them by $a_*\delta$ and $c^*\delta$.

2.1. Definition. *[NP, Definition 2.3] Let $(A,\delta,C),(A',\delta',C')$ be any pair of $\mathbb{E}$-extensions. A morphism*

$$(a,c)\colon (A,\delta,C)\to(A',\delta',C')$$

of $\mathbb{E}$-extensions is a pair of morphisms $a\in\mathscr{C}(A,A')$ and $c\in\mathscr{C}(C,C')$ in $\mathscr{C}$, satisfying the equality

$$a_*\delta=c^*\delta'.$$

Simply we denote it as $(a,c)\colon\delta\to\delta'$.

Let $A,C\in\mathscr{C}$ be any pair of objects. Sequences of morphisms in $\mathscr{C}$

$$A\xrightarrow{x}B\xrightarrow{y}C\quad\text{and}\quad A\xrightarrow{x'}B'\xrightarrow{y'}C$$

are said to be *equivalent* if there exists an isomorphism $b\in\mathscr{C}(B,B')$ which makes the following diagram commutative.

$$\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, equal] & B \arrow[r, "y"] \arrow[d, "b", "\simeq"'] & C \arrow[d, equal] \\
A \arrow[r, "x'"] & B' \arrow[r, "y'"] & C
\end{tikzcd}$$

We denote the equivalence class of $A\xrightarrow{x}B\xrightarrow{y}C$ by $[\,A\xrightarrow{x}B\xrightarrow{y}C\,]$. For any $A,C\in\mathscr{C}$, we denote as $0=[A\xrightarrow{\binom{1}{0}}A\oplus C\xrightarrow{(0,\ 1)}C]$.

2.2. Definition. *[NP, Definition 2.9] Let $\mathfrak{s}$ be a correspondence which associates an equivalence class $\mathfrak{s}(\delta)=[\,A\xrightarrow{x}B\xrightarrow{y}C\,]$ to any $\mathbb{E}$-extension $\delta\in\mathbb{E}(C,A)$. This $\mathfrak{s}$ is called a realization of $\mathbb{E}$, if it satisfies the following condition:*

- *Let $\delta\in\mathbb{E}(C,A)$ and $\delta'\in\mathbb{E}(C',A')$ be any pair of $\mathbb{E}$-extensions, with*

$$\mathfrak{s}(\delta)=[\,A\xrightarrow{x}B\xrightarrow{y}C\,],\quad \mathfrak{s}(\delta')=[\,A'\xrightarrow{x'}B'\xrightarrow{y'}C'\,].$$

  Then, for any morphism $(a,c)\colon\delta\to\delta'$, there exists $b\in\mathscr{C}(B,B')$ which makes the following diagram commutative.

$$\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "a"] & B \arrow[r, "y"] \arrow[d, "b"] & C \arrow[d, "c"] \\
A' \arrow[r, "x'"] & B' \arrow[r, "y'"] & C'
\end{tikzcd} \tag{2.1}$$

In this case, we say that sequence $A\xrightarrow{x}B\xrightarrow{y}C$ realizes δ, whenever it satisfies $\mathfrak{s}(\delta)=[\,A\xrightarrow{x}B\xrightarrow{y}C\,]$. We call such sequence a conflation, x inflation, y deflation. Remark that this condition does not depend on the choices of the representatives of the equivalence classes. In the above situation, we say that (2.1) (or the triplet (a,b,c)) realizes (a,c).

2.3. DEFINITION. *[NP, Definition 2.10] A realization* $\mathfrak{s}$ *of* $\mathbb{E}$ *is called* additive *if it satisfies the following conditions.*

- *For any* $A, C \in \mathscr{C}$, *the split* $\mathbb{E}$*-extension* $0 \in \mathbb{E}(C, A)$ *satisfies* $\mathfrak{s}(0) = 0$.
- *For any pair of* $\mathbb{E}$*-extensions* $\delta \in \mathbb{E}(C, A)$ *and* $\delta' \in \mathbb{E}(C', A')$, *we have*

$$\mathfrak{s}(\delta \oplus \delta') = \mathfrak{s}(\delta) \oplus \mathfrak{s}(\delta').$$

2.4. DEFINITION. *[NP, Definition 2.12] Let* $\mathscr{C}$ *be an additive category. We call the pair* $(\mathbb{E}, \mathfrak{s})$ *an external triangulation of* $\mathscr{C}$ *if it satisfies the following conditions:*

(ET1) $\mathbb{E}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Ab$ *is a biadditive functor.*

(ET2) $\mathfrak{s}$ *is an additive realization of* $\mathbb{E}$.

(ET3) *Let* $\delta \in \mathbb{E}(C, A)$ *and* $\delta' \in \mathbb{E}(C', A')$ *be any pair of* $\mathbb{E}$*-extensions, realized as*

$$\mathfrak{s}(\delta) = [\, A \xrightarrow{x} B \xrightarrow{y} C \,], \quad \mathfrak{s}(\delta') = [\, A' \xrightarrow{x'} B' \xrightarrow{y'} C' \,].$$

For any commutative square

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"] & B \arrow[r, "y"] \arrow[d, "b"] & C \\ A' \arrow[r, "x'"] & B' \arrow[r, "y'"] & C' \end{tikzcd}$$

in $\mathscr{C}$, *there exists a morphism* $(a, c)\colon \delta \to \delta'$ *which is realized by* (a, b, c).

(ET3)$^{\mathrm{op}}$ *Let* $\delta \in \mathbb{E}(C, A)$ *and* $\delta' \in \mathbb{E}(C', A')$ *be any pair of* $\mathbb{E}$*-extensions, realized by*

$$A \xrightarrow{x} B \xrightarrow{y} C \quad \text{and} \quad A' \xrightarrow{x'} B' \xrightarrow{y'} C'$$

respectively. For any commutative square

$$\begin{tikzcd} A \arrow[r, "x"] & B \arrow[r, "y"] \arrow[d, "b"] & C \arrow[d, "c"] \\ A' \arrow[r, "x'"] & B' \arrow[r, "y'"] & C' \end{tikzcd}$$

in $\mathscr{C}$, *there exists a morphism* $(a, c)\colon \delta \to \delta'$ *which is realized by* (a, b, c).

(ET4) *Let* (A, δ, D) *and* (B, δ', F) *be* $\mathbb{E}$*-extensions realized by*

$$A \xrightarrow{f} B \xrightarrow{f'} D \quad \text{and} \quad B \xrightarrow{g} C \xrightarrow{g'} F$$

respectively. Then there exist an object $E \in \mathscr{C}$, a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "f'"] \arrow[d, "g"] & D \arrow[d, "d"] \\ A \arrow[r, "h"] & C \arrow[r, "h'"] \arrow[d, "g'"] & E \arrow[d, "e"] \\ & F \arrow[r, equal] & F \end{tikzcd}$$

in $\mathscr{C}$, and an $\mathbb{E}$-extension $\delta'' \in \mathbb{E}(E, A)$ realized by $A \xrightarrow{h} C \xrightarrow{h'} E$, which satisfy the following compatibilities.

(i) $D \xrightarrow{d} E \xrightarrow{e} F$ *realizes* $\mathbb{E}(F, f')\delta'$,

(ii) $\mathbb{E}(d, A)\delta'' = \delta$,

(iii) $\mathbb{E}(E, f)\delta'' = \mathbb{E}(e, B)\delta'$.

(ET4)$^{\mathrm{op}}$ *Let (D, δ, B) and (F, δ', C) be $\mathbb{E}$-extensions realized by*

$$D \xrightarrow{f'} A \xrightarrow{f} B \quad \textit{and} \quad F \xrightarrow{g'} B \xrightarrow{g} C$$

respectively. Then there exist an object $E \in \mathscr{C}$, a commutative diagram

$$\begin{tikzcd} D \arrow[r, "d"] \arrow[d, equal] & E \arrow[r, "e"] \arrow[d, "h'"] & F \arrow[d, "g'"] \\ D \arrow[r, "f'"] & A \arrow[r, "f"] \arrow[d, "h"] & B \arrow[d, "g"] \\ & C \arrow[r, equal] & C \end{tikzcd}$$

in $\mathscr{C}$, and an $\mathbb{E}$-extension $\delta'' \in \mathbb{E}(C, E)$ realized by $E \xrightarrow{h'} A \xrightarrow{h} C$, which satisfy the following compatibilities.

(i) $D \xrightarrow{d} E \xrightarrow{e} F$ *realizes* $\mathbb{E}(g', D)\delta$,

(ii) $\delta' = \mathbb{E}(C, e)\delta''$,

(iii) $\mathbb{E}(B, d)\delta = \mathbb{E}(g, E)\delta''$.

In this case, we call $\mathfrak{s}$ an $\mathbb{E}$-triangulation of $\mathscr{C}$, and call the triplet $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ an externally triangulated category, or for short, extriangulated category $\mathscr{C}$.

For an extriangulated category $\mathscr{C}$, we use the following notation:

- If a conflation $A \xrightarrow{x} B \xrightarrow{y} C$ realizes $\delta \in \mathbb{E}(C,A)$, we call the pair $(A \xrightarrow{x} B \xrightarrow{y} C, \delta)$ an $\mathbb{E}$-*triangle*, and write it in the following way.

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$$

- Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \overset{\delta'}{\dashrightarrow}$ be any pair of $\mathbb{E}$-triangles. If a triplet (a,b,c) realizes $(a,c)\colon \delta \to \delta'$, then we write it as

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"'] & B \arrow[r, "y"] \arrow[d, "b"] & C \arrow[r, dashed, "\delta"] \arrow[d, "c"] & {} \\ A' \arrow[r, "x'"] & B' \arrow[r, "y'"] & C' \arrow[r, dashed, "\delta'"] & {} \end{tikzcd}$$

and call (a,b,c) a *morphism of* $\mathbb{E}$-*triangles*.

We recall some concepts from [NP]. Let $\mathscr{C}$ be an extriangulated category.

- An object $P \in \mathscr{C}$ is called *projective* if for any $\mathbb{E}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ and any morphism $c \in \mathscr{C}(P,C)$, there exists $b \in \mathscr{C}(P,B)$ satisfying $yb = c$. We denote the full subcategory of projective objects in $\mathscr{C}$ by $\mathcal{P}$. Dually, the full subcategory of injective objects in $\mathscr{C}$ is denoted by $\mathcal{I}$.
- We say $\mathscr{C}$ *has enough projectives*, if for any object $C \in \mathscr{C}$, there exists an $\mathbb{E}$-triangle

$$A \xrightarrow{x} P \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$$

satisfying $P \in \mathcal{P}$. We can define the notion of having enough injectives dually.

- $\mathscr{C}$ is said to be *Frobenius* if $\mathscr{C}$ has enough projectives and enough injectives and if moreover the projectives coincide with the injectives.

We now give some examples of extriangulated categories.

2.5. Example.

(1) An exact category $\mathcal{B}$ can be viewed as an extriangulated category in the usual way. In fact, we take the biadditive functor $\mathbb{E} := \mathrm{Ext}^1_{\mathcal{B}}(-,-)$ and the realization $\mathfrak{s}$ is defined by associating equivalence classes of short exact sequences to itself. A triangulated category $\mathscr{C}$ with shift functor $[1]$ also can be viewed as an extriangulated category. In fact, we put $\mathbb{E} := \mathscr{C}(-,-[1])$. For any $\delta \in \mathbb{E}(C,A) = \mathscr{C}(C,A[1])$, take a triangle

$$A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} A[1]$$

and define as $\mathfrak{s}(\delta) = [\,A \xrightarrow{x} B \xrightarrow{y} C\,]$. For more details, see [NP, Example 2.13] and [NP, Proposition 3.22].

(2) Let $\mathscr{C}$ be an extriangulated category, and $\mathcal{J}$ a subcategory of $\mathscr{C}$. If $\mathcal{J} \subseteq \mathcal{P} \cap \mathcal{I}$, then $\mathscr{C}/[\mathcal{J}]$ is an extriangulated category. This construction gives extriangulated categories which are not exact nor triangulated in general. For more details, see [NP, Proposition 3.30].

(3) Let $\mathscr{C}$ be a triangulated category with Auslander-Reiten translation τ, and $\mathscr{X}$ a functorially finite subcategory of $\mathscr{C}$, which satisfies $\tau\mathscr{X} = \mathscr{X}$. For any $A, C \in \mathscr{C}$, define $\mathbb{E}'(C,A) \subseteq \mathscr{C}(C, A[1])$ to be the collection of all equivalence classes of triangles of the form $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\delta} A[1]$ where f is $\mathscr{X}$-monic, and $\mathfrak{s}'(\delta) = [A \xrightarrow{f} B \xrightarrow{g} C]$, for any $\delta \in \mathbb{E}'(C,A)$. Then $(\mathscr{C}, \mathbb{E}', \mathfrak{s}')$ is a Frobenius extriangulated category whose projective objects are precisely $\mathscr{X}$. This construction gives extriangulated categories which are not exact nor triangulated in general. For more details, see [ZhZ, Theorem 4.8] and [ZhZ, Corollary 4.10].

2.6. Remark.

(1) If $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is an exact category, then enough projectives and enough injectives agree with the usual definitions.

(2) If $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is a triangulated category, then $\mathcal{P}$ and $\mathcal{I}$ consist of zero objects. Moreover it is Frobenius as an extriangulated category.

Assume that $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is an extriangulated category. By Yoneda's lemma, any $\mathbb{E}$-extension $\delta \in \mathbb{E}(C,A)$ induces natural transformations

$$\delta_\sharp \colon \mathscr{C}(-,C) \Rightarrow \mathbb{E}(-,A) \quad \text{and} \quad \delta^\sharp \colon \mathscr{C}(A,-) \Rightarrow \mathbb{E}(C,-).$$

For any $X \in \mathscr{C}$, these $(\delta_\sharp)_X$ and $\delta^\sharp_X$ are given as follows:
(1) $(\delta_\sharp)_X \colon \mathscr{C}(X,C) \to \mathbb{E}(X,A);\ f \mapsto f^*\delta$.
(2) $\delta^\sharp_X \colon \mathscr{C}(A,X) \to \mathbb{E}(C,X);\ g \mapsto g_*\delta$.

2.7. Lemma. *Let $\mathscr{C}$ be an extriangulated category,*

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$$

an $\mathbb{E}$-triangle. Then we have the following long exact sequence:

$$\mathscr{C}(-,A) \xrightarrow{\mathscr{C}(-,x)} \mathscr{C}(-,B) \xrightarrow{\mathscr{C}(-,y)} \mathscr{C}(-,C) \xrightarrow{\delta^\sharp_-} \mathbb{E}(-,A) \xrightarrow{\mathbb{E}(-,x)} \mathbb{E}(-,B) \xrightarrow{\mathbb{E}(-,y)} \mathbb{E}(-,C),$$

$$\mathscr{C}(C,-) \xrightarrow{\mathscr{C}(y,-)} \mathscr{C}(B,-) \xrightarrow{\mathscr{C}(x,-)} \mathscr{C}(A,-) \xrightarrow{\delta^-_\sharp} \mathbb{E}(C,-) \xrightarrow{\mathbb{E}(y,-)} \mathbb{E}(B,-) \xrightarrow{\mathbb{E}(x,-)} \mathbb{E}(A,-).$$

Proof. This follows from Proposition 3.3 and Proposition 3.11 in [NP]. ■

2.8. Definition. *[ZhZ, Definition 3.21] Let $\mathscr{C}$ be an extriangulated category. A subcategory $\mathscr{X}$ of $\mathscr{C}$ is called* strongly contravariantly finite*, if for any object $C \in \mathscr{C}$, there exists an $\mathbb{E}$-triangle*

$$K \longrightarrow X \xrightarrow{g} C \overset{\delta}{\dashrightarrow},$$

where g is a right $\mathscr{X}$-approximation of C.

Dually, a subcategory $\mathscr{X}$ of $\mathscr{C}$ is called strongly covariantly finite*, if for any object $C \in \mathscr{C}$, there exists an $\mathbb{E}$-triangle*

$$C \xrightarrow{f} X \longrightarrow L \overset{\delta'}{\dashrightarrow},$$

where f is a left $\mathscr{X}$-approximation of C.

A strongly contravariantly finite and strongly covariantly finite subcategory is called strongly functorially finite.

2.9. Remark. Let $\mathscr{C}$ be an extriangulated category, $\mathscr{X}$ a subcategory of $\mathscr{C}$.

- If $\mathscr{C}$ has enough projectives $\mathcal{P}$, then $\mathscr{X}$ is strongly contravariantly finite in $\mathscr{C}$ if and only if $\mathscr{X}$ is contravariantly finite in $\mathscr{C}$ containing $\mathcal{P}$. The dual statement holds for strongly covariantly finiteness.
- If $\mathscr{C}$ has enough projectives $\mathcal{P}$ and enough injectives $\mathcal{I}$, then $\mathscr{X}$ is strongly functorially finite in $\mathscr{C}$ if and only if $\mathscr{X}$ is functorially finite in $\mathscr{C}$ containing $\mathcal{P}$ and $\mathcal{I}$.
- If $\mathscr{C}$ is a triangulated category, then $\mathscr{X}$ is strongly contravariantly (or convariantly, or functorially) finite in $\mathscr{C}$ if and only if $\mathscr{X}$ is contravariantly (covariantly, functorially respectively) finite in $\mathscr{C}$.

2.10. Definition. *Let $\mathscr{C}$ be an extriangulated category, $\mathscr{X}$ a subcategory of $\mathscr{C}$.*

- *$\mathscr{X}$ is called* rigid *if $\mathbb{E}(\mathscr{X}, \mathscr{X}) = 0$, i.e., $\mathbb{E}(A, B) = 0$, for any $A, B \in \mathscr{X}$.*
- *$\mathscr{X}$ is called* cluster-tilting *(see [CZZ]) if it satisfies the following conditions:*
  (1) $\mathscr{X}$ is a strongly functorially finite in $\mathscr{C}$;
  (2) $M \in \mathscr{X}$ if and only if $\mathbb{E}(M, \mathscr{X}) = 0$;
  (3) $M \in \mathscr{X}$ if and only if $\mathbb{E}(\mathscr{X}, M) = 0$.
- *$\mathscr{X}$ is called* maximal rigid *if $\mathscr{X}$ is rigid and is maximal with respect to this property, i.e., $\mathbb{E}(\mathscr{X} \cup \mathsf{add}\, M, \mathscr{X} \cup \mathsf{add}\, M) = 0$, then $M \in \mathscr{X}$.*
- *An object X is called* rigid, cluster tilting *or* maximal rigid *if $\mathsf{add}\, X$ is rigid, cluster-tilting or maximal rigid subcategory respectively.*

By the definition of a cluster-tilting subcategory, we can immediately conclude:

2.11. Remark. Let $\mathscr{C}$ be an extriangulated category.

- If $\mathscr{X}$ is a cluster-tilting subcategory of $\mathscr{C}$, then $\mathcal{P} \subseteq \mathscr{X}$ and $\mathcal{I} \subseteq \mathscr{X}$.
- $\mathscr{X}$ is a cluster-tilting subcategory of $\mathscr{C}$ if and only if
  (1) $\mathscr{X}$ is rigid;
  (2) For any $C \in \mathscr{C}$, there exists an $\mathbb{E}$-triangle $C \xrightarrow{a} X_1 \xrightarrow{b} X_2 \overset{\delta}{\dashrightarrow}$, where $X_1, X_2 \in \mathscr{X}$;
  (3) For any $C \in \mathscr{C}$, there exists an $\mathbb{E}$-triangle $X_3 \xrightarrow{c} X_4 \xrightarrow{d} C \overset{\eta}{\dashrightarrow}$, where $X_3, X_4 \in \mathscr{X}$.
- Any cluster-tilting subcategory is strongly functorially finite maximal rigid.

3. Quotients of extriangulated categories by rigid subcategories

Let $\mathscr{C}$ be an extriangulated category with enough projectives and enough injectives, and $\mathscr{X}$ a subcategory of $\mathscr{C}$. If $\mathcal{P} \subseteq \mathscr{X}$ (resp. $\mathcal{I} \subseteq \mathscr{X}$), the (co-)stable category $\underline{\mathscr{X}}$ (resp. $\overline{\mathscr{X}}$) of $\mathscr{X}$ is the quotient category $\mathscr{X}/[\mathcal{P}]$ (resp. $\mathscr{X}/[\mathcal{I}]$), i.e. the category which has the same objects as $\mathscr{X}$ and morphisms are defined as follows

$$\underline{\mathrm{Hom}}_{\mathscr{X}}(A,B) := \mathrm{Hom}_{\mathscr{X}}(A,B)/[\mathcal{P}](A,B)$$

$$(\text{resp. } \overline{\mathrm{Hom}}_{\mathscr{X}}(A,B) := \mathrm{Hom}_{\mathscr{X}}(X,Y)/[\mathcal{I}](A,B)).$$

The following lemma was proved in [DL, Lemma 2.3], when $\mathscr{C}$ is an exact category. However, it can be easily extended to our setting. For the convenience of the readers, we give a simple proof in the following.

3.1. Proposition. *For any contravariantly finite subcategory $\mathscr{X}$ of $\mathscr{C}$ which contains $\mathcal{P}$, $\mathsf{mod}\,\underline{\mathscr{X}}$ is an abelian category.*

Proof. It suffices to show that $\underline{\mathscr{X}}$ has pseudokernels (see [Au, Section 2]). Consider a morphism $f \in \mathrm{Hom}_{\mathscr{C}}(X_1, X_0)$ where $X_0, X_1 \in \mathscr{X}$. Since $\mathscr{C}$ has enough projectives, there exists an $\mathbb{E}$-triangle

$$N \xrightarrow{a} P_0 \xrightarrow{b} X_0 \overset{\eta}{\dashrightarrow},$$

where P_0 is a projective object. By Corollary 3.16 in [NP], there exists an $\mathbb{E}$-triangle

$$K \xrightarrow{\binom{g}{c}} X_1 \oplus P_0 \xrightarrow{(f,\ b)} X_0 \overset{\theta}{\dashrightarrow}.$$

Let $k\colon X_2 \to K$ be a right $\mathscr{X}$-approximation of K. We claim that $\underline{gk}$ is a pseudokernel of $\underline{f}$. If $\underline{fd} = 0$ for a morphism $\underline{d} \in \underline{\mathrm{Hom}}_{\mathscr{C}}(X, X_1)$ where $X \in \mathscr{X}$, then there exists the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "d"] & P \arrow[d, "v"] \\ X_1 \arrow[r, "f"] & X_0, \end{tikzcd}$$

where P is projective. Since P is projective, there exists a morphism $e\colon P \to P_0$ such that $v = be$. Since

$$\begin{pmatrix} d \\ -eu \end{pmatrix} \in \mathrm{Hom}_{\mathscr{C}}(X, X_1 \oplus P_0)$$

and

$$(f,\ b) \circ \begin{pmatrix} d \\ -eu \end{pmatrix} = fh - beu = fd - vu = 0,$$

By Lemma 2.7, there exists a morphism $h\colon X \to K$ such that $\binom{d}{-eu} = \binom{g}{c} h$. In particular, we have $d = gh$. Since k is a right $\mathscr{X}$-approximation of K and $X \in \mathscr{X}$, there exists a morphism $\ell\colon X \to X_2$ such that $k\ell = h$. It follows that $(gk) \circ \ell = gh = d$. Hence $(\underline{gk}) \circ \underline{\ell} = \underline{d}$. ∎

3.2. Quotients of extriangulated categories (I). Let $\mathscr{X}$ be a rigid subcategory of $\mathscr{C}$ which contains $\mathcal{P}$. We denote by $\mathscr{X}_L$ the subcategory of objects M of $\mathscr{C}$ which admits an $\mathbb{E}$-triangle

$$M \xrightarrow{a} X_0 \xrightarrow{b} X_1 \overset{\delta}{\dashrightarrow},$$

where $X_0, X_1 \in \mathscr{X}$. Now we consider the restriction of functor $\mathbb{E}(-,-)$ to $\mathscr{X}_L$, denoted by $\mathbb{H}$:

$$\begin{aligned} \mathbb{H}\colon \mathscr{X}_L &\longrightarrow \mathsf{Mod}\,\underline{\mathscr{X}} \\ M &\longmapsto \mathbb{E}(-, M)|_{\mathscr{X}}. \end{aligned}$$

We will consider the quotient category $\mathscr{X}_L/[\mathscr{X}]$ and denote by $[f]$ the residue class in $\mathscr{X}_L/[\mathscr{X}]$ of a morphism f of $\mathscr{X}_L$.

Let $\pi\colon \mathscr{X}_L \to \mathscr{X}_L/[\mathscr{X}]$ be the projection functor. By the definition of a rigid subcategory, $\mathbb{H}(M) = 0$ if $M \in \mathscr{X}$. Hence, by the universal property of π, there exists a functor

$$\mathbb{F}\colon \mathscr{X}_L/[\mathscr{X}] \to \mathsf{Mod}\,\underline{\mathscr{X}}$$

such that the following diagram commutes:

$$\begin{tikzcd} \mathscr{X}_L \arrow[d, "\pi"] \arrow[dr, "\mathbb{H}"] & \\ \mathscr{X}_L/[\mathscr{X}] \arrow[r, dashed, "\mathbb{F}"] & \mathsf{Mod}\,\underline{\mathscr{X}} \end{tikzcd}$$

3.3. LEMMA. *For any* $\mathbb{E}$*-triangle*

$$M \xrightarrow{x} X_0 \xrightarrow{y} X_1 \overset{\delta}{\dashrightarrow},$$

where $X_0, X_1 \in \mathscr{X}$, *there exists an exact sequence in* $\mathsf{Mod}\,\underline{\mathscr{X}}$

$$\underline{\mathrm{Hom}}_{\mathscr{X}}(-, X_0) \to \underline{\mathrm{Hom}}_{\mathscr{X}}(-, X_1) \to \mathbb{H}(M) \to 0.$$

Thus, $\mathbb{F}(M) = \mathbb{H}(M) \in \mathsf{mod}\,\underline{\mathscr{X}}$.

PROOF. The proof is similar to the proof of Lemma 3.1 in [DL]. ∎

3.4. THEOREM. *The restriction of functor* $\mathbb{E}(-,-)$ *to* $\mathscr{X}_L$ *induces an equivalence of categories* $\mathbb{F}\colon \mathscr{X}_L/[\mathscr{X}] \simeq \mathsf{mod}\,\underline{\mathscr{X}}$.

PROOF. • We prove that $\mathbb{F}$ is dense:

For any object $C \in \mathsf{mod}\,\underline{\mathscr{X}}$, there exists an exact sequence in $\mathsf{mod}\,\underline{\mathscr{X}}$:

$$\underline{\mathrm{Hom}}_{\mathscr{X}}(-, X_1) \xrightarrow{\alpha} \underline{\mathrm{Hom}}_{\mathscr{X}}(-, X_0) \to C \to 0,$$

where $X_0, X_1 \in \mathscr{X}$. Since

$$\mathrm{Hom}_{\mathsf{mod}\,\underline{\mathscr{X}}}(\underline{\mathrm{Hom}}_{\mathscr{X}}(-, X_1),\ \underline{\mathrm{Hom}}_{\mathscr{X}}(-, X_0)) \simeq \underline{\mathrm{Hom}}_{\mathscr{X}}(X_1, X_0)$$

by Yoneda's Lemma, there exists a morphism $f\colon X_1 \to X_0$ such that $\alpha = \underline{\mathrm{Hom}}_{\mathscr{X}}(-, \underline{f})$. Since $\mathscr{C}$ has enough projectives, there exists an $\mathbb{E}$-triangle

$$N \xrightarrow{a} P_0 \xrightarrow{b} X_0 \overset{\eta}{\dashrightarrow},$$

where P_0 is projective object. By Corollary 3.16 in [NP], there exists an $\mathbb{E}$-triangle

$$K \xrightarrow{c} X_1 \oplus P_0 \xrightarrow{(f,\ b)} X_0 \overset{\theta}{\dashrightarrow}.$$

Therefore $K \in \mathscr{X}_L$. By Lemma 3.3, there exists an exact sequence:

$$\underline{\mathrm{Hom}}_{\mathscr{X}}(-, X_1) \xrightarrow{\alpha} \underline{\mathrm{Hom}}_{\mathscr{X}}(-, X_0) \longrightarrow \mathbb{F}(K) \to 0.$$

Since both $\mathbb{F}(K)$ and C are cokernels of α, we have $C \simeq \mathbb{F}(K)$.

• We prove that $\mathbb{F}$ is full:

Let $M, N \in \mathscr{X}_L$ and $\beta \in \mathrm{Hom}_{\mathsf{mod}\,\underline{\mathscr{X}}}(\mathbb{F}(M), \mathbb{F}(N))$. By the definition of $\mathscr{X}_L$, we have the following two $\mathbb{E}$-triangles:

$$M \xrightarrow{a} X_0 \xrightarrow{b} X_1 \overset{\delta}{\dashrightarrow}$$

$$N \xrightarrow{c} Y_0 \xrightarrow{d} Y_1 \overset{\eta}{\dashrightarrow},$$

where $X_0, X_1, Y_0, Y_1 \in \mathscr{X}$. By Lemma 3.3, and since $\underline{\mathrm{Hom}}_{\mathscr{X}}(-, X_0)$ and $\underline{\mathrm{Hom}}_{\mathscr{X}}(-, X_1)$ are projective in $\mathsf{mod}\underline{\mathscr{X}}$, we obtain the following diagram with exact rows in $\mathsf{mod}\underline{\mathscr{X}}$:

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathscr{X}}(-, X_0) \arrow[r, "\underline{\mathrm{Hom}}_{\mathscr{X}}(-,\, \underline{b})"] \arrow[d, "\mu_0"] & \underline{\mathrm{Hom}}_{\mathscr{X}}(-, X_1) \arrow[r] \arrow[d, "\mu_1"] & \mathbb{F}(M) \arrow[r] \arrow[d, "\beta"] & 0 \\
\underline{\mathrm{Hom}}_{\mathscr{X}}(-, Y_0) \arrow[r, "\underline{\mathrm{Hom}}_{\mathscr{X}}(-,\, \underline{d})"] & \underline{\mathrm{Hom}}_{\mathscr{X}}(-, Y_1) \arrow[r] & \mathbb{F}(N) \arrow[r] & 0.
\end{tikzcd}$$

By Yoneda's Lemma, there exist morphisms $\underline{f_0} \in \underline{\mathrm{Hom}}_{\mathscr{X}}(X_0, Y_0)$ and $\underline{f_1} \in \underline{\mathrm{Hom}}_{\mathscr{X}}(X_1, Y_1)$ such that $\mu_0 = \underline{\mathrm{Hom}}_{\mathscr{X}}(-, \underline{f_0})$, $\mu_1 = \underline{\mathrm{Hom}}_{\mathscr{X}}(-, \underline{f_1})$ and $\underline{df_0} = \underline{f_1 b}$. Then there exists a projective object $P \in \mathscr{X}$, $s \in \mathrm{Hom}_{\mathscr{X}}(X_0, P)$ and $t \in \mathrm{Hom}_{\mathscr{X}}(P, Y_1)$ such that $f_1 b - df_0 = ts$. Since P is projective object, there exists a morphism $k \in \mathrm{Hom}_{\mathscr{X}}(P, Y_0)$ such that $t = dk$. Put $g_0 = f_0 + ks$, then $\underline{g_0} = \underline{f_0}$ and $dg_0 = df_0 + ts = f_1 b$. By (ET3), we obtain a morphism of $\mathbb{E}$-triangles

$$\begin{tikzcd}
M \arrow[r, "a"] \arrow[d, dashed, "f"] & X_0 \arrow[r, "b"] \arrow[d, "g_0"] & X_1 \arrow[r, dashed, "\delta"] \arrow[d, "f_1"] & {} \\
N \arrow[r, "c"] & Y_0 \arrow[r, "d"] & Y_1 \arrow[r, dashed, "\eta"] & {} .
\end{tikzcd}$$

Therefore, there exists a morphism $f \colon M \to N$ such that $g_0 a = cf$, and $\beta = \mathbb{F}([f])$.

• We prove that $\mathbb{F}$ is faithful:

Let $f \colon M \to N$ be a morphism in $\mathscr{X}_L$ such that $\mathbb{F}([f]) = 0$. Then $\mathbb{H}(f) = 0$. Since $\mathscr{C}$ has enough injectives, there exists an $\mathbb{E}$-triangle

$$M \xrightarrow{a} I \xrightarrow{b} L \overset{\eta}{\dashrightarrow},$$

where I is an injective object. By Corollary 3.16 in [NP], there exists an $\mathbb{E}$-triangle

$$M \xrightarrow{\binom{f}{a}} N \oplus I \xrightarrow{c} K \overset{\delta}{\dashrightarrow} .$$

Applying $\mathrm{Hom}_{\mathscr{C}}(\mathscr{X}, -)$ to the above $\mathbb{E}$-triangle, we get a long exact sequence

$$\mathrm{Hom}_{\mathscr{C}}(\mathscr{X}, N \oplus I) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(\mathscr{X}, K) \xrightarrow{\delta_\sharp} \mathbb{E}(\mathscr{X}, M) \xrightarrow{\mathbb{E}(\mathscr{X},\, \binom{f}{a}) = 0} \mathbb{E}(\mathscr{X}, N \oplus I).$$

In particular, we have that $\mathrm{Hom}_{\mathscr{C}}(\mathscr{X}, K) \xrightarrow{\delta_\sharp} \mathbb{E}(\mathscr{X}, M)$ is an epimorphism. Since $M \in \mathscr{X}_L$, there exists an $\mathbb{E}$-triangle

$$M \xrightarrow{d} X_0 \xrightarrow{e} X_1 \overset{\theta}{\dashrightarrow} ,$$

where $X_0, X_1 \in \mathscr{X}$. Since $\delta_\sharp$ is an epimorphism and $\mathfrak{s}$ is an additive realization of $\mathbb{E}$, we obtain a morphism of $\mathbb{E}$-triangles

$$\begin{tikzcd}
M \arrow[r, "d"] \arrow[d, equal] & X_0 \arrow[r, "e"] \arrow[d, "u"] & X_1 \arrow[r, dashed, "\theta"] \arrow[d, "v"] & {} \\
M \arrow[r, "\binom{f}{a}"] & N \oplus I \arrow[r, "c"] & K \arrow[r, dashed, "\delta"] & {} .
\end{tikzcd}$$

It follows that $\binom{f}{a} = ud$, thus $[f] = 0$ in $\mathscr{X}_L / [\mathscr{X}]$. ■

This theorem immediately yields the following.

3.5. Corollary. *Let $\mathscr{C}$ be an extriangulated category with enough projectives and enough injectives, and $\mathscr{X}$ a cluster-tilting subcategory of $\mathscr{C}$. Then* $\mathscr{C}/[\mathscr{X}] \overset{\mathbb{E}(\mathscr{X},-)}{\simeq} \operatorname{\mathsf{mod}}\underline{\mathscr{X}}$.

3.6. Corollary. [BMR, KR, KZ, IY] *Let $\mathscr{C}$ be a triangulated category and $\mathscr{X}$ a cluster-tilting subcategory of $\mathscr{C}$. Then* $\mathscr{C}/[\mathscr{X}] \overset{\mathbb{E}(\mathscr{X},-)}{\simeq} \operatorname{\mathsf{mod}}\mathscr{X}$.

3.7. Corollary. [DL] *Let $\mathcal{B}$ be an exact category with enough projectives and enough injectives, and $\mathscr{X}$ a cluster-tilting subcategory of $\mathcal{B}$. Then* $\mathcal{B}/[\mathscr{X}] \overset{\mathbb{E}(\mathscr{X},-)}{\simeq} \operatorname{\mathsf{mod}}\underline{\mathscr{X}}$.

3.8. Corollary. *Let $\mathscr{C}$ be an extriangulated category with enough projectives and enough injectives, and $\mathscr{X}$ a contravariantly finite rigid subcategory of $\mathscr{C}$. Then $\mathscr{X}_L/[\mathscr{X}]$ is abelian.*

Proof. This follows from Theorem 3.4 and Proposition 3.1. ■

Motivated by the definition of 2-Calabi-Yau triangulated categories and exact stably 2-Calabi-Yau categories, Chang-Zhou-Zhu [CZZ] introduced the following definition.

3.9. Definition. *[CZZ, Definition 1.10] Let $\mathscr{C}$ be an extriangulated category, which is a Hom-finite Krull-Schmidt k-linear category, where k is a field. $\mathscr{C}$ is called* 2-Calabi-Yau *if there exists a bifunctorial isomorphism*

$$\mathbb{E}(A,B) \simeq \mathsf{D}\mathbb{E}(B,A),$$

for any $A, B \in \mathscr{C}$, where $\mathsf{D} = Hom_k(-,k)$ is the usual k-duality.

3.10. Example. Let $\mathscr{C}$ be a 2-Calabi-Yau extriangulated category with a cluster-tilting object, and $(\mathscr{X},\mathscr{Y})$ a cotorsion pair with core $\mathcal{M} = \mathscr{X} \cap \mathscr{Y}$ in $\mathscr{C}$. Suppose that there is a cluster tilting subcategory $\mathcal{T}$ contains $\mathcal{M}$ as a subcategory. By Proposition 4.6 in [CZZ], $\mathcal{T}$ can be written uniquely as $\mathcal{T} = \mathcal{T}_{\mathscr{X}} \oplus \mathcal{M} \oplus \mathcal{T}_{\mathscr{Y}}$ such that $\mathcal{T}_{\mathscr{X}} \oplus \mathcal{M}$ is $\mathscr{X}$-cluster tilting and $\mathcal{T}_{\mathscr{Y}} \oplus \mathcal{M}$ is $\mathscr{Y}$-cluster tilting, where $\mathcal{T}_{\mathscr{X}} \in \mathscr{X}$ and $\mathcal{T}_{\mathscr{Y}} \in \mathscr{Y}$. By Theorem 3.4, we have

$$\mathscr{C}/[\mathcal{T}_{\mathscr{X}} \oplus \mathcal{M} \oplus \mathcal{T}_{\mathscr{Y}}] \simeq \operatorname{\mathsf{mod}}(\underline{\mathcal{T}_{\mathscr{X}} \oplus \mathcal{M} \oplus \mathcal{T}_{\mathscr{Y}}}),$$

$$\mathscr{X}/[\mathcal{T}_{\mathscr{X}} \oplus \mathcal{M}] \simeq \operatorname{\mathsf{mod}}(\underline{\mathcal{T}_{\mathscr{X}} \oplus \mathcal{M}}),$$

$$\mathscr{Y}/[\mathcal{T}_{\mathscr{Y}} \oplus \mathcal{M}] \simeq \operatorname{\mathsf{mod}}(\underline{\mathcal{T}_{\mathscr{Y}} \oplus \mathcal{M}}).$$

3.11. Quotients of extriangulated categories (II). In this subsection, we assume that $\mathscr{X}$ is a rigid subcategory of $\mathscr{C}$ which contains $\mathcal{I}$. We denote by $\mathscr{X}_R$ the subcategory of objects M of $\mathscr{C}$ which admits an $\mathbb{E}$-triangle

$$X_1 \xrightarrow{a} X_0 \xrightarrow{b} M \overset{\delta}{\dashrightarrow},$$

where $X_0, X_1 \in \mathscr{X}$ and $\Omega^{-1}\mathscr{X}$ the subcategory of objects L of $\mathscr{C}$ such that there exists an $\mathbb{E}$-triangle

$$X \longrightarrow I \longrightarrow L \dashrightarrow,$$

where $X \in \mathscr{X}, I \in \mathcal{I}$. Now we consider the functor

$$\begin{aligned} \mathbb{K}\colon \mathscr{X}_R &\longrightarrow \mathsf{Mod}\,\overline{\mathscr{X}} \\ M &\longmapsto \overline{\mathrm{Hom}}_{\mathscr{C}}(-,M)|_{\overline{\mathscr{X}}}. \end{aligned}$$

We will consider the quotient category $\mathscr{X}_R/[\Omega^{-1}\mathscr{X}]$ and denote by $[f]$ the residue class in $\mathscr{X}_R/[\Omega^{-1}\mathscr{X}]$ of any morphism f of $\mathscr{X}_R$. Let $\pi'\colon \mathscr{X}_R \to \mathscr{X}_R/[\Omega^{-1}\mathscr{X}]$ be the projection functor. By the universal property of π', there exists a functor

$$\mathbb{G}\colon \mathscr{X}_R/[\Omega^{-1}\mathscr{X}] \to \mathsf{Mod}\,\overline{\mathscr{X}}$$

such that the following diagram commutes:

$$\begin{tikzcd} \mathscr{X}_R \arrow[d, "\pi'"] \arrow[dr, "\mathbb{K}"] & \\ \mathscr{X}_R/[\Omega^{-1}\mathscr{X}] \arrow[r, dashed, "\mathbb{G}"] & \mathsf{Mod}\,\overline{\mathscr{X}} \end{tikzcd}$$

3.12. Lemma. *For any $M \in \mathscr{X}_R$, we have $\mathbb{G}(M) = \mathbb{K}(M) \in \mathsf{mod}\,\overline{\mathscr{X}}$. That is to say, for any $\mathbb{E}$-triangle*

$$X_1 \xrightarrow{x} X_0 \xrightarrow{y} M \overset{\delta}{\dashrightarrow}\ ,$$

where $Y_0, Y_1 \in \mathscr{X}$, there exists an exact sequence in $\mathsf{mod}\,\overline{\mathscr{X}}$

$$\mathbb{K}(X_1) \xrightarrow{\mathbb{K}(x)} \mathbb{K}(X_0) \xrightarrow{\mathbb{K}(y)} \mathbb{K}(M) \to 0.$$

Proof. It is similar to the proof of Lemma 3.4 in [DL]. ■

3.13. Theorem. *The functor $\mathbb{G}\colon \mathscr{X}_R/[\Omega^{-1}\mathscr{X}] \to \mathsf{mod}\,\overline{\mathscr{X}}$ is an equivalence of categories.*

Proof. • Let us prove that $\mathbb{G}$ is dense:

For any object $C \in \mathsf{mod}\,\overline{\mathscr{X}}$, there exists an exact sequence in $\mathsf{mod}\,\overline{\mathscr{X}}$:

$$\mathbb{K}(X_1) \xrightarrow{\mathbb{K}(\alpha)} \mathbb{K}(X_0) \to C \to 0,$$

where $X_0, X_1 \in \mathscr{X}$. Since $\mathscr{C}$ has enough injectives, there exists an $\mathbb{E}$-triangle

$$X_1 \xrightarrow{i} I_1 \longrightarrow N \dashrightarrow,$$

where I_1 is injective object. By Corollary 3.16 in [NP], there exists an $\mathbb{E}$-triangle

$$X_1 \xrightarrow{\binom{i}{\alpha}} I_1 \oplus X_0 \longrightarrow L \dashrightarrow .$$

Therefore $L \in \mathscr{X}_R$. By Lemma 3.12, we get an exact sequence

$$\mathbb{K}(X_1) \xrightarrow{\mathbb{K}(\alpha)} \mathbb{K}(X_0) \to \mathbb{K}(L) \to 0.$$

Since both $\mathbb{K}(L)$ and C are cokernels of $\mathbb{K}(\alpha)$, we have $C \simeq \mathbb{K}(L) \simeq \mathbb{G}(L)$.

• Let us prove that $\mathbb{K}$ (and therefore $\mathbb{G}$) is full:

Let $M, N \in \mathscr{X}_R$ and $\beta \in \mathrm{Hom}_{\mathsf{mod}\overline{\mathscr{X}}}(\mathbb{K}(M), \mathbb{K}(N))$. We have the following two $\mathbb{E}$-triangles:

$$X_1 \xrightarrow{a} X_0 \xrightarrow{b} M \overset{\delta}{\dashrightarrow}$$

$$Y_1 \xrightarrow{c} Y_0 \xrightarrow{d} N \overset{\eta}{\dashrightarrow} ,$$

where $X_0, X_1, Y_0, Y_1 \in \mathscr{X}$. It follows that there exists a commutative diagram with morphisms $f_0\colon X_0 \to Y_0$ and $f_1\colon X_1 \to Y_1$ such that $\mu_0 = \mathbb{K}(f_0)$ and $\mu_1 = \mathbb{K}(f_1)$:

$$\begin{tikzcd}
\mathbb{K}(X_1) \arrow[r, "\mathbb{K}(a)"] \arrow[d, "\mu_1"] & \mathbb{K}(X_0) \arrow[r, "\mathbb{K}(b)"] \arrow[d, "\mu_0"] & \mathbb{K}(M) \arrow[r] \arrow[d, "\beta"] & 0 \\
\mathbb{K}(Y_1) \arrow[r, "\mathbb{K}(c)"] & \mathbb{K}(Y_0) \arrow[r, "\mathbb{K}(d)"] & \mathbb{K}(N) \arrow[r] & 0.
\end{tikzcd}$$

Hence $\overline{f_0 a} = \overline{c f_1}$. There exists an injective object $I_0 \in \mathscr{X}$, $s \in \mathrm{Hom}_{\mathscr{X}}(X_1, I_0)$ and $t \in \mathrm{Hom}_{\mathscr{X}}(I_0, Y_0)$ such that $f_0 a - c f_1 = ts$. Since I_0 is an injective object, there exists a morphism $k \in \mathrm{Hom}_{\mathscr{X}}(X_0, I_0)$ such that $s = ka$. Put $g_0 = f_0 - tk$, then $\underline{g_0} = \underline{f_0}$ and $g_0 a = f_0 a - tka = c f_1$. By (ET3), we obtain a morphism of $\mathbb{E}$-triangles

$$\begin{tikzcd}
X_1 \arrow[r, "a"] \arrow[d, "f_1"] & X_0 \arrow[r, "b"] \arrow[d, "g_0"] & M \arrow[r, dashed, "\delta"] \arrow[d, dashed, "h"] & {} \\
Y_1 \arrow[r, "c"] & Y_0 \arrow[r, "d"] & N \arrow[r, dashed, "\eta"] & {}.
\end{tikzcd}$$

Therefore, there exists a morphism $h\colon M \to N$ such that $hb = dg_0$, and $\beta = \mathbb{K}([h])$.

• Let us prove that $\mathbb{G}$ is faithful:

Let $f\colon M \to N$ be a morphism in $\mathscr{X}_R$ such that $\mathbb{G}([f]) = 0$. For $M \in \mathscr{X}_R$, we have two $\mathbb{E}$-triangles:

$$X_1 \xrightarrow{a} X_0 \xrightarrow{b} M \overset{\delta}{\dashrightarrow}$$

where $X_0, X_1 \in \mathscr{X}$, and

$$X_0 \xrightarrow{u} I_2 \xrightarrow{v} Q \overset{\eta}{\dashrightarrow},$$

where I_2 is injective object. By (ET4), we have commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r, "a"] \arrow[d, equal] & X_0 \arrow[r, "b"] \arrow[d, "u"] & M \arrow[r, dashed, "\delta"] \arrow[d, "c"] & {} \\
X_1 \arrow[r] & I_2 \arrow[r, "k"] \arrow[d, "v"] & L \arrow[r, dashed] \arrow[d, "d"] & {} \\
& Q \arrow[r, equal] \arrow[d, dashed, "\eta"] & Q \arrow[d, dashed] & \\
& {} & {} &
\end{tikzcd}$$

of $\mathbb{E}$-triangles. From $\mathbb{G}([f]) = 0$, we have $\mathbb{K}(f) = 0$. It follows that fb factors through an injective object I_3: i.e. we have a commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r, "b"] \arrow[d, "g'"'] & X_0 \arrow[d, "f"] \\
I_3 \arrow[r, "g"] & N.
\end{tikzcd}$$

Therefore there exists a morphism $h\colon I_2 \to I_3$ such that $g' = hu$. Hence we have a commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r, "b"] \arrow[d, "u"'] & X_0 \arrow[d, "f"] \\
I_2 \arrow[r, "gh"] & N.
\end{tikzcd}$$

By Lemma 3.13 in [NP], there exists a morphism $\phi\colon L \to N$ which makes the following diagram commutative

$$\begin{tikzcd}
X_0 \arrow[r, "b"] \arrow[d, "u"'] & M \arrow[d, "c"] \arrow[ddr, bend left, "f"] & \\
I_2 \arrow[r, "k"] \arrow[drr, bend right, "gh"'] & L \arrow[dr, dashed, "\phi"] & \\
& & N
\end{tikzcd}$$

In particular, we have $f = \phi c$. From the $\mathbb{E}$-triangle:

$$X_1 \longrightarrow I_2 \xrightarrow{k} L \dashrightarrow,$$

we have $L \in \Omega^{-1}\mathscr{X}$. This shows that $[f] = 0$ in $\mathscr{X}_R/[\Omega^{-1}\mathscr{X}]$. ∎

3.14. Corollary. *$\Omega^{-1}\mathscr{X} = \overline{\mathscr{X}}^{\perp}$, where $\overline{\mathscr{X}}^{\perp} = \{M \in \mathscr{X}_R \mid \overline{\mathrm{Hom}}_{\mathscr{C}}(\mathscr{X}, M) = 0\}$.*

Proof. For any $M \in \Omega^{-1}\mathscr{X}$, there exists an $\mathbb{E}$-triangle

$$X \longrightarrow I \longrightarrow M \dashrightarrow,$$

where $X \in \mathscr{X}, I \in \mathcal{I}$. Applying the functor $\mathrm{Hom}_{\mathscr{C}}(\mathscr{X}, -)$ to the above $\mathbb{E}$-triangle, we have the following exact sequence:

$$\mathrm{Hom}_{\mathscr{C}}(\mathscr{X}, I) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(\mathscr{X}, M) \longrightarrow \mathbb{E}(\mathscr{X}, X) = 0.$$

It follows that $M \in \overline{\mathscr{X}}^{\perp}$. Conversely, if $M \in \overline{\mathscr{X}}^{\perp}$, then $\mathbb{G}(M) = \mathbb{K}(M) = 0$. Since $\mathbb{G}$ is faithful, we have $M \in \Omega^{-1}\mathscr{X}$. ■

3.15. Remark. Let $\mathscr{C}$ be a Hom-finite Krull-Schmidt k-linear extriangulated category with Auslander-Reiten translation τ and τ^{-1}, and k a field. Let $\mathscr{X}$ be a cluster-tilting subcategory of $\mathscr{C}$. Assume that there is a functorial isomorphism

$$\mathbb{E}(A, B) \simeq \mathsf{D}\overline{\mathrm{Hom}}_{\mathscr{C}}(B, \tau A) \simeq \mathsf{D}\underline{\mathrm{Hom}}_{\mathscr{C}}(\tau^{-1}B, A),$$

for any $A, B \in \mathscr{C}$, where $\mathsf{D} = \mathrm{Hom}_k(-, k)$ denotes duality over k. Using similar arguments as in the proof of Proposition 3.11 and Proposition 3.13 in [DL], the Auslander-Reiten translation τ induces an equivalence from $\underline{\mathscr{X}}$ to $\overline{\mathscr{X}}$, and then an equivalence from $\mathsf{mod}\,\underline{\mathscr{X}}$ to $\mathsf{mod}\,\overline{\mathscr{X}}$.

4. Relations between cluster tilting and maximal rigid

In this section, all categories are assumed Hom-finite Krull-Schmidt k-linear for a field k. Let $\mathscr{C}$ be an extriangulated category. Any cluster-tilting subcategory in $\mathscr{C}$ is strongly functorially finite maximal rigid, but the converse is not true in general, even when $\mathscr{C}$ is 2-Calabi-Yau. For example, we take an extension closed subcategory $\mathscr{X}$ in a cluster tube $\mathscr{C}$ [BMV], $\mathscr{X}$ is a 2-Calabi-Yau extriangulated category, and has strongly maximal rigid objects, which are not cluster tilting. We will prove that any strongly functorially finite maximal rigid subcategory is cluster tilting in 2-Calabi-Yau extriangulated categories with a cluster tilting subcategory.

The following lemma plays an important role in this section.

4.1. Lemma. *Let $\mathscr{C}$ be a 2-Calabi-Yau extriangulated category, and $\mathscr{X}$ a maximal rigid subcategory of $\mathscr{C}$. Given an $\mathbb{E}$-triangle*

$$A \xrightarrow{f} X \xrightarrow{g} B \overset{\delta}{\dashrightarrow}. \tag{4.1}$$

(1) *If g is a right $\mathscr{X}$-approximation of B and $\mathbb{E}(B, B) = 0$, then $A \in \mathscr{X}$.*

(2) *If f is a left $\mathscr{X}$-approximation of A and $\mathbb{E}(A, A) = 0$, then $B \in \mathscr{X}$.*

PROOF. (1). Applying the functor $\mathrm{Hom}_{\mathscr{C}}(\mathscr{X},-)$ to the $\mathbb{E}$-triangle (4.1), we have the following exact sequence

$$\mathrm{Hom}_{\mathscr{C}}(\mathscr{X},X) \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(\mathscr{X},g)} \mathrm{Hom}_{\mathscr{C}}(\mathscr{X},B) \longrightarrow \mathbb{E}(\mathscr{X},A) \longrightarrow \mathbb{E}(\mathscr{X},X)=0.$$

Since g is a right $\mathscr{X}$-approximation of B, we have that $\mathrm{Hom}_{\mathscr{C}}(\mathscr{X},g)$ is an epimorphism. It follows that $\mathbb{E}(\mathscr{X},\mathsf{add}\,A)=0$ and, by the 2-Calabi-Yau property, $\mathbb{E}(\mathsf{add}\,A,\mathscr{X})=0$.

Applying the functor $\mathrm{Hom}_{\mathscr{C}}(-,B)$ to the $\mathbb{E}$-triangle (4.1), we have the following exact sequence

$$\mathrm{Hom}_{\mathscr{C}}(X,B) \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(f,\ B)} \mathrm{Hom}_{\mathscr{C}}(A,B) \longrightarrow \mathbb{E}(B,B)=0.$$

Applying the functor $\mathrm{Hom}_{\mathscr{C}}(A,-)$ to the $\mathbb{E}$-triangle (4.1), we have the following exact sequence

$$\mathrm{Hom}_{\mathscr{C}}(A,X) \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(A,\ g)} \mathrm{Hom}_{\mathscr{C}}(A,B) \longrightarrow \mathbb{E}(A,A) \longrightarrow \mathbb{E}(A,X)=0.$$

We claim that $\mathrm{Hom}_{\mathscr{C}}(A,g)$ is an epimorphism. Indeed, for any morphism $a\colon A\to B$, there exists a morphism $b\colon X\to B$ such that $bf=a$. Since g is a right $\mathscr{X}$-approximation of B and $X\in\mathscr{X}$, there exists a morphism $c\colon X\to X$ such that $gc=b$ and then $a=g(cf)$. This shows that $\mathrm{Hom}_{\mathscr{C}}(A,g)$ is an epimorphism. Thus we have $\mathbb{E}(A,A)=0$. Since $\mathscr{X}$ is a maximal rigid subcategory, we have $A\in\mathscr{X}$.

(2). The proof is dual. ■

4.2. COROLLARY. *Let $\mathscr{C}$ be a 2-Calabi-Yau extriangulated category, and $\mathscr{X}$ a strongly functorially finite maximal rigid subcategory of $\mathscr{C}$. If $\mathbb{E}(A,A)=0$, then there exist two $\mathbb{E}$-triangles*

$$X_1 \longrightarrow X_0 \xrightarrow{g} A \overset{\delta}{\dashrightarrow},$$

where $X_1,X_0\in\mathscr{X}$ and g is a right $\mathscr{X}$-approximation of A;

$$A \xrightarrow{f} X_2 \longrightarrow X_3 \overset{\delta'}{\dashrightarrow},$$

where $X_2,X_3\in\mathscr{X}$ and f is a left $\mathscr{X}$-approximation of A.

4.3. THEOREM. *Let $\mathscr{C}$ be a 2-Calabi-Yau extriangulated category with a cluster-tilting subcategory $\mathcal{T}$. Then every strongly functorially finite maximal rigid subcategory is cluster-tilting.*

PROOF. Assume that $\mathscr{X}$ is a strongly functorially finite maximal rigid subcategory in $\mathscr{C}$. Given an object $M\in\mathscr{X}$ satisfying $\mathbb{E}(M,\mathscr{X})=0$. By definition of cluster tilting subcategory, there exists an $\mathbb{E}$-triangle

$$T_1 \xrightarrow{f} T_0 \xrightarrow{g} M \overset{\delta}{\dashrightarrow}, \tag{4.2}$$

where $T_0, T_1 \in \mathcal{T}$. Since $\mathbb{E}(T_0, T_0) = 0$, by Corollary 4.2, there exists an $\mathbb{E}$-triangle

$$T_0 \xrightarrow{u} X_1 \xrightarrow{v} X_2 \overset{\delta'}{\dashrightarrow} ,$$

where $X_1, X_2 \in \mathscr{X}$ and u is a left $\mathscr{X}$-approximation of T_0. By (ET4), we have a commutative diagram

$$\begin{tikzcd}
T_1 \arrow[r, "f"] \arrow[d, equal] & T_0 \arrow[r, "g"] \arrow[d, "u"] & M \arrow[r, dashed, "\delta"] \arrow[d, "a"] & {} \\
T_1 \arrow[r, "x=uf"] & X_1 \arrow[r, "y"] \arrow[d, "v"] & N \arrow[r, dashed] \arrow[d, "b"] & {} \\
 & X_2 \arrow[r, equal] \arrow[d, dashed, "\delta'"] & X_2 \arrow[d, dashed] & \\
 & {} & {} &
\end{tikzcd}$$

of $\mathbb{E}$-triangles. We claim that x is a left $\mathscr{X}$-approximation of T_1. Indeed, let $\alpha\colon T_1 \to X$ be any morphism, where $X \in \mathscr{X}$. By applying the functor $\operatorname{Hom}_{\mathscr{C}}(-, X)$ to the $\mathbb{E}$-triangle (4.2), we have the following exact sequence

$$\operatorname{Hom}_{\mathscr{C}}(T_0, X) \xrightarrow{\operatorname{Hom}_{\mathscr{C}}(f,\ X)} \operatorname{Hom}_{\mathscr{C}}(T_1, X) \longrightarrow \mathbb{E}(M, X) = 0.$$

So there exists a morphism $\beta\colon T_0 \to X$ such that $\alpha = \beta f$. Since u is a left $\mathscr{X}$-approximation of T_0 and $X \in \mathscr{X}$, there exists a morphism $\gamma\colon X_1 \to X$ such that $\beta = \gamma u$ and then $\alpha = \gamma(uf) = \gamma x$. This shows that x is a left $\mathscr{X}$-approximation of T_1.

Since $\mathbb{E}(T_1, T_1) = 0$, by Lemma 4.1, we have $N \in \mathscr{X}$. Since $\mathbb{E}(M, \mathscr{X}) = 0$ and $X_2 \in \mathscr{X}$, by the 2-Calabi-Yau property, we have $\mathbb{E}(X_2, M) = 0$. This shows that the $\mathbb{E}$-triangle

$$M \xrightarrow{a} N \xrightarrow{b} X_2 \dashrightarrow$$

splits. By [NP, Corollary 3.5], we have that a is section. This shows that M is a direct summand of N and then $M \in \mathscr{X}$. This completes the proof. ∎

4.4. Corollary. [ZZ, Theorem 2.6] *Let $\mathscr{C}$ be a 2-Calabi-Yau triangulated category with a cluster tilting subcategory $\mathcal{T}$. Then every functorially finite maximal rigid subcategory is cluster tilting.*

Proof. This follows from Theorem 4.3 and Remark 2.9. ∎

Recall that an exact category $\mathcal{B}$ is *stably 2-Calabi-Yau* if it is Frobenius, that is, $\mathcal{B}$ has enough projectives and enough injectives, which coincide, and the stable category $\overline{\mathcal{B}}$, which is triangulated [Ha], is 2-Calabi-Yau. Examples of exact stably 2-Calabi-Yau categories are categories of maximal Cohen-Macaulay modules $\mathsf{CM}(R)$ for a three-dimensional complete local commutative isolated Gorenstein singularity [BIKR] and $\mathsf{mod}\Lambda$ for Λ being the preprojective algebra of a Dynkin quiver [GLS].

4.5. Corollary. [BIRS, Theorem II.1.8(a)] *Let $\mathcal{B}$ be an exact stably 2-Calabi-Yau category with a cluster tilting subcategory $\mathcal{T}$. Then every functorially finite maximal rigid subcategory is cluster tilting.*

Proof. It is easy to see that $\mathrm{Ext}^1_{\mathcal{B}}(A, B) \simeq \mathrm{Ext}^1_{\underline{\mathcal{B}}}(A, B)$. This follows from Theorem 4.3. ■

Acknowledgement

The authors thank the anonymous referee for his/her helpful comments and useful suggestions to improve this article. They also thank Hiroyuki Nakaoka for helpful discussions on the subject. They are indebted to Professor Steve Lack for his useful suggestions to improve the English expressions.

References

[Au] M. Auslander. Coherent functors. In Proceedings Conference Categorical Algebra (La Jolla). Springer-Verlag, Berlin-Heidelberg-NewYork, 189-231, 1966.

[AR] M. Auslander, I. Reiten. Applications of contravariantly finite subcategories. Adv. Math. 86(1), 111-152, 1991.

[BIKR] I. Burban, O. Iyama, B. Keller, I. Reiten. Cluster tilting for one-dimensional hypersurface singularities, Adv. Math. 217, 2443-2484, 2008.

[BIRS] B. Buan, O. Iyama, I. Reiten, J. Scott. Cluster structures for 2-Calabi-Yau categories and unipotent groups. Compo. Math. 145(4), 1035-1079, 2009.

[BMRRT] B. Buan, R. Marsh, M. Reineke, I. Reiten, G. Todorov. Tilting theory and cluster combinatorics. Adv. Math. 204(2), 572-618, 2006.

[BMR] B. Buan, R. Marsh, Robert, I. Reiten. Cluster-tilted algebras. Trans. Amer. Math. Soc. 359(1), 323-332, 2007.

[BMV] B. Buan, R. Marsh, D. Vatne. Cluster structure from 2-Calabi-Yau categories with loops. Math. Z. 265, 951-970, 2010.

[CCS] P. Caldero, F. Chapoton, R. Schiffler. Quivers with relations arising from clusters (A_n case). Trans. Am. Math. Soc. 358(3), 1347-1364, 2006.

[CZZ] W. Chang, P. Zhou, B. Zhu. Cluster subalgebras and cotorsion pairs in Frobenius extriangulated categories, 2016, arXiv: 1610.05006v1. To appear in Algebr. and Rep. Theory.

[DL] L. Demonet, Y. Liu. Quotients of exact categories by cluster tilting subcategories as module categories. J. Pure Appl. Algebra 217(12), 2282-2297, 2013.

[FZ] S. Fomin, A. Zelevinsky. Cluster algebras I: Foundations. J. Amer. Math. Soc. 15(2), 497-529, 2002.

[GLS] C. Geiß, B. Leclerc, J. Shröer. Rigid modules over preprojective algebras. Invent. Math. 165, 589-632, 2006.

[Ha] D. Happel. Triangulated categories in the representation of finite dimensional algebras. Vol. 119, Cambridge University Press, 1988.

[IY] O. Iyama, Y. Yoshino. Mutation in triangulated categories and rigid Cohen-Macaulay modules. Invent. Math. 172(1), 117-168, 2008.

[KR] B. Keller, I. Reiten. Cluster-tilted algebras are Gorenstein and stably Calabi-Yau. Adv. Math. 211, 123-151, 2007.

[KZ] S. Koenig, B. Zhu. From triangulated categories to abelian categories: cluster tilting in a general framework. Math. Z. 258, 143-160, 2008.

[NP] H. Nakaoka, Y. Palu. Mutation via Hovey twin cotorsion pairs and model structures in extriangulated categories. arXiv: 1605.05607v2, 2016.

[ZhZ] P. Zhou, B. Zhu. Triangulated quotient categories revisited. J.Algebra. 502, 196-232, 2018.

[ZZ] Y. Zhou, B. Zhu, Maximal rigid subcategories in 2-Calabi-Yau triangulated categories, J. Algebra. 348, 49-60, 2011.

College of Mathematics, Hunan Institute of Science and Technology,
414000 Yueyang, Hunan, P. R. China.
and
Department of Mathematical Sciences, Tsinghua University,
100084 Beijing, P. R. China.

Email: panyuezhou@163.com
bzhu@math.tsinghua.edu.cn

This article may be accessed at http://www.tac.mta.ca/tac/

THEORY AND APPLICATIONS OF CATEGORIES will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.
Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.

Subscription information Individual subscribers receive abstracts of articles by e-mail as they are published. To subscribe, send e-mail to tac@mta.ca including a full name and postal address. Full text of the journal is freely available at http://www.tac.mta.ca/tac/.

Information for authors LaTeX2e is required. Articles may be submitted in PDF by email directly to a Transmitting Editor following the author instructions at
http://www.tac.mta.ca/tac/authinfo.html.

Managing editor. Robert Rosebrugh, Mount Allison University: rrosebrugh@mta.ca

TeXnical editor. Michael Barr, McGill University: michael.barr@mcgill.ca

Assistant TeX editor. Gavin Seal, Ecole Polytechnique Fédérale de Lausanne:
gavin_seal@fastmail.fm

Transmitting editors.
Clemens Berger, Université de Nice-Sophia Antipolis: cberger@math.unice.fr
Julie Bergner, University of Virginia: jeb2md (at) virginia.edu
Richard Blute, Université d' Ottawa: rblute@uottawa.ca
Gabriella Böhm, Wigner Research Centre for Physics: bohm.gabriella (at) wigner.mta.hu
Valeria de Paiva: Nuance Communications Inc: valeria.depaiva@gmail.com
Richard Garner, Macquarie University: richard.garner@mq.edu.au
Ezra Getzler, Northwestern University: getzler (at) northwestern(dot)edu
Kathryn Hess, Ecole Polytechnique Fédérale de Lausanne: kathryn.hess@epfl.ch
Dirk Hoffman, Universidade de Aveiro: dirk@ua.pt
Pieter Hofstra, Université d' Ottawa: phofstra (at) uottawa.ca
Anders Kock, University of Aarhus: kock@math.au.dk
Joachim Kock, Universitat Autònoma de Barcelona: kock (at) mat.uab.cat
Stephen Lack, Macquarie University: steve.lack@mq.edu.au
F. William Lawvere, State University of New York at Buffalo: wlawvere@buffalo.edu
Tom Leinster, University of Edinburgh: Tom.Leinster@ed.ac.uk
Matias Menni, Conicet and Universidad Nacional de La Plata, Argentina: matias.menni@gmail.com
Ieke Moerdijk, Utrecht University: i.moerdijk@uu.nl
Susan Niefield, Union College: niefiels@union.edu
Robert Paré, Dalhousie University: pare@mathstat.dal.ca
Kate Ponto, University of Kentucky: kate.ponto (at) uky.edu
Jiri Rosicky, Masaryk University: rosicky@math.muni.cz
Giuseppe Rosolini, Università di Genova: rosolini@disi.unige.it
Alex Simpson, University of Ljubljana: Alex.Simpson@fmf.uni-lj.si
James Stasheff, University of North Carolina: jds@math.upenn.edu
Ross Street, Macquarie University: ross.street@mq.edu.au
Tim Van der Linden, Université catholique de Louvain: tim.vanderlinden@uclouvain.be
R. J. Wood, Dalhousie University: rjwood@mathstat.dal.ca