LAX LIMITS OF MODEL CATEGORIES

YONATAN HARPAZ

ABSTRACT. For a diagram of simplicial combinatorial model categories, we show that the associated lax limit, endowed with the projective model structure, is a presentation of the lax limit of the underlying ∞-categories. Our approach can also allow for the indexing category to be simplicial, as long as the diagram factors through its homotopy category. Analogous results for the associated homotopy limit (and other intermediate limits) directly follow.

Contents

1. Introduction

In ordinary category theory, the *lax colimit* of a presheaf $\mathcal{F} : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathrm{Cat}$ of categories is given by the total category of the cartesian fibration $\pi : \int^{\mathcal{I}} \mathcal{F} \longrightarrow \mathcal{I}$ classified by $\mathcal{F}$, while its *oplax limit* is given by the category of sections of π. Both of these constructions are fairly fundamental and appear in a wide variety of circumstances. In homotopy theory one often works in a higher categorical setting, where categories are replaced with ∞-*categories*. In this case, the cartesian fibration $\pi : \int^{\mathcal{I}} \mathcal{F} \longrightarrow \mathcal{I}$ acquires an even more prominent role: indeed, if $\mathcal{I}$ is now an ∞-category then presheaves $\mathcal{F} : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty$ valued in ∞-categories are often hard to write down explicitly, and are hence usually encoded directly as cartesian fibrations over $\mathcal{I}$ via the straightening-unstraightening equivalence (in which the unstraightening construction is the ∞-categorical analogue of the Grothendieck construction). As was established by Gepner–Haugseng–Nikolaus [7], in the higher categorical setting the total ∞-category of the cartesian fibration classified by $\mathcal{F}$ is again a model for the lax colimit of $\mathcal{F}$, while the ∞-category of sections of this cartesian fibration is a model for the corresponding oplax limit. To avoid confusion, we point out that the notation conventions of loc. cit. for lax versus oplax are different than the ones we use here, and which follow the conventions of ordinary category theory.

A closely related invariant is given by the *homotopy* limit and colimit of a given functor $\mathcal{F} : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty$. These are related to the (op)lax limit and colimit as follows: the homotopy colimit of $\mathcal{F}$ is given by localizing the total ∞-category of the associated cartesian fibration $\pi \colon \int^{\mathcal{I}} \mathcal{F} \longrightarrow \mathcal{I}$ by the collection of π-cartesian edges [15, Corollary 3.3.4.3], while the homotopy limit is given by the full sub-∞-category of sections $s : \mathcal{I} \longrightarrow \int^{\mathcal{I}} \mathcal{F}$ which send every edge to a cartesian edge [15, Corollary 3.3.3.2]. The (op)lax and homotopy (co)limit constructions can be put on an equal footing if one considers more generally limits of diagrams indexed by *marked ∞-categories*, that is, ∞-categories $\mathcal{I}$ equipped with a collection of marked edges E (which are not necessarily equivalences in $\mathcal{I}$ and are not necessarily sent to equivalences in Cat_∞). The E-colimit of a presheaf $\mathcal{F} : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty$ is then the localization of $\int^{\mathcal{I}} \mathcal{F}$ by the collection of π-cartesian edges lying over edges in E, while the E-limit is given by the full sub-∞-category of sections $s : \mathcal{I} \longrightarrow \int^{\mathcal{I}} \mathcal{F}$ which send every edge in E to a π-cartesian edge. In this paper we will simply take this as the *definition* of E-limits and E-colimits, though we note that the notion of an E-(co)limit can be defined abstractly for diagrams taking values in an arbitrary $(\infty, 2)$-category: this is part of current work in progress [8] to construct a convenient framework for (op)lax (co)limits in an $(\infty, 2)$-categorical setting which exhibits properties similar to those familiar from the $(\infty, 1)$-categorical context, such as preservation by restriction along a suitable analogue of cofinal maps.

We mentioned above that presheaves of ∞-categories are rarely given explicitly. A notable exception to this statement is the situation in which the ∞-categories in question are all presented by *model categories*, and our presheaf $\mathcal{F} : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty$ comes from a (pseudo-)functor $\mathcal{M} : \mathcal{I} \longrightarrow \mathrm{ModCat}$ to the $(2, 1)$-category of model categories and Quillen adjunctions. Such functors are also known as *Quillen presheaves* (see, e.g., [12], [2]). Given a Quillen presheaf $\mathcal{M} : \mathcal{I} \longrightarrow \mathrm{ModCat}$, we may associate to it an ∞-categorical presheaf $\mathcal{M}_\infty^{\mathrm{R}} : \mathrm{N}(\mathcal{I})^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty$ by post-composing with the functor $\mathrm{ModCat} \longrightarrow \mathrm{Cat}_\infty$ which associates to each model category $\mathcal{M}$ its underlying ∞-category $\mathcal{M}_\infty$, and to each Quillen adjunction $\mathcal{L} : \mathcal{M} \rightleftarrows \mathcal{N} : \mathcal{R}$ the associated right derived functor $\mathbb{R}\mathcal{R} : \mathcal{N}_\infty \longrightarrow \mathcal{M}_\infty$. By the work of Hinich [10, §2] one can construct the cartesian fibration $\int^{\mathcal{I}} \mathcal{M}_\infty^{\mathrm{R}} \longrightarrow \mathrm{N}(\mathcal{I})$ classifying this presheaf by taking the nerve of the full subcategory of $\int^{\mathcal{I}} \mathcal{M}$ spanned by the fiberwise fibrant objects and localizing it with respect to fiberwise weak equivalences. In this case it is natural to ask if one can present the oplax limit ∞-category of $\mathcal{M}_\infty^{\mathrm{R}}$ via a suitable model structure on the 1-categorical oplax limit of the functor $\mathcal{M}^{\mathrm{R}} : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathrm{Cat}$ obtained by forgetting the model structure and keeping just the underlying right adjoints. We note that this oplax limit is also the lax limit of the functor $\mathcal{M}^{\mathrm{L}} : \mathcal{I} \longrightarrow \mathrm{Cat}$ obtained by keeping only the underlying left adjoints. Indeed, the diagram of categories and adjunctions underlying $\mathcal{M}$ can be encoded by a functor

$$\pi : \int \mathcal{M} := \int^{\mathcal{I}} \mathcal{M}^{\mathrm{R}} = \int_{\mathcal{I}} \mathcal{M}^{\mathrm{L}} \longrightarrow \mathcal{I}$$

which is simultaneously the cartesian fibration classified by $\mathcal{M}^{\mathrm{R}}$ and the cocartesian fi-

bration classified by $\mathcal{M}^{\mathrm{L}}$. The category

$$\mathrm{Sec}_{\mathcal{I}}(\mathcal{M}) = \left\{ s : \mathcal{I} \longrightarrow \int \mathcal{M} \,\middle|\, \pi \circ s = \mathrm{Id} \right\}$$

of sections of π can be explicitly described as follows: the objects of $\mathrm{Sec}_{\mathcal{I}}(\mathcal{M})$ are given by collections $s(i) \in \mathcal{M}(i)$ for each $i \in \mathcal{I}$ together with maps $s_\alpha : \alpha_! s(i) \longrightarrow s(i')$ (equivalently, maps $s^\alpha : s(i) \longrightarrow \alpha^* s(i')$) for every morphism $\alpha : i \longrightarrow i'$ in $\mathcal{I}$, where $\alpha_! \dashv \alpha^*$ is the Quillen adjunction associated to α. This data is required to satisfy the usual compatibility conditions for each commutative triangle
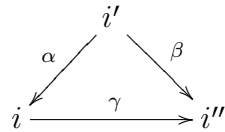
in $\mathcal{I}$. As shown in [2, Theorem 2.28], when each $\mathcal{M}(i)$ is a *combinatorial model category* one can endow $\mathrm{Sec}_{\mathcal{I}}(\mathcal{M})$ with the *projective model structure* $\mathrm{Sec}_{\mathcal{I}}^{\mathrm{proj}}(\mathcal{M})$, in which a map $T : s \longrightarrow s'$ is a weak equivalence/fibration if and only if $T(i) : s(i) \longrightarrow s'(i)$ is a weak equivalence/fibration for every i (here it is actually enough to assume that each $\mathcal{M}(i)$ is cofibrantly generated, see Proposition 3.2 below). One may then phrase the question eluded to above more formally as follows: is the model category $\mathrm{Set}_{\mathcal{I}}^{\mathrm{proj}}(\mathcal{M})$ a presentation of the ∞-categorical lax limit of $\mathcal{M}_\infty^{\mathrm{R}}$? We note that this would imply in particular that every ∞-categorical section of $\int^{\mathcal{I}} \mathcal{M}_\infty^{\mathrm{R}} \longrightarrow \mathrm{N}(\mathcal{I})$ can be represented by an honest section of $\int^{\mathcal{I}} \mathcal{M} \longrightarrow \mathcal{I}$, a phenomenon also known as *strictification*. In the setting of Quillen presheaves, the strictification question was first posed and conjectured to admit a positive answer in the work of Simpson–Hirschowitz, see [12, Conjecture 18.3].

A slightly more structured case which is more convenient to handle is when $\mathcal{M}$ is a *simplicial Quillen presheaf*, that is, a diagram taking values in the $(2,1)$-category ModCat_Δ of simplicial model categories and simplicial Quillen adjunctions. In this case, if each $\mathcal{M}(i)$ is also combinatorial, then the model category $\mathrm{Sec}_{\mathcal{I}}^{\mathrm{proj}}(\mathcal{M})$ inherits a (fiberwise) simplicial structure. We then have a relatively direct access to the underlying ∞-category of $\mathrm{Sec}_{\mathcal{I}}^{\mathrm{proj}}(\mathcal{M})$ by taking the coherent nerve of the full simplicial subcategory $\mathrm{Sec}_{\mathcal{I}}^\circ(\mathcal{M}) \subseteq \mathrm{Sec}_{\mathcal{I}}(\mathcal{M})$ spanned by fibrant-cofibrant sections.

Our main result in this paper is that for simplicial Quillen presheaves taking values in combinatorial model categories, the model category $\mathrm{Sec}_{\mathcal{I}}^{\mathrm{proj}}(\mathcal{M})$ is indeed a presentation of the ∞-categorical oplax limit of $\mathcal{M}_\infty^{\mathrm{R}}$ (equivalently, the lax limit of $\mathcal{M}_\infty^{\mathrm{L}}$). In fact, we prove a more general statement where $\mathcal{I}$ is allowed to be a *simplicial category*. In this case it is not a-priori clear what a diagram of model categories indexed by $\mathcal{I}$ actually means. Though we have an idea of how this should be defined in general, we chose in this paper to restrict attention to diagrams which factor through the *homotopy category* of $\mathcal{I}$. In particular, given a simplicial Quillen presheaf $\mathcal{M} : \mathrm{Ho}(\mathcal{I}) \longrightarrow \mathrm{ModCat}_\Delta$, a simplicial section s of $\mathcal{M}$ along $\mathcal{I}$ is given by the data of:

- an object $s(i) \in \mathcal{M}(i)$ for every $i \in \mathcal{I}$;

- a map $s_\alpha : \mathrm{Map}_{\mathbb{J}}^\alpha(i, i') \otimes \alpha_! s(i) \longrightarrow s(i')$ in $\mathcal{M}(i')$ for every $i, i' \in \mathbb{J}$ and every morphism $\alpha : i \longrightarrow i'$ in $\mathrm{Ho}(\mathbb{J})$, where $\mathrm{Map}_{\mathbb{J}}^\alpha(i, i') \subseteq \mathrm{Map}_{\mathbb{J}}(i, i')$ is the component of $\mathrm{Map}_{\mathbb{J}}(i, i')$ determined by α.

One can then show that the projective model structure on the category $\mathrm{Sec}_{\mathbb{J}}(\mathcal{M})$ of simplicial sections still exists in this more generalized setting (see Proposition 3.2 below). Our main result in this paper can then be formulated as follows:

1.1. THEOREM. *Let $\mathbb{J}$ be a fibrant simplicial category. Let $\mathcal{M} : \mathbb{J} \longrightarrow \mathrm{Ho}(\mathbb{J}) \longrightarrow \mathrm{ModCat}_\Delta$ a simplicial Quillen presheaf which factors through $\mathrm{Ho}(\mathbb{J})$ and takes values in combinatorial model categories. Then $\mathrm{Sec}_{\mathbb{J}}^{\mathrm{proj}}(\mathcal{M})$ is a presentation of the ∞-categorical oplax limit of $\mathcal{M}_\infty^{\mathrm{R}} : \mathrm{N}(\mathbb{J})^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty$ (or, equivalently, the lax limit of $\mathcal{M}_\infty^{\mathrm{L}} : \mathrm{N}(\mathbb{J}) \longrightarrow \mathrm{Cat}_\infty$).*

1.2. REMARK. In Theorem 1.1, the condition that $\mathbb{J}$ is fibrant is only needed to ensure that $\mathrm{N}(\mathbb{J})$ has the correct type, but is otherwise superfluous (see Proposition 3.7).

Given a subset E of morphisms in $\mathbb{J}$, Theorem 1.1 suggests a potential model for the E-limit of $\mathcal{M}_\infty^{\mathrm{R}}$ via a suitable left Bousfield localization of $\mathrm{Set}^{\mathrm{proj}}(\mathcal{M})$, where the new fibrant objects are the old fibrant objects for which in addition the map

$$(1) \qquad\qquad\qquad s^\alpha : s(i) \longrightarrow \alpha^* s(i')$$

is a weak equivalence in $\mathcal{M}(i)$ for every $\alpha : i \longrightarrow i'$ which belongs to E. When this left Bousfield localization exists we call the resulting model structure the *E-cartesian model structure*, and denote it by $\mathrm{Sec}_{\mathbb{J},E}^{\mathrm{proj}}(\mathcal{M})$. In general, the desired localization will not exist as a model category, except in special circumstances, such as when $\mathrm{Sec}_{\mathbb{J}}^{\mathrm{proj}}(\mathcal{M})$ is left proper. In that respect we note that since colimits in $\mathrm{Sec}_{\mathbb{J}}(\mathcal{M})$ are computed levelwise and every projective cofibration is also a levelwise cofibration, the model category $\mathrm{Set}^{\mathrm{proj}}(\mathcal{M})$ is left proper as soon as each $\mathcal{M}(i)$ is left proper.

Even when the E-cartesian model structure does not exist, one may always consider the full simplicial subcategory $\mathrm{Sec}_{\mathbb{J},E}^\circ(\mathcal{M}) \subseteq \mathrm{Sec}_{\mathbb{J}}^\circ(\mathcal{M})$ spanned by those fibrant-cofibrant sections such that (1) is a weak equivalence for every $\alpha \in E$. We may then ask if the coherent nerve of $\mathrm{Sec}_{\mathbb{J},E}^\circ(\mathcal{M})$ is equivalent to the ∞-categorical E-limit of $\mathcal{M}_\infty^{\mathrm{R}}$. When the localized model structure exists this is equivalent to asking whether the model category $\mathrm{Set}_E^{\mathrm{proj}}(\mathcal{M})$ is a presentation of this ∞-category.

1.3. THEOREM. *In the situation of Theorem 1.1, suppose that E is any collection of maps in $\mathbb{J}$. Then the simplicial category $\mathrm{Sec}_{\mathbb{J},E}^\circ(\mathcal{M})$ is a model for the ∞-categorical E-limit of $\mathcal{M}_\infty^{\mathrm{R}}$. In particular, if $\mathcal{M}(i)$ is left proper for every $i \in \mathbb{J}$ then the E-cartesian model structure $\mathrm{Sec}_{\mathbb{J},E}^{\mathrm{proj}}(\mathcal{M})$ presents the ∞-categorical E-limit of $\mathcal{M}_\infty^{\mathrm{R}}$.*

We note that Theorem 1.1 essentially corresponds to Theorem 1.3 in the case where E is empty. Both these theorems are summarized in Corollary 4.4, whose proof occupies §4.

1.4. REMARK. Any combinatorial model category is Quillen equivalent to a simplicial left proper one [5], though not canonically. It is hence not a-priori clear if every diagram of combinatorial model categories can be replaced with a Quillen equivalent diagram of simplicial model categories and simplicial Quillen adjunction (though it seems rather likely that the argument of [5] can be made to work "in families", at least in special cases). By contrast, left proper combinatorial model categories can be functorially replaced with simplicial (and left proper) ones [6], and so if we already know that each $\mathcal{M}(i)$ is left proper than we may replace $\mathcal{M}$ with a Quillen equivalent diagram taking values in ModCat$_\Delta$.

1.5. EXAMPLE. Let $\mathcal{M}$ be a simplicial combinatorial model category and $\mathcal{T} : \mathcal{M} \rightleftarrows \mathcal{M} : \mathcal{U}$ a Quillen adjunction. To this data one can associate a model category $\mathrm{Sp}(\mathcal{M}, \mathcal{T})$ of spectrum objects with repsect to $\mathcal{T}$, see [13, Definition 1.1]. Its objects consist of sequences $(X_0, X_1, ...)$ in $\mathcal{M}$ together with structure maps $\mathcal{T}X_n \longrightarrow X_{n+1}$ for every $n \geq 0$. From the point of view of the present paper we may identify $\mathrm{Sp}(\mathcal{N}, \mathcal{T})$ with the category of sections of a Quillen presheaf $\mathbb{N} \longrightarrow$ ModCat which associates the model category $\mathcal{M}$ to every $n \in \mathbb{N}$ and the left Quillen functor $\mathcal{T}$ to each generating arrow $n \longrightarrow n+1$ in $\mathbb{N}$. This category can be endowed with the projective model structure defined as above, which also coincides with the model structure of the same name in [13]. By Theorem 1.1 above the underlying ∞-category associated to this model structure is the oplax limit of the tower

$$\cdots \xrightarrow{\mathcal{U}} \mathcal{M} \xrightarrow{\mathcal{U}} \mathcal{M} \xrightarrow{\mathcal{U}} \mathcal{M}.$$

If $\mathcal{M}$ is left proper then one can consider the left Bousfield localization of the projective model structure, in which the new fibrant objects are the levelwise fibrant cartesian sections, namely, those $\mathcal{T}$-spectrum objects $(X_0, X_1, ...)$ for which each X_n is fibrant and the adjoint structure maps $X_n \longrightarrow \mathcal{U}(X_{n+1})$ are weak equivalences. Such spectrum objects are also known as $\mathcal{U}$-spectra, and the resulting localized model structure is referred to as the associated *stable model structure* in [13]. This formalism generalizes the setting of classical stable homotopy theory in which $\mathcal{M}$ is the category of pointed simplicial sets (or topological spaces) and $\mathcal{T}$ is given by suspension, and can be used to describe similar type of homotopy theories, such as motivic spectra, see [16]. By Theorem 1.1 above the underlying ∞-category associated to the stable model structure is the homotopy limit of the tower

$$\cdots \xrightarrow{\mathcal{U}} \mathcal{M} \xrightarrow{\mathcal{U}} \mathcal{M} \xrightarrow{\mathcal{U}} \mathcal{M},$$

thus endowing it in particular with a universal characterization (cf. [16, Proposition 4.15]).

1.6. RELATION TO OTHER WORK. Results similar to Theorem 1.1 have appeared before in the literature. When the diagram $\mathcal{M} : \mathcal{I} \longrightarrow$ ModCat$_\Delta$ is constant the model category $\mathrm{Sec}_{\mathcal{I}}^{\mathrm{proj}}(\mathcal{M})$ reduces to the category $\mathcal{M}^{\mathcal{I}}$ of simplicial functors with the projective model structure, in which case it was proved by Lurie [15, Proposition 4.2.4.4] that $\mathcal{M}^{\mathcal{I}}$ models the ∞-category of functors $\mathrm{N}(\mathcal{I}) \longrightarrow \mathcal{M}_\infty$. Our proof of Theorem 1.1 is based on a similar approach to that of [15]. On the other hand, when $\mathcal{M}$ is not necessarily constant (nor simplicial or combinatorial) but $\mathcal{I}$ is an ordinary category and E contains all edges (i.e.,

the case of homotopy limits) then the coincidence with the ∞-categorical limit was proved by Bergner [4], though with very different methods from the present paper.

When $\mathcal{I}$ is an ordinary category and $\mathcal{M}$ is not necessarily simplicial or combinatorial a result similar to our main theorem was also recently established by Balzin [1] using yet another approach, based on earlier work of Simpson–Hirschowitz [12] about the strictification problem (though some errors were later discovered and corrected by Balzin, see the introduction of [1]). Balzin's main theorem concerns certain families of model categories indexed by a Reedy category, and his result for Quillen presheaves is obtained by passing to the category of simplices of $\mathcal{I}$. Our approach is somewhat more direct and yields, in particular, a shorter proof of that statement in the simplicial combinatorial case. On the other hand, when $\mathcal{I}$ is Reedy Balzin's result covers more general types of families of model categories, which are not necessarily Quillen presheaves.

1.7. Notation and convention. In this paper we use employ the higher categorical notations and conventions of [15]. We will denote by Set_Δ the category of simplicial sets and by Cat_Δ the category of simplicial categories. Both of these can be endowed with model structures presenting the theory of ∞-categories: the categorical model structure on Set_Δ due to Joyal [14] and the Dwyer-Kan model structure on Cat_Δ due to Bergner [3]. These two model structures are related via a Quillen equivalence

$$\mathfrak{C} : \mathrm{Set}_\Delta \rightleftarrows \mathrm{Cat}_\Delta : \mathrm{N},$$

where N is the coherent nerve functor and $\mathfrak{C}$ is its left adjoint, see [15, §2]. The weak equivalences in Cat_Δ are the *Dwyer-Kan equivalences*, that is, the simplicial functors which induce weak homotopy equivalences on mapping simplicial sets and are essentially surjective on homotopy categories. The fibrant objects in Cat_Δ are the simplicial categories whose mapping objects are Kan complexes. On the side of the categorical model structure on Set_Δ, the weak equivalences, also known as categorical equivalences, are the maps sent to Dwyer-Kan equivalences by $\mathfrak{C}$, and the fibrant objects are the ∞-*categories*, that is, the simplicial sets $\mathcal{C}$ which admit extensions along inner horns inclusions $\Lambda^n_i \hookrightarrow \Delta^n$. We will denote by Cat_∞ the coherent nerve of the (large) simplicial category whose objects are the ∞-categories and whose mapping spaces from $\mathcal{C}$ to $\mathcal{D}$ are the maximal Kan complexes of the usual mapping simplicial set from $\mathcal{C}$ to $\mathcal{D}$. Then Cat_∞ is a model for the ∞-category of small ∞-categories. It is also equivalent to the localization of Set_Δ by the collection of categorical equivalences and to the localization of Cat_Δ by the Dwyer-Kan equivalences.

2. Simplicial Quillen presheaves

Throughout this section we let $\mathcal{I}$ denote a fixed *ordinary* category. By a *simplicial Quillen presheaf* on $\mathcal{I}$ we will mean a (pseudo-)functor $\mathcal{M} : \mathcal{I} \longrightarrow \mathrm{ModCat}_\Delta$ to the category of simplicial model categories and simplicial Quillen adjunctions. This data is equivalent via the Grothendieck construction to the data of a functor $\int \mathcal{M} \longrightarrow \mathcal{I}$ which is both a cartesian

and cocartesian fibration, together with a simplicial structure on each fiber such that the cocartesian transition maps are simplicial left Quillen functors (this automatically implies that the cartesian transition maps are simplicial right Quillen functors). We note that in this situation the category $\int \mathcal{M}$ inherits a natural enrichment over the category Set_Δ of simplicial sets: for $i, j \in \mathcal{I}$ and objects $X \in \mathcal{M}(i), Y \in \mathcal{M}(j)$, the simplicial mapping space $\mathrm{Map}((i, X), (j, Y))$ is given by

$$(2) \qquad \mathrm{Map}((i, X), (j, Y)) = \coprod_{\alpha: i \to j} \mathrm{Map}_{\mathcal{M}(j)}(\alpha_! X, Y)$$

where the coproduct is taken over all maps $\alpha : i \longrightarrow j$ in $\mathcal{I}$ and $\alpha_! : \mathcal{M}(i) \longrightarrow \mathcal{M}(j)$ is the left Quillen transition functor associated to α. The resulting simplicial category is not fibrant in general. It will hence be useful to consider instead the full subcategory

$$\int^\circ \mathcal{M} \subseteq \int \mathcal{M}$$

consisting of all objects (i, X) such that X is fibrant and cofibrant in $\mathcal{M}(i)$. The simplicial category $\int^\circ \mathcal{M}$ is then fibrant in Cat_Δ with respect to the Dwyer-Kan model structure and we can pass to its coherent nerve

$$\mathcal{E} := \mathrm{N}\left(\int^\circ \mathcal{M} \right).$$

Then $\mathcal{E}$ is a (large) ∞-category carrying a natural map

$$p : \mathcal{E} \longrightarrow \mathrm{N}(\mathcal{I}).$$

Since $\mathcal{E}$ is an ∞-category and $\mathrm{N}(\mathcal{I})$ is the nerve of a 1-category the map p is automatically an inner fibration.

2.1. LEMMA. *Let* $(Y, j), (Z, k) \in \int^\circ \mathcal{M}$ *be objects. Let* $\beta : j \longrightarrow k$ *be a morphism in* $\mathcal{I}$ *and* $f : \beta_! Y \longrightarrow Z$ *a morphism in* $\mathcal{M}(k)$*. Then the edge of* $\mathcal{E}$ *corresponding to* (β, f) *is p-cartesian if and only if the adjoint map* $f^{\mathrm{ad}} : Y \longrightarrow \beta^* Z$ *is a weak equivalence in* $\mathcal{M}(j)$.

PROOF. First assume that f^{ad} is a weak equivalence. By the mapping space criteria for p-cartesian edges [15, Proposition 2.4.1.10] we need to show that the commutative diagram

$$(3) \qquad \begin{array}{ccc} \mathrm{Hom}_{\int \mathcal{M}}((i, X), (j, Y)) & \xrightarrow{(f, \beta)_*} & \mathrm{Hom}_{\int \mathcal{M}}((i, X), (k, Z)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{I}}(i, j) & \xrightarrow{\quad \beta_* \quad} & \mathrm{Hom}_{\mathcal{I}}(i, k) \end{array}$$

is homotopy cartesian for every $i \in \mathcal{I}$ and fibrant-cofibrant object $X \in \mathcal{M}(i)$. Considering the homotopy fibers of the vertical maps and using (2) it will suffice to show that for every morphism $\alpha : i \longrightarrow j$ in $\mathcal{I}$, the induced map

$$(\beta, f)_* : \mathrm{Map}_{\mathcal{M}(j)}(\alpha_! X, Y) \longrightarrow \mathrm{Map}_{\mathcal{M}(k)}(\beta_! \alpha_! X, Z)$$

is a weak homotopy equivalence of simplicial sets. We now observe that under the adjunction isomorphism

$$\text{(4)} \qquad \text{Map}_{\mathcal{M}(k)}(\beta_!\alpha_!X, Z) \cong \text{Map}_{\mathcal{M}(j)}(\alpha_!X, \beta^*Z)$$

the map $(\beta, f)_*$ is given by post-composing with $f^{\text{ad}} : Y \longrightarrow \beta^*Z$. Since f^{ad} is a weak equivalence between fibrant objects and $\alpha_!X$ is cofibrant we get that $(\beta, f)_*$ is indeed a weak homotopy equivalence of simplicial sets.

In the other direction, assume that the edge associated to (β, j) is p-cartesian. Then for every $i \in \mathcal{I}$ and every fibrant-cofibrant $X \in \mathcal{M}(i)$ the square (3) is homotopy cartesian, and hence for every $\alpha : i \longrightarrow j$ the map $(\beta, f)_*$ above is a weak homotopy equivalence of simplicial sets. Taking $i = j$ and $\alpha = \text{Id}$ we conclude that the map

$$\text{Map}_{\mathcal{M}(j)}(X, Y) \longrightarrow \text{Map}_{\mathcal{M}(j)}(X, \beta^*Z)$$

obtained by post-composing with $f^{\text{ad}} : Y \longrightarrow \beta^*Z$ is a weak homotopy equivalence of simplicial sets for every fibrant-cofibrant $X \in \mathcal{M}(j)$. Since Y and β^*Z are fibrant it follows that f^{ad} is a weak equivalence, as desired. ∎

2.2. COROLLARY. *The map p is a cartesian fibration.*

PROOF. In light of Lemma 2.1 it will suffice to show that for every morphism $\beta : j \longrightarrow k$ in $\mathcal{I}$ and for every fibrant-cofibrant object $Z \in \mathcal{M}(k)$ there exists a fibrant-cofibrant object $Y \in \mathcal{M}(j)$ admitting a weak equivalence $Y \xrightarrow{\simeq} \beta^*Z$. But this is clear since we can choose a trivial fibration $Y \xrightarrow{\simeq} \beta^*Z$ such that Y is cofibrant. ∎

2.3. REMARK. Using a dual argument one can show that the map $p : \mathcal{E} \longrightarrow \text{N}(\mathcal{I})$ is also a *cocartesian fibration*.

3. Categories of simplicial sections

In this paper we are interested in simplicial Quillen presheaves on simplicial categories, but restrict attention (mostly for simplicity) to those which factor through the corresponding homotopy category. In such a situation, it is convenient to allow for a bit of extra flexibility by considering an arbitrary functor $\varphi : \mathcal{J} \longrightarrow \mathcal{I}$, where $\mathcal{J}$ is a simplicial category and $\mathcal{I}$ is an ordinary category (which is not necessarily the homotopy category of $\mathcal{J}$). We may then consider simplicial Quillen presheaves parameterized by $\mathcal{I}$, and take simplicial sections along $\mathcal{J}$. We note that this does not result in true additional generality, since in any case the map φ factors as $\mathcal{J} \longrightarrow \text{Ho}(\mathcal{J}) \longrightarrow \mathcal{I}$, and if we start with a Quillen presheaf on $\mathcal{I}$ then its sections along $\mathcal{J}$ are the same whether we consider them as parameterized by $\mathcal{I}$ or $\text{Ho}(\mathcal{J})$.

To set up the stage let us hence fix an ordinary category $\mathcal{I}$ and a simplicial category $\mathcal{J}$ equipped with a functor $\varphi : \mathcal{J} \longrightarrow \mathcal{I}$. Given a simplicial Quillen presheaf $\mathcal{M} : \mathcal{I} \longrightarrow \text{ModCat}_\Delta$, we define $\text{Sec}_{\mathcal{J}}(\mathcal{M})$ to be the category of simplicial functors $s : \mathcal{J} \longrightarrow \int \mathcal{M}$ over $\mathcal{I}$, with respect to the simplicial enrichment of $\int \mathcal{M}$ described in §2. Here we use the

notation $\mathrm{Sec}_{\mathcal{J}}(\mathcal{M})$ to indicate that we think of these functors as *sections* of $\mathcal{M}$ along $\mathcal{J}$. More explicitly, an $s \in \mathrm{Sec}_{\mathcal{J}}(\mathcal{M})$ is given by the data of an object $s(j) \in \mathcal{M}(\varphi(j))$ for each $j \in \mathcal{J}$, and for every $j, j' \in \mathcal{J}$ and map $\alpha : \varphi(j) \longrightarrow \varphi(j')$ in $\mathcal{I}$, a map in $\mathcal{M}(\varphi(j'))$ of the form

$$(5) \qquad\qquad s_\alpha : \mathrm{Map}_{\mathcal{J}}^\alpha(j, j') \otimes \alpha_! s(j) \longrightarrow s(j'),$$

where $\mathrm{Map}_{\mathcal{J}}^\alpha(j, j') \subseteq \mathrm{Map}_{\mathcal{J}}(j, j')$ denotes the pre-image of $\alpha : \varphi(j) \longrightarrow \varphi(j')$ in $\mathrm{Map}_{\mathcal{J}}(j, j')$ (which is a union of connected components since $\mathcal{I}$ is mapping-wise discrete). Equivalently, this data can also be encoded via the adjoint map

$$(6) \qquad\qquad s^\alpha : s(j) \longrightarrow \alpha^* s(j')^{\mathrm{Map}_{\mathcal{J}}^\alpha(j,j')}.$$

Given a vertex $\beta \in \mathrm{Map}_{\mathcal{J}}(j, j')$ lying above $\alpha : \varphi(j) \longrightarrow \varphi(j')$, we will denote by s_β the composed map

$$(7) \qquad\qquad s_\beta : \alpha_! s(j) \cong \Delta^0 \otimes \alpha_! s(j) \xrightarrow{\beta_*} \mathrm{Map}_{\mathcal{J}}^\alpha(j, j') \otimes \alpha_! s(j) \longrightarrow s(j').$$

3.1. DEFINITION. Let $T : s \longrightarrow t$ be a map in $\mathrm{Sec}_{\mathcal{J}}(\mathcal{M})$. We will say that T is a *levelwise weak equivalence* (resp. *fibration*, *cofibration*) if $T(j) : s(j) \longrightarrow t(j)$ is a weak equivalence (resp. fibration, cofibration) in $\mathcal{M}(j)$ for every j.

3.2. PROPOSITION. *Let $\mathcal{M} : \mathcal{I} \longrightarrow \mathrm{ModCat}_\Delta$ be a simplicial Quillen presheaf such that each $\mathcal{M}(i)$ is compactly generated. Then there exists a compactly generated simplicial model structure $\mathrm{Sec}_{\mathcal{J}}^{\mathrm{proj}}(\mathcal{M})$, which we will call the projective model structure, such that the weak equivalences/fibrations are the levelwise weak equivalences/fibrations, and cofibrations are the maps which satisfy the left lifting property with respect to levelwise trivial fibrations. In addition, if each $\mathcal{M}(i)$ is combinatorial then so is $\mathrm{Sec}_{\mathcal{J}}^{\mathrm{proj}}(\mathcal{M})$.*

PROOF. The proof is completely standard, but we recall the main details for the convenience of the reader. For each $j \in \mathcal{J}$ and $X \in \mathcal{M}(\varphi(j))$ let us denote by $s_{j,X} \in \mathrm{Sec}_{\mathcal{J}}(\mathcal{M})$ the section given by

$$(8) \qquad\qquad s_{j,X}(j') = \coprod_{\alpha : \varphi(j) \to \varphi(j')} \mathrm{Map}_{\mathcal{J}}^\alpha(j, j') \otimes \alpha_! X \in \mathcal{M}(\varphi(j'))$$

where the coproduct is taken over all maps $\alpha : \varphi(j) \longrightarrow \varphi(j')$ in $\mathcal{I}$ and $\mathrm{Map}_{\mathcal{J}}^\alpha(j, j')$ is as above. Then for a fixed $j \in \mathcal{J}$ the association $X \mapsto s_{j,X}$ is left adjoint to the evaluation functor $t \mapsto t(j)$. In addition, if $X \longrightarrow Y$ is a cofibration (resp. trivial cofibration) in $\mathcal{M}(\varphi(j))$ then the induced map

$$s_{j,X} \longrightarrow s_{j,Y}$$

is a levelwise cofibration (resp. trivial cofibration). Now for each $j \in \mathcal{J}$, let I_j, J_j be sets of generating cofibrations and trivial cofibrations respectively for $\mathcal{M}(\varphi(j))$ which permit the small object argument. Define I to be the union of the images of I_j under the functors $X \mapsto s_{j,X}$, and J to be the union of the images of J_j under the functors $X \mapsto s_{j,X}$. By adjunction we then have that:

1. a map $T : s \longrightarrow t$ satisfies the right lifting property with respect to J if and only if it is a levelwise fibration;

2. a map $T : s \longrightarrow t$ satisfies the right lifting property with respect to I if and only if it is a levelwise trivial fibration.

We now apply [11, Theorem 11.3.1] (with 4(b)) to deduce that $\mathrm{Sec}_{\mathcal{J}}(\mathcal{M})$ carries a model structure cofibrantly generated by I and J, whose weak equivalences are the levelwise weak equivalences. We endow $\mathrm{Sec}_{\mathcal{J}}(\mathcal{M})$ with the levelwise simplicial structure $(K \otimes s)(i) = K \otimes s(i)$ and $(s^K)(i) = s(i)^K$, with the induced action on the structure maps s_α / s^α for $\alpha : i \longrightarrow i'$ in $\mathcal{I}$ (see (5) and (6)) using the simplicial compatibility of the Quillen adjunction $\alpha_! \dashv \alpha^*$. The resulting simplicial structure on $\mathrm{Sec}_{\mathcal{J}}(\mathcal{M})$ is then seen to be compatible with the model structure just constructed by inspecting the cotensoring axiom and using the fact that fibrations and trivial fibrations are defined levelwise.

To obtain the last statement, we note that since the evaluation functors $t \mapsto t(j)$ preserve all colimits, we have that $\mathrm{Sec}_{\mathcal{J}}(\mathcal{M})$ is monadic over $\prod_{j \in \mathrm{Ob}(J)} \mathcal{M}(j)$ with monad $t \mapsto \oplus_{j \in \mathcal{J}} s_{j, t(j)}$. In particular, if each $\mathcal{M}(i)$ is presentable as an ordinary category then $\mathrm{Sec}_{\mathcal{J}}(\mathcal{M})$ is presentable as well.   ■

We shall now establish a few basic properties of the categories $\mathrm{Sec}_{\mathcal{J}}(\mathcal{M})$. We begin with some terminology.

3.3. NOTATION. Let $\mathcal{C}$ be a simplicial category. By a morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ we will simply mean a vertex $f \in (\mathrm{Map}_{\mathcal{C}}(X, Y))_0$. We will say that two morphisms $f, g : X \longrightarrow Y$ in $\mathcal{C}$ are *weakly homotopic* if they are in the same connected component of $\mathrm{Map}_{\mathcal{C}}(X, Y)$. We will say that $f : X \longrightarrow Y$ is a *weak homotopy equivalence* if there exists a morphism $g : Y \longrightarrow X$ such that $f \circ g$ and $g \circ f$ are weakly homotopic to the respective identities. This notion coincides with the notion of equivalence in the ∞-category $\mathrm{N}(\mathcal{C}^{\mathrm{fib}})$ where $\mathcal{C}^{\mathrm{fib}}$ denotes a fibrant replacement for $\mathcal{C}$ and N is the coherent nerve functor.

3.4. REMARK. Let $\mathcal{C}$ be a simplicial *model* category and $f, g : X \longrightarrow Y$ a pair of weakly homotopic maps in the sense of Notation 3.3 (with respect to the underlying simplicial structure). If either X is cofibrant or Y is fibrant then f, g will have the same image in $\mathrm{Ho}(\mathcal{C})$. Indeed, in the former case $X \times \Delta^1$ constitutes a cylinder object for X and in the latter case Y^{Δ^1} constitutes a path object for Y. Similarly, if X is cofibrant *and* Y is fibrant then f, g are weakly homotopic if and only if they have the same image in $\mathrm{Ho}(\mathcal{C})$. Since any model category is saturated as a relative category, we get that if $f : X \longrightarrow Y$ is a weak homotopy equivalence between fibrant objects (or cofibrant objects) then f is a weak equivalence.

3.5. REMARK. Any functor of simplicial categories $\mathcal{C} \longrightarrow \mathcal{D}$ sends weakly homotopic pairs of maps to weakly homotopic pairs of maps. In particular, if $\mathcal{C} \underset{\mathcal{R}}{\overset{\mathcal{L}}{\rightleftarrows}} \mathcal{D}$ is a simplicial Quillen adjunction between simplicial model categories then both $\mathcal{L}$ and $\mathcal{R}$ refine to simplicial functors (since $\mathcal{L}$ preserves tensoring with simplicial sets and $\mathcal{R}$ cotensoring) and hence preserve weakly homotopic pairs of maps.

3.6. Lemma. *Keeping the notations above, let $s \in \mathrm{Sec}^{\mathrm{proj}}_{\mathcal{J}}(\mathcal{M})$ a fibrant object. Let $\gamma : j \longrightarrow j'$ be a weak homotopy equivalence in $\mathcal{J}$ and let $\alpha = \varphi(\gamma)$ be the corresponding map in $\mathcal{I}$. Then the map*

$$s^\gamma : s(j) \longrightarrow \alpha^* s(j'),$$

adjoint to the map $s_\gamma : \alpha_! s(j) \longrightarrow s(j')$ of (7), is a weak equivalence in $\mathcal{M}(j)$.

Proof. Since both $s(j)$ and $\alpha^* s(j')$ are fibrant it will be enough to prove that s^γ is a weak homotopy equivalence (see Remark 3.4). Since γ is a weak homotopy equivalence there exists a $\delta : j' \longrightarrow j$ such that $\delta \circ \gamma$ and $\gamma \circ \delta$ are weakly homotopic to the corresponding identity maps. In particular, $\beta = \varphi(\delta)$ is an inverse for α and so both α and β are isomorphisms. This implies that $\alpha_! \dashv \alpha^*$ and $\beta_! \dashv \beta^*$ are Quillen equivalences such that both their composites are (naturally isomorphic to) the identity Quillen equivalence. Now let

$$s^\delta : s(j') \longrightarrow \beta^* s(j)$$

be the adjoint to the map $s_\delta : \beta_! s(j') \longrightarrow s(j)$ determined by the vertex $\delta \in \mathrm{Map}^\beta_{\mathcal{J}}(j', j)$. Since $\delta \circ \gamma : j \longrightarrow j$ and $\gamma \circ \delta : j' \longrightarrow j'$ are weakly homotopic to the respective identity maps, we get from Remark 3.5 that the composites

$$(9) \qquad s^{\delta \circ \gamma} : s(j) \xrightarrow{\ s^\gamma\ } \alpha^* s(j') \xrightarrow{\ \alpha^* s^\delta\ } \alpha^* \beta^* s(j) \cong s(j)$$

and

$$(10) \qquad s^{\gamma \circ \delta} : s(j') \xrightarrow{\ s^\delta\ } \beta^* s(j) \xrightarrow{\ \beta^* s^\gamma\ } \beta^* \alpha^* s(j') \cong s(j')$$

are also weakly homotopic to the respective identities. Applying α^* to (10) and using again Remark 3.5 we may now conclude that $\alpha^* s^\delta : \alpha^* s(j') \longrightarrow \alpha^* \beta^* s(j) \cong s(j)$ is a weak homotopy inverse to s^γ, and so s^γ is a weak homotopy equivalence, as desired. ∎

With $\varphi : \mathcal{J} \longrightarrow \mathcal{I}$ as above, consider now a diagram of simplicial categories of the form

$$
\begin{array}{ccc}
\mathcal{J} & \xrightarrow{\ s\ } & \int \mathcal{M} \\
\downarrow{\scriptstyle \psi} & & \downarrow \\
\mathcal{J}' & \xrightarrow{\ \varphi'\ } & \mathcal{I}
\end{array}
$$

such that $\varphi = \varphi' \circ \psi$. The *enriched relative left Kan extension* $\psi_! s : \mathcal{J}' \longrightarrow \int \mathcal{M}$ of s is the coequalizer

$$\psi_! s(j') = \mathrm{coeq}\left[\coprod_{\substack{j_1, j_2 \in \mathcal{J} \\ \beta : \varphi(j_1) \to \varphi(j_2) \\ \gamma : \varphi(j_2) \to \varphi'(j')}} \left[\mathrm{Map}^\beta_{\mathcal{J}}(j_1, j_2) \times \mathrm{Map}^\gamma_{\mathcal{J}'}(\psi(j_2), j') \otimes \gamma_! \beta_! s(j_1) \right] \rightrightarrows \right.
$$

$$\coprod_{\substack{j \in \mathcal{J} \\ \alpha: \varphi(j) \to \varphi'(j')}} \mathrm{Map}^{\alpha}_{\mathcal{J}'}(\psi(j), j') \otimes \alpha_! s(j) \Bigg].$$

We note that the functor $\psi_! : \mathrm{Sec}_{\mathcal{J}}(\mathcal{M}) \longrightarrow \mathrm{Sec}_{\mathcal{J}'}(\mathcal{M})$ is left adjoint to the restriction functor $\psi^* : \mathrm{Sec}_{\mathcal{J}'}(\mathcal{M}) \longrightarrow \mathrm{Sec}_{\mathcal{J}}(\mathcal{M})$. In particular, for every $s : \mathcal{J} \longrightarrow \int \mathcal{M}$ and $t : \mathcal{J}' \longrightarrow \int \mathcal{M}$ one has a canonical isomorphism
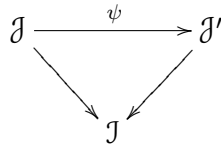
$$\mathrm{Map}_{\mathrm{Sec}_{\mathcal{J}'}(\mathcal{M})}(\psi_! s, t) \cong \mathrm{Map}_{\mathrm{Sec}_{\mathcal{J}}(\mathcal{M})}(s, \psi^* t).$$

In addition, $\psi_!$ preserves the "free sections" (8) in the sense that one has a canonical isomorphism

(11) $$\psi_! s_{j,X} \cong s_{\psi(j),X}$$

for every $j \in \mathcal{J}$ and $X \in \mathcal{M}(j)$. The following proposition is a generalization of [15, A.3.3.8].

3.7. PROPOSITION. *Let*

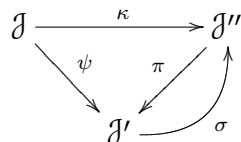*be a map of simplicial categories over the ordinary category* $\mathfrak{I}$. *Then the adjunction*

$$\mathrm{Sec}_{\mathcal{J}}^{\mathrm{proj}}(\mathcal{M}) \underset{\psi^*}{\overset{\psi_!}{\rightleftarrows}} \mathrm{Sec}_{\mathcal{J}'}^{\mathrm{proj}}(\mathcal{M}).$$

is a Quillen adjunction. Furthermore, if ψ is a Dwyer-Kan equivalence of simplicial categories then this adjunction is a Quillen *equivalence.*

PROOF. The fact that ψ^* is a right Quillen functor is immediate since fibrations and trivial fibrations are defined levelwise. The main part is checking that $\psi_! \dashv \psi^*$ is a Quillen equivalence when ψ is a Dwyer-Kan equivalence. Let us say that ψ is a *local trivial cofibration over* $\mathfrak{I}$ if for every $j, k \in \mathcal{J}$ and $\alpha : \varphi(j) \longrightarrow \varphi(k)$ in $\mathfrak{I}$, the induced map

$$\mathrm{Map}^{\alpha}_{\mathcal{J}}(j, k) \hookrightarrow \mathrm{Map}^{\alpha}_{\mathcal{J}'}(\psi(j), \psi(k))$$

is a trivial cofibration of simplicial sets. As in the proof of [15, A.3.3.8], we begin by reducing to the case where ψ is a local trivial cofibration over $\mathfrak{I}$. To perform this reduction, factor the induced map $\mathcal{J} \coprod \mathcal{J}' \longrightarrow \mathcal{J}'$ as a cofibration $\kappa \coprod \sigma : \mathcal{J} \coprod \mathcal{J}' \longrightarrow \mathcal{J}''$ in Cat_{Δ} followed by a trivial fibration $\pi : \mathcal{J}'' \longrightarrow \mathcal{J}'$. Note that by construction the map π is equipped with a section $\sigma : \mathcal{J}' \longrightarrow \mathcal{J}''$. We obtain a commutative diagram

of simplicial categories over $\mathcal{I}$, and by the 2-out-of-3 rule one can deduce that all maps appearing in this diagram are Dwyer-Kan equivalences. Since Quillen equivalences are closed under 2-out-of-3, it will be enough to prove the theorem for κ and σ. But κ and σ are both local trivial cofibrations over $\mathcal{I}$. Hence we can assume without loss of generality that ψ is a local trivial cofibration over $\mathcal{I}$.

Our next step is to observe that the functor

$$\psi^* : \mathrm{Sec}_{\mathcal{I}'}^{\mathrm{proj}}(\mathcal{M}) \longrightarrow \mathrm{Sec}_{\mathcal{I}}^{\mathrm{proj}}(\mathcal{M})$$

preserves all weak equivalences, and, in view of Lemma 3.6 and the assumption that ψ is a Dwyer-Kan equivalence (and in particular essentially surjective), also detects weak equivalences between fibrant objects. It is hence enough to show that for every cofibrant object $s \in \mathrm{Sec}_{\mathcal{I}}^{\mathrm{proj}}(\mathcal{M})$ the unit map

$$s \longrightarrow \psi^* \psi_! s$$

is a weak equivalence. Proceeding as in the proof of Proposition A.3.3.8 of [15], we will say that a map $T : s \longrightarrow t$ in $\mathrm{Sec}_{\mathcal{I}}^{\mathrm{proj}}(\mathcal{M})$ is *good* if for every $k \in \mathcal{I}$ the induced map

$$t(k) \coprod_{s(k)} \psi^* \psi_! s(k) \longrightarrow \psi^* \psi_! t(k)$$

is a trivial cofibration in $\mathcal{M}(\varphi(k))$. It will then be enough to prove that every cofibration is good. We now note that since both $\psi_!$ and ψ^* both preserves colimits (which are computed levelwise for sections) the collection of good maps is weakly saturated. It will hence suffice to prove that every generating cofibration is good. Let $j \in \mathcal{I}$ be an object and $X \hookrightarrow Y$ a generating cofibration of $\mathcal{M}(\varphi(j))$. We wish to show that the map $s_{j,X} \hookrightarrow s_{j,Y}$ in $\mathrm{Sec}_{\mathcal{I}}(\mathcal{M})$ is good. In light of (11), what we need to check is that for every $k \in \mathcal{I}$ and every $\alpha : \varphi(j) \longrightarrow \varphi(k)$ the induced map

$$\left[\mathrm{Map}_{\mathcal{I}}^{\alpha}(j,k) \otimes \alpha_! Y \right] \coprod_{\mathrm{Map}_{\mathcal{I}}^{\alpha}(j,k) \otimes \alpha_! X} \left[\mathrm{Map}_{\mathcal{I}'}^{\alpha}(\psi(j), \psi(k)) \otimes \alpha_! X \right] \longrightarrow \mathrm{Map}_{\mathcal{I}'}^{\alpha}(\psi(j), \psi(k)) \otimes \alpha_! Y$$

is a trivial cofibration in $\mathcal{M}(\varphi(k))$. But this follows from the pushout-product axiom for the simplicial structure on $\mathcal{M}(\varphi(k))$ since $\alpha_! X \hookrightarrow \alpha_! Y$ is a cofibration in $\mathcal{M}(\varphi(k))$ and $\mathrm{Map}_{\mathcal{I}}^{\alpha}(j,k) \longrightarrow \mathrm{Map}_{\mathcal{I}'}^{\alpha}(\psi(j), \psi(k))$ is assumed to be a trivial cofibration of simplicial sets (that is, ψ is assumed to be a local trivial cofibration). $\blacksquare$

4. Proof of the main theorem

In this final section we will formulate and prove our main theorem in the somewhat more flexible setting of §3. We hence fix an ordinary category $\mathcal{I}$, a simplicial Quillen presheaf $\mathcal{M} : \mathcal{I} \longrightarrow \mathrm{ModCat}_{\Delta}$ valued in combinatorial model categories, and a map of simplicial categories $\varphi : \mathcal{J} \longrightarrow \mathcal{I}$. We will denote by $\mathrm{Sec}_{\mathcal{J}}^{\circ}(\mathcal{M}) \subseteq \mathrm{Sec}_{\mathcal{J}}(\mathcal{M})$ the full subcategory spanned

by the objects that are fibrant and cofibrant with respect to the projective model structure of Proposition 3.2.

Let $(\mathrm{Set}_\Delta^+)_{/\mathrm{N}(\mathcal{J})}$ denote the category of marked simplicial sets over $\mathrm{N}(\mathcal{J})$ (whose objects consist of marked simplicial sets (X, E) equipped with an unmarked map $X \longrightarrow \mathrm{N}(\mathcal{J})$). Given a cartesian fibration $q : X \longrightarrow \mathrm{N}(\mathcal{J})$ we will denote by $X^\natural$ the marked simplicial set whose underlying simplicial set is X is whose marked edges are the q-cartesian edges. We may then naturally consider $X^\natural$ as an object of $(\mathrm{Set}_\Delta^+)_{/\mathrm{N}(\mathcal{J})}$. Following Lurie, we will endow $(\mathrm{Set}_\Delta^+)_{/\mathrm{N}(\mathcal{J})}$ with the *cartesian model structure*, in which the fibrant objects are precisely those of the form $X^\natural$ for some Cartersian fibration $X \longrightarrow \mathrm{N}(\mathcal{J})$ [15, Proposition 3.1.3.7, Proposition 3.1.4.1]. In light of Corollary 2.2 we may then view $\mathcal{E}^\natural = (\mathrm{N}(\int^\circ \mathcal{M}))^\natural$ as a fibrant object in $(\mathrm{Set}_\Delta^+)_{/\mathrm{N}(\mathcal{J})}$. We note that this is slightly abusive since $\mathcal{E}^\natural$ is a *large* marked simplicial set. We will address this subtlety more carefully below.

The cartesian model structure $(\mathrm{Set}_\Delta^+)_{/\mathrm{N}(\mathcal{J})}$ is tensored and cotensored over the category of marked simplicial sets Set_Δ^+ endowed the marked categorical model structure (that is, the cartesian model structure over the point). We note that with this model structure the category of marked simplicial sets presents the ∞-category Cat_∞, and the functor which forgets the marked edges provides a right Quillen equivalence to the Joyal model structure on simplicial sets. Given two objects $X, Y \in (\mathrm{Set}_\Delta^+)_{/\mathrm{N}(\mathcal{J})}$ such that Y is fibrant we have a fibrant mapping object $\mathrm{Map}_{\mathrm{N}(\mathcal{J})}^+(X, Y) \in \mathrm{Set}_\Delta^+$. Adapting the notation of [15, §3.1.3], we will denote by $\mathrm{Map}_{\mathrm{N}(\mathcal{J})}^\flat(X, Y)$ the underlying simplicial set of $\mathrm{Map}_{\mathrm{N}(\mathcal{J})}^+(X, Y)$, which is an ∞-category (see [15, Remark 3.1.3.1]). We note that the simplicial set $\mathrm{Map}_{\mathrm{N}(\mathcal{J})}^\flat(X, Y)$ is determined by the "exponential rule"
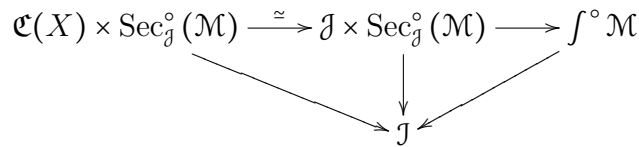
$$(12) \qquad \mathrm{Hom}_{\mathrm{Set}_\Delta}(K, \mathrm{Map}_{\mathrm{N}(\mathcal{J})}^\flat(X, Y)) \cong \mathrm{Hom}_{(\mathrm{Set}_\Delta^+)_{/\mathrm{N}(\mathcal{J})}}(K^\flat \times X, Y),$$

where $K^\flat$ denotes the marked simplicial set K with only degenerate edges marked.

Now suppose we are given a simplicial set X and a commutative diagram

$$
\begin{array}{ccc}
\mathfrak{C}(X) & \xrightarrow[\simeq]{\psi} & \mathcal{J} \\
 & \searrow \quad \swarrow & \\
 & \mathcal{J} &
\end{array}
$$
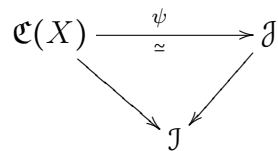
of simplicial categories in which ψ is a Dwyer-Kan equivalence. For example, if $\mathcal{J}$ is fibrant then we may take X to be $\mathrm{N}(\mathcal{J})$ and ψ to be the counit map. To this data we may associate a diagram of simplicial categories

$$
\begin{array}{ccccc}
\mathfrak{C}(X) \times \mathrm{Sec}_{\mathcal{J}}^\circ(\mathcal{M}) & \xrightarrow{\simeq} & \mathcal{J} \times \mathrm{Sec}_{\mathcal{J}}^\circ(\mathcal{M}) & \longrightarrow & \int^\circ \mathcal{M} \\
 & \searrow & \downarrow & \swarrow & \\
 & & \mathcal{J} & &
\end{array}
$$

in which the right horizontal arrow is the evaluation map $(j, s) \mapsto s(j)$. Passing to coherent nerves and pre-composing with the unit map $X \mapsto \mathrm{N}(\mathfrak{C}(X))$ we obtain a diagram

of simplicial sets

$$(13) \qquad X \times \mathrm{N}\left(\mathrm{Sec}^\circ_{\mathcal{J}}(\mathcal{M})\right) \longrightarrow \mathcal{E}$$
$$\searrow \qquad \swarrow$$
$$\mathrm{N}(\mathcal{J})$$

which, in light of Lemma 2.1, refines to a map

$$(14) \qquad u_\psi : X^\flat \times \mathrm{N}^+\left(\mathrm{Sec}^\circ_{\mathcal{J}}(\mathcal{M})\right) \longrightarrow \mathcal{E}^\natural$$

in the model category $(\mathrm{Set}^+_\Delta)_{/\mathrm{N}(\mathcal{J})}$. Here, the marked simplicial set $\mathrm{N}^+\left(\mathrm{Sec}^\circ_{\mathcal{J}}(\mathcal{M})\right)$ is the coherent nerve of $\mathrm{Sec}^\circ_{\mathcal{J}}(\mathcal{M})$ considered as a marked simplicial set in which the marked edges are those which correspond to equivalences in $\mathrm{Sec}^\circ_{\mathcal{J}}(\mathcal{M})$. The core part of our main theorem is then given by the following assertion.

4.1. THEOREM. *Keeping the assumptions and notations above, the map*

$$(15) \qquad v_\psi : \mathrm{N}\left(\mathrm{Sec}^\circ_{\mathcal{J}}(\mathcal{M})\right) \xrightarrow{\simeq} \mathrm{Map}^\flat_{\mathrm{N}(\mathcal{J})}\left(X^\flat, \mathcal{E}^\natural\right),$$

adjoint to (14) by (12), is an equivalence of ∞-categories.

We pause to note that the map (15) is a map of *large* ∞-categories (though they are both locally small), and so some caution is required. In what follows we will use the following terminology: for a (possibly large) ∞-category $\mathcal{C}$ we will denote by $\pi_0\mathcal{C}$ the collection of equivalence classes of objects of $\mathcal{C}$. We note that if $\mathcal{C}$ is locally small and K is a small simplicial set then the ∞-category $\mathcal{C}^K$ of functors from X to $\mathcal{C}$ is locally small as well. To accommodate the size issue we will need the following lemma.

4.2. LEMMA. *Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between possibly large locally small ∞-categories. Suppose that for every small simplicial set K the induced map*

$$\pi_0\mathcal{C}^K \longrightarrow \pi_0\mathcal{D}^K$$

is bijective. Then $\mathcal{F}$ is an equivalence.

PROOF. Applying the assumption for $K = \Delta^0$ implies that $\mathcal{F}$ is essentially surjective. It will hence suffice to show that $\mathcal{F}$ is fully-faithful. Let $X, Y \in \mathcal{C}$ be objects and consider the map of spaces

$$(16) \qquad \mathcal{F}_* : \mathrm{Map}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Map}_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Y)).$$

Since $\mathcal{D}$ is locally small there exists a small full subcategory $\mathcal{U} \subseteq \mathcal{D}$ which contains $\mathcal{F}(X), \mathcal{F}(Y) \in \mathcal{D}$. Let $\mathcal{F}^{-1}\mathcal{U} \subseteq \mathcal{C}$ be the inverse image of $\mathcal{U}$ in $\mathcal{C}$. Since $\pi_0\mathcal{C} \longrightarrow \pi_0\mathcal{D}$ is injective we can find inside $\mathcal{F}^{-1}\mathcal{U}$ a small full subcategory $\mathcal{V}$ containing X, Y and such

that the inclusion $\mathcal{V} \subseteq \mathcal{F}^{-1}\mathcal{U}$ is an equivalence. Consider the resulting homotopy cartesian square of ∞-categories

(17)
$$
\begin{array}{ccc}
\mathcal{V} & \longrightarrow & \mathcal{U} \\
\downarrow & & \downarrow \\
\mathcal{C} & \longrightarrow & \mathcal{D}
\end{array}
$$

in which the vertical maps are fully-faithful inclusions. Then for every small simplicial set K the resulting square

(18)
$$
\begin{array}{ccc}
\pi_0\mathcal{V}^K & \longrightarrow & \pi_0\mathcal{U}^K \\
\downarrow & & \downarrow \\
\pi_0\mathcal{C}^K & \longrightarrow & \pi_0\mathcal{D}^K
\end{array}
$$

is cartesian as well: indeed, the map $\pi_0\mathcal{V}^K \longrightarrow \pi_0\mathcal{C}^K \times_{\pi_0\mathcal{D}^K} \pi_0\mathcal{U}^K$ is surjective because (17) is homotopy cartesian (and hence remains homotopy cartesian after applying $(-)^K$) and is injective because the vertical maps in (18) are injective. Since the bottom horizontal map in (18) is bijective it now follows that the top horizontal map is bijective. Since this is true for any small simplicial set K and $\mathcal{V}, \mathcal{U}$ are small we get that the map $\mathcal{V} \longrightarrow \mathcal{U}$ is necessarily an equivalence. We may then conclude that (16) is an equivalence of spaces and so $\mathcal{F}$ is fully-faithful. ∎

4.3. REMARK. In Lemma 4.2 the assumption that $\mathcal{C}$ and $\mathcal{D}$ are locally small is essential. To see this, observe that if we could prove the claim without this assumption then we could also prove using Grothendieck universes that for a sufficiently large regular cardinal κ the corepresentable homotopy functors $\mathrm{Ho}(\mathrm{Cat}_\infty) \longrightarrow \mathrm{Set}$ associated to the collection of κ-small ∞-categories are jointly conservative in $\mathrm{Ho}(\mathrm{Cat}_\infty)$. Since the ∞-category $\mathcal{S}$ of space is reflective inside Cat_∞ this would mean that $\mathrm{Ho}(\mathcal{S})$ contains a set of objects whose corepresentable functors are jointly conservative. But this is known to be false, see [9, Corollary 2.3].

PROOF OF THEOREM 4.1. In light of Lemma 4.2 it will suffice to show that for every small simplicial set K, the map

(19)
$$
\pi_0\, \mathrm{N}(\mathrm{Sec}_{\mathcal{J}}(\mathcal{M})^\circ)^K \longrightarrow \pi_0\, \mathrm{Map}^\flat_{\mathrm{N}(\mathcal{J})}(X^\flat, \mathcal{E}^\natural)^K
$$

induced by v_ψ is bijective. We first note that by comparing the universal mapping property (12) with the analogous property for the exponentiation by K we see that

(20)
$$
\pi_0\, \mathrm{Map}^\flat_{\mathrm{N}(\mathcal{J})}(X^\flat, \mathcal{E}^\natural)^K \cong \pi_0\, \mathrm{Map}^\flat_{\mathrm{N}(\mathcal{J})}(K^\flat \times X^\flat, \mathcal{E}^\natural).
$$

Now consider the category $\mathrm{Sec}_{\mathcal{J}}(\mathcal{M})^{\mathfrak{C}(K)}$ of simplicial functors $\mathfrak{C}(K) \longrightarrow \mathrm{Sec}_{\mathcal{J}}(\mathcal{M})$, equipped with the projective model structure, and let $(\mathrm{Sec}_{\mathcal{J}}(\mathcal{M})^{\mathfrak{C}(K)})^\circ \subseteq \mathrm{Sec}_{\mathcal{J}}(\mathcal{M})^{\mathfrak{C}(K)}$ denote the full

simplicial subcategory spanned by the fibrant-cofibrant functors. We may then apply [15, Proposition 4.2.4.4] to the model category $\mathbf{A} = \mathrm{Sec}_{\mathcal{J}}(\mathcal{M})$ to deduce that the map

$$(21) \qquad \pi_0 \, \mathrm{N}\left(\mathrm{Sec}_{\mathcal{J}}(\mathcal{M})^{\mathfrak{C}(K)}\right)^{\circ} \longrightarrow \pi_0 \, \mathrm{N}\left(\mathrm{Sec}_{\mathcal{J}}^{\circ}(\mathcal{M})\right)^{K}$$

is bijective. In addition, we also have a canonical equivalence of categories

$$(22) \qquad \mathrm{Sec}_{\mathcal{J}}(\mathcal{M})^{\mathfrak{C}(K)} \simeq \mathrm{Sec}_{\mathfrak{C}(K) \times \mathcal{J}}(\mathcal{M})$$

which identifies the projective model structure on the right with the twice nested projective model structure on the left. It will hence suffice to show that the map

$$(23) \qquad \pi_0 \, \mathrm{N}\left(\mathrm{Sec}_{\mathfrak{C}(K) \times \mathcal{J}}^{\circ}(\mathcal{M})\right) \longrightarrow \pi_0 \, \mathrm{Map}_{\mathrm{N}(\mathcal{J})}^{\flat}\left(K^{\flat} \times X^{\flat}, \mathcal{E}^{\natural}\right)$$

obtained by composing (21), (19) and (20), and using the identification (22), is bijective. Unwinding the definitions, we see that (23) is the map induced on connected components by the map

$$v_{\psi'} : \mathrm{N}\left(\mathrm{Sec}_{\mathfrak{C}(K) \times \mathcal{J}}^{\circ}(\mathcal{M})\right) \longrightarrow \mathrm{Map}_{\mathrm{N}(\mathcal{J})}^{\flat}\left(K^{\flat} \times X^{\flat}, \mathcal{E}^{\natural}\right)$$

associated to the composed Dwyer-Kan equivalence

$$\psi' : \mathfrak{C}(K \times X) \xrightarrow{\simeq} \mathfrak{C}(K) \times \mathfrak{C}(X) \xrightarrow{\psi_*} \mathfrak{C}(K) \times \mathcal{J}$$

as in (15). Replacing X with $K \times X$, $\mathcal{J}$ with $\mathfrak{C}(K) \times \mathcal{J}$ and ψ with ψ' we may simply assume that $K = \Delta^0$. It remains to show that the map

$$(v_{\psi})_* : \pi_0 \, \mathrm{N}\left(\mathrm{Sec}_{\mathcal{J}}^{\circ}(\mathcal{M})\right) \longrightarrow \pi_0 \, \mathrm{Map}_{\mathrm{N}(\mathcal{J})}^{\flat}\left(X^{\flat}, \mathcal{E}^{\natural}\right)$$

induced on π_0 by (15) is a bijective. In light of Proposition 3.7 we may furthermore assume that $\mathcal{J} = \mathfrak{C}(X)$ and $\psi = \mathrm{Id} : \mathfrak{C}(X) \longrightarrow \mathfrak{C}(X)$ is the identity. In this case, the map $(v_{\psi})_*$ admits a particularly simple description. Indeed, every projectively fibrant/cofibrant section $s : \mathfrak{C}(X) \longrightarrow \int \mathcal{M}$ factors through

$$\mathfrak{C}(X) \xrightarrow{s} \int^{\circ} \mathcal{M} \hookrightarrow \int \mathcal{M}$$

and one can identify $(v_{\mathrm{Id}})_*([s]) \in \pi_0 \, \mathrm{Map}_{\mathrm{N}(\mathcal{J})}^{\flat}\left(X^{\flat}, \mathcal{E}^{\natural}\right)$ with the homotopy class of the marked map $s_+^{\mathrm{ad}} : X^{\flat} \longrightarrow \mathcal{E}^{\natural}$ determined by the adjoint $s^{\mathrm{ad}} : X \longrightarrow \mathcal{E}$ of s. We start by showing that $(v_{\mathrm{Id}})_*$ is surjective. Let $s_+^{\mathrm{ad}} : X^{\flat} \longrightarrow \mathcal{E}^{\natural}$ be a marked map over $\mathrm{N}(\mathcal{J})$. It corresponds by adjunction to a map

$$s : \mathfrak{C}(X) \longrightarrow \int^{\circ} \mathcal{M}$$

over $\mathcal{J}$, determining a fibrant object in $\mathrm{Sec}_{\mathfrak{C}(X)}^{\mathrm{proj}}(\mathcal{M})$. Let $s' \xrightarrow{\sim} s$ be a trivial fibration from a cofibrant s', so that $s' \in \mathrm{Sec}_{\mathfrak{C}(X)}^{\circ}(\mathcal{M})$. The map s' in turn corresponds to some other

map $(s')^{\mathrm{ad}}_+ : X^\flat \longrightarrow \mathcal{E}^\natural$ over $\mathrm{N}(\mathcal{J})$. We now claim that s^{ad}_+ and $(s')^{\mathrm{ad}}_+$ are equivalent in the ∞-category $\mathrm{Map}^\flat_{\mathrm{N}(\mathcal{J})}(X^\flat, \mathcal{E}^\natural)$. Indeed, the weak equivalence $s' \longrightarrow s$ can be encoded by a simplicial functor

$$h : [1] \times \mathfrak{C}(X) \longrightarrow \int^\circ \mathcal{M}$$

over $\mathcal{J}$, where $[1] = \bullet \longrightarrow \bullet$ is considered as a (mapping-wise discrete) simplicial category. Then $\mathrm{N}([1]) \cong \Delta^1$ and the composed map $\Delta^1 \times X \longrightarrow \Delta^1 \times \mathrm{N}(\mathfrak{C}(X)) \longrightarrow \mathcal{E}$ refines to a marked map $(\Delta^1)^\sharp \times X^\flat \longrightarrow \mathcal{E}^\natural$ by Lemma 2.1. The latter determines an invertible edge in $\mathrm{Map}^\flat_{\mathrm{N}(\mathcal{J})}(X^\flat, \mathcal{E}^\natural)$ from $(s')^{\mathrm{ad}}_+$ to s^{ad}_+ and so

$$[s^{\mathrm{ad}}_+] = [(s')^{\mathrm{ad}}_+] = (v_{\mathrm{Id}})_*[s'] \in \pi_0 \mathrm{Map}^\flat_{\mathrm{N}(\mathcal{J})}(X^\flat, \mathcal{E}^\natural)$$

is in the image of $(v_{\mathrm{Id}})_*$. It remains to show that $(v_{\mathrm{Id}})_*$ is injective. Let $s, t : \mathcal{J} \longrightarrow \int \mathcal{M}$ be fibrant-cofibrant objects such that $s^{\mathrm{ad}}_+, t^{\mathrm{ad}}_+ : X^\flat \longrightarrow \mathcal{E}^\natural$ are equivalent in the ∞-category $\mathrm{Map}^\flat_{\mathrm{N}(\mathcal{J})}(X^\flat, \mathcal{E}^\natural)$. Since $\mathcal{E}^\natural$ is fibrant in $(\mathrm{Set}^+_\Delta)_{/\mathrm{N}(\mathcal{J})}$ there exists a direct homotopy

$$h : (\Delta^1)^\sharp \times X^\flat \longrightarrow \mathcal{E}^\natural$$

from s^{ad}_+ to t^{ad}_+. By adjunction we obtain a map

$$h_{\mathrm{ad}} : \mathfrak{C}(\Delta^1 \times X^\flat) \longrightarrow \int^\circ \mathcal{M}$$

whose restriction to $\mathfrak{C}(\{0\} \times X)$ is s and whose restriction to $\mathfrak{C}(\{1\} \times X)$ is t. Furthermore, since the marked edges in $\mathcal{E}^\natural$ are exactly the p-cartesian edges, Lemma 2.1 implies that the composed map

$$\mathfrak{C}(\Delta^1 \times \{x\}) \longrightarrow \mathfrak{C}(\Delta^1 \times X^\flat) \overset{h_{\mathrm{ad}}}{\longrightarrow} \int^\circ \mathcal{M}$$

determines a weak equivalence from $s(x)$ to $t(x)$ in $\mathcal{M}(x)$ for every vertex $x \in X$ (i.e., for every object $x \in \mathcal{J} = \mathfrak{C}(X)$). Note that the map h_{ad} is not yet an honest natural equivalence from s to t but only a *homotopy coherent* one. In order to strictify it we will need to employ Proposition 3.7 again. We can consider the map h_{ad} as a fibrant object in $\mathrm{Sec}_{\mathfrak{C}(\Delta^1 \times X^\flat)}(\mathcal{M})$. We have a natural map

$$\phi : \mathfrak{C}(\Delta^1 \times X^\flat) \overset{\simeq}{\longrightarrow} \mathfrak{C}(\Delta^1) \times \mathfrak{C}(X^\flat)$$

which is a Dwyer-Kan equivalence of simplicial categories. From Proposition 3.7 it follows that there exists a fibrant-cofibrant object $h'_{\mathrm{ad}} \in \mathrm{Sec}_{\mathfrak{C}(\Delta^1) \times \mathfrak{C}(X^\flat)}(\mathcal{M})$ such that $\phi^* h'_{\mathrm{ad}}$ is weakly equivalent to h_{ad}. This implies, in particular, that the restriction $s' := h'_{\mathrm{ad}}|_{\{0\} \times \mathfrak{C}(X)}$ is weakly equivalent to s and the restriction $t' := h'_{\mathrm{ad}}|_{\{1\} \times \mathfrak{C}(X)}$ is weakly equivalent to t. The map h'_{ad} determines an honest weak equivalence from s' to t'. We may hence conclude that s is weakly equivalent to t in the model category $\mathrm{Sec}_{\mathfrak{C}(X)}(\mathcal{M})$ and hence weakly equivalent to t in the simplicial category $\mathrm{Sec}^\circ_{\mathfrak{C}(X)}(\mathcal{M})$, as desired. ∎

Now suppose that E is a set of maps in $\mathcal{J}$ (i.e., a set of vertices in the various mapping simplicial sets of $\mathcal{J}$). We will denote by $\mathrm{Sec}^{\circ}_{\mathcal{J},E}(\mathcal{M}) \subseteq \mathrm{Sec}^{\circ}_{\mathcal{J}}(\mathcal{M})$ the full simplicial subcategory spanned by those fibrant-cofibrant sections $s : \mathcal{J} \longrightarrow \int \mathcal{M}$ such that for every $\beta : j \longrightarrow j'$ in E, which lies above a map $\alpha : \varphi(j) \longrightarrow \varphi(j')$ in $\mathcal{I}$, the map

$$s(j) \xrightarrow{s^{\beta}} \alpha^{*}s(j')$$

is a weak equivalence in $\mathcal{M}(j)$, where s^{β} is the adjoint of the map (7) determined by the vertex $\beta \in \mathrm{Map}^{\alpha}(j,j')$. We may now finally deduce the main result of this paper:

4.4. COROLLARY. *Let $\mathcal{I}$ be an ordinary category, $\mathcal{M} : \mathcal{I} \longrightarrow \mathrm{ModCat}_{\Delta}$ a simplicial Quillen presheaf taking values in combinatorial model categories, and $\mathcal{J}$ a fibrant simplicial category equipped with a map $\varphi : \mathcal{J} \longrightarrow \mathcal{I}$. Let E be a set of maps in $\mathcal{J}$ and denote by $\mathrm{N}(\mathcal{J},E)$ the marked simplicial set consisting of the coherent nerve of $\mathcal{J}$ with the edges corresponding to E marked. Then there is a natural equivalence of ∞-categories*

$$\mathrm{N}\left(\mathrm{Sec}^{\circ}_{\mathcal{J},E}(\mathcal{M})\right) \simeq \mathrm{Map}^{\flat}\left(\mathrm{N}(\mathcal{J},E), \mathcal{E}^{\natural}\right).$$

PROOF. We have a diagram of ∞-categories

$$\begin{array}{ccc}
\mathrm{N}\left(\mathrm{Sec}^{\circ}_{\mathcal{J},E}(\mathcal{M})\right) & \longrightarrow & \mathrm{Map}^{\flat}_{\mathrm{N}(\mathcal{J})}\left(\mathrm{N}(\mathcal{J},E), \mathcal{E}^{\natural}\right) \\
\downarrow & & \downarrow \\
\mathrm{N}\left(\mathrm{Sec}^{\circ}_{\mathcal{J}}(\mathcal{M})\right) & \xrightarrow{\;\simeq\;} & \mathrm{Map}^{\flat}_{\mathrm{N}(\mathcal{J})}\left(\mathrm{N}(\mathcal{J})^{\flat}, \mathcal{E}^{\natural}\right)
\end{array}$$

where the vertical maps are fully-faithful inclusions and the lower horizontal map is the equivalence of Theorem 4.1 associated to the counit map $\psi : \mathfrak{C}(\mathrm{N}(\mathcal{J})) \xrightarrow{\;\simeq\;} \mathcal{J}$. It will be enough to verify that this diagram is homotopy cartesian. For this, it will suffice to show that a fibrant-cofibrant section $s : \mathcal{J} \longrightarrow \int^{\circ} \mathcal{M}$ lies in $\mathrm{Sec}^{\circ}_{\mathcal{J},E}(\mathcal{M})$ if and only if the corresponding map $\mathrm{N}(\mathcal{J}) \longrightarrow \mathcal{E}$ sends every marked edge of $\mathrm{N}(\mathcal{J},E)$ to a p-cartesian edge. But this is a direct consequence of the characterization of p-cartesian edges in $\int^{\circ} \mathcal{M}$ given by Lemma 2.1, and so the proof is complete. ∎

References

[1] Balzin E., Reedy Model Structures in Families, preprint arXiv:1803.00681, 2018.

[2] Barwick C., On left and right model categories and left and right Bousfield localizations, *Homology, Homotopy and Applications*, 12.2, 2010, p. 245-320

[3] Bergner J, A model category structure on the category of simplicial categories, *Transactions of the American Mathematical Society*, 359.5, 2007, p. 2043–2058.

[4] Bergner J., Homotopy limits of model categories and more general homotopy theories, *Bulletin of the London Mathematical Society*, 44.2, 2011, p. 311–322.

[5] Dugger, D., Combinatorial model categories have presentations, *Advances in Mathematics*, 164.1, 2001, p. 177–201.

[6] Dugger, D., Replacing model categories by simplicial ones, to appear in *Transactions of the American Mathematical Society*.

[7] Gepner D., Haugseng R., Nikolaus T., Lax colimits and free fibrations in ∞-categories, *Documenta Mathematica*, 22, 2017, p. 1255–1266.

[8] Gagna A., Harpaz Y., Lanari E., Lax limits in $(\infty, 2)$-categories, in preparation.

[9] Heller A., On the representability of homotopy functors, *Journal of the London Mathematical Society*, 2.3, 1981, p. 551–562.

[10] Hinich V., Dwyer–Kan localization revisited, *Homology, Homotopy and Applications*, 18.1, 2016, p. 27–48.

[11] Hirschhorn P. S., *Model categories and their localizations*, American Mathematical Society, 2009.

[12] Hirschowitz A., Simpson C, Descente pour les n-champs, preprint arXiv:math/9807049, 2001.

[13] Hovey M., Spectra and symmetric spectra in general model categories, *Journal of Pure and Applied Algebra*, 165(1), (2001), p. 63–127.

[14] Joyal A., The theory of quasi-categories and its applications, 2008, `http://mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern45-2.pdf`.

[15] Lurie J., *Higher Topos Theory*, Annals of Mathematics Studies, 170, Princeton University Press, 2009, `http://www.math.harvard.edu/~lurie/papers/highertopoi.pdf`.

[16] Robalo M, Noncommutative motives I: A universal characterization of the motivic stable homotopy theory of schemes, preprint arXiv:1206.3645, 2012.

Yonatan Harpaz
Institut Galilée
Université Sorbonne Paris Nord
99 avenue Jean-Baptiste Clément
93430 Villetaneuse
France

THEORY AND APPLICATIONS OF CATEGORIES will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.

Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.

SUBSCRIPTION INFORMATION Individual subscribers receive abstracts of articles by e-mail as they are published. To subscribe, send e-mail to `tac@mta.ca` including a full name and postal address. Full text of the journal is freely available at `http://www.tac.mta.ca/tac/`.

INFORMATION FOR AUTHORS LaTeX2e is required. Articles may be submitted in PDF by email directly to a Transmitting Editor following the author instructions at
`http://www.tac.mta.ca/tac/authinfo.html`.

MANAGING EDITOR. Geoff Cruttwell, Mount Allison University: `gcruttwell@mta.ca`

TEXNICAL EDITOR. Michael Barr, McGill University: `michael.barr@mcgill.ca`

ASSISTANT TEX EDITOR. Gavin Seal, Ecole Polytechnique Fédérale de Lausanne:
`gavin_seal@fastmail.fm`

TRANSMITTING EDITORS.
Clemens Berger, Université de Nice-Sophia Antipolis: `cberger@math.unice.fr`
Julie Bergner, University of Virginia: `jeb2md (at) virginia.edu`
Richard Blute, Université d' Ottawa: `rblute@uottawa.ca`
Gabriella Böhm, Wigner Research Centre for Physics: `bohm.gabriella (at) wigner.mta.hu`
Valeria de Paiva: Nuance Communications Inc: `valeria.depaiva@gmail.com`
Richard Garner, Macquarie University: `richard.garner@mq.edu.au`
Ezra Getzler, Northwestern University: `getzler (at) northwestern(dot)edu`
Kathryn Hess, Ecole Polytechnique Fédérale de Lausanne: `kathryn.hess@epfl.ch`
Dirk Hofmann, Universidade de Aveiro: `dirk@ua.pt`
Pieter Hofstra, Université d' Ottawa: `phofstra (at) uottawa.ca`
Anders Kock, University of Aarhus: `kock@math.au.dk`
Joachim Kock, Universitat Autònoma de Barcelona: `kock (at) mat.uab.cat`
Stephen Lack, Macquarie University: `steve.lack@mq.edu.au`
F. William Lawvere, State University of New York at Buffalo: `wlawvere@buffalo.edu`
Tom Leinster, University of Edinburgh: `Tom.Leinster@ed.ac.uk`
Matias Menni, Conicet and Universidad Nacional de La Plata, Argentina: `matias.menni@gmail.com`
Ieke Moerdijk, Utrecht University: `i.moerdijk@uu.nl`
Susan Niefield, Union College: `niefiels@union.edu`
Kate Ponto, University of Kentucky: `kate.ponto (at) uky.edu`
Robert Rosebrugh, Mount Allison University: `rrosebrugh@mta.ca`
Jiri Rosicky, Masaryk University: `rosicky@math.muni.cz`
Giuseppe Rosolini, Università di Genova: `rosolini@disi.unige.it`
Alex Simpson, University of Ljubljana: `Alex.Simpson@fmf.uni-lj.si`
James Stasheff, University of North Carolina: `jds@math.upenn.edu`
Ross Street, Macquarie University: `ross.street@mq.edu.au`
Tim Van der Linden, Université catholique de Louvain: `tim.vanderlinden@uclouvain.be`