

# HIGHER SEGAL SPACES VIA HIGHER EXCISION

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ABSTRACT. We show that the various higher Segal conditions of Dyckerhoff and Kapranov can all be characterized in purely categorical terms by higher excision conditions (in the spirit of Goodwillie–Weiss manifold calculus) on the simplex category  $\Delta$  and the cyclic category  $\Lambda$ .

## 1. Introduction

Starting with Segal’s [1974] original definition of special  $\Gamma$ - and  $\Delta$ -spaces, many related notions have been used as an efficient tool to describe monoids and categories which are not strictly associative but only associative up to a coherent system of higher homotopies. Important examples are the Segal categories of Dwyer et al. [1989] and Hirschowitz and Simpson [1998], and the complete Segal spaces of Rezk [2001]. The unifying theme of these *Segal objects* is that they are simplicial objects in an  $(\infty)$ -category  $\mathcal{C}$ —*i.e.*, functors  $\mathcal{X}: \Delta^{\text{op}} \rightarrow \mathcal{C}$  on the simplex category  $\Delta$ —satisfying a certain family of descent conditions.

The starting point for this work is the easy but little-known observation that Segal objects can be concisely characterized by a condition that does not refer to the internal workings of  $\Delta$  but only to its abstract category-theoretic properties.

1.0.1. OBSERVATION. A simplicial object  $\Delta^{\text{op}} \rightarrow \mathcal{C}$  is Segal if and only if it sends bicartesian squares in  $\Delta$  to cartesian squares in  $\mathcal{C}$ .

Dyckerhoff and Kapranov [2019] generalized Rezk’s Segal condition and introduced what they call *higher Segal spaces*<sup>1</sup>. Their definition is very geometric in nature: They consider the so called *cyclic polytopes*  $C(n, d)$ , defined as the convex hull of  $n + 1$  points on the  $d$ -dimensional moment curve  $t \mapsto (t, t^2, \dots, t^d)$ . The main feature of these polytopes in this context is that they have two canonical triangulations, called the *lower triangulation* and the *upper triangulation*, respectively. Each of these triangulations defines a simplicial subcomplex  $\mathcal{T}$  of the standard  $n$ -simplex  $\Delta^n$ ; Dyckerhoff and Kapranov then impose conditions on simplicial objects by requiring that the value<sup>2</sup> on the inclusion  $\mathcal{T} \hookrightarrow \Delta^n$

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<sup>1</sup>not to be confused with the  $n$ -fold Segal spaces of Barwick [2005]

<sup>2</sup>Every simplicial object can be canonically evaluated on simplicial sets by Kan extension along the Yoneda embedding; see Section 5.1.

is an equivalence: a simplicial object is called **lower** (resp. **upper**)  $d$ -**Segal** if this is true for the lower (resp. upper) triangulation of  $C(n, d)$  and  $d$ -**Segal** if this is true for all triangulations of  $C(n, d)$ .

The purpose of this article is to characterize the various flavors of higher Segal conditions in terms of purely categorical notions of higher excision. We first do this for lower  $(2k - 1)$ -Segal spaces, since they are the most fundamental<sup>3</sup> amongst all versions of higher Segal spaces. The following is the first main result of this paper:

1.0.2. **THEOREM.** [Theorem 7.2.2] *Let  $\mathcal{X}: \Delta^{\text{op}} \rightarrow \mathcal{C}$  be a simplicial object in an  $\infty$ -category  $\mathcal{C}$  with finite limits. The following are equivalent:*

1. *the simplicial object  $\mathcal{X}$  is lower  $(2k - 1)$ -Segal;*
2. *the functor  $\mathcal{X}$  sends every strongly bicartesian<sup>4</sup>  $(k + 1)$ -dimensional cube in  $\Delta$  to a limit diagram in  $\mathcal{C}$ .*

We call a functor  $\mathcal{Z}^{\text{op}} \rightarrow \mathcal{C}$  satisfying condition 2 of Theorem 1.0.2 **weakly  $k$ -excisive**; compare this with Goodwillie’s [1992] calculus of functors, where a (covariant) functor  $\mathcal{Z} \rightarrow \mathcal{C}$  is called  $k$ -excisive if it sends strongly *cocartesian*  $(k + 1)$ -dimensional cubes in  $\mathcal{Z}$  to limit diagrams in  $\mathcal{C}$ .

We illustrate Theorem 1.0.2 with some examples.

- The cyclic polytope  $C(n, 1)$  is just the interval  $\Delta^{\{0,n\}}$ ; its lower triangulation (see Figure 1) yields the simplicial complex

$$\text{Sp}[n] := \Delta^{\{0,1\}} \cup \dots \cup \Delta^{\{n-1,n\}} \subset \Delta^n. \tag{1}$$

Rezk’s Segal condition for a simplicial object says precisely that the inclusion  $\text{Sp}[n] \hookrightarrow \Delta^n$  needs to be sent to an equivalence; this is what Dyckerhoff and Kapranov call the lower 1-Segal condition. For  $n = 1$ , this condition says precisely that the bicartesian square

$$\begin{array}{ccc} 1 & \longrightarrow & 12 \\ \downarrow & \square & \downarrow \\ 01 & \longrightarrow & 012 \end{array} \tag{2}$$

in  $\Delta$  needs to be sent to a limit diagram. More generally, every square of the form

$$\begin{array}{ccc} \{i\} & \longrightarrow & \{i, \dots, n\} \\ \downarrow & \square & \downarrow \\ \{0, \dots, i\} & \longrightarrow & \{0, \dots, n\} \end{array} \tag{3}$$

<sup>3</sup>This vague assertion is made precise by the path space criterion (see Poguntke [2017, Proposition 2.7]) which expresses all higher Segal conditions in terms of lower  $(2k - 1)$ -Segal conditions.

<sup>4</sup>A cube is strongly bicartesian if each of its 2-dimensional faces is bicartesian; see Definition 3.3.4.



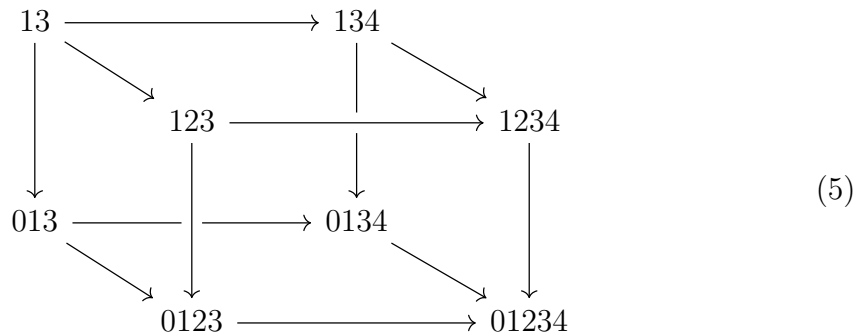
Figure 1: The lower triangulation of the cyclic polytope  $C(n, 1)$ , here depicted with  $n = 5$ .

(for  $0 < i < n$ ) is bicartesian in  $\Delta$ ; it is in fact an often used characterization of Segal objects to require these squares to be sent to pullbacks.

- The cyclic polytope  $C(4, 3)$  is a double triangular pyramid; its lower triangulation (see Figure 2) induces the simplicial complex

$$\mathcal{T} = \Delta^{\{1,2,3,4\}} \cup \Delta^{\{0,1,3,4\}} \cup \Delta^{\{0,1,2,3\}} \subset \Delta^4. \tag{4}$$

By definition, a simplicial object satisfies the first lower 3-Segal condition if it sends the canonical inclusion  $\mathcal{T} \hookrightarrow \Delta^4$  to an equivalence; this is equivalent to sending the cube



which is strongly bicartesian in  $\Delta$ , to a limit diagram.

In general, the first non-trivial lower  $(2k - 1)$ -Segal condition (*i.e.*, the one for  $n = 2k$ ) can always be expressed in terms of a strongly bicartesian cube in  $\Delta$  of dimension  $k + 1$  and this cube is the unique such cube which is in a certain sense “basic”. However, for bigger  $n$  both the number of simplices in the lower triangulation of  $C(n, 2k - 1)$  and the number of basic strongly bicartesian cubes grows very rapidly so that, *a priori*, the behavior of weakly  $k$ -excisive simplicial objects and lower  $(2k - 1)$ -Segal objects diverges dramatically.

Since the introduction of higher Segal spaces, most interest in the area was garnered by 2-Segal spaces; more precisely by 2-Segal spaces that satisfy an additional condition called *unitality*. For example, unital 2-Segal spaces were studied by Dyckerhoff [2018] from the perspective of Hall algebras and by Gálvez-Carrillo, Kock and Tonks [2018a; 2018b; 2018c] from the perspective of bialgebras arising in combinatorics<sup>5</sup>. The  $\infty$ -category of unital 2-Segal spaces was identified in a previous paper [Walde, 2017] as a certain

<sup>5</sup>Unital 2-Segal spaces are called *decomposition spaces* in this context.

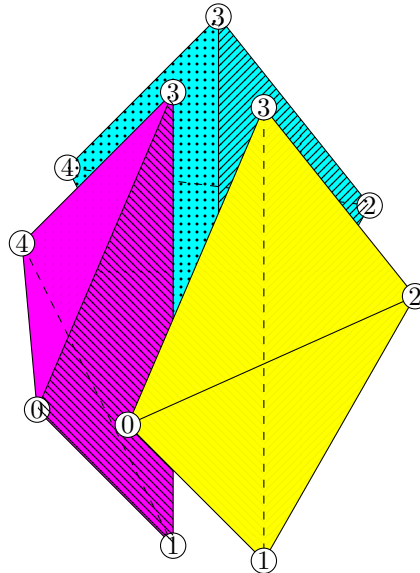


Figure 2: The three 3-simplices  $\Delta^{\{1234\}}$ ,  $\Delta^{\{0134\}}$  and  $\Delta^{\{0123\}}$  (depicted in cyan, magenta and yellow, respectively) assemble into the lower triangulation of the double triangular pyramid  $C(4, 3)$ .

sub- $\infty$ -category of the  $\infty$ -category of  $\infty$ -operads and recently by Stern [2019] as a certain  $\infty$ -category of algebras in correspondences. The main source of examples for unital 2-Segal objects is Waldhausen’s [1985] S-construction from algebraic K-theory; Bergner, Osorno, Ozornova, Rovelli and Scheimbauer [2018] showed that in a certain sense every unital 2-Segal space arises this way; Poguntke [2017] generalized Waldhausen’s construction to higher dimensions, thus providing many examples for  $2k$ -Segal spaces. Furthermore, *cyclic* unital 2-Segal spaces—which can be identified with certain *cyclic*  $\infty$ -operads (see Walde [2017]) or with *Calabi–Yau* algebras in correspondences (see Stern [2019])—play a central role in the construction of topological Fukaya categories of marked surfaces due to Dyckerhoff and Kapranov [2018].

We show that 2-Segal spaces, and more generally  $2k$ -Segal spaces, can be characterized by a relative version of higher weak excision which involves Connes’ cyclic category  $\Lambda$ .

1.0.3. THEOREM. [Theorem 7.2.2] *Let  $\mathcal{X}: \Delta^{\text{op}} \rightarrow \mathcal{C}$  be a simplicial object in an  $\infty$ -category  $\mathcal{C}$  with finite limits. The following are equivalent:*

1. *the simplicial object  $\mathcal{X}$  is  $2k$ -Segal;*
2. *the functor  $\mathcal{X}$  sends to cartesian cubes in  $\mathcal{C}$  those  $(k + 1)$ -dimensional cubes in  $\Delta$  which become strongly bicartesian in  $\Lambda$  (under the canonical functor  $\Delta \rightarrow \Lambda$ ).*

We again illustrate the theorem with some examples:

- The square (2) encoding the first Segal condition is typically not sent to a cartesian

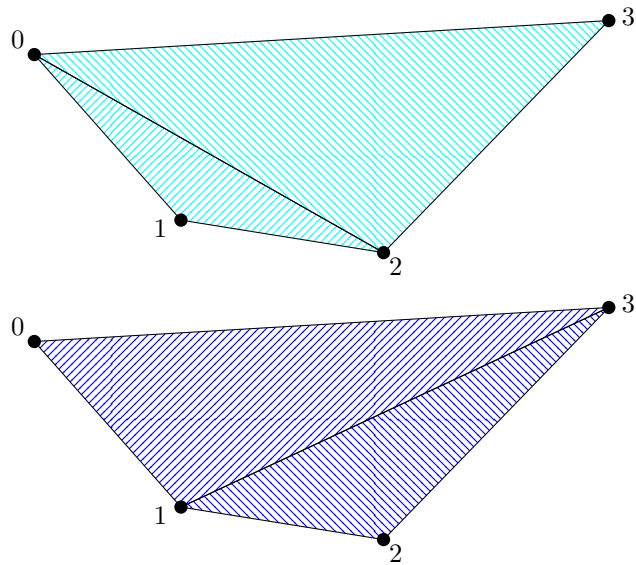


Figure 3: The lower and the upper triangulations of the cyclic polytope  $C(4, 2)$ .

square by 2-Segal objects. This is explained by Theorem 1.0.3: while the square (2) is bicartesian in  $\Delta$ , it is no longer a pushout square in  $\Lambda$ .

- The 2-dimensional cyclic polytope  $C(4, 2)$  is a square. It has the two triangulations (see Figure 3) whose corresponding Segal condition expresses that the two squares

$$\begin{array}{ccc}
 13 & \longrightarrow & 123 \\
 \downarrow & & \downarrow \\
 013 & \longrightarrow & 0123
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 02 & \longrightarrow & 012 \\
 \downarrow & & \downarrow \\
 023 & \longrightarrow & 0123
 \end{array}
 \tag{6}$$

in  $\Delta$  are sent to a limit diagram. Both of the squares (6) are bicartesian in  $\Lambda$ .

- The squares

$$\begin{array}{ccc}
 11' & \xrightarrow{d^0} & 011' \\
 s^0 \downarrow & & \downarrow s^1 \\
 1 & \xrightarrow{d^0} & 01
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 0'0 & \xrightarrow{d^2} & 0'01 \\
 s^0 \downarrow & & \downarrow s^0 \\
 0 & \xrightarrow{d^1} & 01
 \end{array}
 \tag{7}$$

are bicartesian both in  $\Delta$  and in  $\Lambda$ . Hence they need to be sent to pullback squares by every Segal object (by Theorem 1.0.2) and by every 2-Segal object (by Theorem 1.0.3). While the first of these facts is easy, the second is non-trivial; it is precisely the statement that 2-Segal spaces are automatically unital, which was discovered only very recently by Feller, Garner, Kock, Proulx and Weber [2019].

We would like to point out the following corollary of Theorem 1.0.3, which cements the importance of cyclic  $2k$ -Segal objects and might help explain the particular usefulness of cyclic 2-Segal objects.

1.0.4. COROLLARY. [Corollary 7.2.3] *Let  $\mathcal{C}$  be an  $\infty$ -category with finite limits. The cyclic  $2k$ -Segal objects in  $\mathcal{C}$  are precisely the weakly  $k$ -excisive functors  $\Lambda^{\text{op}} \rightarrow \mathcal{C}$ .*

Finally, we remark that our main theorem implies a non-trivial bound (Proposition 7.3.1) on how many values of a higher Segal object can be trivial without the whole object collapsing. Whether this bound is sharp is still unknown (at least to the author) and remains to be investigated in future research.

1.1. METHODS AND STRUCTURE OF THE PAPER. The main conceptual framework which informs our approach is a version for the simplex category of the Goodwillie–Weiss manifold calculus (see Weiss [1999]; Goodwillie and Weiss [1999]). In Section 2 we explain a system of heuristic analogies between manifold calculus (in its version described by Boavida de Brito and Weiss [2013]) and a “manifold calculus” on  $\Delta$ . While the mathematics in the rest of the paper stands on its own, it is the author’s opinion that these informal analogies to manifold calculus can be very helpful when digesting the definitions and building intuition. Interestingly, they also explain how one might have guessed the definition of higher Segal spaces without knowing about cyclic polytopes. One practical upshot of the analogy to manifold calculus is that it inspires the definition of *polynomial* simplicial objects, a notion which is implied by higher weak excision (while being, *a priori*, weaker) and which can be compared more easily to the higher Segal conditions.

In Section 3 we recall basic definitions and facts about the categories  $\Delta$  and  $\Lambda$ , (co)cartesian and strongly (co)cartesian cubes, as well as general notions of excision, weak excision and descent. In Section 4, we explicitly classify strongly cartesian and bicartesian cubes in  $\Delta$  and in  $\Lambda$ . In Section 5 we explain a descent theory on  $\Delta$  and study polynomial simplicial objects in this framework<sup>6</sup>. In Section 6 we show that polynomial simplicial objects agree with weakly excisive ones; our key arguments here are a version of the ones of Feller et al. [2019] repackaged in a way which directly generalizes to arbitrary dimensions. The main theorem (Theorem 7.2.2)—comparing higher Segal conditions with weak excision—is proved in the last section (Section 7) by considering a series of descent conditions which interpolate between the higher Segal conditions and the conditions of being polynomial.

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<sup>6</sup>This framework has already proven its worth in the classification of higher Segal objects with values in stable or additive  $\infty$ -categories; see Dyckerhoff et al. [2019] and Walde [2020].

1.3.  $\infty$ -CATEGORICAL CONVENTIONS. Throughout this article, we will freely use the theory of  $\infty$ -categories as developed by Lurie [2009]; most relevant will be the theory of limits and Kan extensions developed in Chapter 4. We silently identify each ordinary category with its nerve so that each ordinary category is in particular an  $\infty$ -category. Given two ( $\infty$ -)categories  $\mathcal{C}$  and  $\mathcal{Z}$ , we write  $\text{Fun}(\mathcal{Z}, \mathcal{C})$  for the ( $\infty$ -)category of functors between them; for instance,  $\text{Fun}(\Delta^{\text{op}}, \mathcal{C})$  denotes the  $\infty$ -category of simplicial objects in the  $\infty$ -category  $\mathcal{C}$ . When we talk about commutative diagrams in an  $\infty$ -category we will usually only depict objects and arrows while leaving the higher coherence data implicit. All limits and colimits are always meant to be taken in the homotopy-coherent (*i.e.*,  $\infty$ -categorical) sense.

## 2. A “manifold calculus” for the simplex category

A contravariant functor  $\mathcal{X}$  defined on the topological (*i.e.*,  $\infty$ -) category  $\mathbf{Man}$  of smooth  $d$ -manifolds and smooth embeddings is usually called *polynomial of degree  $\leq 1$*  if its value on a manifold  $M$  can be computed by cutting  $M$  up into smaller open pieces, evaluating  $\mathcal{X}$  piece by piece and then reassembling the values. More precisely, for each pair of disjoint closed subsets  $A_0, A_1 \subset M$ , one requires the canonical map

$$\mathcal{X}(M) \longrightarrow \mathcal{X}(M \setminus A_0) \times_{\mathcal{X}(M \setminus A_0 \cup A_1)} \mathcal{X}(M \setminus A_1)$$

to be an equivalence.

Boavida de Brito and Weiss [2013] show that polynomial functors of degree  $\leq 1$  are precisely the (homotopy) sheaves on  $\mathbf{Man}$  for the Grothendieck topology  $\mathcal{J}_1$  of open covers. More generally, they consider a hierarchy  $\mathcal{J}_k$  of Grothendieck topologies on  $\mathbf{Man}$  (with  $k \geq 1$ ), where  $\mathcal{J}_k$  consists of those open covers (called *k-covers*) which have the property that every set of  $k$  (or fewer) points is contained in some open set of the cover. The manifold calculus of Boavida de Brito and Weiss is concerned with the systematic study of sheaves on  $(\mathbf{Man}, \mathcal{J}_k)$ . They introduce the following classes of open covers:

1. the class  $\mathcal{J}_k^{\text{h}}$  consists of open covers of the form

$$\{M \setminus A_i \hookrightarrow M \mid i = 0, \dots, k\} \tag{8}$$

for pairwise disjoint closed subsets  $A_0, \dots, A_k \subset M$  of  $M$ .

2. the class  $\mathcal{J}_k^{\circ}$  consists of *good k-covers*, *i.e.*,  $k$ -covers with the property that every finite intersection of open sets is diffeomorphic to a disjoint union of at most  $k$  balls.

While the classes  $\mathcal{J}_k^{\text{h}}$  and  $\mathcal{J}_k^{\circ}$  are not Grothendieck topologies anymore, they are so called *coverages*, hence they admit a well-behaved theory of descent and sheaves. Sheaves for the coverage  $\mathcal{J}_k^{\text{h}}$  are called *polynomial functors of degree  $\leq k$* . One of the main results of Boavida de Brito and Weiss in this context is the following theorem:

2.0.1. THEOREM. [Boavida de Brito and Weiss, 2013, Theorem 5.2 and Theorem 7.2] The coverages  $\mathcal{J}_k$ ,  $\mathcal{J}_k^h$  and  $\mathcal{J}_k^\circ$  define the same class of sheaves on **Man**.

We shall now describe a similar theory for simplicial objects, *i.e.*, presheaves on the simplex category  $\Delta$  (see Section 3.1 for the notation). It turns out that the following list of analogies is useful; we put terms coming from the language of manifold in quotes to emphasize that they should be thought of heuristically:

- We think of the object  $[n] = \{0, \dots, n\} \in \Delta$  as a “manifold” with “points” given by pairs  $(x - 1, x)$  with  $x = 1, \dots, n$ .
- An “open subset” of  $[n]$  is simply an ordinary subset  $U \subseteq \{0, \dots, n\}$ ; it contains the “points”  $(x - 1, x)$  such that  $\{x - 1, x\} \subseteq U$ .
- We say that two “open subsets”  $U, U'$  of the “manifold”  $[n]$  are “disjoint” if they are disjoint as subsets of  $[n]$ ; note that this is a stronger condition than requiring  $U$  and  $U'$  to share no “point”.
- A “closed set”  $A$  of  $[n]$  is an ordinary subset of  $A \subseteq [n]$ ; it contains all the points not contained in its complement  $[n] \setminus A \subseteq [n]$  (viewed as an “open set”); explicitly,  $A$  contains all “points”  $(x - 1, x)$  with  $x \in A$  or  $x - 1 \in A$ .
- We say that two “closed sets”  $A, A' \subseteq [n]$  are “disjoint” if they share no “point”; note that this is stronger than being disjoint as subsets of  $[n]$ .
- Each “point”  $p = (x - 1, x)$  has a unique minimal “open neighborhood” given by the subset  $U^p = \{x - 1, x\} \subseteq [n]$ , which we think of as a very small “open ball” around the “point”  $p$ .

Armed with this intuition, we can define analogs of the coverings  $\mathcal{J}_k^h$  and  $\mathcal{J}_k^\circ$  in the simplex category:

1. The open covers (8) can be translated to  $\Delta$  by putting everything in quotation marks: For every collection  $A_0, \dots, A_k$  of “nonempty and pairwise disjoint closed subsets” of the “manifold”  $[n]$ , we can define the “open cover”

$$\{[n] \setminus A_i \hookrightarrow [n] \mid i = 0, \dots, k\} \tag{9}$$

of  $[n]$ . See also Section 5.2.

2. Heuristically<sup>7</sup>, one way to produce good  $k$ -covers of a manifold  $M$  is as follows: Fix a Riemannian metric on  $M$  and, for every tuple  $p = (p_1, \dots, p_k)$  of  $k$  points in  $M$ , choose very small (with respect to the geodesic distance between the points  $p_i$ ) balls  $U_i^p \ni p_i$ . Then the collection  $\left\{ \bigcup_{i=1}^k U_i^p \mid p \in M^k \right\}$  is a  $k$ -good cover of  $M$ .

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<sup>7</sup>For an actual proof, see for instance Boavida de Brito and Weiss [2013, Proposition 2.10].



In our analogy, every “point”  $p$  of a “manifold”  $[n] \in \Delta$  has a canonical/minimal “open ball”  $U^p$  surrounding it. Hence each  $[n] \in \Delta$  has a canonical “good  $k$ -cover” containing all those “open subsets” of  $[n] \in \Delta$  that can be written as union of the form

$$\bigcup_{i=1}^k U^{p_i},$$

where  $p_1, \dots, p_k$  are “points” of the “manifold”  $[n]$  with “pairwise disjoint neighborhoods”  $U^{p_i}$ . See also Section 7.1.

Inspired by the analogy, we call a functor  $\Delta^{\text{op}} \rightarrow \mathcal{C}$  **polynomial of degree  $\leq k$**  if it is a sheaf for the “open covers” of type 1 (see Definition 5.2.1).

The following easy observation was the author’s original motivation for this line of inquiry because it shows on one hand that the canonical “good  $k$ -covers” are a meaningful concept and on the other hand that a “manifold calculus” of  $\Delta$  can be a powerful organizational principle for higher Segal spaces.

2.0.2. **OBSERVATION.** Sheaves on  $\Delta$  with respect to the canonical “good  $k$ -covers” of 2 are precisely the lower  $(2k - 1)$ -Segal spaces of Dwyerhoff and Kapranov.

The notion of polynomial simplicial objects might be a bit unsatisfying because its very definition relies on an informal analogy to manifold calculus; without this analogy, the “open covers” (8) might seem a bit mysterious and devoid of intrinsic meaning. We will clarify this issue by showing that a functor  $\Delta^{\text{op}} \rightarrow \mathcal{C}$  is polynomial of degree  $\leq k$  if and only if it is weakly  $k$ -excisive (see Theorem 6.1.1). In this light, our main result (Theorem 7.2.2) relating lower  $(2k - 1)$ -Segal objects with weakly  $k$ -excisive functors should be seen as a discrete analog of Theorem 2.0.1 of Boavida de Brito and Weiss.

We will not spell out the whole story for  $2k$ -Segal objects since it is very similar. Let us just say that one should now consider a “manifold calculus” not on the simplex category  $\Delta$  but on Connes’ cyclic category  $\Lambda$ , where the “manifold”  $[n] = \{0, \dots, n\}$  now has an additional “point” given by  $(n, 0)$ .

### 3. Preliminaries

3.1. **THE SIMPLEX CATEGORY.** The **augmented simplex category**  $\Delta_+$  is the category of finite linearly ordered sets and order preserving (*i.e.*, weakly monotone) maps between them. The **simplex category**  $\Delta \subset \Delta_+$  is the full subcategory spanned by the *nonempty* finite linearly ordered sets. Every object in  $\Delta$  is isomorphic, by a unique isomorphism, to a standard ordinal of the form  $[n] := \{0 < 1 < \dots < n\}$  for some  $n \in \mathbb{N}$ ; when convenient can we therefore identify  $\Delta$  with its skeleton spanned by  $\{[n] \mid n \in \mathbb{N}\}$ .

3.1.1. **DEFINITION.** A **simplicial object** in an  $(\infty)$ -category  $\mathcal{C}$  is a functor  $\Delta^{\text{op}} \rightarrow \mathcal{C}$ .

The augmented simplex category has a monoidal structure

$$\star: \Delta_+ \times \Delta_+ \longrightarrow \Delta_+, \tag{10}$$

given by left-to-right concatenation or **join** of linearly ordered sets. Explicitly we have

$$\{a_0 < \dots < a_n\} \star \{b_0 < \dots < b_m\} := \{a_0 < \dots < a_n < b_0 < \dots < b_m\};$$

the monoidal unit for  $\star$  is the empty set  $\emptyset \in \Delta_+$ . We use the convention  $[-1] := \emptyset \in \Delta_+$  and  $[n \setminus i] := \{i + 1 < \dots < n\}$  for all  $-1 \leq i \leq n$  so that we always have  $[n] = [i] \star [n \setminus i]$ . Given a simplicial object  $\mathcal{X}: \Delta^{\text{op}} \rightarrow \mathcal{C}$ , the **left path object**  $P^{\triangleleft} \mathcal{X}$  and the **right path object**  $P^{\triangleright} \mathcal{X}$  are defined as the compositions

$$P^{\triangleleft} \mathcal{X}: \Delta^{\text{op}} \xrightarrow{[0] \star -} \Delta^{\text{op}} \xrightarrow{\mathcal{X}} \mathcal{C} \quad \text{and} \quad P^{\triangleright} \mathcal{X}: \Delta^{\text{op}} \xrightarrow{- \star [0]} \Delta^{\text{op}} \xrightarrow{\mathcal{X}} \mathcal{C},$$

respectively.

A morphism  $f: [m] \rightarrow [n]$  in  $\Delta$  is called **left active** if it preserves the minimal element (*i.e.*,  $f(0) = 0$ ) and **right active** if it preserves the maximal element (*i.e.*,  $f(m) = n$ ). We call  $f$  **active** if it is both left and right active. Denote by  $\Delta^{\text{min}}$ ,  $\Delta^{\text{max}}$  and  $\Delta^{\text{act}} := \Delta^{\text{min}} \cap \Delta^{\text{max}}$  the wide subcategories of  $\Delta$  containing the left active, right active and active morphisms, respectively. Call a morphism  $f: [m] \rightarrow [n]$  **left strict** (resp. **right strict**) if it satisfies  $f^{-1}\{0\} = \{0\}$  (resp.  $f^{-1}\{n\} = \{m\}$ ). For each  $n \in \mathbb{N}$ , we denote by  $a_n: [1] \rightarrow [n]$  the unique active map; explicitly given as  $a_n(0) = 0$  and  $a_n(1) = n$ .

**3.2. THE CYCLIC CATEGORY.** A **finite cyclic set** is a pair  $(N, T)$  consisting of a finite set  $N$  together with an endomorphism  $T: N \rightarrow N$  which is transitive, *i.e.*, for each  $x, y \in N$  there is some  $i \in \mathbb{N}$  such that  $T^i x = y$ . A **linearly ordered subset**  $L = (L_0, \prec)$  of  $(N, T)$  is a subset  $L_0$  of  $N$  (called the **underlying set** of  $L$ ) equipped with a linear order  $\prec$  such that the restriction of  $T$  to  $L$  agrees with the successor function induced by  $\prec$ . A morphism  $(f, f^*): (N', T') \rightarrow (N, T)$  of finite cyclic sets consists of

- a map of sets  $N' \rightarrow N$  which we also denote by  $f$  and
- an assignment  $f^*$ , which for each linearly ordered subset  $L \subset N$  produces a linearly ordered subset  $f^*L \subset N'$  with underlying set  $f^{-1}L$  such that  $f^*L = f^*L' \star f^*L''$  whenever the linearly ordered subset  $L \subset N$  is decomposed as  $L = L' \star L''$ .

Composition of morphisms  $N'' \xrightarrow{(f', f'^*)} N' \xrightarrow{(f, f^*)} N$  between finite cyclic set is given by the usual composition of underlying set maps and  $(f \circ f')^* = f'^* \circ f^*$ .

**3.2.1. DEFINITION.** [*Connes, 1983*] **Connes' cyclic category**  $\Lambda$  consists of nonempty finite cyclic sets and morphisms between them. A **cyclic object** in some  $(\infty)$ -category  $\mathcal{C}$  is a functor  $\mathcal{X}: \Lambda^{\text{op}} \rightarrow \mathcal{C}$ .

**3.2.2. REMARK.** Following the usual naming convention, a cyclic object in the category of sets would also be called a cyclic set, hence produce a naming clash with the finite cyclic sets introduced above. This will not be an issue since cyclic objects in the category of sets never explicitly appear in this article.

For each  $n \in \mathbb{N}$ , we have the standard finite cyclic set

$$\langle n \rangle := \left( \mathbb{Z} / (n + 1), +1 \right).$$

It is easy to see that every nonempty finite cyclic set is (non-canonically) isomorphic to exactly one such standard cyclic set. Motivated by this, we use the notation  $+m := T^m$  and  $-m := T^{-m}$  for arbitrary finite cyclic sets  $(N, T)$  and omit  $T$  from the notation entirely.

For every finite cyclic set  $(N, +1)$ , the automorphism group  $\text{Aut}_\Lambda(N)$  is cyclic of order  $|N|$  and is generated by the structure morphism  $+1: N \rightarrow N$  where  $(+1)^* := -1$  is given by

$$N \supset L \mapsto L - 1 := \{x - 1 \mid x \in L\} \subset N.$$

Specifying a morphism  $f: N \rightarrow \langle 0 \rangle$  amounts to the choice of what we call a **linear order on the cyclic set  $N$** , namely a linearly ordered subset  $f^*\{0\} \subset N$  with underlying set  $f^{-1}\{0\} = N$ . A commutative triangle

$$\begin{array}{ccc} N' & \xrightarrow{\quad} & N \\ & \searrow f' & \swarrow f \\ & & \langle 0 \rangle \end{array}$$

corresponds precisely to an order preserving map  $f'^*\{0\} \rightarrow f^*\{0\}$ . We conclude that the assignment  $f \mapsto f^*\{0\}$  describes a functor

$$\Lambda_{/\langle 0 \rangle} \xrightarrow{\cong} \Delta,$$

which is easily seen to be an isomorphism of categories between  $\Delta$  and the slice of  $\Lambda$  over  $\langle 0 \rangle$ . Under this identification, the object  $[n] \in \Delta$  corresponds to  $\langle n \rangle \in \Lambda$  which is equipped with the structure map  $[n]: \langle n \rangle \rightarrow \langle 0 \rangle$  induced by the standard linear order  $0 < 1 < \dots < n$  on  $\mathbb{Z} / (n + 1)$ .

Composition in  $\Lambda$  induces a free and transitive right group action

$$\begin{aligned} \Lambda(N, \langle 0 \rangle) \times \text{Aut}_\Lambda(\langle n \rangle) &\longrightarrow \Lambda(N, \langle 0 \rangle); \\ (f, +m) &\longmapsto f^{+m} \end{aligned}$$

which corresponds to cyclic rotation of linear orders: if  $[n]: \langle n \rangle \rightarrow \langle 0 \rangle$  is the structure map corresponding to the standard order  $<$  on  $[n]$ , then  $[n]^{+m}$  corresponds to the linear order  $\prec$  on the set  $\{0, 1, \dots, n\}$  given by  $n - m + 1 \prec \dots \prec n \prec 0 \prec \dots \prec n - m$ .

**3.3. CARTESIAN AND COCARTESIAN CUBES.** Fix a finite set  $S$  and denote by  $\mathbf{P}(S)$  the powerset of  $S$ , partially ordered by inclusion.

**3.3.1. DEFINITION.** An *S-cube* in some  $(\infty\text{-})$ category  $\mathcal{C}$  is a functor  $Q: \mathbf{P}(S) \rightarrow \mathcal{C}$ .

3.3.2. REMARK. Since the poset  $\mathbf{P}(S)$  is canonically isomorphic to its opposite (via the assignment  $S \supseteq T \mapsto S \setminus T$ ), we will often write cubes in an  $(\infty)$ -category  $\mathcal{Z}$  as functors  $\mathbf{P}^{\text{op}}(S) \rightarrow \mathcal{Z}$ . This is convenient when studying contravariant functors  $\mathcal{X}: \mathcal{Z}^{\text{op}} \rightarrow \mathcal{C}$ , where we can then say that the cube  $\mathbf{P}^{\text{op}}(S) \rightarrow \mathcal{Z}$  in  $\mathcal{Z}$  is sent by  $\mathcal{X}$  to the composite  $\mathbf{P}(S) \rightarrow \mathcal{Z}^{\text{op}} \xrightarrow{\mathcal{X}} \mathcal{C}$ ; the main example in this paper is of course the case where  $\mathcal{Z} = \Delta$  and  $\mathcal{X}: \Delta^{\text{op}} \rightarrow \mathcal{C}$  is a simplicial object in  $\mathcal{C}$ .

Let  $s \in S$  and write  $S' := S \setminus \{s\}$ . We have an isomorphism of posets

$$\Delta^1 \times \mathbf{P}(S') \xrightarrow{\cong} \mathbf{P}(S) \tag{11}$$

given by  $(0, T) \mapsto T$  and  $(1, T) \mapsto T \dot{\cup} \{s\}$ . For every  $\infty$ -category  $\mathcal{C}$  we get an induced equivalence

$$\text{Fun}(\mathbf{P}(S), \mathcal{C}) \xrightarrow{\cong} \text{Fun}(\Delta^1, \text{Fun}(\mathbf{P}(S'), \mathcal{C})) \tag{12}$$

of  $\infty$ -categories, which we denote by  $Q \mapsto Q^s$ . We say that a cube  $Q$  is the pasting in  $s$ -direction of two cubes  $Q'$  and  $Q''$  if we have an identification  $Q^s = Q'^s \circ Q''^s$ .

Denote by  $\mathbf{P}_*(S) := \mathbf{P}(S) \setminus \{\emptyset\}$  the poset of *nonempty* subsets of  $S$ .

3.3.3. DEFINITION. An  $S$ -cube  $Q: \mathbf{P}(S) \rightarrow \mathcal{C}$  is called

- **cartesian** if it is a limit diagram in  $\mathcal{C}$ ; i.e., if  $Q$  is the right Kan extension of its restriction to  $\mathbf{P}_*(S)$ .
- **cocartesian** if it is a colimit diagram in  $\mathcal{C}$ ; i.e., if  $Q$  is the left Kan extension of its restriction to  $\mathbf{P}(S) \setminus \{S\}$ .

3.3.4. DEFINITION. An  $S$ -cube  $Q: \mathbf{P}^{\text{op}}(S) \rightarrow \mathcal{Z}$  is called **strongly cartesian** or **strongly cocartesian** if, for each  $T \subset S$  and  $s, s' \in S \setminus T$  with  $s \neq s'$ , the 2-dimensional face

$$\begin{array}{ccc} T & \longrightarrow & T \dot{\cup} \{s\} \\ \downarrow & & \downarrow \\ T \dot{\cup} \{s'\} & \longrightarrow & T \dot{\cup} \{s, s'\} \end{array} \tag{13}$$

is sent by  $Q$  to a pullback square or a pushout square in  $\mathcal{Z}$ , respectively. A cube is called **strongly bicartesian** if it is both strongly cartesian and strongly cocartesian.

3.3.5. REMARK. Denote by  $\mathbf{P}_{\leq 1}^{\text{op}}(S)$  and by  $\mathbf{P}_{\geq |S|-1}^{\text{op}}(S)$  the subposet of  $\mathbf{P}^{\text{op}}(S)$  spanned by the subsets  $T \subset S$  of cardinality  $|T| \leq 1$  and  $|T| \geq |S| - 1$ , respectively. It is easy to see that a cube  $Q: \mathbf{P}^{\text{op}}(S) \rightarrow \mathcal{Z}$  is strongly cartesian if and only if it is the right Kan extension of its restriction to  $\mathbf{P}_{\leq 1}^{\text{op}}(S)$ ; it is strongly cocartesian if and only if it is the left Kan extension of its restriction to  $\mathbf{P}_{\geq |S|-1}^{\text{op}}(S)$ .

3.3.6. REMARK. If  $|S| \geq 2$ , then every strongly (co)cartesian cube is also (co)cartesian; thus justifying the terminology. Beware however, that for  $|S| = 1$  an  $S$ -cube is just an arrow; it is always strongly bicartesian and is (co)cartesian if and only if it is an equivalence.

3.3.7. LEMMA. Let  $\mathcal{C}$  be an  $\infty$ -category. Let  $s \in S$  and put  $S' := S \setminus \{s\}$ . The restriction functor

$$p: \text{Fun}(\mathbf{P}(S'), \mathcal{C}) \longrightarrow \text{Fun}(\mathbf{P}_*(S'), \mathcal{C}) \tag{14}$$

is a cocartesian fibration which is cartesian if and only if  $\mathcal{C}$  admits limits of shape  $\mathbf{P}_*(S)$ . An  $S$ -cube  $Q: \mathbf{P}(S) \rightarrow \mathcal{C}$  is cartesian if and only if the corresponding edge  $Q^s: \Delta^1 \rightarrow \text{Fun}(\mathbf{P}(S'), \mathcal{C})$  is  $p$ -cartesian.

PROOF. Lemma 3.3.7 is the higher dimensional analog of [Lurie, 2009, Lemma 6.1.1.1]; the proof is essentially the same. ■

We say that an  $S$ -cube  $Q$  is **degenerate in direction**  $s \in S$  if the corresponding natural transformation  $Q^s$  of  $S \setminus \{s\}$ -cubes is an equivalence. It follows directly from Lemma 3.3.7 that **degenerate cubes**—cubes that are degenerate in at least one direction—are automatically cartesian and cocartesian.

The following lemma is a standard argument which is useful to compare cartesian cubes of different dimensions.

3.3.8. LEMMA. Let  $Q: \mathbf{P}(S) \rightarrow \mathcal{C}$  be an  $S$ -cube in an  $\infty$ -category  $\mathcal{C}$  with finite limits. Fix  $s \in S$  and write  $S' := S \setminus \{s\}$ . Assume that the  $S'$ -cube  $Q^s(1): T \mapsto Q(T \cup \{s\})$  is cartesian. Then the canonical map

$$\lim Q|_{\mathbf{P}_*(S)} \longrightarrow \lim Q|_{\mathbf{P}_*(S')} \tag{15}$$

is an equivalence. In particular, the original  $S$ -cube  $Q$  is cartesian if and only if the restricted  $S'$ -cube  $Q|_{\mathbf{P}(S')} = Q^s(0): T \mapsto Q(T)$  is cartesian.

PROOF. Consider the following commutative diagram in  $\mathcal{C}$

$$\begin{array}{ccccc} Q(\emptyset) & \longrightarrow & \lim Q|_{\mathbf{P}_*(S)} & \longrightarrow & Q(\{s\}) \\ & \searrow & \downarrow & \lrcorner & \downarrow \simeq \\ & & \lim Q|_{\mathbf{P}_*(S')} & \longrightarrow & \lim Q^s(1)|_{\mathbf{P}_*(S')} \end{array} \tag{16}$$

which is induced by the universal properties of the various limits. By a standard decomposition argument for limits, the rightmost square in the diagram (16) is cartesian; moreover, the rightmost vertical map is an equivalence by assumption. It follows that the left vertical map is also an equivalence; the result follows. ■

3.4. ČECH CUBES, DESCENT AND WEAK EXCISION. Let  $\mathcal{Z}$  be an  $\infty$ -category.

3.4.1. DEFINITION. Let  $S$  be a finite set. An  $S$ -**pronged claw** (or just  $S$ -**claw**, for short)  $\mathcal{F}$  on an object  $N$  in  $\mathcal{Z}$  is an  $S$ -indexed tuple  $\mathcal{F} = (f_s: I_s \rightarrow N \mid s \in S)$  of maps  $f_s$  in  $\mathcal{Z}$  with common codomain  $N \in \mathcal{Z}$  or, equivalently, a diagram  $\mathcal{F}: \mathbf{P}_{\leq 1}^{\text{op}}(S) \rightarrow \mathcal{Z}$  with  $\mathcal{F}(\emptyset) = N$ .

3.4.2. EXAMPLE. A [2]-pronged claw  $\mathcal{F} \models N$  looks as follows

$$\begin{array}{ccccc}
 I_0 & & I_1 & & I_2 \\
 & \searrow & \downarrow & \swarrow & \\
 & f_0 & f_1 & f_2 & \\
 & & N & & 
 \end{array} \tag{17}$$

(recall that  $[2] \in \Delta$  has three elements).

Given an  $S$ -claw  $\mathcal{F} = (f_s: I_s \rightarrow N \mid s \in S)$  on  $N \in \mathcal{Z}$ , we write  $\mathcal{F} \models N$  to make the codomain  $N$  explicit in the notation while keeping the  $f_s$ , the  $I_s$  and sometimes even the  $S$  anonymous. In a similar spirit we will use the symbol  $f \in \mathcal{F}$  to mean  $f_s$  for some  $s$ . With this convention  $f_s$  and  $f_{s'}$  should be considered distinct if  $s \neq s'$ , even if they are the same map in  $\mathcal{Z}$ . Each subset  $T \subset S$  induces a restricted  $T$ -claw of  $\mathcal{F}$  given by  $\mathcal{F}|_T := (f_t \mid t \in T) \models N$ .

3.4.3. DEFINITION. An  $S$ -claw  $\mathcal{F} \models N$  in  $\mathcal{Z}$  is called a **candidate  $S$ -covering** if it can be extended to a strongly cartesian  $S$ -cube  $\check{\mathcal{C}}\mathcal{F}: \mathbf{P}^{\text{op}}(S) \rightarrow \mathcal{Z}$ . In this case we call  $\check{\mathcal{C}}\mathcal{F}$  the **Čech cube** associated to  $\mathcal{F}$ .

If it exists, the Čech cube  $\check{\mathcal{C}}\mathcal{F}$  is given by the formula

$$S \supseteq T \longmapsto \lim \mathcal{F}|_T. \tag{18}$$

We shall sometimes think of the prongs  $f_s: I_s \rightarrow N$  as generalized subobjects of  $N$ ; the values (18) of the Čech cube should then be thought of as generalized intersections. In this spirit it is sometimes convenient to use the notation  $\bigcap_{t \in T} f_t := \check{\mathcal{C}}\mathcal{F}(T) = \lim \mathcal{F}|_T$  and denote, for instance, the Čech square of two maps  $f: I \rightarrow N$  and  $f': I' \rightarrow N$  as follows:

$$\begin{array}{ccc}
 I \cap I' & \xrightarrow{f \cap I'} & I' \\
 I \cap f' \downarrow & \searrow f \cap f' & \downarrow f' \\
 I & \xrightarrow{f} & N
 \end{array} \tag{19}$$

3.4.4. DEFINITION. Let  $\mathcal{F}$  be a candidate covering in  $\mathcal{Z}$ . A functor  $\mathcal{X}: \mathcal{Z}^{\text{op}} \rightarrow \mathcal{C}$  is said to **satisfy descent with respect  $\mathcal{F}$**  if it sends the Čech cube  $\check{\mathcal{C}}\mathcal{F}$  to a cartesian cube in  $\mathcal{C}$ ; in this case we also say that  $\mathcal{F}$  is  **$\mathcal{X}$ -local**.

Following Boavida de Brito and Weiss we say that a **coverage**  $\tau$  on  $\mathcal{Z}$  is a collection of candidate coverings. If  $\mathcal{F} \models N$  is an element of  $\tau$  then we say that  $\mathcal{F}$  is a  **$\tau$ -covering**; if the coverage  $\tau$  is implicit from the context then we say that  $\mathcal{F}$  is a **covering** of  $N$ .

3.4.5. DEFINITION. A  $\mathcal{C}$ -valued **sheaf** for the coverage  $\tau$  is a functor  $\mathcal{X}: \mathcal{Z}^{\text{op}} \rightarrow \mathcal{C}$  which satisfies descent with respect to all  $\tau$ -coverings.

3.4.6. **REMARK.** For each  $k \geq 0$ , there is a canonical coverage  $\tau_k$  on  $\mathcal{Z}$  which consists of all candidate  $[k]$ -coverings. A presheaf  $\mathcal{Z}^{\text{op}} \rightarrow \mathcal{C}$  is a sheaf for this coverage  $\tau_k$  if and only if it is an  $k$ -excisive (covariant) functor in the sense of Goodwillie [1992], i.e., if it sends strongly cocartesian  $[k]$ -cubes in  $\mathcal{Z}^{\text{op}}$  to cartesian cubes in  $\mathcal{C}$ .

We say that an  $S$ -claw is **strongly bicartesian** if it is a candidate covering and if its Čech cube is strongly cocartesian (hence strongly bicartesian).

3.4.7. **DEFINITION.** A functor  $\mathcal{Z}^{\text{op}} \rightarrow \mathcal{C}$  is called **weakly  $S$ -excisive** if it is a sheaf for the coverage of strongly bicartesian  $S$ -claws, i.e., if it sends all strongly bicartesian  $S$ -cubes to cartesian cubes in  $\mathcal{C}$ .

We will also need the following relative notion:

3.4.8. **DEFINITION.** Let  $\mathcal{Z} \rightarrow \mathcal{Z}'$  be a limit-preserving functor. We call a functor  $\mathcal{X}: \mathcal{Z}^{\text{op}} \rightarrow \mathcal{C}$  **weakly  $S$ - $\mathcal{Z}'$ -excisive** (with the functor  $\mathcal{Z} \rightarrow \mathcal{Z}'$  left implicit) if it is a sheaf with respect to those candidate  $S$ -coverings which become strongly bicartesian in  $\mathcal{Z}'$ .

Clearly the property of being weakly  $S$ -excisive (both in the relative and in the absolute sense) only depends on the cardinality of  $S$ . For  $k \in \mathbb{N}$ , we say that  $\mathcal{X}: \Delta^{\text{op}} \rightarrow \mathcal{C}$  is **weakly  $k$ -excisive** if it is weakly  $[k]$ -excisive. We will stick to  $S$ -cubes instead of  $[k]$ -cubes whenever possible, because the latter might suggest a dependency on the linear order of the coordinates.

3.4.9. **REMARK.** In the setting of Definition 3.4.8, if every candidate covering in  $\mathcal{Z}'$  admits a lift to a candidate covering in  $\mathcal{Z}$  then a functor  $\mathcal{Z}'^{\text{op}} \rightarrow \mathcal{C}$  is weakly  $S$ -excisive if and only if its restriction to  $\mathcal{Z}$  is weakly  $S$ - $\mathcal{Z}'$ -excisive.

## 4. Strongly bicartesian cubes in $\Delta$ and $\Lambda$

The goal of this section is to classify and explicitly describe the strongly bicartesian cubes in the simplex category and the cyclic category.

### 4.1. STRONGLY BICARTESIAN CUBES IN THE SIMPLEX CATEGORY.

4.1.1. **DEFINITION.** An  $S$ -claw  $\mathcal{F} = (f_s \mid s \in S)$  on  $[n]$  in  $\Delta_+$  is called

- **backwards compatible** if for each  $i \in [n]$  there is at most one  $s \in S$  such that the preimage  $f_s^{-1}\{i\}$  has more than one element;

- **compatible** if it satisfies the following two conditions:

**BC1** for each  $i \in [n]$ , there is at most one  $s \in S$  such that the preimage  $f_s^{-1}\{i\}$  is not a singleton;

**BC2** for each  $0 < i \leq n$ , there is at most one  $s \in S$  such that the subset  $\{i-1, i\} \subseteq [n]$  is not contained in the image of  $f_s$ .

4.1.2. **REMARK.** The  $S$ -claw  $\mathcal{F}$  satisfies condition **BC1** if and only if it is backwards compatible and: if the preimage  $f_s^{-1}\{i\}$  is empty for some  $i \in [n]$  and  $s \in S$  then the preimage  $f_{s'}^{-1}\{i\}$  is a singleton for all  $s' \in S \setminus s$ . In the language of Section 2, condition **BC2** says precisely that the images of the maps  $f_s$  are of the form  $[n] \setminus A_s$ , where the  $(A_s \mid s \in S)$  are “pairwise disjoint closed subsets” of the “manifold”  $[n]$ .

We call a diagram in  $\Delta_+$  **left active** or **right active** if it takes values in the subcategory of  $\Delta$  spanned by the left active or right active morphisms, respectively.

4.1.3. **REMARK.** It will be useful to visualize  $S$ -claws  $\mathcal{F} \models [n]$  in  $\Delta_+$  as arrays as in the following example (with  $n = 9$  and  $S = [3]$ ):

	0	1	2	3	4	5	6	7	8	9	
0	*	*	*	*	*	3	2	*	*	*	
1	*	$\emptyset$	*	*	*	*	*	*	*	*	(20)
2	*	*	*	$\emptyset$	$\emptyset$	*	*	*	*	*	
3	*	*	*	*	*	*	*	$\emptyset$	2	$\emptyset$	

There is one row for each prong  $f_s: I_s \rightarrow [n]$  of  $\mathcal{F}$  and one column for each  $i \in [n]$ ; in the entry  $(s, i)$  we draw:

- a star  $*$  if the preimage  $f_s^{-1}\{i\}$  is a singleton,
- the symbol  $\emptyset$  if the preimage  $f_s^{-1}\{i\}$  is empty or
- a number  $l$  if the preimage  $f_s^{-1}\{i\}$  has  $l > 1$  many elements.

A claw is backwards compatible if and only if in each column there is at most one entry with more than one star. It is compatible if and only if it satisfies the following two conditions:

- in each column there is at most one “special” entry, *i.e.*, a cell which is not a star  $*$ ;
- each pair of two empty cells is either in the same row or separated by a column with no empty cells.

The example (20) depicts the left active compatible claw

$$\begin{array}{ccc}
 \{0, 2, 3, 4, 5, 6, 7, 8, 9\} & \{0, 1, 2, 5, 6, 7, 8, 9\} & \{0, 1, 2, 3, 4, 5, 6, 8, 8'\} \\
 \swarrow & \downarrow & \swarrow \\
 \{0, 1, 2, 3, 4, 5, 5', 5'', 6, 6', 7, 8, 9\} & \rightarrow [9] & \leftarrow
 \end{array}
 \tag{21}$$

defined by  $i, i', i'' \mapsto i \in [9]$ .



4.1.4. PROPOSITION. Let  $\mathcal{F} \models [n]$  be an  $S$ -claw in  $\Delta_+$ .

- (a) The claw  $\mathcal{F}$  is a candidate  $S$ -covering in  $\Delta_+$  if and only if  $\mathcal{F}$  is backwards compatible. The Čech cube  $\check{\mathcal{C}}\mathcal{F}: \mathbf{P}^{\text{op}}(S) \rightarrow \Delta_+$  is given explicitly by the formula

$$\check{\mathcal{C}}\mathcal{F}: T \longmapsto \star_{i \in [n]} \prod_{t \in T} f_t^{-1}\{i\}. \tag{22}$$

- (b) The  $S$ -claw  $\mathcal{F}$  is strongly bicartesian (i.e., the Čech cube  $\check{\mathcal{C}}\mathcal{F}$  of  $\mathcal{F}$  is strongly bicartesian) if and only if  $\mathcal{F}$  is compatible.

4.1.5. COROLLARY. A claw in  $\Delta$  is strongly bicartesian if and only if it is compatible.

PROOF. Corollary 4.1.5 follows directly from Proposition 4.1.4 and the easy observation that the whole Čech cube of a compatible claw  $\mathcal{F} \models [n]$  in  $\Delta_+$  lies in  $\Delta$  provided that  $n \neq -1$ . ■

4.1.6. EXAMPLE. The  $[1]$ -claw

$$\begin{array}{ccc} 0 & 1 & 2 \\ \hline \emptyset & * & * \\ * & * & \emptyset \end{array} \tag{23}$$

is compatible and gives rise to the bicartesian square

$$\begin{array}{ccc} 1 & \longrightarrow & 12 \\ \downarrow & \square & \downarrow \\ 01 & \longrightarrow & 012 \end{array} \tag{24}$$

in  $\Delta$  which encodes the lowest instance of Rezk’s Segal conditions.

PROOF OF PROPOSITION 4.1.4.

- (a) *A priori*, the formula (22) describes a strongly cartesian extension  $\check{\mathcal{C}}\mathcal{F}: \mathbf{P}^{\text{op}}(S) \rightarrow \mathbf{Pos}$  of  $\mathcal{F}$  in the category of posets. Since the canonical inclusion  $\Delta_+ \hookrightarrow \mathbf{Pos}$  preserves limits, we conclude that  $\check{\mathcal{C}}\mathcal{F}$  is a strongly cartesian extension of  $\mathcal{F}$  in  $\Delta_+$  if and only if  $\check{\mathcal{C}}\mathcal{F}$  takes values in linearly ordered posets. This happens if and only if each product  $\prod_{t \in T} f_t^{-1}\{i\}$  has at most one factor which is not empty or a singleton; this is precisely the backwards compatibility condition on  $\mathcal{F}$ .
- (b) Assume that  $\mathcal{F}$  is backwards compatible so that the Čech cube  $\check{\mathcal{C}}\mathcal{F} := \mathbf{P}^{\text{op}}(S) \rightarrow \Delta_+$  is well defined by part (a). We need to understand when  $\check{\mathcal{C}}\mathcal{F}$  is additionally strongly cocartesian. By definition, the cube  $\check{\mathcal{C}}\mathcal{F}$  is strongly cocartesian if and only if for every subset  $T \subset S$  and every pair of distinct elements  $s, s' \in S \setminus T$ , the square

$$\begin{array}{ccc} \star_{i \in [n]} (f_s^{-1}\{i\} \times f_{s'}^{-1}\{i\} \times \prod_{t \in T} f_t^{-1}\{i\}) & \longrightarrow & \star_{i \in [n]} (f_{s'}^{-1}\{i\} \times \prod_{t \in T} f_t^{-1}\{i\}) =: B' \\ \downarrow & & \downarrow \\ B := \star_{i \in [n]} (f_s^{-1}\{i\} \times \prod_{t \in T} f_t^{-1}\{i\}) & \longrightarrow & \star_{i \in [n]} (\prod_{t \in T} f_t^{-1}\{i\}) =: N \end{array} \tag{25}$$

is a pushout in  $\Delta_+$ .

To show “if” in the claimed equivalence, assume that  $\mathcal{F}$  is compatible; we will show that then each square (25) is a pushout in  $\Delta_+$ . Condition **BC1** implies that, for every  $i \in [n]$ , if one amongst  $f_s^{-1}\{i\}$  and  $f_{s'}^{-1}\{i\}$  is empty then the other is a singleton; it follows that the square (25) is a pushout on the level of underlying sets. It remains to show that a map of sets  $\beta: N \rightarrow M$  is weakly monotone if it is weakly monotone when composed with  $B \rightarrow N$  and  $B' \rightarrow N$ ; for this it is sufficient to show that each pair of adjacent elements in  $N$  is contained in the image of  $B \rightarrow N$  or in the image of  $B' \rightarrow N$ . Let  $x < x + 1 =: x'$  be two adjacent elements of  $N$  and denote by  $i$  and  $i'$  their respective images in  $[n]$ . It is enough to show that the subset  $\{i, i'\} \subseteq [n]$  is contained in the image of  $f_s$  or in the image of  $f_{s'}$ . If  $i = i'$  then this follows from condition **BC1**; if  $i' = i + 1$  then this follows from condition **BC2**. We may therefore assume  $i < i + 1 \leq i' - 1 < i'$ . For each  $i < i'' < i'$  the product  $\prod_{t \in T} f_t^{-1}\{i''\}$  must be empty by adjacency of  $x$  and  $x'$ . Hence there must be  $t, t' \in T$  such that  $f_t^{-1}\{i + 1\}$  and  $f_{t'}^{-1}\{i' - 1\}$  are empty; in particular the subsets  $\{i, i + 1\}$  and  $\{i' - 1, i'\}$  of  $[n]$  are not contained in the image of  $f_t$  and  $f_{t'}$ , respectively. Condition **BC2** implies that the sets  $\{i, i + 1\}$ ,  $\{i' - 1, i'\}$  and, *a fortiori*,  $\{i, i'\}$  are contained in the image of both  $f_s$  and  $f_{s'}$ .

To show “only if”, assume that the cube  $\check{\mathcal{C}}\mathcal{F}$  is strongly bicartesian. We show that conditions **BC1** and **BC2** hold, *i.e.*, that  $\mathcal{F}$  is compatible.

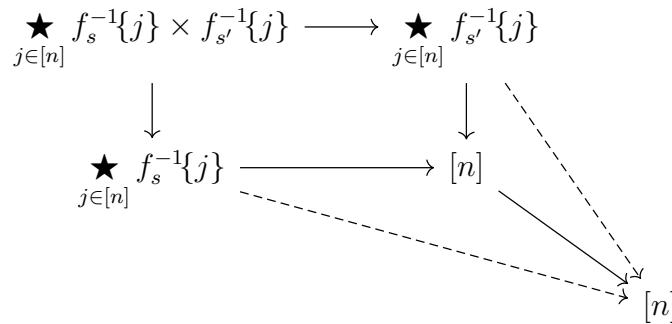
**BC1** Let  $i \in [n]$  and  $s \in S$  be such that  $f_s^{-1}\{i\}$  is empty. For each  $s' \in S \setminus \{s\}$  consider the following commutative diagram, where the inner solid square is the pushout square (25) (for  $T = \emptyset$ ):

$$\begin{array}{ccc}
 \star_{j \in [n]} f_s^{-1}\{j\} \times f_{s'}^{-1}\{j\} & \longrightarrow & \star_{j \in [n]} f_{s'}^{-1}\{j\} \\
 \downarrow & & \downarrow \\
 \star_{j \in [n]} f_s^{-1}\{j\} & \longrightarrow & [n] \\
 \parallel & & \searrow \text{dashed} \\
 \star_{j \in [n] \setminus \{i\}} f_s^{-1}\{j\} & \longrightarrow & [n] \setminus \{i\} \hookrightarrow [i - 1] \star f_{s'}^{-1}\{j\} \star \{i + 1, \dots, n\}
 \end{array}$$

The dashed arrow—which exists by the pushout property—exhibits  $f_{s'}^{-1}\{i\}$  as a retract of the singleton  $\{i\}$ , hence as a singleton itself.

**BC2** Fix  $0 < i \leq n$  and distinct elements  $s, s' \in S$ . Consider the commutative

diagram



where  $[n] \rightarrow [n]$  is the (not order preserving) map that exchanges  $i - 1$  and  $i$ . By the pushout property of the solid square, at least one of the dashed composites must be not order preserving; this can only happen if least one of the maps  $f_s$  and  $f_{s'}$  contains the subset  $\{i - 1, i\} \subseteq [n]$  in its image. ■

4.1.7. REMARK. An  $S$ -claw  $\mathcal{F} = (f_s \mid s \in S)$  is backwards compatible if and only if for each pair of distinct elements  $s, s' \in S$  the induced  $\{s, s'\}$ -subclaw is backwards compatible. Hence it follows from Proposition 4.1.4, that  $\mathcal{F}$  admits a Čech cube in  $\Delta_+$  if and only if each pair  $f_s, f_{s'}$  (for distinct  $s, s' \in S$ ) admits pullback in  $\Delta_+$ . Similarly, an  $S$ -claw admits a strongly bicartesian Čech cube if and only if each two-pronged subclaw is compatible.

4.2. STRONGLY BICARTESIAN CUBES IN THE CYCLIC CATEGORY. In this section, we characterize strongly bicartesian cubes in  $\Lambda$ . To this end, we introduce the cyclic analog of a compatible claw. Heuristically, this corresponds to adding the new “point”  $(n, 0)$  to the “manifold”  $[n] \in \Delta$ .

4.2.1. DEFINITION. An  $S$ -claw  $\mathcal{F} \models [n]$  in  $\Delta$  is called **cyclically compatible** if the claw  $\mathcal{F}$  is compatible and all but at most one  $f \in \mathcal{F}$  have the set  $\{0, n\} \subseteq [n]$  in their image.

4.2.2. REMARK. Let  $\iota: I'' \hookrightarrow I_0 \star I'' \star I_1 = I$  and  $\alpha: I'' \rightarrow I'$  be an inert map and an active map in  $\Delta$ , respectively. Define  $[n] := I_0 \star I' \star I_1$ . It is easy to see that the  $[1]$ -claw  $(I' \hookrightarrow [n], \text{Id} \star \alpha \star \text{Id}: I \rightarrow [n])$  is cyclically compatible and that  $I''$  is the associated pullback. By definition, the *decomposition spaces* of Gálvez-Carrillo, Kock and Tonks [2018a; 2018b; 2018c] are precisely those simplicial objects which send to cartesian squares the bicartesian squares that arise this way.

4.2.3. EXAMPLE. The  $[1]$ -claws

$$\begin{array}{ccc}
 \begin{array}{cccc}
 0 & 1 & 2 & 3 \\
 \hline
 \emptyset & * & * & * \\
 * & * & \emptyset & *
 \end{array} & \text{and} & \begin{array}{cc}
 0 & 1 \\
 \hline
 * & 2 \\
 \emptyset & *
 \end{array}
 \end{array} \tag{26}$$

are cyclically compatible and arise as the pushouts of the inert map  $d^0: [1] \rightarrow [2]$  along the active maps  $d^1: [1] \rightarrow [2]$  and  $s^0: [1] \rightarrow [0]$ , respectively. They encode the first upper 2-Segal condition and an instance of unitality. The  $[1]$ -claw (23) of Example 4.1.6 is not cyclically compatible because the “point”  $(2, 0)$  of the “manifold”  $[2]$  is not covered by any prong; the corresponding Čech square (24) is not cocartesian in the cyclic category.

The following is the main result of this section:

4.2.4. PROPOSITION. *An  $S$ -claw  $\mathcal{F} \models [n]$  in  $\Delta$  has a strongly bicartesian image in  $\Lambda$  if and only if it is cyclically compatible.*

4.2.5. COROLLARY. *The following three classes of  $S$ -cubes in  $\Lambda$  agree:*

- *strongly bicartesian  $S$ -cubes in  $\Lambda$*
- *images of left active strongly bicartesian  $S$ -cubes in  $\Delta$*
- *images of right active strongly bicartesian  $S$ -cubes in  $\Delta$ .*

Before we can prove Proposition 4.2.4 and Corollary 4.2.5 we need a couple of lemmas.

4.2.6. LEMMA. *Let  $\mathcal{F} = (f_s: I_s \rightarrow [n] \mid s \in S)$  be an  $S$ -claw in  $\Delta$ . If  $\mathcal{F}$  is compatible and either left active or right active then  $\mathcal{F}$  is cyclically compatible. Moreover, the following are equivalent:*

1. *the claw  $\mathcal{F}$  is cyclically compatible;*
2. *for every  $m \in [n]$ , the cyclic rotation  $\mathcal{F}^{+m} := (f_s^{+m}: I_s^{+m} \rightarrow [n]^{+m} \mid s \in S)$  of the claw  $\mathcal{F}$  is compatible;*
3. *there is an  $m \in [n]$  such that the cyclic rotation  $\mathcal{F}^{+m}$  of the claw  $\mathcal{F}$  is left active and compatible;*
4. *there is an  $m \in [n]$  such that the cyclic rotation  $\mathcal{F}^{+m}$  of the claw  $\mathcal{F}$  is right active and compatible.*

PROOF. The first statement follows directly from the definitions. It is clear from the definition that the property of being cyclically compatible is preserved under cyclic rotation; hence we have the implications  $(1 \implies 2)$ ,  $(3 \implies 1)$  and  $(4 \implies 1)$ . Given a compatible  $S$ -claw  $\mathcal{F} = (f_s \mid s \in S)$  on  $[n]$  in  $\Delta$ , there is an element  $m \in [n]$  which is in the image of all the  $f_s$ . Then for any such  $m$ , the rotated claws  $\mathcal{F}^{-m}$  and  $\mathcal{F}^{-m-1}$  are left active and right active, respectively. We thus obtain the implications  $(2 \implies 3)$  and  $(2 \implies 4)$ . ■

4.2.7. LEMMA. Let  $Q: \mathbf{P}^{\text{op}}(S) \rightarrow \Lambda$  be an  $S$ -cube in the cyclic category. The following are equivalent:

1. the cube  $Q$  is strongly cartesian;
2. there is a strongly cartesian  $S$ -cube in  $\Delta$  which is mapped to  $Q$  under the canonical functor  $\Delta \rightarrow \Lambda$ ;
3. every  $S$ -cube  $Q'$  in  $\Delta$  which maps to  $Q$  is strongly cartesian.

PROOF. The implications  $2 \implies 1 \implies 3$  follow from the general fact about slice categories that the projection  $\Delta \cong \Lambda_{/\langle 0 \rangle} \rightarrow \Lambda$  preserves and reflects pullbacks. The implication  $3 \implies 2$  holds because the cube  $Q$  lifts to a cube in  $\Delta \cong \Lambda_{/\langle 0 \rangle}$  by choosing any map  $Q(\emptyset) \rightarrow \langle 0 \rangle$ . ■

4.2.8. LEMMA. Let

$$\begin{array}{ccc}
 I \cap I' & \xrightarrow{f \cap I'} & I' \\
 I \cap f' \downarrow & \square & \downarrow f' \\
 I & \xrightarrow{f} & [n]
 \end{array} \tag{27}$$

be the left active bicartesian Čech square associated to a left active compatible claw  $(f, f') \models [n]$  in  $\Delta$ . Then the image in  $\Lambda$  of the square (27) is a pushout.

PROOF. Consider a solid commutative diagram

$$\begin{array}{ccc}
 I \cap I' & \xrightarrow{f \cap I'} & I' \\
 I \cap f' \downarrow & & \downarrow f' \\
 I & \xrightarrow{f} & \langle n \rangle
 \end{array}
 \begin{array}{l}
 \nearrow p' \\
 \dashrightarrow p \\
 \searrow \\
 \end{array}
 \begin{array}{l}
 \\
 \\
 N
 \end{array} \tag{28}$$

in  $\Lambda$ , where the top left square is the image of the square (27). We need to show that there is a unique dashed morphism  $p: \langle n \rangle \rightarrow N$  of cyclic sets making the diagram (28) commute.

- First, we treat the case  $N = \langle 0 \rangle$ . In this case the maps  $p: I \rightarrow \langle 0 \rangle$ ,  $p': I' \rightarrow \langle 0 \rangle$  and  $p'': I \cap I' \rightarrow \langle 0 \rangle$  correspond to cyclic rotations  $\prec$  of the linear order on  $I$ ,  $I'$  and  $I'' := I \cap I'$ , respectively; we have to show that there is a unique linear order  $\prec$  on the cyclic set  $\langle n \rangle$  such that both  $f$  and  $f'$  are order preserving with respect to  $\prec$ . Uniqueness is clear, because by compatibility of  $(f, f')$  each set  $\{i - 1, i\}$  (for  $i \in [n]$ ) is in the image of  $f$  or of  $f'$ .

To construct the linear order  $\prec$  on  $[n]$ , denote by  $x$  and  $x'$  the maximal elements in the linearly ordered sets  $(I, \prec)$  and  $(I', \prec)$ , respectively, *i.e.*, the unique elements

with  $x + 1 \prec x$  and  $x' + 1 \prec x'$ . Without loss of generality, assume  $i' := f(x') \leq f(x) =: i$ . Define  $\prec$  to be the unique linear order on the cyclic set  $\langle n \rangle$  which has  $i$  as its maximum. We need to show that  $f$  and  $f'$  preserve the orders  $\prec$ ; for this it is enough to verify that  $i < f(x + 1)$  and  $i < f'(x' + 1)$  (because  $f(x) \leq i$  and  $f'(x') \leq i$ ).

Denote by  $z'', z'$  and  $z$  the  $\prec$ -minimal elements of  $I'', I'$  and  $I$ , respectively; they satisfy  $(f \cap I')(z'') = z'$ ,  $(I \cap f')(z'') = z$  and  $f(z) = 0 = f'(z')$  because the square (27) was assumed to be left active.

- Assume that  $i = f(x) = f(x + 1)$ . Then by backwards compatibility of  $(f, f')$  we must have a unique  $y' \in I'$  with  $f'(y') = i$ . By the explicit formula for Čech cubes we deduce that the order preserving map (with respect to both  $\prec$  and  $<$ )  $I \cap f': I'' \rightarrow I$  restricts to a bijection  $I'' \cap \{i\} \xrightarrow{\cong} I \cap \{i\}$  which is therefore an isomorphism (with respect to  $\prec$  and  $<$ ). Denote by  $\bar{x}, \bar{x} + 1 \in I''$  the (unique) preimages under  $I \cap f'$  of  $x$  and  $x + 1$ , respectively; they satisfy  $\bar{x} + 1 = \overline{x + 1} \prec \bar{x}$  by the isomorphism property, which means they are the maximal and minimal element of the linearly ordered set  $(I'', \prec)$ , respectively. Since both  $\bar{x}$  and  $\bar{x} + 1$  are mapped to  $y'$  by  $f \cap I'$  we deduce that  $f \cap I': I'' \rightarrow I'$  is constant. This can only happen if  $f$  was already constant and  $f'$  was an equivalence. Hence the square (27) is degenerate and therefore trivially a pushout in  $\Lambda$ .
- The case  $i' = f'(x') = f(x' + 1)$  is analogous.

We may therefore assume that  $x$  and  $x'$  are the maximal elements (with respect to both  $<$  and  $\prec$ ) of their corresponding preimages  $f^{-1}\{i\}$  and  $f'^{-1}\{i'\}$ . It follows directly that  $f(x + 1) > i$  and  $f'(x' + 1) > i'$ ; it remains to show  $f'(x' + 1) > i$  and we may assume that  $i' < i$ . Next, we show that there is no  $j \in [n]$  with  $i' < j \leq i$  which is in the image of  $f'' := f \cap f': I'' \rightarrow [n]$ :

- Otherwise, choose  $w'' \in I''$  with  $f''(w'') = j$ . Set  $w' := (f \cap I')(w'') \in I'$  and  $w := (I \cap f')(w'') \in I$ . We have  $z < w$  and  $z' \leq x' < w'$  by construction and  $w \leq x$  because  $x$  is maximal for  $<$  in the preimage  $f^{-1}\{i\}$ . Hence we have (after cyclic rotation and using that  $x$  and  $x'$  are  $\prec$ -maximal)  $z \prec w \preceq x$  and  $w' \prec z' \preceq x'$ , which implies  $z'' \prec w''$  and  $w'' \prec z''$ , respectively. Contradiction.

Since  $i$  is in the image of  $f$  (by definition) and each  $j$  with  $i' < j \leq i$  is not in the image of  $f''$ , it follows from the compatibility of  $(f, f')$  that each such  $j$  is not in the image of  $f'$ . Since we already know  $f'(x' + 1) > i'$  we obtain  $f'(x' + 1) > i$ , as desired; this concludes the case  $N = \langle 0 \rangle$ .

- We prove the case of a general  $N$ . To see the existence of the dashed map in the diagram (28), choose any map  $N \rightarrow \langle 0 \rangle$ . By the case  $N = \langle 0 \rangle$  which we have just shown, we can fill the dotted morphism  $\langle n \rangle \rightarrow \langle 0 \rangle$  of cyclic sets in the following

commutative diagram

$$\begin{array}{ccc}
 I \cap I' & \xrightarrow{f \cap I'} & I' \\
 I \cap f' \downarrow & & f \downarrow \\
 I & \xrightarrow{f} & \langle n \rangle \\
 & \searrow p & \nearrow p' \\
 & & N \longrightarrow \langle 0 \rangle
 \end{array} \tag{29}$$

Thus we have constructed a diagram in the overcategory  $\Lambda_{/\langle 0 \rangle}$ . Under the canonical identification  $\Delta \cong \Lambda_{/\langle 0 \rangle}$ , the top left square of the diagram (29) gets identified with a cyclic rotation of the original diagram (27). Since any cyclic rotation of a left active compatible claw is compatible, we deduce from Corollary 4.1.5 that the corresponding Čech square is a pushout in  $\Delta \cong \Lambda_{/\langle 0 \rangle}$ . We conclude by the pushout property that the desired dashed map  $\langle n \rangle \rightarrow N$  in (29) and *a fortiori* in (28) exists.

To prove uniqueness, recall that the square (27) is a pushout on the level of underlying sets, so that the dashed map is unique as a function of underlying sets. If  $\langle n \rangle \rightarrow N$  is constant then it factors uniquely as  $\langle n \rangle \rightarrow \langle 0 \rangle \rightarrow N$ , hence is unique by the case  $N = \langle 0 \rangle$ . If  $\langle n \rangle \rightarrow N$  is not constant then it is uniquely determined by its underlying function of sets.

■

**PROOF OF PROPOSITION 4.2.4.** If  $\mathcal{F}$  is cyclically compatible then by Lemma 4.2.6 there is a cyclic rotation  $\mathcal{F}^{-m}$  of  $\mathcal{F}$  which is left active and compatible. Since  $\mathcal{F}$  and  $\mathcal{F}^{-m}$  have isomorphic images in  $\Lambda$ , it is enough to show that the latter image is strongly bicartesian. Since the Čech cube  $\check{\mathcal{C}}\mathcal{F}^{-m}$  is left active and strongly bicartesian, it follows from Lemma 4.2.7 and Lemma 4.2.8 (applied to each 2-dimensional face of the cube) that its image in  $\Lambda$  is still strongly bicartesian.

Conversely, let  $Q$  be a strongly bicartesian cube in  $\Lambda$  extending  $\mathcal{F}$ . Then every choice of  $m \in [n]$  yields a structure map  $[n]^{+m}: Q(\emptyset) = \langle n \rangle \rightarrow \langle 0 \rangle$  which gives rise to a cube  $Q_m$  in  $\Lambda_{/\langle 0 \rangle} \cong \Delta$  that maps to  $Q$  and extends the claw  $\mathcal{F}^{+m}$ . Since the slice projection  $\Delta \rightarrow \Lambda$  reflects pullbacks and pushouts, we deduce that each of these cubes  $Q_m$  is strongly bicartesian. Hence by Corollary 4.1.5 the corresponding claw  $\mathcal{F}^{+m}$  is compatible. We conclude by Lemma 4.2.6 that the original claw  $\mathcal{F}$  is cyclically compatible.

■

**PROOF OF COROLLARY 4.2.5.** Recall from Corollary 4.1.5 that strongly bicartesian  $S$ -cubes in  $\Delta$  are precisely the Čech cubes of compatible  $S$ -claws. Hence Corollary 4.2.5 follows directly from Proposition 4.2.4 and Lemma 4.2.6.

■

**4.3. PRIMITIVE DECOMPOSITION OF BICARTESIAN CUBES.** In this section we show how a strongly bicartesian cube in  $\Delta$  can be decomposed into simpler building blocks.

4.3.1. DEFINITION. A map  $f: I \rightarrow [n]$  in  $\Delta$  is called **primitive** if there is exactly one  $i \in [n]$  such that  $f^{-1}\{i\}$  is not a singleton; the map  $f$  is called **preprimitive** if it is primitive or an isomorphism. A candidate covering  $\mathcal{F}$  in  $\Delta_+$  (and the corresponding Čech cube  $\check{C}\mathcal{F}$ ) is called **(pre)primitive** if the claw  $\mathcal{F}$  consists only of (pre)primitive maps.

4.3.2. CONSTRUCTION. Let  $f: I \rightarrow [n]$  be a map in  $\Delta$ . For each  $i \in \{-1, 0, \dots, n\}$ , we define objects

$$I_i := f^{-1}[i] \star [n \setminus i]$$

in  $\Delta$ . Then  $f$  admits a canonical factorization

$$f: I = I_n \xrightarrow{\bar{f}_n} \dots \xrightarrow{\bar{f}_{i+1}} I_i \xrightarrow{\bar{f}_i} \dots \xrightarrow{\bar{f}_1} I_0 \xrightarrow{\bar{f}_0} I_{-1} = [n] \tag{30}$$

where each map  $\bar{f}_i: I_i \rightarrow I_{i-1}$  is given as

$$\bar{f}_i := \text{Id}_{f^{-1}[i-1]} \star (f \cap \{i\} : f^{-1}\{i\} \rightarrow \{i\}) \star \text{Id}_{[n \setminus i]}.$$

Observe that each map  $\bar{f}_i$  is preprimitive.

4.3.3. LEMMA. Let  $(f: I \rightarrow [n], f': I' \rightarrow [n])$  be backwards compatible and factorize  $f$  as in Construction 4.3.2.

1. For each  $i \in [n]$ , the composition  $I_i \rightarrow [n]$  in (30) is backwards compatible with  $f'$  so that by Proposition 4.1.4 we can form the pullbacks

$$\begin{array}{ccccccccc}
 I \cap I' & \longrightarrow & I_{n-1} \cap I' & \longrightarrow & \dots & \longrightarrow & I_1 \cap I' & \longrightarrow & I_0 \cap I' & \longrightarrow & I' \\
 \downarrow I \cap f' & & \downarrow I_{n-1} \cap f' & & & & \downarrow I_1 \cap f' & & \downarrow I_0 \cap f' & & \downarrow f' \\
 I & \xrightarrow{\bar{f}_n} & I_{n-1} & \xrightarrow{\bar{f}_{n-1}} & \dots & \xrightarrow{\bar{f}_2} & I_1 & \xrightarrow{\bar{f}_1} & I_0 & \xrightarrow{\bar{f}_0} & [n]
 \end{array} \tag{31}$$

$f$

which factorize the Čech square of  $f$  and  $f'$  into smaller Čech squares.

2. The original claw  $(f, f')$  is compatible if and only if the claw  $(\bar{f}_i, I_{i-1} \cap f') \models I_{i-1}$  is compatible for each  $i \in [n]$ .
3. The original claw  $(f, f')$  is cyclically compatible if and only if the claw  $(\bar{f}_i, I_{i-1} \cap f') \models I_{i-1}$  is cyclically compatible for each  $i \in [n]$ .

PROOF. Follows by direct inspection of the explicit constructions. ■



4.3.4. LEMMA.

1. Every strongly bicartesian cube  $Q$  in  $\Delta$  can be decomposed into a pasting of preprimitive strongly bicartesian cubes. If  $Q$  was left active then each of these cubes can be chosen to be left active. If  $Q$  was right active then each of these cubes can be chosen to be right active.
2. Every cube in  $Q$  in  $\Delta$  which becomes strongly bicartesian in  $\Lambda$  can be decomposed into a pasting of preprimitive strongly bicartesian cubes, each of which is left active or right active.
3. If the original cube  $Q$  in 1 or 2 is non-degenerate then the pastings can be chosen to consist of primitive cubes.

PROOF. By Corollary 4.1.5, each strongly bicartesian cube in  $\Delta$  is the Čech cube  $\check{C}\mathcal{F}$  of some compatible  $S$ -claw  $\mathcal{F} = (f_s \mid s \in S)$ . By Proposition 4.2.4, each cube in  $\Delta$  which becomes strongly bicartesian in  $\Lambda$  is of this form  $\check{C}\mathcal{F}$  where  $\mathcal{F}$  is cyclically compatible. For each  $s \in S$ , consider the factorization of  $f_s$  into preprimitive maps from Construction 4.3.2. By a repeated application of Lemma 4.3.3, we can decompose the cube  $\check{C}\mathcal{F}$  into a pasting of Čech cubes of compatible claws which are cyclically compatible if  $\mathcal{F}$  was. Parts 1 and 2 of Lemma 4.3.4 now follow by applying Corollary 4.1.5, Proposition 4.2.4 and by the observing that preprimitive cyclically compatible claws are automatically either left active or right active. Part 3 follows with the same procedure by dropping all identities appearing in the factorizations produced by Construction 4.3.2. ■

5. Precovers and intersection cubes

Let  $\mathcal{F} \models [n]$  be a  $S$ -claw on  $[n]$  in  $\Delta$ . If all of the maps in the claw  $\mathcal{F}$  are injective then we call  $\mathcal{F}$  an **( $S$ -)precover** on  $[n]$ . Since precovers are trivially backwards compatible, Proposition 4.1.4 guarantees the existence of the Čech cube  $\check{C}\mathcal{F}$ ; we call it the **intersection cube** of  $\mathcal{F}$ . If we view the injective maps  $\mathcal{F} \ni f_s: I_s \hookrightarrow [n]$  as subsets  $I_s \subseteq [n]$  of  $[n]$  then the intersection cube of  $\mathcal{F}$  is given explicitly by the intersections

$$T \longmapsto \bigcap_{t \in T} I_t, \tag{32}$$

(where the empty intersection is  $[n]$  by convention); thus the terminology “intersection cube” is justified. A **cover** (not to be confused with *covering* as in Section 3.4) is a precover  $\mathcal{F}$  whose prongs are jointly surjective, *i.e.*,  $\bigcup \mathcal{F} = [n]$  when we identify the prongs of  $\mathcal{F}$  with subsets of  $[n]$ .

5.1. MEMBRANE SPACES AND REFINEMENTS. By right Kan extension along the Yoneda embedding  $\Delta \hookrightarrow \text{Fun}(\Delta^{\text{op}}, \mathbf{Set})$ , we can extend any simplicial object  $\mathcal{X}: \Delta^{\text{op}} \rightarrow \mathcal{C}$  to a functor

$$\mathcal{X}: \text{Fun}(\Delta^{\text{op}}, \mathbf{Set})^{\text{op}} \longrightarrow \mathcal{C}, \tag{33}$$

which we still denote by  $\mathcal{X}$ . Given any simplicial set  $K$ , we can calculate the value of  $\mathcal{X}$  at  $K$ —which Dyckerhoff and Kapranov call the **object of  $K$ -membranes in  $\mathcal{X}$** —by the pointwise formula for Kan extensions:

$$\mathcal{X}_K \simeq \lim \left( (\Delta_{/K})^{\text{op}} \rightarrow \Delta^{\text{op}} \xrightarrow{\mathcal{X}} \mathcal{C} \right) \tag{34}$$

The inclusion  $\Delta \hookrightarrow \text{Fun}(\Delta^{\text{op}}, \mathbf{Set})$  factors as  $\Delta \hookrightarrow \Delta_+ \hookrightarrow \text{Fun}(\Delta^{\text{op}}, \mathbf{Set})$ , where the second map sends the initial object  $\emptyset$  to the initial presheaf. We can therefore evaluate any simplicial object  $\mathcal{X}: \Delta^{\text{op}} \rightarrow \mathcal{C}$  at  $\emptyset$  and the value will be a terminal object in  $\mathcal{C}$ .

Given a candidate covering  $\mathcal{F} = (f_s: I_s \rightarrow [n] \mid s \in S)$  in  $\Delta$ , we obtain a simplicial set  $\tilde{\mathcal{F}}$  as the colimit

$$\tilde{\mathcal{F}} := \text{colim} \left( \mathbf{P}_*^{\text{op}}(S) \xrightarrow{\check{\mathcal{C}}\mathcal{F}} \Delta \hookrightarrow \text{Fun}(\Delta^{\text{op}}, \mathbf{Set}) \right) \tag{35}$$

which comes equipped with a canonical map  $\tilde{\mathcal{F}} \rightarrow \Delta^n$ . It is easy to see that if  $\mathcal{F}$  is a precover (*i.e.*, if all maps  $f_s$  are injective) then  $\tilde{\mathcal{F}} \subseteq \Delta^n$  can be identified with the simplicial subset  $\tilde{\mathcal{F}} := \bigcup_{I_s \in S} \Delta^{I_s}$  of the  $n$ -simplex. We say that a precover  $\mathcal{F}' \models [n]$  is a **refinement** of  $\mathcal{F} \models [n]$ —written  $\mathcal{F}' \preceq \mathcal{F}$ —if and only if  $\tilde{\mathcal{F}}'$  is a simplicial subset of  $\tilde{\mathcal{F}}$ ; explicitly, this means that for every  $I' \in \mathcal{F}'$  there is at least one  $I \in \mathcal{F}$  such that  $I' \subseteq I$  (as subobjects of  $[n]$ ). We say the refinement  $\mathcal{F}' \preceq \mathcal{F}$  is **degenerate** if  $\tilde{\mathcal{F}}' = \tilde{\mathcal{F}}$ . For each  $[n] \in \Delta$  the assignment  $\mathcal{F} \mapsto \tilde{\mathcal{F}}$  describes an equivalence of categories between the category (which is just a preorder) of precovers and refinements on  $[n]$  and the full subcategory of the overcategory  $\text{Fun}(\Delta^{\text{op}}, \mathbf{Set})_{/\Delta^n}$  spanned by the simplicial subsets of  $\Delta^n$ . An explicit inverse is given by identifying each simplicial subset  $K \subseteq \Delta^n$  with the precover given by the maximal simplices of  $K$ . We will implicitly use this identification and write

$$\tilde{\mathcal{F}} := \left( I \mid \Delta^I \hookrightarrow \tilde{\mathcal{F}} \text{ maximal} \right) \models [n] \tag{36}$$

for the precover obtained from a precover  $\mathcal{F}$  by “removing redundant subsets”.

5.1.1. REMARK. For every precover  $\mathcal{F}$ , the restriction  $\check{\mathcal{C}}\mathcal{F}|_{\mathbf{P}_*^{\text{op}}(S)}: \mathbf{P}_*^{\text{op}}(S) \rightarrow \Delta_{+/\tilde{\mathcal{F}}}$  of the Čech cube of  $\mathcal{F}$  has a left adjoint given by

$$([m], \alpha: \Delta^m \rightarrow \tilde{\mathcal{F}}) \longmapsto \{s \in S \mid \alpha(\Delta^m) \subseteq \Delta^{I_s}\} \tag{37}$$

which becomes a right adjoint after passing to opposite categories. Since left adjoints are homotopy initial<sup>8</sup>, the canonical map

$$\mathcal{X}_{\tilde{\mathcal{F}}} \simeq \lim \mathcal{X}|_{(\Delta_{+/\tilde{\mathcal{F}}})^{\text{op}}} \xrightarrow{\simeq} \lim \mathcal{X} \circ \check{\mathcal{C}}\mathcal{F}|_{\mathbf{P}_*(S)} \tag{38}$$

is an equivalence. In particular,  $\mathcal{X}$  satisfies descent with respect to  $\mathcal{F}$  if and only if  $\mathcal{X}$  sends the inclusion  $\tilde{\mathcal{F}} \hookrightarrow \Delta^n$  to an equivalence.

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<sup>8</sup>Here we use the terminology of Dugger [2008]: He calls homotopy terminal what Joyal and Lurie would call cofinal; homotopy initial is then the dual notion.

5.1.2. DEFINITION. We say that a refinement  $\mathcal{F}' \preceq \mathcal{F}$  of precovers  $[n]$  is  $\mathcal{X}$ -local if the induced morphism  $\widetilde{\mathcal{F}'} \rightarrow \widetilde{\mathcal{F}}$  of simplicial sets is sent by  $\mathcal{X}$  to an equivalence in  $\mathcal{C}$ .

The following lemma (which is essentially [Dyckerhoff et al., 2019, Corollary 3.16]) is the main tool to compare to one another descent conditions with respect to various precovers.

5.1.3. LEMMA. Let  $\mathcal{F} \models [n]$  be a precover in  $\Delta$  and  $I \subset [n]$  a subset. Let  $\mathcal{X}: \Delta_+^{\text{op}} \rightarrow \mathcal{C}$  be an augmented simplicial object and assume that the restricted precover

$$\mathcal{F} \cap I := (I' \cap I \mid I' \in \mathcal{F}) \models I$$

on  $I$  is  $\mathcal{X}$ -local. Then the refinement  $\mathcal{F} \preceq \widetilde{\mathcal{F} \cup \{I\}}$  is  $\mathcal{X}$ -local. In particular, the original precover  $\mathcal{F}$  is  $\mathcal{X}$ -local if and only if the extended precover  $\widetilde{\mathcal{F} \cup \{I\}}$  is  $\mathcal{X}$ -local.

PROOF. The refinement  $\mathcal{F} \preceq \widetilde{\mathcal{F} \cup \{I\}}$  can be written as the composition of refinements

$$\mathcal{F} \preceq \mathcal{F} \cup \{I\} \preceq \widetilde{\mathcal{F} \cup \{I\}}. \tag{39}$$

The first refinement in the composition (39) is  $\mathcal{X}$ -local by Lemma 3.3.8 (due to the assumption of the Lemma 5.1.3 and using the identification of Remark 5.1.1); the second refinement is degenerate, hence always local. The claim follows. ■

5.2. POLYNOMIAL SIMPLICIAL OBJECTS. Recalling the analogy to manifold calculus described in Section 2, we observe that compatible precovers can be identified precisely with the “open covers” of the form (9). Indeed, an  $S$ -precover  $\mathcal{F}$  on  $[n] \in \Delta$  is compatible if and only if every “point”  $(x - 1, x)$  of the “manifold”  $[n]$  is contained in all but at most one of the elements of  $\mathcal{F}$ , which we think of as “open subsets” of  $[n]$ ; in other words,  $\mathcal{F}$  consists precisely of “open subsets” with “pairwise disjoint closed complements”. The analogy thus motivates the following definition:

5.2.1. DEFINITION. We call a functor  $\Delta^{\text{op}} \rightarrow \mathcal{C}$  **polynomial of degree**  $\leq |S|$  (or  **$S$ -polynomial**, for short) if  $\mathcal{X}$  satisfies descent with respect to all compatible  $S$ -covers in  $\Delta$ .

5.2.2. EXAMPLE. We depict, for  $k = 1, 2, 3$ , the unique non-degenerate compatible  $[k]$ -cover on  $[2k]$ :

$$\begin{array}{ccc|ccc|cccc}
 0 & 1 & 2 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
 \hline
 \emptyset & * & * & \emptyset & * & * & * & * & \emptyset & * & * & * & * & * & * \\
 \emptyset & * & * & * & * & \emptyset & * & * & * & * & \emptyset & * & * & * & * \\
 * & * & \emptyset & * & * & * & * & \emptyset & * & * & * & * & * & * & \emptyset
 \end{array} \tag{40}$$

Note that for  $n < 2k$ , there are no non-degenerate compatible  $[k]$ -covers on  $[n]$ .

The number of compatible  $S$ -covers on  $[n] \in \Delta$  grows quite rapidly in  $n$ . Thus *a priori* to determine that a simplicial object is  $S$ -polynomial, there is an increasing number of conditions to check in each dimension. We show now that it suffices to check *any one* non-trivial condition in each dimension.

5.2.3. PROPOSITION. *Let  $\mathcal{X}: \Delta^{\text{op}} \rightarrow \mathcal{C}$  be a simplicial object in some  $\infty$ -category with finite limits. Assume that for each  $n \geq 2k$  there exists a non-degenerate compatible  $[k]$ -cover  $\mathcal{F} \models [n]$  in  $\Delta$  which is  $\mathcal{X}$ -local. Then all compatible  $[k]$ -covers are  $\mathcal{X}$ -local.*

PROOF. Assume the assumption of Proposition 5.2.3. Recall that degenerate covers are automatically local. Hence there is nothing to show for  $n < 2k$  because in this case there are no non-degenerate compatible  $[k]$ -covers on  $[n]$ . We prove by induction on  $n \geq 2k$  that all non-degenerate compatible  $[k]$ -covers are  $\mathcal{X}$ -local. The induction start is the case  $n = 2k$ , which is trivial because there is a unique non-degenerate compatible  $[k]$ -cover on  $[2k]$ . For the induction step consider the following directed graph:

- Vertices are non-degenerate compatible  $[k]$ -covers on  $[n]$ .
- Let  $\mathcal{F}$  be a non-degenerate compatible  $[k]$ -cover and let  $I \in \mathcal{F}$  and  $x \in [n] \setminus I$  such that  $I' := I \cup \{x\} \neq [n]$ . Then the cover  $\mathcal{F}' := \widetilde{\mathcal{F} \cup \{I'\}}$  is easily seen to be again  $[k]$ -pronged, compatible and non-degenerate. We add the refinement

$$\mathcal{F} \preceq \widetilde{\mathcal{F} \cup \{I'\}}$$

to the graph as an arrow  $\mathcal{F} \rightarrow \mathcal{F}'$ . Observe that in the language of Remark 4.1.3, the cover  $\mathcal{F}'$  arises from the cover  $\mathcal{F}$  by choosing a row with at least two  $\emptyset$ 's and replacing one of them by  $*$ .

With the notation above it is easy to see that the restricted  $[k]$ -cover  $\mathcal{F} \cap I' \models I'$  is still compatible, hence  $\mathcal{X}$ -local by the induction hypothesis (since  $I' \subsetneq [n]$ ). It follows from Lemma 5.1.3 that every arrow in the graph corresponds to an  $\mathcal{X}$ -local refinement. The proof of Proposition 5.2.3 is concluded by the easy combinatorial observation that the graph is connected as an undirected graph, *i.e.*, one can connect every pair of non-degenerate compatible  $[k]$ -covers by a zigzag of  $\mathcal{X}$ -local refinements as above. ■

5.2.4. REMARK. The directed graph constructed in the proof of Proposition 5.2.3 is just the Hasse diagram of the poset of non-degenerate compatible  $[k]$ -covers under refinement. Our proof therefore shows that if there is an  $n \geq 2k$  such that  $\mathcal{X}$  satisfies descent with respect to all compatible  $[k]$ -covers in  $\Delta_{<n}$  then all refinements between non-degenerate compatible  $[k]$ -covers on  $[n]$  are  $\mathcal{X}$ -local.

## 6. Weakly excisive and weakly $\Lambda$ -excisive simplicial objects

Fix an  $\infty$ -category  $\mathcal{C}$  with finite limits. Recall from Section 3.4 that a simplicial object  $\mathcal{X}: \Delta^{\text{op}} \rightarrow \mathcal{C}$  is

- weakly  $S$ -excisive if it sends strongly bicartesian  $S$ -cubes in  $\Delta$  to cartesian cubes in  $\mathcal{C}$ .
- weakly  $S$ - $\Lambda$ -excisive if it sends to cartesian cubes in  $\mathcal{C}$  those  $S$ -cubes in  $\Delta$  which become strongly bicartesian in  $\Lambda$  after applying the canonical functor  $\Delta \rightarrow \Lambda$ .

6.0.1. **REMARK.** It follows from Remark 3.4.9 that a cyclic object  $\Lambda^{\text{op}} \rightarrow \mathcal{C}$  is weakly  $S$ -excisive if and only if its restriction to  $\Delta$  is weakly  $S$ - $\Lambda$ -excisive.

We can refine the notion of weak  $\Lambda$ -excision as follows:

6.0.2. **DEFINITION.** A simplicial object  $\mathcal{X}: \Delta^{\text{op}} \rightarrow \mathcal{C}$  in  $\mathcal{C}$  is called

- **lower weakly  $S$ - $\Lambda$ -excisive** if  $\mathcal{X}$  sends every left active strongly bicartesian  $S$ -cube in  $\Delta$  to a cartesian cube in  $\mathcal{C}$ ;
- **upper weakly  $S$ - $\Lambda$ -excisive** if  $\mathcal{X}$  sends every right active strongly bicartesian  $S$ -cube in  $\Delta$  to a cartesian cube in  $\mathcal{C}$ .

The terminology is justified by the following easy lemma.

6.0.3. **LEMMA.** A simplicial object is weakly  $S$ - $\Lambda$ -excisive if and only if it is both lower weakly  $S$ - $\Lambda$ -excisive and upper weakly  $S$ - $\Lambda$ -excisive.

**PROOF.** By Lemma 4.3.4, every  $S$ -cube in  $\Delta$  with strongly bicartesian image in  $\Lambda$  can be decomposed into a pasting of strongly bicartesian cubes each of which is left active or right active; thus we have “if”. The converse “only if” follows from the fact (Corollary 4.2.5) that every strongly bicartesian in  $\Delta$  which is left active or right active has a strongly bicartesian image in  $\Lambda$ . ■

6.1. **WEAKLY EXCISIVE = POLYNOMIAL.** As explained in Section 5.2, a polynomial functor of degree  $\geq k$  is a simplicial object  $\Delta^{\text{op}} \rightarrow \mathcal{C}$  which sends all strongly bicartesian intersection  $[k]$ -cubes to cartesian cubes in  $\mathcal{C}$ . *A priori*, this does not agree with weak  $k$ -excision, because it only takes into account strongly bicartesian cubes which consist of *injective* maps. The next theorem states that this discrepancy is illusory both for weak ( $\Delta$ -)excision and for (lower and/or upper) weak  $\Lambda$ -excision.

6.1.1. **THEOREM.** Let  $\mathcal{C}$  be an  $\infty$ -category with all finite limits. A simplicial object  $\mathcal{X}: \Delta^{\text{op}} \rightarrow \mathcal{C}$  is

- (a) weakly  $S$ -excisive if and only if it sends primitive strongly bicartesian intersection  $S$ -cubes in  $\Delta$  to cartesian cubes in  $\mathcal{C}$ ;
- (b) lower weakly  $S$ - $\Lambda$ -excisive if and only if it sends primitive strongly bicartesian left active intersection  $S$ -cubes in  $\Delta$  to cartesian cubes in  $\mathcal{C}$ ;
- (c) upper weakly  $S$ - $\Lambda$ -excisive if and only if it sends primitive strongly bicartesian right active intersection  $S$ -cubes in  $\Delta$  to cartesian cubes in  $\mathcal{C}$ .

Before we prove Theorem 6.1.1, we deduce the following criterion for detecting weak  $\Lambda$ -excision of a simplicial object in terms of weak ( $\Delta$ -)excision of its path objects.

6.1.2. COROLLARY. [Path space criterion] *A simplicial object  $\mathcal{X}: \Delta^{\text{op}} \rightarrow \mathcal{C}$  in an  $\infty$ -category with all finite limits is*

- *lower weakly  $S$ - $\Lambda$ -excisive if and only if the left path object  $P^\triangleleft \mathcal{X} := \mathcal{X} \circ ([0] \star -)$  is weakly  $S$ -excisive;*
- *upper weakly  $S$ - $\Lambda$ -excisive if and only if the right path object  $P^\triangleright \mathcal{X} := \mathcal{X} \circ (- \star [0])$  is weakly  $S$ -excisive.*

PROOF. Observe that composition with the functor  $[0] \star -: \Delta \rightarrow \Delta$  identifies compatible  $S$ -covers in  $\Delta$  with left active compatible  $S$ -covers in  $\Delta$ ; hence by Corollary 4.1.5 it identifies strongly bicartesian intersection  $S$ -cubes in  $\Delta$  with left active strongly bicartesian intersection  $S$ -cubes  $\Delta$ . The first statement of Corollary 6.1.2 now follows directly from Theorem 6.1.1; the proof of the second statement is analogous. ■

6.1.3. REMARK. The proof of Corollary 6.1.2 makes crucial use of Theorem 6.1.1 because in general a left active diagram in  $\Delta$  need not factor through the functor  $[0] \star -: \Delta \rightarrow \Delta$ . It is the fact that we can reduce to diagrams of *injective* maps that makes this argument work.

To prove Theorem 6.1.1 we isolate the following key lemma which we prove separately below. Recall that, for each  $m \geq 0$ , we denote the unique active maps  $[1] \rightarrow [m]$  in  $\Delta$  by  $a_m$ .

6.1.4. LEMMA. [Key lemma] *Let  $p: \mathcal{C} \rightarrow \mathcal{B}$  be a cartesian fibration of  $\infty$ -categories. Let  $\mathcal{X}: \Delta^{\text{op}} \rightarrow \mathcal{C}$  be a simplicial object. Assume that, for all  $m \geq 1$ , the edge  $\mathcal{X}(a_m)$  of  $\mathcal{C}$  is  $p$ -cartesian. Then the edge  $\mathcal{X}(\alpha)$  is also  $p$ -cartesian for every active morphism  $\alpha$  in  $\Delta$ .*

PROOF OF THEOREM 6.1.1. We will prove part (a); the proof for (b) or (c) is the same, word by word, by only considering cubes which are left or right active, respectively. The direction “only if” is trivial.

To prove “if” let  $\mathcal{X}: \Delta^{\text{op}} \rightarrow \mathcal{C}$  be a simplicial object which sends primitive strongly bicartesian intersection  $S$ -cubes in  $\Delta$  to cartesian cubes in  $\mathcal{C}$ . Assume that there is a counterexample to Theorem 6.1.1, *i.e.*, a compatible  $S$ -claw  $\mathcal{F} = (f_s \mid s \in S)$  on  $[n] \in \Delta$  such that the corresponding Čech cube  $\check{\mathcal{C}}\mathcal{F}$  is not sent by  $\mathcal{X}$  to a cartesian cube in  $\mathcal{C}$ . By Lemma 4.3.4 we may choose  $\mathcal{F}$  to be preprimitive. We may assume that  $\mathcal{F}$  is primitive because otherwise it would be degenerate; and degenerate cubes are always sent to cartesian cubes. By induction we may additionally assume that the number

$$d\mathcal{F} := |\{s \in S \mid f_s \text{ is not injective}\}| \tag{41}$$

is minimal amongst all counterexamples. The number  $d\mathcal{F}$  has to be at least one, because otherwise  $\check{\mathcal{C}}\mathcal{F}$  would be an intersection  $S$ -cube which is not a counterexample by assumption. Choose an  $s \in S$  such that  $f_s$  is not injective and write  $S' := S \setminus \{s\}$ . Since  $f_s$  is primitive, it is of the form

$$f_s = \text{Id}_{[i-1]} \star (f_s^{-1}\{i\} \rightarrow \{i\}) \star \text{Id}_{[n \setminus i]}.$$

for some  $i \in [n]$ , where  $|f_s^{-1}\{i\}| > 1$  by assumption. Denote by  $L$ ,  $A$  and  $R$  the  $S$ -claws obtained by restricting the  $S$ -claw  $\mathcal{F}$  to  $[i - 1]$ ,  $\{i\}$  and  $[n \setminus i]$ , respectively. Hence we have  $\mathcal{F} = L \star A \star R$ . Denote by  $L'$  and  $R'$  the  $S'$ -claws induced from  $L$  and  $R$ , respectively. Since the restriction of  $f_s$  to both  $[i - 1]$  and  $[n \setminus i]$  is the identity, the edges

$$\check{C}^s L: \Delta^1 \longrightarrow \text{Fun}(\mathbf{P}^{\text{op}}(S'), \Delta) \quad \text{and} \quad \check{C}^s R: \Delta^1 \longrightarrow \text{Fun}(\mathbf{P}^{\text{op}}(S'), \Delta),$$

corresponding to the Čech cubes  $\check{C}L$  and  $\check{C}R$ , are the identity on the objects  $\check{C}L'$  and  $\check{C}R'$  of  $\text{Fun}(\mathbf{P}^{\text{op}}(S'), \Delta)$ , respectively. Denote by  $\text{const}: \Delta \rightarrow \text{Fun}(\mathbf{P}^{\text{op}}(S'), \Delta)$  the constant-diagram functor and define a cosimplicial object  $Y$  in  $\text{Fun}(\mathbf{P}^{\text{op}}(S'), \Delta)$  by

$$Y: \Delta \xrightarrow{\text{const}} \text{Fun}(\mathbf{P}^{\text{op}}(S'), \Delta) \xrightarrow{\check{C}L' \star (-) \star \check{C}R'} \text{Fun}(\mathbf{P}^{\text{op}}(S'), \Delta)$$

Denote by  $\mathcal{Y}$  the simplicial object

$$\mathcal{Y}: \Delta^{\text{op}} \xrightarrow{Y^{\text{op}}} \text{Fun}(\mathbf{P}^{\text{op}}(S'), \Delta)^{\text{op}} = \text{Fun}(\mathbf{P}(S'), \Delta^{\text{op}}) \xrightarrow{\mathcal{X} \circ -} \text{Fun}(\mathbf{P}(S'), \mathcal{C})$$

and by

$$p: \text{Fun}(\mathbf{P}(S'), \mathcal{C}) \longrightarrow \text{Fun}(\mathbf{P}_*(S'), \mathcal{C})$$

the cartesian fibration of Lemma 3.3.7. Observe, that the value of  $Y$  at the (active) edge  $f_s \cap \{i\}: (f_s^{-1}\{i\} \rightarrow \{i\})$  is precisely the edge  $\check{C}^s \mathcal{F}$  in  $\text{Fun}(\mathbf{P}(S'), \Delta)$  associated to the Čech cube  $\check{C}\mathcal{F}$ . By Lemma 3.3.7, the simplicial object  $\mathcal{X}$  sends the cube  $\check{C}\mathcal{F}$  to a cartesian cube if and only if the edge  $\mathcal{Y}(f_s \cap \{i\})$  is  $p$ -cartesian.

To complete the proof we set up an application of the key lemma (Lemma 6.1.4) to show that this edge  $\mathcal{Y}(f_s \cap \{i\})$  is  $p$ -cartesian, so that the cube  $\check{C}\mathcal{F}$  was not a counterexample after all. Let  $m \geq 1$  and consider the  $S$ -claw  $\mathcal{F}^m = (f_{s'}^m \mid s' \in S)$  on  $[i - 1] \star [m] \star [n \setminus i]$  given by

$$f_{s'}^m := (f_{s'} \cap [i - 1]) \star \text{Id}_{[m]} \star (f_{s'} \cap [n \setminus i])$$

for all  $s' \neq s$  and by

$$f_s^m := \text{Id}_{[i-1]} \star (a_m: [1] \rightarrow [m]) \star \text{Id}_{[n \setminus i]}.$$

It is clear that the  $S$ -claw  $\mathcal{F}^m$  inherits compatibility from  $\mathcal{F}$  and that the Čech cube  $\check{C}\mathcal{F}^m$  corresponds precisely to the edge

$$Y(a_m): \Delta^1 \xrightarrow{a_m} \Delta \xrightarrow{Y} \text{Fun}(\mathbf{P}^{\text{op}}(S'), \Delta).$$

Since we have  $|f_s^{-1}\{i\}| > 1$  by assumption, the compatibility of  $\mathcal{F}$  implies that for each  $s' \in S \setminus \{s\}$  the preimage  $f_{s'}^{-1}\{i\}$  is a singleton; hence the map  $f_{s'}^m$  is injective if and only if  $f_{s'}$  is injective. Furthermore, the map  $f_s^m$  is injective (this is where we use the condition  $m \neq 0$ ); hence the number  $d\mathcal{F}^m$  is smaller than  $d\mathcal{F}$ . By the minimality assumption on the counterexample  $\mathcal{F}$ , we conclude that the simplicial object  $\mathcal{X}$  sends the Čech cube  $\check{C}\mathcal{F}^m$  to a cartesian cube. By Lemma 3.3.7 this translates to the fact that the corresponding edge  $\mathcal{X} \circ \check{C}^s \mathcal{F}^m = \mathcal{Y}(a_m)$  in  $\text{Fun}(\mathbf{P}(S'), \mathcal{C})$  is  $p$ -cartesian. Finally, we apply the key lemma (Lemma 6.1.4) to the cartesian fibration  $p$  and the simplicial object  $\mathcal{Y}$  to deduce that  $\mathcal{Y}$  sends all active maps in  $\Delta$  to  $p$ -cartesian edges; in particular this is true for the active map  $f_s \cap \{i\}: f_s^{-1}\{i\} \rightarrow \{i\}$ . This completes the proof. ■



6.2. PROOF OF THE KEY LEMMA.

6.2.1. CONSTRUCTION. *Via the functor*

$$J \longmapsto J \dot{\cup} \{\infty\}$$

*we identify the augmented simplex category  $\Delta_+$  with the wide subcategory  $\Delta^{\text{rstr}} \subset \Delta^{\text{max}}$  spanned by the right strict morphisms. For every right active morphism  $f: [m] \rightarrow [n]$  in  $\Delta$  we define a left active morphism  $f^-: [n] \rightarrow [m]$  by the formula*

$$f^-: j \longmapsto \min f^{-1} \{j, \dots, n\}.$$

*For every left active morphism  $g: [n] \rightarrow [m]$  in  $\Delta$  we define a left active morphism  $g^+: [m] \rightarrow [n]$  by the formula*

$$g^+: i \longmapsto \max g^{-1} \{0, \dots, i\}.$$

6.2.2. LEMMA. [Joyal duality] *The assignments  $f \mapsto f^-$  and  $g \mapsto g^+$  of Construction 6.2.1 are mutually inverse and assemble to an isomorphism of categories*

$$\Delta^{\text{max}} \xleftarrow{\cong} \Delta^{\text{min,op}}$$

*(given by the identity on objects) which restricts to an isomorphism*

$$\Delta_+ \cong \Delta^{\text{rstr}} \xleftarrow{\cong} \Delta^{\text{act,op}}.$$

PROOF. This is a straightforward calculation. ■

The category  $\Delta^{\text{act}}$  has an initial object  $[1]$  and a terminal object  $[0]$  which, under the identification  $\Delta_+ \cong \Delta^{\text{act,op}}$  of Lemma 6.2.2 correspond to the objects  $[0]$  and  $\emptyset$  of  $\Delta_+$ , respectively.

6.2.3. LEMMA. *Let  $\mathcal{X}: \Delta^{\text{op}} \rightarrow \mathcal{C}$  be a simplicial object in any  $\infty$ -category  $\mathcal{C}$ . Then the restriction of  $\mathcal{X}$  to the subcategory  $\Delta^{\text{act,op}} \subset \Delta^{\text{op}}$  is a limit cone.*

PROOF. [Lurie, 2009, Lemma 6.1.3.16] states (after passing to opposite categories) that every augmented cosimplicial object  $\Delta_+ \cong \Delta^{\text{rstr}} \rightarrow \mathcal{C}$  which extends to a diagram  $\Delta^{\text{max}} \rightarrow \mathcal{C}$  is automatically a limit diagram. Hence by Lemma 6.2.2 every diagram  $\Delta^{\text{min,op}} \rightarrow \mathcal{C}$  and, *a fortiori*, every simplicial object  $\Delta^{\text{op}} \rightarrow \mathcal{C}$  restricts to a limit diagram  $\Delta^{\text{act,op}} \rightarrow \mathcal{C}$ . ■



PROOF OF THE KEY LEMMA, LEMMA 6.1.4. For each active map  $\alpha: [m] \rightarrow [n]$  in  $\Delta$  we have  $a_n \circ \alpha = a_m$ ; hence by the left cancellation property of  $p$ -cartesian edges (see Lurie [2009, Proposition 2.4.1.7]) it is enough to prove that the edge  $\mathcal{X}(a_m)$  is  $p$ -cartesian for all  $m \geq 0$ . For  $m \geq 1$  this is precisely our assumption, so it remains to show that the edge  $\mathcal{X}(a_0: [1] \rightarrow [0])$  is  $p$ -cartesian.

Denote by  $\mathcal{X}^{\text{act}}$  the restriction of  $\mathcal{X}$  to  $\Delta^{\text{act}}$ . Denote by  $\Delta_{\geq 1}^{\text{act}}$  the full subcategory of  $\Delta^{\text{act}}$  spanned by the objects  $[m]$  with  $m \geq 1$ . Applying Lemma 6.2.3 twice we deduce that  $\mathcal{X}^{\text{act}}$  and  $p \circ \mathcal{X}^{\text{act}}$  are limit cones; it follows from [Lurie, 2009, Proposition 4.3.1.5] that  $\mathcal{X}^{\text{act}}$  is also a  $p$ -limit cone, *i.e.*, a right  $p$ -Kan extensions of its restriction to  $\Delta_{\geq 1}^{\text{act,op}}$ . Since the object  $[1] \in \Delta^{\text{act}}$  is initial, the assumption of Lemma 6.1.4 expresses precisely that the restriction of  $\mathcal{X}^{\text{act}}$  to  $\Delta_{\geq 1}^{\text{act,op}}$  is the right  $p$ -Kan extension of its restriction to  $\{[1]\} \subset \Delta^{\text{act}}$ . We conclude by transitivity of  $p$ -Kan extensions (see Lurie [2009, Proposition 4.3.2.8]) that  $\mathcal{X}^{\text{act}}$  is a right  $p$ -Kan extension of its restriction to  $\{[1]\}$ , which implies by the pointwise formula at  $[0] \in \Delta^{\text{act}}$  that the edge  $\mathcal{X}(a_0: [1] \rightarrow [0])$  is  $p$ -cartesian. ■

## 7. Higher Segal conditions

In this last section, we explain the relationship between the higher Segal spaces of Dyckerhoff and Kapranov and the notions of higher weak excision studied in Section 6.

7.1. HIGHER SEGAL COVERS. Fix a positive natural number  $k \geq 1$ . Given a subset  $I \subseteq [n]$ , a **gap** of  $I$  (with  $[n]$  implicit) is an element  $x \in [n]$  with  $x \notin I$ . A gap  $x$  of  $I \subseteq [n]$  is called **even** if the cardinality  $|\{y \in I \mid x < y\}|$  is even. A subset  $I \subseteq [n]$  is called **even** if all its gaps are even. Note that even subsets  $I \subseteq [n]$  of cardinality  $2k$  are precisely those which can be written as a disjoint union of the form

$$I = \bigcup_{i=1}^k \{x_i - 1, x_i\},$$

with  $0 \leq x_1 - 1 < x_1 < x_2 - 1 < \dots < x_{k-1} < x_k - 1 < x_k \leq n$ .

7.1.1. DEFINITION. For each  $n \geq 2k$ , the **lower  $(2k - 1)$ -Segal cover** on  $[n] \in \Delta$  is defined as follows:

$$\text{lSeg}_n^k := \{I \subset [n] \mid I \text{ even with of cardinality } |I| = 2k\} \models [n]$$

Observe that the lower  $(2k - 1)$ -Segal covers are precisely the canonical “good  $k$ -covers” described in Section 2. The first lower  $(2k - 1)$ -Segal cover  $\text{lSeg}_n^k$ , *i.e.*, the one for  $n = 2k$ , is the unique non-degenerate compatible  $[k]$ -cover on  $[n]$ . As  $n$  grows bigger, the behavior of lower  $(2k - 1)$ -Segal covers on  $[n]$  and non-degenerate compatible  $[k]$ -covers on  $[n]$  diverges dramatically: In the first case the number of prongs increasingly rapidly with  $[n]$ , but each subset of  $[n]$  remains of constant size  $2k$ ; in the second case it is the number of prongs  $(k + 1)$  that stays constant, while most of the subsets appearing in a compatible  $[k]$ -cover are large. This dichotomy should remind the reader of the analogous behavior of  $\mathcal{J}_k^\circ$  and  $\mathcal{J}_k^h$  described in Section 2:

- Good  $k$ -covers of a manifold typically consist of a large number of open subsets; however, each of these subsets is simple and small (just a disjoint union of at most  $k$  balls)
- The open covers in  $\mathcal{J}_k^h$  always contain exactly  $k + 1$  open subsets  $M \setminus A_i$ ; however, each of these open subsets is usually big and complicated.

7.1.2. EXAMPLE. The following is a depiction of the first two lower 3-Segal covers:

$$\begin{array}{ccccc}
 & & & & 0 & 1 & 2 & 3 & 4 & 5 \\
 & & & & \hline
 & & & & \emptyset & \emptyset & * & * & * & * \\
 & & & & \emptyset & * & * & \emptyset & * & * \\
 & & & & * & * & \emptyset & \emptyset & * & * \\
 & & & & \emptyset & * & * & * & * & \emptyset \\
 & & & & * & * & \emptyset & * & * & \emptyset \\
 & & & & * & * & * & * & \emptyset & \emptyset \\
 \hline
 0 & 1 & 2 & 3 & 4 & & & & & \\
 \hline
 \emptyset & * & * & * & * & & & & & \\
 * & * & \emptyset & * & * & & & & & \\
 * & * & * & * & \emptyset & & & & & 
 \end{array}
 \quad \text{and} \quad (42)$$

Observe that the left cover is the unique non-degenerate compatible  $[2]$ -cover on  $[4] = [2k]$ .

We now come to the definition of higher Segal objects. The definition we will use is not the original one, but rather a reformulation called the *path space criterion*; see Poguntke [2017, Proposition 2.7].

7.1.3. DEFINITION. A simplicial object  $\mathcal{X} : \Delta^{\text{op}} \rightarrow \mathcal{C}$  is called

- **lower  $(2k - 1)$ -Segal** if, for each  $n \geq 2k$ , it satisfies descent with respect to the lower  $(2k - 1)$ -Segal cover  $\text{lSeg}_n^k$ ;
- **lower  $2k$ -Segal** if the left path object  $P^{\triangleleft} \mathcal{X}$  is lower  $(2k - 1)$ -Segal;
- **upper  $2k$ -Segal** if the right path object  $P^{\triangleright} \mathcal{X}$  is lower  $(2k - 1)$ -Segal;
- **$2k$ -Segal** if  $\mathcal{X}$  is both lower and upper  $2k$ -Segal.

7.2. SEGAL = POLYNOMIAL = WEAKLY EXCISIVE. We come now to the main result of this article, the comparison of higher Segal conditions and weak excision. The key ingredient is the following theorem, which identifies the hierarchy of lower odd Segal objects with the hierarchy of polynomial functors.

7.2.1. THEOREM. Let  $\mathcal{C}$  be an  $\infty$ -category with finite limits. The lower  $(2k - 1)$ -Segal objects in  $\mathcal{C}$  are precisely the polynomial functors  $\Delta^{\text{op}} \rightarrow \mathcal{C}$  of degree  $\leq k$ .

Before we prove Theorem 7.2.1, we use it to deduce our main theorem.

7.2.2. THEOREM. A simplicial object in an  $\infty$ -category with finite limits is

1. lower  $(2k - 1)$ -Segal if and only if it is weakly  $k$ -excisive.
2. lower  $2k$ -Segal if and only if it is lower weakly  $k$ - $\Lambda$ -excisive.
3. upper  $2k$ -Segal if and only if it is upper weakly  $k$ - $\Lambda$ -excisive.
4.  $2k$ -Segal if and only if it is weakly  $k$ - $\Lambda$ -excisive.

PROOF OF THEOREM 7.2.2. In Theorem 6.1.1 we have seen that a functor  $\Delta^{\text{op}} \rightarrow \mathcal{C}$  is polynomial of degree  $\leq k$  if and only if it is weakly  $k$ -excisive; thus part 1 is an immediate consequence of Theorem 7.2.1. The rest of Theorem 7.2.2 then follows immediately from the path space criterion for weak  $\Lambda$ -excision (Corollary 6.1.2). ■

Recall that a cyclic object  $\Lambda^{\text{op}} \rightarrow \mathcal{C}$  is defined to be  $2k$ -Segal if the underlying simplicial object  $\Delta^{\text{op}} \rightarrow \Lambda^{\text{op}} \rightarrow \mathcal{C}$  is  $2k$ -Segal.

7.2.3. COROLLARY. *A cyclic object in an  $\infty$ -category with finite limits is  $2k$ -Segal if and only if it is weakly  $k$ -excisive.*

PROOF. Corollary 7.2.3 follows directly from Theorem 7.2.2 and Remark 6.0.1. ■

We now give the proof of Theorem 7.2.1.

PROOF OF THEOREM 7.2.1. Fix a simplicial object  $\mathcal{X}: \Delta^{\text{op}} \rightarrow \mathcal{C}$  in an  $\infty$ -category  $\mathcal{C}$  with finite limits. By the characterization of strongly bicartesian intersection cubes in  $\Delta$  (Corollary 4.1.5) we only need to show that  $\mathcal{X}$  satisfies descent with respect to all lower  $(2k - 1)$ -Segal covers if and only if  $\mathcal{X}$  satisfies descent with respect to all compatible  $[k]$ -covers. In view of Proposition 5.2.3, we only have to relate, for each  $n \geq 2k$ , the lower  $(2k - 1)$ -Segal cover to *one* non-degenerate compatible  $k$ -cover. For each  $n \geq 2k$  and each  $j \in \{-1, 0, \dots, k\}$ , we define a cover  $\mathcal{F}_j^n \models [n]$  (with the  $k$  left implicit since it is fixed throughout the proof) to consist of the following subsets of  $[n]$ :

- $I_i^n := [n] \setminus \{2i\}$  for  $i = 0, \dots, j$
- those  $I \in \text{lSeg}_k^n$  that satisfy  $[2j] = \{0, 1, \dots, 2j\} \subset I$ .

Clearly  $\mathcal{F}_{-1}^n$  is nothing but the lower  $(2k - 1)$ -Segal cover  $\text{lSeg}_k^n \models [n]$ . Moreover, we have a chain of refinements

$$\text{lSeg}_k^n = \mathcal{F}_{-1}^n \preceq \mathcal{F}_0^n \preceq \dots \preceq \mathcal{F}_k^n \tag{43}$$

because every  $I \in \text{lSeg}_k^n$  with  $[2(j - 1)] \subset I$  must either satisfy  $[2j] \subset I$  or  $2j \notin I$ . The last cover  $\mathcal{F}_k^n = (I_i^n \mid i \in k)$  in the refinement (43) is a non-degenerate compatible  $[k]$ -claw; in this sense, the chain (43) is an interpolation between the Segal condition and the descent condition with respect to the family  $\{\mathcal{F}_k^n \mid n \geq 2k\}$  of non-degenerate compatible  $[k]$ -covers in  $\Delta$ .

We establish the following two facts:

1. If  $n = 2k$  then the chain (43) of refinements collapses, *i.e.*, we have

$$\text{lSeg}_k^{2k} = \mathcal{F}_{-1}^{2k} = \mathcal{F}_0^{2k} = \dots = \mathcal{F}_k^{2k}.$$

2. For every  $n > 2k$  and every  $j = 0, \dots, n$  the refinement  $\mathcal{F}_{j-1}^n \preceq \mathcal{F}_j^n$  is  $\mathcal{X}$ -local provided that the cover  $\mathcal{F}_{j-1}^{n-1} \models [n - 1]$  is  $\mathcal{X}$ -local.

Fact 1 is immediate from the definition. For each  $j = 0, \dots, k$  we have  $\mathcal{F}_j^n = \widetilde{\mathcal{F}_{j-1}^n} \cup \{I_j^n\}$  and the cover  $\mathcal{F}_{j-1}^n \cap I_j^n \models I_j^n$  is easily seen to be isomorphic (under the unique isomorphism  $I_j^n \cong [n-1]$ ) to the cover  $\mathcal{F}_{j-1}^{n-1} \models [n-1]$ ; hence fact 2 follows from Lemma 5.1.3.

By a straightforward inductive argument, facts 1 and 2 imply that the following three conditions are equivalent:

- For all  $n \geq 2k$ , the cover  $\text{lSeg}_k^n = \mathcal{F}_{-1}^n \models [n]$  is  $\mathcal{X}$ -local.
- For all  $n \geq 2k$  and all  $j = -1, \dots, k$ , the cover  $\text{lSeg}_k^n = \mathcal{F}_j^n \models [n]$  is  $\mathcal{X}$ -local.
- For all  $n \geq 2k$ , the (nondegenerate, compatible,  $[k]$ -pronged) cover  $\mathcal{F}_k^n \models [n]$  is  $\mathcal{X}$ -local.

We have therefore related the Segal conditions to one hierarchy of descent conditions with respect to non-degenerate compatible  $[k]$ -covers; Proposition 5.2.3 precisely states that this is enough, hence the proof is concluded. ■

**7.3. TRIVIALITY BOUNDS FOR HIGHER SEGAL OBJECTS.** Let  $\mathcal{X}: \Delta^{\text{op}} \rightarrow \mathcal{C}$  be a lower or upper  $d$ -Segal object in  $\mathcal{C}$ . Since for each  $m > d$  the  $d$ -Segal conditions express the value  $\mathcal{X}_m$  as a cubical limit of the values  $\mathcal{X}_n$  with  $n \leq d$ , it is obvious that  $\mathcal{X}$  is trivial (*i.e.*,  $\mathcal{X}_n$  is a terminal object in  $\mathcal{C}$  for each  $[n] \in \Delta$ ) as soon as  $\mathcal{X}$  is trivial when restricted to  $\Delta_{\leq d}$ . From the comparison with weak excision we can deduce the following sharper bounds:

**7.3.1. PROPOSITION.** *Fix  $d \geq 2$  and let  $\mathcal{X}: \Delta^{\text{op}} \rightarrow \mathcal{C}$  be a lower or upper  $d$ -Segal object in an  $\infty$ -category  $\mathcal{C}$  with finite limits. If  $\mathcal{X}$  is trivial when restricted to  $\Delta_{< d}$  then  $\mathcal{X}$  is trivial.*

**7.3.2. REMARK.** Since not every monoid is trivial, it is not true that a lower 1-Segal object (*i.e.*, a Segal object in the sense of Rezk) is trivial as soon as its restriction to  $\Delta_{\leq 0}$  is trivial. Hence the assumption  $d \geq 2$  in Proposition 7.3.1 is necessary.

**PROOF OF PROPOSITION 7.3.1.** First, we prove the case of lower odd Segal objects. Let  $k \geq 2$  and assume that  $\mathcal{X}: \Delta^{\text{op}} \rightarrow \mathcal{C}$  is lower  $(2k-1)$ -Segal and trivial on  $\Delta_{< 2k-1}$ . It suffices to show that  $\mathcal{X}_{[2k-1]}$  is trivial. Consider the following compatible  $[k]$ -claw  $\mathcal{F}$  on  $[2k-2]$ :

	0	1	2	3	4	⋯	$2k-4$	$2k-3$	$2k-2$
0	*	2	*	*	*	⋯	*	*	*
1	∅	*	*	*	*	⋯	*	*	*
2	*	*	∅	*	*	⋯	*	*	*
3	*	*	*	*	∅	⋯	*	*	*
⋮									
$k-1$	*	*	*	*	*	⋯	∅	*	*
$k$	*	*	*	*	*	⋯	*	*	∅

(44)

The corresponding bicartesian Čech cube  $\check{\mathcal{C}}\mathcal{F}: \mathbf{P}^{\text{op}}([k]) \rightarrow \Delta$  satisfies  $\check{\mathcal{C}}\mathcal{F}(\{0\}) \cong [2k-1]$  and  $\check{\mathcal{C}}\mathcal{F}(T) \in \Delta_{< 2k-1}$  for all  $T \neq \{0\}$ . It follows that the  $[k]$ -cube  $\mathcal{X} \circ \check{\mathcal{C}}\mathcal{F}$  sends every

$T \subseteq [k]$ , except possibly  $T = \{0\}$ , to a terminal object in  $\mathcal{C}$ . Since  $\mathcal{X}$  is weakly  $k$ -excisive by Theorem 7.2.2, this cube in  $\mathcal{C}$  is cartesian. It then follows that we have a cartesian square

$$\begin{array}{ccc}
 (\mathcal{X} \circ \check{\mathcal{F}})(\emptyset) & \longrightarrow & \lim_{\substack{0 \notin T \subseteq [k] \\ \emptyset \neq T}} (\mathcal{X} \circ \check{\mathcal{F}})(T) \\
 \downarrow & \lrcorner & \downarrow \\
 (\mathcal{X} \circ \check{\mathcal{F}})(\{0\}) & \longrightarrow & \lim_{\substack{0 \in T \subseteq [k] \\ \{0\} \neq T}} (\mathcal{X} \circ \check{\mathcal{F}})(T)
 \end{array}$$

in  $\mathcal{C}$ , where all but the lower left corner are trivial; we conclude that  $(\mathcal{X} \circ \check{\mathcal{F}})(\{0\}) \simeq \mathcal{X}_{[2k-1]}$  is also trivial.

If  $d = 2k$  is even with  $k \geq 1$  then the same proof works for lower or upper  $2k$ -Segal objects by considering instead of (44) the left active compatible  $k$ -claw

	0	1	2	3	...	$2k - 3$	$2k - 2$	$2k - 1$
0	2	*	*	*	...	*	*	*
1	*	$\emptyset$	*	*	...	*	*	*
2	*	*	*	$\emptyset$	...	*	*	*
$\vdots$								
$k - 1$	*	*	*	*	...	$\emptyset$	*	*
$k$	*	*	*	*	...	*	*	$\emptyset$

on  $[2k - 1]$  or its obvious right active analog.

Poguntke [2017, Proposition 2.7] proved that a simplicial object is upper  $(2k + 1)$ -Segal if and only if its left path object is upper  $2k$ -Segal (or, equivalently, if its right path object is lower  $2k$ -Segal); the result for upper odd Segal objects thus follows immediately from the one for (lower or upper) even Segal objects. ■

It is not known to the author if the bounds in Proposition 7.3.1 are sharp. More precisely, the author does not know the answer to the following question, which remains to be investigated in future work:

7.3.3. QUESTION. Let  $k \geq 1$  and let  $\mathcal{X}$  be a simplicial object which is lower  $(2k - 1)$ -Segal, or upper  $2k$ -Segal or lower  $2k$ -Segal. If  $\mathcal{X}$  is trivial when restricted to  $\Delta_{\leq k}$ , does it follow that  $\mathcal{X}$  is trivial?

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