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WEIGHTED NORMAL COMMUTATOR AS THE HUQ COMMUTATOR IN POINTS

VAINO TUHAFENI SHAUMBWA

Abstract. We show that the weighted normal commutator is obtained by applying the kernel functor to the Huq commutator of certain morphisms in a category of points over a fixed object. In addition, we compare the local representation (that is, an equivalence relation considered as a subobject in a category of points over a fixed object) of the Smith commutator of a pair of equivalence relations and the Huq commutator of the corresponding local representations, showing that they coincide in a normal Mal'tsev category with finite colimits.

Introduction

The notion of Huq commutator, introduced by Huq in [6], is defined for a pair of morphisms having the same codomain, and it measures how far are two morphisms from *commuting* in the sense of [6].

The *weighted normal commutator* due to M. Gran, G. Janelidze, and A. Ursini [5] is a more general notion of commutator, defined for a pair of morphisms having the same codomain A, and depends on a "weight", which is just a subobject of A. As observed in [5], the Huq commutator of a pair of subobjects of an object A coincides with the weighted normal commutator when the "weight" is the zero morphism $0 : 0 \longrightarrow A$.

The weighted normal commutator is derived from the notion of *weighted centrality* also introduced in [5], and they relate in the same way as the concept of commuting morphisms relates to Huq commutator. As shown in [5], the weighted normal commutator can also be obtained as the normal closure of what is called *weighted subobject commutator*, introduced and studied in [5]. Weighted normal commutator and weighted subobject commutator are together called weighted commutators.

The aim of the present paper is to prove that the weighted normal commutator can be expressed in terms of the Huq commutator, and further show that some relationships between different commutators follow from this fact.

Recall that in a Mal'tsev category $\mathbb{C}$ an equivalence relation (R, r_1, r_2) on an object A can be identified with the diagram

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$$\begin{array}{ccc} R & \overset{\langle r_1, r_2\rangle}{\rightarrowtail} & A \times A \\ {\scriptstyle \triangle_R}\uparrow\downarrow{\scriptstyle r_1} & & {\scriptstyle \langle 1,1\rangle}\uparrow\downarrow{\scriptstyle \pi_1} \\ A & \underset{1}{\longrightarrow} & A \end{array}$$

(with $\triangle_R : A \longrightarrow R$ denoting the morphism arising from reflexivity of the relation), which represents the subobject $\langle r_1, r_2\rangle : (R, r_1, \triangle_R) \rightarrowtail (A \times A, \pi_1, \langle 1, 1\rangle)$ in the category of points over A (that is, the category of split epimorphisms with a fixed choice of a splitting and with a codomain A). Such subobject is called the local representation of (R, r_1, r_2) by D. Bourn, N. Martins-Ferreira, and T. Van der Linden in [4].

D. Bourn [2] showed that two equivalence relations on an object A centralize each other (in the sense of J.D.H. Smith [12] and M. C. Pedicchio [14]), if and only if their corresponding local representations commute in the category of points over A. In a similar way, N. Martins-Ferreira and T. Van der Linden [11] showed that weighted centrality can be reformulated in terms of commuting morphisms in a category of points over a fixed object. In Section 2 we unified the above-mentioned facts from [2] and [11], and this led us to investigate further relationships between weighted normal commutator, Huq commutator, and Smith commutator [12] [14] in Section 3.

Since in a normal Mal'tsev category $\mathbb{C}$ with finite colimits both Smith commutator of a pair of equivalence relations (R, r_1, r_2) and (S, s_1, s_2) on an object A and the Huq commutator of the corresponding local representations can be constructed, we prove (Section 3) that the local representation of the Smith commutator of (R, r_1, r_2) and (S, s_1, s_2) is the Huq commutator of their corresponding local representations. This result is an application of a more general fact about weighted normal commutator and Huq commutator (Theorem 3.6): the weighted normal commutator of subobjects (X, x) and (Y, y) of A over a "weight" (W, w) is obtained by applying the *kernel functor* to the Huq commutator of certain morphisms in the category of points over W.

1. Preliminaries

For convenience, we will begin by recalling some necessary definitions, and also fix some notation. In a pointed category $\mathbb{C}$, we will write 0 to denote the null (zero) morphism between any two objects, and just 1 (instead of 1_X) to denote the identity morphism on any object X. For morphisms $f : A \longrightarrow B$ and $g : A \longrightarrow C$ in a category $\mathbb{C}$ with finite products and coproducts, $\langle f, g\rangle$ will denote the unique morphism $A \longrightarrow B \times C$ such that $f = \pi_1\langle f, g\rangle$ and $g = \pi_2\langle f, g\rangle$, where π_1 and π_2 are the first and second product projections respectively. Dually, for morphisms $u : U \longrightarrow W$ and $v : V \longrightarrow W$ in $\mathbb{C}$, $[u, v]$ will denote the unique morphism $U + V \longrightarrow W$ such that $u = [u, v]i_1$ and $v = [u, v]i_2$, with i_1 and i_2 denoting the coproduct inclusions.

Recall that a pointed finitely complete category $\mathbb{C}$ is unital (in the sense of D. Bourn [3]) if for each pair of objects X and Y in $\mathbb{C}$, the pair of morphisms $\langle 1, 0\rangle : X \longrightarrow X \times Y$

and $\langle 0,1\rangle : Y \longrightarrow X \times Y$ is jointly extremal-epimorphic. It can be easily shown that a pointed finitely complete category $\mathbb{C}$ is unital if and only if for each commutative diagram

$$\begin{array}{ccccc} & \overset{u}{\nearrow} & R & \overset{v}{\nwarrow} & \\ & & \downarrow{\scriptstyle \langle r_1, r_2\rangle} & & \\ A & \underset{\langle f,0\rangle}{\longrightarrow} & X \times Y & \underset{\langle 0,g\rangle}{\longleftarrow} & B, \end{array} \tag{1}$$

the morphism $f \times g : A \times B \longrightarrow X \times Y$ factors through $\langle r_1, r_2\rangle$.

According to Z. Janelidze [8], a pointed finitely complete category $\mathbb{C}$ is subtractive if and only if for every relation $\langle r_1, r_2\rangle : R \rightarrowtail X \times Y$ and a pair of morphisms $f : A \longrightarrow X$ and $g : A \longrightarrow Y$, if $\langle f, g\rangle$ and $\langle f, 0\rangle$ factor through $\langle r_1, r_2\rangle$, then $\langle 0, g\rangle$ factors through $\langle r_1, r_2\rangle$ as well . It is shown in [8] that a pointed finitely complete category $\mathbb{C}$ is strongly unital [3] if and only if it is both unital and subtractive.

Following Z. Janelidze [9], we will define a normal category to be a pointed regular category where every regular epimorphism is a normal epimorphism. Also in this paper, by a normal subobject we will mean a kernel of some morphism.

For each object A in a category $\mathbb{C}$, we write $\mathsf{Pt}(A) \cong ((A, 1) \downarrow (\mathbb{C} \downarrow A))$ to denote the category of points (split epimorphisms) over A, whose objects are triples (X, r, s), with X an object in $\mathbb{C}$ and $r : X \longrightarrow A, s : A \longrightarrow X$ are morphisms such that $rs = 1$. A morphism $f : (X, r, s) \longrightarrow (Y, q, p)$ in $\mathsf{Pt}(A)$ is a morphism $f : X \longrightarrow Y$ in $\mathbb{C}$ such that $qf = r$ and $fs = p$. If $\mathbb{C}$ is a pointed finitely complete category, then for each object A in $\mathbb{C}$ one can define the "kernel functor" from $\mathsf{Pt(A)}$ to $\mathbb{C}$; that is, the functor

$$\mathrm{Ker} : \mathsf{Pt}(A) \longrightarrow \mathbb{C}$$

assigning to every object (X, r, s) the kernel $\mathrm{Ker}(r)$ of r, and every morphism $f : (X, r, s) \to (Y, u, v)$ as in the diagram

$$\begin{array}{ccc} \mathrm{Ker}(r) & \dashrightarrow & \mathrm{Ker}(u) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle r} & & {\scriptstyle v}\uparrow\downarrow{\scriptstyle u} \\ A & \xrightarrow[\;1\;]{} & A \end{array}$$

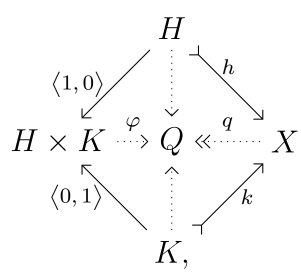
is assigned to the induced morphism $\mathrm{Ker}(r) \dashrightarrow \mathrm{Ker}(u)$. When coproducts also exist in $\mathbb{C}$, the kernel functor above has a left adjoint

$$A + (-) : \mathbb{C} \longrightarrow \mathsf{Pt}(A),$$

which assigns to every object X and every morphism $f : X \longrightarrow Y$ in $\mathbb{C}$, the object $(A + X, [1, 0], i_1)$ and the morphism $1 + f : (A + X, [1, 0], i_1) \longrightarrow (A + Y, [1, 0], i_1)$ respectively.

Recall that a category $\mathbb{C}$ is Mal'tsev if it has finite limits, and every reflexive relation is an equivalence relation. It is well known (see [3]) that a category $\mathbb{C}$ with finite limits is Mal'tsev if and only if, for each object A in $\mathbb{C}$, $\mathsf{Pt}(A)$ is unital. Our main result is stated for a normal Mal'tsev category $\mathbb{C}$ with finite colimits, and one of the reasons is that for every object A in $\mathbb{C}$, $\mathsf{Pt}(A)$ is a normal unital category with finite colimits, which allows, among other things, to construct the Huq commutator in $\mathsf{Pt}(A)$ (see the construction of the Huq commutator below).

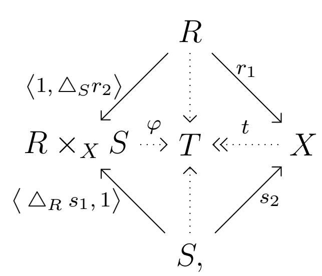
Commuting morphisms and Huq commutator. A pair of morphisms $f : A \longrightarrow X$ and $g : B \longrightarrow X$ in a unital category $\mathbb{C}$ is said to **commute** [6] if there exists a morphism $\varphi : A \times B \longrightarrow X$ such that $f = \varphi\langle 1, 0\rangle$ and $g = \varphi\langle 0, 1\rangle$. For a pair of subobjects (H, h) and (K, k) of an object X in a normal unital category $\mathbb{C}$, the Huq commutator [6] of (H, h) and (K, k) is the smallest normal subobject $\kappa : [H, K]_Q \rightarrowtail X$ such that qh and qk, where q is the cokernel of κ, commute. In a normal unital category $\mathbb{C}$ with finite colimits, the Huq commutator $[H, K]_Q$ always exists, and it is constructed (see e.g [2]) as the kernel of q in the diagram



where Q is the colimit of the outer morphisms.

In a regular unital category $\mathbb{C}$, for composites fr and gs, where r and s are regular epimorphisms, fr and gs commute if and only if f and g commute (see F. Borceux and D. Bourn [1], Proposition 1.6.4). For this reason, the Huq commutator is the same when constructed for a pair of morphisms f and g or for their respective regular images.

Centrality of equivalence relations and Smith commutator. In a regular Mal'tsev category $\mathbb{C}$ with finite colimits, two equivalence relations (R, r_1, r_2) and (S, s_1, s_2) on an object X are said to **centralize** [12][14] each other when there exists a morphism $\phi : R \times_X S \longrightarrow X$ such that $r_1 = \phi\big\langle 1, \triangle_S r_2\big\rangle$ and $s_2 = \phi\big\langle \triangle_R\, s_1, 1\big\rangle$, with $R \times_X S$ denoting the pullback of s_1 along r_2. The Smith commutator $[R, S]_{\mathsf{S}}$ [12][14] of (R, r_1, r_2) and (S, s_1, s_2) is the kernel pair relation of the regular epimorphism t in the diagram



where T is the colimit of the outer morphisms.

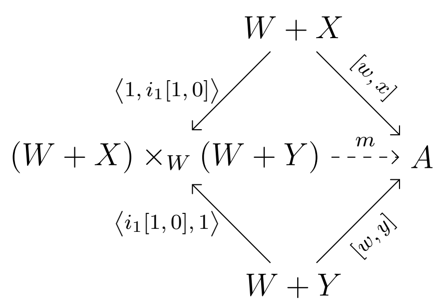
2. Weighted centrality

For morphisms $w : W \longrightarrow A, x : X \longrightarrow A$, and $y : Y \longrightarrow A$ in a pointed category $\mathbb{C}$ with finite limits and colimits, the object $(W+X)\times_W (W+Y) = (W+X)\times_{\langle[1,0],[1,0]\rangle} (W+Y)$ denotes the pullback of $[1,0] : W+X \longrightarrow W$ along $[1,0] : W+Y \longrightarrow W$.

2.1. Definition. [5] Let $w : W \longrightarrow A, x : X \longrightarrow A$, and $y : Y \longrightarrow A$ be morphisms in a pointed category $\mathbb{C}$ with finite limits and colimits. The morphisms x and y **commute over** w if there exists a morphism

$$m : (W+X)\times_W (W+Y) \longrightarrow A$$

making the diagram



(2)

commute.

Note that we will use "commute over" to refer to morphisms commuting in the sense of Definition 2.1, and "commute" will be used for morphisms commuting in the sense of Huq. It is easy to see that a pair of morphisms $x : X \longrightarrow A$ and $y : Y \longrightarrow A$ commute if and only if x and y commute over the zero morphism $0 : 0 \longrightarrow A$.

For morphisms $w : W \longrightarrow A,\ x : X \longrightarrow A$, and $y : Y \longrightarrow A$ in a pointed Mal'tsev category $\mathbb{C}$ with finite colimits, we write

$$\begin{bmatrix} 1 & w \\ 0 & x \end{bmatrix} : W+X \longrightarrow W\times A \text{ and } \begin{bmatrix} 1 & w \\ 0 & y \end{bmatrix} : W+Y \longrightarrow W\times A$$

to denote the morphisms $\langle [1,0],[w,x]\rangle = [\langle 1,w\rangle,\langle 0,x\rangle] : W+X \to W\times A$ and $\langle [1,0],[w,y]\rangle = [\langle 1,w\rangle,\langle 0,y\rangle] : W+Y \to W\times A$ respectively, which give rise to the following cospan in $\mathsf{Pt}(W)$

$$\begin{array}{ccccc} W+X & \xrightarrow{\left[\begin{smallmatrix}1&w\\0&x\end{smallmatrix}\right]} & W\times A & \xleftarrow{\left[\begin{smallmatrix}1&w\\0&y\end{smallmatrix}\right]} & W+Y \\ & {\scriptstyle [1,0]}\searrow\nwarrow{\scriptstyle i_1} & {\scriptstyle \langle 1,w\rangle}\uparrow\downarrow{\scriptstyle \pi_1} & {\scriptstyle i_1}\nearrow\swarrow{\scriptstyle [1,0]} & \\ & & W. & & \end{array} \tag{3}$$

As already observed in [11], x and y commute over w if and only if the morphisms

$$(W+X,[1,0],i_1) \xrightarrow{\left[\begin{smallmatrix}1&w\\0&x\end{smallmatrix}\right]} (W\times A,\pi_1,\langle 1,w\rangle) \xleftarrow{\left[\begin{smallmatrix}1&w\\0&y\end{smallmatrix}\right]} (W+Y,[1,0],i_1)$$

in $\mathsf{Pt}(W)$ commute.

Recall that in a Barr-exact (that is, a regular category where every equivalence relation is a kernel pair relation) normal category $\mathbb{C}$, for every object A, normal monomorphisms (kernels) with codomain A are in bijection with equivalence relations on A. Since in this paper we define normal subobjects to be normal monomorphisms, for every normal subobject in a Barr-exact normal category there is (up to isomorphism) a unique equivalence relation associated to it.

We observe in the next lemma that for a normal subobject (X,x) of A in a Barr-exact normal Mal'tsev category $\mathbb{C}$ with finite colimits, its associated equivalence relation (denormalization) (R_x,r_1,r_2) can be given by the regular image of the morphism $[\langle 1,1\rangle,\langle 0,x\rangle] = \langle [1,0],[1,x]\rangle : A+X \longrightarrow A\times A$, which we shall denote by

$$\begin{bmatrix}1&1\\0&x\end{bmatrix} : A+X \longrightarrow A\times A.$$

2.2. Lemma. *For a normal subobject (X,x) of A in a normal Barr-exact Mal'tsev category $\mathbb{C}$ with finite colimits, its associated equivalence relation (R_x,r_1,r_2) can be given by the join of the morphisms $\langle 1,1\rangle : A \longrightarrow A\times A$ and $\langle 0,x\rangle : X \longrightarrow A\times A$.*

Proof. The join of the morphisms $\langle 1,1\rangle : A \longrightarrow A\times A$ and $\langle 0,x\rangle : X \longrightarrow A\times A$ can be computed as the image R_x in the diagram

$$\begin{array}{ccc} & R_x & \\ {\scriptstyle e}\nearrow & & \searrow{\scriptstyle \langle r_1,r_2\rangle} \\ A+X & \xrightarrow{\left[\begin{smallmatrix}1&1\\0&x\end{smallmatrix}\right]} & A\times A. \end{array}$$

It is clear the diagonal $\langle 1,1\rangle : A \longrightarrow A\times A$ factors through $\langle r_1, r_2\rangle$, and because $\mathbb{C}$ is Mal'tsev, (R_x, r_1, r_2) is an equivalence relation on A. Since $\mathbb{C}$ is Barr-exact, let $q : A \longrightarrow Q$ be the quotient of the equivalence relation (R_x, r_1, r_2). It remains to show that q is the cokernel of x, so that x is indeed the associated normal subobject of (R_x, r_1, r_2). Since $qr_1 = qr_2$, and from the diagram $r_1e = [1, 0]$ and $r_2e = [1, x]$, one has $qx = qr_2ei_2 = qr_1ei_2 = q[1, 0]i_2 = 0$. Writing coker$(x)$ for the cokernel of x, from $qx = 0$, we know that q factors through coker(x). On the other hand, since

$$\mathrm{coker}(x)r_1e = \mathrm{coker}(x)[1,0] = \mathrm{coker}(x)[1,x] = \mathrm{coker}(x)r_2e$$

and e is a (regular) epimorphism, one obtains coker$(x)r_1$ = coker$(x)r_2$, which implies that coker(x) factors through q, since q is the coequalizer of r_1 and r_2. Hence q is the cokernel of x. ∎

According to Proposition 2.3 of [2], in a Mal'tsev category $\mathbb{C}$ two equivalence relations (R, r_1, r_2) and (S, s_1, s_2) on an object A centralize each other if and only if their respective local representations $\langle r_1, r_2\rangle : (R, r_1, \triangle_R) \rightarrowtail (A\times A, \pi_1, \langle 1, 1\rangle)$ and $\langle s_1, s_2\rangle : (S, s_1, \triangle_S) \rightarrowtail (A \times A, \pi_1, \langle 1, 1\rangle)$ commute in $\mathsf{Pt}(A)$.

The fact that a pair of equivalence relations centralize each other if and only if their associated normal subobjects commute over the identity morphism was first observed in [5], through internal pregroupoid structures. In the next proposition we show that this can also be deduced by unifying some results from [2] and [11].

2.3. Proposition. *Let (X, x) and (Y, y) be normal subobjects of A in a normal Barr-exact Mal'tsev category $\mathbb{C}$ with finite colimits. The following statements are equivalent:*

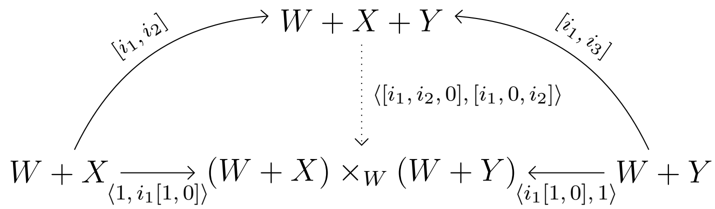
- *(a) x and y commute over $1 : A \longrightarrow A$;*
- *(b) the morphisms*

$$(A + X, [1, 0], i_1) \xrightarrow{\begin{bmatrix} 1 & 1 \\ 0 & x \end{bmatrix}} (A \times A, \pi_1, \langle 1, 1\rangle) \xleftarrow{\begin{bmatrix} 1 & 1 \\ 0 & y \end{bmatrix}} (A + Y, [1, 0], i_1)$$

  in $\mathsf{Pt}(A)$ commute;
- *(c) the local representations corresponding to the associated equivalence relations (R_x, r_1, r_2) and (R_y, r_1, r_2) commute in $\mathsf{Pt}(\mathsf{A})$;*
- *(d) the associated equivalence relations (R_x, r_1, r_2) and (R_y, r_1, r_2) centralize each other.*

Proof. The implication $(a) \Leftrightarrow (b)$ is the case when w is the identity morphism of A in the fact mentioned immediately after diagram (3). Using Lemma 2.2 and the fact that two morphisms commute if and only if their respective regular images also commute, one obtains $(b) \Leftrightarrow (c)$. The implication $(c) \Leftrightarrow (d)$ is Proposition 2.3 of [2] mentioned above. ∎

Weighted commutators. As observed in [5], for subobjects $(X, x), (Y, y)$, and (W, w) of an object A in a normal Mal'tsev category $\mathbb{C}$ with finite colimits, since the diagram

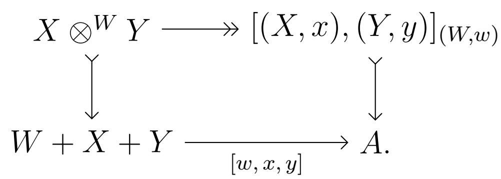


commutes and the pair of morphisms $\langle 1, i_1[1,0]\rangle$ and $\langle i_1[1,0], 1\rangle$ is jointly extremal-epimorphic in $\mathsf{Pt}(W)$ (and so in $\mathbb{C}$), the dotted morphism

$$[\langle 1, i_1[1,0]\rangle, \langle i_1[1,0], 1\rangle] = \langle [i_1, i_2, 0], [i_1, 0, i_2]\rangle : W + X + Y \longrightarrow (W + X) \times_W (W + Y)$$

is a normal epimorphism, and its kernel is denoted by $X \otimes^W Y \rightarrowtail W + X + Y$.

2.4. Definition. [5] For subobjects $(X, x), (Y, y)$, and (W, w) of an object A in a normal Mal'tsev category $\mathbb{C}$ with finite colimits, the **weighted subobject commutator** $[(X, x), (Y, y)]_{(W,w)}$ is obtained as the image under $[w, x, y] : W + X + Y \longrightarrow A$ of the kernel $X \otimes^W Y \rightarrowtail W + X + Y$

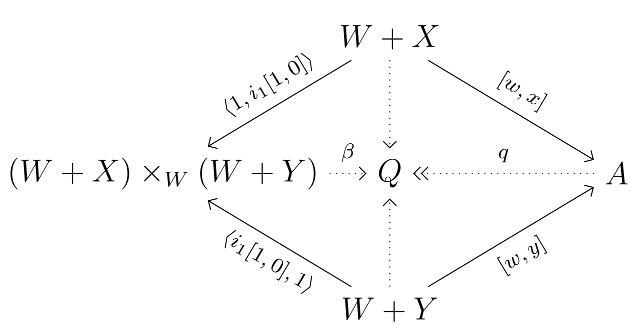


(4)

2.5. Definition. [5] For subobjects $(X, x), (Y, y)$, and (W, w) of an object A in a normal Mal'tsev category $\mathbb{C}$ with finite colimits, the **weighted normal commutator**

$$N[(X, x), (Y, y)]_{(W,w)}$$

is obtained as the kernel of q in the diagram



(5)

where Q is the colimit of the outer morphisms.

When w is the identity morphism of A in Definitions 2.4 and 2.5, the weighted commutators are called 1-weighted subobject commutator and 1-weighted normal commutator respectively, and it is shown in [5] that they always coincide in a Mal'tsev, ideal-determined category [7] (that is, a normal category with finite colimits, where every normal monomorphism is preserved by regular images along regular epimorphisms). Furthermore, it is explained in [5] (see also S. Mantovani [10]) that in a normal Barr-exact Mal'tsev category with finite colimits, the 1-weighted normal commutator (defined on normal subobjects (X, x) and (Y, y) of A) is the associated normal subobject of the Smith commutator of the associated equivalence relations (R_x, r_1, r_2) and (R_y, r'_1, r'_2). We will see in the next section that this fact can also be deduced from a more general result about weighted normal commutator and Huq commutator.

3. Main results

Let $(W, w), (X, x)$, and (Y, y) be subobjects of A in a normal Mal'tsev category $\mathbb{C}$ with finite colimits. We shall establish a relationship between the weighted normal commutator $N[(X, x), (Y, y)]_{(W,w)}$ and the Huq commutator of the pair of morphisms

$$\begin{bmatrix} 1 & w \\ 0 & x \end{bmatrix} : (W + X, [1, 0], i_1) \longrightarrow (W \times A, \pi_1, \langle 1, w\rangle) \text{ and}$$

$$\begin{bmatrix} 1 & w \\ 0 & y \end{bmatrix} : (W + Y, [1, 0], i_1) \longrightarrow (W \times A, \pi_1, \langle 1, w\rangle)$$

in $\mathsf{Pt}(\mathsf{W})$.

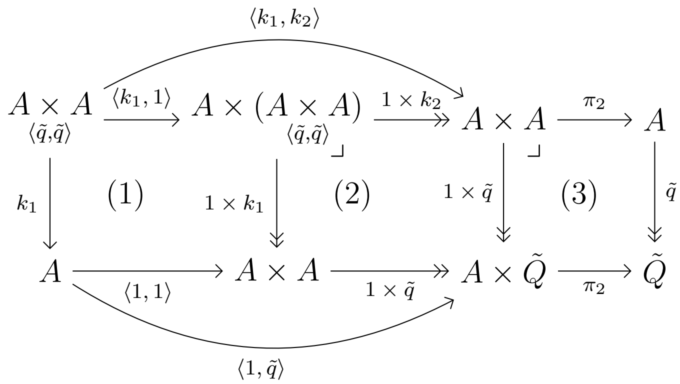
Let us first prove some necessary technical facts.

3.1. Proposition. *Let $\mathbb{C}$ be a regular category, and $\tilde{q} : A \longrightarrow \tilde{Q}$ be a regular epimorphism in $\mathbb{C}$. If $(A \underset{\langle \tilde{q}, \tilde{q}\rangle}{\times} A, k_1, k_2)$ is the kernel pair relation of $\tilde{q}$, then its local representation is the kernel of the morphism*

$$1 \times \tilde{q} : (A \times A, \pi_1, \langle 1, 1\rangle) \longrightarrow (A \times \tilde{Q}, \pi_1, \langle 1, \tilde{q}\rangle)$$

in $\mathsf{Pt}(A)$.

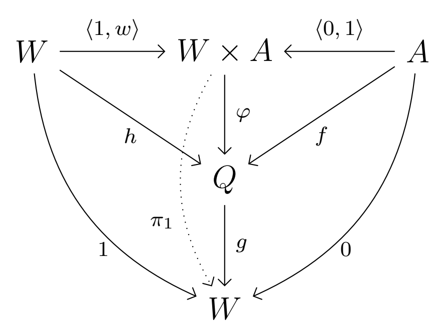
Proof. In the diagram



since diagram (3) and the outer diagram (1) + (2) + (3) are pullbacks, the outer diagram (1) + (2) is also a pullback. The kernel of the morphism $1 \times \tilde{q}$ in $\mathsf{Pt}(A)$ is given by the pullback in $\mathbb{C}$ of $\langle 1, \tilde{q}\rangle$ (that is, the section of the split epimorphism which forms part of the object which is the codomain of $1 \times \tilde{q}$ in $\mathsf{Pt}(A)$) along $1 \times \tilde{q} : A \times A \longrightarrow A \times \tilde{Q}$. Therefore, diagram (1)+(2) being a pullback implies $\langle k_1, k_2\rangle$ is the kernel of the morphism $1 \times \tilde{q}$ in $\mathsf{Pt}(A)$. ■

The following is a slight generalization of Lemma 1.8.18 of [1].

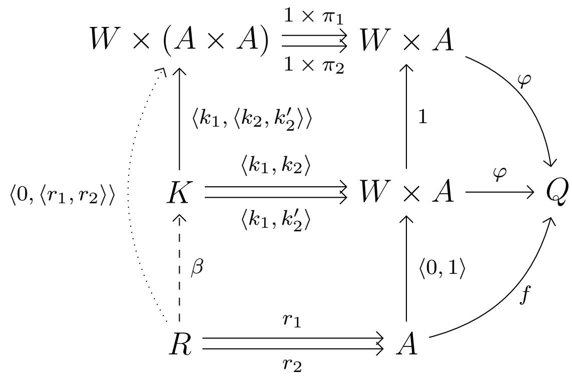
3.2. Proposition. *Let $\mathbb{C}$ be a strongly unital category. Consider the following commutative diagram*



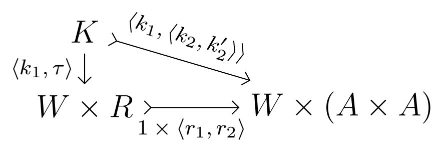
(6)

with $g\varphi = \pi_1$. If (R, r_1, r_2) is the kernel pair relation of f, then $(W \times R, 1 \times r_1, 1 \times r_2)$ is the kernel pair relation of φ.

Proof. Let (K, k, k') be the kernel pair relation of φ. Writing $\langle k, k'\rangle = \langle \langle k_1, k_2\rangle, \langle k_1', k_2'\rangle\rangle : K \rightarrowtail (W \times A) \times (W \times A)$, since $g\varphi = \pi_1$, we see that $k_1 = \pi_1\langle k_1, k_2\rangle = g\varphi\langle k_1, k_2\rangle = g\varphi\langle k_1', k_2'\rangle = \pi_1\langle k_1', k_2'\rangle = k_1'$. Now consider the diagram



where β is the factorization of the pair $\langle 0, r_1 \rangle, \langle 0, r_2 \rangle$ through the kernel pair of φ. It can also be seen that β is the factorization of the morphism $\langle 0, \langle r_1, r_2 \rangle\rangle$ through $\langle k_1, \langle k_2, k_2' \rangle\rangle$. In a similar way, since $\varphi\langle 1, 0 \rangle = \varphi\langle 1, 0 \rangle$, the morphism $\langle 1, \langle 0, 0 \rangle\rangle$ factors through $\langle k_1, \langle k_2, k_2' \rangle\rangle$. Applying the fact about unital categories mentioned here (1), it follows that $1 \times \langle r_1, r_2 \rangle : W \times R \longrightarrow W \times (A \times A)$ factors through $\langle k_1, \langle k_2, k_2' \rangle\rangle$. It remains to show that $\langle k_1, \langle k_2, k_2' \rangle\rangle$ factors through $1 \times \langle r_1, r_2 \rangle : W \times R \longrightarrow W \times (A \times A)$. Since $\varphi\langle k_1, k_2 \rangle = \varphi\langle k_1', k_2' \rangle$ and $\varphi\langle k_1, 0 \rangle = \varphi\langle k_1', 0 \rangle$ (since $k_1 = k_1'$), it follows by subtractivity that $\varphi\langle 0, k_2 \rangle = \varphi\langle 0, k_2' \rangle$. This means $fk_2 = \varphi\langle 0, 1 \rangle k_2 = \varphi\langle 0, 1 \rangle k_2' = fk_2'$, which implies that $\langle k_2, k_2' \rangle$ factors through $\langle r_1, r_2 \rangle$ via a morphism τ. Now it can be seen in the diagram



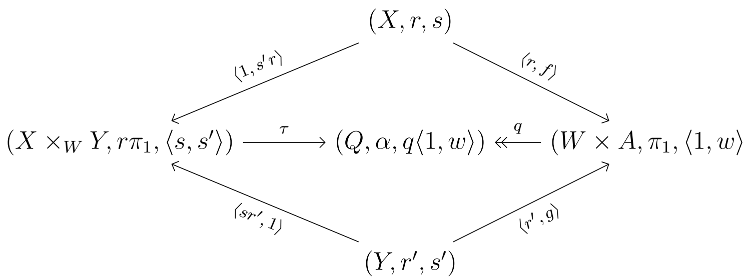
that $\langle k_1, \tau \rangle$ is the factorization of $\langle k_1, \langle k_2, k_2' \rangle\rangle$ through $1 \times \langle r_1, r_2 \rangle$. ∎

As a corollary we obtain the following:

3.3. Corollary. *Let $\mathbb{C}$ be a regular strongly unital category. In diagram (6) of Proposition 3.2, if φ is a regular epimorphism, then it is of the form $1 \times \tilde{q} : W \times A \longrightarrow W \times \tilde{Q}$, where $\tilde{q} : A \longrightarrow \tilde{Q}$ is the regular epimorphism in the (regular epi, mono)-factorization of $\varphi\langle 0, 1 \rangle$.*

3.4. Remark. Corollary 3.3 can be equivalently formulated as follows: For a morphism $w : W \longrightarrow A$ in a regular strongly unital category $\mathbb{C}$, every regular epimorphism φ in $\mathsf{Pt}(W)$ whose domain is $(W \times A, \pi_1, \langle 1, w \rangle)$ is of the form $1 \times \tilde{q} : (W \times A, \pi_1, \langle 1, w \rangle) \longrightarrow (W \times \tilde{Q}, \pi_1, \langle 1, \tilde{q}w \rangle)$, with $\tilde{q} : A \longrightarrow \tilde{Q}$ a regular epimorphism in $\mathbb{C}$.

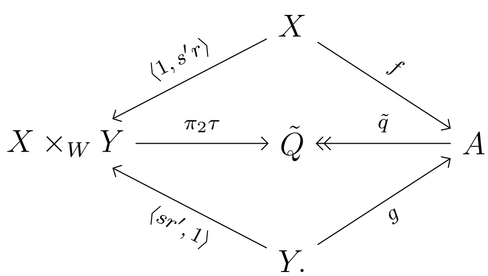
For a morphism $w : W \longrightarrow A$ in a regular Mal'tsev category $\mathbb{C}$ with finite colimits, and for each diagram



(7)

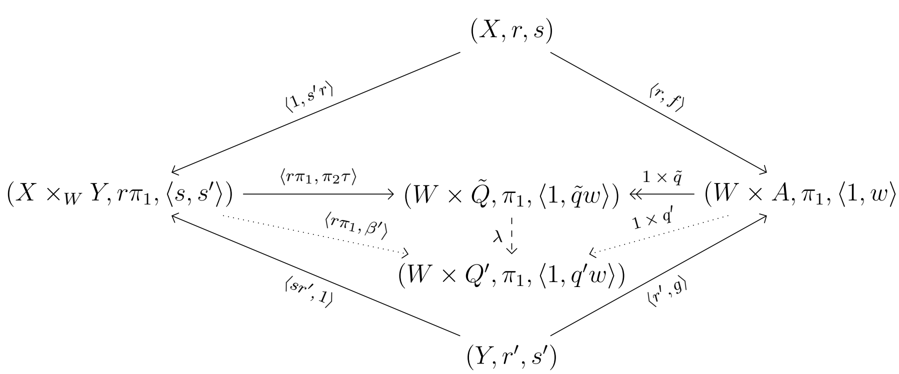
in $\mathsf{Pt}(W)$, where $(Q, \alpha, q\langle 1, w\rangle)$ is the colimit of the outer morphisms and $X \times_W Y$ is the pullback of $r : X \longrightarrow W$ along $r' : Y \longrightarrow W$, the morphism q is a regular epimorphism (see e.g Proposition 1.9 of [2]). So applying Corollary 3.3 through Remark 3.4, Q and q can be chosen to be of the forms $W \times \tilde{Q}$ and $1 \times \tilde{q}$ respectively, where $\tilde{q} : A \longrightarrow \tilde{Q}$ is a regular epimorphism. Furthermore, one can observe the following:

3.5. LEMMA. *Let $\mathbb{C}$ be a normal Mal'tsev category with finite colimits. In diagram (7), writing $(W \times \tilde{Q}, \pi_1, \langle 1, \tilde{q}w\rangle)$ for the colimit of the outer morphisms and $1 \times \tilde{q}$ instead of q, the object $\tilde{Q}$ is the colimit of the outer morphisms in the diagram*



(8)

PROOF. Let $q' : A \longrightarrow Q'$ and $\beta' : X \times_W Y \longrightarrow Q'$ be morphisms making diagram (8) commute. It is not difficult to see that $1 \times q'$ and $\langle r\pi_1, \beta'\rangle$ make the diagram



(9)

commute, and this implies that there is a morphism λ making both lower triangles commute. Applying the kernel functor $\mathrm{Ker} : \mathsf{Pt}(W) \longrightarrow \mathbb{C}$ to the equation $1 \times q' = \lambda(1 \times \tilde{q})$, one obtains $q' = \rho\tilde{q}$, where $\rho : \tilde{Q} \longrightarrow Q'$ is $\mathrm{Ker}(\lambda)$. Since $1 \times \tilde{q}$ is a (normal) epimorphism, we see that $1 \times \rho = \lambda$, and this implies that $\beta' = \rho(\pi_2\tau)$. Thus $\tilde{Q}$ is the colimit of the outer morphisms in diagram (8).

■

Now the main result.

3.6. Theorem. *Let $(X, x), (Y, y)$, and (W, w) be subobjects of an object A in a normal Mal'tsev category $\mathbb{C}$ with finite colimits. The weighted normal commutator*

$$N[(X, x), (Y, y)]_{(W,w)}$$

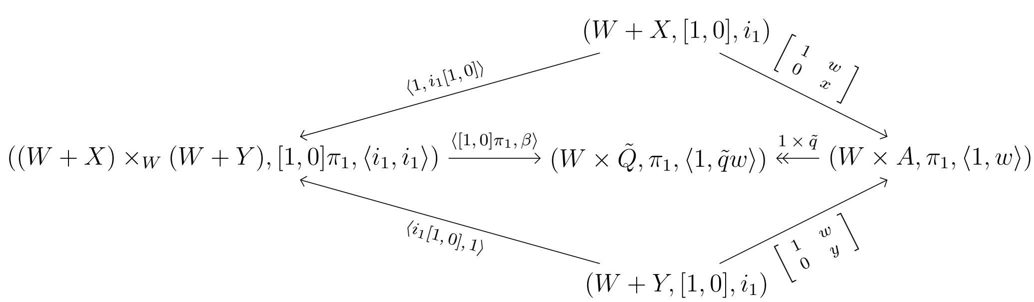
is the image of the kernel functor $\mathrm{Ker} : \mathsf{Pt}(W) \longrightarrow \mathbb{C}$ applied to the Huq commutator of the morphisms

$$\begin{bmatrix} 1 & w \\ 0 & x \end{bmatrix} : (W + X, [1, 0], i_1) \longrightarrow (W \times A, \pi_1, \langle 1, w\rangle) \ \ \textit{and}$$

$$\begin{bmatrix} 1 & w \\ 0 & y \end{bmatrix} : (W + Y, [1, 0], i_1) \longrightarrow (W \times A, \pi_1, \langle 1, w\rangle)$$

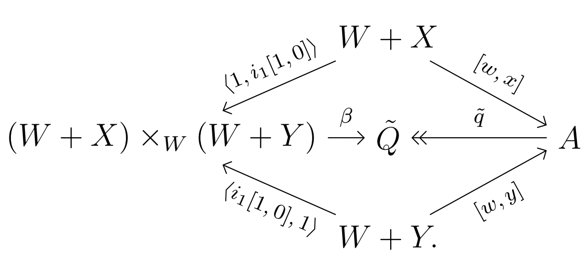
in $\mathsf{Pt}(W)$.

Proof. Consider the diagram



(10)

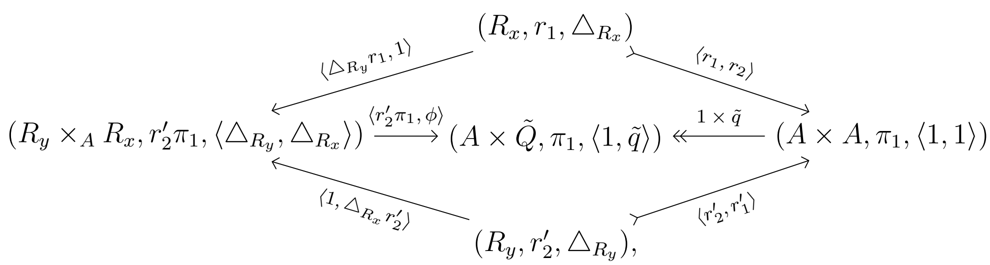
in $\mathsf{Pt(W)}$, where $(W \times \tilde{Q}, \pi_1, \langle 1, \tilde{q}w\rangle)$ is the colimit of the outer morphisms. Applying the previous lemma, $\tilde{Q}$ is the colimit of the outer morphisms in the diagram



(11)

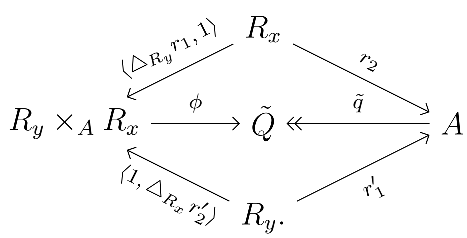
The weighted normal commutator $N[(X,x),(Y,y)]_{(W,w)}$ is the kernel of $\tilde{q}$, and now the result follows immediately from the fact that the kernel functor preserves kernels; the kernel functor sends the kernel of $1 \times \tilde{q}$ (as a morphism in $\mathsf{Pt}(W)$) to the kernel of $\tilde{q}$. ∎

In Theorem 3.6, let us assume $\mathbb{C}$ is a normal Barr-exact Mal'tsev category with finite colimits, (X,x) and (Y,y) are normal subobjects of A, and w is the identity morphism of A. Using Lemma 2.2, the regular images of the morphisms denoted with matrices in diagram (10) (under the above assumptions) are the local representations of the associated equivalence relations (R_x, r_1, r_2) and (R_y, r'_1, r'_2) of normal subobjects (X,x) and (Y,y) respectively. Since $(A \times \tilde{Q}, \pi_1, \langle 1, \tilde{q}\rangle)$ is the colimit of the outer morphisms in diagram (10) (under the above assumptions) if and only if it is the colimit of the outer morphisms in the diagram



(12)

using Lemma 3.5, it follows that $\tilde{Q}$ is the colimit of the outer morphisms in the diagram



(13)

The kernel pair relation of $\tilde{q}$ in diagram (13) is the Smith commutator $[R_y, R_x]_{\mathsf{S}}$, but according to Proposition 3.1, the local representation of the kernel pair relation of $\tilde{q}$ is the kernel (in $\mathsf{Pt}(A)$) of $1 \times \tilde{q}$ in diagram (12), i.e. the local representation of the Smith commutator $[R_y, R_x]_{\mathsf{S}}$ is the Huq commutator of the local representations of (R_x, r_1, r_2) and (R_y, r'_1, r'_2). Note that this observation can be generalized for every pair of equivalence relations on an object in a normal Mal'tsev category $\mathbb{C}$ with finite colimits, by just applying Lemma 3.5 to diagram (12). So we have the following:

3.7. Theorem. *Let (R, r_1, r_2) and (R', r'_1, r'_2) be two equivalence relations on an object A in a normal Mal'tsev category $\mathbb{C}$ with finite colimits. The local representation of the Smith commutator $[R', R]_{\mathsf{S}}$ is the Huq commutator of the local representations of (R, r_1, r_2) and (R', r'_1, r'_2).*

In addition, we also recover the fact that the 1-weighted normal commutator (defined for normal subobjects) is the associated normal subobject of the Smith commutator of their associated equivalence relations (proven independently in [5] and [10], where in [10] the associated normal subobject is called the Ursini commutator): In Theorem 3.6, assuming $\mathbb{C}$ is a normal Barr-exact Mal'tsev category with finite colimits, w is the identity morphism of A, and (X, x), (Y, y) are normal subobjects of A, the $1-$weighted normal commutator $N[(X, x), (Y, y)]_1$ is the kernel of $\tilde{q}$ in diagram (11) (under the above assumptions), but the same $\tilde{q}$ is the quotient of the Smith commutator $[R_x, R_y]_{\mathsf{S}}$ in diagram (13). Thus $N[(X, x), (Y, y)]_1$ is the associated normal subobject of $[R_x, R_y]_{\mathsf{S}}$.

References

[1] F. Borceux and D. Bourn, Mal'tsev, protomodular, homological and semi-abelian categories. Mathematics and its Application 566, Kluwer Academic Publisher, 2004.

[2] D. Bourn, Commutator theory in regular Mal'tsev categories, in Hopf Algebras and Semi-abelian Categories. G. Janelidze, B. Pareigis, W. Tholen eds., the Fields Institute Communication 43, 61-75, 2004.

[3] D. Bourn, Mal'tsev categories and fibration of pointed objects. Applied Categorical Structures 4, 307-327, 1996.

[4] D. Bourn, N. Martins-Ferreira, and T. Van der Linden, A characterization of the "Smith is Huq" condition in the pointed Mal'tsev setting. Cah. Topologie Géom. Différ. Catégoriques 54, 163-183, 2013.

[5] M. Gran, G. Janelidze, and A. Ursini, Weighted commutators in semi-abelian categories. Journal of Algebra 397, 643-665, 2014.

[6] S.A Huq, Commutator, nilpotency and solvability in categories. Quart. J. Math. Oxford Ser. 19, 363-389, 1968.

[7] G. Janelidze, L. Márki, W.Tholen, and A. Ursini, Ideal-determined categories. Cah. Topol. Géom. Différ. Catég. 51, 115-125, 2010.

[8] Z. Janelidze, Subtractive categories. Applied Categorical Structures 13, 343-350, 2005.

[9] Z. Janelidze, The pointed subobject functor, 3×3 lemmas, and subtractivity of spans. Theory and Application of categories 23, 221-242, 2010.

[10] S. Mantovani, The Ursini commutator as normalized Smith-Pedicchio commutator. Theory and Applications of Categories 27, 174-188, 2012.

[11] N. Martins-Ferreira and T. Van der Linden, Further remarks on the "Smith is Huq" condition. Applied Categorical Structures 23, 527-541, 2015.

[12] M.C. Pedicchio, A categorical approach to commutator theory. Journal of Algebra 177, 647-657, 1995.

[13] V.T. Shaumbwa, Weighted centrality, and a further approach to categorical commutativity. PhD thesis, University of Stellenbosch, 2019.

[14] J.D.H. Smith, Mal'tsev varieties. Springer L.N. in Math. 554, 1976.

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