Theory and Applications of Categories, Vol. 35, No. 41, 2020, pp. 1549–1563.

TRANSFER OF A GENERALISED GROUPOID ACTION ALONG A MORITA EQUIVALENCE

GIORGI ARABIDZE

ABSTRACT. Buss and Meyer define fibrations of topological groupoids and interpret a groupoid fibration $L \to H$ with fibre G as a generalised action of H on G by groupoid equivalences. My result shows that a generalised action of H on G may be transported along a Morita equivalence $G \sim K$ to a generalised action of H on K, which is given from a fibration $R \to H$ with fibre K. Furthermore, topological groupoids R and L are Morita equivalent.

1. Introduction

Buss and Meyer explain the classical topological groupoid action by isomorphisms in [1]. A classical action of H on G consists of H-actions on G^0 and G^1 such that the source, range ($r, s\colon G^1 \rightrightarrows G^0$) and multiplication ($m\colon G^1 \times_{s,G^0,r} G^1 \to G^1$) maps are H-equivariant. Then they build a transformation groupoid $L := G \rtimes H$ for this classical action. Also, they construct a special kind of continuous functor $L \to H$, which is briefly called groupoid fibration (Definition 2.1. [1]) such that G is the fibre of this fibration. They interpret any groupoid fibration $L \to H$ with fibre G as an action of H on G by groupoid equivalences. Examples of groupoid fibrations also appear in [3]. Also, examples without a topological structure you can find in [4] and [5].

My result answers the following question. Is it possible to transfer a generalised groupoid action along a Morita equivalence? The answer is yes. If the groupoid that acts is étale, this result is already done by Buss and Meyer in [2], using inverse semigroup language. The construction in this paper works for any topological groupoid. If I have two Morita equivalent topological groupoids G and K and a third topological groupoid H which acts on G, then I can build an action of H on K. Here I mean generalised action in the previous sentence. So it must be understood as follows: If I have a groupoid fibration $L \to H$ with fibre G and G and K are Morita equivalent, then I can construct a topological groupoid R and a groupoid fibration $R \to H$ with fibre K such that L and R are Morita equivalent.

I would like to thank Ralf Meyer for invaluable guidance, as well as Emzar Khmaladze and Niko Inassaridze for various helpful conversations.

Received by the editors 2018-04-05 and, in final form, 2020-08-31.

Transmitted by Anders Kock. Published on 2020-09-10.

2020 Mathematics Subject Classification: 22A99.

Key words and phrases: topological groupoid, Morita equivalence, groupoid fibration, generalised groupoid action, fibre of a groupoid fibration.

2. Construction of R

There are some equivalent ways to define Morita equivalence between topological groupoids. One of them is used in this paper: Two topological groupoids G and K are equivalent if and only if there exist a left G-action and a right K-action on the same topological space and these actions are compatible. In more details:

2.1. Definition. *Two topological groupoids* G *and* K *are Morita equivalent if and only if there are a topological space* X*, anchor maps* $\mathrm{r_X}\colon \mathrm{X} \to \mathrm{G}^0$ *and* $\mathrm{s_X}\colon \mathrm{X} \to \mathrm{K}^0$ *and left and right action maps* $\mathrm{m_G}\colon \mathrm{G}^1 \times_{\mathrm{s},\mathrm{G}^0,\mathrm{r_X}} \mathrm{X} \to \mathrm{X}, \quad (g,x) \mapsto g\cdot x$*, and* $\mathrm{m_K}\colon \mathrm{X} \times_{\mathrm{s_X},\mathrm{K}^0,\mathrm{r}} \mathrm{K}^1 \to \mathrm{X}, \quad (x,k) \mapsto x\cdot k$*, such that the following properties are satisfied:*

1. $\mathrm{r_X}(g\cdot x) = \mathrm{r}(g), \quad \mathrm{s_X}(g\cdot x) = \mathrm{s_X}(x), \quad \forall x \in \mathrm{X}, \quad \forall g \in \mathrm{G}^1$ *with* $\mathrm{s}(g) = \mathrm{r_X}(x)$ *and* $\mathrm{s_X}(x\cdot k) = \mathrm{s}(k), \quad \mathrm{r_X}(x\cdot k) = \mathrm{r_X}(x), \quad \forall x \in \mathrm{X}, \quad \forall k \in \mathrm{K}^1$ *with* $\mathrm{r}(k) = \mathrm{s_X}(x)$;
2. $\mathrm{m_G}$ *and* $\mathrm{m_K}$ *are associative:*
$$g_1\cdot(g_2\cdot x) = (g_1\cdot g_2)\cdot x, \quad \forall x \in \mathrm{X}, \quad \forall g_1,g_2 \in \mathrm{G}^1 \text{ with } \mathrm{s}(g_2) = \mathrm{r_X}(x), \quad \mathrm{s}(g_1) = \mathrm{r}(g_2);$$
$$(x\cdot k_1)\cdot k_2 = x\cdot(k_1\cdot k_2), \quad \forall x \in \mathrm{X}, \quad \forall k_1,k_2 \in \mathrm{K}^1 \text{ with } \mathrm{s_X}(x) = \mathrm{r}(k_1), \quad \mathrm{s}(k_1) = \mathrm{r}(k_2);$$
$$g\cdot(x\cdot k) = (g\cdot x)\cdot k, \quad \forall x \in \mathrm{X}, \quad \forall g \in \mathrm{G}^1, \quad \forall k \in \mathrm{K}^1 \text{ with } \mathrm{s}(g) = \mathrm{r_X}(x), \quad \mathrm{r}(k) = \mathrm{s_X}(x);$$
3. *The following two maps are homeomorphisms:*
$$\psi_\mathrm{G}\colon \mathrm{G}^1 \times_{\mathrm{s},\mathrm{G}^0,\mathrm{r_X}} \mathrm{X} \to \mathrm{X} \times_{\mathrm{s_X},\mathrm{K}^0,\mathrm{s_X}} \mathrm{X}, \qquad (g,x) \mapsto (x, g\cdot x);$$
$$\psi_\mathrm{K}\colon \mathrm{X} \times_{\mathrm{s_X},\mathrm{K}^0,\mathrm{r}} \mathrm{K}^1 \to \mathrm{X} \times_{\mathrm{r_X},\mathrm{G}^0,\mathrm{r_X}} \mathrm{X}, \qquad (x,k) \mapsto (x, x\cdot k);$$
4. $\mathrm{s_X}$ *and* $\mathrm{r_X}$ *are open surjections.*

2.2. Remark. Propositions A.2 and A.5 in [2] say that $\mathrm{m_G}$ and $\mathrm{m_K}$ are open surjections and, also, $x\cdot 1_{\mathrm{s}}(x) = x$ and $x\cdot 1_{\mathrm{r}}(x) = x, \forall x \in \mathrm{X}$.

2.3. Definition. *Let* L *and* H *be topological groupoids. A groupoid* fibration *is a continuous functor* $\mathrm{F}\colon \mathrm{L} \to \mathrm{H}$ *(continuous maps* $\mathrm{F}^i\colon \mathrm{L}^i \to \mathrm{H}^i$ *for* $i = 0,1$ *that intertwine* r, s *and the multiplication maps), such that the map*
$$(\mathrm{F}^1,\mathrm{s})\colon \mathrm{L}^1 \to \mathrm{H}^1 \times_{\mathrm{s},\mathrm{H}^0,\mathrm{F}^0} \mathrm{L}^0 := \{(h,x) \in \mathrm{H}^1 \times \mathrm{L}^0 \mid \mathrm{s}(h) = \mathrm{F}^0(x)\} \tag{1}$$
is an open surjection. Its fibre *is the subgroupoid* G *of* L *defined by* $\mathrm{G}^0 = \mathrm{L}^0$ *and*
$$\mathrm{G}^1 = \{g \in \mathrm{L}^1 \mid \mathrm{F}^1(g) = 1_{\mathrm{F}^0(\mathrm{s}(g))}\},$$
equipped with the subspace topology on $\mathrm{G}^1 \subseteq \mathrm{L}^1$*.* A generalised groupoid action *of* H *on* G *means a groupoid fibration* $\mathrm{F}\colon \mathrm{L} \to \mathrm{H}$ *with fibre* G*.*

2.4. Remark. Lemma 2.5 in [1] says that the fibre of a groupoid fibration is a topological groupoid.

2.5. Example. Let H be any topological groupoid and let X be any topological space equipped with a right H-action. Lemma 2.12 in [1] says that there is a groupoid fibration from a transformation groupoid of this action $\mathrm{X} \rtimes \mathrm{H}$ to H with fibre X,

$$\mathrm{X} \hookrightarrow \mathrm{X} \rtimes \mathrm{H} \to \mathrm{H}.$$

The object space of this transformation groupoid is X and for all x_1 and x_2 belongs to X we have $\mathrm{Hom}_{\mathrm{X}\rtimes\mathrm{H}}(x_1, x_2) = \{h \in \mathrm{H}^1 \mid x_1 \cdot h = x_2\}$. The composition in $\mathrm{X} \rtimes \mathrm{H}$ is the composition in H. The functor from a transformation groupoid to H is defined obviously, and it is shown that this functor is a groupoid fibration between topological groupoids and its fibre is a topological groupoid X without arrows.

2.6. Example. Let L be a topological groupoid and let $\alpha\colon \mathrm{L}^1 \times_{\mathrm{s},\mathrm{L}^0,\mathrm{r_X}} \mathrm{X} \to \mathrm{H}$ be an open and star surjective cocycle with values in the topological group H. Let us call such cocycle an *exact*. The star surjective morphism between groupoids is defined in [4]. It is easy to check that the exact cocycle is a groupoid fibration from the transformation groupoid to the group of values. So we have a generalised groupoid action of the topological group on the kernel of the exact cocycle.

Now we have all information that is needed to construct a topological groupoid R and a groupoid fibration $\mathrm{E}\colon \mathrm{R} \to \mathrm{H}$ with fibre K such that R and L are Morita equivalent. Suppose that we have all data from Definitions 2.1 and 2.3.

We already know that the object space of the fibre and of the source of the fibration are equal. So $\mathrm{R}^0 = \mathrm{K}^0$.

For finding an arrow space R^1 we take the quotient of the topological space

$$\mathrm{X} \times_{\mathrm{r_X},\mathrm{L}^0,\mathrm{s}} \mathrm{L}^1 \times_{\mathrm{r},\mathrm{L}^0,\mathrm{r_X}} \mathrm{X} = \{(x, l, y) | \mathrm{r_X}(x) = \mathrm{s}(l), \mathrm{r_X}(y) = \mathrm{r}(l)\}$$

by the following equivalence relation "$\sim$": The elements (x_1, l_1, y_1) and (x_2, l_2, y_2) of $\mathrm{X} \times_{\mathrm{r_X},\mathrm{L}^0,\mathrm{s}} \mathrm{L}^1 \times_{\mathrm{r},\mathrm{L}^0,\mathrm{r_X}} \mathrm{X}$ are equivalent if and only if there exist $g_1, g_2 \in \mathrm{G}^1$ such that $x_2 = g_1 \cdot x_1, y_2 = g_2 \cdot y_1$ and $l_2 = g_2 \cdot l_1 \cdot g_1^{-1}$. In this case I write: $(x_1, l_1, y_1) \sim_{g_1,g_2} (x_2, l_2, y_2)$.

2.7. Proposition. *The relation "$\sim$" defined above is an equivalence relation.*

Proof. Reflexivity holds: $(x, l, y) \sim_{1_{\mathrm{s}(l)},1_{\mathrm{r}(l)}} (x, l, y)$ because $x = 1_{\mathrm{s}(l)} \cdot x, y = 1_{\mathrm{r}(l)} \cdot y$ and $l = 1_{\mathrm{r}(l)} \cdot l \cdot 1_{\mathrm{s}(l)}^{-1}$. It is symmetric because if $(x_1, l_1, y_1) \sim_{g_1,g_2} (x_2, l_2, y_2)$ then $(x_2, l_2, y_2) \sim_{g_1^{-1},g_2^{-1}} (x_1, l_1, y_1)$, and it is transitive because if $(x_1, l_1, y_1) \sim_{g_1,g_2} (x_2, l_2, y_2)$ and $(x_2, l_2, y_2) \sim_{g_1',g_2'} (x_3, l_3, y_3)$ then $(x_1, l_1, y_1) \sim_{g_1' \cdot g_1, g_2' \cdot g_2} (x_3, l_3, y_3)$. ∎

Because of this lemma, I can consider the quotient of $\mathrm{X} \times_{\mathrm{r_X},\mathrm{L}^0,\mathrm{s}} \mathrm{L}^1 \times_{\mathrm{r},\mathrm{L}^0,\mathrm{r_X}} \mathrm{X}$ by "$\sim$". The quotient space with the quotient topology will be the arrow space for the topological groupoid R:

$$\mathrm{R}^1 = (\mathrm{X} \times_{\mathrm{r_X},\mathrm{L}^0,\mathrm{s}} \mathrm{L}^1 \times_{\mathrm{r},\mathrm{L}^0,\mathrm{r_X}} \mathrm{X})/\sim.$$

Firstly, I have to define a groupoid structure on R^1. Let us start by defining source and range maps.

The source map: $\mathrm{s}\colon \mathrm{R}^1 \to \mathrm{R}^0, \qquad [x, l, y] \mapsto \mathrm{s_X}(x)$.

The range map: $\mathrm{r}\colon \mathrm{R}^1 \to \mathrm{R}^0, \qquad [x, l, y] \mapsto \mathrm{s_X}(y)$.

These maps are well-defined because if $(x_1, l_1, y_1) \sim_{g_1,g_2} (x_2, l_2, y_2)$ then $x_2 = g_1 \cdot x_1$ and therefore $\mathrm{s_X}(x_1) = \mathrm{s_X}(x_2)$ because of Definition 2.1. So the source map and analogously the range map are well-defined.

I have to define the multiplication map from composable pairs of arrows to the arrow space. If the arrows $[x_1, l_1, y_1]$ and $[x_2, l_2, y_2]$ are composable, then $\mathrm{s_X}(y_2) = \mathrm{s_X}(x_1)$. This means that $(y_2; x_1) \in \mathrm{X} \times_{\mathrm{s_X},\mathrm{K}^0,\mathrm{s_X}} \mathrm{X}$, so there is exactly one $g \in \mathrm{G}^1$ such that $\psi_\mathrm{G}(g, y_2) = (y_2; x_1)$, where ψ_G is the homeomorphism from Definition 2.1. Therefore, I can define the composition of arrows in R^1 in the following way:

$$\mathrm{m}\colon \mathrm{R}^1 \times_{\mathrm{s},\mathrm{R}^0,\mathrm{r}} \mathrm{R}^1 \to \mathrm{R}^1, \qquad ([x_1, l_1, y_1]; [x_2, l_2, y_2]) \mapsto [x_2, l_1 \cdot g \cdot l_2, y_1].$$

I have to show that this map is well-defined. Let us consider two composable pair of arrows $([x_1, l_1, y_1]; [x_2, l_2, y_2])$ and $([a_1, l_1', b_1]; [a_2, l_2', b_2])$ in $\mathrm{R}^1 \times_{\mathrm{s},\mathrm{R}^0,\mathrm{r}} \mathrm{R}^1$ and let $(x_1, l_1, y_1) \sim_{g_1,g_2} (a_1, l_1', b_1)$ and $(x_2, l_2, y_2) \sim_{g_1',g_2'} (a_2, l_2', b_2)$. Then it is easy to check that $(x_2, l_1 \cdot g \cdot l_2, y_1) \sim_{g_1',g_2} (a_2, l_1' \cdot g' \cdot l_2', b_1)$, where $x_1 = g \cdot y_2$ and $a_1 = g' \cdot b_2$. This means that the multiplication map is well-defined.

The unit map is the following: $\mathrm{u}\colon \mathrm{R}^0 \to \mathrm{R}^1, \quad \mathrm{K} \mapsto [x, 1_{\mathrm{r_X}(x)}, x]$, where x is any element in X which goes to K by $\mathrm{s_X}$. If $(x; y) \in \mathrm{X} \times_{\mathrm{s_X},\mathrm{K}^0,\mathrm{s_X}} \mathrm{X}$ then $(x, 1_{\mathrm{r_X}(x)}, x) \sim_{g,g^{-1}} (y, 1_{\mathrm{r_X}(y)}, y)$ where $y = g \cdot x$. So the unit map is well-defined.

The inverse map: $\mathrm{i}\colon \mathrm{R}^1 \to \mathrm{R}^1, \quad [x, l, y] \mapsto [y, l^{-1}, x]$.

If $(x_1, l_1, y_1) \sim_{g_1,g_2} (x_2, l_2, y_2)$ then $(y_1, l_1^{-1}, x_1) \sim_{g_2^{-1},g_1^{-1}} (y_2, l_2^{-1}, x_2)$. So the inverse map is well-defined.

It is easy to check that R with the maps $(\mathrm{s}, \mathrm{r}, \mathrm{m}, \mathrm{u}, \mathrm{i})$ is a groupoid.

2.8. Proposition. *The groupoid* R *with the maps* $(\mathrm{s}, \mathrm{r}, \mathrm{m}, \mathrm{u}, \mathrm{i})$ *is a topological groupoid.*

Proof. I have to check that the source and range maps are open surjections and that the multiplication, unit and inverse maps are continuous.

I use the following well known fact: In a fibre product, a projection which is parallel to an open surjection is an open surjection too. In our case, for example, the following projection is an open surjection: $\mathrm{pr}_2\colon \mathrm{L}^1 \times_{\mathrm{r},\mathrm{L}^0,\mathrm{r_X}} \mathrm{X} \to \mathrm{X}$ because it is parallel to the range map of the topological groupoid L. Therefore, the following projection $\mathrm{pr}_2\colon (\mathrm{X} \times_{\mathrm{r_X},\mathrm{L}^0,\mathrm{s}} \mathrm{L}^1) \times_{\mathrm{pr}_2,\mathrm{L}^1,\mathrm{pr}_1} (\mathrm{L}^1 \times_{\mathrm{r},\mathrm{L}^0,\mathrm{r_X}} \mathrm{X}) \to \mathrm{L}^1 \times_{\mathrm{r},\mathrm{L}^0,\mathrm{r_X}} \mathrm{X}$ is an open surjection too. Also there is the standard homeomorphism:

$$\mathrm{X} \times_{\mathrm{r_X},\mathrm{L}^0,\mathrm{s}} \mathrm{L}^1 \times_{\mathrm{r},\mathrm{L}^0,\mathrm{r_X}} \mathrm{X} \cong (\mathrm{X} \times_{\mathrm{r_X},\mathrm{L}^0,\mathrm{s}} \mathrm{L}^1) \times_{\mathrm{pr}_2,\mathrm{L}^1,\mathrm{pr}_1} (\mathrm{L}^1 \times_{\mathrm{r},\mathrm{L}^0,\mathrm{r_X}} \mathrm{X}).$$

The composition of this homeomorphism, the two projections above and the anchor map $[(x,l,y) \mapsto (x,l,l,y) \mapsto (l,y) \mapsto y \mapsto \mathrm{s_X}(y)]$ is an open surjection, and it induces the range map of the groupoid R after factorization by "$\sim$" . Of course, an open surjection induces an open surjection after factorization. Therefore, the range map is an open surjection. Analogously, the source map is an open surjection.

Also, the map $((x_1,l_1,y_1);(x_2,l_2,y_2)) \mapsto (x_2, l_1 \cdot g \cdot l_2, y_1)$, where $x_1 = g \cdot y_2$, which induces the multiplication map in the groupoid R after double fuctorization by "$\sim$" , is continuous because it is induced by projection maps for some fibre product. So the multiplication map is continuous too.

Because of almost the same reason the inverse map is continuous. Also, $\mathrm{i} \cdot \mathrm{i} = \mathrm{id}_{\mathrm{R}^1}$. Therefore, it is an automorphism.

Now I need to show that the unit map is continuous. Let "$\simeq$"be the relation on X such that $x \simeq y$ if and only if there is $g \in \mathrm{G}^1$ such that $y = g \cdot x$. It is easy to check that this is an equivalence relation. So we can consider the qoutient of X by "$\simeq$". Let us show that $\mathrm{X}/\!\simeq$ and R^0 are homeomorphic by the map $\widetilde{\mathrm{s_X}}\colon [x] \mapsto \mathrm{s_X}(x)$. It is well-defined because if $x \simeq y$ then $y = g \cdot x$ and $\mathrm{s_X}(x) = \mathrm{s_X}(y)$ by Definition 2.1. $\widetilde{\mathrm{s_X}}$ is injective because if $\widetilde{\mathrm{s_X}}([x]) = \widetilde{\mathrm{s_X}}([y])$ then $\mathrm{s_X}(x) = \mathrm{s_X}(y)$, therefore $(x,y) \in \mathrm{X} \times_{\mathrm{s_X},\mathrm{K}^0,\mathrm{s_X}} \mathrm{X}$, and this means that there is a $g \in \mathrm{G}^1$ such that $y = g \cdot x$ because of the homeomorphism ψ_{G} from Definition 2.1. $\widetilde{\mathrm{s_X}}$ is an open surjection because it is induced by $\mathrm{s_X}$, which is an open surjection by Definition 2.1. So $\widetilde{\mathrm{s_X}}$ is a homeomorphism. Therefore, the unit map is continuous if the map $[x] \mapsto [x, 1_{\mathrm{r_X}(x)}, x]$ is continuous, and this is so because this map is induced by the composition $x \mapsto (x, 1_{\mathrm{r_X}(x)}, x) \mapsto [x, 1_{\mathrm{r_X}(x)}, x]$, and it is clear that this composition is continuous. So I proved that the source and range maps are open surjections and the multiplication, inverse and unit maps are continuous. So the groupoid R is a topological groupoid. ∎

3. Morita equivalence between R and L

So far, we have a topological groupoid R. Now I need to show that the topological groupoids R and L are Morita equivalent.

3.1. PROPOSITION. *The topological groupoids* R *and* L *are Morita equivalent.*

PROOF. For proving this proposition we need a topological space Y equipped with a left L-action and a right R-action. Firstly, let us construct the topological space Y. Consider the topological space $\mathrm{L}^1 \times_{\mathrm{s},\mathrm{L}^0,\mathrm{r_X}} \mathrm{X}$ with the following relation "$\eqsim$" on it: (l,x) and (l_1,y) are equivalent if and only if there is $g \in \mathrm{G}^1$ such that $l = l_1 \cdot g^{-1}$ and $y = g \cdot x$. It is easy to check that "$\eqsim$" is an equivalence relation. So we can consider the quotient of $\mathrm{L}^1 \times_{\mathrm{s},\mathrm{L}^0,\mathrm{r_X}} \mathrm{X}$ by "$\eqsim$" and let $\mathrm{Y} = (\mathrm{L}^1 \times_{\mathrm{s},\mathrm{L}^0,\mathrm{r_X}} \mathrm{X})/\!\eqsim$ equipped with the quotient topology.

For defining the left L-action on Y we need an anchor map $\mathrm{r_Y}\colon \mathrm{Y} \to \mathrm{L}^0$. Let $\mathrm{r_Y}([l,x]) = \mathrm{r}(l)$. This anchor map is well-defined because if $(l,x) \eqsim (l_1,y)$ then for some $g \in \mathrm{G}^1$ we have $l = l_1 \cdot g^{-1}$ for some $g \in \mathrm{G}^1$ and by the multiplication rule in the groupoid we have that $\mathrm{r}(l_1) = \mathrm{r}(l_1 \cdot g) = \mathrm{r}(l)$. So $\mathrm{r_Y}$ is well-defined. Now, we need an action map:

$$\mathrm{m_L}\colon \mathrm{L}^1 \times_{s,l^0,\mathrm{r_Y}} \mathrm{Y} \to \mathrm{Y}: \quad (l_1, [l, x]) \mapsto [l_1 \cdot l, x].$$

It is well-defined because if $(l, x) \approx (l_0, y)$ by g then $g \cdot x = y$ and $l_0 \cdot g = l$, therefore $l_1 \cdot l_0 \cdot g = l_1 \cdot l$ and this means that $(l_1 \cdot l, x) \approx (l_1 \cdot l_0, y)$ by g. Also, $\mathrm{m_L}$ must be associative and it is so because

$$\begin{aligned} l_2 \cdot (l_1 \cdot [l, x]) &= l_2 \cdot [l_1 \cdot l, x] \\ &= [l_2 \cdot l_1 \cdot l, x] \\ &= (l_2 \cdot l_1) \cdot [l, x]. \end{aligned}$$

We have to check one more property wich is required in the definition of groupoid action. This is

$$\begin{aligned} \mathrm{r_Y}(l_1 \cdot [l, x]) &= \mathrm{r_Y}[l_1 \cdot l, x] \\ &= \mathrm{r}(l_1 \cdot l) \\ &= \mathrm{r}(l_1). \end{aligned}$$

The last step for showing that we have a topological groupoid action is making sure that the action map is continuous. It is clear that this action map $\mathrm{m_L}$ is continuous because it is the composition of some quotient maps and maps which are induced by projections or multiplication maps in some fibre products.

For defining the right R-action on Y we need an anchor map $\mathrm{s_Y}\colon \mathrm{Y} \to \mathrm{R}^0$. Let $\mathrm{s_Y}([l, x]) = \mathrm{s_X}(x)$. This anchor map is well-defined because if $(l, x) \approx (l_1, y)$ then for some $g \in \mathrm{G}^1$ we have: $y = g \cdot x$ and by Definition 2.1 we have that $\mathrm{s_X}(y) = \mathrm{s_X}(g \cdot x) = \mathrm{s_X}(x)$. So $\mathrm{s_X}$ is well-defined. Now, we need an action map:

$$\mathrm{m_R}\colon \mathrm{Y} \times_{\mathrm{s_Y},\mathrm{R}^0,\mathrm{r}} \mathrm{R}^1 \to \mathrm{Y}: \quad ([l_1, x_1], [x, l, y]) \mapsto [l_1 \cdot g \cdot l, x],$$

where g is an element from G^1 such that $x_1 = g \cdot y$, such element exists because for any element $([l_1, x_1], [x, l, y])$ in $\mathrm{Y} \times_{\mathrm{s_Y},\mathrm{R}^0,\mathrm{r}} \mathrm{R}^1$ we have $\mathrm{s_X}(x_1) = \mathrm{s_X}(y)$, therefore $(y, x_1) \in \mathrm{X} \times_{\mathrm{s_X},\mathrm{K}^0,\mathrm{s_X}} \mathrm{X}$ and because of the homeomorphism ψ_G in Definition 2.1 we have such $g \in \mathrm{G}^1$. I have to show that this action map is well-defined. Let $(x, l, y) \sim_{g_1,g_2} (\hat{x}, \hat{l}, \hat{y})$ and $(l_1, x_1) \approx (\hat{l_1}, \hat{x_1})$ by g_3, then

$$\hat{x} = g_1 \cdot x, \quad \hat{y} = g_2 \cdot y, \quad \hat{x_1} = g_3 \cdot x_1, \quad \hat{l} = g_2 \cdot l \cdot {g_1}^{-1}, \quad \hat{l_1} = l_1 \cdot {g_3}^{-1}.$$

The associativity of $\mathrm{m_G}$ implies

$$\begin{aligned} (g_3 \cdot g \cdot {g_2}^{-1}) \cdot \hat{y} &= (g_3 \cdot g) \cdot ({g_2}^{-1} \cdot \hat{y}) \\ &= (g_3 \cdot g) \cdot y \\ &= g_3 \cdot (g \cdot y) \\ &= g_3 \cdot x_1 \\ &= \hat{x_1}. \end{aligned}$$

This means that $[\hat{l_1}, \hat{x_1}] \cdot [\hat{x}, \hat{l}, \hat{y}] = [\hat{l_1} \cdot g_3 \cdot g \cdot {g_2}^{-1} \cdot \hat{l}, \hat{x}]$. Also, $\hat{l_1} \cdot g_3 \cdot g \cdot {g_2}^{-1} \cdot \hat{l} \cdot g_1 = l_1 \cdot g \cdot l$. Therefore, $(l_1 \cdot g \cdot l, x) \approx (\hat{l_1} \cdot g_3 \cdot g \cdot {g_2}^{-1} \cdot \hat{l}, \hat{x})$ by g_1, so $[l_1, x_1] \cdot [x, l, y] = [\hat{l_1}, \hat{x_1}] \cdot [\hat{x}, \hat{l}, \hat{y}]$. The action map $\mathrm{m_R}$ is well-defined. As in case of $\mathrm{m_L}$ we can say the same. This action map $\mathrm{m_R}$ is continuous because it is a composition of some quotient maps and maps which are induced by projections or multiplication maps in some fibre products.

Also, it must be associative:

$$\begin{aligned}
[x, l] \cdot ([x_1, l_1, y_1] \cdot [x_2, l_2, y_2]) &= [x, l] \cdot [x_2, l_1 \cdot g_1 \cdot l_2, y_1] \\
&= [l \cdot g \cdot l_1 \cdot g_1 \cdot l_2, x_2] \\
&= [l \cdot g \cdot l_1, x_1] \cdot [x_2, l_2, y_2] \\
&= ([l, x] \cdot [x_1, l_1, y_1]) \cdot [x_2, l_2, y_2],
\end{aligned}$$

where $x_1 = g_1 \cdot y_2$ and $x = g \cdot y_1$. Also we have one more required property:

$$\begin{aligned}
\mathrm{s_Y}([l, x] \cdot [x_1, l_1, y_1]) &= \mathrm{s_Y}(l \cdot g \cdot l_1, x_1) \\
&= \mathrm{s_X}(x_1) \\
&= \mathrm{s}([x_1, l_1, y_1]).
\end{aligned}$$

So we have a left L-action and a right R-action on the topological space Y. For making sure that the topological groupoids R and L are Morita equivalent we have to show that three more required properties are satisfied:

First: The anchor maps $\mathrm{s_Y}$ and $\mathrm{r_Y}$ must be open surjections. From the fibre product $\mathrm{L}^1 \times_{\mathrm{s}, \mathrm{L}^0, \mathrm{r_X}} \mathrm{X}$ both projections are open surjections because they are parallel to the source map s and the anchor map $\mathrm{r_X}$, which are open surjections by definition. Also, the range map in the topological groupoid L and the anchor map $\mathrm{s_X}$ are open surjections by definition. Therefore, $\mathrm{s_Y}$ and $\mathrm{r_Y}$ are both open surjections because they are induced by the following compositions of open surjections: $\mathrm{s_X} \circ \mathrm{pr}_2$ and $\mathrm{r} \circ \mathrm{pr}_1$, respectively.

Second: The anchor map $\mathrm{s_Y}$ must be L-equivariant, the anchor map $\mathrm{r_Y}$ must be R-equivariant and we need mixed associativity of the action maps. We have

$$\begin{aligned}
\mathrm{s_Y}(l_1 \cdot [l, x]) &= \mathrm{s_Y}([l_1 \cdot l, x]) \\
&= \mathrm{s_X}(x) \\
&= \mathrm{s_Y}([l, x])
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{r_Y}([l, x] \cdot [x_1, l_1, y_1]) &= \mathrm{r_Y}([l \cdot g \cdot l_1, x_1]) \\
&= \mathrm{r}(l \cdot g \cdot l_1) \\
&= \mathrm{r}(l) \\
&= \mathrm{r_Y}([l, x]),
\end{aligned}$$

where $x = g \cdot y_1$. Also,

$$\begin{aligned} \hat{l} \cdot ([l,x] \cdot [x_1, l_1, y_1]) &= \hat{l} \cdot [l \cdot g \cdot l_1, x_1] \\ &= [\hat{l} \cdot l \cdot g \cdot l_1, x_1] \\ &= [\hat{l} \cdot l, x] \cdot [x_1, l_1, y_1] \\ &= (\hat{l} \cdot [l,x]) \cdot [x_1, l_1, y_1]. \end{aligned}$$

Third: We need to show that the following two maps are homeomorphisms:

$$\psi_{\mathrm{L}} \colon \mathrm{L}^1 \times_{\mathrm{s},\mathrm{L}^0,\mathrm{r}_\mathrm{Y}} \mathrm{Y} \to \mathrm{Y} \times_{\mathrm{s}_\mathrm{Y},\mathrm{R}^0,\mathrm{s}_\mathrm{Y}} \mathrm{Y}, \qquad (l_1, [l,x]) \mapsto ([l,x], l_1 \cdot [l,x]).$$

$$\psi_{\mathrm{R}} \colon \mathrm{Y} \times_{\mathrm{s}_\mathrm{Y},\mathrm{R}^0,\mathrm{r}} \mathrm{R}^1 \to \mathrm{Y} \times_{\mathrm{r}_\mathrm{Y},\mathrm{L}^0,\mathrm{r}_\mathrm{Y}} \mathrm{Y}, \qquad ([l,x],[x_1,l_1,y_1]) \mapsto ([l,x],[l,x]\cdot[x_1,l_1,y_1]).$$

First of all we, ψ_{L} and ψ_{R} are continuous because of the same reasons as in many cases above, they are induced by various continuous maps. Also, we can directly name inverses of both maps, which are continuous for the same reasons.

$$\psi_{\mathrm{L}}{}^{-1} \colon \mathrm{Y} \times_{\mathrm{s}_\mathrm{Y},\mathrm{R}^0,\mathrm{s}_\mathrm{Y}} \mathrm{Y} \to \mathrm{L}^1 \times_{\mathrm{s},\mathrm{L}^0,\mathrm{r}_\mathrm{Y}} \mathrm{Y}, \qquad ([l,x],[l_1,x_1]) \mapsto (l_1 \cdot g \cdot l^{-1}, [l,x]),$$

where $x_1 = g \cdot x$. Also we have inverse of ψ_{R}

$$\psi_{\mathrm{R}}{}^{-1} \colon \mathrm{Y} \times_{\mathrm{r}_\mathrm{Y},\mathrm{L}^0,\mathrm{r}_\mathrm{Y}} \mathrm{Y} \to \mathrm{Y} \times_{\mathrm{s}_\mathrm{Y},\mathrm{R}^0,\mathrm{r}} \mathrm{R}^1, \qquad ([l,x],[l_1,x_1]) \mapsto ([l,x],[x_1, l^{-1} \cdot l_1, x]).$$

Let us consider the compositions:

$$\begin{aligned} \psi_{\mathrm{L}}(\psi_{\mathrm{L}}{}^{-1}([l,x],[l_1,x_1])) &= \psi_{\mathrm{L}}(l_1 \cdot g \cdot l^{-1}, [l,x]) \\ &= ([l,x], l_1 \cdot g \cdot l^{-1} \cdot [l,x]) \\ &= ([l,x], [l_1 \cdot g \cdot l^{-1} \cdot l, x]) \\ &= ([l,x], [l_1 \cdot g, x]). \end{aligned}$$

It is clear that $(l_1, x_1) \approx (l_1 \cdot g, x)$ by g. Thus $\psi_{\mathrm{L}} \circ \psi_{\mathrm{L}}{}^{-1} = \mathrm{id}_{(\mathrm{Y} \times_{\mathrm{r}_\mathrm{Y},\mathrm{L}^0,\mathrm{r}_\mathrm{Y}} \mathrm{Y})}$.

Also,

$$\begin{aligned} \psi_{\mathrm{L}}{}^{-1}(\psi_{\mathrm{L}}(l_1,[l,x])) &= \psi_{\mathrm{L}}{}^{-1}([l,x], l_1 \cdot [l,x]) \\ &= \psi_{\mathrm{L}}{}^{-1}([l,x],[l_1 \cdot l, x]) \\ &= (l_1 \cdot l \cdot 1_{\mathrm{r}_\mathrm{X}(x)} \cdot l^{-1}, [l,x]) \\ &= (l_1, [l,x]). \end{aligned}$$

So $\psi_{\mathrm{L}}{}^{-1} \circ \psi_{\mathrm{L}} = \mathrm{id}_{(\mathrm{L}^1 \times_{\mathrm{s},\mathrm{L}^0,\mathrm{r}_\mathrm{Y}} \mathrm{Y})}$ and we have that ψ_{L} is a homeomorphism.

Now, let us check the same in the case of ψ_{R}.

$$\begin{aligned} \psi_{\mathrm{R}}(\psi_{\mathrm{R}}{}^{-1}([l,x],[l_1,x_1])) &= \psi_{\mathrm{R}}([l,x],[x_1, l^{-1} \cdot l_1, x]) \\ &= ([l,x],[l,x] \cdot [x_1, l^{-1} \cdot l_1, x]) \\ &= ([l,x],[l \cdot 1_{\mathrm{r}_\mathrm{X}(x)} \cdot l^{-1} \cdot l_1, x_1]) \\ &= ([l,x],[l_1,x_1]). \end{aligned}$$

So $\psi_{\mathrm{R}} \circ {\psi_{\mathrm{R}}}^{-1} = \mathrm{id}_{(\mathrm{Y} \times_{\mathrm{r_Y}, \mathrm{L}^0, \mathrm{r_Y}} \mathrm{Y})}$.

Also,

$$\begin{aligned} {\psi_{\mathrm{R}}}^{-1}(\psi_{\mathrm{R}}([l,x],[x_1,l_1,y_1])) &= {\psi_{\mathrm{R}}}^{-1}([l,x],[l,x]\cdot[x_1,l_1,y_1]) \\ &= {\psi_{\mathrm{R}}}^{-1}([l,x],[l\cdot g\cdot l_1,x_1]) \\ &= ([l,x],[x_1,l^{-1}\cdot l\cdot g\cdot l_1,x]) \\ &= ([l,x],[x_1,g\cdot l_1,x]). \end{aligned}$$

It is cleare that $(x_1,l_1,y_1) \sim_{1_{\mathrm{r_X}(x_1)},g} (x_1, g\cdot l_1, x)$. Therefore ${\psi_{\mathrm{L}}}^{-1} \circ \psi_{\mathrm{L}} = \mathrm{id}_{(\mathrm{Y} \times_{\mathrm{s_Y}, \mathrm{R}^0, \mathrm{r}} \mathrm{R}^1)}$.

So ψ_{L} and ψ_{R} are homeomorphisms. We proved all required properties and, therefore, the topological groupoids R and L are Morita equivalent. ∎

4. Construction of fibration from R to H

Our main goal is to transfer a generalised action of H on G to a generalised action of H on K. We already have the topological groupoid R. The last step is to construct a fibration E from R to H with fibre K.

We need a continuous map E^1 from R^1 to H^1. Suppose that we have all data from Definition 2.3. Define E^1 by:

$$\mathrm{E}^1\colon \mathrm{R}^1 \to \mathrm{H}^1 \qquad [x,l,y] \mapsto \mathrm{F}^1(l).$$

E^1 is well-defined because if $(x_1,l_1,y_1) \sim_{g_1,g_2} (x_2,l_2,y_2)$ then

$$\begin{aligned} \mathrm{E}^1([x_2,l_2,y_2]) &= \mathrm{F}^1(l_2) \\ &= \mathrm{F}^1(g_2\cdot l_1\cdot g_1^{-1}) \\ &= \mathrm{F}^1(g_2)\cdot \mathrm{F}^1(l_1)\cdot \mathrm{F}^1(g_1^{-1}) \\ &= \mathrm{F}^1(l_1) \\ &= \mathrm{E}^1([x_1,l_1,y_1]). \end{aligned}$$

Here I use that the elements g_1 and g_2 belongs to G^1, the arrow space of fibre of fibration F. Therefore, they go to the identity elements in H^1.

E^1 is continuous because it is a composition of the continuous map F^1 and a map which is induced by a projection map.

The map $\mathrm{E}^0\colon \mathrm{R}^0 \to \mathrm{H}^0$ is defined by the following way: $\mathrm{E}^0 = \mathrm{s} \circ \mathrm{E}^1 \circ \mathrm{u}$, where u is the unit map in R and s is the source map in H. This automatically means that E^1 and E^0 intertwine the sorce maps of the groupoids R and H. Also, we have to check that they intertwine the multiplication maps.

$$\begin{aligned} \mathrm{E}^1([x_1, l_1, y_1] \cdot [x_2, l_2, y_2]) &= \mathrm{E}^1([x_2, l_1 \cdot g \cdot l_2, y_1]) \\ &= \mathrm{F}^1(l_1 \cdot g \cdot l_2) \\ &= \mathrm{F}^1(l_1) \cdot \mathrm{F}^1(g) \cdot \mathrm{F}^1(l_2) \\ &= \mathrm{F}^1(l_1) \cdot \mathrm{F}^1(l_2) \\ &= \mathrm{E}^1([x_1, l_1, y_1]) \cdot \mathrm{E}^1([x_2, l_2, y_2]), \end{aligned}$$

where g is an element from the fibre of F such that $x_1 = g \cdot y_2$.

So we have a continuous functor $\mathrm{E}\colon \mathrm{R} \to \mathrm{H}$.

4.1. PROPOSITION. *The continuous functor* $\mathrm{E}\colon \mathrm{R} \to \mathrm{H}$ *defined above is a fibration between topological groupoids.*

PROOF. For proving this lemma we have to show that the map

$$(\mathrm{E}^1, \mathrm{s})\colon \mathrm{R}^1 \to \mathrm{H}^1 \times_{\mathrm{s},\mathrm{H}^0,\mathrm{E}^0} \mathrm{R}^0 := \{(h, x) \in \mathrm{H}^1 \times \mathrm{R}^0 \mid \mathrm{s}(h) = \mathrm{E}^0(x)\} \tag{2}$$

is an open surjection.

Firstly, let us show that the map

$$(\mathrm{F}^1, \mathrm{id}_\mathrm{X})\colon \mathrm{L}^1 \times_{\mathrm{s},\mathrm{L}^0,\mathrm{r}_\mathrm{X}} \mathrm{X} \to \mathrm{H}^1 \times_{\mathrm{s},\mathrm{H}^0,\mathrm{F}^0 \mathrm{o} \mathrm{r}_\mathrm{X}} \mathrm{X}, \qquad (l, x) \mapsto (\mathrm{F}^1(l), x),$$

is an open surjection.

It is easy to check that there is a fibre product diagram:

$$\begin{CD} \mathrm{L}^1 \times_{\mathrm{s},\mathrm{L}^0,\mathrm{r}_\mathrm{X}} \mathrm{X} @>{(\mathrm{F}^1,\mathrm{id}_\mathrm{X})}>> \mathrm{H}^1 \times_{\mathrm{s},\mathrm{H}^0,\mathrm{F}^0 \mathrm{o} \mathrm{r}_\mathrm{X}} \mathrm{X} \\ @V{\mathrm{pr}_1}VV @VV{(\mathrm{id}_{\mathrm{H}^1},\mathrm{r}_\mathrm{X})}V \\ \mathrm{L}^1 @>{(\mathrm{F}^1,\mathrm{s})}>> \mathrm{H}^1 \times_{\mathrm{s},\mathrm{H}^0,\mathrm{F}^0} \mathrm{L}^0 \end{CD}$$

Therefore $(\mathrm{F}^1, \mathrm{id}_\mathrm{X})$ is an open surjection because it is a pull-back of $(\mathrm{F}^1, \mathrm{s})$ which is an open surjection because of Definition 2.3.

Also, we have one more fibre product diagram:

$$\begin{CD} \mathrm{H}^1 \times_{\mathrm{s},\mathrm{H}^0,\mathrm{F}^0 \mathrm{o} \mathrm{r}_\mathrm{X}} \mathrm{X} @>{\mathrm{pr}_2}>> X \\ @V{(\mathrm{id}_{\mathrm{H}^1},\mathrm{s}_\mathrm{X})}VV @VV{\mathrm{s}_\mathrm{X}}V \\ \mathrm{H}^1 \times_{\mathrm{s},\mathrm{H}^0,\mathrm{E}^0} \mathrm{R}^0 @>{\mathrm{pr}_2}>> \mathrm{R}^0 \end{CD}$$

So $(\mathrm{id}_{\mathrm{H}^1}, \mathrm{s}_\mathrm{X})$ is an open surjection because it is a pull-back of s_X which is an open surjection because of Definition 2.1.

Also, the projection $\mathrm{pr}_1\colon \mathrm{X} \times_{\mathrm{r_X},\mathrm{L}^0,\mathrm{s}} \mathrm{L}^1 \times_{\mathrm{r},\mathrm{L}^0,\mathrm{r_X}} \mathrm{X} \to \mathrm{X} \times_{\mathrm{r_X},\mathrm{L}^0,\mathrm{s}} \mathrm{L}^1$ is an open surjection because it is a pull-back of $\mathrm{r_X}$, which is an open surjection. There is an obvious homeomorphism $c\colon \mathrm{X} \times_{\mathrm{r_X},\mathrm{L}^0,\mathrm{s}} \mathrm{L}^1 \to \mathrm{L}^1 \times_{\mathrm{s},\mathrm{L}^0,\mathrm{r_X}} \mathrm{X}$, $(x,l) \mapsto (l,x)$, which is an open surjection, of course. So the composition

$$\alpha = (\mathrm{id}_{\mathrm{H}^1}, \mathrm{s_X}) \circ (\mathrm{F}^1, \mathrm{id_X}) \circ c \circ \mathrm{pr}_1\colon \mathrm{X} \times_{\mathrm{r_X},\mathrm{L}^0,\mathrm{s}} \mathrm{L}^1 \times_{\mathrm{r},\mathrm{L}^0,\mathrm{r_X}} \mathrm{X} \to \mathrm{H}^1 \times_{\mathrm{s},\mathrm{H}^0,\mathrm{E}^0} \mathrm{R}^0$$
$$(x,l,y) \mapsto (x,l) \mapsto (l,x) \mapsto (\mathrm{F}^1(l),x) \mapsto (\mathrm{F}^1(l),\mathrm{s_X}(x)),$$

is an open surjection. It is easy to see that the map $(\mathrm{E}^1,\mathrm{s})\colon \mathrm{R}^1 \to \mathrm{H}^1 \times_{\mathrm{s},\mathrm{H}^0,\mathrm{E}^0} \mathrm{R}^0$ is induced by α. So $(\mathrm{E}^1,\mathrm{s})\colon \mathrm{R}^1 \to \mathrm{H}^1 \times_{\mathrm{s},\mathrm{H}^0,\mathrm{E}^0} \mathrm{R}^0$ is an open surjection. Therefore, the continuous functor $\mathrm{E}\colon \mathrm{R} \to \mathrm{H}$ is a fibration between topological groupoids. ∎

Now, we need to show that the fibre of the groupoid fibration $\mathrm{E}\colon \mathrm{R} \to \mathrm{H}$ is isomorphic to the topological groupoid K. The element $[x,l,y]$ of R^1 goes to $\mathrm{F}^1(l)$ in H^1 by the fibration E^1. This means that the element $[x,l,y]$ of R^1 goes to an identity element in H^1 by the fibration E^1 if and only if l goes to an identity element in H^1 by F^1. But the fibre of F^1 is the topological groupoid G. Therefore, the element $[x,l,y]$ of R^1 goes to an identity element in H^1 by the fibration E^1 if and only if l belongs to G^1. So the fibre of the groupoid fibration $\mathrm{E}\colon \mathrm{R} \to \mathrm{H}$ is the subgroupoid of R with the following arrow space:

$$(\mathrm{X} \times_{\mathrm{r_X},\mathrm{L}^0,\mathrm{s}} \mathrm{G}^1 \times_{\mathrm{r},\mathrm{L}^0,\mathrm{r_X}} \mathrm{X})/\sim,$$

where the equivalence relation "$\sim$" is the same as in Proposition 2.7. We already know that this fibre is a topological groupoid.

4.2. Theorem. *The groupoid fibration* $\mathrm{E}\colon \mathrm{R} \to \mathrm{H}$ *constructed above gives a generalised groupoid action of* H *on* K*. Farthermore,* R *and* L *are Morita equivalent.*

Proof. we need to show the following lemma:

4.3. Lemma. *The fibre of the topological groupoid fibration* $\mathrm{E}\colon \mathrm{R} \to \mathrm{H}$ *is homeomorphic to the topological groupoid* K*.*

Proof. Let us consider any element $[x,g,y]$ in the arrow space of the fibre of the groupoid fibration $\mathrm{E}\colon \mathrm{R} \to \mathrm{H}$. We know that $\mathrm{r_X}(x) = \mathrm{s}(g)$ and $\mathrm{r_X}(y) = \mathrm{r}(g)$. Therefore $\mathrm{r_X}(g \cdot x) = \mathrm{r_X}(y)$. Thus $(g \cdot x, y)$ belongs to $\mathrm{X} \times_{\mathrm{r_X},\mathrm{G}^0,\mathrm{r_X}} \mathrm{X}$. Let ${\psi_\mathrm{K}}^{-1}(g \cdot x, y) = (g \cdot x, k)$, where ψ is the homeomorphism from Definition 2.1. It is useful to see that ${\psi_\mathrm{K}}^{-1}(g \cdot x, y) = (g \cdot x, k)$ if and only if $y = g \cdot x \cdot k$. We need this element k to construct a homeomorphism between the arrow space of the fibre of the groupoid fibration $\mathrm{E}\colon \mathrm{R} \to \mathrm{H}$ and the arrow space of the topological groupoid K. So we have the following composition:

$$\mathrm{pr}_2 \circ {\psi_\mathrm{K}}^{-1} \circ (\mathrm{m_G} \circ (\mathrm{pr}_2, \mathrm{pr}_1), \mathrm{pr}_3)\colon \mathrm{X} \times_{\mathrm{r_X},\mathrm{L}^0,\mathrm{s}} \mathrm{G}^1 \times_{\mathrm{r},\mathrm{L}^0,\mathrm{r_X}} \mathrm{X} \to \mathrm{K}^1,$$
$$(x,g,y) \mapsto (g \cdot x, y) \mapsto (g \cdot x, k) \mapsto k,$$

we need to show that this composition is an open surjection. ${\psi_\mathrm{K}}^{-1}$ is open surjection because it is a homeomorphism. $\mathrm{pr}_2\colon \mathrm{X} \times_{\mathrm{s_X},\mathrm{K}^0,\mathrm{r}} \mathrm{K}^1 \to \mathrm{K}^1$ is an open surjection because it is a pull-back of $\mathrm{s_X}\colon \mathrm{X} \to \mathrm{K}^0$, which is an open surjection by Definition 2.1.

For proving that the following map

$$(\mathrm{m_G} \circ (\mathrm{pr}_2, \mathrm{pr}_1), \mathrm{pr}_3) \colon \mathrm{X} \times_{\mathrm{r_X}, \mathrm{L}^0, \mathrm{s}} \mathrm{G}^1 \times_{\mathrm{r}, \mathrm{L}^0, \mathrm{r_X}} \mathrm{X} \to \mathrm{X} \times_{\mathrm{r_X}, \mathrm{G}^0, \mathrm{r_X}} \mathrm{X}, \quad (x, g, y) \mapsto (g \cdot x, y)$$

is an open surjection we need to check that the following diagram is a fibre product:

$$\begin{array}{ccc} \mathrm{X} \times_{\mathrm{r_X}, \mathrm{L}^0, \mathrm{s}} \mathrm{G}^1 \times_{\mathrm{r}, \mathrm{L}^0, \mathrm{r_X}} \mathrm{X} & \xrightarrow{(\mathrm{m_G} \circ (\mathrm{pr}_2, \mathrm{pr}_1), \mathrm{pr}_3)} & \mathrm{X} \times_{\mathrm{r_X}, \mathrm{K}^0, \mathrm{r_X}} \mathrm{X} \\ \Big\downarrow{\scriptstyle (\mathrm{pr}_2, \mathrm{pr}_1)} & & \Big\downarrow{\scriptstyle \mathrm{pr}_1} \\ \mathrm{G}^1 \times_{\mathrm{s}, \mathrm{G}^0, \mathrm{r_X}} \mathrm{X} & \xrightarrow{\mathrm{m_G}} & X \end{array}$$

and we will make sure that $(\mathrm{m_G} \circ (\mathrm{pr}_2, \mathrm{pr}_1), \mathrm{pr}_3)$ is an open surjection because it is a pull-back of $\mathrm{m_G} \colon \mathrm{G}^1 \times_{\mathrm{s}, \mathrm{G}^0, \mathrm{r_X}} \mathrm{X} \to \mathrm{X}$, which is an open surjection because of Remark 2.2. So the composition $\mathrm{pr}_2 \circ {\psi_\mathrm{K}}^{-1} \circ (\mathrm{m_G} \circ (\mathrm{pr}_2, \mathrm{pr}_1), \mathrm{pr}_3)$ is an open surjection.

This composition induces the map ω between the arrow space of the fibre of the groupoid fibration $\mathrm{E} \colon \mathrm{R} \to \mathrm{H}$ and the arrow space of the groupoid K. So we have

$$\omega \colon (\mathrm{X} \times_{\mathrm{r_X}, \mathrm{L}^0, \mathrm{s}} \mathrm{G}^1 \times_{\mathrm{r}, \mathrm{L}^0, \mathrm{r_X}} \mathrm{X})/\sim \to \mathrm{K}^1 \qquad [x, g, y] \mapsto k.$$

We have to show that ω is a homeomorphism. Firstly, let us show that ω is well-defined. Let $(x, g, y) \sim_{g_1, g_2} (\hat{x}, \hat{g}, \hat{y})$ and $\omega([x, g, y]) = k$, therefore, we have the following equalities:

$$x = {g_1}^{-1} \cdot \hat{x}, \quad g_2 \cdot y = \hat{y}, \quad g_2 \cdot g \cdot {g_1}^{-1} = \hat{g}, \quad g \cdot x = y \cdot k^{-1}.$$

Thus

$$\begin{aligned} \hat{y} \cdot k^{-1} &= (g_2 \cdot y) \cdot k^{-1} \\ &= g_2 \cdot (y \cdot k^{-1}) \\ &= g_2 \cdot (g \cdot x) \\ &= (g_2 \cdot g) \cdot x \\ &= (g_2 \cdot g) \cdot ({g_1}^{-1} \cdot \hat{x}) \\ &= (g_2 \cdot g \cdot {g_1}^{-1}) \cdot \hat{x} \\ &= \hat{g} \cdot \hat{x}. \end{aligned}$$

So $\hat{y} \cdot k^{-1} = \hat{g} \cdot \hat{x}$, and this means that $\omega([\hat{x}, \hat{g}, \hat{y}]) = k$. So ω is well-defined. ω is an open surjection because it is induced by an open surjection. If we prove that it is an injection, then it will be a homeomorphism automaticaly.

Let $\omega([x, g, y]) = \omega([\hat{x}, \hat{g}, \hat{y}]) = k$. So $y = g \cdot x \cdot k$ and $\hat{y} = \hat{g} \cdot \hat{x} \cdot k$. Thus

$$\begin{aligned} \mathrm{s_X}(x) &= \mathrm{r}(k) \\ &= \mathrm{s_X}(\hat{x}) \end{aligned}$$

and $(x, \hat{x}) \in \mathrm{X} \times_{\mathrm{s_X}, \mathrm{K}^0, \mathrm{s_X}} \mathrm{X}$. Because of the homeomorphism ψ_{K} there is exactly one element $g_1 \in \mathrm{G}^1$ such that $\hat{x} = g_1 \cdot x$. Also,

$$\begin{aligned} \mathrm{s_X}(y) &= \mathrm{s}(k) \\ &= \mathrm{s_X}(\hat{y}), \end{aligned}$$

hence $(y, \hat{y}) \in \mathrm{X} \times_{\mathrm{s_X}, \mathrm{K}^0, \mathrm{s_X}} \mathrm{X}$. Because of the homeomorphism ψ_{K} there is exactly one element $g_2 \in \mathrm{G}^1$ such that $\hat{y} = g_2 \cdot y$. So we have

$$\begin{aligned} \hat{g} \cdot \hat{x} \cdot k &= \hat{y} \\ &= g_2 \cdot y \\ &= g_2 \cdot (g \cdot x \cdot k) \\ &= (g_2 \cdot g) \cdot (x \cdot k) \\ &= (g_2 \cdot g) \cdot ((g_1{}^{-1} \cdot \hat{x}) \cdot k) \\ &= (g_2 \cdot g \cdot g_1{}^{-1}) \cdot \hat{x} \cdot k. \end{aligned}$$

So $\hat{g} \cdot \hat{x} \cdot k = (g_2 \cdot g \cdot g_1{}^{-1}) \cdot \hat{x} \cdot k$. Hence $\hat{g} \cdot \hat{x} = (g_2 \cdot g \cdot g_1{}^{-1}) \cdot \hat{x}$. Therefore

$$\begin{aligned} (\hat{g}, \hat{x}) &= \psi_{\mathrm{G}}{}^{-1}(\hat{x}, \hat{g} \cdot \hat{x}) \\ &= \psi_{\mathrm{G}}{}^{-1}(\hat{x}, (g_2 \cdot g \cdot g_1{}^{-1}) \cdot \hat{x}) \\ &= (g_2 \cdot g \cdot g_1{}^{-1}, \hat{x}). \end{aligned}$$

So $\hat{g} = g_2 \cdot g \cdot g_1{}^{-1}$ and therefore $(x, g, y) \sim_{g_1, g_2} (\hat{x}, \hat{g}, \hat{y})$, thus $[x, g, y] = [\hat{x}, \hat{g}, \hat{y}]$. So ω is an injection and, therefore, it is a homeomorphism.

So we have the homeomorphism ω between the arrow space of the fibre of the groupoid fibration $\mathrm{E} \colon \mathrm{R} \to \mathrm{H}$ and the arrow space of the groupoid K. We need a homeomorphism which is compatible with the groupoid structures in both groupoids. Such a homeomorphism is the composition $W = \mathrm{i} \circ \omega$ where i is the inverse map in the groupoid K. It is clear that this composition is a homeomorphism too. Let us show that it intertwines the range, source and multiplication maps.

Let $W([x, g, y]) = k$. This means that $y = g \cdot x \cdot k^{-1}$. Therefore,

$$\begin{aligned} \mathrm{s}(k) &= \mathrm{r}(k^{-1}) \\ &= \mathrm{s_X}(x) \\ &= \mathrm{s}([x, g, y]). \end{aligned}$$

So W intertwines the source map. Analogously,

$$\begin{aligned} \mathrm{r}(k) &= \mathrm{s}(k^{-1}) \\ &= \mathrm{s_X}(y) \\ &= \mathrm{r}([x, g, y]) \end{aligned}$$

and the range map is intertwined too.

Let us check the same for the multiplication map. Let $([x,g,y],[x_1,g_1,y_1])$ be any composable pair of arrows in R and $[x,g,y]\circ[x_1,g_1,y_1]=[x_1,g\circ g_2\circ g_1,y]$ where $x=g_2\cdot y_1$. Let $W([x,g,y])=k$ and $W([x_1,g_1,y_1])=k_1$. So we have the following: $y\cdot k=g\cdot x$ and $y_1\cdot \mathrm{K}^1=g_1\cdot x_1$. Therefore,

$$\begin{aligned} y\cdot(k\cdot k_1) &= (y\cdot k)\cdot k_1 \\ &= (g\cdot x)\cdot k_1 \\ &= (g\cdot(g_2\cdot y_1))\cdot k_1 \\ &= g\cdot g_2\cdot(y_1\cdot k_1) \\ &= g\cdot g_2\cdot(g_1\cdot x_1) \\ &= (g\cdot g_2\cdot g_1)\cdot x_1. \end{aligned}$$

So we have $y\cdot(k\cdot k_1)=(g\cdot g_2\cdot g_1)\cdot x_1$ and this means that $W([x_1,g\circ g_2\circ g_1,y])=k\cdot k_1$. So the multiplication map is intertwined.

Finally, we have a natural homeomorphism between the arrow space of the fibre of the groupoid fibration $\mathrm{E}\colon \mathrm{R}\to\mathrm{H}$ and the arrow space of the groupoid K and therefore, these topological groupoids are homeomorphic.

■

So because of this lemma we can say that we have a generalised groupoid action of a topological groupoid H on a topological groupoid K.

The second part of this theorem is proved in Proposition 3.1, which finishes the main theorem. ■

5. Examples

There are some interesting examples which show us how we can construct a generalised groupoid action through a groupoid fibration in some special cases.

5.1. Example. Let H be any topological groupoid and let $p\colon \mathrm{X}\to\mathrm{Y}$ be an open surjection onto a topological space Y equipped with a right H-action. Then we can construct a generalised groupoid action of H on the Čech groupoid of p in the following way. Example 2.5 shows a groupoid fibration from the transformation groupoid of this action to H with fibre Y. Also, we know that the topological groupoid Y and the Čech groupoid of p are Morita equivalent by the topological space X. The main construction in this paper gives a topological groupoid R which is Morita equivalent to the tranformation groupoid and a fibration from R to H such that the Čech groupoid of p is the fibre of this fibration. By construction the object space of R is X and for all x_1 and x_2 in X we have

$$\mathrm{Hom}_{\mathrm{R}}(x_1,x_2)=\{h\in\mathrm{H}^1\mid h\cdot p(x_1)=p(x_2)\}.$$

5.2. Example. Let H be any topological group and X be any topological space. Then we can construct a generalised groupoid action of H on the pair groupoid of X. It is easy to check that the identity map id: H → H is a groupoid fibration with fibre $\{1\}$. We know that the pair groupoid of X is Morita equivalent to $\{1\}$ by the space X. The main construction in this paper gives a topological groupoid R Morita equivalent to H with a functor from R to H which is a fibration between topological groupoids with fibre the pair groupoid of X. The object space of R is X and for all x_1 and x_2 in X we have $\mathrm{Hom}_{\mathrm{R}}(x_1, x_2) = \mathrm{H}$.

References

[1] A. Buss and R. Meyer, Iterated crossed products for groupoid fibrations. accepted, arXiv: 1604.02015, 2016.

[2] A. Buss and R. Meyer, Inverse semigroup actions on groupoids. Rocky Mountain Journal of Mathematics. accepted, arXiv: 1410.2051, 2016.

[3] V. Deaconu, A. Kumjian and B. Ramazan, Fell bundles associated to groupoid morphisms, Mathematica Scandinavica (Vol: 102, No: 2), 2008.

[4] R. Brown, Fibrations of groupoids, Journal of Algebra (vol: 15), 1970.

[5] R. Brown, G. Danesh-Naruie and J. P. L. Hardy, Topological groupoids. II. Covering morphisms and *G*-spaces, Mathematische Nachrichten (vol: 74), 1976.

A. Razmadze Mathematical Institute of Tbilisi State University,
Tamarashvili Str. 6, Tbilisi 0186, Georgia

Email: g.arabidze@gmail.com
g.arabidze@freeuni.edu.ge