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THE LAW OF EXCLUDED MIDDLE IN THE SIMPLICIAL MODEL OF TYPE THEORY

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ABSTRACT. We show that the law of excluded middle holds in Voevodsky's simplicial model of type theory. As a corollary, excluded middle is compatible with univalence.

Since [Kapulkin and Lumsdaine, 2020] first appeared in 2012, various readers have wondered whether Voevodsky's model of type theory in simplicial sets validates the law of excluded middle. This fact is by now folklore within the field (implicitly appealed to in [Univalent Foundations Program, 2013, §3.4], for instance, for the relative consistency of LEM); but since it has still not appeared in the literature, we set it down here for the record.

We assume [Kapulkin and Lumsdaine, 2020] as background throughout, and follow its notational conventions, with a few shorthands for readability: we omit Scott brackets, write $\Gamma \models A\ \mathtt{Type}$ to mean that A is a type of the simplicial model (i.e., a Kan fibration $p_A : \Gamma.A \to \Gamma$), and write $\Gamma \models A$ to mean that p_A admits a section, i.e., A is inhabited.

As required for constructing the simplicial model as in [Kapulkin and Lumsdaine, 2020, Cor. 2.3.5], we assume throughout an inaccessible cardinal α, and later another $\beta < \alpha$ to give a universe U_β in the model.

For $\Gamma \models A\ \mathtt{Type}$, define $\mathtt{isProp}\, A := \prod_{x,y:A} \mathtt{Id}_A(x, y)$. Our main goal is:

1. THEOREM. [Schema of Excluded Middle] *Let* $\Gamma \models A\ \mathtt{Type}$*, and suppose* $\Gamma \models \mathtt{isProp}\, A$*. Then* $\Gamma \models A + \neg A$*.*

We write $i_n \colon \partial\Delta^n \hookrightarrow \Delta^n$ for the boundary inclusion of the standard n-simplex, and $f \pitchfork g$ to indicate that f has the left lifting property with respect to g.

2. LEMMA. *The following are equivalent for a Kan fibration* p*:*

1. $i_1 \widehat{\times} i_n \pitchfork p$ *for all* $n \geq 0$*;*

2. $i_n \pitchfork p$ *for all* $n \geq 1$*.*

PROOF. Standard combinatorics of prisms, similar to [Joyal and Tierney, 2008, proof of Thm. 1.5.3]. ∎

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3. Lemma. *Given a Kan fibration $p\colon Y \to X$, the image of p is complemented: that is, the sets $\{X_n \setminus p(Y_n)\}_{n\in\mathbb{N}}$ form a simplicial set $X \setminus p(X) \subseteq X$.*

Proof. For any n-simplex $x \in X_n$, note that $x \in X_n \setminus p(Y_n)$ exactly when all vertices of x lie in $X_0 \setminus p(Y_0)$. The claim follows directly. ∎

Proof of Theorem 1. Suppose $\Gamma \models \mathtt{isProp}\, A$. Unwinding the interpretation of isProp in the simplicial model, this says just that the two projections $\pi_1, \pi_2 \colon \Gamma.A.A \to \Gamma.A$ are homotopic over Γ; equivalently, the fibration $p_{\mathtt{Id}_A}\colon \Gamma.A.A.\mathtt{Id}_A \to \Gamma.A.A$ is trivial. But $p_{\mathtt{Id}_A}$ is a pullback of the Leibniz exponential $i_1 \triangleright p_A$ along a weak equivalence, so the latter is also trivial:

$$\begin{array}{ccc}
\Gamma.A.A.\mathtt{Id}_A & \xrightarrow{\sim} & (\Gamma.A)^{\Delta^1} \\
{\scriptstyle p_{\mathtt{Id}_A}}\downarrow \lrcorner & & \downarrow{\scriptstyle i_1 \triangleright p_A} \\
\Gamma.A.A & \xrightarrow{\sim} & (\Gamma.A)^2 \times_{\Gamma^2} \Gamma^{\Delta^1} \\
\downarrow \lrcorner & & \downarrow \\
\Gamma & \xrightarrow{\sim} & \Gamma^{\Delta^1}
\end{array}$$

This is in turn equivalent to $i_1 \mathbin{\widehat{\times}} i_n \pitchfork p_A$ for all n; so by Lemma 2, $i_n \pitchfork p_A$ for all $n \geq 1$.

Now to give a section of $p_{A+\neg A}$, we decompose Γ according to Lemma 3 as $\Gamma = \Gamma_0 + \Gamma_1$, where $\Gamma_0 = p_A(\Gamma.A)$ and $\Gamma_1 = \Gamma \setminus \Gamma_0$, and work over each component separately. The pullback of p_A to Γ_0 is orthogonal to i_0 by definition of Γ_0, and higher i_n since p_A was; so it is a trivial fibration, so admits a section. Over Γ_1, the pullback of p_A is empty, so we have a section of $p_{\neg A}$. Together they give the desired section $\Gamma \to \Gamma.A + \neg A$ of $p_{A+\neg A}$. ∎

Theorem 1 gave the law of excluded middle in the form of a global scheme. This immediately implies other forms of LEM, e.g. quantified over an universe as in [Univalent Foundations Program, 2013, (3.4.1)]. Let $\mathtt{U}_\beta$ be a universe in the model, and define $\mathtt{Prop}_\beta \coloneqq \sum_{A:\mathtt{U}_\beta} \mathtt{isProp}\, A$.

4. Corollary. *The universe $\mathtt{U}_\beta$ satisfies LEM: that is,*

$$\models \prod_{A:\mathtt{Prop}_\beta} (\mathtt{El}(\pi_1(A)) + \neg\mathtt{El}(\pi_1(A)))\,.$$

Proof. Apply Theorem 1 to the type $A : \mathtt{Prop}_\beta \models \mathtt{El}(\pi_1(A))\ \mathtt{Type}$. ∎

5. Corollary. *It is consistent, over Martin-Löf Type Theory with Π-, Σ-, Id-, 1-, 0-, and +-types (as set out in [Kapulkin and Lumsdaine, 2020, App. A, B]), for a universe to simultaneously satisfy the univalence axiom, the law of excluded middle, and closure under all the listed type formers.*

Proof. By Corollary 4 together with [Kapulkin and Lumsdaine, 2020, Cor. 2.3.5]. ∎

6. Corollary. *In each simplicial universe β, the type of propositions is equivalent to a discrete simplicial set with 2 elements, i.e., $\mathtt{Prop}_\beta \simeq 1 + 1$.*

Proof. This follows internally from Corollary 4, by [Univalent Foundations Program, 2013, Ex. 3.9]. ∎

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