

THE LAW OF EXCLUDED MIDDLE IN THE SIMPLICIAL MODEL OF TYPE THEORY

CHRIS KAPULKIN AND PETER LEFANU LUMSDAINE

ABSTRACT. We show that the law of excluded middle holds in Voevodsky’s simplicial model of type theory. As a corollary, excluded middle is compatible with univalence.

Since [Kapulkin and Lumsdaine, 2020] first appeared in 2012, various readers have wondered whether Voevodsky’s model of type theory in simplicial sets validates the law of excluded middle. This fact is by now folklore within the field (implicitly appealed to in [Univalent Foundations Program, 2013, §3.4], for instance, for the relative consistency of LEM); but since it has still not appeared in the literature, we set it down here for the record.

We assume [Kapulkin and Lumsdaine, 2020] as background throughout, and follow its notational conventions, with a few shorthands for readability: we omit Scott brackets, write $\Gamma \models A$ **Type** to mean that A is a type of the simplicial model (i.e., a Kan fibration $p_A : \Gamma.A \rightarrow \Gamma$), and write $\Gamma \models A$ to mean that p_A admits a section, i.e., A is inhabited.

As required for constructing the simplicial model as in [Kapulkin and Lumsdaine, 2020, Cor. 2.3.5], we assume throughout an inaccessible cardinal α , and later another $\beta < \alpha$ to give a universe U_β in the model.

For $\Gamma \models A$ **Type**, define $\text{isProp } A := \prod_{x,y:A} \text{Id}_A(x,y)$. Our main goal is:

1. **THEOREM.** [Schema of Excluded Middle] *Let $\Gamma \models A$ **Type**, and suppose $\Gamma \models \text{isProp } A$. Then $\Gamma \models A + \neg A$.*

We write $i_n : \partial\Delta^n \hookrightarrow \Delta^n$ for the boundary inclusion of the standard n -simplex, and $f \pitchfork g$ to indicate that f has the left lifting property with respect to g .

2. **LEMMA.** *The following are equivalent for a Kan fibration p :*

1. $i_1 \widehat{\times} i_n \pitchfork p$ for all $n \geq 0$;
2. $i_n \pitchfork p$ for all $n \geq 1$.

PROOF. Standard combinatorics of prisms, similar to [Joyal and Tierney, 2008, proof of Thm. 1.5.3]. ■

Received by the editors 2020-04-29 and, in final form, 2020-09-02.

Transmitted by Richard Garner. Published on 2020-09-03.

2020 Mathematics Subject Classification: 03B15 (primary), 55U10, 18C50.

Key words and phrases: Univalent foundations, homotopy type theory, simplicial sets, law of excluded middle, dependent type theory.

© Chris Kapulkin and Peter LeFanu Lumsdaine, 2020. Permission to copy for private use granted.

3. **LEMMA.** *Given a Kan fibration $p: Y \rightarrow X$, the image of p is complemented: that is, the sets $\{X_n \setminus p(Y_n)\}_{n \in \mathbb{N}}$ form a simplicial set $X \setminus p(X) \subseteq X$.*

PROOF. For any n -simplex $x \in X_n$, note that $x \in X_n \setminus p(Y_n)$ exactly when all vertices of x lie in $X_0 \setminus p(Y_0)$. The claim follows directly. ■

PROOF OF THEOREM 1. Suppose $\Gamma \models \mathbf{isProp} A$. Unwinding the interpretation of \mathbf{isProp} in the simplicial model, this says just that the two projections $\pi_1, \pi_2: \Gamma.A.A \rightarrow \Gamma.A$ are homotopic over Γ ; equivalently, the fibration $p_{\text{Id}_A}: \Gamma.A.A.\text{Id}_A \rightarrow \Gamma.A.A$ is trivial. But p_{Id_A} is a pullback of the Leibniz exponential $i_1 \triangleright p_A$ along a weak equivalence, so the latter is also trivial:

$$\begin{array}{ccc}
 \Gamma.A.A.\text{Id}_A & \xrightarrow{\sim} & (\Gamma.A)^{\Delta^1} \\
 p_{\text{Id}_A} \downarrow \lrcorner & & \downarrow i_1 \triangleright p_A \\
 \Gamma.A.A & \xrightarrow{\sim} & (\Gamma.A)^2 \times_{\Gamma^2} \Gamma^{\Delta^1} \\
 \downarrow \lrcorner & & \downarrow \\
 \Gamma & \xrightarrow{\sim} & \Gamma^{\Delta^1}
 \end{array}$$

This is in turn equivalent to $i_1 \widehat{\times} i_n \pitchfork p_A$ for all n ; so by Lemma 2, $i_n \pitchfork p_A$ for all $n \geq 1$.

Now to give a section of $p_{A+\neg A}$, we decompose Γ according to Lemma 3 as $\Gamma = \Gamma_0 + \Gamma_1$, where $\Gamma_0 = p_A(\Gamma.A)$ and $\Gamma_1 = \Gamma \setminus \Gamma_0$, and work over each component separately. The pullback of p_A to Γ_0 is orthogonal to i_0 by definition of Γ_0 , and higher i_n since p_A was; so it is a trivial fibration, so admits a section. Over Γ_1 , the pullback of p_A is empty, so we have a section of $p_{\neg A}$. Together they give the desired section $\Gamma \rightarrow \Gamma.A + \neg A$ of $p_{A+\neg A}$. ■

Theorem 1 gave the law of excluded middle in the form of a global scheme. This immediately implies other forms of LEM, e.g. quantified over an universe as in [Univalent Foundations Program, 2013, (3.4.1)]. Let \mathbf{U}_β be a universe in the model, and define $\mathbf{Prop}_\beta := \sum_{A:\mathbf{U}_\beta} \mathbf{isProp} A$.

4. **COROLLARY.** *The universe \mathbf{U}_β satisfies LEM: that is,*

$$\models \prod_{A:\mathbf{Prop}_\beta} (\mathbf{El}(\pi_1(A)) + \neg \mathbf{El}(\pi_1(A))).$$

PROOF. Apply Theorem 1 to the type $A : \mathbf{Prop}_\beta \models \mathbf{El}(\pi_1(A)) \text{ Type}$. ■

5. **COROLLARY.** *It is consistent, over Martin-Löf Type Theory with Π -, Σ -, Id -, 1 -, 0 -, and $+$ -types (as set out in [Kapulkin and Lumsdaine, 2020, App. A, B]), for a universe to simultaneously satisfy the univalence axiom, the law of excluded middle, and closure under all the listed type formers.*

PROOF. By Corollary 4 together with [Kapulkin and Lumsdaine, 2020, Cor. 2.3.5]. ■

6. **COROLLARY.** *In each simplicial universe β , the type of propositions is equivalent to a discrete simplicial set with 2 elements, i.e., $\mathbf{Prop}_\beta \simeq 1 + 1$.*

PROOF. This follows internally from Corollary 4, by [Univalent Foundations Program, 2013, Ex. 3.9]. ■

ACKNOWLEDGEMENTS. We are grateful to Christian Sattler for catching an error in an earlier version of this paper.

References

A. Joyal and M. Tierney. Notes on simplicial homotopy theory. Unpublished manuscript, 2008. URL: <http://mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern47.pdf>.

K. Kapulkin and P. LeF. Lumsdaine. The simplicial model of univalent foundations (after Voevodsky). *Journal of the European Mathematical Society*, 2020. To appear. E-print: [arXiv:1211.2851](https://arxiv.org/abs/1211.2851).

The Univalent Foundations Program. *Homotopy type theory: Univalent foundations of mathematics*. Institute for Advanced Study, Princeton, 2013. URL: <https://homotopytypetheory.org/book>.

This article may be accessed at <http://www.tac.mta.ca/tac/>

THEORY AND APPLICATIONS OF CATEGORIES will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods. Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.

SUBSCRIPTION INFORMATION Individual subscribers receive abstracts of articles by e-mail as they are published. To subscribe, send e-mail to tac@mta.ca including a full name and postal address. Full text of the journal is freely available at <http://www.tac.mta.ca/tac/>.

INFORMATION FOR AUTHORS \LaTeX 2 ϵ is required. Articles may be submitted in PDF by email directly to a Transmitting Editor following the author instructions at <http://www.tac.mta.ca/tac/authinfo.html>.

MANAGING EDITOR. Geoff Cruttwell, Mount Allison University: gcruttwell@mta.ca

TEXNICAL EDITOR. Michael Barr, McGill University: michael.barr@mcgill.ca

ASSISTANT TEX EDITOR. Gavin Seal, Ecole Polytechnique Fédérale de Lausanne: gavin_seal@fastmail.fm

TRANSMITTING EDITORS.

Clemens Berger, Université de Nice-Sophia Antipolis: cberger@math.unice.fr

Julie Bergner, University of Virginia: jeb2md@virginia.edu

Richard Blute, Université d' Ottawa: rblute@uottawa.ca

Gabriella Böhm, Wigner Research Centre for Physics: bohm.gabriella@wigner.mta.hu

Valeria de Paiva, Nuance Communications Inc: valeria.depaiva@gmail.com

Richard Garner, Macquarie University: richard.garner@mq.edu.au

Ezra Getzler, Northwestern University: getzler@northwestern.edu

Kathryn Hess, Ecole Polytechnique Fédérale de Lausanne: kathryn.hess@epfl.ch

Dirk Hofmann, Universidade de Aveiro: dirk@ua.pt

Pieter Hofstra, Université d' Ottawa: phofstra@uottawa.ca

Anders Kock, University of Aarhus: kock@math.au.dk

Joachim Kock, Universitat Autònoma de Barcelona: kock@mat.uab.cat

Stephen Lack, Macquarie University: steve.lack@mq.edu.au

Tom Leinster, University of Edinburgh: Tom.Leinster@ed.ac.uk

Matias Menni, Conicet and Universidad Nacional de La Plata, Argentina: matias.menni@gmail.com

Ieke Moerdijk, Utrecht University: i.moerdijk@uu.nl

Susan Niefield, Union College: niefiels@union.edu

Kate Ponto, University of Kentucky: kate.ponto@uky.edu

Robert Rosebrugh, Mount Allison University: rrosebrugh@mta.ca

Jiri Rosicky, Masaryk University: rosicky@math.muni.cz

Giuseppe Rosolini, Università di Genova: rosolini@disi.unige.it

Alex Simpson, University of Ljubljana: Alex.Simpson@fmf.uni-lj.si

James Stasheff, University of North Carolina: jds@math.upenn.edu

Ross Street, Macquarie University: ross.street@mq.edu.au

Tim Van der Linden, Université catholique de Louvain: tim.vanderlinden@uclouvain.be