

THE 2-CHU-DIALECTICA CONSTRUCTION AND THE POLYCATEGORY OF MULTIVARIABLE ADJUNCTIONS

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ABSTRACT. Cheng, Gurski, and Riehl constructed a cyclic double multicategory of multivariable adjunctions. We show that the same information is carried by a poly double category, in which opposite categories are polycategorical duals. Moreover, this poly double category is a full substructure of a double Chu construction, whose objects are a sort of polarized category, and which is a natural home for 2-categorical dualities.

We obtain the double Chu construction using a general “Chu-Dialectica” construction on polycategories, which includes both the Chu construction and the categorical Dialectica construction of de Paiva. The Chu and Dialectica constructions each impose additional hypotheses making the resulting polycategory representable (hence *-autonomous), but for different reasons; this leads to their apparent differences.

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1. Introductions

I have written two introductions to this paper, each of which can be read independently. If you are interested in Dialectica and Chu constructions, please continue with section 1.1;

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but if you are more interested in multivariable adjunctions, I suggest skipping ahead to read section 1.4 first.

1.1. FIRST INTRODUCTION: UNIFYING THE DIALECTICA AND CHU CONSTRUCTIONS. The categorical Dialectica constructions were introduced by [dP89a, dP89b] as an abstraction of Gödel’s “Dialectica interpretation” [Göd58]. Although Gödel’s interpretation modeled intuitionistic logic, de Paiva’s categorical analysis revealed that it factors naturally through Girard’s *classical linear* logic [Gir87], which categorically means a $*$ -autonomous category [Bar91, CS97b].

On the other hand, the Chu construction [Chu78, Chu79, Bar06] was introduced specifically as a way to produce $*$ -autonomous categories. Anyone familiar with both constructions can tell that they have a very similar feel, and one formal functorial comparison was given in [dP06]. In this paper we compare them in a new way, by giving such a general construction that includes both Dialectica and Chu constructions as special cases.

One reason it is hard to compare the Dialectica and Chu constructions is that while their underlying categories are defined very similarly, their monoidal structures are defined rather differently. This suggests that a fruitful way to compare them would be to perform them both in a more general context where these monoidal structures need not exist, but can be characterized up to isomorphism by universal properties. In other words, instead of monoidal categories we will use *multicategories*, and instead of $*$ -autonomous categories we will use *polycategories* [Sza75].¹

To define a multi- or polycategorical version of the Dialectica or Chu constructions, we need to start by asking what universal property is possessed by their tensor products, i.e. what functor they represent, in the way that the tensor product of abelian groups represents bilinear maps. In other words, if \boxtimes denotes these tensor products, then what does a morphism $A \boxtimes B \rightarrow C$ look like if we “beta-reduce away” the definition of \boxtimes ?

First, let us consider the Chu construction, which in its basic form applies to a closed symmetric monoidal category \mathbf{C} with pullbacks, equipped with an arbitrary object Ω . In the resulting category $\text{Chu}(\mathbf{C}, \Omega)$,

- The objects are triples $A = (A^+, A^-, \underline{A})$ where A^+, A^- are objects of \mathbf{C} and $\underline{A} : A^+ \otimes A^- \rightarrow \Omega$ is a morphism in \mathbf{C} .
- The morphisms $f : (A^+, A^-, \underline{A}) \rightarrow (B^+, B^-, \underline{B})$ are pairs (f^+, f^-) where $f^+ : A^+ \rightarrow B^+$ and $f^- : B^- \rightarrow A^-$ are morphisms in \mathbf{C} such that

$$\underline{A} \circ (1 \otimes f^-) = \underline{B} \circ (f^+ \otimes 1).$$

¹A polycategorical viewpoint on the Chu construction, in the case of non-symmetric “cyclic” polycategories, can also be found in [CKS03, Example 1.8(2)]. In this paper we will consider only the symmetric case.

The tensor product of two objects $A, B \in \mathbf{Chu}(\mathbf{C}, \Omega)$ is defined by

$$\begin{aligned} (A \boxtimes B)^+ &= A^+ \otimes B^+ \\ (A \boxtimes B)^- &= [A^+, B^-] \times_{[A^+ \otimes B^+, \Omega]} [B^+, A^-] \\ \underline{A \boxtimes B} &= \left(([A^+, B^-] \times_{[A^+ \otimes B^+, \Omega]} [B^+, A^-]) \otimes_{A^+ \otimes B^+ \rightarrow [A^+ \otimes B^+, \Omega]} \otimes_{A^+ \otimes B^+ \rightarrow \Omega} \right) \end{aligned}$$

By definition of the morphisms in $\mathbf{Chu}(\mathbf{C}, \Omega)$, a morphism $A \boxtimes B \rightarrow C$ consists of \mathbf{C} -morphisms $f^+ : A^+ \otimes B^+ \rightarrow C^+$ and $f^- : C^- \rightarrow [A^+, B^-] \times_{[A^+ \otimes B^+, \Omega]} [B^+, A^-]$ such that some diagram commutes. But by the universal property of pullbacks and internal-homs, to give f^- is equivalent to giving $f_1^- : A^+ \otimes C^- \rightarrow B^-$ and $f_2^- : B^+ \otimes C^- \rightarrow A^-$ such that $\underline{B} \circ (1_{B^+} \otimes f_1^-) = \underline{A} \circ (1_{A^+} \otimes f_2^-)$ (modulo a symmetry in the domain). And the commutative diagram then simply asserts that this joint composite is also equal (again, modulo symmetry) to $\underline{C} \circ (f^+ \otimes 1_{C^-})$.

Thus, in total a morphism $A \boxtimes B \rightarrow C$ in $\mathbf{Chu}(\mathbf{C}, \Omega)$ consists of morphisms

$$\begin{aligned} f^+ &: A^+ \otimes B^+ \rightarrow C^+ \\ f_1^- &: A^+ \otimes C^- \rightarrow B^- \\ f_2^- &: B^+ \otimes C^- \rightarrow A^- \end{aligned}$$

such that

$$\underline{B} \circ (1_{B^+} \otimes f_1^-) = \underline{A} \circ (1_{A^+} \otimes f_2^-) = \underline{C} \circ (f^+ \otimes 1_{C^-}).$$

There are several things to note about this:

- It is certainly a “two-variable” generalization of the definition of ordinary morphisms $A \rightarrow B$ in $\mathbf{Chu}(\mathbf{C}, \Omega)$.
- It makes sense even if \mathbf{C} is only a multicategory, with $f^+ : (A^+, B^+) \rightarrow C^+$ and so on.
- With a little thought, one can guess the correct n -variable version, dualize to describe maps $A \rightarrow B \diamond C$, where $B \diamond C = (B^* \boxtimes C^*)^*$ is the dual “cotensor product” (the “par” of linear logic), and then generalize to maps from an n -ary tensor to an m -ary cotensor. This leads to our polycategorical definition.
- If we write the above equalities in the internal type theory of \mathbf{C} , using formal variables $a : A^+$, $b : B^+$, and $c : C^-$, they become

$$\underline{C}(f^+(a, b), c) = \underline{B}(b, f_1^-(a, c)) = \underline{A}(a, f_2^-(b, c)),$$

which is highly reminiscent of the hom-set isomorphisms in a *two-variable adjunction*. We will pick up this thread in section 1.4.

Moving on to the Dialectica construction, we will describe the version from [dP06], which looks the most like the Chu construction. This Dialectica construction applies to a closed symmetric monoidal category \mathbf{C} with finite products, equipped with an object Ω that internally has the structure of a closed monoidal poset. In the resulting category $\mathbf{Dial}(\mathbf{C}, \Omega)$,

- The objects are the same as those of $\text{Chu}(\mathbf{C}, \Omega)$: triples $A = (A^+, A^-, \underline{A})$ where A^+, A^- are objects of \mathbf{C} and $\underline{A} : A^+ \otimes A^- \rightarrow \Omega$ is a morphism in \mathbf{C} .
- The morphisms $f : (A^+, A^-, \underline{A}) \rightarrow (B^+, B^-, \underline{B})$ are pairs (f^+, f^-) where $f^+ : A^+ \rightarrow B^+$ and $f^- : B^- \rightarrow A^-$ are morphisms in \mathbf{C} such that

$$\underline{A} \circ (1 \otimes f^-) \leq \underline{B} \circ (f^+ \otimes 1)$$

in the internal order of Ω (applied pointwise to morphisms $A^+ \otimes B^- \rightarrow \Omega$).

The tensor product of two objects $A, B \in \text{Dial}(\mathbf{C}, \Omega)$ is defined by

$$\begin{aligned} (A \boxtimes B)^+ &= A^+ \otimes B^+ \\ (A \boxtimes B)^- &= [A^+, B^-] \times [B^+, A^-] \end{aligned}$$

with $\underline{A} \boxtimes \underline{B}$ being the tensor product (in the internal monoidal structure of Ω) of the two morphisms

$$\begin{aligned} A^+ \otimes B^+ \otimes ([A^+, B^-] \times [B^+, A^-]) &\rightarrow A^+ \otimes B^+ \otimes [A^+, B^-] \rightarrow B^+ \otimes B^- \xrightarrow{\underline{B}} \Omega \\ A^+ \otimes B^+ \otimes ([A^+, B^-] \times [B^+, A^-]) &\rightarrow A^+ \otimes B^+ \otimes [B^+, A^-] \rightarrow A^+ \otimes A^- \xrightarrow{\underline{A}} \Omega \end{aligned}$$

Now by definition of the morphisms in $\text{Dial}(\mathbf{C}, \Omega)$, a morphism $A \boxtimes B \rightarrow C$ consists of \mathbf{C} -morphisms $f^+ : A^+ \otimes B^+ \rightarrow C^+$ and $f^- : C^- \rightarrow [A^+, B^-] \times [B^+, A^-]$ such that some inequality holds. But by the universal property of products and internal-homs, to give f^- is equivalent to giving $f_1^- : A^+ \otimes C^- \rightarrow B^-$ and $f_2^- : B^+ \otimes C^- \rightarrow A^-$, and the inequality then asserts that

$$(\underline{A} \circ (1_{A^+} \otimes f_2^-)) \boxtimes (\underline{B} \circ (1_{B^+} \otimes f_1^-)) \leq (\underline{C} \circ (f^+ \otimes 1_{C^-})) \quad (1.2)$$

in the internal order of Ω (applied pointwise to morphisms $A^+ \otimes B^+ \otimes C^- \rightarrow \Omega$). We now note similarly that:

- This is also certainly a “two-variable” generalization of the definition of ordinary morphisms $A \rightarrow B$ in $\text{Dial}(\mathbf{C}, \Omega)$.
- It also makes sense if \mathbf{C} is only a multicategory, with $f^+ : (A^+, B^+) \rightarrow C^+$ etc.
- In fact, it makes sense even if Ω is only a *multi-poset* (a multicategory having at most one morphism with any given domain and codomain, just as a poset is a category with this property), with (1.2) replaced by

$$(\underline{A} \circ (1_{A^+} \otimes f_2^-), \underline{B} \circ (1_{B^+} \otimes f_1^-)) \leq (\underline{C} \circ (f^+ \otimes 1_{C^-})) \quad (1.3)$$

- One can again guess the correct n -to- m -variable version and write down a polycategorical definition, with Ω replaced by a *poly-poset* (a polycategory having at most one morphism in each hom-set).

Furthermore, the descriptions of morphisms $A \boxtimes B \rightarrow C$ in $\text{Chu}(\mathbf{C}, \Omega)$ and $\text{Dial}(\mathbf{C}, \Omega)$ are very similar, indeed they are related in essentially the same way as the descriptions of ordinary morphisms $A \rightarrow B$. Specifically, the Chu construction asks for an *equality*, while the Dialectica construction asks for an *inequality* — where an “inequality” between more than two elements is interpreted with respect to a multi-poset or poly-poset structure.

This leads to our common generalization: just as equalities $\phi = \psi$ are inequalities in a discrete poset (where $\phi \leq \psi$ is defined to mean $\phi = \psi$), “multi-variable equalities” $\phi = \psi = \xi$ can be regarded as “multi-variable inequalities” in a “discrete poly-poset”, where an inequality $(\phi, \psi) \leq (\xi)$ is *defined to mean* $\phi = \psi = \xi$. Thus, *the polycategorical Dialectica construction includes the polycategorical Chu construction*. The reason the original constructions look different is that they make different additional assumptions, each of which implies that the polycategorical result is “representable” and hence defines a $*$ -autonomous category — but this representability happens in *different ways* for the original Dialectica and Chu constructions.

In fact, we will generalize further in a few ways:

- We allow Ω to be a *polycategory* rather than a *polyposet*, i.e. our construction will be “proof-relevant” in the strongest sense.
- We will replace the object Ω by a not-necessarily-representable presheaf with the same structure. This allows us to include the original Dialectica constructions [dP89a, dP89b], where instead of morphisms into Ω we use subobjects, without supposing \mathbf{C} to have a subobject classifier.
- We will generalize the output of the construction to be a \mathbf{C} -indexed family of polycategories rather than a single one, as in [Bie08, Hof11]. This amounts to building a model of first-order rather than merely propositional linear logic.

Taken together, these generalizations imply that the output of our “Chu-Dialectica construction” is the same kind of thing as its input: a multicategory equipped with a presheaf of polycategories, which we call a *virtual linear hyperdoctrine*. I do not know whether this endomorphism of the category of virtual linear hyperdoctrines has a universal property (see [Pav93, Hof11] for universal properties of the Chu and Dialectica constructions respectively).

From the perspective of higher category theory, we can regard our construction as a categorification. In the original Chu construction, Ω is a discrete object, i.e. a 0-category. In the original Dialectica construction, Ω is a posetal object, a.k.a. a $(0, 1)$ -category (where a set or 0-category is more verbosely called a $(0, 0)$ -category). Our construction (as well as other categorified Dialectica constructions, e.g. [Bie08, Hof11]) allows Ω to be a categorical object, i.e. a $(1, 1)$ -category.

This suggests that our construction should also specialize to a version involving $(1, 0)$ -categories, i.e. groupoids. It seems appropriate to call this a *2-Chu construction*, since it replaces the equalities in the ordinary Chu constructions by *isomorphisms*. The “prototypical” 2-Chu construction $\text{Chu}(\text{Cat}, \text{Set})$ (which directly categorifies the prototypical

Chu construction $\text{Chu}(\text{Set}, 2)$) is particularly interesting as its morphisms are a “polarized” sort of *multivariable adjunction*.

The second introduction to the paper, which follows, reverses the flow of motivation by *starting* with multivariable adjunctions.

1.4. SECOND INTRODUCTION: THE POLYCATAGORY OF MULTIVARIABLE ADJUNCTIONS. In view of the well-known importance of adjunctions in category theory, it is perhaps surprising that it has taken so long for *multivariable adjunctions* to be systematically studied. To be sure, *two*-variable adjunctions have a long history, and include some of the earliest examples of adjunctions. For instance, in a biclosed monoidal category each functor $(A \otimes -)$ has a right adjoint $[A, -]^l$, and each functor $(- \otimes B)$ has a right adjoint $[B, -]^r$; but this is more symmetrically expressed by saying that the two-variable functor \otimes has $[-, -]^l$ and $[-, -]^r$ as two-variable right adjoints. To be precise, in this case we have three functors

$$\otimes : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A} \quad [-, -]^l : \mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \mathcal{A} \quad [-, -]^r : \mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \mathcal{A}$$

with natural isomorphisms

$$\mathcal{A}(A \otimes B, C) \cong \mathcal{A}(B, [A, C]^l) \cong \mathcal{A}(A, [B, C]^r).$$

In general, a two-variable adjunction $(\mathcal{A}, \mathcal{B}) \rightarrow \mathcal{C}$ consists of functors

$$f : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C} \quad g : \mathcal{A}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{B} \quad h : \mathcal{B}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{A}$$

and natural isomorphisms

$$\mathcal{C}(f(a, b), c) \cong \mathcal{B}(b, g(a, c)) \cong \mathcal{A}(a, h(b, c)).$$

In addition to biclosed monoidal structures, another well-known example is the “tensor-hom-cotensor” (or “copower-hom-power”) situation of an enriched category, which inspired the terminology *THC-situation* for the general case in [Gra80]. The name *adjunction of two variables* from [Hov99] was shortened in [Rie13, CGR14] to *two-variable adjunction*; in [Gui13] the term used is *trijunction* (though see below).

Of course when we have one-variable and two-variable versions of something, it is natural to expect an n -variable version. If we go back to the fact that the functors g and h in a two-variable adjunction are determined up to unique isomorphism by f , we can define an *n -variable adjunction* $(\mathcal{A}_1, \dots, \mathcal{A}_n) \rightarrow \mathcal{B}$ to be a functor $\mathcal{A}_1 \times \dots \times \mathcal{A}_n \rightarrow \mathcal{B}$ such that if we fix its value on all but one (say \mathcal{A}_i) of the input categories, the resulting functor $\mathcal{A}_i \rightarrow \mathcal{B}$ has a right adjoint. Each such right adjoint then automatically becomes contravariantly functorial on the categories \mathcal{A}_j for $j \neq i$.

Unsurprisingly, multivariable adjunctions of this sort can be assembled into a multicategory: we can compose a two-variable adjunction $(\mathcal{A}, \mathcal{B}) \rightarrow \mathcal{C}$ with another one $(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{E}$ to obtain a three-variable adjunction $(\mathcal{A}, \mathcal{B}, \mathcal{D}) \rightarrow \mathcal{E}$, and so on. However, one-variable adjunctions are the morphisms not only of a category but of a 2-category Adj , whose

2-cells are mate-pairs of natural transformations. More generally, one-variable adjunctions form the horizontal morphisms in a *double* category \mathbf{Adj} , whose vertical morphisms are single functors, and whose 2-cells are a more general kind of mate-pairs. (Recall that if $f \dashv g$ and $h \dashv k$ are adjunctions, then the “mate correspondence” is a bijection between natural transformations $fu \rightarrow vh$ and $uk \rightarrow gv$ obtained by pasting with the adjunction unit and counit. The functoriality of this bijection is conveniently expressed in terms of the double category \mathbf{Adj} ; see [KS74].)

The first step towards a similar calculus for multivariable adjunctions was taken by [CGR14], who exhibited them as the horizontal² morphisms in a *cyclic multi double category*³ $\mathbb{M}\mathbf{Adj}$ (i.e. an internal category in the category of cyclic multicategories). The multicategory structure of $\mathbb{M}\mathbf{Adj}$ is unsurprising. Its vertical arrows of $\mathbb{M}\mathbf{Adj}$ are functors and its 2-cells are natural transformations, while its cyclic structure encodes a calculus of *multivariable mates*, describing the behavior of multivariable adjunctions with respect to passage to opposite categories. In general, a cyclic structure on a multicategory consists of an involution $(-)^{\bullet}$ on objects together with a cyclic action on morphism sets

$$\mathcal{M}(A_1, \dots, A_n; B) \rightarrow \mathcal{M}(A_2, \dots, A_n, B^{\bullet}; A_1^{\bullet})$$

satisfying appropriate axioms. In $\mathbb{M}\mathbf{Adj}$ we define $\mathcal{A}^{\bullet} = \mathcal{A}^{\text{op}}$, and the cyclic action generalizes the observation that a two-variable adjunction $(\mathcal{A}, \mathcal{B}) \rightarrow \mathcal{C}$ is essentially the same as a two-variable adjunction $(\mathcal{C}^{\text{op}}, \mathcal{A}) \rightarrow \mathcal{B}^{\text{op}}$ or $(\mathcal{B}, \mathcal{C}^{\text{op}}) \rightarrow \mathcal{A}^{\text{op}}$. The extension of this cyclic action to 2-cells then encodes the mate correspondence.

In practice, three- and higher-variable adjunctions seem to arise mainly as composites of two-variable adjunctions. But the whole multicategory structure is nevertheless useful, because it gives an abstract context in which to express conditions and axioms regarding such composites. For instance, the associativity of the tensor product in a closed monoidal category has an equivalent form involving the internal-hom [EK66]; they are 2-cells in $\mathbb{M}\mathbf{Adj}$ related by the mate correspondence. Put differently, just as a monoidal category can be defined as a pseudomonoid in the 2-category \mathbf{Cat} , a *closed* monoidal category can be defined as a pseudomonoid in $\mathcal{M}\mathbf{Adj}$, the horizontal 2-multicategory of $\mathbb{M}\mathbf{Adj}$. Similarly, a module over a pseudomonoid A in $\mathcal{M}\mathbf{Adj}$ is an A -enriched category with powers and copowers, and so on.

In this paper I will propose a different viewpoint on $\mathbb{M}\mathbf{Adj}$: rather than a cyclic multicategory, we can regard it as a *polycategory*. A polycategory is like a multicategory, but it allows the *codomain* of a morphism to contain multiple objects, as well as the domain; thus we have morphisms like $f : (A, B) \rightarrow (C, D)$. Such morphisms can be composed only “along single objects”, with the “leftover” objects in the codomain of f and the domain of g surviving into the codomain and domain of $g \circ f$. For instance, given $f : (A, B) \rightarrow (C, D)$ and $g : (E, C) \rightarrow (F, G)$ we get $g \circ_C f : (E, A, B) \rightarrow (F, G, D)$.

²Actually, their double categories are transposed from ours, so for them the multivariable adjunctions are the *vertical* morphisms.

³They called it a *cyclic double multicategory*, but the phrase “double multicategory” may suggest an internal multicategory in multicategories rather than the intended meaning of an internal category in multicategories, so we have chosen to order the modifiers differently.

What is a multivariable adjunction $(\mathcal{A}_1, \dots, \mathcal{A}_m) \rightarrow (\mathcal{B}_1, \dots, \mathcal{B}_n)$? There are several ways to figure out the answer. One is to inspect the definition of a multivariable adjunction $(\mathcal{A}_1, \dots, \mathcal{A}_m) \rightarrow \mathcal{B}_1$ and rephrase it in a way that doesn't depend on the assumption $n = 1$. The functors involved in such an adjunction are

$$\begin{aligned} f_1 &: \mathcal{A}_1 \times \cdots \times \mathcal{A}_m \rightarrow \mathcal{B}_1 \\ g_i &: \mathcal{A}_1^{\text{op}} \times \cdots \widehat{\mathcal{A}_i^{\text{op}}} \cdots \times \mathcal{A}_m^{\text{op}} \times \mathcal{B}_1 \rightarrow \mathcal{A}_i \quad (1 \leq i \leq m) \end{aligned}$$

where $\widehat{\mathcal{A}_i^{\text{op}}}$ indicates that $\mathcal{A}_i^{\text{op}}$ is omitted. This can be described as “for each category \mathcal{A}_i or \mathcal{B}_j , a functor with that codomain, whose domain is the product of all the other categories, with opposites applied to those denoted by the same letter as the codomain”. That is, the functor g_i with codomain \mathcal{A}_i depends contravariantly on all the other \mathcal{A} 's and covariantly on the (single) \mathcal{B} , while the functor f with codomain \mathcal{B}_1 depends contravariantly on the (zero) other \mathcal{B} 's and covariantly on all the \mathcal{A} 's. If we apply this description in the case $n > 1$ as well, we see that a multivariable adjunction $(\mathcal{A}_1, \dots, \mathcal{A}_m) \rightarrow (\mathcal{B}_1, \dots, \mathcal{B}_n)$ should involve functors

$$\begin{aligned} f_j &: \mathcal{A}_1 \times \cdots \times \mathcal{A}_m \times \mathcal{B}_1^{\text{op}} \times \cdots \widehat{\mathcal{B}_j^{\text{op}}} \cdots \times \mathcal{B}_n^{\text{op}} \rightarrow \mathcal{B}_j \quad (1 \leq j \leq n) \\ g_i &: \mathcal{A}_1^{\text{op}} \times \cdots \widehat{\mathcal{A}_i^{\text{op}}} \cdots \times \mathcal{A}_m^{\text{op}} \times \mathcal{B}_1 \times \cdots \times \mathcal{B}_n \rightarrow \mathcal{A}_i \quad (1 \leq i \leq m) \end{aligned}$$

with an appropriate family of natural isomorphisms. For instance, a multivariable adjunction $(\mathcal{A}, \mathcal{B}) \rightarrow (\mathcal{C}, \mathcal{D})$ consists of four functors

$$\begin{aligned} f &: \mathcal{C}^{\text{op}} \times \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{D} & g &: \mathcal{A} \times \mathcal{B} \times \mathcal{D}^{\text{op}} \rightarrow \mathcal{C} \\ h &: \mathcal{A}^{\text{op}} \times \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{B} & k &: \mathcal{C} \times \mathcal{D} \times \mathcal{B}^{\text{op}} \rightarrow \mathcal{A} \end{aligned}$$

and natural isomorphisms

$$\mathcal{D}(f(c, a, b), d) \cong \mathcal{C}(g(a, b, d), c) \cong \mathcal{B}(b, h(a, c, d)) \cong \mathcal{A}(a, k(c, d, b)).$$

I find this definition quite illuminating already. One of the odd things about a two-variable adjunction, as usually defined, is the asymmetric placement of opposites. The polycategorical perspective reveals that this arises simply from the asymmetry of having a 2-ary domain but a 1-ary codomain: a “(2, 2)-variable adjunction” as above looks much more symmetrical.

With this definition of (m, n) -variable adjunctions in hand, it is a nice exercise to write down a composition law making them into a polycategory. For instance, suppose in addition to $(f, g, h, k) : (\mathcal{A}, \mathcal{B}) \rightarrow (\mathcal{C}, \mathcal{D})$ as above, we have a two-variable adjunction $(\ell, m, n) : (\mathcal{D}, \mathcal{E}) \rightarrow \mathcal{Z}$ with $\mathcal{Z}(\ell(d, e), z) \cong \mathcal{D}(d, m(e, z)) \cong \mathcal{E}(e, n(d, z))$. Then we have a composite multivariable adjunction $(\mathcal{A}, \mathcal{B}, \mathcal{E}) \rightarrow (\mathcal{C}, \mathcal{Z})$ defined by

$$\mathcal{C}(g(a, b, m(e, z)), c) \cong \mathcal{Z}(\ell(f(c, a, b), e), z) \cong \mathcal{A}(a, k(c, m(e, z), b)) \cong \cdots .$$

Of course, a $(1, 1)$ -variable adjunction is an ordinary adjunction, while a $(2, 1)$ -variable adjunction is a two-variable adjunction as above. A $(2, 0)$ -variable adjunction $(\mathcal{A}, \mathcal{B}) \rightarrow ()$ consists of functors $f : \mathcal{A}^{\text{op}} \rightarrow \mathcal{B}$ and $g : \mathcal{B}^{\text{op}} \rightarrow \mathcal{A}$ and a natural isomorphism $\mathcal{B}(b, f(a)) \cong \mathcal{A}(a, g(b))$. This is sometimes called a **mutual right adjunction** or **dual adjunction**, and arises frequently in examples, such as Galois connections between posets or the self-adjunction of the contravariant powerset functor. Similarly, a $(0, 2)$ -variable adjunction $() \rightarrow (\mathcal{A}, \mathcal{B})$ is a mutual left adjunction $\mathcal{B}(f(a), b) \cong \mathcal{A}(g(b), a)$. Of course a mutual right or left adjunction can also be described as an ordinary adjunction between \mathcal{A}^{op} and \mathcal{B} , or between \mathcal{A} and \mathcal{B}^{op} , but the choice of which category to oppositize is arbitrary; the polycategorical approach respects mutual right and left adjunctions as independent objects.⁴

More generally, an $(n, 0)$ -variable adjunction $(\mathcal{A}_1, \dots, \mathcal{A}_n) \rightarrow ()$ is a “mutual right multivariable adjunction” between n contravariant functors

$$f_i : \mathcal{A}_{i+1} \times \dots \times \mathcal{A}_n \times \mathcal{A}_1 \times \dots \times \mathcal{A}_{i-1} \rightarrow \mathcal{A}_i^{\text{op}}.$$

In fact, the “mutual right” version is the formal definition of n -variable adjunction given in [CGR14] (and, in the case $n = 3$, of “trijunction” in [Gui13]). This makes the cyclic structure more apparent, but the enforced contravariance makes for a mismatch with many standard examples.

A further advantage of the polycategorical framework is the way that opposite categories enter the picture: rather than imposed by the *structure* of a cyclic action, they are characterized by a universal *property*. Specifically, they are duals in the polycategorical sense: we have multivariable adjunctions $\eta : () \rightarrow (\mathcal{A}, \mathcal{A}^{\text{op}})$ and $\varepsilon : (\mathcal{A}^{\text{op}}, \mathcal{A}) \rightarrow ()$ satisfying analogues of the triangle identities. Opposite categories are also dual objects in the monoidal bicategory of profunctors, but the polycategory of multivariable adjunctions provides a new perspective, which in particular characterizes them up to equivalence (not just Morita equivalence).

In fact, the characterization of \mathcal{A}^{op} as a polycategorical dual of \mathcal{A} encodes almost exactly the same information as the cyclic action of [CGR14]. Any polycategory \mathcal{P} with strict duals (a.k.a. a “*-polycategory” [Hyl02]) has an underlying cyclic symmetric multicategory, in which the cyclic action

$$\mathcal{P}(A_1, \dots, A_n; B) \rightarrow \mathcal{P}(A_2, \dots, A_n, B^\bullet; A_1^\bullet)$$

is obtained by composing with ε_B and η_{A_1} . Conversely, any cyclic symmetric multicategory \mathcal{M} can be extended to a polycategory by defining

$$\mathcal{M}(A_1, \dots, A_m; B_1, \dots, B_n) = \mathcal{M}(A_1, \dots, A_m, B_1^\bullet, \dots, \widehat{B_j^\bullet}, \dots, B_n^\bullet; B_j).$$

The cyclic structure ensures that this is independent, up to isomorphism, of j . The polycategorical composition can then be induced from the multicategorical one and the

⁴At this point I encourage the reader to stop and think for a while about what a $(0, 0)$ -variable adjunction should be. The answer will be given in Remark 1.5.

cyclic action, and the cyclic “duals” A^\bullet indeed turn out to be abstract polycategorical duals.

Thus symmetric polycategories with duals are almost⁵ equivalent to cyclic symmetric multicategories, and our polycategorical $\mathbb{M}\text{Adj}$ corresponds under this almost-equivalence to the cyclic $\mathbb{M}\text{Adj}$ of [CGR14]. This provides another *a posteriori* explanation of the definition of (m, n) -variable adjunctions: they are exactly the morphisms in the polycategory we obtain by passing the cyclic multicategory $\mathbb{M}\text{Adj}$ across this equivalence. For instance, the reader may check that a $(2, 2)$ -variable adjunction $(\mathcal{A}, \mathcal{B}) \rightarrow (\mathcal{C}, \mathcal{D})$ could equivalently be defined to be simply a three-variable adjunction $(\mathcal{A}, \mathcal{B}, \mathcal{C}^{\text{op}}) \rightarrow \mathcal{D}$ (or, equivalently, $(\mathcal{A}, \mathcal{B}, \mathcal{D}^{\text{op}}) \rightarrow \mathcal{C}$).

Finally, like the cyclic multicategory $\mathbb{M}\text{Adj}$ of [CGR14], the polycategory $\mathbb{M}\text{Adj}$ is in fact a poly *double* category (meaning an internal category in the category of polycategories), whose vertical arrows are functors and whose 2-cells are an appropriate sort of multivariable mate tuple. Thus, it is equally appropriate for studying the multivariable mate correspondence. It also suggests new applications: for instance, in a 2-polycategory we can define *pseudo-comonoids* and *Frobenius pseudomonoids*, and in a future paper [Shu19] (building on [DS04, Str04, Egg10]) I will show that Frobenius pseudomonoids in $\mathcal{M}\text{Adj}$ are $*$ -autonomous categories.

However, there is still something unsatisfying about the picture. The double category Adj of ordinary adjunctions can actually be constructed out of internal adjunctions in *any* 2-category \mathcal{K} instead of Cat ; but it is unclear exactly what the analogous statement should be for multivariable adjunctions. In particular, the definition of multivariable adjunction involves the notion of *opposite category*, which despite its apparent simplicity is actually one of the more mysterious and difficult-to-abstract properties of Cat . At the “one-variable” level it is simply a 2-contravariant involution $\text{Cat}^{\text{co}} \cong \text{Cat}$ [Shu18], but its multivariable nature is still not fully understood (despite important progress such as [DS97, Web07]).

However, it turns out that we can avoid this question entirely if we are willing to settle for constructing something rather *larger* than $\mathbb{M}\text{Adj}$. Upon inspection, the definition of multivariable adjunction uses very little information about the relation of a category to its opposite: basically nothing other than the existence of the hom-functors $\mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \text{Set}$, and nothing at all about the structure of their codomain Set . Thus, instead of trying to *characterize* the opposite of a category, we can simply consider “categories equipped with a formal opposite”.

Let \mathcal{K} be a symmetric monoidal 2-category with a specified object Ω . We define an Ω -*polarized object* to be a triple $(A^+, A^-, \underline{A})$ where A^+, A^- are objects of \mathcal{K} and $\underline{A} : A^+ \otimes A^- \rightarrow \Omega$. If $\mathcal{K} = \text{Cat}$ and $\Omega = \text{Set}$, every category \mathcal{A} induces a *representable Set*-polarized object $[\mathcal{A}]$ with $[\mathcal{A}]^+ = \mathcal{A}^{\text{op}}$, $[\mathcal{A}]^- = \mathcal{A}$, and $[\underline{\mathcal{A}}] = \text{hom}_{\mathcal{A}}$ (but see Remark 1.6).

A *polarized adjunction* $f : A \rightarrow B$ between polarized objects consists of morphisms $f^+ : A^+ \rightarrow B^+$ and $f^- : B^- \rightarrow A^-$ and an isomorphism $\underline{A} \circ (1 \otimes f^-) \cong \underline{B} \circ (f^+ \otimes 1)$. Similarly, a *polarized two-variable adjunction* $(A, B) \rightarrow C$ consists of morphisms

$$f : A^+ \otimes B^+ \rightarrow C^+ \quad g : A^+ \otimes C^- \rightarrow B^- \quad h : B^+ \otimes C^- \rightarrow A^-$$

⁵See Remark 1.5 for why the “almost”.

and isomorphisms (modulo appropriate symmetric actions)

$$\underline{A} \circ (1 \otimes h) \cong \underline{B} \circ (1 \otimes g) \cong \underline{C} \circ (f \otimes 1).$$

We can similarly define polarized (n, m) -variable adjunctions and assemble them into a polycategory. More generally, we can take them to be the horizontal morphisms in a poly *double* category $\mathbb{P}\text{olMAdj}(\mathcal{K}, \Omega)$; its vertical morphisms are *polarized functors* $h : A \rightarrow B$ consisting of morphisms $f^+ : A^+ \rightarrow B^+$ and $f^- : A^- \rightarrow B^-$ (note that both go in the same direction) and a 2-cell $\underline{A} \Rightarrow \underline{B} \circ (f^+ \otimes f^-)$, and its 2-cells are families of 2-cells in \mathcal{K} satisfying a “polarized mate” relationship.

In the case $\mathcal{K} = \mathcal{C}at$, $\Omega = \mathbf{Set}$, a polarized adjunction between representable polarized categories $[\mathcal{A}] \rightarrow [\mathcal{B}]$ reduces to an ordinary adjunction, and likewise a polarized two-variable adjunction $([\mathcal{A}], [\mathcal{B}]) \rightarrow [\mathcal{C}]$ reduces to an ordinary two-variable adjunction. More generally, we can say that $\mathbb{P}\text{olMAdj}(\mathcal{C}at, \mathbf{Set})$ contains our original \mathbf{MAdj} as a “horizontally full” subcategory (but see Remark 1.6). So there is a general 2-categorical construction that at least comes close to reproducing \mathbf{MAdj} .

On the other hand, $\mathbb{P}\text{olMAdj}(\mathcal{C}at, \mathbf{Set})$ is also interesting in its own right! Its objects and vertical arrows are (modulo replacement of A^+ by its opposite) the “polarized categories” and functors of [CS07], which were studied as semantics for polarized logic and games. It also provides a formal context for *relative adjunctions*, in which one or both adjoints are only defined on a subcategory of their domain. Furthermore, at least if \mathcal{K} is closed monoidal with pseudo-pullbacks (like $\mathcal{C}at$), the polycategory $\mathbb{P}\text{olMAdj}(\mathcal{K}, \Omega)$ has (bicategorical) tensor and cotensor products (the appropriate sort of “representability” condition for a polycategory).

For instance, for polarized objects A, B there is a polarized object $A \boxtimes B$ such that polarized two-variable adjunctions $(A, B) \rightarrow C$ are naturally equivalent to polarized *one*-variable adjunctions $A \boxtimes B \rightarrow C$. This universal property, like most others, tells us how to construct $A \boxtimes B$, as follows. A polarized adjunction $A \boxtimes B \rightarrow C$ consists of morphisms $(A \boxtimes B)^+ \rightarrow C^+$ and $C^- \rightarrow (A \boxtimes B)^-$ together with a certain isomorphism; whereas in a polarized two-variable adjunction $(A, B) \rightarrow C$ as above we can apply the internal-hom isomorphism to obtain

$$f : A^+ \otimes B^+ \rightarrow C^+ \quad \tilde{g} : C^- \rightarrow [A^+, B^-] \quad \tilde{h} : C^- \rightarrow [B^+, A^-].$$

Comparing the two suggests $(A \boxtimes B)^+ = A^+ \otimes B^+$ and $(A \boxtimes B)^- = [A^+, B^-] \times [B^+, A^-]$. The first is correct, but the second is not quite right: to incorporate the *two* isomorphisms of a two-variable adjunction, we have to let $(A \boxtimes B)^-$ be the pseudo-pullback $[A^+, B^-] \times_{[A^+ \otimes B^+, \Omega]}^{\text{ps}} [B^+, A^-]$. The third datum is the composite

$$\underline{A \boxtimes B} = \left(([A^+, B^-] \times_{[A^+ \otimes B^+, \Omega]}^{\text{ps}} [B^+, A^-]) \otimes_{A^+ \otimes B^+} \rightarrow_{[A^+ \otimes B^+, \Omega]} [A^+ \otimes B^+ \rightarrow \Omega] \right).$$

There is a similar “cotensor product” \boxtimes such that polarized $(1, 2)$ -variable adjunctions $A \rightarrow (B, C)$ are equivalent to polarized adjunctions $A \rightarrow B \boxtimes C$. We also have duals defined

by $(A^+, A^-, \underline{A})^\bullet = (A^-, A^+, \underline{A}\sigma)$, where σ is transposition of inputs; note that $[\mathcal{A}^{\text{op}}] = [\mathcal{A}]^\bullet$. Thus, the horizontal 2-category of $\text{PolMAdj}(\mathcal{K}, \Omega)$ is actually a **-autonomous 2-category*⁶ [Bar79].

It turns out that this structure is a categorification of a well-studied one. If \mathcal{K} is a closed symmetric monoidal 1-category with pullbacks, then all the isomorphisms degenerate to equalities, and the *-autonomous category of “ Ω -polarized objects” is precisely the *Chu construction* [Chu78, Chu79, Bar06] $\text{Chu}(\mathcal{K}, \Omega)$. Thus, the horizontal 2-category of $\text{PolMAdj}(\mathcal{K}, \Omega)$ is a *2-Chu construction* $\text{Chu}(\mathcal{K}, \Omega)$, while the whole double category $\text{PolMAdj}(\mathcal{K}, \Omega)$ can be called a *double Chu construction*; we denote it by $\text{Chu}(\mathcal{K}, \Omega)$. Thus in particular we have $\text{Chu}(\text{Cat}, \text{Set}) = \text{PolMAdj}(\text{Cat}, \text{Set})$.

This connection also suggests other applications of $\text{Chu}(\text{Cat}, \text{Set})$. As a categorification of the prototypical 1-Chu construction $\text{Chu}(\text{Set}, 2)$, which is an abstract home for many concrete dualities, we may expect $\text{Chu}(\text{Cat}, \text{Set})$ to be an abstract home for concrete 2-categorical dualities. For instance, Gabriel-Ulmer duality [GU71] between finitely complete categories and locally finitely presentable categories sits inside $\text{Chu}(\text{Cat}, \text{Set})$ just as Stone duality between Boolean algebras and Stone spaces sits inside $\text{Chu}(\text{Set}, 2)$ [PBB06]. There are other applications as well; see section 6.

There remains, however, the problem of constructing $\text{Chu}(\mathcal{K}, \Omega) = \text{PolMAdj}(\mathcal{K}, \Omega)$ in general: we need a systematic way to deal with all the isomorphisms. For instance, in defining a (2, 2)-variable adjunction we wrote

$$\mathcal{D}(f(c, a, b), d) \cong \mathcal{C}(g(a, b, d), c) \cong \mathcal{B}(b, h(a, c, d)) \cong \mathcal{A}(a, k(c, d, b))$$

but there is no justifiable reason for privileging these three isomorphisms over all the $\binom{4}{2} = 6$ possible pairwise isomorphisms; what we really mean is that these four profunctors are “all coherently isomorphic to each other”. There are many ways to deal with this, but a particularly elegant approach is to first formulate a “lax” version of the structure in which the isomorphisms are replaced by directed transformations. This clarifies exactly how the isomorphisms ought to be composed, since the directedness imposes a discipline that allows only certain composites.

In our case, we choose to regard the above family of coherent isomorphisms as a “morphism” relating the four profunctors, and the natural way to separate the four into domain and codomain is by copying the analogous division for the multivariable adjunction itself, with \mathcal{A}, \mathcal{B} in the domain and \mathcal{C}, \mathcal{D} in the codomain:

$$(\mathcal{A}(a, k(c, d, b)), \mathcal{B}(b, h(a, c, d))) \rightarrow (\mathcal{C}(g(a, b, d), c), \mathcal{D}(f(c, a, b), d)).$$

⁶The tensor product is only bicategorically associative and unital. Fortunately, we can avoid specifying all the coherence axioms involved in an explicit up-to-isomorphism *-autonomous structure on a monoidal bicategory by simply noting that we have a 2-polycategory with tensor and cotensor products that satisfy an up-to-equivalence universal property. As usual, structure that is characterized by a universal property is automatically “fully coherent”.

Our *-autonomous 2-categories are unrelated to the “linear bicategories” of [CKS00], which are instead a “horizontal” or “many-objects” categorification.

Thus, these morphisms must themselves live in some polycategory. This suggests that the “lax 2-Chu construction” should apply to a 2-category \mathcal{K} containing an object Ω that is an internal polycategory, with the ordinary 2-Chu construction recovered by giving Ω a sort of “discrete” polycategory structure in which a morphism $(\phi, \psi) \rightarrow (\xi, \zeta)$ consists of a coherent family of isomorphisms between ϕ, ψ, ξ, ζ .

This is indeed what we will do. (We will also generalize in a couple of other ways, replacing Ω by a not-necessarily-representable presheaf, and enhancing the output to an indexed family of polycategories rather than a single one.) Intriguingly, it turns out that while the 2-Chu construction yields a polycategory that is representable under certain assumptions on \mathcal{K} , the *lax* 2-Chu construction yields a polycategory that can naturally be shown to be representable under *different* assumptions on \mathcal{K} . Moreover, it is also well-known under a different name: it is one of the categorical *Dialectica constructions* [dP89a, dP89b, dP06].

From a higher-categorical perspective, our lax 2-Chu construction has categorified the ordinary Chu construction in two ways. The latter involves equalities, a 0-categorical structure. We first replaced these by isomorphisms, a groupoidal or “(1,0)-categorical” structure. Then we made them directed, yielding a 1-categorical or (1,1)-categorical structure. By contrast, the Dialectica construction is usually formulated at the other missing vertex involving *posets*, a.k.a. (0,1)-categories (though 1-categorical versions do appear in the literature, e.g. [Bie08, Hof11]).

Because the representability conditions on the lax and pseudo 2-Chu constructions are different, the Dialectica and Chu constructions, though obviously bearing a family resemblance [dP06], have not previously been placed in the same abstract context. The polycategorical perspective allows us to exhibit them as both instances of one “2-Chu-Dialectica construction”, which moreover includes the polycategory of (polarized) multivariable adjunctions at the other vertex. The first introduction to this paper in section 1.1, which you can go back and read now if you skipped it the first time, reverses the flow of motivation by *starting* with the question of how to compare the Chu and Dialectica constructions.

1.5. REMARK. There is one small fly in the ointment. The “lax 2-Chu-Dialectica” construction that we will describe is *strict*: it expects its input to involve strict 2-multicategories and 2-polycategories and produces a similarly strict output. This is convenient not just because it is easier, but because we can obtain the double-polycategorical version by applying it directly to internal categories. However, there is one place where it is not fully satisfactory, involving the question of what a “(0,0)-variable adjunction” should be.

This question is not answered by [CGR14]: the (0,0)-ary morphisms are the one place where a polycategory with duals contains more information than a cyclic multicategory. Duals allow representing any (n, m) -ary morphism as an $(n + m - 1, 1)$ -ary morphism, but only if $n + m > 0$. Thus, the underlying cyclic multicategory of a polycategory only remembers the (n, m) -ary morphisms for $n + m > 0$.

I claim that a (0,0)-variable adjunction should be simply *a set*. There are many ways to argue for this, including the following:

- The only way to produce a $(0, 0)$ -ary morphism in a polycategory is to compose a $(0, 1)$ -ary morphism with a $(1, 0)$ -ary one. Now a $(0, 1)$ -variable adjunction $a_1 : () \rightarrow \mathcal{A}$ and a $(1, 0)$ -variable adjunction $a_2 : \mathcal{A} \rightarrow ()$ are both just objects of \mathcal{A} , one “regarded covariantly” and the other “regarded contravariantly”. What can we get naturally from two such objects? Obviously, the hom-set $\mathcal{A}(a_2, a_1)$.
- The unit object of the $*$ -autonomous 2-category $\mathcal{Chu}(\mathcal{Cat}, \mathbf{Set})$ is $(1, \mathbf{Set}, \text{id}_{\mathbf{Set}})$, and its counit is $(\mathbf{Set}, 1, \text{id}_{\mathbf{Set}})$. This can be seen by analogy to the 1-Chu construction, or by checking their universal property with respect to (n, m) -ary morphisms for $n + m > 0$. But if these universal properties extend to $(0, 0)$ -ary morphisms, then a $(0, 0)$ -ary morphism must be the same as a polarized adjunction $(1, \mathbf{Set}, \text{id}_{\mathbf{Set}}) \rightarrow (\mathbf{Set}, 1, \text{id}_{\mathbf{Set}})$, which is (up to equivalence) a set.
- A multivariable adjunction $(A_1, \dots, A_n) \rightarrow (B_1, \dots, B_m)$ can equivalently be defined as a profunctor $A_1 \times \dots \times A_n \nrightarrow B_1 \times \dots \times B_m$ that is representable in each variable, and a profunctor $1 \nrightarrow 1$ is just a set.

However, I do not know of any way to define a *strict* 2-polycategory of multivariable adjunctions in which the $(0, 0)$ -ary morphisms are sets. The problem can be seen as follows: suppose we have a $(0, 1)$ -variable adjunction $a : () \rightarrow \mathcal{A}$ (i.e. an object $a \in \mathcal{A}$), a $(1, 1)$ -variable adjunction $f : \mathcal{A} \rightarrow \mathcal{B}$ (notated $f^+ \dashv f^-$), and a $(1, 0)$ -variable adjunction $b : \mathcal{B} \rightarrow ()$ (i.e. an object $b \in \mathcal{B}$). The composite $f \circ a : () \rightarrow \mathcal{B}$ can seemingly only be the object $f^+(a) \in \mathcal{B}$, and hence $b \circ (f \circ a)$ must be the hom-set $\mathcal{B}(f^+(a), b)$. But the composite $b \circ f : \mathcal{A} \rightarrow ()$ can seemingly only be the object $f^-(b) \in \mathcal{A}$, and hence $(b \circ f) \circ a$ must be the hom-set $\mathcal{A}(a, f^-(b))$, which is only *isomorphic* to $\mathcal{B}(f^+(a), b)$ rather than equal to it.⁷

In principle, it should be possible to give a “pseudo” version of the 2-Chu-Dialectica construction. However, for now we simply ignore this question by defining the $(0, 0)$ -ary hom-category “incorrectly” to be the terminal category rather than \mathbf{Set} . Since $(0, 0)$ -ary morphisms in a polycategory cannot be composed with anything else (they have no objects to compose along), it is always possible to brutalize a polycategory by declaring there to be exactly one $(0, 0)$ -ary morphism without changing anything else (which, as we will see, can also be described as following a round-trip pair of adjoint functors through cyclic multicategories). For the same reasons, I do not know of any real use for $(0, 0)$ -ary morphisms; so however unsatisfying this cop-out is philosophically, it has little practical import.

1.6. REMARK. I have been rather cavalier about variance in this informal introduction. In fact there are *two* natural ways to define a “representable” polarized category corresponding

⁷One of the referees pointed out that at the level of the underlying 1-category $\mathcal{Chu}_0(\mathcal{Cat}, \mathbf{Set})$ we could define the set of $(0, 0)$ -ary morphisms to be the set of *isomorphism classes* of sets. But this wouldn’t work for the full 2-category $\mathcal{Chu}(\mathcal{Cat}, \mathbf{Set})$, since the composition functors yielding $(0, 0)$ -ary output must also act on non-invertible 2-cells.

to an ordinary category \mathcal{A} :

$$[\mathcal{A}]_{\text{L}} = (\mathcal{A}^{\text{op}}, \mathcal{A}, \text{hom}_{\mathcal{A}}) \qquad [\mathcal{A}]_{\text{R}} = (\mathcal{A}, \mathcal{A}^{\text{op}}, \text{hom}_{\mathcal{A}}).$$

(Of course, the two functors denoted $\text{hom}_{\mathcal{A}}$ above take their arguments in opposite orders.) The difference is that a polarized adjunction $f : [\mathcal{A}]_{\text{L}} \rightarrow [\mathcal{B}]_{\text{L}}$ is an adjunction $f^+ : \mathcal{A} \rightleftarrows \mathcal{B} : f^-$ in which $f^+ : \mathcal{A} \rightarrow \mathcal{B}$ is the *left* adjoint, while a polarized adjunction $g : [\mathcal{A}]_{\text{R}} \rightarrow [\mathcal{B}]_{\text{R}}$ is an adjunction $g^+ : \mathcal{A} \rightleftarrows \mathcal{B} : g^-$ in which $g^+ : \mathcal{A} \rightarrow \mathcal{B}$ is the *right* adjoint. However, in *both* cases a 2-cell between polarized adjunctions is a mate-pair of natural transformations considered as pointing in the direction of the transformation between the *right* adjoints: $f^- \rightarrow (f')^-$ or $g^+ \rightarrow (g')^+$.

Similarly, a polarized two-variable adjunction $([\mathcal{A}]_{\text{L}}, [\mathcal{B}]_{\text{L}}) \rightarrow [\mathcal{C}]_{\text{L}}$ is an ordinary two-variable adjunction $(\mathcal{A}, \mathcal{B}) \rightarrow \mathcal{C}$ as described above, with a functor $f^+ : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$ equipped with a pair of two-variable right adjoints; but the 2-cells between these go in the direction of the induced mates between the right adjoints. A polarized two-variable adjunction $([\mathcal{A}]_{\text{R}}, [\mathcal{B}]_{\text{R}}) \rightarrow [\mathcal{C}]_{\text{R}}$, by contrast, is a functor $g^+ : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$ equipped with a pair of two-variable *left* adjoints, with the 2-cells pointing in the direction of the mates between the “forwards” functors, $g^+ \rightarrow (g')^+$.

Of course, the two conventions carry the same information, and are interchanged by duality: $[\mathcal{A}^{\text{op}}]_{\text{L}} = [\mathcal{A}]_{\text{R}}$. In the above introduction I wrote $[\mathcal{A}]$ for $[\mathcal{A}]_{\text{L}}$, since the most familiar examples of multivariable adjunctions (e.g. closed monoidal structures) are generally considered to point in the direction of their left adjoints. However, the fact that this choice flips the 2-cells makes it seem less natural from an abstract point of view, so *in the rest of the paper I will change notation and write $[\mathcal{A}]$ for $[\mathcal{A}]_{\text{R}}$* ; this also has the advantage of coinciding with the orientation of multivariable adjunctions chosen by [CGR14]. In particular, this means that a $(2, 0)$ -variable adjunction $(\mathcal{A}, \mathcal{B}) \rightarrow ()$ is now a mutual *left* adjunction, and dually.

Another way to “fix” the orientation of 2-cells would be to use $\text{Chu}(\text{Cat}, \text{Set}^{\text{op}})$ instead of $\text{Chu}(\text{Cat}, \text{Set})$. Then we could define $[\mathcal{A}]$ to be $(\mathcal{A}, \mathcal{A}^{\text{op}}, \text{hom}_{\mathcal{A}^{\text{op}}})$ and have adjunctions point in the direction of left adjoints and 2-cells in the direction of transformations between these left adjoints. However, this would have the unaesthetic consequence that the “correct” category of $(0, 0)$ -ary morphisms, as in Remark 1.5, would be Set^{op} rather than Set . There seems to be no perfect solution.

1.7. **OUTLINE.** We begin in section 2 by defining the abstract input (and also the output!) of our 2-Chu-Dialectica construction. In section 3 we give the construction itself (the general $(1, 1)$ -categorical case). Then in section 4 we show how it specializes to one of the Dialectica constructions (the $(0, 1)$ -categorical case), while in section 5 we show how it specializes to the Chu construction (the $(0, 0)$ -categorical case). Finally, in section 6 we specialize to the 2-Chu construction (the $(1, 0)$ -categorical case) and enhance the result to a poly double category of polarized multivariable adjunctions, and in section 7 we connect this construction to the cyclic multi double category of [CGR14].

1.8. **ACKNOWLEDGMENTS.** I would like to thank Andrej Bauer and Valeria de Paiva for helping me to understand the Dialectica construction; Sam Staton for pointing out that relative adjunctions appear as morphisms in $\mathcal{C}hu(\mathcal{C}at, \mathbf{Set})$; and Emily Riehl, Philip Hackney, and René Guitart for useful conversations and feedback.

2. Presheaves of polycategories

We first recall Szabo’s [Sza75] definition of polycategory. On the logical side, polycategories are a categorical abstraction of the structural rules of classical linear logic [Gir87] (identity, cut, and exchange), while on the categorical side they are related to $*$ -autonomous categories [Bar79] (and more generally linearly distributive categories [CS97b]) roughly in the same way that multicategories are related to monoidal categories.

All of our polycategories and multicategories will be symmetric, so we often omit the adjective. If Γ and Γ' are finite lists of the same length, by an *isomorphism* $\sigma : \Gamma \xrightarrow{\sim} \Gamma'$ we mean a permutation of $|\Gamma|$ that maps the objects in Γ to those in Γ' , i.e. if $\Gamma' = (A_1, \dots, A_n)$ then $\Gamma = (A_{\sigma_1}, \dots, A_{\sigma_n})$.

2.1. **DEFINITION.** A **symmetric polycategory** \mathcal{P} consists of

- A set of objects.
- For each pair (Γ, Δ) of finite lists of objects, a set $\mathcal{P}(\Gamma; \Delta)$ of “polyarrows”, which we may also write $f : \Gamma \rightarrow \Delta$.
- For each $\Gamma, \Gamma', \Delta, \Delta'$, and isomorphisms $\rho : \Gamma \xrightarrow{\sim} \Gamma'$ and $\tau : \Delta \xrightarrow{\sim} \Delta'$, an action

$$\mathcal{P}(\Gamma; \Delta) \rightarrow \mathcal{P}(\Gamma'; \Delta')$$

written $f \mapsto \tau f \rho$, that is functorial on composition of permutations.

- For each object A , an identity polyarrow $1_A \in \mathcal{P}(A; A)$.
- For finite lists of objects $\Gamma, \Delta_1, \Delta_2, \Lambda_1, \Lambda_2, \Sigma$, and object A , composition maps

$$\mathcal{P}(\Lambda_1, A, \Lambda_2; \Sigma) \times \mathcal{P}(\Gamma; \Delta_1, A, \Delta_2) \rightarrow \mathcal{P}(\Lambda_1, \Gamma, \Lambda_2; \Delta_1, \Sigma, \Delta_2).$$

We write this operation infix as \circ_A , if there is no risk of confusion.

- Axioms of associativity and equivariance:

$$1_A \circ_A f = f \tag{2.2}$$

$$f \circ_A 1_A = f \tag{2.3}$$

$$(h \circ_B g) \circ_A f = h \circ_B (g \circ_A f) \tag{2.4}$$

$$(h \circ_B g) \circ_A f = \sigma((h \circ_A f) \circ_B g) \tag{2.5}$$

$$h \circ_B (g \circ_A f) = (g \circ_A (h \circ_B f)) \sigma \tag{2.6}$$

$$\tau_1 g \rho_1 \circ_A \tau_2 f \rho_2 = \tau_3 (g \circ_A f) \rho_3 \tag{2.7}$$

where the associativity axioms (2.4), (2.5), and (2.6) apply whenever both sides make sense, with σ in (2.5) and (2.6) chosen to make the equation well-typed, and in (2.7) the six permutations are related in a straightforward way that makes the equation well-typed.

2.8. **REMARK.** We can, and will, identify (symmetric) multicategories with (symmetric) polycategories that are **co-unary**, i.e. $\mathcal{P}(\Gamma; \Delta)$ is empty unless $|\Delta| = 1$.

2.9. **DEFINITION.** Let A, B be objects of a (symmetric) polycategory \mathcal{P} .

- A **tensor product** of A, B is an object $A \boxtimes B$ with a morphism $(A, B) \rightarrow (A \boxtimes B)$ such that the following precomposition maps⁸ are isomorphisms:

$$\mathcal{P}(\Gamma, A \boxtimes B; \Delta) \xrightarrow{\cong} \mathcal{P}(\Gamma, A, B; \Delta).$$

- A **unit** is an object \top with a morphism $() \rightarrow (\top)$ such that the following precomposition maps are isomorphisms:

$$\mathcal{P}(\Gamma, \top; \Delta) \xrightarrow{\cong} \mathcal{P}(\Gamma; \Delta).$$

- A **cotensor product**⁹ of A, B is an object $A \diamond B$ with a morphism $(A \diamond B) \rightarrow (A, B)$ such that the following postcomposition maps are isomorphisms:

$$\mathcal{P}(\Gamma; A \diamond B, \Delta) \xrightarrow{\cong} \mathcal{P}(\Gamma; A, B, \Delta).$$

- A **counit** is an object \perp with a morphism $(\perp) \rightarrow ()$ such that the following postcomposition maps are isomorphisms:

$$\mathcal{P}(\Gamma; \perp, \Delta) \xrightarrow{\cong} \mathcal{P}(\Gamma; \Delta).$$

- A **dual** of A is an object A^\bullet with morphisms $\eta : () \rightarrow (A, A^\bullet)$ and $\varepsilon : (A^\bullet, A) \rightarrow ()$ such that $\varepsilon \circ_A \eta = 1_{A^\bullet}$ and $\varepsilon \circ_{A^\bullet} \eta = 1_A$.

- A **strong hom** of A, B is an object $A \multimap B$ with a morphism $(A \multimap B, A) \rightarrow (B)$ such that the following precomposition maps are isomorphisms:

$$\mathcal{P}(\Gamma; \Delta, A \multimap B) \xrightarrow{\cong} \mathcal{P}(\Gamma, A; \Delta, B).$$

It is a **weak hom** if this holds only when $\Delta = \emptyset$.

⁸We take advantage of the symmetry of \mathcal{P} to place the objects with universal properties last in the domain or first in the codomain. In the non-symmetric case, the tensor product isomorphism should be $\mathcal{P}(\Gamma_1, A \boxtimes B, \Gamma_2; \Delta) \cong \mathcal{P}(\Gamma_1, A, B, \Gamma_2; \Delta)$, and so on.

⁹I agree with [CS97b] that the notations for tensor and cotensor products should be visually dual, but I find \oplus has too strong a connotation of direct sums to use it for the cotensor product. Hence \boxtimes and \diamond .

A polycategory is **representable** if it has all tensor products, units, cotensor products, and counits; then it is equivalently a **linearly distributive category** [CS97b], while if it also has duals then it is a ***-autonomous category** [Bar79]. Strong homs can be defined in terms of duals and cotensors, if both exist, as $(A \multimap B) = (A^\bullet \boxplus B)$; while duals can be defined in terms of strong homs and a counit as $A^\bullet = (A \multimap \perp)$. Finally, if duals exist, then tensors and cotensors are interdefinable by $(A \boxplus B) = (A^\bullet \boxtimes B^\bullet)^\bullet$ and dually.

Our basic structure will be a multicategory equipped with a presheaf of polycategories. It may not be immediately clear what should be meant by a *presheaf* on a multicategory; the following definition is obtained by abstracting the structure of the “representable presheaf” $\mathcal{C}(-; A)$ for an object A .

2.10. DEFINITION. Let \mathcal{C} be a symmetric multicategory and \mathcal{A} a category. An **\mathcal{A} -valued presheaf on \mathcal{C}** consists of:

- (i) An object $\mathcal{M}(\Gamma) \in \mathcal{A}$ for each finite list Γ of objects of \mathcal{C} .
- (ii) For all isomorphisms $\rho : \Gamma \xrightarrow{\sim} \Gamma'$, an action $\mathcal{M}(\Gamma) \rightarrow \mathcal{M}(\Gamma')$, written $x \mapsto x\rho$, that is functorial on composition of permutations.
- (iii) For each morphism $f \in \mathcal{C}(\Gamma; A)$ and finite lists of objects Δ_1, Δ_2 , an action morphism

$$f^* : \mathcal{M}(\Delta_1, A, \Delta_2) \rightarrow \mathcal{M}(\Delta_1, \Gamma, \Delta_2)$$

- (iv) Any 1_A^* is the identity, and the following diagrams commute for any morphisms $f \in \mathcal{C}(\Gamma; A)$, $g \in \mathcal{C}(\Lambda; B)$, and $h \in \mathcal{C}(\Sigma_1, A, \Sigma_2; C)$.

$$\begin{array}{ccc} \mathcal{M}(\Delta_1, A, \Delta_2, B, \Delta_3) & \xrightarrow{f^*} & \mathcal{M}(\Delta_1, \Gamma, \Delta_2, B, \Delta_3) \\ g^* \downarrow & & \downarrow g^* \\ \mathcal{M}(\Delta_1, A, \Delta_2, \Lambda, \Delta_3) & \xrightarrow{f^*} & \mathcal{M}(\Delta_1, \Gamma, \Delta_2, \Lambda, \Delta_3) \end{array}$$

$$\begin{array}{ccc} \mathcal{M}(\Delta_1, C, \Delta_2) & \xrightarrow{h^*} & \mathcal{M}(\Delta_1, \Sigma_1, A, \Sigma_2, \Delta_2) \\ & \searrow (h \circ_A f)^* & \downarrow f^* \\ & & \mathcal{M}(\Delta_1, \Sigma_1, \Gamma, \Sigma_2, \Delta_2) \end{array}$$

- (v) The following diagrams commute for any morphism $f \in \mathcal{C}(\Gamma; A)$ and permutations $\rho : \Gamma \xrightarrow{\sim} \Gamma'$, and $\tau : (\Delta_1, A, \Delta_2) \xrightarrow{\sim} (\Delta'_1, A, \Delta'_2)$ where τ sends one of the notated copies of A to the other one, and τ' treats Γ as a block replacing A in τ :

$$\begin{array}{ccc} \mathcal{M}(\Delta_1, A, \Delta_2) & \xrightarrow{f^*} & \mathcal{M}(\Delta_1, \Gamma, \Delta_2) & \mathcal{M}(\Delta_1, A, \Delta_2) & \xrightarrow{\tau} & \mathcal{M}(\Delta'_1, A, \Delta'_2) \\ & \searrow (f\rho)^* & \downarrow \rho & f^* \downarrow & & \downarrow f^* \\ & & \mathcal{M}(\Delta_1, \Gamma', \Delta_2) & \mathcal{M}(\Delta_1, \Gamma, \Delta_2) & \xrightarrow{\tau'} & \mathcal{M}(\Delta'_1, \Gamma, \Delta'_2) \end{array}$$

2.11. EXAMPLES.

- (i) As suggested above, any $A \in \mathcal{C}$ gives rise to a **representable** **Set**-valued presheaf defined by $\mathfrak{L}_A(\Gamma) = \mathcal{C}(\Gamma; A)$.
- (ii) For any presheaf \mathcal{M} and any finite list of objects Δ , there is a **shifted** presheaf $\mathcal{M}[\Delta]$ defined by $\mathcal{M}[\Delta](\Gamma) = \mathcal{M}(\Gamma, \Delta)$.

2.12. REMARK. Presheaves on multicategories can be reformulated in several ways:

- (i) When a multicategory \mathcal{C} is regarded as a co-unary polycategory, a **Set**-valued presheaf on \mathcal{C} is equivalently a *module* over \mathcal{C} in the sense of [Hyl02] whose nonempty values are all of the form $\mathcal{M}(\Gamma;)$.
- (ii) A multicategory \mathcal{C} equipped with a **Set**-valued presheaf \mathcal{M} can equivalently be considered as a polycategory that is *co-subunary*, i.e. where morphisms have codomain arity 0 or 1: we define $\mathcal{C}(\Gamma;) = \mathcal{M}(\Gamma)$. The presheaf \mathcal{M} is representable if and only if this polycategory has a “counit in the co-subunary sense” $\mathcal{C}(\Gamma; \perp) \cong \mathcal{C}(\Gamma;)$.
- (iii) An \mathcal{A} -valued presheaf on a multicategory \mathcal{C} can equivalently be defined as an ordinary functor $(\mathbf{F}_{\otimes}\mathcal{C})^{\text{op}} \rightarrow \mathcal{A}$, where $\mathbf{F}_{\otimes}\mathcal{C}$ is the free symmetric strict monoidal category generated by \mathcal{C} , whose objects are finite lists of objects of \mathcal{C} .

Formulation (iii) implies that the category $\text{Psh}(\mathcal{C})$ of **Set**-valued presheaves on \mathcal{C} admits a Day-convolution monoidal structure [Day70]. By the monoidal Yoneda lemma, it follows that morphisms $(\mathfrak{L}_{A_1}, \dots, \mathfrak{L}_{A_n}) \rightarrow \mathcal{M}$ in the underlying multicategory of $\text{Psh}(\mathcal{C})$ are in natural bijection with elements of $\mathcal{M}(A_1, \dots, A_n)$. Accordingly, we will sometimes abuse notation by writing $x : (A_1, \dots, A_n) \rightarrow \mathcal{M}$ instead of $x \in \mathcal{M}(A_1, \dots, A_n)$, with the presheaf action similarly denoted by composition, $f^*x = x \circ f$.

Logically, a presheaf on a multicategory represents a “logic over a type theory”: we have terms¹⁰ $\Gamma \vdash t : A$ for the morphisms of \mathcal{C} , together with an additional judgment form “ $\Gamma \vdash \phi$ **prop**” for the elements of the presheaf, with a substitution action by the terms. The expected structure of *entailment* between such propositions can be modeled by choosing an appropriate target category \mathcal{A} other than **Set**, depending on the desired kind of logic.

In our case, we want the logic to be classical linear logic, so we consider multicategories \mathcal{C} equipped with a *presheaf of polycategories*, which we generally denote Ω . Note that Ω is equivalently an internal polycategory object in $\text{Psh}(\mathcal{C})$. Logically, the objects of \mathcal{C} correspond to *types*, the morphisms correspond to *terms*

$$x_1 : A_1, \dots, x_n : A_n \vdash t : B,$$

¹⁰For an ordinary symmetric multicategory as in our case, the base type theory in question is an intuitionistic linear one; by instead using a *cartesian* multicategory with an appropriate notion of presheaf we would model an intuitionistic nonlinear type theory.

the elements of $\Omega(\Gamma)$ correspond to *predicates*

$$x_1 : A_1, \dots, x_n : A_n \vdash \phi \text{ prop},$$

and the morphisms in $\Omega(\Gamma)$ correspond to *sequents* or *entailments* in context:

$$x_1 : A_1, \dots, x_n : A_n \mid \phi_1, \dots, \phi_m \vdash \psi_1, \dots, \psi_k.$$

Note that each ϕ_i and ψ_j depends *separately* linearly on the context: each variable x_k is “used exactly once” in *each* ϕ_i and ψ_j .

Following this intuition, we refer to such a pair (\mathcal{C}, Ω) as a **virtual linear hyperdoctrine**. In general a hyperdoctrine [Law06, Law70] is an indexed category whose base category represents the types and terms in a type theory and whose fibers represent the predicates and sequents in a first-order logic over that type theory. The word “virtual” is used by analogy to [CS10] and indicates that nothing corresponding to the type constructors or logical connectives or quantifiers is present; we have only the structural rules. (The lack of even finite products of types is what forces us to allow predicates to depend on finite lists of types rather than single ones.) Note that we do not assume our polycategories $\Omega(\Gamma)$ to be poly-posets; as for Lawvere, the fibers of our hyperdoctrine can distinguish between different “proofs” with the same domain and codomain.

3. Dimension $(1, 1)$: the 2-Chu-Dialectica construction

Let Ω be a presheaf of polycategories on a multicategory \mathcal{C} ; we will describe another presheaf of polycategories $\text{Adj}_{\mathcal{C}}(\Omega)$ on \mathcal{C} . For a finite list of objects Γ , an element of $\text{Adj}_{\mathcal{C}}(\Omega)(\Gamma)$ is a triple $(\phi^+, \phi^-, \underline{\phi})$, where ϕ^+, ϕ^- are objects of \mathcal{C} and $\underline{\phi} \in \Omega(\Gamma, \phi^-, \phi^+)$. The presheaf action is induced in the obvious way from that of Ω .

A morphism $(\phi_1, \dots, \phi_m) \rightarrow (\psi_1, \dots, \psi_n)$ in $\text{Adj}_{\mathcal{C}}(\Omega)(\Gamma)$ consists of:

(i) Morphisms in \mathcal{C} :

$$\begin{aligned} f_j &: (\phi_1^+, \dots, \phi_m^+, \psi_1^-, \dots, \widehat{\psi_j^-}, \dots, \psi_n^-) \longrightarrow \psi_j^+ \\ g_i &: (\phi_1^+, \dots, \widehat{\phi_i^+}, \dots, \phi_m^+, \psi_1^-, \dots, \psi_n^-) \longrightarrow \phi_i^- \end{aligned}$$

for $1 \leq j \leq n$ and $1 \leq i \leq m$ (where $\widehat{\chi}$ means that χ is omitted from the list). We call these the *primary components*.

(ii) A morphism in $\Omega(\Gamma, \phi_1^+, \dots, \phi_m^+, \psi_1^-, \dots, \psi_n^-)$:

$$\alpha : (g_1^* \underline{\phi}_1, \dots, g_m^* \underline{\phi}_m) \rightarrow (f_1^* \underline{\psi}_1, \dots, f_n^* \underline{\psi}_n)$$

We call this the *secondary component*.

We have omitted to notate the action of symmetric groups needed to make all the above composites live in the right place. We will continue to do the same below; in all cases there is only one possible permutation that could be meant. Note that Γ appears in the domain of α , but *not* in the domains of f_j and g_i . Also, if $n = m = 0$, the only datum is the $(0, 0)$ -ary morphism α in $\Omega(\Gamma)$.

The symmetric and presheaf actions on morphisms in $\text{Adj}_C(\Omega)(\Gamma)$ is obvious. To define composition in the polycategory $\text{Adj}_C(\Omega)(\Gamma)$, suppose we have another such morphism $(\xi_1, \dots, \xi_p) \rightarrow (\zeta_1, \dots, \zeta_q)$, where $\psi_{j_0} = \xi_{k_0}$, with primary components

$$\begin{aligned} r_l &: (\xi_1^+, \dots, \xi_p^+, \zeta_1^-, \dots, \widehat{\zeta_l^-}, \dots, \zeta_q^-) \longrightarrow \zeta_l^+ \\ s_k &: (\xi_1^+, \dots, \widehat{\xi_k^+}, \dots, \xi_p^+, \zeta_1^-, \dots, \zeta_q^-) \longrightarrow \xi_k^- \end{aligned}$$

for $1 \leq k \leq p$ and $1 \leq l \leq q$, and secondary component

$$\beta : (s_1^* \xi_1, \dots, s_p^* \xi_p) \rightarrow (r_1^* \zeta_1, \dots, r_q^* \zeta_q).$$

For conciseness we write $\vec{\phi}^+ = (\phi_1^+, \dots, \phi_m^+)$ and $\vec{\phi}_{\neq j}^+ = (\phi_1^+, \dots, \widehat{\phi_j^+}, \dots, \phi_m^+)$.

The desired composite should be (up to symmetric action) a morphism $(\vec{\phi}, \vec{\xi}_{\neq k_0}) \rightarrow (\vec{\psi}_{\neq j_0}, \vec{\zeta})$. We take its primary components to be (up to symmetric action)

$$\begin{aligned} f_j \circ_{\psi_{j_0}^- = \xi_{k_0}^-} s_{k_0} &: (\vec{\phi}^+, \vec{\xi}_{\neq k_0}^+, \vec{\psi}_{\neq j, j_0}^-, \vec{\zeta}^-) \longrightarrow \psi_j^+ \quad (j \neq j_0) \\ r_l \circ_{\xi_{k_0}^+ = \psi_{j_0}^+} f_{j_0} &: (\vec{\phi}^+, \vec{\xi}_{\neq k_0}^+, \vec{\psi}_{\neq j_0}^-, \vec{\zeta}_{\neq l}^-) \longrightarrow \zeta_l^+ \\ g_i \circ_{\psi_{j_0}^- = \xi_{k_0}^-} s_{k_0} &: (\vec{\phi}_{\neq i}^+, \vec{\xi}_{\neq k_0}^+, \vec{\psi}_{\neq j_0}^-, \vec{\zeta}^-) \longrightarrow \phi_i^- \\ s_k \circ_{\xi_{k_0}^+ = \psi_{j_0}^+} f_{j_0} &: (\vec{\phi}^+, \vec{\xi}_{\neq k, k_0}^+, \vec{\psi}_{\neq j_0}^-, \vec{\zeta}^-) \longrightarrow \xi_k^- \quad (k \neq k_0). \end{aligned}$$

For the secondary component, first note that we have

$$\begin{aligned} s_{k_0}^* \alpha &: (s_{k_0}^* g_1^* \phi_1, \dots, s_{k_0}^* g_m^* \phi_m) \rightarrow (s_{k_0}^* f_1^* \psi_1, \dots, s_{k_0}^* f_n^* \psi_n) \\ f_{j_0}^* \beta &: (f_{j_0}^* s_1^* \xi_1, \dots, f_{j_0}^* s_p^* \xi_p) \rightarrow (f_{j_0}^* r_1^* \zeta_1, \dots, f_{j_0}^* r_q^* \zeta_q) \end{aligned}$$

Now since $\psi_{j_0} = \xi_{k_0}$, by associativity of the presheaf action we have $s_{k_0}^* f_{j_0}^* \psi_{j_0} = f_{j_0}^* s_{k_0}^* \xi_{k_0}$. Thus, we can compose these two morphisms along this common object to get a morphism

$$\begin{aligned} & (s_{k_0}^* g_1^* \phi_1, \dots, s_{k_0}^* g_m^* \phi_m, f_{j_0}^* s_1^* \xi_1, \dots, \widehat{f_{j_0}^* s_{k_0}^* \xi_{k_0}}, \dots, f_{j_0}^* s_p^* \xi_p) \\ & \longrightarrow (s_{k_0}^* f_1^* \psi_1, \dots, s_{k_0}^* \widehat{f_{j_0}^* \psi_{j_0}}, \dots, s_{k_0}^* f_n^* \psi_n, f_{j_0}^* r_1^* \zeta_1, \dots, f_{j_0}^* r_q^* \zeta_q) \end{aligned}$$

This is the secondary component of our desired composite in $\text{Adj}_C(\Omega)(\Gamma)$. The associativity, equivariance, and so on of this operation follow from the analogous properties in Ω .

4. Dimension $(0, 1)$: the Dialectica construction

Dialectica and Chu constructions generally yield a *monoidal* category (perhaps linearly distributive, closed, or $*$ -autonomous) or a fibration of such. Our construction produces a fully virtual (multi/poly-categorical) structure, so to compare it to the usual constructions we need to consider its representability conditions, which are induced from similar conditions on both \mathcal{C} and Ω .

4.1. **DEFINITION.** A presheaf of polycategories Ω on \mathcal{C} has **tensors**, a **unit**, **cotensors**, a **counit**, **duals**, or **homs** if the polycategories $\Omega(\Gamma)$ have the relevant structure and it is preserved by the presheaf action (up to isomorphism).

4.2. **LEMMA.** *Let Ω be a presheaf of polycategories on a multicategory \mathcal{C} .*

- (i) *If Ω is co-unary with tensors and a unit, it is equivalently a presheaf of symmetric monoidal categories.*
- (ii) *If Ω is co-unary with tensors, a unit, and homs, it is equivalently a presheaf of closed symmetric monoidal categories.*
- (iii) *If Ω has tensors, a unit, cotensors, and a counit, it is equivalently a presheaf of linearly distributive categories.*
- (iv) *If Ω has tensors, a unit, cotensors, a counit, and weak homs, it is equivalently a presheaf of “full multiplicative categories” [CS97a]: linearly distributive categories whose tensor (but not cotensor) monoidal structure is closed.*
- (v) *If Ω has tensors, a unit, cotensors, a counit, and duals, it is equivalently a presheaf of $*$ -autonomous categories. ■*

We keep the notations $\boxtimes, \top, \boxplus, \perp, -\circ$ for such structures on Ω , but to avoid confusion we will instead denote tensor products, units, and internal-homs in the multicategory \mathcal{C} by $A \otimes B, I$, and $[A, B]$ respectively. We also require the following assumption:

4.3. **DEFINITION.** A tensor product $(A, B) \rightarrow A \otimes B$ in a multicategory \mathcal{C} is **preserved** by an \mathcal{A} -valued presheaf Ω if the induced maps $\Omega(\Gamma, A \otimes B) \xrightarrow{\sim} \Omega(\Gamma, A, B)$ are all isomorphisms. Similarly, Ω preserves a unit $() \rightarrow (I)$ if it induces isomorphisms $\Omega(\Gamma, I) \xrightarrow{\sim} \Omega(\Gamma)$.

4.4. **LEMMA.** *If \mathcal{C} has all tensor products and a unit, then an \mathcal{A} -valued presheaf Ω that preserves all such tensors and the unit is equivalently an ordinary presheaf on the underlying ordinary category of \mathcal{C} . ■*

4.5. **EXAMPLE.** Just as an internal category in \mathcal{C} induces a representable presheaf of categories on \mathcal{C} , an internal polycategory induces a presheaf of polycategories, and similarly for any other such structure. Such presheaves always preserve all tensor products and units in \mathcal{C} .

4.6. **THEOREM.** *Let \mathcal{C} be a multicategory and Ω a presheaf of polycategories on \mathcal{C} .*

- (i) *$\text{Adj}_{\mathcal{C}}(\Omega)$ always has duals.*
- (ii) *If \mathcal{C} has a unit and a terminal object, and Ω preserves the unit of \mathcal{C} and has a unit (resp. a counit), then $\text{Adj}_{\mathcal{C}}(\Omega)$ has a unit (resp. a counit).*
- (iii) *If \mathcal{C} has tensor products, homs, and binary cartesian products, and Ω preserves the tensor products of \mathcal{C} and has tensor products, cotensor products, or strong or weak homs, then $\text{Adj}_{\mathcal{C}}(\Omega)$ also has tensor products, cotensor products, or strong or weak homs respectively.*

PROOF. The dual of $(\phi^+, \phi^-, \underline{\phi})$ is

$$(\phi^{\pm})^+ = \phi^- \quad (\phi^{\pm})^- = \phi^+ \quad (\Gamma, \phi^+, \phi^-) \simeq (\Gamma, \phi^-, \phi^+) \xrightarrow{\underline{\phi}} \Omega.$$

The unit is defined by $\top^+ = I$ and $\top^- = 1$ (the terminal object), with $\perp = \top$ in $\Omega(\Gamma, I, 1)$, while the counit similarly has $\perp^+ = 1$ and $\perp^- = I$.

The tensor product of $(\phi^+, \phi^-, \underline{\phi})$ and $(\psi^+, \psi^-, \underline{\psi})$ is

$$\begin{aligned} (\phi \boxtimes \psi)^+ &= (\phi^+ \otimes \psi^+) \\ (\phi \boxtimes \psi)^- &= [\phi^+, \psi^-] \times [\psi^+, \phi^-] \end{aligned}$$

with $\underline{\phi \boxtimes \psi} \in \Omega(\Gamma, [\phi^+, \psi^-] \times [\psi^+, \phi^-], \phi^+ \otimes \psi^+)$ induced by the universal property of $\phi^+ \otimes \psi^+$ from the following tensor product in Ω

$$\begin{aligned} & \left((\Gamma, [\phi^+, \psi^-] \times [\psi^+, \phi^-], \phi^+, \psi^+) \rightarrow (\Gamma, [\phi^+, \psi^-], \phi^+, \psi^+) \rightarrow (\Gamma, \psi^-, \psi^+) \xrightarrow{\underline{\psi}} \Omega \right) \\ & \quad \boxtimes \\ & \left((\Gamma, [\phi^+, \psi^-] \times [\psi^+, \phi^-], \phi^+, \psi^+) \rightarrow (\Gamma, [\psi^+, \phi^-], \phi^+, \psi^+) \rightarrow (\Gamma, \phi^-, \phi^+) \xrightarrow{\underline{\phi}} \Omega \right) \end{aligned} \tag{4.7}$$

The universal morphism $(\phi, \psi) \rightarrow (\phi \boxtimes \psi)$ has primary components

$$\begin{aligned} & (\phi^+, \psi^+) \rightarrow (\phi^+ \otimes \psi^+) \\ & ([\phi^+, \psi^-] \times [\psi^+, \phi^-], \phi^+) \rightarrow ([\phi^+, \psi^-], \phi^+) \rightarrow \psi^- \\ & ([\phi^+, \psi^-] \times [\psi^+, \phi^-], \psi^+) \rightarrow ([\psi^+, \phi^-], \psi^+) \rightarrow \phi^- \end{aligned}$$

and its secondary component exhibits the universal property of the tensor product (4.7).

We check the universal property of $\phi \boxtimes \psi$ in the case of a morphism $(\phi \boxtimes \psi, \xi) \rightarrow (\zeta)$ in $\text{Adj}_{\mathcal{C}}(\Omega)(\Gamma)$; the general case is the same but the notation is more tedious. Such a morphism has primary components

$$\begin{aligned} f &: (\phi^+ \otimes \psi^+, \xi^+) \rightarrow \zeta^+ \\ g &: (\phi^+ \otimes \psi^+, \zeta^-) \rightarrow \xi^- \\ h &: (\xi^+, \zeta^-) \rightarrow [\phi^+, \psi^-] \times [\psi^+, \phi^-] \end{aligned}$$

and a secondary component

$$(\underline{\phi} \boxtimes \underline{\psi} \circ h, \xi \circ g) \rightarrow (\zeta \circ f). \quad (4.8)$$

Composing f, g, h with the components of $(\phi, \psi) \rightarrow (\phi \boxtimes \psi)$ exactly implements the universal properties of $\phi^+ \otimes \psi^+$ and $[\phi^+, \psi^-] \times [\psi^+, \phi^-]$, yielding a bijective correspondence to quadruples of morphisms

$$\begin{aligned} f' : (\phi^+, \psi^+, \xi^+) &\rightarrow \zeta^+ & h' : (\phi^+, \xi^+, \zeta^-) &\rightarrow \psi^- \\ g' : (\phi^+, \psi^+, \zeta^-) &\rightarrow \xi^- & h'' : (\psi^+, \xi^+, \zeta^-) &\rightarrow \phi^- \end{aligned}$$

which are exactly as required for a morphism $(\phi, \psi, \xi) \rightarrow (\zeta)$. Similarly, composing (4.8) with the secondary component of $\phi \boxtimes \psi$ simply composes $\underline{\phi} \boxtimes \underline{\psi}$ with the universal map $(\phi^+, \psi^+) \rightarrow (\phi^+ \otimes \psi^+)$, exposing the tensor product (4.7), and then composes with the morphism exhibiting the universal property of the latter.

Dually, the cotensor product of ϕ and ψ is

$$\begin{aligned} (\phi \boxplus \psi)^+ &= [\phi^-, \psi^+] \times [\psi^-, \phi^+] \\ (\phi \boxplus \psi)^- &= (\phi^- \otimes \psi^-) \end{aligned}$$

with $\underline{\phi} \boxplus \underline{\psi}$ defined similarly using \boxplus in Ω instead of \boxtimes , while the counit has $\perp^+ = 1$ and $\perp^- = I$, with $\underline{\perp} = \perp$. And the hom $\phi \multimap \psi$ (strong or weak according to that of Ω) is

$$\begin{aligned} (\phi \multimap \psi)^+ &= [\phi^+, \psi^+] \times [\psi^-, \phi^-] \\ (\phi \multimap \psi)^- &= (\phi^+ \otimes \psi^-) \end{aligned}$$

with $\underline{\phi} \multimap \underline{\psi}$ defined using \multimap in Ω . ■

4.9. EXAMPLE. The original Dialectica construction focused on what in our notation is the “empty context” component $\text{Adj}_{\mathcal{C}}(\Omega)(\cdot)$. For instance, applying Theorem 4.6 to Example 4.5 we see that if Ω is an internal closed monoidal poset in a closed symmetric monoidal category \mathcal{C} , then $\text{Adj}_{\mathcal{C}}(\Omega)(\cdot)$ is a closed symmetric monoidal category. This reproduces the general Dialectica construction from [dP91, dP06].

4.10. EXAMPLES. The original construction from [dP89b] (called **GC** in [dP89a]) is the case when we have a *cartesian* closed category \mathcal{C} , with $\Omega = \mathbf{Sub}(\mathcal{C})$ its subobject fibration where $\mathbf{Sub}(\mathcal{C})(A)$ is the poset of subobjects of A , with additional structure induced from that of \mathcal{C} :

- (i) As long as \mathcal{C} has finite limits, $\mathbf{Sub}(\mathcal{C})$ is a presheaf of meet-semilattices, hence in particular symmetric monoidal posets, so we can regard it as a presheaf of multicategories (i.e. co-unary polycategories) with tensors and units. Thus, $\text{Adj}_{\mathcal{C}}(\mathbf{Sub}(\mathcal{C}))(\cdot)$ is a symmetric monoidal category.

- (ii) If \mathcal{C} is a Heyting category, then $\mathbf{Sub}(\mathcal{C})$ is a presheaf of Heyting algebras, i.e. cartesian closed posets, so we can regard it as a presheaf of multicategories with tensors, units, and homs. Thus, in this case $\mathbf{Adj}_{\mathcal{C}}(\mathbf{Sub}(\mathcal{C}))()$ is a closed symmetric monoidal category.
- (iii) If \mathcal{C} is a coherent category, then $\mathbf{Sub}(\mathcal{C})$ is a presheaf of distributive lattices. Since a distributive lattice can be regarded as a linearly distributive category, we can also regard it as a polycategory that is *not* co-unary, and has tensors, a unit, cotensors, and a counit. Thus, in this case $\mathbf{Adj}_{\mathcal{C}}(\mathbf{Sub}(\mathcal{C}))()$ is a linearly distributive category.
- (iv) If in (iii) \mathcal{C} is furthermore a Heyting category, then the polycategories $\mathbf{Sub}(\mathcal{C})(\Gamma)$ also have weak homs, so that $\mathbf{CD}_{\mathbf{Sub}(\mathcal{C})}()$ is a full multiplicative category, as in [dP89b]. (In this paper we will not consider the additive fragment, i.e. the cartesian products and coproducts in $\mathbf{CD}_{\mathcal{T}}$, or the exponential modalities $!$ and $?$.)
- (v) If \mathcal{C} is furthermore a *Boolean* category, then $\mathbf{Sub}(\mathcal{C})$ is a presheaf of Boolean algebras, which as linearly distributive categories are $*$ -autonomous; thus in this case $\mathbf{Adj}_{\mathcal{C}}(\mathbf{Sub}(\mathcal{C}))()$ is also $*$ -autonomous. More generally, we can restrict to the sub-Boolean-algebras of $\neg\neg$ -closed subobjects in $\mathbf{Sub}(\mathcal{C})$; this produces the $*$ -autonomous category $\mathbf{Dec GC}$ from [dP89b].

5. Dimension $(0, 0)$: the Chu construction

The Chu construction is generally defined as an operation on closed symmetric monoidal categories equipped with an arbitrary object Ω ; see [Chu78, Chu79, Bar06, Pav93]. We fit this into our context with the following construction.

5.1. **DEFINITION.** Any set X is the set of objects of a **Frobenius-discrete**¹¹ polycategory X_{fd} , for which a polyarrow $(x_1, \dots, x_m) \rightarrow (y_1, \dots, y_n)$ consists of an element $z \in X$ such that $x_i = z$ and $y_j = z$ for all i, j .

The Frobenius-discrete polycategories are equivalently the coproducts of copies of the terminal polycategory; this motivates the name, since the terminal (symmetric) polycategory is freely generated by a (commutative) Frobenius algebra. Note that a $(0, 0)$ -ary arrow in a Frobenius-discrete polycategory is still determined by a single object, even though there is no domain or codomain for that object to appear in.

The construction $X \mapsto X_{\text{fd}}$ is functorial, so any **Set**-valued presheaf Ω on a multicategory \mathcal{C} induces a presheaf of polycategories Ω_{fd} . Applying the construction of section 3, we obtain another presheaf of polycategories $\mathbf{Adj}_{\mathcal{C}}(\Omega_{\text{fd}})$, whose objects are triples $(\phi^+, \phi^-, \underline{\phi})$, where ϕ^+, ϕ^- are objects of \mathbf{C} and $\underline{\phi} : (\Gamma, \phi^-, \phi^+) \rightarrow \Omega$. A morphism $(\phi_1, \dots, \phi_m) \rightarrow (\psi_1, \dots, \psi_n)$

¹¹A *discrete* polycategory, in my preferred terminology, would be one in the image of the left adjoint to the forgetful functor from polycategories to sets, i.e. one containing only identity arrows.

in $\mathbf{Adj}_{\mathcal{C}}(\Omega_{\text{fd}})(\Gamma)$ consists of

$$\begin{aligned} f_j &: (\phi_1^+, \dots, \phi_m^+, \psi_1^-, \dots, \widehat{\psi_j^-}, \dots, \psi_n^-) \longrightarrow \psi_j^+ \\ g_i &: (\phi_1^+, \dots, \widehat{\phi_i^+}, \dots, \phi_m^+, \psi_1^-, \dots, \psi_n^-) \longrightarrow \phi_i^- \\ \alpha &: (\phi_1^+, \dots, \phi_m^+, \psi_1^-, \dots, \psi_n^-) \longrightarrow \Omega \end{aligned}$$

such that

$$(\underline{\phi}_1 \circ_{\phi_1^-} g_1) = \dots = (\underline{\phi}_m \circ_{\phi_m^-} g_m) = (\underline{\psi}_1 \circ_{\psi_1^+} f_1) = \dots = (\underline{\psi}_n \circ_{\psi_n^+} f_n) = \alpha.$$

(Of course, if $m+n > 0$ then α is uniquely determined by the f 's and g 's, but if $m=n=0$ then α is the only datum.)

A Frobenius-discrete polycategory always has duals; in fact each object is its own dual. Thus, $\mathbf{Adj}_{\mathcal{C}}(\Omega_{\text{fd}})$ also has duals, and in particular $\mathbf{Adj}_{\mathcal{C}}(\Omega_{\text{fd}})(\Gamma)$ is a polycategory with duals (in fact it is a $*$ -polycategory; see section 7). However, a Frobenius-discrete polycategory almost never has tensors or cotensors (see Remark 5.6). So we cannot obtain tensors and cotensors in $\mathbf{Adj}_{\mathcal{C}}(\Omega_{\text{fd}})$ from Theorem 4.6, but we can construct them in a different way (coinciding with the usual Chu construction).

5.2. THEOREM. *Suppose \mathcal{C} is a closed symmetric monoidal category with pullbacks, and Ω is an object of \mathcal{C} (identified with its representable presheaf \mathfrak{K}_{Ω}). Then $\mathbf{Adj}_{\mathcal{C}}(\Omega_{\text{fd}})$ has tensors, a unit, cotensors, and a counit (and hence is a presheaf of $*$ -autonomous categories).*

PROOF. The tensor product of $(\phi^+, \phi^-, \underline{\phi})$ and $(\psi^+, \psi^-, \underline{\psi})$ is now

$$\begin{aligned} (\phi \boxtimes \psi)^+ &= (\phi^+ \otimes \psi^+) \\ (\phi \boxtimes \psi)^- &= [\phi^+, \psi^-] \times_{[\Gamma \otimes_{\phi^+ \otimes \psi^+}, \Omega]} [\psi^+, \phi^-] \end{aligned}$$

(where Γ denotes abusively the tensor product of all the objects in Γ), with

$$\underline{\phi \boxtimes \psi} : (\Gamma, [\phi^+, \psi^-] \times_{[\Gamma \otimes_{\phi^+ \otimes \psi^+}, \Omega]} [\psi^+, \phi^-], \phi^+ \otimes \psi^+) \rightarrow \Omega$$

induced by the universal property of $\phi^+ \otimes \psi^+$ from the *common value* of the following two morphisms

$$\begin{aligned} (\Gamma, [\phi^+, \psi^-] \times_{[\Gamma \otimes_{\phi^+ \otimes \psi^+}, \Omega]} [\psi^+, \phi^-], \phi^+, \psi^+) &\rightarrow (\Gamma, [\phi^+, \psi^-], \phi^+, \psi^+) \rightarrow (\Gamma, \psi^-, \psi^+) \xrightarrow{\underline{\psi}} \Omega \\ (\Gamma, [\phi^+, \psi^-] \times_{[\Gamma \otimes_{\phi^+ \otimes \psi^+}, \Omega]} [\psi^+, \phi^-], \phi^+, \psi^+) &\rightarrow (\Gamma, [\psi^+, \phi^-], \phi^+, \psi^+) \rightarrow (\Gamma, \phi^-, \phi^+) \xrightarrow{\underline{\phi}} \Omega. \end{aligned} \tag{5.3}$$

Its universal morphism is defined similarly:

$$\begin{aligned} (\phi^+, \psi^+) &\rightarrow (\phi^+ \otimes \psi^+) \\ ([\phi^+, \psi^-] \times_{[\Gamma \otimes_{\phi^+ \otimes \psi^+}, \Omega]} [\psi^+, \phi^-], \phi^+) &\rightarrow ([\phi^+, \psi^-], \phi^+) \rightarrow \psi^- \\ ([\phi^+, \psi^-] \times_{[\Gamma \otimes_{\phi^+ \otimes \psi^+}, \Omega]} [\psi^+, \phi^-], \psi^+) &\rightarrow ([\psi^+, \phi^-], \psi^+) \rightarrow \phi^- \end{aligned}$$

plus the fact that the latter two of these, when composed with $\underline{\psi}$ and $\underline{\phi}$ respectively, yield (5.3). For the universal property, a morphism $(\phi \boxtimes \psi, \xi) \rightarrow (\zeta)$ in $\text{Adj}_{\mathcal{C}}(\Omega_{\text{fd}})$ now consists of morphisms in \mathcal{C} :

$$\begin{aligned} f &: (\phi^+ \otimes \psi^+, \xi^+) \rightarrow \zeta^+ \\ g &: (\phi^+ \otimes \psi^+, \zeta^-) \rightarrow \xi^- \\ h &: (\xi^+, \zeta^-) \rightarrow [\phi^+, \psi^-] \times_{[\Gamma \otimes \phi^+ \otimes \psi^+, \Omega]} [\psi^+, \phi^-] \end{aligned}$$

such that

$$\phi \boxtimes \psi \circ (h, 1) = \xi \circ (1, g) = \zeta \circ (f, 1). \quad (5.4)$$

Composing with the universal morphism again implements the universal property of $\phi^+ \otimes \psi^+$ and $[\phi^+, \psi^-]$ and $[\psi^+, \phi^-]$ to get

$$\begin{aligned} f' &: (\phi^+, \psi^+, \xi^+) \rightarrow \zeta^+ & h' &: (\phi^+, \xi^+, \zeta^-) \rightarrow \psi^- \\ g' &: (\phi^+, \psi^+, \zeta^-) \rightarrow \xi^- & h'' &: (\psi^+, \xi^+, \zeta^-) \rightarrow \phi^- \end{aligned}$$

as required for a morphism $(\phi, \psi, \xi) \rightarrow (\zeta)$; but now h is only determined by h' and h'' subject to a compatibility condition of agreement in $[\Gamma \otimes \phi^+ \otimes \psi^+, \Omega]$, which means equivalently that $\underline{\phi} \circ h'' = \underline{\psi} \circ h'$. This is ensured by the equality condition for a morphism $(\phi, \psi, \xi) \rightarrow (\zeta)$:

$$\underline{\phi} \circ h'' = \underline{\psi} \circ h' = \xi \circ g' = \zeta \circ f'.$$

For the rest of the equality conditions, composing the morphisms in (5.4) with the universal morphism $u : (\phi^+, \psi^+) \rightarrow (\phi^+ \otimes \psi^+)$, which preserves and reflects equalities since it is a bijection, yields

$$\phi \boxtimes \psi \circ h \circ u = \xi \circ g' = \zeta \circ f'. \quad (5.5)$$

and $\phi \boxtimes \psi \circ h \circ u$ is exactly the common value $\underline{\phi} \circ h'' = \underline{\psi} \circ h'$.

As before, the general case is analogous. The unit is

$$\top^+ = I \qquad \top^- = [\Gamma, \Omega]$$

with $\underline{\top} : (\Gamma, [\Gamma, \Omega], I) \rightarrow \Omega$ induced by the universal property of I from the evaluation map $(\Gamma, [\Gamma, \Omega]) \rightarrow \Omega$. The cotensors and the counit are dual. \blacksquare

Thus, the reason the Dialectica and Chu constructions look different is that while they are both instances of a single abstract construction at the virtual level, they are representable for different reasons.

5.6. REMARK. It is natural to ask what the intersection of the Dialectica and Chu constructions is, i.e. when do both Theorem 4.6 and Theorem 5.2 apply? The reader can check that a Frobenius-discrete polycategory can only have tensors and a unit, or cotensors and a counit, when it has exactly one object. Thus, this happens if and only if $\Omega = 1$ is a terminal object, in which case the underlying ordinary category of $\text{Adj}_{\mathcal{C}}(1)$ is $\mathcal{C} \times \mathcal{C}^{\text{op}}$.

5.7. **REMARK.** For an arbitrary multicategory \mathcal{C} with **Set**-valued presheaf Ω , even if the hypotheses of Theorem 5.2 fail, it still makes sense to refer to the polycategory $\text{Adj}_{\mathcal{C}}(\Omega_{\text{fd}})$ as a **Chu construction** $\text{Chu}(\mathcal{C}, \Omega)$. A similar generalized Chu construction taking multi-bicategories to “cyclic” poly-bicategories appears in [CKS03, Example 1.8(2)], but the symmetric case does not appear to be in the literature. (The symmetric Chu construction is not simply obtained by applying the non-symmetric one to a symmetric input.)

The universal property of the Chu construction described in [Pav93] also generalizes cleanly to the polycategorical version: **Chu** is a right adjoint to the forgetful functor from $*$ -polycategories to co-subunary polycategories (i.e. multicategories equipped with a **Set**-valued presheaf). The special case of this for $\text{Chu}(-, 1)$, namely that it is a right adjoint to the forgetful functor from $*$ -polycategories to multicategories, appears in [DCH18].

6. Dimension (1, 0): the 2-Chu and double Chu constructions

Having recovered the classical Dialectica and Chu constructions, we now categorify the latter.

6.1. **DEFINITION.** Let $\vec{\phi} = (\phi_1, \dots, \phi_n)$ be a list of objects of a category \mathcal{X} . A **clique** on $\vec{\phi}$ is a functor from the chaotic category on n objects to \mathcal{X} that picks out the objects of $\vec{\phi}$, i.e. a family of isomorphisms $\theta_{ij} : \phi_i \xrightarrow{\sim} \phi_j$ such that $\theta_{ii} = 1$ and $\theta_{jk}\theta_{ij} = \theta_{ik}$.

Note there is a unique clique on the empty list (this is the cop-out from Remark 1.5).

6.2. **LEMMA.** *Given cliques on (ϕ_1, \dots, ϕ_m) and (ξ_1, \dots, ξ_n) with $\phi_{j_0} = \xi_{k_0}$, there is an induced clique on $(\vec{\phi}_{\neq j_0}, \vec{\xi}_{\neq k_0})$, and this operation is associative.*

PROOF. We take the isomorphisms among the ϕ 's and the ξ 's to be the given ones, and the isomorphism $\phi_j \xrightarrow{\sim} \xi_k$ to be the composite $\phi_j \xrightarrow{\sim} \phi_{j_0} = \xi_{k_0} \xrightarrow{\sim} \xi_k$. ■

6.3. **DEFINITION.** For any category \mathcal{X} , the **Frobenius pseudo-discrete** polycategory \mathcal{X}_{fpd} has the same objects as \mathcal{X} , with $\mathcal{X}_{\text{fpd}}(\Gamma; \Delta)$ the set of cliques on (Γ, Δ) .

This defines a functor $(-)\text{fpd}$ from the 1-category of categories to the 1-category of polycategories. Thus, a presheaf of categories Ω on a multicategory \mathcal{C} gives rise to a presheaf of polycategories Ω_{fpd} . We write

$$\text{Chu}_0(\mathcal{C}, \Omega) = \text{Adj}_{\mathcal{C}}(\Omega_{\text{fpd}}).$$

This polycategory will be the underlying 1-dimensional structure of our 2-Chu construction: its objects are triples $(A^+, A^-, \underline{A})$ with $\underline{A} : (A^+, A^-) \rightarrow \Omega$, and a polyarrow $(A_1, \dots, A_m) \rightarrow (B_1, \dots, B_n)$ in $\text{Chu}_0(\mathcal{C}, \Omega)$ consists of morphisms in \mathcal{C} :

$$\begin{aligned} f_j^+ &: (A_1^+, \dots, A_m^+, B_1^-, \dots, \widehat{B_j^-}, \dots, B_n^-) \longrightarrow B_j^+ \\ f_i^- &: (A_1^+, \dots, \widehat{A_i^+}, \dots, A_m^+, B_1^-, \dots, B_n^-) \longrightarrow A_i^- \end{aligned}$$

together with a clique on

$$((\underline{A}_1 \circ_{A_1^-} f_1^-), \dots, (\underline{A}_m \circ_{A_m^-} f_m^-), (\underline{B}_1 \circ_{B_1^+} f_1^+), \dots, (\underline{B}_n \circ_{B_n^+} f_n^+)).$$

That is, $\text{Chu}_0(\mathcal{C}, \Omega)$ is the polycategory of Ω -polarized objects and polarized multivariable adjunctions described in section 1.4 (of the strict sort having exactly one $(0,0)$ -ary morphism, as in Remark 1.5).

In particular, if $\Omega = \text{Set} \in \text{Cat}$ and each A, B is of the form $[A] = (\mathcal{A}, \mathcal{A}^{\text{op}}, \text{hom}_{\mathcal{A}})$ (recall Remark 1.6), then we can write the functors involved in a morphism $(A_1, \dots, A_m) \rightarrow (B_1, \dots, B_n)$

$$\begin{aligned} f_j^+ &: (\mathcal{A}_1, \dots, \mathcal{A}_m, \mathcal{B}_1^{\text{op}}, \dots, \widehat{\mathcal{B}_j^{\text{op}}}, \dots, \mathcal{B}_n^{\text{op}}) \longrightarrow \mathcal{B}_j \\ (f_i^-)^{\text{op}} &: (\mathcal{A}_1^{\text{op}}, \dots, \widehat{\mathcal{A}_i^{\text{op}}}, \dots, \mathcal{A}_m^{\text{op}}, \mathcal{B}_1, \dots, \mathcal{B}_n) \longrightarrow \mathcal{A}_i. \end{aligned}$$

and the clique becomes the family of adjunction isomorphisms

$$\mathcal{A}_1(f_1^-(a_2, \dots, a_m, b_1, \dots, b_n), a_1) \cong \dots \cong \mathcal{B}_n(b_n, f_n^+(a_1, \dots, a_m, b_1, \dots, b_{n-1})).$$

Thus the sub-polycategory of $\text{Chu}_0(\text{Cat}, \text{Set})$ determined by objects of this form is the polycategory of multivariable adjunctions.¹²

Now we want to incorporate the 2-categorical structure of \mathcal{C} into the output as well, obtaining a 2-polycategory and even a poly double category. By a **2-polycategory** we mean a polycategory (strictly) enriched over Cat , so that the hom-objects $\mathcal{C}(\Gamma; \Delta)$ are categories and all operations are functorial. Similarly, a **2-multicategory** can be defined as a co-unary 2-polycategory, while a **2-presheaf** on a 2-multicategory is a Cat -valued presheaf whose actions are 2-functorial. For instance, any object Ω of a 2-multicategory represents a 2-presheaf. As in Remark 2.12(ii), a 2-multicategory equipped with a 2-presheaf can equivalently be regarded as a co-subunary 2-polycategory.

Of course, if \mathcal{C} is an ordinary multicategory regarded as a locally discrete 2-multicategory, a 2-presheaf on \mathcal{C} is just a Cat -valued presheaf. In particular, Chu_0 can be regarded as a functor defined on the category of subunary 2-polycategories with only identity 2-cells between co-unary arrows.

Next, recall that any category \mathcal{A} is the object-of-objects of a canonical internal category in Cat whose object-of-morphisms is \mathcal{A}^2 , the category of arrows in \mathcal{A} . Put differently, this is a double category $\mathbb{Q}(\mathcal{A})$ whose vertical and horizontal arrows are both those of \mathcal{A} , and whose 2-cells are commutative squares. Similarly, any 2-category \mathcal{C} can be enhanced to an internal category $\mathbb{Q}(\mathcal{C})$ in 2-Cat (a ‘‘cylindrical’’ 3-dimensional structure) whose object-of-morphisms is $\text{Lax}(2, \mathcal{C})$ (the 2-category whose objects are arrows of \mathcal{C} , whose morphisms are squares in \mathcal{C} inhabited by a 2-cell, and whose 2-cells are commuting cylinders in \mathcal{C}). The underlying double category of this structure consists of squares or ‘‘quintets’’ in \mathcal{C} .

¹²We can also exclude Set from Cat for size reasons, allowing the latter to consist of only *small* categories, and still have Ω be a non-representable presheaf of categories.

The same idea works for polycategories: any 2-polycategory \mathcal{P} can be enhanced to an internal category $\mathbb{Q}(\mathcal{P})$ in the category 2-Poly of 2-polycategories. This gives a 3-dimensional structure containing:

- Objects: those of \mathcal{P} .
- Horizontal poly-arrows: those of \mathcal{P} .
- Horizontal 2-cells between parallel poly-arrows: those of \mathcal{P} .
- Vertical arrows: the unary co-unary arrows of \mathcal{P} .
- 2-cells of the following shape:

$$\begin{array}{ccc} (A_1, \dots, A_m) & \xrightarrow{f} & (B_1, \dots, B_n) \\ h_1 \downarrow & \cdots & \downarrow h_m \quad \Downarrow \quad k_1 \downarrow & \cdots & \downarrow k_n \\ (C_1, \dots, C_m) & \xrightarrow{g} & (D_1, \dots, D_n) \end{array}$$

coming from 2-cells $k_1 \circ_{B_1} \cdots \circ_{B_n} f \Longrightarrow g \circ_{C_1} h_1 \circ \cdots \circ_{C_m} h_m$ in \mathcal{P} .

- “Poly-cylinders”: commutativity relations in \mathcal{P} .

$$\begin{array}{ccc} (A_1, \dots, A_m) & \xrightarrow{\Downarrow} & (B_1, \dots, B_n) \\ \downarrow \cdots \downarrow & \Downarrow & \downarrow \cdots \downarrow \\ (C_1, \dots, C_m) & \longrightarrow & (D_1, \dots, D_n) \end{array} = \begin{array}{ccc} (A_1, \dots, A_m) & \longrightarrow & (B_1, \dots, B_n) \\ \downarrow \cdots \downarrow & \Downarrow & \downarrow \cdots \downarrow \\ (C_1, \dots, C_m) & \xrightarrow{\Downarrow} & (D_1, \dots, D_n) \end{array}$$

In particular, when \mathcal{P} is co-subunary, the 2-cells of $\mathbb{Q}(\mathcal{P})$ are all horizontally co-unary or co-nullary:

$$\begin{array}{ccc} (A_1, \dots, A_m) & \longrightarrow & B \\ \downarrow \cdots \downarrow & \Downarrow & \downarrow \\ (C_1, \dots, C_m) & \longrightarrow & D \end{array} \quad \text{or} \quad \begin{array}{ccc} (A_1, \dots, A_m) & \longrightarrow & () \\ \downarrow \cdots \downarrow & \Downarrow & \parallel \\ (C_1, \dots, C_m) & \longrightarrow & () \end{array}$$

Now let \mathcal{C} be a 2-multicategory equipped with a 2-presheaf Ω , as before. Regarding (\mathcal{C}, Ω) as a co-subunary 2-polycategory, we form the internal category $\mathbb{Q}(\mathcal{C}, \Omega)$ in 2-Poly , which is also co-subunary. Now we forget the nonidentity horizontal 2-cells between co-unary arrows, obtaining an internal category $\mathbb{Q}'(\mathcal{C}, \Omega)$ in the category of ordinary multicategories equipped with $\mathcal{C}at$ -valued presheaves. Finally, this latter category is the domain of the above functor Chu_0 , which preserves pullbacks and hence internal categories. Thus, we can define:

6.4. DEFINITION. The **double Chu construction** of (\mathcal{C}, Ω) is

$$\mathbf{Chu}(\mathcal{C}, \Omega) = \mathbf{Chu}_0(\mathbb{Q}'(\mathcal{C}, \Omega)).$$

It is an internal category in polycategories, which we call a **poly double category**.

Tracing through the constructions, we see that $\mathbf{Chu}(\mathcal{C}, \Omega)$ can be described more explicitly as follows.

- Its objects are triples $(A^+, A^-, \underline{A})$, with $\underline{A} : (A^+, A^-) \rightarrow \Omega$.
- Its horizontal poly-arrows are families of morphisms

$$\begin{aligned} f_j^+ &: (A_1^+, \dots, A_m^+, B_1^-, \dots, \widehat{B_j^-}, \dots, B_n^-) \longrightarrow B_j^+ \\ f_i^- &: (A_1^+, \dots, \widehat{A_i^+}, \dots, A_m^+, B_1^-, \dots, B_n^-) \longrightarrow A_i^- \end{aligned}$$

equipped with a clique (the ‘‘adjunction isomorphisms’’) on

$$((\underline{A}_1 \circ_{A_1^-} f_1^-), \dots, (\underline{A}_m \circ_{A_m^-} f_m^-), (\underline{B}_1 \circ_{B_1^+} f_1^+), \dots, (\underline{B}_n \circ_{B_n^+} f_n^+)).$$

- A vertical arrow $u : A \rightarrow B$ is a triple $(u^+, u^-, \underline{u})$, where $u^+ : A^+ \rightarrow B^+$ and $u^- : A^- \rightarrow B^-$ are morphisms in \mathcal{C} (note that both go in the forwards direction) and

$$\underline{u} : \underline{A} \Longrightarrow \underline{B} \circ (u^+, u^-)$$

is a morphism in the hom-category $\mathcal{C}(A^+, A^-;)$, i.e. a 2-cell in \mathcal{C} . This comes from a co-nullary 2-cell in $\mathbb{Q}'(\mathcal{C}, \Omega)$.

- A 2-cell

$$\begin{array}{ccc} (A_1, \dots, A_m) & \xrightarrow{f} & (B_1, \dots, B_n) \\ u_1 \downarrow & \dots & \downarrow u_m \quad \Downarrow \mu \quad v_1 \downarrow \quad \dots & \downarrow v_n \\ (C_1, \dots, C_m) & \xrightarrow{g} & (D_1, \dots, D_n) \end{array}$$

consists of a family of 2-cells in \mathcal{K} :

$$\begin{array}{ccc} (A_1^+, \dots, A_m^+, B_1^-, \dots, \widehat{B_j^-}, \dots, B_n^-) & \xrightarrow{f_j^+} & B_j^+ \\ \downarrow \dots (u_1^+, \dots, u_m^+, v_1^-, \dots, \widehat{v_j^-}, \dots, v_n^-) \dots & & \downarrow \Downarrow \mu_j^+ \quad \downarrow v_j^+ \\ (C_1^+, \dots, C_m^+, D_1^-, \dots, \widehat{D_j^-}, \dots, D_n^-) & \xrightarrow{g_j^+} & D_j^+ \end{array}$$

and

$$\begin{array}{ccc} (A_1^+, \dots, \widehat{A_i^+}, \dots, A_m^+, B_1^-, \dots, B_n^-) & \xrightarrow{f_i^-} & A_i^- \\ \downarrow \dots (u_1^+, \dots, \widehat{u_i^+}, \dots, u_m^+, v_1^-, \dots, v_n^-) \dots & & \downarrow \Downarrow \mu_i^- \quad \downarrow u_i^- \\ (C_1^+, \dots, \widehat{C_i^+}, \dots, C_m^+, D_1^-, \dots, D_n^-) & \xrightarrow{g_i^-} & C_i^- \end{array}$$

$$\begin{array}{ccc}
(A_1^+, \dots, A_m^+, B_1^-, \dots, B_n^-) & \xrightarrow{\underline{A}_1 \circ (1, f_1^-)} \Omega & (A_1^+, \dots, A_m^+, B_1^-, \dots, B_n^-) & \xrightarrow{\underline{A}_1 \circ (1, f_1^-)} \Omega \\
\downarrow \dots(u_1^+, \dots, u_m^+, v_1^-, \dots, v_n^-) \dots & \Downarrow \underline{B}_1 \circ (1, f_1^+) & \downarrow \dots(u_1^+, \dots, u_m^+, v_1^-, \dots, v_n^-) \dots & \Downarrow \underline{u}_1 \circ (1, \mu_1^-) \\
(C_1^+, \dots, C_m^+, D_1^-, \dots, D_n^-) & \xrightarrow{\underline{D}_1 \circ (1, g_1^+)} \Omega & (C_1^+, \dots, C_m^+, D_1^-, \dots, D_n^-) & \xrightarrow{\underline{C}_1 \circ (1, g_1^-)} \Omega \\
& & & \Downarrow \underline{D}_1 \circ (1, g_1^+)
\end{array} =$$

Figure 1: A 2-cell compatibility condition for $\mathbf{Chu}(\mathcal{K})$

such that *any two* of these 2-cells satisfy a commutativity condition relating them to the adjunction isomorphisms of f, g and the structure 2-cells $\underline{u}, \underline{v}$. For instance, the condition for μ_1^+ and μ_1^- is shown in Definition 1. The 2-cells μ_j^+, μ_i^- come from co-unary 2-cells in $\mathbb{Q}'(\mathcal{C}, \Omega)$, while the $\binom{n+m}{2}$ commutativity conditions are a “clique of commutative cylinders” therein.

We can now quite easily define:

6.5. DEFINITION. The **2-Chu construction** of (\mathcal{C}, Ω) is the 2-polycategory $\mathbf{Chu}(\mathcal{C}, \Omega)$ obtained by discarding all the non-identity vertical arrows in $\mathbf{Chu}(\mathcal{C}, \Omega)$.

The 1-categorical Chu construction is usually described as a $*$ -autonomous category, under suitable conditions on the input category (closed monoidal with pullbacks). Our 2-categorical version has no such conditions on the input, so it produces only a 2-polycategory. In the presence of suitable structure we expect it to be a “ $*$ -autonomous 2-category”, but in order to prove this we need to define the latter term. Defining it as a particular kind of monoidal 2-category would result in numerous tedious coherence axioms, so instead we take the expected polycategorical characterization as a definition.

6.6. DEFINITION. We say that a 2-polycategory \mathcal{P} has **bicategorical** tensor products, units, cotensor products, and counits if they induce *equivalences* of hom-categories:

$$\begin{aligned}
\mathcal{P}(\Gamma, A \boxtimes B; \Delta) &\simeq \mathcal{P}(\Gamma, A, B; \Delta) \\
\mathcal{P}(\Gamma, \top; \Delta) &\simeq \mathcal{P}(\Gamma; \Delta) \\
\mathcal{P}(\Gamma; A \diamond B, \Delta) &\simeq \mathcal{P}(\Gamma; A, B, \Delta) \\
\mathcal{P}(\Gamma; \perp, \Delta) &\simeq \mathcal{P}(\Gamma; \Delta).
\end{aligned}$$

If a \top or \perp only satisfies this property when $|\Gamma| + |\Delta| > 0$, we call it a **positive** bicategorical unit or counit. We say that \mathcal{P} has **bicategorical duals** if for any A there are morphisms $\eta : () \rightarrow (A, A^\bullet)$ and $\varepsilon : (A^\bullet, A) \rightarrow ()$ with isomorphisms $\varepsilon \circ_A \eta \cong 1_{A^\bullet}$ and $\varepsilon \circ_{A^\bullet} \eta \cong 1_A$.¹³

¹³For a coherent notion of duality, these isomorphisms should also satisfy axioms; but we will not worry about that, since in a 2-Chu construction these isomorphisms are in fact equalities.

The positivity condition on units and counits is because our definition of the $(0, 0)$ -ary morphisms is “wrong”, as noted in Remark 1.5.

6.7. THEOREM. *If \mathcal{C} is a 2-multicategory with bicategorical tensor products, unit, and homs, and also has bipullbacks¹⁴, and Ω is an object of \mathcal{C} , then $\mathcal{C}hu(\mathcal{C}, \Omega)$ has bicategorical tensor products, cotensor products, positive unit and counit, and duals.*

PROOF. As in Theorem 5.2, the tensor product of $(A^+, A^-, \underline{A})$ and $(B^+, B^-, \underline{B})$ is

$$\begin{aligned} (A \boxtimes B)^+ &= (A^+ \otimes B^+) \\ (A \boxtimes B)^- &= [A^+, B^-] \times_{[A^+ \otimes B^+, \Omega]}^b [B^+, A^-] \end{aligned}$$

where \times^b denotes the bipullback. To define $\underline{A \boxtimes B}$, we note that now the following two morphisms are *isomorphic*

$$\begin{aligned} ([A^+, B^-] \times_{[A^+ \otimes B^+, \Omega]}^b [B^+, A^-], A^+, B^+) &\rightarrow ([A^+, B^-], A^+, B^+) \rightarrow (B^-, B^+) \xrightarrow{\underline{B}} \Omega \\ ([A^+, B^-] \times_{[A^+ \otimes B^+, \Omega]}^b [B^+, A^-], A^+, B^+) &\rightarrow ([B^+, A^-], A^+, B^+) \rightarrow (A^-, A^+) \xrightarrow{\underline{A}} \Omega \end{aligned} \quad (6.8)$$

and determine $\underline{A \boxtimes B} : ([A^+, B^-] \times [B^+, A^-], A^+ \otimes B^+) \rightarrow \Omega$, up to isomorphism, by the universal property of $A^+ \otimes B^+$. Its universal morphism $(A, B) \rightarrow (A \otimes B)$

$$\begin{aligned} (A^+, B^+) &\rightarrow (A^+ \otimes B^+) \\ ([A^+, B^-] \times_{[A^+ \otimes B^+, \Omega]} [B^+, A^-], A^+) &\rightarrow ([A^+, B^-], A^+) \rightarrow B^- \\ ([A^+, B^-] \times_{[A^+ \otimes B^+, \Omega]} [B^+, A^-], B^+) &\rightarrow ([B^+, A^-], B^+) \rightarrow A^- \end{aligned}$$

plus the isomorphism between the two maps in (6.8) and the defining isomorphism of $\underline{A \boxtimes B}$. For the universal property, a morphism $(A \boxtimes B, C) \rightarrow D$ in $\mathcal{C}hu(\mathcal{C}, \Omega)$ now consists of morphisms in \mathcal{K} :

$$\begin{aligned} f &: (A^+ \otimes B^+, C^+) \rightarrow D^+ \\ g &: (A^+ \otimes B^+, D^-) \rightarrow C^- \\ h &: (C^+, D^-) \rightarrow [A^+, B^-] \times_{[A^+ \otimes B^+, \Omega]}^b [B^+, A^-] \end{aligned}$$

together with a clique on

$$\underline{A \boxtimes B} \circ (1, h) \quad \underline{C} \circ (1, g) \quad \underline{D} \circ (f, 1). \quad (6.9)$$

Composing with the universal morphism implements the universal properties of $A^+ \otimes B^+$ and $[A^+, B^-]$ and $[B^+, A^-]$, yielding an equivalence to the category of quadruples

$$\begin{aligned} f' &: (A^+, B^+, C^+) \rightarrow D^+ & h' &: (A^+, C^+, D^-) \rightarrow B^- \\ g' &: (A^+, B^+, D^-) \rightarrow C^- & h'' &: (B^+, C^+, D^-) \rightarrow A^- \end{aligned}$$

¹⁴I.e. bicategorical pullbacks, whose universal property is an equivalence of hom-categories.

together with a clique corresponding to (6.8), plus an additional *isomorphism* between $\underline{A} \circ (1, h'')$ and $\underline{B} \circ (1, h')$ coming from the bipullback. This yields the desired clique on

$$(\underline{A} \circ (1, h''), \underline{B} \circ (1, h'), \underline{C} \circ (1, g'), \underline{D} \circ (f', 1)),$$

and hence the desired morphism $(A, B, C) \rightarrow D$. The general case is analogous, as is the cotensor product.

As before, we define the unit by $\top^+ = I$ and $\top^- = \Omega$, with $\top : (I, \Omega) \rightarrow \Omega$ induced by the universal property of I . Its universal property is straightforward to check; the case of $(0, 0)$ -ary morphisms fails because morphisms $\top \rightarrow ()$ in $\mathcal{C}hu(\mathcal{C}, \Omega)$ are equivalent to morphisms $I \rightarrow \Omega$ in \mathcal{C} , whereas there is only one morphism $() \rightarrow ()$ in $\mathcal{C}hu(\mathcal{C}, \Omega)$. ■

6.10. **REMARK.** If (as in $\mathcal{C}at$) the tensor products, units, and homs in \mathcal{K} satisfy a strict universal property, and the bipullbacks are strict *iso-comma objects* (not strict pullbacks!), then the tensor and cotensor products in $\mathcal{C}hu(\mathcal{C}, \Omega)$ are again strict. But the unit and counit of $\mathcal{C}hu(\mathcal{C}, \Omega)$ are not strict even in this case.

6.11. **REMARK.** When we construct a monoidal 2-category from a 2-polycategory, the positivity condition should be irrelevant. That is, once given a definition of “*-autonomous 2-category” as a monoidal 2-category with extra structure, any 2-polycategory with bicategorical tensors, cotensors, and duals and *positive* bicategorical unit and counit should still have an underlying *-autonomous 2-category. Moreover, this should give the correct “monoidal” version of $\mathcal{C}hu(\mathcal{C}, \Omega)$, despite our incorrect definition of the $(0, 0)$ -ary morphisms in the polycategorical version.

Our primary interest is in the case $\mathcal{K} = \mathcal{C}at$ and $\Omega = \mathbf{Set}$. In section 7 we will show that $\mathbf{Chu}(\mathcal{C}at, \mathbf{Set})$ contains the cyclic multi double category \mathbf{MAdj} of multivariable adjunctions, by restricting to the “representable”¹⁵ objects $[\mathcal{A}] = (\mathcal{A}, \mathcal{A}^{\text{op}}, \text{hom}_{\mathcal{A}})$. Here we instead mention a few applications of the full structure $\mathbf{Chu}(\mathcal{C}at, \mathbf{Set})$.

6.12. **EXAMPLE.** Any (poly) double category has an underlying *vertical 2-category* consisting of the objects, vertical arrows, and 2-cells whose vertical source and target are identity horizontal arrows. The vertical 2-category of $\mathbf{Chu}(\mathcal{C}at, \mathbf{Set})$ is isomorphic to the 2-category \mathbf{PolCat} of **polarized categories** from [CS07]. (Since an object of \mathbf{PolCat} is by definition two categories with a profunctor between them, i.e. a functor $\mathbf{X}_o^{\text{op}} \times \mathbf{X}_p \rightarrow \mathbf{Set}$, this isomorphism has to dualize one of the categories.) The term “polarized” comes from a logical perspective, with \mathcal{A}^+ and \mathcal{A}^- as the “positive” and “negative” types that can occur on the left or right sides of a sequent¹⁶, and the elements of $\underline{\mathcal{A}}(A, B)$ as the set of sequents $A \vdash B$.

¹⁵Another name might be “discrete”, since these are analogous to sets regarded as “discrete Chu spaces” in $\mathbf{Chu}(\mathbf{Set}, 2)$.

¹⁶As noted in Remark 1.6, in most of the paper we consider (multivariable) adjunctions to point in the direction of their *right* adjoints. But in Example 6.12, Example 6.13, and Example 6.14 it is more natural to orient them in the other direction.

6.13. **EXAMPLE.** The horizontal morphisms of $\mathbf{Chu}(\mathcal{C}at, \mathbf{Set})$ are not the same as the “inner/outer adjoints” of [CS07], but they are a different sensible notion of “(multivariable) adjunction” for polarized categories. For instance, just as a horizontal pseudomonoid in \mathbf{MAdj} is a closed monoidal category, a horizontal pseudomonoid in $\mathbf{Chu}(\mathcal{C}at, \mathbf{Set})$ is a natural notion of “closed monoidal polarized category”: it has a tensor product $\otimes : \mathcal{A}^+ \times \mathcal{A}^+ \rightarrow \mathcal{A}^+$ and internal-homs $- \circ : \mathcal{A}^+ \times \mathcal{A}^- \rightarrow \mathcal{A}^-$ and $\circ - : \mathcal{A}^- \times \mathcal{A}^+ \rightarrow \mathcal{A}^-$ with natural bijections between sequents

$$\frac{A_1 \otimes A_2 \vdash B}{\frac{A_1 \vdash A_2 \circ B}{A_2 \vdash B \circ A_1}}$$

This allows us to take any of the above sets as a *definition* of a set of sequents $A_1, A_2 \vdash B$. We also have coherent associativity isomorphisms of all sorts — not just $(A_1 \otimes A_2) \otimes A_3 \cong A_1 \otimes (A_2 \otimes A_3)$ but also $A_1 \circ (A_2 \circ B) \cong (A_1 \otimes A_2) \circ B$ etc. (in the polarized case none of these is determined by the others) — giving a consistent definition of a set of sequents $A_1, A_2, A_3 \vdash B$, and so on for higher arity as well. (The fact that closed monoidal categories are particular pseudomonoids in $\mathbf{Chu}(\mathcal{C}at, \mathbf{Set})$ was observed by [Gar09].) Similarly, just as it can be shown that a Frobenius pseudomonoid in \mathbf{MAdj} is a $*$ -autonomous category [DS04, Str04, Egg10, Shu19], a Frobenius pseudomonoid in $\mathbf{Chu}(\mathcal{C}at, \mathbf{Set})$ is a “ $*$ -autonomous polarized category”, with an additional “co-closed monoidal structure” \wp allowing a consistent definition of $A_1, \dots, A_m \vdash B_1, \dots, B_n$ in terms of $A_1 \otimes \dots \otimes A_m \vdash B_1 \wp \dots \wp B_n$.

6.14. **EXAMPLE.** Intuitively, a polarized category should have “binary products” if its diagonal functor $A \rightarrow A \times A$ has a “right adjoint”. However, as noted in [CS07], right adjoints in the vertical 2-category \mathbf{PolCat} are not the correct notion. The inner/outer adjoints of [CS07] give one possible solution, but the double category $\mathbf{Chu}(\mathcal{C}at, \mathbf{Set})$ gives another. The diagonal $A \rightarrow A \times A$ only exists as a *vertical* arrow in this double category, but [GP04] have defined a notion of “adjunction” between a vertical arrow and a horizontal arrow in a double category, called a **conjunction**.

In our case, for $A, B \in \mathbf{Chu}(\mathcal{C}at, \mathbf{Set})$, a “right conjoint” of a vertical arrow $u : A \rightarrow B$ with components $u^+ : A^+ \rightarrow B^+$ and $u^- : A^- \rightarrow B^-$ consists essentially of an ordinary right adjoint f^- to u^- together with a compatible bijection between sequents $u^+(\Gamma) \vdash \Delta$ and $\Gamma \vdash f^-(\Delta)$. In the case when $u : A \rightarrow A \times A$ is the diagonal, this means that A^- has binary products in the ordinary sense, and we also have a compatible natural bijection between sequents $\Gamma \vdash \Delta_1 \times \Delta_2$ and pairs of sequents $\Gamma \vdash \Delta_1$ and $\Gamma \vdash \Delta_2$.

6.15. **EXAMPLE.** Let $k : \mathcal{A} \rightarrow \mathcal{B}$ be a functor, and write $[\mathcal{B}_k]$ for the object $(\mathcal{B}, \mathcal{A}^{\text{op}}, \underline{\mathcal{B}}_k) \in \mathbf{Chu}(\mathcal{C}at, \mathbf{Set})$ where $\underline{\mathcal{B}}_k(a, b) = \mathcal{B}(k(a), b)$. Then a horizontal morphism $[\mathcal{C}] \rightarrow [\mathcal{B}_k]$, with $[\mathcal{C}] = (\mathcal{C}, \mathcal{C}^{\text{op}}, \text{hom}_{\mathcal{C}})$ representable, is known as a **relative adjunction**: a pair of functors $f : \mathcal{A} \rightarrow \mathcal{C}$ and $g : \mathcal{C} \rightarrow \mathcal{B}$ with a natural isomorphism $\mathcal{C}(f(a), b) \cong \mathcal{B}(k(a), g(b))$.

6.16. **EXAMPLE.** For any category \mathcal{A} , we have a “maximal” object $[\mathcal{A}] = (\mathcal{A}, \mathbf{Set}^{\mathcal{A}}, \text{ev})$ of $\mathbf{Chu}(\mathcal{C}at, \mathbf{Set})$. A horizontal morphism $[\mathcal{A}] \rightarrow [\mathcal{B}]$ is just a functor $\mathcal{A} \rightarrow \mathcal{B}$, and similarly a two-variable morphism $([\mathcal{A}], [\mathcal{B}]) \rightarrow [\mathcal{C}]$ is just a two-variable functor $\mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$.

If \mathcal{A} has finite limits, we also have $[\mathcal{A}]_{\text{lex}} = (\mathcal{A}, \text{Lex}(\mathcal{A}, \text{Set}), \text{ev})$, where $\text{Lex}(\mathcal{A}, \text{Set})$ denotes the category of finite-limit-preserving functors. Then a horizontal morphism $[\mathcal{A}]_{\text{lex}} \rightarrow [\mathcal{B}]_{\text{lex}}$ is equivalent to a finite-limit-preserving functor $\mathcal{A} \rightarrow \mathcal{B}$, but also to a finitary right adjoint $\text{Lex}(\mathcal{B}, \text{Set}) \rightarrow \text{Lex}(\mathcal{A}, \text{Set})$. This is essentially Gabriel–Ulmer duality [GU71] for locally finitely presentable categories, and generalizes to many other doctrines of limits (the maximal case $[\mathcal{A}]$ corresponds to the empty doctrine of no limits). A two-variable morphism $([\mathcal{A}]_{\text{lex}}, [\mathcal{B}]_{\text{lex}}) \rightarrow [\mathcal{C}]_{\text{lex}}$ is a two-variable functor $\mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$ that preserves finite limits in each variable separately.

Thus, just as the 1-Chu construction gives abstract homes for 1-categorical concrete dualities like Stone duality and Pontryagin duality, the 2-Chu construction gives abstract homes for 2-categorical concrete dualities like Gabriel–Ulmer duality [PBB06].

6.17. EXAMPLE. In [Ave17], objects of the 1-Chu construction $\text{Chu}(\text{Cat}, \mathcal{C})$, for an arbitrary category \mathcal{C} , are called **aritations**. In §4.7 thereof a structure-semantics adjunction is phrased in terms of a universal morphism in $\text{Chu}(\text{Cat}, \mathcal{C})$, and in Chapter 5 our $[\mathcal{B}] \in \text{Chu}(\text{Cat}, \text{Set})$ is called the **canonical aritation**. The possibility of the weaker notion of morphism in the 2-Chu construction $\text{Chu}(\text{Cat}, \mathcal{C})$ (reducing to adjunctions between canonical aritations with $\mathcal{C} = \text{Set}$) is considered in §11.1 of [Ave17].

6.18. REMARK. We can also “iterate” the Chu construction in various ways. For instance, from the 2-polycategory \mathcal{MAdj} we can construct $\text{Chu}(2\text{-Cat}, \mathcal{MAdj})$. Since the objects of \mathcal{MAdj} are categories, every 2-category with its hom-functor yields a representable object of $\text{Chu}(2\text{-Cat}, \mathcal{MAdj})$. A horizontal morphism between such objects consists of functors $f^+ : \mathcal{A} \rightarrow \mathcal{B}$ and $f^- : \mathcal{B} \rightarrow \mathcal{A}$ with an *adjunction* $\mathcal{A}(f^-(b), a) \rightleftarrows \mathcal{B}(b, f^+(a))$. Such **local adjunctions** were studied by [BP88] in the more general context of bicategories and (op)lax functors.

6.19. REMARK. The method of categorifying a construction by applying a pullback-preserving functor to internal categories in its domain can also be applied to the general construction $\text{Adj}_{\mathcal{C}}(\Omega)$ from section 3. I do not know whether there are interesting examples of “double Dialectica constructions”.

7. Cyclic multicategories and parametrized mates

Finally, as promised in section 1.4, we can define the poly double category of multivariable adjunctions as a subcategory of $\text{Chu}(\text{Cat}, \text{Set})$.

7.1. DEFINITION. The poly double category MAdj is the sub-double-polycategory of $\text{Chu}(\text{Cat}, \text{Set})$ determined by:

- The objects of the form $[\mathcal{A}] = (\mathcal{A}, \mathcal{A}^{\text{op}}, \text{hom}_{\mathcal{A}})$ for a category \mathcal{A} .
- The vertical arrows of the form $[f] = (f, f^{\text{op}}, \text{hom}_f)$ for a functor $f : \mathcal{A} \rightarrow \mathcal{B}$.
- All the horizontal arrows and 2-cells relating these.

We want to compare this with the cyclic multi double category of multivariable adjunctions from [CGR14]. This requires making precise the relationship between polycategories and cyclic multicategories; as suggested in section 1, we will show that cyclic symmetric multicategories are almost equivalent to polycategories with strict duals. In fact, there are multiple ways of defining each of these notions, which we compare with the following omnibus definition.

7.2. DEFINITION. Let $X \subseteq \mathbb{N} \times \mathbb{N}$. A **(symmetric) X -ary $*$ -polycategory** \mathcal{P} consists of the following:

- A set of objects equipped with a strict involution $(-)^{\bullet}$, so that $(A^{\bullet})^{\bullet} = A$ strictly. If Γ is a list of objects, we write Γ^{\bullet} for applying $(-)^{\bullet}$ to each object in Γ .
- For each pair (Γ, Δ) of finite lists of objects such that $(|\Gamma|, |\Delta|) \in X$, a set $\mathcal{P}(\Gamma; \Delta)$ of polyarrows.
- For any $\Gamma, \Lambda, \Delta, \Sigma$ and any isomorphism of lists (i.e. object-preserving permutation) $\sigma : \Gamma, \Delta^{\bullet} \xrightarrow{\sim} \Lambda, \Sigma^{\bullet}$, an action $(-)^{\sigma} : \mathcal{P}(\Gamma; \Delta) \xrightarrow{\sim} \mathcal{P}(\Lambda; \Sigma)$, whenever both hom-sets exist, which is functorial on composition of permutations.
- Each object A has identities $1_A^m \in \mathcal{P}(A; A)$, $1_A^l \in \mathcal{P}(A^{\bullet}, A;)$, and/or $1_A^r \in \mathcal{P}(; A, A^{\bullet})$, each existing whenever the relevant hom-set does. Moreover, any two of these that exist simultaneously are each other's image under the relevant permutation.
- For finite lists of objects $\Gamma, \Delta, \Lambda, \Sigma$, and object A , composition maps

$$\begin{aligned} \circ_A^m &: \mathcal{P}(\Lambda_1, A, \Lambda_2; \Sigma) \times \mathcal{P}(\Gamma; \Delta_1, A, \Delta_2) \rightarrow \mathcal{P}(\Lambda_1, \Gamma, \Lambda_2; \Delta_1, \Sigma, \Delta_2) \\ \circ_A^l &: \mathcal{P}(\Lambda_1, A, \Lambda_2; \Sigma) \times \mathcal{P}(\Gamma_1, A^{\bullet}, \Gamma_2; \Delta) \rightarrow \mathcal{P}(\Lambda_1, \Lambda_2, \Gamma_1, \Gamma_2; \Delta, \Sigma) \\ \circ_A^r &: \mathcal{P}(\Lambda; \Sigma_1, A^{\bullet}, \Sigma_2) \times \mathcal{P}(\Gamma; \Delta_1, A, \Delta_2) \rightarrow \mathcal{P}(\Lambda, \Gamma; \Delta_1, \Delta_2, \Sigma_1, \Sigma_2) \end{aligned}$$

in each case presuming that all three hom-sets exist. Moreover, any two of these that exist simultaneously are each other's image under the relevant permutations, as are the corresponding composites along A^{\bullet} ; in other words any two of the composites

$$g \circ_A^m f \quad g^{\rho} \circ_A^r f \quad g \circ_A^l f^{\sigma} \quad (f^{\sigma} \circ_A^m \cdot g^{\rho})^{\tau} \quad (f \circ_A^r \cdot g^{\rho})^{\tau} \quad (f^{\sigma} \circ_A^l \cdot g)^{\tau}$$

that exist are equal.

- Axioms of associativity and equivariance for all choices of $i, j \in \{m, l, r\}$ and whenever both sides make sense and the permutations make everything well-typed:

$$\begin{aligned} 1_A^i \circ_A^i f &= f \\ f \circ_A^i 1_A^i &= f \\ (h \circ_B^i g) \circ_A^j f &= h \circ_B^i (g \circ_A^j f) \\ (h \circ_B^i g) \circ_A^j f &= ((h \circ_A^j f) \circ_B^i g)^{\sigma} \\ h \circ_B^i (g \circ_A^j f) &= (g \circ_A^i (h \circ_B^i f))^{\sigma} \\ g^{\rho} \circ_A^i f^{\sigma} &= (g \circ_A^i f)^{\tau} \end{aligned}$$

We write $*\text{Poly}_X$ for the category of X -ary $*$ -polycategories.

7.3. **EXAMPLE.** A $\{(1, 1)\}$ -ary $*$ -polycategory is just an ordinary category equipped with a strict contravariant involution, since none of the l or r data exists. Even more trivially, a $\{(0, 0)\}$ -ary $*$ -polycategory is just a set $\mathcal{P}(\cdot)$, with no operations.

7.4. **DEFINITION.** We define a **cyclic symmetric multicategory** to be a “co-unary $*$ -polycategory”, i.e. an $(\mathbb{N} \times \{1\})$ -ary one. To see that this is sensible, note firstly that it ensures that none of the l and r data exist. Thus a $(\mathbb{N} \times \{1\})$ -ary $*$ -polycategory is just a symmetric multicategory with a strict involution on its objects and an extended action on the homsets $\mathcal{P}(A_1, \dots, A_n; B)$ indexed by the symmetric group S_{n+1} . But S_{n+1} is generated by its two subgroups S_n (permuting the first n objects A_1, \dots, A_n) and C_{n+1} (the cyclic group of order $n + 1$, permuting the objects cyclically). The resulting action of S_n is just that of a symmetric multicategory, while the action of C_{n+1} says that the underlying non-symmetric multicategory of \mathcal{P} is a *cyclic* multicategory in the sense of [CGR14], and the relations in S_{n+1} between these subgroups say that the symmetric and cyclic structure are compatible in a natural way.

7.5. **EXAMPLE.** If $X = \mathbb{N} \times \mathbb{N}$, then all the composites and identities exist, and each pair of operations 1_A^i and \circ_A^i uniquely determine the others. In particular, if we look at 1_A^m and \circ_A^m , we see that an $(\mathbb{N} \times \mathbb{N})$ -ary $*$ -polycategory reduces to an ordinary **$*$ -polycategory** as defined in [Hyl02, §5.3]; for emphasis we may call it a **bi-infinitary $*$ -polycategory**.

7.6. **REMARK.** Note that in a bi-infinitary $*$ -polycategory, A^\bullet is indeed a dual of A : the identities 1_A^l and 1_A^r supply the unit and counit of the duality. Conversely, any polycategory equipped with “strictly involutive duals” can be made into a $*$ -polycategory.

7.7. **EXAMPLE.** If $X = \mathbb{N} \times \{0\}$, then *only* 1_A^l and \circ_A^l exist. Thus an $(\mathbb{N} \times \{0\})$ -ary $*$ -polycategory may be called an **entries-only $*$ -polycategory**, by analogy with “entries-only” cyclic operads (which are the positive-ary one-object case) — since there is no codomain, the objects in the domain are simply called “entries”. In [DCH18], entries-only $*$ -polycategories are called “colored cyclic operads”, but I prefer the terminology of [GK95, CGR14, HRY19] whereby “cyclic multicategories” and “cyclic operads” can be regarded as ordinary multicategories or operads equipped only with the structure of an involution and a compatible cyclic action, rather than additionally with the stuff of an extra hom-set $\mathcal{P}(\cdot)$.

7.8. **EXAMPLE.** With $X = \{0, 1, \dots, n\} \times \{1\}$ (or $X = \{0, 1, \dots, n + 1\} \times \{0\}$ for the entries-only version) we obtain **n -truncated cyclic symmetric multicategories**, which include n -truncated cyclic operads (for truncated operads see e.g. [SNPR05]).

Of course, if $Y \subseteq X$ we have a functor $U_Y^X : *\text{Poly}_X \rightarrow *\text{Poly}_Y$ that forgets the morphisms with undesired arities and the operations relating to them. As we will now see, these functors often do not forget very much.

Given a fixed set \mathcal{O} of objects, let $\text{Seq}_X(\mathcal{O})$ be the groupoid whose objects are pairs $(\Gamma; \Delta)$ of finite lists of elements of \mathcal{O} with $(|\Gamma|, |\Delta|) \in X$, and whose morphisms

are isomorphisms $\Gamma^\bullet, \Delta \xrightarrow{\sim} \Lambda^\bullet, \Sigma$. An inclusion $Y \subseteq X$ yields a fully faithful inclusion $\text{Seq}_Y(\mathcal{O}) \hookrightarrow \text{Seq}_X(\mathcal{O})$. By an X -ary collection over \mathcal{O} we mean a functor $\mathcal{P} : \text{Seq}_X(\mathcal{O}) \rightarrow \text{Set}$; thus an X -ary $*$ -polycategory consists of an X -ary collection over a set of objects together with identities and composition operations.

7.9. THEOREM. *If $Y \subseteq X \subseteq \mathbb{N} \times \mathbb{N}$ and for any $(m, n) \in X$ there exists $(k, \ell) \in Y$ such that $k + \ell = m + n$, then the forgetful functor $U_Y^X : *Poly_X \rightarrow *Poly_Y$ is an equivalence.*

PROOF. The assumption ensures that the corresponding inclusion $\text{Seq}_Y(\mathcal{O}) \hookrightarrow \text{Seq}_X(\mathcal{O})$, for any set \mathcal{O} , is essentially surjective and hence an equivalence. For if $(\Gamma; \Delta) \in \text{Seq}_X(\mathcal{O})$, with say $|\Gamma| = m$ and $|\Delta| = n$ where $(m, n) \in X$, we can choose $(k, \ell) \in Y$ as in the assumption and find an isomorphism $(\Gamma; \Delta) \cong (\Lambda; \Sigma)$ such that $|\Lambda| = k$ and $|\Sigma| = \ell$, hence $(\Lambda; \Sigma) \in \text{Seq}_Y(\mathcal{O})$. Moreover, any hom-set in an X -ary $*$ -polycategory \mathcal{P} is isomorphic to one in $U_Y^X \mathcal{P}$, and any composition operation in \mathcal{P} is related by the corresponding permutation actions to one that exists in $U_Y^X \mathcal{P}$. Thus the structure of \mathcal{P} is uniquely determined by that of $U_Y^X \mathcal{P}$.

Finally, given a Y -ary $*$ -polycategory \mathcal{Q} , its underlying Y -ary collection extends to an X -ary one, uniquely up to unique isomorphism, and we can use these same permutation actions to define the necessary identities and compositions for the latter to be an X -ary $*$ -polycategory. It is straightforward to check that the axioms are then satisfied. ■

Let $\mathbb{N}_{>0} = \{m \in \mathbb{N} \mid m > 0\}$ and $(\mathbb{N} \times \mathbb{N})_{>0} = \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid m + n > 0\}$.

7.10. COROLLARY. *In the following diagram of forgetful functors:*

$$\begin{array}{ccccc}
 *Poly_{\mathbb{N} \times \{0\}} & \xleftarrow{\sim} & *Poly_{\mathbb{N} \times \mathbb{N}} & \xrightarrow{\sim} & *Poly_{\mathbb{N} \times \{0,1\}} \\
 \downarrow & & \downarrow & & \downarrow \\
 *Poly_{\mathbb{N}_{>0} \times \{0\}} & \xleftarrow{\sim} & *Poly_{(\mathbb{N} \times \mathbb{N})_{>0}} & \xrightarrow{\sim} & *Poly_{\mathbb{N} \times \{1\}}
 \end{array} \tag{7.11}$$

all the horizontal functors are equivalences.

PROOF. Each of these inclusions satisfies the hypothesis of Theorem 7.9. ■

Thus, bi-infinitary $*$ -polycategories are equivalent to entries-only $*$ -polycategories and also to co-subunary ones (i.e. $(\mathbb{N} \times \{0, 1\})$ -ary ones). The former equivalence is familiar from the syntax of classical linear logic, which can be presented either with two-sided sequents or one-sided ones (although *right*-sided sequents are more common than left-sided ones, corresponding to $(\{0\} \times \mathbb{N})$ -ary $*$ -polycategories instead of $(\mathbb{N} \times \{0\})$ -ary ones). Co-subunary syntax is less common, but can be found for instance in [Red91].

Similarly, the bottom row shows that cyclic symmetric multicategories (Definition 7.4) are equivalent to positive-ary entries-only ones. This suggests that arbitrary $*$ -polycategories could also be called something like “augmented cyclic symmetric multicategories”.

7.12. **THEOREM.** *Each of the vertical functors in (7.11) has both a left adjoint L and a right adjoint R , each of which is fully faithful (equivalently, the unit $\text{Id} \rightarrow UL$ and counit $UR \rightarrow \text{Id}$ are isomorphisms). Moreover, the counit $LU \rightarrow \text{Id}$ and unit $\text{Id} \rightarrow RU$ are bijective on objects, and fully faithful except on $(0, 0)$ -ary morphisms.*

This lemma makes Remark 1.5 precise: the underlying cyclic symmetric multicategory of a $*$ -polycategory remembers everything but the $(0, 0)$ -ary morphisms. The fully faithful right adjoint of the left-hand vertical functor appears in [DCH18].

PROOF. It suffices to consider the right-hand one $U : *Poly_{\mathbb{N} \times \{0,1\}} \rightarrow *Poly_{\mathbb{N} \times \{1\}}$. To start with, since all the structures in question are essentially algebraic and U simply forgets some of the data, it preserves limits. Thus, by the adjoint functor theorem for locally presentable categories, it has a left adjoint.

For its right adjoint, we define the homsets of $R\mathcal{P}$ by right Kan extending those of \mathcal{P} along the inclusion $\text{Seq}_{\mathbb{N} \times \{1\}}(\mathcal{O}) \hookrightarrow \text{Seq}_{\mathbb{N} \times \{0,1\}}(\mathcal{O})$. This automatically gives the symmetric actions, with $R\mathcal{P}(\cdot) = 1$. The only new composition operations we need to define are those involving co-nullary morphisms:

$$R\mathcal{P}(\Lambda, A; \cdot) \times \mathcal{P}(\Gamma; A) \xrightarrow{\circ_A} R\mathcal{P}(\Lambda, \Gamma; \cdot)$$

Suppose $g \in R\mathcal{P}(\Lambda, A; \cdot) \cong \mathcal{P}(\Lambda; A^\bullet)$ and $f \in \mathcal{P}(\Gamma; A)$. If $|\Lambda| > 0$, say $\Lambda = \Lambda', B$, we can permute B into the codomain of g and A into its domain, and compose in \mathcal{P} along A if instead $|\Gamma| > 0$, say $\Gamma = \Gamma', C$, we can permute A into the domain of f and C into the codomain, and compose in \mathcal{P} along A^\bullet . The unit, equivariance, and associativity axioms follow directly. The remaining composites to define have the form

$$\mathcal{P}(\cdot; A^\bullet) \times \mathcal{P}(\cdot; A) \rightarrow R\mathcal{P}(\cdot) = 1,$$

so they exist uniquely and all axioms about them are true.

Evidently $UR\mathcal{P} \cong \mathcal{P}$ naturally. On the other hand, note that all of the above definitions were forced except for $R\mathcal{P}(\cdot)$ and the compositions having it as codomain. Thus, if \mathcal{P} is given as a co-subunary $*$ -polycategory, it must be isomorphic to RUP except possibly at (\cdot) . Since $RUP(\cdot) = 1$ is terminal, this “isomorphism away from (\cdot) ” extends to a unique functor $\mathcal{P} \rightarrow RUP$ that is, as claimed, bijective on objects and fully faithful except on $(0, 0)$ -ary morphisms. The triangle identities for an adjunction are straightforward.

Finally, full-faithfulness of L follows from that of R by a standard abstract argument, and the fact that U remembers the objects and non- $(0, 0)$ -ary morphisms implies that $LU \rightarrow \text{Id}$ is also bijective on objects and fully faithful except on $(0, 0)$ -ary morphisms. ■

7.13. **REMARK.** The $(0, 0)$ -ary morphisms of $L\mathcal{P}$ are, as befits a left adjoint, “freely generated” by all composites $g \circ_A f$ for $f \in \mathcal{P}(\cdot; A)$ and $g \in \mathcal{P}(\cdot; A^\bullet) \cong L\mathcal{P}(A; \cdot)$, subject to relations imposed to force the necessary associativity axiom.

Now I claim that our poly double category $\mathbb{M}\text{Adj}$ is in fact a $*$ -poly double category, i.e. an internal category in $*$ -polycategories. More generally, we have:

7.14. THEOREM. *If Ω is a presheaf of $*$ -polycategories on a multicategory \mathcal{C} , so is $\text{Adj}_{\mathcal{C}}(\Omega)$.*

PROOF. We take the dual of $(\phi^+, \phi^-, \underline{\phi})$ to be $(\phi^-, \phi^+, \underline{\phi}^\bullet)$, where $\underline{\phi}^\bullet$ is the dual of $\underline{\phi}$ in the $*$ -polycategory $\Omega(\Gamma, \phi^-, \phi^+;)$, acted on by a symmetry to land in $\Omega(\Gamma, \phi^+, \phi^-;)$. The symmetric action on 2-morphisms in $\text{Adj}_{\mathcal{C}}(\Omega)(\Gamma)$ permutes the morphisms f_j and g_i and uses the symmetric action on morphisms in Ω . ■

7.15. COROLLARY. *Any double Chu construction $\text{Chu}(\mathcal{C}, \Omega)$ is a $*$ -poly double category.*

PROOF. Frobenius (pseudo-)discrete polycategories are always naturally $*$ -polycategories. ■

Recall from Definition 7.1 that $\mathbb{M}\text{Adj}$ consists of the objects $[\mathcal{A}] = (\mathcal{A}, \mathcal{A}^{\text{op}}, \text{hom}_{\mathcal{A}})$ and similar vertical arrows in $\text{Chu}(\text{Cat}, \text{Set})$. It is therefore closed under the duality of $\text{Chu}(\text{Cat}, \text{Set})$, so it is also a $*$ -poly double category. Hence it has an underlying cyclic symmetric multi double category, which we can compare to the cyclic multi double category of [CGR14]. In [CGR14] no symmetric structure was considered, but we can of course forget the existence of that symmetric structure and remember only the cyclic one. This enables us to finally state the following theorem.

7.16. REMARK. In fact, [CGR14] work directly with n -variable *mutual left* adjunctions. Thus, in the language of Definition 7.2, what their construction yields most directly is a positive-ary entries-only (i.e. $(\mathbb{N}_{>0} \times \{0\})$ -ary) $*$ -polycategory. However, to facilitate comparison with [CGR14] we will likewise use the notation of the equivalent $(\mathbb{N} \times \{1\})$ -ary version.

7.17. THEOREM. *The underlying cyclic multi double category of the $*$ -poly double category $\mathbb{M}\text{Adj}$ is isomorphic to the cyclic multi double category constructed in [CGR14].*

PROOF. For now, let $\mathbb{M}\text{Adj}_{\mathcal{S}}$ denote our version and $\mathbb{M}\text{Adj}_{\text{CGR}}$ denote theirs. By inspection, the two coincide on objects (categories), vertical arrows (functors), and horizontal arrows (co-unary multivariable adjunctions). (Recall in particular that for $f : [\mathcal{A}] \rightarrow [\mathcal{B}]$ in $\mathbb{M}\text{Adj}_{\mathcal{S}}$, the functor $f^+ : \mathcal{A} \rightarrow \mathcal{B}$ is the *right* adjoint and $f^- : \mathcal{B}^{\text{op}} \rightarrow \mathcal{A}^{\text{op}}$ is the *left* adjoint.)

However, the 2-cells of $\mathbb{M}\text{Adj}_{\mathcal{S}}$ are, like those in $\text{Chu}(\text{Cat}, \text{Set})$, *families* of natural transformations μ_j^+, μ_i^- related by the axioms such as Definition 1. Specifically, a 2-cell

$$\begin{array}{ccc} (A_1, \dots, A_m) & \xrightarrow{f} & B \\ u_1 \downarrow & \cdots & \downarrow u_m \quad \Downarrow \mu \quad \downarrow v \\ (C_1, \dots, C_m) & \xrightarrow{g} & D \end{array}$$

consists of a family of natural transformations

$$\begin{array}{ccc} (A_1, \dots, A_m) & \xrightarrow{f^+} & B \\ \downarrow \cdots (u_1, \dots, u_m) \cdots \downarrow & & \Downarrow \mu^+ \quad \downarrow v \\ (C_1, \dots, C_m) & \xrightarrow{g^+} & D \end{array}$$

and

$$\begin{array}{ccc}
 (A_1, \dots, \widehat{A}_i, \dots, A_m, B^{\text{op}}) & \xrightarrow{f_i^-} & A_i^{\text{op}} \\
 \downarrow \dots (u_1, \dots, \widehat{u}_i, \dots, u_m, v^{\text{op}}) \dots & & \downarrow \Downarrow \mu_i^- \quad \downarrow u_i^{\text{op}} \\
 (C_1, \dots, \widehat{C}_i, \dots, C_m, D^{\text{op}}) & \xrightarrow{g_i^-} & C_i^{\text{op}}
 \end{array} \quad (1 \leq i \leq m)$$

any two of which satisfy a commutativity condition; whereas an analogous 2-cell in $\mathbb{M}\text{Adj}_{\text{CGR}}$ consists *only* of the transformation μ^+ . Thus, we have a multicategory functor $\mathbb{M}\text{Adj}_{\text{S}} \rightarrow \mathbb{M}\text{Adj}_{\text{CGR}}$ that simply forgets the transformations μ_i^- .

We now show that this functor preserves the cyclic action. As before, this is obvious except on the 2-cells. In $\mathbb{M}\text{Adj}_{\text{S}}$, the cyclic action on 2-cells simply rotates the μ^+ and μ_i^- ; whereas in $\mathbb{M}\text{Adj}_{\text{CGR}}$ the cyclic action is defined by constructing mates. The point is that the compatibility axioms on the 2-cells μ^+ and μ_i^- in $\mathbb{M}\text{Adj}_{\text{S}}$ are precisely a way of saying that they are each other's mates. For instance, the condition from Definition 1 for μ^+ and μ_1^- becomes

$$\begin{array}{ccc}
 \mathcal{A}_1(f_1^-(\vec{a}, b), a_1) & \xrightarrow{\cong} & \mathcal{B}(b, f^+(a_1, \vec{a})) \\
 u_1 \downarrow & & \downarrow v \\
 \mathcal{C}_1(u_1(f_1^-(\vec{a}, b)), u_1(a_1)) & & \mathcal{D}(v(b), v(f^+(a_1, \vec{a}))) \\
 \mu_1^- \downarrow & & \downarrow \mu^+ \\
 \mathcal{C}_1(g_1^-(u\vec{a}, v(b)), u_1(a_1)) & \xrightarrow{\cong} & \mathcal{D}(v(b), g^+(u_1(a_1), u\vec{a}))
 \end{array}$$

where $\vec{a} = (a_2, \dots, a_m)$ and $u\vec{a} = (u_2(a_2), \dots, u_m(a_m))$. The Yoneda lemma implies that this is equivalent to

$$\begin{array}{ccc}
 v(b) & \xrightarrow{\quad} & g^+(g_1^-(u\vec{a}, v(b)), u\vec{a}) \\
 \downarrow & & \downarrow \mu_1^- \\
 v(f^+(f_1^-(\vec{a}, b), \vec{a})) & \xrightarrow{\mu^+} & g^+(u_1(f_1^-(\vec{a}, b)), u\vec{a})
 \end{array}$$

If we fix \vec{a} and write

$$\begin{array}{lll}
 F^+(a) = f^+(a, \vec{a}) & G^+(c) = g^+(c, u\vec{a}) & U(a) = u_1(a) \\
 F^-(b) = f_1^-(\vec{a}, b) & G^-(d) = g_1^-(u\vec{a}, d) & V(b) = v(b)
 \end{array}$$

then this becomes

$$\begin{array}{ccc}
 Vb & \xrightarrow{\quad} & G^+G^-Vb \\
 \downarrow & & \downarrow G^+\mu^- \\
 VF^+F^-b & \xrightarrow{\mu^+F^-} & G^+UF^-b
 \end{array} \quad (7.18)$$

which is a standard condition characterizing μ^+ and μ^- as mates under the one-variable adjunctions $F^- \dashv F^+$ and $G^- \dashv G^+$. Explicitly, if we apply G^- on the outside and postcompose with the counit of $G^- \dashv G^+$, we get

$$\begin{array}{ccccc} G^-Vb & \longrightarrow & G^-G^+G^-Vb & \longrightarrow & G^-Vb \\ \downarrow & & \downarrow_{G^-G^+\mu^-} & & \downarrow_{\mu^-} \\ G^-VF^+F^-b & \xrightarrow{G^-\mu^+F^-} & G^-G^+UF^-b & \longrightarrow & UF^-b \end{array}$$

where the right-hand square is naturality and the top composite is 1_{G^-Vb} by a triangle identity. Thus, μ^- is characterized as the left-bottom composite, i.e. as a mate of μ^+ . We can dually characterize μ^+ as a mate of μ^- ; while conversely if either is defined as a mate of the other in such a way then (7.18) commutes.

One does have to check that such a definition is natural in the *other* variables, but this was done in [CGR14, Prop. 2.11]. Thus, the functor $\mathbb{M}\text{Adj}_S \rightarrow \mathbb{M}\text{Adj}_{\text{CGR}}$ preserves the cyclic action. Moreover, this also shows that it is *faithful* on 2-cells, since all the μ_i^- 's are determined as mates of μ^+ .

To show that it is also full on 2-cells, we need to know that if μ^+ is given and we define all the μ_i^- 's as its mates, the resulting μ_i^- 's satisfy their *own* pairwise conditions (Definition 1), and therefore define a 2-cell in $\mathbb{M}\text{Adj}_S$. But this is the content of [CGR14, Prop. 2.13 and Theorem 2.16]. Thus, the functor $\mathbb{M}\text{Adj}_S \rightarrow \mathbb{M}\text{Adj}_{\text{CGR}}$ is an isomorphism. ■

7.19. COROLLARY. *A 2-cell in $\mathbb{M}\text{Adj}$ is uniquely determined by any one of the transformations μ_j^+ or μ_i^- .* ■

Corollary 7.19 is not true of more general 2-cells in $\text{Chu}(\text{Cat}, \text{Set})$: a transformation between “polarized adjunctions” must be “equipped with specified mates”.

Recall also (Remark 1.6) that our conventions were chosen to agree with those of [CGR14], so that a 2-cell $f \rightarrow g$ in $\mathbb{M}\text{Adj}$ is determined by transformations in the *same* direction between the *right* adjoints $f_i^+ \rightarrow g_i^+$ and in the *opposite* direction between the *left* adjoints $g_j^- \rightarrow f_j^-$. But this is a fairly arbitrary choice.

7.20. REMARK. Since we chose to “incorrectly” give our $*$ -polycategory $\mathbb{M}\text{Adj}$ exactly one $(0,0)$ -ary morphism (recall Remark 1.5), it happens to be in the image of the right adjoint R from Theorem 7.12. Thus, it is R of its underlying cyclic symmetric multicategory, which by Theorem 7.17 is that of [CGR14]. Thus, we could equivalently have constructed it by (adding a symmetric action and) applying Theorem 7.17 to the construction in [CGR14]; but the relationship to the Chu and Dialectica constructions would then be obscured.

7.21. REMARK. We have focused on multivariable adjunctions between ordinary categories and $\text{Chu}(\text{Cat}, \text{Set})$, mainly for simplicity and to match [CGR14]. However, multivariable adjunctions exist much more generally, e.g. for enriched, internal, and indexed categories, as well as the “enriched indexed categories” of [Shu13]; the only requirement is that in the enriched cases the enriching category must apparently be symmetric. Each of these

contexts gives rise to a similar poly double category of multivariable adjunctions that embeds into an appropriate double Chu construction.

There ought to be a general theorem encompassing all these cases, applying to any 2-category \mathcal{K} containing an object Ω satisfying some sort of “Yoneda lemma”, but it is not clear exactly what this should mean. Existing contexts for formal Yoneda lemmas such as [SW78, Str74, Web07, Woo82] are either too closely tied to the one-variable case, lack a notion of “opposite”, or consider only “cartesian” situations at the expense of enriched ones.

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